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Games Between Players With Dual-Selves

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ABSTRACT

Human decision making often seems to be determined by the resolution of intrapersonal conflict. This paper conceptualizes the analysis of decisions governed by such dual-self processes in individual decision contexts and strategic interactions. For this purpose, we define the solution concept Dual-Selves Equilibrium and derive sufficient conditions for the existence of such equilibria. We show that our framework also helps to study the strategic interaction between boundedly rational players that have only one-self. In particular, our results extend several equilibrium existence results in Behavioral Game Theory and provide a simple sufficient condition for equilibrium existence.

JEL Classification: C72, D01, D91

1 | Introduction

There is growing consensus that individuals' beliefs and actions often seem to be determined by the interaction of multiple selves that pursue potentially different objectives (e.g., Brocas and Carrillo 2008). For example, we engage in wishful thinking to improve subsequent behavior (Bénabou and Tirole 2006a; Schwardmann and Van der Weele 2019; Schwardmann et al. 2022), make identity investments to maintain self-serving beliefs (Brunnermeier and Parker 2005; Bénabou and Tirole 2011), exhibit intertemporal behavior that seems to be determined by the resolution of conflict between a farsighted planner and a myopic doer (Thaler and Shefrin 1981; Fudenberg and Levine 2006), and display ambiguity preferences that can be rationalized via a dual-self representation of risk preferences (Trautmann and van de Kuilen 2015; Chandrasekher et al. 2022).

This paper proposes an approach to facilitate the analysis of strategic interaction when the decisions of each player are determined by intrapersonal conflict between two selves. For this purpose, we conceptualize dual-self decision processes as the noncooperative interaction between one-self that chooses an individual's beliefs and another self that chooses the payoff-relevant actions. We define the solution concept *Dual-Selves Equilibrium*, which subsumes Nash Equilibrium as a subcase. To illustrate the key idea underlying this concept, consider a scenario in which beliefs and actions follow from different objectives. In a Dual-Selves Equilibrium, beliefs are not just mere consequences of equilibrium strategy profiles but rather optimally chosen according to the corresponding self's objectives given all action selves' strategies in the game. Actions then constitute a best response to the arising system of beliefs. Note that, with dual-selves, every decision process—no matter whether it is taken in isolation or in a strategic situation—is inherently strategic because different selves may follow opposing objectives. Accordingly, our solution concept is also helpful to study individual decision making of individuals with multiple selves.

To illustrate Dual-Selves Equilibrium, consider the following simplistic example. Suppose there is an ex ante uncertain state of the world $s \in \{A, B\}$ with p being the prior probability that $s = A$. Furthermore, there are two politicians $i \in \{1, 2\}$, each represented by two selves—an action-self and a belief-self. The action-self chooses a policy $x_i \in \{A, B\}$, which improves voter welfare if and only if $x_i = s$. The objective of the action-self is to win an election, which the player will achieve if he is the only candidate who proposes a welfare-improving policy. Each belief-self chooses a belief $\hat{p}_i$ about the likelihood that the state of the world is A. Its objective is to rationalize the taken action (see also Eyster et al. 2021), that is, to choose $\hat{p}_i$ to maximize expected utility given x_i, subject to a cost function that increases in the difference between $\hat{p}_i$ and p. In such a game, a Dual-Selves Equilibrium would be given by a combination of beliefs and strategy profiles that are mutually best responses. In equilibrium, politicians may thus not only advocate different policies to maximize their chance of winning the election but may also develop polarized beliefs, as their belief-selves ex post rationalize the proposed policies.

Our first contribution lies in demonstrating the existence of a Dual-Selves Equilibrium under relatively mild conditions on the strategic interaction between the two selves. This finding holds significance for two primary reasons. First, it demonstrates that individual decision problems with dual-selves are generally well-defined. While some prior contributions have shown that decision making with dual-selves can be well-behaved (Loewenstein and O'Donoghue 2004; Brunnermeier and Parker 2005; Bracha and Brown 2012), they have confined themselves to specific objective functions of belief-selves and particular strategy spaces.[1] By establishing existence for a broad spectrum of games and objective functions, we provide an overarching concept facilitating the general analysis of dual-self processes in individual decision contexts. Second, and more importantly, we establish existence when players with dual-selves engage in strategic interactions. This result paves the way for analyzing applications where outcomes are shaped by both intrapersonal and interpersonal conflicts. Examples of such applications encompass decisions related to exerting effort in promotion tournaments when self-esteem is a pivotal factor, financial advice if beliefs are optimally chosen as in Brunnermeier and Parker (2005), bargaining about risky choices if individuals fall victim to overoptimism as in Bracha and Brown (2012), the formulation of political platforms where politicians manipulate their beliefs regarding the prevailing state of affairs, and purchasing choices of consumers with self-control problems.

Our second contribution is to show that several existing equilibrium concepts in Behavioral Game Theory are subcases of Dual-Selves Equilibrium. These concepts are employed to study the strategic interaction of one-self players with cognitive limitations or belief-dependent utility (for an overview see: Eyster 2019; Battigalli and Dufwenberg 2022). However, each of them reflects one particular manifestation of conflicting objectives between two selves. We formally demonstrate that psychological equilibrium (Geanakoplos et al. 1989), Cursed Equilibrium (Eyster and Rabin 2005), Berk–Nash Equilibrium (Esponda and Pouzo 2016), and Personal Nash Equilibrium (PNE, Dato et al. 2017) are subcases of Dual-Selves Equilibrium. More specifically, Cursed Equilibrium and Psychological Equilibrium can be interpreted as a Dual-Selves Equilibrium in which each belief-self chooses the expectation about the other players' strategies. In addition to these beliefs, each belief-self also chooses the expectation about the action self's strategy of the same player in the case of PNE. Similarly, Berk–Nash Equilibrium corresponds to a situation in which each belief-self chooses a mental model of the game in order to maximize a given consistency criterion. As we derive the existence of Dual-Selves equilibrium for nonfinite games, we thus also extend the previously established existence results (Geanakoplos et al. 1989; Eyster and Rabin 2005; Esponda and Pouzo 2016; Dato et al. 2017), which are restricted to finite games. Showing existence in nonfinite games facilitates the analysis of how limited cognitive capabilities affect strategic interactions in economically relevant situations. For instance, noise, signals or pure strategies might stem from an infinite set as it is often assumed when modeling price or quantity competition, rank-order tournaments, and auctions.

Finally, we provide a simple sufficient condition that allows deriving equilibrium existence in games between behavioral players with only one-self. A large body of literature documents that people hold misspecified beliefs as they update in a non-Bayesian way (Enke and Zimmermann 2019; Benjamin et al. 2019), are overconfident (De la Rosa 2011) or misperceive the mechanism through which signals are generated (Rabin 2002; Rabin and Vayanos 2010). However, when analyzing strategic interactions between players, such deviations from standard rationality assumptions render equilibrium existence unclear. Our sufficient condition essentially requires that players' (incorrect) beliefs about the information structure and the strategy profile can be represented by a continuous mapping of the true objects, that is, the true underlying parameters of the game and the strategy profile that is being played. As this condition is easily tested, it provides a ready to use toolbox for deriving equilibrium existence when studying the consequences of distorted beliefs or suboptimal decision making in strategic interaction.

The paper is organized as follows: We present the general model setup and state our main result in Section 2. Section 3 provides an example of a Dual-Selves game and illustrates equilibrium derivation. In Section 4, we discuss how our existence result relates and contributes to Behavioral Game Theory. Section 5 concludes.

2 | A General Dual-Selves Game

This section describes a general model for games in which each player has a dual-self—one-self chooses actions, and one chooses beliefs. In the following, we will refer to the individual as a whole when we use the term *player*, while the term *self* refers to the entity that chooses either the action or the belief. We study games with the following timing. First, a state and a profile of signals are drawn. Second, selves may privately observe signals. Third, all selves simultaneously make their choices. Fourth, the profile of actions in combination with the state of the world determines consequences, which, in turn, determine payoffs.

Throughout the paper, we denote the action-self of a player by i and the same player's belief-self by $\hat{\imath}$. We assume that all selves interact noncooperatively. A game between a finite number, N, of players whose behavior is defined by the decisions of their selves can be described by a tuple

$$\tilde{\mathcal{G}} = \langle \tilde{I}, \tilde{\Omega}, \tilde{\mathbb{S}}, \tilde{p}, \tilde{\mathbb{X}}, \tilde{\mathbb{Y}}, \tilde{f}, \tilde{\pi} \rangle,$$

with $\tilde{I}$ as the finite set of selves. Let $I = \{1, 2, \ldots, N\}$ denote the set of *action-selves* and $\hat{I} = \{\hat{1}, \hat{2}, \ldots, \hat{N}\}$ the set of *belief-selves* with $\tilde{I} = I \cup \hat{I}$. The complete and separable metric space $\tilde{\Omega}$ is the set of payoff-relevant states. Furthermore, all action-selves may privately observe a signal about the state of the world before taking actions. We collect the profiles of possible signals in the space $\tilde{\mathbb{S}} = \times_{i \in I} \mathbb{S}_i$, where $\mathbb{S}_i$ is the set of signals of self i. For all i, $\mathbb{S}_i$ is a complete and separable metric space. In our setup, only action-selves receive signals about the state of the world. The signal realizations are private knowledge and are not observed by belief-selves.[2]

The signal structure is represented by a probability measure $\tilde{p}$ over the Borel subsets[3] of $\tilde{\Omega} \times \tilde{\mathbb{S}}$, and, for simplicity, it is assumed to have marginals over $\tilde{\Omega}$ and $\tilde{\mathbb{S}}$ with full support; $p^{\tilde{\Omega}|S_i}(\cdot|s_i)$ denotes a probability measure over the subsets of $\tilde{\Omega}$ given $S_i = s_i$, and p^{S_i} denotes the marginal probability measure over the subsets of $\mathbb{S}_i$.

The action set of action-self i is denoted by a compact metric space $\mathbb{X}_i$. Hence, $\mathbb{X} = \times_{i \in I} \mathbb{X}_i$ is the set of action profiles of all action-selves. Each belief-self chooses at least a part of the beliefs that the corresponding action-self holds about the game's parameters or the strategies of her opponents. We denote the set of action profiles of the belief-selves by $\hat{\mathbb{X}} = \times_{\hat{i} \in \hat{I}} \mathbb{X}_{\hat{i}}$, where $\mathbb{X}_{\hat{i}}$ is a compact metric space and contains the actions of belief-self $\hat{i}$. Intuitively, the set $\mathbb{X}_{\hat{i}}$ contains those beliefs about the game of action-self i, which can be influenced by belief-self $\hat{i}$. If belief-selves can manipulate an action-self's belief about her opponents' strategies, this set would contain the strategy set of all other action-selves. $\tilde{\mathbb{X}} = \mathbb{X} \times \hat{\mathbb{X}}$ is then the set of action profiles of all selves in the game.

