

Siebert, Horst

**Book Part — Digitized Version**

## Negative externalities, environmental quality and the transformation space

**Provided in Cooperation with:**

Kiel Institute for the World Economy – Leibniz Center for Research on Global Economic Challenges

*Suggested Citation:* Siebert, Horst (1982) : Negative externalities, environmental quality and the transformation space, In: Eichhorn, Wolfgang Henn, R. Neumann, K. Shephard, R. W. (Ed.): Economic theory of natural resources, ISBN 3-7908-0274-3, Physica-Verlag, Würzburg, pp. 489-506

This Version is available at:

<https://hdl.handle.net/10419/3570>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

## Negative Externalities, Environmental Quality and the Transformation Space<sup>1</sup> )

Horst Siebert

The purpose of this paper is to analyze the properties of the transformation space of an economy with private goods, environmental quality, and negative externalities. The property of the transformation space determines the level of permissible decentralization [Laffont]. If nonconvexities prevail, only milder forms of decentralization (such as taxation instead of artificial markets) are permitted for optimal allocation results; or, as Baumol/Oates [1975, p. 103] point out, “prices may give the wrong signals — directing the economy away from the social optimum.”

Technological externalities are defined as interrelations between economic activities outside the market system. Let  $y_i$  represent the output of activity  $i$  (or an output vector) such as production quantities or utility, let  $x_i$  represent an input to activity  $i$  (or an input vector) and let  $y_j$  and  $x_j$  denote the output and input of activity  $j$ , so that

$$y_i = \pi_i(x_i, x_j, y_j). \quad (1)$$

Then, a negative externality exists if  $\partial y_i / \partial x_j < 0$  or  $\partial y_i / \partial y_j < 0$ . We may have negative externalities in production (output of activity  $i$  and inputs in activity  $j$  or output of activity  $j$ ) or in consumption (utility from consumption activity of individual  $i$  and input [or output] of consumption activity of individual  $j$ ). Furthermore, externalities may exist between consumption and production activities.

When externalities exist, the two basic propositions of welfare economics can no longer be established. This result is due to the fact that externalities cause nonconvexities to arise. In this paper we will analyze the problem to what extent the introduction of externalities will affect the convexity condition. We can show that the introduction of some aspects of externalities still allows the concavity of the transformation space. Also, the paper generates some insight into the question what type of externalities may destroy the concavity of the transformation space.

The starting point of the analysis is that in the literature on technological externalities, the technological systems were not discussed through which economic activities are

---

<sup>1</sup>) This paper was written in the Sonderforschungsbereich 5 “Allocation policy in market economies” sponsored by the Deutsche Forschungsgemeinschaft. The author acknowledges support from the Center of Energy Policy Research and the Energy Laboratory at the Massachusetts Institute of Technology. He received helpful comments to a first draft from F. Dudenhöffer, H. Gebauer, H. Meder, S. Toussaint, and two anonymous referees.

linked. Such systems are ground-water systems, river systems, meteorological systems or other natural and environmental media.<sup>2)</sup> The innovation of environmental economics consists in explicitly considering these systems and introducing intervening variables between different activities that explain the technological links. This procedure has an important advantage: By introduction intervening variables such as emissions, pollutants ambient in the environment and environmental quality we get new starting points for economic policy. For instance, we are able to fix a target in terms of environmental quality by defining the tolerable quantity of pollutants. And we are also able to indicate the policy instruments to be used such as emission taxes or pollution rights (defined as a permissible quantity of emissions) not yet known to the Pigouvian analysis (1920). Also, some type of externality may exist, and the convexity condition may not be violated.

In the following analysis, we consider an economy with two private outputs and a public good: environmental quality. Production generates pollutants, and pollutants ambient in the environment determine environmental quality. Pollutants in the environment also have a negative effect on output. Each sector or a government agency can abate pollutants. The abatement of pollutants uses up resources and consequently competes with resource use in production. In this context, the concavity of the transformation space may be destroyed by the negative effect of pollutants on production. Also, we introduce a convex emission function for each firm.

In Section I, the assumptions for our analysis are specified. In Section II, we study the properties of the transformation space. Whereas in Section II the production function is concave, we consider nonconvexities in the production function and their possible influence on the transformation space in Section III. In the appendix, we establish some of the formal properties of the transformation space.

## I. Assumptions

A 1. Output  $Q$  depends on resource input  $R$  and on the quantity of pollutants  $S$  ambient in the environment. In order to keep the model as simple as possible, we assume only one type of resource. We have declining marginal productivities of the resource and a negative impact of pollutants on output. With a higher level of pollution, the impact increases in absolute terms. In order to simplify the presentation, we assume that the production function is separable in  $R$  and  $S$ . Diagram 1 shows the production function

$$Q^i \leq F^i(R^i, S)$$

$$F_R^i > 0, F_{RR}^i < 0; F_S^i < 0, F_{SS}^i < 0; F_{RS}^i = F_{SR}^i = 0. \quad (2)$$

<sup>2)</sup> Another system linking economic activities is the social system. For instance, in a developing country a firm with high technical knowledge may have a positive effect on the labour force in a region and thus on other firms located in the area. This technological effect runs via the social system.

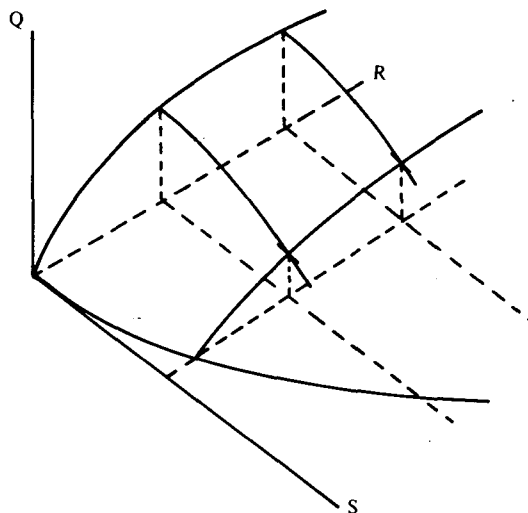


Diagram 1

The production function is concave.<sup>3)</sup> Since pollutants reduce output, increasing pollutants require more resources if a given output is to be maintained.<sup>4)</sup>

The production function 2 defines an inverse

$$R^i = \phi^i(Q^i, S) \quad (2i)$$

that can be interpreted as an input requirement function for alternative  $Q^i$ ,  $S$ . Since we make use of this function in our argument, the properties of the input requirement function are of interest to us. These properties are completely determined by the properties of the production function. Diagrammatically, the input requirement function is obtained by turning the production function around the  $S$ -axis in Diagram 1 by 90 degrees. Diagram 2a shows the input requirement function. For convenience of interpretation, the same function is shown in Diagram 2b.