We allow all selves to play distributional strategies: a strategy of action-self i is a probability measure σ_i on the subsets of $\mathbb{S}_i \times \mathbb{X}_i$, for which the marginal distribution on $\mathbb{S}_i$ is p^{S^i}. Formally, this restriction on the marginal distribution is that for all $T \subset \mathbb{S}_i$, $\sigma_i(T \times \mathbb{X}_i) = p^{S_i}(T)$. Moreover, the behavioral strategy of action-self i is the conditional distribution denoted by $\sigma(dx_i|s_i)$. Let Σ_i denote the space of distributional strategies of action-self i and $\Sigma = \times_{i \in I} \Sigma_i$ the space of all strategy profiles of the action-selves. A strategy profile of all action-selves is a vector of strategies $\sigma = (\sigma_i)_{i \in I} \in \Sigma$. For the belief-selves, a strategy of $\hat{i}$ is a probability measure $\mu_{\hat{i}} \in \mathcal{M}(\mathbb{X}_{\hat{i}})$ over $\mathbb{X}_{\hat{i}}$. A strategy profile of all belief-selves is a vector of strategies $\mu = (\mu_{\hat{i}})_{\hat{i} \in \hat{I}}$.

In order to capture a broad spectrum of objectives that the selves may have, we explicitly model how actions and a state of the world determine consequences, which then determine payoffs.[4] Let the finite set $\mathbb{Y}_j$ denote the set of (observable) consequences for self $j \in \tilde{I}$, such that $\mathbb{Y} = \times_{i \in I} \mathbb{Y}_i$ is the set of consequences for all action-selves and $\hat{\mathbb{Y}} = \times_{i \in \hat{I}} \mathbb{Y}_i$ the set of consequences for all belief-selves. Accordingly, $\tilde{\mathbb{Y}} = \mathbb{Y} \times \hat{\mathbb{Y}}$.

For action-self i, the consequence function $f_i : \mathbb{X}_i \times \mathbb{X}_{\hat{i}} \times \Omega \to \mathbb{Y}_i$ assigns a consequence to every combination of own action, the action of the associated belief-self and the realized state of the world. Note that action-self i's realized consequence depends on the choice of her belief-self $\hat{i}$, but does not directly depend on the other action-selves' choices. More specifically, which consequence the action-self anticipates does not immediately follow from the true strategy profile that is being played in equilibrium but rather from the beliefs about that strategy profile, which, in turn, are chosen by the belief-self. This feature allows us to represent applications in which a player disregards or at least mispredicts other players' strategies.

For a given action and signal of action-self i, each action $x_{\hat{i}} \in \mathbb{X}_{\hat{i}}$ of the belief-self induces a distribution over consequences $Q_i^{x_{\hat{i}}}(\cdot|s_i, x_i)$, which is assumed to be continuous on $\mathbb{X}_{\hat{i}} \times \mathbb{S}_i \times \mathbb{X}_i$. Note that we allow the belief-self $\hat{i}$'s choice to affect the perceived distribution over outcomes directly. Accordingly, belief-self $\hat{i}$'s choice can not only distort action-self i's perception about which actions are being played, but it might as well affect the likelihood with which action-self i expects a specific consequence to occur given all opponents' actions. The bounded payoff function is given by $\pi_i : \mathbb{X}_i \times \mathbb{X}_{\hat{i}} \times \mathbb{Y}_i \to \mathbb{R}$, which is assumed to be continuous on $\mathbb{X}_i \times \mathbb{X}_{\hat{i}}$. Here, the dependence of π_i on $x_{\hat{i}}$ allows us to capture objectives where players' payoffs are directly influenced by their beliefs. All payoff functions are common knowledge among all selves.

We allow the payoff of each belief-self to depend on her own action and the strategies of all other action-selves. For example, an objective of a belief-self may be to correctly anticipate the strategies of all action players or a distorted functional of these strategies (see Section 4 for details). Formally, each belief-self $\hat{i}$ has the same consequence function given by $f_{\hat{i}} : \Sigma \to \Sigma$, with $f_{\hat{i}}(\sigma) = \sigma$. Hence, the observable consequence for the belief player is the strategy profile that is being played by all action-selves. The payoff function of $\hat{i}$ is given by $\pi_{\hat{i}} : \Sigma \times \mathbb{X}_{\hat{i}} \to \mathbb{R}$ with $\pi_{\hat{i}}(\sigma, x_{\hat{i}})$ being bounded from above.

The described payoff functions allow us to derive under which conditions a strategy is optimal for an action-self and a belief-self. A strategy of a belief-self, $\mu_{\hat{i}}$ is optimal if and only if $\mu_{\hat{i}}(x_{\hat{i}}) > 0$ implies

$$x_{\hat{i}} \in \arg\max_{\bar{x}_{\hat{i}} \in X_{\hat{i}}} \pi_{\hat{i}}\left(\sigma, \bar{x}_{\hat{i}}\right).$$

Similarly, a strategy σ_i of action-self i is optimal if and only if $\sigma_i(x_i, s_i) > 0$ implies

$$x_i \in \arg\max_{\bar{x}_i \in \mathbb{X}_i} E_{\bar{Q}_i^{\mu_{\hat{i}}}(\cdot|s_i, \bar{x}_i)}[\pi_i(\bar{x}_i, x_{\hat{i}}, y_i)], \tag{1}$$

where $\bar{Q}_i^{\mu_{\hat{i}}}(\cdot|s_i, x_i) = \int_{\mathbb{X}_{\hat{i}}} Q_i^{x_{\hat{i}}}(\cdot|s_i, x_i)\, \mu_{\hat{i}}(dx_{\hat{i}})$ is the distribution over consequences for action-self i, conditional on $(s_i, x_i) \in \mathbb{S}_i \times \mathbb{X}_i$, induced by the strategy of the belief-self $\mu_{\hat{i}}$. We define a Dual-Selves Equilibrium as follows:

Definition 1. A strategy profile (σ^*, μ^*) is a Dual-Selves Equilibrium of game $\tilde{\mathcal{G}}$ if and only if, (i) σ_i^* is optimal given $\mu_{\hat{i}}^*$ for all i, and (ii) $\mu_{\hat{i}}^*$ is optimal given σ^* for all $\hat{i}$.

Our definition embraces the notion that in decision processes with dual-selves, each self optimizes her actions taken the other selves' behavior as given. This approach is in line with evidence on the brain's functioning when resolving intrapersonal conflict (Brocas and Carrillo 2008; Alonso et al. 2014). The notion that (i) all strategies are chosen simultaneously and (ii) the action- and belief-self of each player best respond to each other implies that the equilibrium concept does not inherently equip one of the two selves with more influence on the outcome of the game. Nevertheless, it is also possible to apply the concept to dynamic games if these have a strategic form representation (as demonstrated by the example presented in the next section). The concept is therefore useful to study games with various timings and various relative influences of the selves on the final outcomes. Applying the concept to dynamic games comes with the same drawbacks as applying Nash Equilibrium to dynamic games: some of the potentially multiple equilibria may be less plausible than others.

The definition of a Dual-Selves Equilibrium puts clear boundaries on which kind of processes we capture in our analysis. First, we exclude decision processes in which actions immediately affect beliefs. In such processes, individuals choose actions while being aware that these actions will immediately also change their beliefs about future outcomes (see, e.g., Kőszegi and Rabin 2007; Masatlioglu and Raymond 2016). Second, we do not allow the belief-selves' payoffs to directly depend on the actions of other belief-selves. This assumption captures the idea that beliefs are private information of any given player. Instead, the framework could capture situations in which the payoffs of belief-self i depend on her own beliefs about the beliefs of another player j. Situations like this could, for example, arise if players have a preference for conformity and therefore would like to hold beliefs that they think are congruent with the beliefs of other players.

The definition of a Dual-Selves Equilibrium presumes that all action-selves best respond to their beliefs and all belief-selves play a best response to the equilibrium strategy profile. Note that the definition does not directly require all action-selves to play mutually best responses. In particular, each action-self's belief about the other action-selves' strategies is induced by the choice of the corresponding belief-self. Therefore, action-selves best respond to their possibly incorrect beliefs. Clearly, whenever belief-selves choose the correct belief about the strategy profile on the equilibrium path, action-selves best respond to the equilibrium strategies of the other action-selves. Consequently, Dual-Selves Equilibrium is a generalization of Nash Equilibrium. We prove the following claim:

Theorem 1. *Suppose that* $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu_{\hat{i}}(dx_{\hat{i}})$ *is continuous on* $\mathcal{M}(\mathbb{X}_{\hat{i}}) \times \Sigma$ *for all* $\hat{i}$. *Then there exists a Dual-Selves Equilibrium in* $\tilde{\mathcal{G}}$.

We relegate the proof of the theorem to the Appendix. In Dual-Selves games, players do not interact immediately with each other but rather through the belief-selves' choices. In contrast to Milgrom and Weber (1985), who show the existence of Nash Equilibrium in nonfinite games, we thus do not have to ensure continuity of all players' expected payoffs in their opponents' strategies. Instead we prove continuity of the action-selves' expected payoffs in the corresponding belief-self's strategy.

We consider the assumptions we have imposed on the action-selves' action sets and the signal generating mechanism to be relatively mild as they resemble conditions that are often invoked to ensure the existence of a Nash Equilibrium in Single-Self games (see also Milgrom and Weber 1985). Instead, the crucial additional requirement that Dual-Selves games have to fulfill to ensure the existence of a Dual-Selves Equilibrium is the continuity of the belief-selves' objective functions. As the applications of Theorem 1 in Section 4 in and the Appendix show, these requirements restrict the classes of games captured by our existence result. However, the theorem also shows for which kind of Dual-Selves games equilibrium existence is less problematic: If, for example, the set of possible beliefs $\hat{\mathbb{X}}$ is finite or the objectives of belief-selves is to match one particular belief according to a continuous scoring rule, the condition in Theorem 1 is ensured. While the theorem explicitly addresses situations with many players, it is also directly applicable to an individual decision problem with dual-selves. In this case, the decision of the individual is the outcome of the resolution of the conflict between the two selves. The decision problem is thus only well-defined if a Dual-Selves Equilibrium in the game between the two selves exists for which the continuity restriction in Theorem 1 is essential.

When a player's decisions are governed by multiple selves with different objectives, welfare considerations are not straightforward: it is a priori unclear how each of the selves' objectives should influence welfare. The Pareto criterion, which is used, for instance, in some settings of intertemporal choice (see, e.g., Goldman 1979; Laibson 1994), is noncontroversial but also restrictive. To study welfare in Dual-Selves games even if there is no unambiguous ranking of alternatives in the sense of Pareto-dominance, authors will have to take a stance on how welfare is determined. While the plausibility of any given welfare concept will depend on the setting under consideration, there are some points worth discussing. First, in some Dual-Selves games, it is reasonable that welfare is identical to one of the selves' payoff functions such as it is sometimes assumed in the context of intertemporal choice or personal motivation (O'donoghue and Rabin 1999; Bénabou and Tirole 2002). Second, in the class of potential games (Monderer and Shapley 1996), where the selves' objective functions can be replaced with a common payoff function, the latter can serve as the player's overall welfare.[5] Third, in our setup where one-self selects beliefs and the other actions, a possible interpretation is that the belief-self's payoff captures mental and the action-self's payoff represents "economic" well-being. Therefore, it can be reasonable to employ a convex combination of the belief and the action-selves' payoffs as the overall players' welfare (see also Loewenstein 1987). The weights attached to the selves' payoff functions then reflect the individual relative importance of mental and economic well-being.

3 | An Example of a Dual-Selves Game

This section works out a simple example of a Dual-Selves game serving as an antidote to the abstractness of the preceeding section. We follow Schwardmann et al. (2022), who show that debating competitions are a prime example for the impact of dual-self processes in games. Consider a debate in which persuasion goals (pro or contra a motion) are assigned to two players. Each player consists of an action-self and a belief-self. The former chooses costly effort to win the debate, where the effort costs to defend the assigned persuasion goal depends on the cognitive dissonance that she faces because her own opinion deviates from the persuasion goal, that is, it is cognitively easier to argue for a motion that is more conveniently aligned with one's own opinion. Prior to the debate, the belief-self can engage in costly self-persuasion to reduce cognitive dissonance. Jointly, the two selves thus determine the player's behavior in the debate.

Formally, each player, $i \in \{1,2\}$ consists of an action-self i and a corresponding belief-self $\hat{\imath}$. Moreover, the state of the world $\omega \in \Omega$ is represented by the realizations ε_i, $i \in \{1,2\}$ of two i.i.d random variables, which are uniformly distributed over the interval $[-x, x]$. Action-self i chooses effort $e_i \in [0, x]$ determining her persuasiveness in the debate, which is given by $e_i + \varepsilon_i$. The costs of effort depend on the distance between her own opinion and the assigned persuasion goal. We denote this distance by $\theta_i^T \in [0, \bar{\theta}^T]$, with $\bar{\theta}^T < b/4x$. However, the belief-self can engage in costly self-persuasion such that the action-self perceives the difference to be only $\hat{\theta}_i \geq 0$. Self-persuasion is costly according to $\hat{c}(|\hat{\theta}_i - \theta_i^T|) = \left(\hat{\theta}_i - \theta_i^T\right)^2/2$. The costs of effort for the action-self are then given by $c(e_i) = ke_i^2/2 + \hat{\theta}_i e_i$, where the first term represents standard effort cost and the second term the costs that the player faces due to cognitive dissonance.

The timing of the game is as follows: first, belief-selves simultaneously choose perceived cognitive dissonance, anticipating the own action-self's effort choice. Thereafter, action-selves simultaneously choose effort. The more persuasive player wins the debate. Formally, player i wins if and only if $e_i + \varepsilon_i > e_j + \varepsilon_j$. For each action-self, the tournament thus results in consequence $y_i \in \{0,1\}$, where i wins the tournament and receives benefit b if and only if $y_i = 1$. To ensure nonzero winning probabilities in equilibrium, we assume $k > b/x^2$. We can then construct a consequence function $f_i : \Omega \times [0,x]^2 \to \{0,1\}$ for each action-self that maps a combination of a state of the world and choices of all selves into consequences.[6] The payoff $\pi_i : [0,x] \times \{0,1\} \to \mathbb{R}$ is thus given by

$$\pi_i(e_i, y_i) = \begin{cases} b - c(e_i) & \text{if } y_i = 1 \\ -c(e_i) & \text{if } y_i = 0. \end{cases}$$

The belief-self $\hat{\imath}$ chooses the action-self's perceived cognitive dissonance $\hat{\theta}_i$ in order to raise the action-self's effort, so that the payoff function is

$$\pi_{\hat{\imath}} = e_i(\hat{\theta}_i, e_j) - \hat{c}\left(|\hat{\theta}_i - \theta_i^T|\right).$$

Overall, the behavior in the debating competition of the players is thus the resolution of an intrapersonal conflict. The (opportunistic) action-self chooses effort in order to maximize her expected payoff. She profits from self-persuasion but seeks to keep effort levels low. In contrast, the belief-self (noncooperatively) chooses costly self-persuasion to induce maximal effort levels. This example demonstrates that in Dual-Selves games, beliefs and actions may influence each other in a nontrivial way. For example, the degree of self-persuasion, which determines effort choices, may depend on the underlying opinion of both players as well as their original cognitive dissonance.

3.1 | Analysis of Example

Our paper proposes Dual-Selves Equilibrium as a solution concept for this type of strategic interaction with dual-selves. Since the described game is dynamic, we restrict attention to Dual-Selves Equilibria in which all selves are sequentially rational implying that the belief-self correctly anticipates the action-self's choice. To illustrate the derivation of a Dual-Selves Equilibrium and its properties, we briefly derive the solution to the game described above and discuss some empirical predictions that such a setup would generate. In the main text, we will focus on the case in which the original cognitive dissonance of player j is small, $\theta_j^T \leq \min\left\{\theta_i^T, 4x^2/\left(4x^2k + b\right)\right\}$. The appendix derives the solution for all other values of θ_j^T. The game yields the following Dual-Selves Equilibrium:

Proposition 1. *Assume $\theta_j^T \leq 4x^2/\left(4x^2k + b\right)$. If $\theta_i^T \leq 4x^2/\left(4x^2k - b\right)$ holds, the strategy profile $(e_i^*, e_j^*, \hat{\theta}_i^*, \hat{\theta}_j^*) = (b/2xk, b/2xk, 0, 0)$ constitutes the unique Dual-Selves Equilibrium. If $\theta_i^T \geq 4x^2/\left(4x^2k - b\right)$, the unique Dual-Selves Equilibrium is given by*

$$e_i^* = \frac{b}{2xk} - \frac{4x^2k + b}{4x^2k^2}\theta_i^T + \frac{4x^2k + b}{(4x^2k - b)k^2} \qquad \hat{\theta}_i^* = \theta_i^T - \frac{4x^2}{4x^2k - b}$$

$$e_j^* = \frac{b}{2xk} - \frac{b}{4x^2k^2}\theta_i^T + \frac{b}{(4x^2k - b)k^2} \qquad \hat{\theta}_j^* = 0.$$

Consider first the scenario in which initial cognitive dissonance is relatively small, that is, $\theta_i^T \leq 4x^2/\left(4x^2k - b\right)$. Intuitively, the belief-self is then able to eliminate cognitive dissonance at a moderate cost. As the elimination of cognitive dissonance implies a high effective ability, exertion of relatively high effort is then attractive for the action-self. The resulting high winning probability, in turn, justifies incurring self-persuasion costs for the belief-self. If, however, initial cognitive dissonance is substantial, $\theta_i^T \geq 4x^2/\left(4x^2k - b\right)$ for one player, completely eliminating cognitive dissonance via self-persuasion is prohibitively costly. Accordingly, the belief-self chooses an intermediate level of self-persuasion, to which the action-self best responds by choosing an intermediate effort level. In this case, the opinion of the player thus (only) partially eliminates cognitive dissonance. Moreover, the opponent also reacts to the increase in cognitive dissonance for player i. In particular, player j would reduce her effort in the debate, because the increase in cognitive dissonance for player i leads to a less competitive situation.

The described Dual-Selves Equilibrium features two interesting properties. First, a player will eliminate all cognitive dissonance if the distance θ_j^T between the goal and the true underlying position is sufficiently small. As a consequence, the model predicts that assigning goals to participants of the debating competition causes a squeeze in the distribution of stated opinions because all players that originally hold opinions close to the assigned goal will convince themselves that their opinion is perfectly aligned with the persuasion goal.[7]

Second, consider how the resolution of intrapersonal conflict responds to changing persuasion goals. To generate an intuition, fix θ_2^T. If the persuasion goal is close to the underlying opinion of player 1, that is, θ_1^T is small, a slight shift away from the player's opinion immediately implies an increase in self-persuasion since all cognitive dissonance is eliminated. If, however, the assigned motion is already distant from the underlying opinion, it becomes prohibitively costly to completely eliminate cognitive dissonance. As a reaction to the arising cognitive dissonance, the action-self reduces effort which in turn attenuates the incentives for self-persuasion. In equilibrium, the marginal impact of an increase in the original cognitive dissonance on self-persuasion activities will thus decrease. In the case of our particular cost function, the belief-self in fact holds self-persuasion activities constant with increasing θ_1^T, that is, the distance between the original cognitive dissonance θ_i^T and the perceived cognitive dissonance $\hat{\theta}_i$ is independent of the persuasion goal if the goal is sufficiently distant from the player's own opinion.[8] Empirically, one would expect the correlation between players' stated opinions after the debate and their assigned persuasion goals to be significantly stronger when the assigned motions are moderate rather than extreme.

These insights imply that the Dual-Selves framework predicts greater effort and success in debating competitions when assigned motions are less extreme or more widely accepted. Conversely, defending more radical positions may induce cognitive dissonance, leading to reduced effort and lower performance. As a result, even within the structured setting of debating competitions, asymmetric levels of cognitive dissonance may produce self-confirming dynamics—arguments that align with preexisting popular views may be articulated more effectively by participants.

Overall, this application demonstrates how players in a Dual-Selves Equilibrium resolve intrapersonal conflict. Furthermore, it shows how the concept can be used to generate novel empirical predictions in situations in which intrapersonal conflict is a key driver of decision making in strategic settings.

4 | Relation to Behavioral Game Theory

This section illustrates how our setup and results contribute to the literature that studies games in which each player only has one-self but may have misspecified beliefs or belief-dependent utility. First, we show how Dual-Selves Equilibria correspond to equilibria in such Single-Self games. The corresponding arguments demonstrate that many commonly suggested equilibrium concepts in Behavioral Game Theory can be reinterpreted as a Dual-Selves Equilibrium. Accordingly, Dual-Selves Equilibrium constitutes an overarching solution concept for the analysis of strategic interaction between boundedly rational players or those with belief-dependent utility. Second, we discuss several existence results in the literature that we extend to nonfinite games as corollaries of Theorem 1. Third, we derive a simple sufficient condition that can be applied to test equilibrium existence if behavioral players interact strategically.