<sup>3)</sup> We have  $d^2Q < 0$  since in the Hessian determinant

$$H: \begin{vmatrix} F_{RR} & F_{SR} \\ F_{RS} & F_{SS} \end{vmatrix}$$

$|H_1| < 0$  and  $|H_2| = F_{RR}F_{SS} - F_{RS}F_{SR} > 0$  for  $F_{RS} = 0$ .

This property also holds for the isoquant  $Q^i = 0$  so that the output space is limited by a convex function  $R^i = \phi^i(0, S)$ .

<sup>4)</sup> The assumption of separability implies that for a cut through the production surface for given  $S$ ,  $F_S$  remains constant for all  $R$ . Similarly  $F_R$  is constant for all  $S$ , if  $R$  is given.

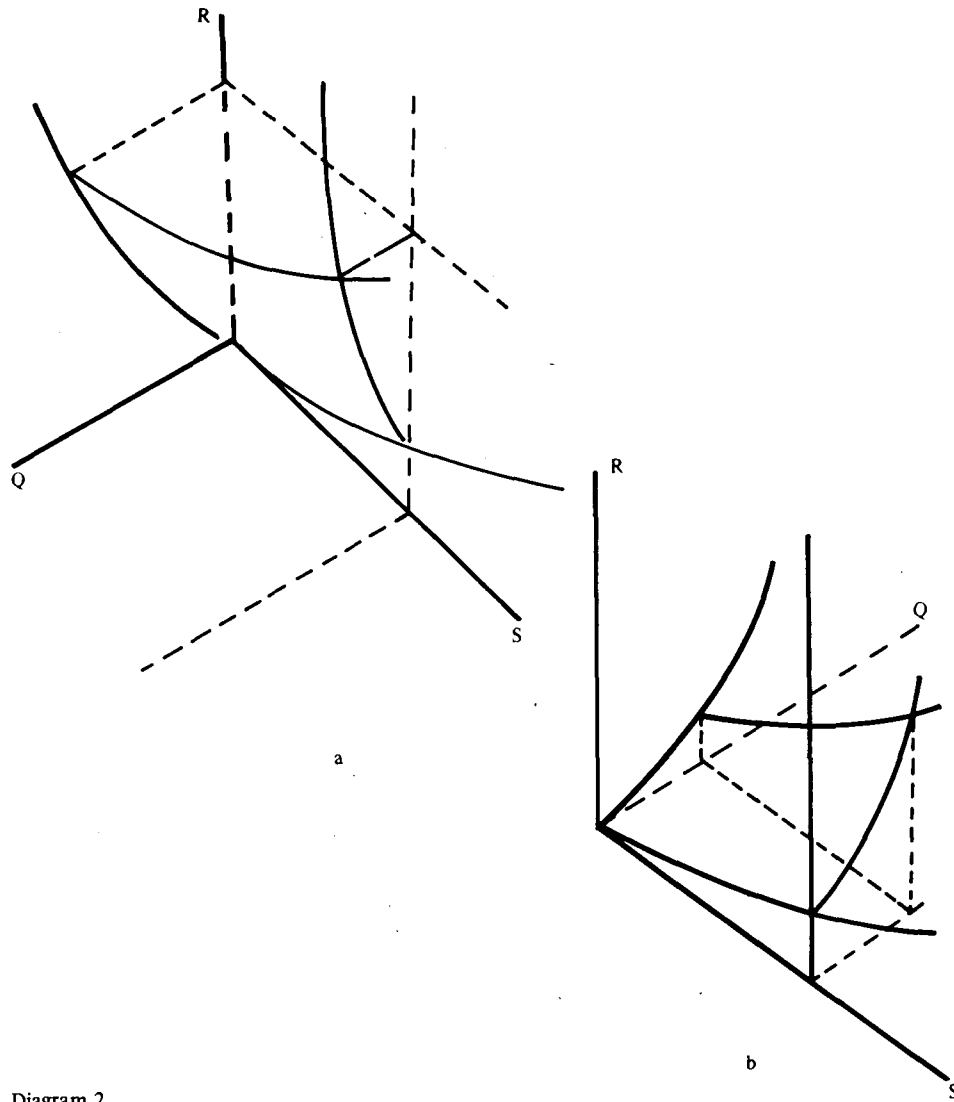


Diagram 2

The properties of the input requirement function can be determined as follows. Take the total derivative of the production function, rearrange it into

$$dR^i = \frac{1}{F_R^i} dQ^i - \frac{F_S^i}{F_R^i} dS \quad (2ii)$$

and compare it with the total derivative of Equation (2i)

$$dR^i = \phi_Q^i dQ^i + \phi_S^i dS. \quad (2iii)$$

Then, we have

$$\phi_Q^i = \frac{1}{F_R^i} > 0$$

$$\phi_S^i = -F_S^i \phi_Q^i - \frac{-F_S^i}{F_R^i} > 0.$$

Taking the second total derivative<sup>5)</sup> of the input requirement function and the production function we establish

$$\phi_{QQ}^i = -\frac{F_{RR}^i}{(F_R^i)^3} > 0$$

$$\phi_{SS}^i = -\frac{F_{SS}^i (F_R^i)^2 + F_{RR}^i (F_S^i)^2}{(F_R^i)^3} > 0$$

$$\phi_{QS}^i = \frac{F_{RR}^i F_S^i}{(F_R^i)^3} > 0.$$

A 2. Production generates pollutants  $S^{pi}$  as a joint product. For purposes of simplification, there is only one type of pollutants. It is assumed that pollutants emitted rise proportionally or progressively with output

$$S^{pi} = H^i(Q^i) \text{ with } H_Q^i > 0, H_{QQ}^i \geq 0. \quad (3)$$

A 3. Resources may also be used for abatement purposes. Let  $S^r$  indicate the quantity of pollutants reduced in the economy. The abatement function is given by

$$S^r \leq F^r(R^r) \text{ with } F_R^r > 0, F_{RR}^r < 0. \quad (4)$$

<sup>5)</sup> The total derivative of Equation (2iii) is

$$d^2R = \phi_{QQ} dQ^2 + \phi_{SS} dS^2 + 2\phi_{QS} dQdS. \quad (I)$$

The total derivative of Equation (2ii) is (with  $F_{RS} = F_{SR} = 0$ )

$$d^2R = -\frac{F_{RR}}{F_R^2} dRdQ - \frac{F_{SS} F_R}{F_R^2} ds^2 + \frac{F_{RR} F_S}{F_R^2} dRdS.$$

Substituting Equation (2iii) for  $dR$  we have

$$d^2R = -\frac{F_{RR}}{F_R^3} dQ^2 - \frac{(F_{SS} F_R^2 + F_{RR} F_S^2)}{F_R^3} ds^2 + 2 \frac{F_{RR} F_S}{F_R^3} dQdS. \quad (II)$$

From Equations (I) and (II), the properties for  $\phi_{QQ}$ ,  $\phi_{SS}$  and  $\phi_{QS}$  follow.