4.1 | Relation to Other Equilibrium Concepts

Theorem 1 proves the existence of Dual-Selves Equilibria. Notably, many equilibrium concepts in Behavioral Game Theory that feature behavioral players with only one-self are equivalent to a Dual-Selves Equilibrium. To delineate this correspondence, we will start this section by showing how one specific equilibrium concept—Cursed Equilibrium by Eyster and Rabin (2005)—can be accommodated as a Dual-Selves Equilibrium. Afterward, we will show that the arguments for this specific example also hold for several other widely applied equilibrium concepts.

4.1.1 | Cursed Equilibrium and Dual-Selves Equilibrium

Recall that Cursed Equilibrium assumes that each player correctly predicts all opponents' actions but may underestimate the extent to which these actions are correlated with the privately observed signals. We will first use our notation to describe a game in which every player has only one-self. In a second step, we show the analogy. Consider game $\tilde{\mathcal{G}}$ defined in Section 2. Let game $\mathcal{G}_{OS}$ be equivalent to game $\tilde{\mathcal{G}}$ except for the following two changes. First, assume that the set of belief-selves $\hat{I}$ is empty. Second, adapt the consequence function of action-self i such that it is given by $f_i : \mathbb{X} \times \Omega \to \mathbb{Y}_i$. Hence, which consequence materializes depends on the state of the world and the actions of all players. Game $\mathcal{G}_{OS}$ is, therefore, a standard game in which each player has only one-self (see also Esponda and Pouzo 2016).

According to Eyster and Rabin (2005), a Cursed Equilibrium of $\mathcal{G}_{OS}$ can be defined as follows. Let σ_{-i} be the strategy profile of all opponents' strategies. Then a cursed player expects with probability $\mathcal{X} \in [0, 1]$ that her opponents play

$$\bar{\sigma}_{-i}(x_{-i}|s_i) = \int_{\mathbb{S}_{-i}} \prod_{j\neq i} \sigma_j(dx_j|s_j) \times \int_{\Omega} p_{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i).$$

For player i with signal s_i, $\bar{\sigma}_{-i}(x_{-i}|s_i)$ denotes the average strategy of her opponents, where the average is taken over their signals. The perceived probability distribution over outcomes of a cursed player might therefore be incorrect and is given by

$$Q_i^{\bar{\sigma}}(y_i|s_i, x_i) = \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \Bigg[\mathcal{X}\bar{\sigma}_{-i}(dx_{-i}|s_i) + (1-\mathcal{X})\prod_{j\neq i}\sigma_j(dx_j|s_j)\Bigg] \times p_{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i).$$

A *Cursed Equilibrium* of $\mathcal{G}_{OS}$ is defined as follows:

Definition 2. A strategy profile σ is a Cursed Equilibrium of game $\mathcal{G}_{OS}$ if, for all players $i \in I$ and $s_i \in \mathbb{S}_i$, and any x_i with $\sigma_i(x_i|s_i) > 0$, we have

$$x_i \in \arg\max_{x_i\in\mathbb{X}_i} E_{Q_i^{\bar{\sigma}}(\cdot|s_i,x_i)}[\pi(x_i, y_i)].$$

To show that this concept is a subcase of a Dual-Selves Equilibrium, consider game $\tilde{\mathcal{G}}_{OS}$, which is identical to $\mathcal{G}_{OS}$ except for three changes. First, for every action-self i, there exists a belief-self $\hat{i}$. Second, the consequence that self i anticipates depends on the action of her belief-self instead of the strategies of her opponents. However, the action space of the belief-self is given by Σ. Essentially, the belief-self, thus, selects what the player thinks about the strategy profile that is being played. More specifically, the consequence function is specified exactly as before but maps from the belief-self's choice and the state of the world to outcomes, $f_i : \mathbb{X}_{\hat{i}} \times \Omega \to \mathbb{Y}_i$. Third, let the objective of the belief-self be given by

$$\pi_{\hat{i}}(\sigma, x_{\hat{i}}) = -\sum_{y_i \in \mathbb{Y}_i} \left[Q_i^{\tilde{\sigma}}(y_i | x_i, s_i) - Q_i^{x_{\hat{i}}}(y_i | x_i, s_i) \right]^2.$$

By Theorem 1, we know that a Dual-Selves Equilibrium in this game exists, whenever $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}}) \mu_{\hat{i}}(dx_{\hat{i}})$ is continuous. Corollary 2, which we relegate to the appendix, derives sufficient conditions for this to be the case. In this Dual-Selves Equilibrium, every belief-self will play a best response to all action-selves and all other belief-selves. In equilibrium, each action-self will, therefore, anticipate exactly the probability distribution over consequences that a cursed player would anticipate (see also Equation A.2). Accordingly, action-self i will play a best response given that her beliefs are cursed (cp. Equation A.1). Hence, the strategy profile played by the action-selves in the Dual-Selves Equilibrium of $\tilde{\mathcal{G}}_{OS}$ constitutes a *Cursed Equilibrium* in $\mathcal{G}_{OS}$. We conclude that the concept of Cursed Equilibrium is identical to a Dual-Selves Equilibrium with specific objective functions of the belief-selves.

4.1.2 | Other Equilibrium Concepts and Dual-Selves Equilibrium

The correspondence between Dual-Selves Equilibria and different equilibrium concepts in games with behavioral players is not restricted to Cursed Equilibrium. The following section intuitively delineates the types of equilibrium concepts that are subcases of Dual-Selves Equilibrium. To formally corroborate these arguments, Corollaries 3–5 in the Appendix demonstrate how PNE (Dato et al. 2017), Psychological Equilibrium (Geanakoplos et al. 1989), and Berk–Nash Equilibrium (Esponda and Pouzo 2016) can be subsumed as a Dual-Selves Equilibrium. There are four main classes of concepts that we can grasp. First, concepts in which behavioral players systematically misperceive their opponents' strategies as, for example, in Jehiel (2005), Eyster and Rabin (2005), or Grunewald et al. (2025). Our setup allows us to capture such concepts by letting the belief-self choose the expectation about the opponent's strategies. Hence, $\mathbb{X}_{\hat{i}} = \Sigma_{-i}$. Via an appropriate payoff function $\pi_{\hat{i}}$, the belief-self is then incentivized to select the biased belief specified by the specific equilibrium concept.

Second, Dual-Selves Equilibria also subsume concepts in which behavioral players hold fully rational beliefs in equilibrium, but their utility directly depends on beliefs as, for example, in a Psychological Game Theory (Geanakoplos et al. 1989; Battigalli and Dufwenberg 2009;2022) or in a PNE (Dato et al. 2017). We can represent such concepts by specifying the belief-selves' objectives such that it is optimal to choose correct beliefs about the equilibrium strategy profile. A belief-dependent objective function of the action-self can then be captured via the payoff function π_i, which is allowed to directly depend on beliefs in our setup (for a formal derivation that these concepts can be considered to be Dual-Selves Equilibria, see Corollaries 3 and 4).

Third, our setup can incorporate concepts in which players correctly anticipate their opponents' behavior but may hold a wrong view about the fundamentals of the game. Such biases are prominently discussed and well-documented in the domain of individual decision making. Possible sources for such misperceptions are if signals are processed in a non-Bayesian way (see, e.g., Phillips and Edwards 1966; Kahneman and Tversky 1973; Rabin and Schrag 1999; Benjamin et al. 2016; Bénabou and Tirole 2011), or individuals believe in a wrong signal generating mechanism (see, e.g., Barberis et al. 1998; Rabin 2002; Rabin and Vayanos 2010). Strategic interaction between behavioral players that fall victim to such belief biases can be studied by using Dual-Selves Equilibria. To capture such biases, each belief-self not only chooses a player's belief about her opponent's strategies but also about primitives of the game (such as players' own types as in the example from Section 3). This choice can be incorporated in the action set of belief-selves, which is only restricted to be a metric space. Again, $\pi_{\hat{i}}(\cdot)$ then specifies according to which objective a belief-self chooses the corresponding beliefs.[9]

Fourth, we can also capture concepts that allow for misperceptions concerning both the strategy profile played in equilibrium and primitives of the game. Embracing the notion that both of these components may be misperceived typically implies that beliefs about the game's strategies and primitives are jointly determined in equilibrium according to an optimality or consistency criterion as in Esponda (2008), Esponda and Pouzo (2016), or Massari and Newton (2020). We can incorporate such criteria in the belief-selves' objectives (see also the proof of Corollary 5) such that it is optimal for the belief-self to determine beliefs according to the corresponding consistency criterion.

4.2 | Generalization of Equilibrium Existence

As argued in the previous section, various existing equilibrium concepts correspond to Dual-Selves Equilibria. Corollaries 2–5 in the Appendix show that Cursed Equilibrium by Eyster and Rabin (2005), PNE by Dato et al. (2017), Psychological Equilibrium by Geanakoplos et al. (1989), and Berk–Nash Equilibrium by Esponda and Pouzo (2016), for example, can be reinterpreted as Dual-Selves Equilibria. A key virtue of Theorem 1 is that it derives equilibrium existence also for nonfinite games. Essentially, the reinterpretation allows to capture the behavioral component as a choice rather than a mistake, which in turn enables us to leverage existing tools from game theory for nonfinite games (such as the result by Milgrom and Weber 1985) to derive the existence proof. As all of the above contributions restrict themselves to show equilibrium existence in games with finite action and signal spaces, Theorem 1 together with Corollaries 2–5, show under which conditions the corresponding equilibrium existence generalizes to nonfinite games.

Theorem 1 can thus be applied to ensure equilibrium existence for a large variety of nonfinite settings and various equilibrium concepts in Behavioral Game Theory. Nevertheless, the assumptions that we employ to prove Theorem 1 are clearly not innocuous. Most importantly, we impose continuity of the beliefs selves' expected payoffs. This assumption excludes concepts postulating that beliefs follow discontinuously from the game's primitives, for example, according to a threshold rule. For example, partially attentive players might realize that specific actions are played in equilibrium only if their likelihood of being on the equilibrium path exceeds some threshold (Gabaix 2012;2014).

4.3 | Sufficient Condition for Equilibrium Existence

When modeling individual decisions, a number of papers in behavioral economics have deviated from standard rationality assumptions that are also necessary prerequisites to apply the prominent equilibrium existence results in game theory. Building upon this work, many applications set out to study how particular forms of irrationality affect decisions in strategic contexts. Since Dual-Selves Equilibrium can represent a broad variety of decision processes, Theorem 1 lends itself to study equilibrium existence for incorrect beliefs on a more general level. For this purpose, we will first define a concept that allows players to hold incorrect beliefs in a game. We will then show for which type of belief formation processes Theorem 1 guarantees equilibrium existence in strategic situations.