The abatement function describes a technology that prevents pollutants from entering the environment. Alternatively, we can introduce an index  $i$  and assume a technology to reduce pollutants that is specific to each sector.

A 4. Net emissions or pollutants ambient in the environment are defined as emissions produced minus emissions abated. A diffusion function is not explicitly introduced.

$$S = \sum S^{pi} - S^r. \quad (5)$$

A 5. The resource can be used for production and abatement. Resource endowment is given.

$$R^1 + R^2 + R^r \leq \bar{R}. \quad (6)$$

A 6. The public good: environmental quality  $U$ , is determined by the quantity of pollutants ambient in the environment ( $S$ )

$$U = G(S) \text{ with } G' < 0, G'' \leq 0. \quad (7)$$

Note that equation (7) may be understood as a linear transformation ( $G'' = 0$ ), but it may also be understood as damage function in a physical sense ( $G'' < 0$ ). If equation (7) is interpreted as a linear transformation, then environmental quality is defined in terms of pollutants ambient in the environment. If equation (7) is interpreted as a damage function, the stock of pollutants negatively affects environmental quality, that is, the stock of pollutants influences such variables as the height of trees, abundance of natural systems, ecological equilibria etc.

## II. Properties of the Transformation Space

Inserting Equations (2i), (3), (4), (5) and (6) into Equation (7) we have

$$U = G[\sum H^i(Q^i) - F^r\{\bar{R} - \sum \phi^i(Q^i, G^{-1}(U))\}]. \quad (8)$$

If we leave the damage function (7) out of consideration, Equation (8) reduces to

$$S = \sum H^i(Q^i) - F^r\{\bar{R} - \sum \phi^i(Q^i, S)\}. \quad (8')$$

Equations (8) and (8') describe two different versions of the transformation space. We now discuss the properties of these equations. We consider three different cases, namely i)  $F_S = 0$  and  $G'' < 0$ , ii)  $F_S < 0$  and  $G'' < 0$  and iii)  $F_S < 0$  and  $G'' = 0$ .

*Case 1.* As a frame of reference, consider a case where  $F_S^i = 0$  and  $G'' < 0$ . Then the transformation space has the property shown in Diagram 3 [Siebert, 1978, 1981].

At a zero production in both sectors, the maximal environmental quality (0A) is reached, so to speak the natural original condition. Let  $Q^2 = 0$  and expand production of commodity 1. Then one can imagine such a resource allocation ( $R^1, R^r$ ) at which all pollutants occurring in the production of 1 are abated (distance AG). Analogously, AH indicates those production quantities of commodity 2 for  $Q^1 = 0$  at which the environmental quality remains at a maximum. Except for the curve GH, the horizontal ceiling

represents a situation with maximal environmental quality and underemployment.

Expand production of commodity 1 at point  $G$  for  $Q^2 = 0$  by one unit. Then the quantity of emissions increases progressively due to  $H_{QQ}^1 > 0$ . Because environmental quality decreases overproportionally with emissions, environmental quality has to fall overproportionally as consequence of the expanded production of commodity 1. With an increase in production of commodity 1, additional resources are used in production that have to be withdrawn from abatement. Therefore, the quantity of emissions abated falls (and environmental quality declines). We even know that due to every further input unit withdrawn from abatement, the emissions not abated increase overproportionally. This is explained by the decreasing marginal productivities in abatement. Finally, the law of declining marginal returns demands that every further unit of commodity 1 produced requires an increasing resource input. Consequently, for a movement from  $G$  to  $B$  the quantity of pollutants has to increase progressively because inputs are reallocated from abatement to the production of commodity 1, and the environmental quality has to decrease progressively. The curve  $GB$  is concave. A similar reasoning holds for the curve  $HC$  and other cuts through the transformation space. We can establish that the transformation space is concave for  $F_S^i = 0$  [Siebert, 1978, 1981]. Note that the projection of  $BC$  into the  $Q^1 - Q^2$  plane is the traditional transformation curve  $Q^1 = \pi(Q^2)$  without taking into account environmental quality.

**Proposition 1:** If there is no negative productivity effect of the stock of pollutants on production, i.e.,  $F_S^i = 0$ , the transformation space is concave for  $G'' < 0$  as well as  $G'' = 0$ .

Compare this result with an intuitive reasoning for a Pigouvian type concept of externalities between production and consumption as specified in equation (1). In a Pigouvian world one would expect prima vista that the externality destroys the convexity conditions. This, however, does not hold.

**Case 2.** Consider now the case where pollutants ambient in the environment affect output negatively, i.e.,  $F_S^i < 0$  and  $G'' < 0$ . Note that point  $G$  is identical to Diagram 3. This is due to the fact that at  $G$ , the level of pollution is zero and there cannot be a negative effect on output. Moving from  $G$  and increasing  $Q^1$  (Diagram 4), output is reduced by the negative productivity effect so that the transformation space [ $F_S^i < 0$ ] lies inside the transformation space [ $F_S^i = 0$ ]. With a reduction of environmental quality (or an increase in pollutants), the curve  $G\tilde{B}$  (diagram 4) will have a stronger curvature than curve  $GB$  (Diagram 3). If the negative effect of pollutants is large enough, the transformation space will bend inward. The slope  $\partial U / \partial Q^i$  becomes positive due to a stronger impact of pollutants on output. From Equation (3) in the appendix we have

$$\frac{\partial U}{\partial Q^i} \leq 0 \Leftrightarrow \phi_S^1 + \phi_S^2 \leq \frac{1}{F_R^*}. \quad (9)$$

On the right side of Equation (9), the term  $1/F_R^*$  specifies the input requirement for reducing one unit of pollutants. Note that from  $S^r = F^r(R^r)$  and the inverse  $R^r = F^{r-1}(S^r)$ , we have  $dR^r/dS = 1/F_R^r$  (inverse function rule). Consequently,  $1/F_R^r$  denotes the input requirement for reducing one unit of pollutants. On the left side of



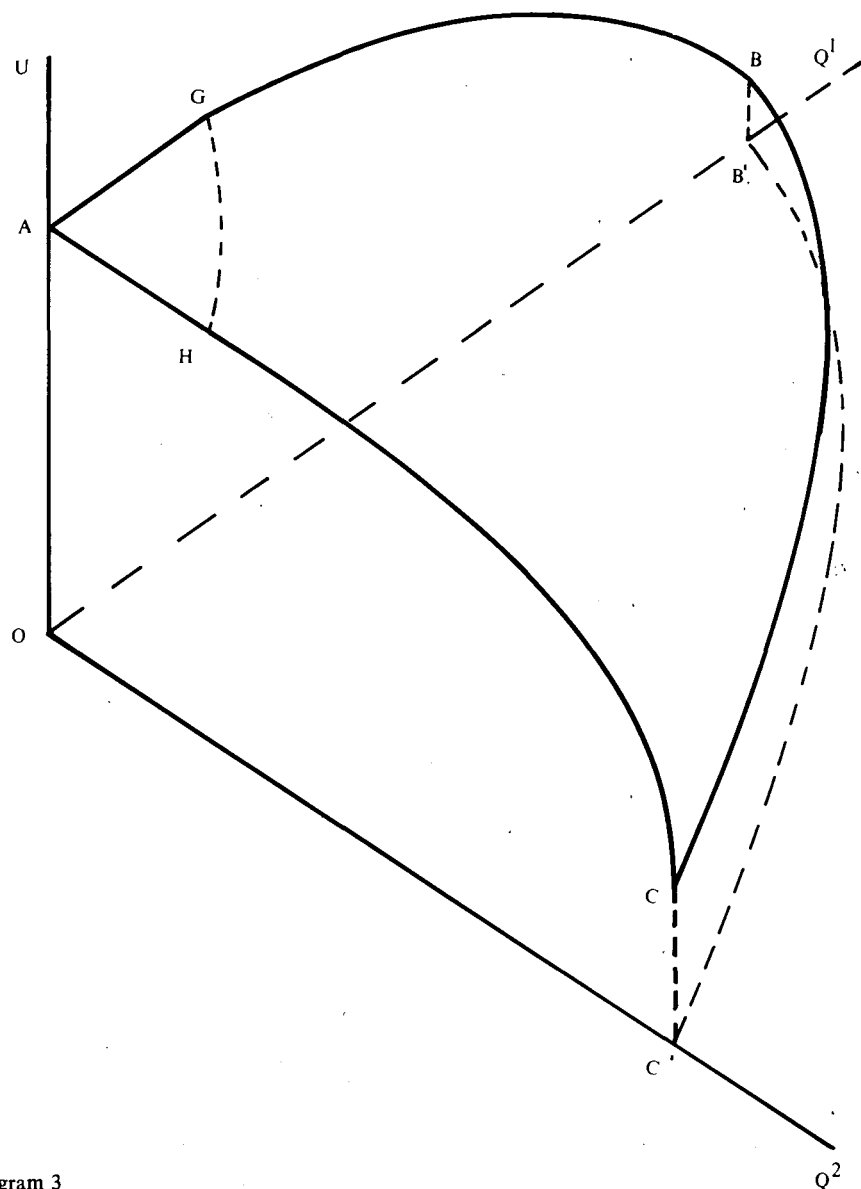


Diagram 3

Equation (9), the term  $\phi_S^1 + \phi_S^2$  specifies the inputs required to compensate for the negative productivity effect of one unit of pollutants on output ( $\phi_S^1 + \phi_S^2$ ). If resources are less productive in abating one unit of pollutants than in compensating its negative effect, the transformation space has a negative slope. If, however, it requires more resources to compensate the negative effect of one unit of pollutants in production than it requires resources to abate one unit of pollutants, the slope of the transformation space is positive.