Suppose there is a game $\mathcal{G}_{OS}$ in which each player has one-self (see also Section 4.1). Furthermore, collect the strategies and the joint distributions over signals and states of the world in the set $\Theta = \Sigma \times \mathcal{M}(\Omega \times \mathbb{S})$. To define an equilibrium in which players hold incorrect beliefs, we postulate that each player plays a best response given her beliefs about the parameters of the game and the played strategy profile. These beliefs, however, are governed by a particular deviation from rationality. Formally, denote by C an equilibrium concept. Then C describes the beliefs that each player i holds given the strategy profile σ and the true signal structure $\tilde{p}$. We assume that this process can be represented by some function $F_i^C : \Theta \to \Theta$, which maps the true $\theta \in \Theta$ to $\hat{\theta}_i \in \Theta$ which represents i's beliefs.[10] More specifically, player i anticipates the distribution over outcomes to be $Q_i^{\hat{\theta}_i}(y_i|s_i, x_i)$ (cp. Section 2). We define a C-Equilibrium in game $\mathcal{G}_{OS}$ as follows:

Definition 3. A C-Equilibrium in $\mathcal{G}_{OS}$ is a strategy profile σ such that:

(i) Every player i plays a best response given $\hat{\theta}_i$.

(ii) If $\mathcal{G}_{OS}$ specifies $\tilde{p}$ and the players play strategy profile σ, then each player i holds the belief $\hat{\theta}_i = F_i^C(\sigma, \tilde{p})$.

In a C-Equilibrium, every player may hold incorrect beliefs that depend on the strategy profile being played and the underlying signal generating mechanism. This formulation may, for example, reflect that players are non-Bayesian updaters due to, for example, correlation neglect (Enke and Zimmermann 2019), base rate neglect (Benjamin et al. 2019), or overconfidence with respect to the own type (De la Rosa 2011). To illustrate more broadly what kind of concepts can be reflected by this specification, note two points. First, recall that space Σ is the space of profiles of distributional strategies. Hence, a misperception about opponents' strategies can be represented by some collection of F_i^C»s by assuming that the belief about the profile of distributional strategies is distorted. Second, concepts assuming that players misperceive their own or other players' types or other parts of the information structure can be reflected by F_i^C specifying an incorrect belief about $\tilde{p}$. Exploiting Theorem 1, we can then derive the following statement.

Corollary 1. *If $F_i^C(\cdot)$ is continuous on Θ for all i, there exists a C-Equilibrium in $\mathcal{G}_{OS}$.*

Corollary 1 follows immediately from Theorem 1. To facilitate a better comprehension of the restrictions imposed by Corollary 1, let us first discuss what kind of settings are ruled out by them. For instance, the corollary does not cover misperceptions of (i) the signal realization, (ii) the utility function of another player given her type, (iii) the order of play, or (iv) the set of players. All these misperceptions are excluded, because the formulation assumes that beliefs only impact the distribution over consequences, given the signal realization and the choice of the action-self. Furthermore, as discussed in Section 4.2, the continuity of F_i^C, excludes concepts with partially attentive players who only realize that specific actions are played in equilibrium if their likelihood of being on the equilibrium path exceeds some threshold (Gabaix 2012;2014). Moreover, the sufficient condition assumes that the function F_i^C maps into Θ. This rules out every concept in which players expect types or strategies that, in fact, do not exist in the game.

Nevertheless, it lends itself to be applied in various settings of strategic interaction. There are two common subcases of Corollary 1: First, modeling non-Bayesian updating oftentimes relies on formulations in which individuals' beliefs correspond to a continuous distortion of their true signal or type (see, e.g., Enke and Zimmermann 2019; Benjamin et al. 2019; De la Rosa 2011). In these cases, also the corresponding F_i^C is going to be continuous, and a C-Equilibrium exists. Second, recall that our setup allows payoffs to depend on beliefs directly. Hence, equilibrium existence is rarely an issue if all players have belief-dependent utility but correctly anticipate the distribution of types and the strategy profile. Such situations arise, for example, if players have image concerns as in Bénabou and Tirole (2006b) or anticipatory utility as in Caplin and Leahy (2001). In our notation, these concepts correspond to F_i^C being the identity for all players.[11]

The sufficient condition allows players to hold distorted priors on some characteristics of the game and, thus, also to hold different priors on parameters of the game. In contrast to the literature on games without common priors (see Mertens and Zamir 1985; Sákovics 2001), the condition assumes that the objectives to distort beliefs are common knowledge and thus players have to agree to disagree on their priors.

5 | Conclusion

This paper conceptualizes the strategic interaction between players whose beliefs and actions follow from different or even opposing objectives, that is, players with dual-selves. To account for the fact that the players themselves and the dual-selves of each player interact noncooperatively, we define the solution concept Dual-Selves Equilibrium, which can be viewed as a generalization of Nash equilibrium to study Dual-Selves games. Our main theorem shows that the continuity of the belief-selves' objective functions ensures existence of a Dual-Selves Equilibrium.

While we explicitly allow for multiple players and for the dual-selves of each player to interact at the same time, our results also provide useful insights for (i) the analysis of games with single-self players who are boundedly rational or have belief-dependent utility and (ii) individual decisions of agents with a dual-self. First, we show that a variety of solution concepts proposed in Behavioral Game Theory, where players have only one-self but are characterized by limited cognitive capabilities or nonstandard preferences (e.g., Geanakoplos et al. 1989; Eyster and Rabin 2005; Esponda and Pouzo 2016; Dato et al. 2017), constitute subcases of Dual-Selves Equilibrium. By proving the existence of Dual-Selves Equilibrium in nonfinite games, we extend the previously established existence results, which are restricted to finite games. We further contribute to this strand of the literature by providing a simple sufficient condition that allows deriving equilibrium existence in games between behavioral players with only one-self. Second, Dual-Selves Equilibrium can be applied to study decisions that arise as a resolution to the intrapersonal conflict in individual decision making. Even these decisions are inherently strategic due to the opposing objectives of the selves.

We introduce an overarching framework designed to encompass a wide array of dual-self processes and to facilitate their analysis within contexts of strategic interaction. This contribution also establishes a platform to investigate why dual-self processes seem to be pervasive in our brain functioning (Thaler and Shefrin 1981; Brocas and Carrillo 2008; Alonso et al. 2014). Specifically, there is still a lack of understanding whether these dual-selves processes emerge due to inherently limited cognitive ability of players or if they induce optimal behavior in interactions with other individuals. Heller and Winter (2020) have presented a toolkit for exploring the persistence and optimality of belief distortions in finite normal form games between two players with single selves. By integrating their methodology with our framework, one could systematically assess and compare the plausibility and stability of diverse dual-self processes, shedding light on instances where such processes might outperform decision making of single-self players.

Our framework focuses on the prominent case of conflict between beliefs and actions. For this case, we assume that the two corresponding selves interact noncooperatively. An alternative approach would be to model their interaction with tools from cooperative game theory such as Nash Bargaining, and only use noncooperative game theory for the interaction between players. In addition, the framework could also be adapted to feature more selves, for example, for different regions of the brain, reflecting different motives with respect to the actions that are eventually taken, or representing different moods.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Endnotes

[1] Dual-selves models have also found application in the analysis of self-control within intertemporal choices (e.g., Thaler and Shefrin 1981; Bernheim and Rangel 2004; Benhabib and Bisin 2005; Fudenberg and Levine 2006). However, unlike our study and the aforementioned contributions, these decision problems are typically represented by an individual maximization problem. Accordingly, the resolution of intrapersonal conflict in these contexts does not require a solution concept for the strategic interaction between the selves.

[2] While the assumption is mainly made for tractability, it also captures the idea that the action-self may hold private information about the consequences of his choices that are not observable by the belief-self. For instance, a far-sighted belief-self may be able to shape beliefs in the long term but may not be able to react to short-term shocks to the decision environment. At the same time, the assumption excludes setups in which both selves hold identical information that is private knowledge of the player but unknown to all other players.

[3] As long as not stated otherwise, we will only consider Borel subsets for the rest of the paper.

[4] Parts of this notation are borrowed from Esponda and Pouzo (2016). It allows us to study situations in which belief-selves do not only wish to manipulate beliefs about expected payoffs but also care about how payoffs come about, that is, through which combination of strategies and state of the world. Such objectives may, for example, be relevant if belief-selves try to anticipate a correct distribution over consequences, in line with the ideas in Esponda and Pouzo (2016).

[5] For further discussion of why it is reasonable to use the common payoff function to represent the utility of the composite agent, see Bracha and Brown (2012), who study individual decision making under risk and uncertainty within a dual-self process.

[6] For ease of exposition, the action-selves' consequence functions directly depend on the actions of other action-selves and not on the beliefs chosen by the belief-self. It is straightforward to modify the belief-selves' action sets such that they also choose beliefs about the opponent's actions. To ensure that these beliefs are correct in equilibrium, the belief-selves' objective functions then have to be complemented by an additive part that constitutes a scoring rule incentivizing them to choose correct beliefs. More generally, this shortcut to simplify notation can be used in applications of the proposed Dual-Selves framework as long as each player's belief about other action-selves' strategies has to be correct in equilibrium.

[7] It would also be feasible to derive an empirical prediction with respect to the shift in moments of the distribution of stated opinions in response to receiving the persuasion goal. This exercise, however, would require knowledge about the initial distribution of underlying opinions.

[8] Formally, the derivation in Appendix B shows that for the comparative static $d\hat{\theta}_i/d\theta_i^T = 1$ holds for this case.

[9] Recall that the payoff function of the belief-self in Section 4.1.1 is artificial in the sense that it directly incentivizes her to choose cursed beliefs, so that the belief-self is to be interpreted as an as-if construct, that is, players behave in equilibrium as if a third party chose their beliefs according to the specified payoff function. In contrast, payoff-functions of belief-selves can be psychologically plausible in settings where players hold incorrect perceptions of the game's primitives. As an example, confirmatory bias could result from a payoff function that incentivizes the belief-self to choose correct beliefs about, for example, primitives of the game, but equips her with a cost function according to which it is more costly to hold nonconfirming beliefs.

[10] As an easy example consider a player who receives a correct signal about her ability $\theta \in [0, 1]$. However, the player is overconfident and believes

to have ability $\hat{\theta} = f^C(\theta)$, where $f^C : [0,1] \to [0,1]$ is a continuous function. Then her perceived ability is a continuous function of the underlying true ability.

[11] Notably, our concept covers additional cases compared to psychological games as discussed in Geanakoplos et al. (1989) even if we restrict F_i^C to be the identity for all i. In particular, we can allow for belief-dependent utility also for own strategies, which may arise if players are expectation-based loss averse as in Dato et al. (2017) or with respect to their type, which may, for example, arise if players have image concerns with respect to their ability.