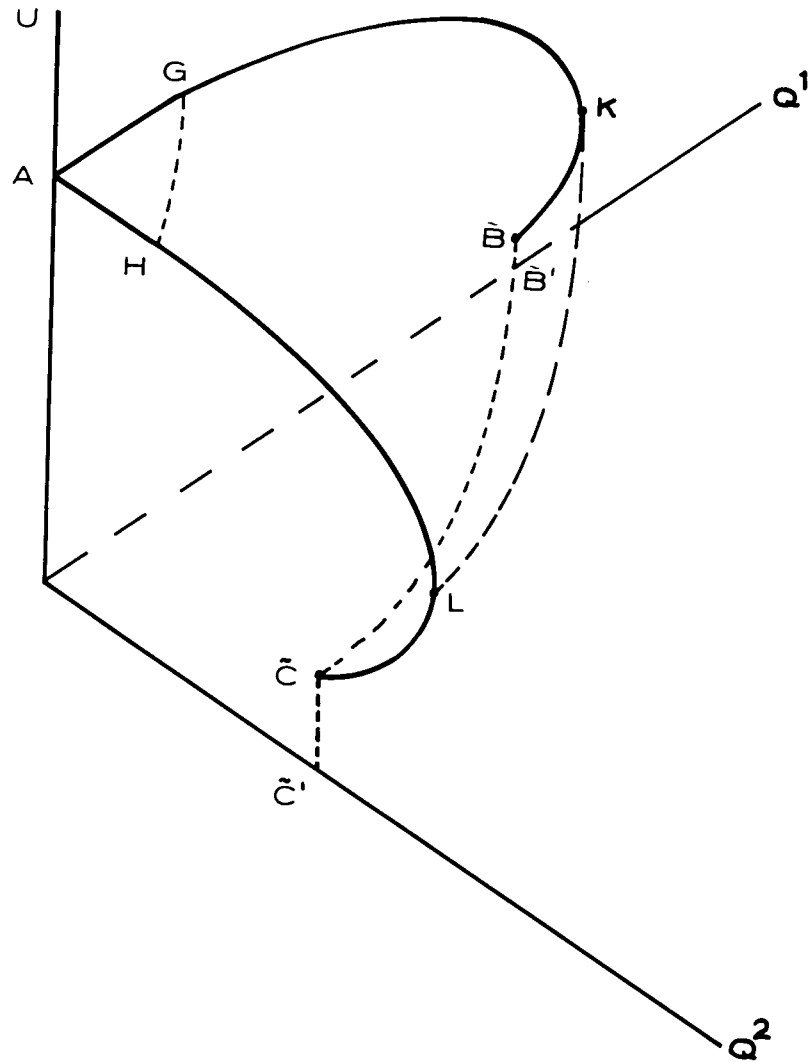


Diagram 4

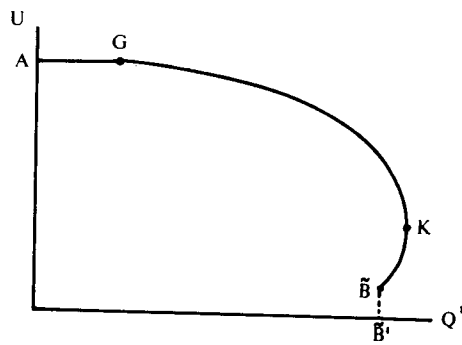


Diagram 5

Diagram 5 represents a cut through the transformation space for given  $Q_2$ . In the section  $\tilde{BK}$  of the transformation curve,  $\partial U/\partial Q^1 > 0$ . This means that in this situation, we can increase output and environmental quality. Alternatively we can state that there are no costs of environmental policy, but only gains. This argument was established by *Pethig* [1977]. Another interpretation is that the area  $\tilde{BK}$  is inefficient. By withdrawing resources from production, the stock of pollutants can be reduced, and this reduction leads to an increase of output.

Note that the curve  $\tilde{BC}$  in Diagram 4 or its projection into the  $Q_1$ - $Q_2$ -plane is the traditional transformation curve. It is defined for  $R^r = 0$ , i.e., no resources are used in abatement. It can be established that this curve is concave.<sup>6)</sup>

We also can establish that the curve  $\bar{U} = \pi(Q^1, Q^2)$  is concave. This can be seen from Equations (8i) and (8ii) in the appendix. Another interesting problem would be to determine the property of the curve  $KL$  in Diagram 4 which is defined by  $\partial U/\partial Q_i = \infty$  or by

$$\begin{aligned} \phi_S^1 [Q^1, G^{-1}(U)] + \phi_S^2 [Q^2, G^{-1}(U)] = \\ = \frac{1}{F_R^r [\bar{R} - \phi^1(Q^1, G^{-1}(U)) - \phi^2(Q^2, G^{-1}(U))]} \end{aligned} \quad (10)$$

<sup>6)</sup> Let  $\rho(Q^1, Q^2) = \phi^1(Q^1, S) + \phi^2(Q^2, S)$  where  $S = \Sigma H^i(Q^i)$ . Then it can be shown that the set  $Z = \{(Q^1, Q^2): \rho(Q^1, Q^2) \leq \bar{R}\}$  is convex. I owe this suggestion independently to S. Toussaint and an anonymous referee.

Note that the discussion of the first and second derivative of the  $\tilde{BC}$  curve, however, only can establish the concavity under a specific condition. From Equation (6) in the appendix we have that the marginal rate of transformation is negative. From Equation (7ii) in the appendix

$$\frac{d^2 Q^1}{dQ^2{}^2} < 0 = \begin{cases} H_Q^1 F_R^1 > H_Q^2 F_R^2 & \text{i)} \\ \alpha \phi_{SQ}^1 F_R^1 \geq \phi_{SQ}^2 F_R^2 & \text{ii)} \end{cases}$$

with  $\alpha < 1$ .  $\alpha$  is defined in the appendix.

Concavity can be established by the usual second derivative argument if the pollution-intensive sector experiences a stronger negative productivity effect relative to the other sector. Note that condition i) compares the relative pollution-intensities of both sectors. Condition ii) can be interpreted as

$$\alpha \frac{\partial^2 R^1}{\partial S \partial Q^1} \frac{\partial Q^1}{\partial R^1} > \frac{\partial^2 R^2}{\partial S \partial Q^2} \frac{\partial Q^2}{\partial R^2}.$$

This expression tells us how many resources are needed to compensate the negative productivity effect arising from using one unit of the resource in production. The condition states that the pollution-intensive Sector 1 needs more resources to compensate the negative productivity effects of pollutants (per unit of resources used in production).

According to this approach, the transformation curve  $\tilde{BC}$  will be concave if the pollution-intensive sector experiences a strong negative productivity effect. This means, that the pollution-intensive sector hurts itself. If, however, Sector 1 is the pollution-intensive sector and if Sector 2 experiences a high negative productivity effect of pollutants,  $d^2 Q^1/dQ^2{}^2 < 0$  cannot be established. This result points into the direction of the analysis of *Baumol/Oates* [1975]. They can show in a different model that if a parameter denoting the intensity of a negative externality is strong enough, the transformation space may not be concave.

From the discussion in section 1. iii) of the appendix we cannot establish that the transformation space is concave. Also, if we look at a cut  $GK\tilde{B}$  in Diagram 4 through the transformation space we cannot establish a negative second derivative for the section  $K\tilde{B}$  (compare the discussion of Equation (4) in the appendix). Consequently, we have:

*Proposition 2.* With a negative productivity effect ( $F_S^i < 0$ ) and a negative effect on environmental quality ( $G'' < 0$ ), the concavity of the transformation space cannot be established. However, a truncated part of the transformation space is concave.

It is apparent that the relevance of nonconvexities depends on the magnitude of  $|F_S|$ . If  $|F_S|$  is relatively small, a large part of the transformation space is concave.

*Case 3.* If we only consider the set  $X := \{Q^1, Q^2, S\}$ , then we can establish that the set  $X$  is convex (compare the discussion in section 1.i) of the appendix. The set  $X$  is not influenced by the damage function  $G$  and its properties. We can also show that the set  $Y := \{Q^1, Q^2, U\}$  is convex if  $G$  is interpreted as a linear relationship, i.e., if  $G'' = 0$  (see section 1. ii) of the appendix.

*Proposition 3.* With a negative productivity effect ( $F_S^i < 0$ ), the transformation space is concave for  $G'' = 0$ .

All three propositions can be summarized in the following statement.

*Proposition 4.* The concavity of the transformation space can be established if the negative effect of the stock of pollutants relates to the private goods or the public goods only. It cannot be established, if the negative productivity effects influences both private production and the public good environmental quality.

Note that in this section we have assumed that all individual functions in the model are concave so that the model is defined by a set of concave functions. Yet, the overall relationship specified by this set of relationships produces convexity only under special conditions. Compared to a world with a more intuitive Pigouvian interpretation of externalities such as in equation (1), we can show that some type of externalities do not destroy the convexity condition.

### III. Nonconvexities in the Production Function

In Equation (2) it was assumed that  $F_{SS}^i < 0$  so that the production function is concave. This assumption implies that the negative productivity effect of additional pollutants increases (in absolute terms). It may be more realistic that with an increasing stock of pollutants,  $F_S^i$  decreases in absolute terms. The first unit of pollutants has done already such a damage that the damage of an additional unit of pollutants will be smaller. This assumption implies  $F_{SS}^i > 0$ , i.e., the production function shown in Diagram 1 is no longer concave (compare Diagram 6).

From

$$\phi_{SS} = -\frac{F_{SS}F_R^2 + F_{RR}F_S^2}{F_R^3} \geq 0,$$

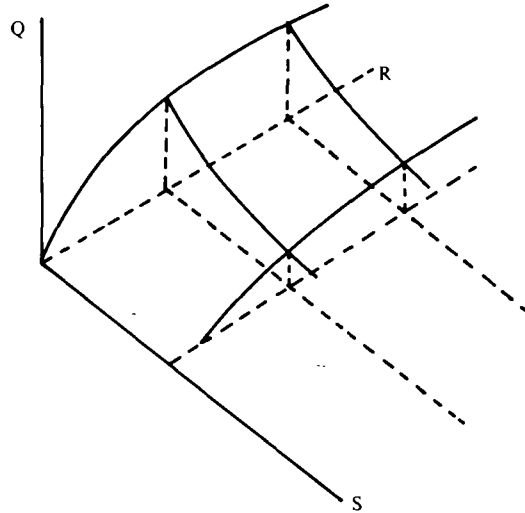


Diagram 6

we can see that  $\partial^2 U / \partial Q^2 > 0$  (Equation (4) in the appendix) cannot be ruled out. In Diagram 7 the transformation curve is shown for the one-commodity case, i.e., for  $Q^2 = 0$ . The curve  $GB$  illustrates the case  $F_S = 0$ . If  $F_S < 0$  and  $F_{SS} > 0$ , we cannot rule out that the transformation curve  $\tilde{GB}$  is convex. This result seems to be consistent with the analysis of nonconvexities by *Baumol/Oates* [1975]. Note that  $|F_S|$  is relatively large for a small stock of pollutants and becomes smaller for a larger quantity of pollutants. Consequently, nonconvexities may arise relatively early in the game. Therefore, the idea to truncate a concave section of the transformation curve may not be too helpful for the analysis [compare assumption A 2.5 in *Pethig*, 1979].

The transformation curve may have the form shown in Diagram 7. Whereas in the case of a concave production function we can establish the concavity of the transformation curve  $\tilde{BC}$  for  $R^r = 0$  (no abatement), we now cannot establish the concavity of  $\tilde{BC}$  (compare equation (7i) in the appendix).

Note that from Equations (8i) and (8ii) in the appendix we have that a cut through the transformation space for  $\bar{U}$  or  $\bar{S}$  is concave, even if  $F_{SS} > 0$ .

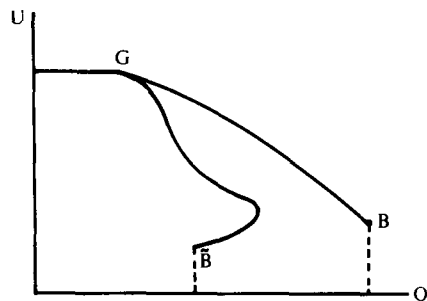


Diagram 7

The result of this section may be summarized in

*Proposition 5.* If the production function is not concave, the concavity of the transformation space cannot be established. Only a truncated part of the production function is concave. But this truncated part may be relatively small since non-convexities may arise “early in the game”.

#### IV. Summary

The paper shows that the property of concavity of the transformation space varies with the assumptions on the negative effect of the stock of pollutants. The propositions discussed are summarized in Table 1.

Proposition	Production function ( $i = 1, 2$ )	Damage function	$X := \{Q^1, Q^2, S\}$ <sup>7)</sup>	$U = \xi(Q^1, Q^2)$ <sup>8)</sup>
1.	Concave; $F_S^i = 0$	$G'' < 0$	Convex	Concave
2./3.	Concave; $F_S^i < 0$	$G'' < 0$	Convex	Not concave Truncated part concave
		$G'' = 0$	Convex	Concave
5.	Not concave; $F_S^i < 0$	$G'' < 0$	Not convex	Not concave Truncated part concave

Tab. 1

If the production function is not concave, the transformation space will not be concave. Concavity in the individual functions of the model is not sufficient to establish the concavity of the transformation space. If the stock of pollutants has a negative effect on private goods only or on the public good only, concavity can be established. If the negative effect influences both private and public goods, concavity cannot be shown.<sup>9)</sup>

#### Appendix

1. i) It can be shown<sup>10)</sup> that set  $X := \{Q^1, Q^2, S\}$  is convex. Define the function  $\psi$
- $$\psi(Q^1, Q^2, S) = \phi^1(Q^1, S) + \phi^2(Q^2, S) + F^{r-1}(H^1(Q^1) + H^2(Q^2) - S)$$

<sup>7)</sup> The set  $X$  is defined in the appendix.

<sup>8)</sup> Function  $\xi$  is implicitly defined in equation (8).

<sup>9)</sup> We have not analyzed the case in which  $G'' < 0$ ,  $F_S^i < 0$  and  $F_S^j = 0$  for  $i \neq j$ .

<sup>10)</sup> I owe this proof to S. Toussaint.

and define the set

$$X := \{(Q^1, Q^2, S): \psi(Q^1, Q^2, S) \leq \bar{R}\}. \quad (1)$$

In order to show that  $X$  is a convex set we establish the convexity of the function  $\psi$ .

Since  $\phi^i$  is convex, we have

$$\phi^i[\lambda Q^i + (1 - \lambda) \tilde{Q}^i, \lambda S + (1 - \lambda) \tilde{S}] \leq \lambda \phi^i(Q^i, S) + (1 - \lambda) \phi^i(\tilde{Q}^i, \tilde{S}).$$

Since  $H^i$  is convex, we have

$$H^i(\lambda Q^i + (1 - \lambda) \tilde{Q}^i) \leq \lambda H^i(Q^i) + (1 - \lambda) H^i(\tilde{Q}^i).$$

Due to  $d(F^r)^{-1}/dS^r = 1/F_R^r > 0$  and  $d^2(F^r)^{-1}/dS^{r^2} = -F_{RR}^r/(F_R^r)^3 > 0$ , we have

$$\begin{aligned} (F^r)^{-1} [\sum_i H^i(\lambda Q^i + (1 - \lambda) \tilde{Q}^i) - (\lambda S + (1 - \lambda) \tilde{S})] &\leq \\ (F^r)^{-1} [\sum_i (\lambda H^i(Q^i) + (1 - \lambda) H^i(\tilde{Q}^i)) - (\lambda S + (1 - \lambda) \tilde{S})] &= \\ (F^r)^{-1} [\lambda (\sum_i H^i(Q^i) - S) + (1 - \lambda) (\sum_i H^i(\tilde{Q}^i) - \tilde{S})] &\leq \\ \lambda (F^r)^{-1} (\sum_i H^i(Q^i) - S) + (1 - \lambda) (F^r)^{-1} (\sum_i H^i(\tilde{Q}^i) - \tilde{S}). \end{aligned}$$