[12] Specifically, we will use the following result: Let the players' strategy spaces be nonempty compact, convex subsets of convex Hausdorff linear topological spaces. Let the payoff functions be continuous on the product of the strategy spaces, and let each player's payoff function be quasi-concave in her strategy. Then an equilibrium point exists.

[13] A set of probability measures on a metric space is called tight if for every $\epsilon > 0$ there is a compact set K such that for every P in the set of measures, $P(K) > 1 - \epsilon$.

[14] The Cartesian product of separable and complete metric spaces is separable itself (see De la Fuente 2000, 82)

[15] Billingsley (1968, Theorem 6, 240).

[16] See Theorem 20.1 in Munkres (2000).

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Appendix A: Proof of the Theorem

Proof of Theorem 1. To prove the claim, we show that $\tilde{\mathcal{G}}$ satisfies the requirements of the existence result by Glicksberg (1952).[12]

Step 1: Compactness and Convexity of Strategy Spaces.

Step 1a: Action-selves

We will prove the existence of a Dual-Selves Equilibrium in distributional strategies in $\tilde{\mathcal{G}}$. For this purpose, we start by arguing that the strategy spaces are compact and convex metric spaces in the weak topology. Note that $\tilde{p}$ is tight since it is a measure over complete separable metric spaces (see Parthasarathy 1967, Theorem 3.2, 29). In addition, action spaces are assumed to be compact, and hence they are also complete (see De la Fuente 2000, Theorem 8.16, 96) such that also the Cartesian products $\mathbb{X}_i \times \mathbb{S}_i$ are complete. By Theorem 3.2 in Parthasarathy (1967), each action-self's set of distributional strategies is then a tight set of probability measures;[13] also, since it is a set of measures over the Cartesian product of separable and complete metric spaces,[14] it is complete itself by Theorem 6.5 in Parthasarathy (1967) and hence closed in the weak topology. By Prohorov's Theorem,[15] it follows that the strategy sets are compact metric spaces in the weak topology. Furthermore, the set of distributional strategies is clearly convex.

Step 1b: Belief-selves

By the same arguments as in *Step 1a*, the belief-selves' strategy sets are compact metric spaces in the weak topology. Furthermore, the set of strategies is clearly convex.

Step 2: Continuity of the action-selves' expected payoffs.

Note that the expected payoff of an action-self i is given by

$$\begin{aligned} U_i(\mu_{\hat{i}}, \sigma_i) &= \int E^{\bar{Q}_i^{\mu_{\hat{i}}}(\cdot|s_i, x_i)}\left[\pi_i(x_i, x_{\hat{i}}, y_i)\right]\sigma_i(dx_i|s_i)p^{\mathbb{S}_i}(ds_i) \\ &= \int_{\mathbb{X}_i\times\mathbb{S}_i\times\mathbb{X}_{\hat{i}}} \sum_{y_i\in\mathbb{Y}_i} \pi_i(x_i, x_{\hat{i}}, y_i) Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) d\mu_{\hat{i}}(x_{\hat{i}}) d\sigma_i(x_i, s_i), \end{aligned}$$

where we make use of Fubini's theorem. To prove continuity under the weak topology, we need to prove the following: For any sequence $\left(\mu_{\hat{i}}^n, \sigma_i^n\right)$ such that $\left(\mu_{\hat{i}}^n, \sigma_i^n\right) \to (\mu_{\hat{i}}, \sigma_i)$, it holds that $\lim_{n\to\infty} U_i\left(\mu_{\hat{i}}^n, \sigma_i^n\right) = U_i\left(\mu_{\hat{i}}, \sigma_i\right)$. Define the probability measure $p_i^n\left(x_i, s_i, x_{\hat{i}}\right)$ such that $\int g d\mu_{\hat{i}}^n\left(x_{\hat{i}}\right) d\sigma_i^n\left(x_i, s_i\right) = \int g dp_i^n\left(x_i, s_i, x_{\hat{i}}\right)$. By Theorem 1.1 in Feinberg et al. (2014), $\lim_{n\to\infty} \int g dp_i^n\left(x_i, s_i, x_{\hat{i}}\right) = \int g dp_i\left(x_i, s_i, x_{\hat{i}}\right)$ for all bounded and continuous real valued functions g if $p_i^n \to p_i$. Note that by the continuity and boundedness of payoffs and $Q^{x_{\hat{i}}}$ also $\sum_{y_i\in\mathbb{Y}_i} \pi_i\left(x_i, x_{\hat{i}}, y_i\right) Q_i^{x_{\hat{i}}}\left(y_i|s_i, x_i\right)$ is continuous and bounded implying that $\lim_{n\to\infty} U_i\left(\mu_{\hat{i}}^n, \sigma_i^n\right) = U_i\left(\mu_{\hat{i}}, \sigma_i\right)$ if $\left(\mu_{\hat{i}}^n, \sigma_i^n\right) \to (\mu_{\hat{i}}, \sigma_i)$. Hence, U_i is continuous. Note that $\sum_{y_i\in\mathbb{Y}_i} \pi_i\left(x_i, x_{\hat{i}}, y_i\right) Q_i^{x_{\hat{i}}}\left(y_i|s_i, x_i\right)$ is bounded, such that $U_i\left(\mu_{\hat{i}}, \sigma_i\right)$ is an affine function of $\sigma_i\left(x_i, s_i\right)$.

Step 3: Existence of Nash Equilibrium

Since $U_{\hat{i}}\left(\sigma, \mu_{\hat{i}}\right)$ is an integral over a bounded function, it is an affine function on the set of strategies of the belief-self $\hat{i}$, and hence quasi-concave. In summary, the selves' strategy sets are compact, convex metric spaces and the payoff functions are continuous and affine. According to the version of Glicksberg's theorem that is also applied by Milgrom and Weber (1985) these properties guarantee the existence of a Bayes' Nash Equilibrium. In each Bayes' Nash Equilibrium, the selves play mutually best responses. As a consequence, $\sigma_i\left(x_i, s_i\right) > 0$ only if

$$x_i \in \arg\max_{\bar{x}_i\in\mathbb{X}_i} E^{\bar{Q}_i^{\mu_{\hat{i}}}(\cdot|s_i, \bar{x}_i)}\left[\pi_i\left(\bar{x}_i, x_{\hat{i}}, y_i\right)\right]. \tag{A.1}$$

Similarly, $\mu_{\hat{i}}\left(x_{\hat{i}}\right) > 0$ only if

$$x_{\hat{i}} \in \mathbb{X}_{\hat{i}}(\sigma) \equiv \arg\max_{x_{\hat{i}}\in\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}\left(\sigma, x_{\hat{i}}\right). \tag{A.2}$$

□

Appendix B: Derivations for the Example

This appendix derives the Dual-Selves Equilibrium and the corresponding comparative static for the example in Section 3.1. While the proposition focuses on the special case of $\theta_j^T \leq 4x^2/\left(4x^2k - b\right)$, the appendix derives the equilibrium for all values of θ_j^T.

Derivation of Equilibrium and Proof of Proposition 1. We start with the action-selves' effort choices. Each action-self i chooses effort e_i, taking the own perceived cognitive dissonance $\hat{\theta}_i$ and the effort e_j of the opponent as given. In equilibrium, the action-self's correct perception of the opponent's effort will be determined by the belief-self's choice. Denote the cdf of the convolution of the uniform distributions by $G(\varepsilon_j - \varepsilon_i)$ and the probability of winning the debate by $P_i(e_i, e_j)$. Action-self i's objective function is then given by

$$\mathbb{E}[\pi_i] = P_i(e_i, e_j)b - \frac{k}{2}e_i^2 - \hat{\theta}_i e_i,$$

which is concave in e_i. The first-order condition is

$$g(e_1 - e_2)b - ke_i - \hat{\theta}_i = 0.$$

The first-order approach is valid if the solution to the first-order condition e_i^* is interior, that is, $e_i^* \in [0, x]$. This is the case if, for all $\hat{\theta}_i, e_j$, it holds that:

$$\left.\frac{\partial \mathbb{E}[\pi_i]}{\partial e_i}\right|_{e_i=0} > 0 \;\Leftarrow\; \bar{\theta}^T < \frac{b}{4x} \text{ and } \left.\frac{\partial \mathbb{E}[\pi_i]}{\partial e_i}\right|_{e_i=x} < 0, \;\Leftarrow\; k > \frac{b}{x^2},$$

which holds by assumption.

To derive action-self i's optimal effort, we need to distinguish the cases $e_i \le e_j$ and $e_i \ge e_j$. First, assume $e_i \le e_j$, then the maximization problem is

$$\max_{e_i \in [0, e_j]} \frac{\left(2x + e_i - e_j\right)^2}{8x^2} b - \frac{k}{2}e_i^2 - \hat{\theta}_i e_i$$

implying

$$e_i = \frac{b}{4x^2k - b}\left(2x - e_j\right) - \frac{4x^2}{4x^2k - b}\hat{\theta}_i \equiv \underline{e}_i(\hat{\theta}_i, e_j).$$

As

$$\underline{e}_i(\hat{\theta}_i, e_j) > e_j \;\Leftrightarrow\; \hat{\theta}_i > \frac{b}{2x} - ke_j,$$

the payoff-maximizing effort choice is given by

$$e_i(\hat{\theta}_i, e_j) = \begin{cases} e_j & \text{if } \hat{\theta}_i < \frac{b}{2x} - ke_j \\ \underline{e}_i(\hat{\theta}_i, e_j) & \text{otherwise.} \end{cases}$$

In analogy, we get for the case $e_i \le e_j$ that the payoff-maximizing effort is given by

$$e_i(\hat{\theta}_i, e_j) = \begin{cases} \bar{e}_i(\hat{\theta}_i, e_j) & \text{if } \hat{\theta}_i \le \frac{b}{2x} - ke_j \\ e_j & \text{otherwise.} \end{cases}$$

If $\hat{\theta}_i < b/2x - ke_j$ ($\hat{\theta}_i \ge b/2x - ke_j$), action-self i prefers to choose $\bar{e}_i(\hat{\theta}_i, e_j)$ ($\underline{e}_i(\hat{\theta}_i, e_j)$) over e_j, so that the best-response function is given by

$$e_i^*(\hat{\theta}_i, e_j) = \begin{cases} \bar{e}_i(\hat{\theta}_i, e_j) & \text{if } \hat{\theta}_i < \frac{b}{2x} - ke_j \\ \underline{e}_i(\hat{\theta}_i, e_j) & \text{otherwise.} \end{cases}$$