It follows that

$$\begin{aligned} \phi[\lambda(Q^1, Q^2, S) + (1 - \lambda)(\tilde{Q}^1, \tilde{Q}^2, \tilde{S})] &\leq \\ \lambda \psi(Q^1, Q^2, S) + (1 - \lambda) \psi(\tilde{Q}^1, \tilde{Q}^2, \tilde{S}) &\leq \bar{R} \\ \text{if } \psi(Q^1, Q^2, S) \leq \bar{R} \text{ and } \psi(\tilde{Q}^1, \tilde{Q}^2, \tilde{S}) \leq \bar{R}. \end{aligned}$$

Consequently,  $X$  is convex.

ii) If the damage function is linear, we have

$$\begin{aligned} G^{-1}(\lambda U + (1 - \lambda) \tilde{U}) &= \lambda G^{-1}(U) + (1 - \lambda) G^{-1}(\tilde{U}) = \lambda S + (1 - \lambda) \tilde{S} \\ \text{if } S &= G^{-1}(U) \text{ and } \tilde{S} = G^{-1}(\tilde{U}). \end{aligned}$$

Therefore the set

$$Y := \{(Q^1, Q^2, U): \psi(Q^1, Q^2, S) \leq \bar{R}\} \quad (2)$$

is convex, if  $G(S)$  is linear.

iii) If  $G'' < 0$ , the convexity of  $Y$  cannot be established. Then

$(d^2(G^{-1}))/dU^2 = -G''/(G')^3 < 0$  and  $G^{-1}(U)$  is concave. That implies that

$$G^{-1}(\lambda U + (1 - \lambda) \tilde{U}) \geq \lambda G^{-1}(U) + (1 - \lambda) G^{-1}(\tilde{U}) \text{ and}$$

$$\phi^i[\lambda Q^i + (1 - \lambda) \tilde{Q}^i, G^{-1}(\lambda U + (1 - \lambda) \tilde{U})] \geq$$

$$\phi^i[\lambda Q^i + (1 - \lambda) \tilde{Q}^i, \lambda G^{-1}(U) + (1 - \lambda) G^{-1}(\tilde{U})].$$

Thus,  $\phi^i(Q^i, G^{-1}(U))$  is no longer a convex function of  $Q^i$  and  $U$ .

2. An alternative approach is to analyze whether for Equation 8 we can establish that  $d^2 U < 0$ . For this we have to show that the Hessian matrix is negative definite, i.e., that  $|H_1| < 0$  and  $|H_2| > 0$ . The result  $|H_2| > 0$  cannot be established since the expression is too complex. However, some results allow an economic interpretation. We have

$$\frac{\partial U}{\partial Q^1} = \frac{-G' (H_Q^1 + F_R^r \phi_Q^1)}{F_R^r (\phi_S^1 + \phi_S^2) - 1} = G' \frac{dS}{dQ^1} = -\frac{A_1}{B} \quad (3)$$

with  $A_i = G' (H_Q^i + F_R^r \phi_Q^i) < 0$

$$B = F_R^r (\phi_S^1 + \phi_S^2) - 1$$

$$\begin{aligned} \frac{\partial^2 U}{\partial Q^{1^2}} = & -\frac{1}{B^2} \left\{ -BG' F_{RR}^r \left[ \frac{A_1}{BG'} (\phi_S^1 + \phi_S^2) - \phi_Q^1 \right]^2 \right. \\ & + \frac{A_1^2}{G'} \left[ \frac{-G''}{G'} + \frac{F_R^r (\phi_{SS}^1 + \phi_{SS}^2)}{B} \right] \\ & \left. + BG' (H_{QQ}^1 + F_R^r \phi_{QQ}^1) \right\}. \end{aligned} \quad (4)$$

Equations (3) and (4) define the properties of the curve  $GK\tilde{B}$  and  $HL\tilde{C}$  in Diagram 4.

From Equation (3) we have

$$\frac{\partial U}{\partial Q^1} < 0 \Leftrightarrow B < 0.$$

In this case, Equation (4) yields  $(\partial^2 U)/(\partial Q^{1^2}) < 0$ . If, however,  $B > 0$ ,  $(\partial^2 U)/(\partial Q^{1^2}) > 0$  cannot be ruled out, i.e., beneath the curve  $KL$  in Diagram 4 non-convexities may prevail.

3. The curve  $\tilde{B}\tilde{C}$  or its projection onto the  $Q_1$ - $Q_2$ -plane (Diagram 4) is the traditional transformation curve. This curve is characterized by the property that no resources are used for abatement so that  $R^r = 0$ . Therefore we have

$$\bar{R} = R^1 + R^2$$

$$\bar{R} = \phi^1 [Q^1, H^1(Q^1) + H^2(Q^2)] + \phi^2 [Q^2, H^1(Q^1) + H^2(Q^2)] \quad (5)$$

$$\frac{dQ^1}{dQ^2} = -\frac{\phi_Q^2 + H_Q^2 (\phi_S^1 + \phi_S^2)}{\phi_Q^1 + H_Q^1 (\phi_S^1 + \phi_S^2)} < 0 \quad (6)$$

$$\begin{aligned} \frac{d^2 Q^1}{dQ^{2^2}} = & -\frac{1}{(\phi_Q^1 + aH_Q^1)} \left\{ \phi_{QQ}^2 + b\phi_{QS}^2 + aH_{QQ}^2 + eH_Q^2 \right. \\ & \left. - \frac{(\phi_Q^2 + aH_Q^2)}{(\phi_Q^1 + aH_Q^1)} \left( \phi_{QQ}^1 \frac{dQ^1}{dQ^2} + b\phi_{QS}^1 + aH_{QQ}^1 \frac{dQ^1}{dQ^2} + eH_Q^1 \right) \right\} \end{aligned} \quad (7)$$



where

$$a = \phi_S^1 + \phi_S^2 > 0$$

$$b = H_Q^1 \frac{dQ^1}{dQ^2} + H_Q^2$$

$$c = \phi_{QS}^2 + \phi_{QS}^1 \frac{dQ^1}{dQ^2}$$

$$e = \phi_{SQ}^1 \frac{dQ^1}{dQ^2} + \phi_{SQ}^2 + b (\phi_{SS}^1 + \phi_{SS}^2).$$