We now turn to the self-persuasion choices of the belief-selves. First, if $e_j < b/2xk - \theta_i^T/k$, then $e_i^*(\hat{\theta}_i, e_j) = \bar{e}_i(\hat{\theta}_i, e_j)\; \forall \hat{\theta}_i \in \left[0, \theta_i^T\right]$: no matter belief-self $\hat{\imath}$'s choice, action-self i will choose $e_i \ge e_j$. Accordingly, the maximization problem is

$$\max_{\hat{\theta}_i \in \left[0, \theta_i^T\right]} \bar{e}_i(\hat{\theta}_i, e_j) - c\left(|\hat{\theta}_i - \theta_i^T|\right),$$

for which the first-order condition is

$$\hat{\theta}_i = \theta_i^T - \frac{4x^2}{4x^2k + b}.$$

The optimal level of cognitive dissonance is

$$\hat{\theta}_i^*(e_j) = \begin{cases} 0 & \text{if } \theta_i^T \le \frac{4x^2}{4x^2k+b} \\ \theta_i^T - \frac{4x^2}{4x^2k+b} & \text{otherwise.} \end{cases}$$

Second, $e_i^*(\hat{\theta}_i, e_j) = \underline{e}_i(\hat{\theta}_i, e_j)\, \forall \hat{\theta}_i \in \left[0, \theta_i^T\right]$ if $e_j > b/2xk$: no matter belief-self $\hat{\imath}$'s choice, action-self i will choose $e_i \le e_j$. In analogy to before, the optimal level of cognitive dissonance then is

$$\hat{\theta}_i^*(e_j) = \begin{cases} 0 & \text{if } \theta_i^T \le \frac{4x^2}{4x^2k-b} \\ \theta_i^T - \frac{4x^2}{4x^2k-b} & \text{otherwise.} \end{cases}$$

Third, belief-self $\hat{\imath}$ can induce $e_i \le e_j$ by choosing $\hat{\theta}_i \ge b/2x - ke_j$, or $e_i \ge e_j$ by choosing $\hat{\theta}_i \le b/2x - ke_j$ if $b/2xk - \theta_i^T/k \le e_j \le b/2xk$. Among all $\hat{\theta}_i \in \left[b/2x - ke_j, \theta_i^T\right]$, the payoff-maximizing level of cognitive dissonance for belief-self $\hat{\imath}$ is $\hat{\theta}_i = \max\left\{b/2x - ke_j, \theta_i^T - 4x^2/\left(4x^2k - b\right)\right\}$. The candidate for an optimal choice is

$$\hat{\theta}_i = \frac{b}{2x} - ke_j \text{ if } \theta_i^T \le \frac{4x^2}{4x^2k - b}$$

and

$$\hat{\theta}_i(e_j) = \begin{cases} \theta_i^T - \frac{4x^2}{4x^2k-b} & \text{if } e_j \ge \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k-b)k} \\ \frac{b}{2x} - ke_j & \text{otherwise.} \end{cases}$$

for the case $\theta_i^T > 4x^2/\left(4x^2k - b\right)$.

Among all $\hat{\theta}_i \in \left[0, b/2x - ke_j\right]$, the payoff-maximizing level of cognitive dissonance for belief-self $\hat{\imath}$ is $\hat{\theta}_i = 0$ if $\theta_i^T \le 4x^2/\left(4x^2k + b\right)$ and $\hat{\theta}_i = \min\left\{\theta_i^T - 4x^2/\left(4x^2k + b\right), b/2x - ke_j\right\}$ if $\theta_i^T > 4x^2/\left(4x^2k + b\right)$. Hence, the candidate for an optimal choice is

$$\hat{\theta}_i(e_j) = 0 \;\text{ if }\; \theta_i^T \le \frac{4x^2}{4x^2k + b}$$

and

$$\hat{\theta}_i(e_j) = \begin{cases} \theta_i^T - \frac{4x^2}{4x^2k+b} & \text{if } e_j \le \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \\ \frac{b}{2x} - ke_j & \text{otherwise} \end{cases}$$

for the case $\theta_i^T > 4x^2/\left(4x^2k + b\right)$.

We have analyzed the two cases $\hat{\theta}_i \le b/2x - ke_j$ and $\hat{\theta}_i \ge b/2x - ke_j$ separately. As the cognitive dissonance level $\hat{\theta}_i = b/2x - ke_j$ is included in both cases, it is the optimal choice if and only if it is payoff-maximizing in both cases.

It follows that

$$\hat{\theta}_i^* = 0\, \forall e_j \in \left[\frac{b}{2xk} - \frac{\theta_i^T}{k}, \frac{b}{2xk}\right]$$

if $\theta_i^T \le 4x^2/\left(4x^2k + b\right)$. Furthermore,

$$\hat{\theta}_i^*(e_j) = \begin{cases} \theta_i^T - \frac{4x^2}{4x^2k+b} & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} \le e_j < \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \\ \frac{b}{2x} - ke_j & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \le e_j \le \frac{b}{2xk} \end{cases}$$

if $\theta_i^T \in \left(4x^2/\left(4x^2k + b\right), 4x^2/\left(4x^2k - b\right)\right]$. If $\theta_i^T > 4x^2/\left(4x^2k - b\right)$

$$\hat{\theta}_i^*(e_j) = \begin{cases} \theta_i^T - \frac{4x^2}{4x^2k+b} & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} \le e_j < \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \\ \frac{b}{2x} - ke_j & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \le e_j \le \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k-b)k} \\ \theta_i^T - \frac{4x^2}{4x^2k-b} & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k-b)k} < e_j \le \frac{b}{2xk}. \end{cases}$$

Belief-self $\hat{\imath}$'s optimal degree of self-persuasion, dependent on (the belief about) action-self i's effort choice, is then given by

$$\hat{\theta}_i^* = 0\, \forall e_j \in [0, x]$$

if $\theta_i^T \leq 4x^2/\left(4x^2k+b\right)$, that is, the initial cognitive dissonance is sufficiently small. If $\theta_i^T \in \left(4x^2/\left(4x^2k+b\right), 4x^2/\left(4x^2k-b\right)\right]$,

$$\hat{\theta}_i^*(e_j) = \begin{cases} \theta_i^T - \frac{4x^2}{4x^2k+b} & \text{if } e_j < \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \\ \frac{b}{2x} - ke_j & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \leq e_j \leq \frac{b}{2xk} \\ \theta_i^T - \frac{4x^2}{4x^2k-b} & \text{if } e_j > \frac{b}{2xk}. \end{cases}$$

Finally, for the case $\theta_i^T > 4x^2/\left(4x^2k-b\right)$,

$$\hat{\theta}_i^*(e_j) = \begin{cases} \theta_i^T - \frac{4x^2}{4x^2k+b} & \text{if } e_j < \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \\ \frac{b}{2x} - ke_j & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \leq e_j \leq \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k-b)k} \\ \theta_i^T - \frac{4x^2}{4x^2k-b} & \text{if } e_j > \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k-b)k}. \end{cases}$$

We can now write action-self i's optimal reaction to belief-self $\hat{i}$'s degree of self-persuasion, as a function of the other actions self j's effort.

First, if $\theta_i^T \leq 4x^2/\left(4x^2k+b\right)$,

$$e_i^*(e_j) = \begin{cases} \overline{e}_i(0, e_j) = \frac{b}{4x^2k+b}\left(2x + e_j\right) & \text{if } e_j < \frac{b}{2kx} \\ \underline{e}_i(0, e_j) = \frac{b}{4x^2k-b}\left(2x - e_j\right) & \text{otherwise.} \end{cases}$$

Second, if $\theta_i^T \in \left(4x^2/\left(4x^2k+b\right), 4x^2/\left(4x^2k-b\right)\right]$,

$$e_i^*(e_j) = \begin{cases} \frac{b}{4x^2k+b}\left(2x + e_j\right) - \frac{4x^2}{4x^2k+b}\theta_i^T + \frac{16x^4}{\left(4x^2k+b\right)^2} & \text{if } e_j < \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \\ e_j & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \leq e_j \leq \frac{b}{2xk} \\ \frac{b}{4x^2k-b}\left(2x - e_j\right) - \frac{4x^2}{4x^2k-b}\theta_i^T + \frac{16x^4}{\left(4x^2k-b\right)^2} & \text{if } e_j > \frac{b}{2xk}. \end{cases}$$

Third, if $\theta_i^T > 4x^2/\left(4x^2k-b\right)$,

$$e_i^*(e_j) = \begin{cases} \frac{b}{4x^2k+b}\left(2x + e_j\right) - \frac{4x^2}{4x^2k+b}\theta_i^T + \frac{16x^4}{\left(4x^2k+b\right)^2} & \text{if } e_j < \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \\ e_j & \text{if } \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k+b)k} \leq e_j \leq \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k-b)k} \\ \frac{b}{4x^2k-b}\left(2x - e_j\right) - \frac{4x^2}{4x^2k-b}\theta_i^T + \frac{16x^4}{\left(4x^2k-b\right)^2} & \text{if } e_j > \frac{b}{2xk} - \frac{\theta_i^T}{k} + \frac{4x^2}{(4x^2k-b)k}. \end{cases}$$

If $\theta_i^T \leq 4x^2/\left(4x^2k-b\right)$, there are only symmetric equilibria. $(e_i^*, e_j^*) = (b/2xk, b/2xk)$ is always an equilibrium. It is unique as long as $\theta_j^T \leq 4x^2/\left(4x^2k+b\right)$. If $\theta_j^T > 4x^2/\left(4x^2k+b\right)$, then additional equilibria exist in which action-selves exert lower effort. All effort levels $b/2xk - \theta_j^T/k + 4x^2/\left(4x^2k+b\right)k \leq e \leq b/2xk$ can then be supported in equilibrium.

If $\theta_i^T \geq 4x^2/\left(4x^2k-b\right)$, there exists either one asymmetric equilibrium in which $e_i < e_j$, or a continuum of symmetric equilibria. First, if

$$\theta_j^T < \theta_i^T - \frac{8bx^2}{(4x^2k+b)(4x^2k-b)}$$

holds (initial cognitive dissonances are sufficiently different), then only the asymmetric equilibrium exists. Otherwise, all effort levels $b/2xk - \theta_j^T/k + 4x^2/\left(4x^2k+b\right)k \leq e \leq b/2xk - \theta_i^T/k + 4x^2/\left(4x^2k-b\right)k$ can be supported in a symmetric equilibrium.