Define

$$n = \phi_Q^1 + aH_Q^1 > 0.$$

Then, Equation (7) can be simplified into

$$\begin{aligned} \frac{d^2 Q^1}{dQ^2} = & -\frac{1}{n} \left\{ \phi_{QQ}^1 \left( \frac{dQ^1}{dQ^2} \right)^2 + \phi_{QQ}^2 + a \left[ H_{QQ}^1 \left( \frac{dQ^1}{dQ^2} \right)^2 + H_{QQ}^2 \right] \right. \\ & \left. + 2bc + b^2 (\phi_{SS}^1 + \phi_{SS}^2) \right\}. \end{aligned} \quad (7i)$$

We have the following condition

$$\frac{\partial^2 Q^1}{\partial Q^2} < 0: bc \geq 0 \text{ or } \text{sgn } b = \text{sgn } c \text{ or } b = 0 \text{ or } c = 0. \quad (7ii)$$

Equation (7ii) is satisfied if conditions (7iii) and (7iv) are given

$$b = H_Q^1 \frac{dQ^1}{dQ^2} + H_Q^2 < 0 \text{ if } H_Q^1 F_R^1 > H_Q^2 F_R^2 \quad (7iii)$$

$$c = \phi_{QS}^1 \frac{dQ^1}{dQ^2} + \phi_{QS}^2 < 0. \quad (7iv)$$

Equation (7iv) is given, if

$$\frac{\phi_{SQ}^2}{\phi_{SQ}^1} = \frac{\phi_{QS}^2}{\phi_{QS}^1} < -\frac{dQ^1}{dQ^2} = \frac{\phi_Q^2 + H_Q^2 (\phi_S^1 + \phi_S^2)}{\phi_Q^1 + H_Q^1 (\phi_S^1 + \phi_S^2)}.$$

Define

$$\alpha = \frac{1 + H_Q^2 F_R^2 (\phi_S^1 + \phi_S^2)}{1 + H_Q^1 F_R^1 (\phi_S^1 + \phi_S^2)}$$

with  $\alpha < 1$  if  $H_Q^1 F_R^1 > H_Q^2 F_R^2$ .

The condition (7iv) is given if

$$\frac{\phi_{SQ}^2}{\phi_{SQ}^1} < \alpha \frac{F_R^1}{F_R^2} \quad (7v)$$

or

$$\frac{\partial^2 R_2}{\partial S \partial Q_2} \frac{\partial Q_2}{\partial R_2} = \phi_{SQ}^2 F_R^2 < \alpha \frac{\partial^2 R_1}{\partial S \partial Q_1} \frac{\partial Q_1}{\partial R_1} = \alpha \phi_{SQ}^1 F_R^1. \quad (7vi)$$

4. Consider  $\bar{U} := \text{constant}$  so that due to  $S = G^{-1}(U)$  also  $S$  is constant. Then we have from

$$\phi^1(Q^1, \bar{S}) + \phi^2(Q^2, \bar{S}) + (F')^{-1}[H^1(Q^1) + H^2(Q^2) - \bar{S}] = \bar{R} \quad (8)$$

that

$$\frac{dQ^1}{dQ^2} = - \frac{H_Q^2 + F_R' \phi_Q^2}{H_Q^1 + F_R' \phi_Q^1} < 0 \quad (8i)$$

$$\begin{aligned} \frac{d^2 Q^1}{dQ^2{}^2} = & - \frac{1}{(H_Q^1 + F_R' \phi_Q^1)} \left\{ H_{QQ}^1 \left( \frac{dQ^1}{dQ^2} \right)^2 + H_{QQ}^2 \right. \\ & \left. - F_{RR}' \left[ \phi_Q^2 - \phi_Q^1 \frac{dQ^1}{dQ^2} \right]^2 \right. \\ & \left. + F_R' \left( \phi_{QQ}^1 \left( \frac{dQ^1}{dQ^2} \right)^2 + \phi_{QQ}^2 \right) \right\} < 0. \end{aligned} \quad (8ii)$$

## References

- Baumol, W.J.*: On Taxation and the Control of Externalities. *American Economic Review* 62, 1972, 307–322.
- Baumol, W.J., and D.F. Bradford*: Detrimental Externalities and Nonconvexity of the Production Set. *Economica* 39, 1972, 160–176.
- Baumol, W.J., and W.E. Oates*: The Theory of Environmental Policy. Englewood Cliffs N.J., 1975.
- Laffont, J.J.*: Decentralization with Externalities. *European Economic Review* 7, 1976, 359–375.
- Pearson, Ch.*: International Trade and Environmental Controls: Comment. *Weltwirtschaftliches Archiv* 111, 1975, 562–564.
- Pethig, R.*: Die gesamtwirtschaftlichen Kosten der Umweltpolitik. *Zeitschrift für die gesamte Staatswissenschaft* 133, 1977, 322–342.
- : Umweltökonomische Allokation mit Emissionssteuern. Tübingen 1979.
- Pigou, A.C.*: The Economics of Welfare. London 1920.
- Rockafeller, R.T.*: Convex Analysis. Princeton N.J., 1970.
- Schlieper, U.*: Externe Effekte, Handwörterbuch der Wirtschaftswissenschaften zugleich: Neuauflage des Handwörterbuches für Sozialwissenschaften, Bd. 2, Stuttgart–New York; Tübingen; Göttingen–Zürich, 1980, 254–530.

- Siebert, H.*: Externalities, Environmental Quality and Allocation. *Zeitschrift für Wirtschafts- und Sozialwissenschaften* 2, 1975a, 17–32.
- : Resource Withdrawal, Productivity Effect and Environmental Policy: Comment. *Weltwirtschaftliches Archiv* 111, 1975b, 569–572.
- : *Ökonomische Theorie der Umwelt*. Tübingen 1978.
- : *Economic Theory of Environmental Allocation*. To be published in: *The Environment and Natural Resources*. Ed. by P. Kent. Oxford 1980.
- : *Economics of the Environment*. Lexington, Mass., 1981.
- Siebert, H.*, and *A. Antal-Berthoin*: *The Political Economy of Environmental Protection*. Greenwich, Conn., 1979.
- Siebert, H.* et al.: *Trade and Environment. A Theoretical Inquiry*. Amsterdam 1980.
- Starrett, D.*: Fundamental Nonconvexities in the Theory of Externalities. *Journal of Economic Theory* 4, 1972, 180–199.