For the case of Proposition 1, $\theta_j^T \leq 4x^2/\left(4x^2k+b\right)$, the equilibrium is thus unique. We get $e_i = e_j = b/2xk$ and $\hat{\theta}_i = \hat{\theta}_j = 0$ if $\theta_i^T \leq 4x^2/\left(4x^2k-b\right)$. Otherwise,

$$e_i = \frac{b}{4x^2k-b}(2x - e_j) - \frac{4x^2}{4x^2k-b}\theta_i^T + \frac{16x^4}{(4x^2k-b)^2}$$

and

$$e_j = \frac{b}{4x^2k+b}(2x + e_i),$$

which implies

$$e_i^* = \frac{b}{2xk} - \frac{4x^2k+b}{4x^2k^2}\theta_i^T + \frac{4x^2k+b}{(4x^2k-b)k^2}$$

and

$$e_j^* = \frac{b}{2xk} - \frac{b}{4x^2k^2}\theta_i^T + \frac{b}{(4x^2k-b)k^2}.$$

As $\theta_j^T < 4x^2/\left(4x^2k+b\right)$, $\hat{\theta}_j^* = 0$ holds. Furthermore, $\theta_i^T > 4x^2/\left(4x^2k-b\right)$ implies $e_j^* > b/2xk - \theta_i^T/k + 4x^2/\left(4x^2k-b\right)k$, so that

$$\hat{\theta}_i^* = \theta_i^T - \frac{4x^2}{4x^2k-b}.$$

Hence,

$$\frac{de_i^*}{d\theta_i^T} = -\frac{4x^2k+b}{4x^2k^2}, \quad \frac{de_j^*}{d\theta_i^T} = -\frac{b}{4x^2k^2}$$

and

$$\frac{d\hat{\theta}_i^*}{d\theta_i^T} = 1, \quad \frac{d\hat{\theta}_j^*}{d\theta_i^T} = 0.$$

□

Proof of Corollary 1. Take game $\mathcal{G}_{OS}$ and transfer it to a game $\tilde{\mathcal{G}}_{OS}$ in which each player has two selves along the lines laid out in Section 4.1. In particular, the action-self corresponds to the player in $\mathcal{G}_{OS}$. The belief-self $\hat{i}$ receives no signal, has strategy space $\mathbb{X}_{\hat{i}} = \Theta$ and objective

$$\pi_{\hat{i}}(\sigma, x_{\hat{i}}) = -\sum_{y_i \in Y_i}\left[Q^{F_i^C(\sigma,p^\Omega)}(y_i|x_i,s_i) - Q^{x_{\hat{i}}}(y_i|x_i,s_i)\right]^2.$$

As $\pi_{\hat{i}}$ is continuous due to the continuity of F_i^C, according to Theorem 1, there exist a Dual-Selves Equilibrium in $\tilde{\mathcal{G}}_{OS}$. This Dual-Selves Equilibrium corresponds to a C-Equilibrium in $\mathcal{G}_{OS}$. □

Appendix C: Applications

In this section, we are going to highlight the usefulness of Theorem 1 by extending a number of existence results for finite games to nonfinite games. In particular, we extend Cursed Equilibrium by Eyster and Rabin (2005) (Section C.1), Personal Nash Equilibrium (PNE) by Dato et al. (2017) (Section C.2), Psychological Equilibrium by Geanakoplos et al. (1989) (Section C.3), and Berk–Nash Equilibrium by Esponda and Pouzo (2016) (Section C.4). The game in which each player only has one-self $\mathcal{G}_{OS}$ constitutes the starting point for this analysis. In any Nash Equilibrium of $\mathcal{G}_{OS}$, players form correct beliefs according to the true distribution over consequences, which is given by

$$Q^\sigma(y_i|s_i,x_i) = \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \prod_{j\neq i}\sigma_j\left(dx_j|s_j\right) \times p^{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i).$$

Following Esponda and Pouzo (2016), we refer to this distribution as the *objective* distribution over consequences throughout this section. Moreover, we make the following additional assumptions about game $\mathcal{G}_{OS}$:

Assumption 1. For all $i \in I$, $p^{\Omega\times\mathbb{S}_{-i}|\mathbb{S}_i}$ is absolutely continuous with respect to the product measure $p^\Omega \times p^{\mathbb{S}_1} \times \cdots p^{\mathbb{S}_{i-1}} \times p^{\mathbb{S}_{i+1}} \cdots \times p^{\mathbb{S}_N}$.

Assumption 1 is reminiscent of the first requirement in Milgrom and Weber (1985). However, it requires absolute continuity of *conditional* probability measures instead of *unconditional* probability measures. We consider this assumption to be a rather mild restriction on feasible distributions of states and signals: it is trivially fulfilled if for all players (i) Ω and $\mathbb{S}_i$ are finite, (ii) states and the players' signals are independent, or (iii) $p^{\Omega\times\mathbb{S}_{-i}|\mathbb{S}_i}$ has no mass points (see also Milgrom and Weber 1985). Assumption 1 allows us to simplify the expression for the objective distribution of outcomes such that

$$\begin{aligned} Q^\sigma(y_i|s_i,x_i) &= \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} g_i(\omega, s_{-i}|s_i) p^\Omega(d\omega) \prod_{j\neq i}\sigma_j\left(dx_j, ds_j\right) \\ &= \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x,s) \prod_{j\neq i}\sigma_j\left(dx_j, ds_j\right), \end{aligned}$$

where $g_i(\omega, s_{-i}|s_i)$ is the Radon–Nikodym derivative of $p^{\Omega\times\mathbb{S}_{-i}|\mathbb{S}_i}$ with respect to the product measure $p^{\Omega} \times p^{\mathbb{S}_1} \times \cdots \times p^{\mathbb{S}_{i-1}} \times p^{\mathbb{S}_{i+1}} \times \cdots \times p^{\mathbb{S}_N}$. Furthermore, we assume this probability to be a continuous function.

Assumption 2. $\Pr(\cdot|x, s)$ is a continuous function on $\mathbb{X} \times \mathbb{S}$.

Assumptions 1 and 2 together ensure that as a corollary of Theorem 1 there exist a Nash Equilibrium in the game $\mathcal{G}_{OS}$, that is, an equilibrium, when all players form correct beliefs.

C.1 | Existence of Cursed Equilibrium

Consider a Cursed Equilibrium as defined in Section 4.1. Let $g_i(\omega, s_{-i}|s_i)$ be the same Radon–Nikodym derivative as above. To simplify exposition, we use the following notation: $g_i(s_{-i}|s_i) = \int_{\Omega} g_i(\omega, s_{-i}|s_i)\, p^{\Omega}(d\omega)$ and $g_i(\omega|s_i) = \int_{\mathbb{S}_{-i}} g_i(\omega, s_{-i}|s_i)\, p^{S_{-i}|S_i}(ds_{-i})$. We make the following assumptions on the game $\mathcal{G}_{OS}$ to be able to transfer the notion of a Cursed Equilibrium to our setting.

Assumption 3. $\int_{\Omega} \mathbb{1}_{\{f_i(x_i, x_{-i}, \omega)=y_i\}} g_i(s_{-i}|s_i) \times g_i(\omega|s_i)\, p^{\Omega}(d\omega)$ is bounded and continuous on $\mathbb{X} \times \mathbb{S}$.

Assumption 3 restricts the joint information of the players in the game. It ensures, that the probability with which a fully cursed player anticipates a certain consequence to emerge is bounded and continuous. The following corollary extends the existence result in Eyster and Rabin (2005) to nonfinite games.

Corollary 2. *Suppose that Assumptions 1–3 hold for game $\mathcal{G}_{OS}$. Then, a Cursed Equilibrium exists.*

Proof. Take $Q_i^{\bar{\sigma}}(y_i|s_i, x_i)$ as defined in Section 4.1. First note that because of Assumptions 1–3, we can express this distribution over consequences by

$$
\begin{aligned}
Q_i^{\bar{\sigma}}(y_i|s_i, x_i) &= \mathcal{X} \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \bar{\sigma}_{-i}(dx_{-i}|s_i) \times p^{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i) \\
&\quad + (1-\mathcal{X}) \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \prod_{j\neq i} \sigma_j\left(dx_j|s_j\right) \times p^{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i) \\
&= \mathcal{X} \int_{\Omega\times\mathbb{X}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \bar{\sigma}_{-i}(dx_{-i}|s_i) \times \int_{\mathbb{S}_{-i}} p^{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i) \\
&\quad + (1-\mathcal{X}) \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \prod_{j\neq i} \sigma_j\left(dx_j|s_j\right) \times p^{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i) \\
&= \mathcal{X} \int_{\Omega\times\mathbb{X}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \bar{\sigma}_{-i}(dx_{-i}|s_i) \times \int_{\mathbb{S}_{-i}} g_i(\omega, s_{-i}|s_i) \prod_{j\neq i} p^{\mathbb{S}_j}\left(ds_j\right) p^{\Omega}(d\omega) \\
&\quad + (1-\mathcal{X}) \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \prod_{j\neq i} \sigma_j\left(dx_j|s_j\right) \times p^{\Omega\times S_{-i}|S_i}(d\omega, ds_{-i}|s_i) \\
&= \mathcal{X} \int_{\Omega\times\mathbb{X}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \bar{\sigma}_{-i}(dx_{-i}|s_i) \times g_i(\omega|s_i) p^{\Omega}(d\omega) \\
&\quad + (1-\mathcal{X}) \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} g_i(\omega, s_{-i}|s_i) p^{\Omega}(d\omega) \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right) \\
&= \mathcal{X} \int_{\Omega\times\mathbb{X}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} \int_{\mathbb{S}_{-i}} g_i(s_{-i}|s_i) \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right) \times g_i(\omega|s_i) p^{\Omega}(d\omega) \\
&\quad + (1-\mathcal{X}) \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} g_i(\omega, s_{-i}|s_i) p^{\Omega}(d\omega) \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right) \\
&= \mathcal{X} \int_{\Omega\times\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} g_i(s_{-i}|s_i) \times g_i(\omega|s_i) p^{\Omega}(d\omega) \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right) \\
&\quad + (1-\mathcal{X}) \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x, s) \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right) \\
&= \mathcal{X} \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \left[\int_{\Omega} \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} g_i(s_{-i}|s_i) \times g_i(\omega|s_i) p^{\Omega}(d\omega)\right] \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right) \\
&\quad + (1-\mathcal{X}) \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x, s) \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right),
\end{aligned}
$$

where

$$\Pr(y_i|x,s) = \int_\Omega \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} g_i(\omega, s_{-i}|s_i) p^\Omega(d\omega).$$

The application of Theorem 1 requires the following steps. First, we have to define a game $\tilde{\mathcal{G}}_{OS}$, which reflects the structure of game $\mathcal{G}_{OS}$ but features players with dual-selves. As laid out in Section 4.1, this requires us to define (i) the space $\mathbb{X}_{\hat{i}}$, that specifies the action sets of the belief-selves (ii) how these actions relate to the consequences that the action-selves anticipate $Q_i^{x_{\hat{i}}}$, and (iii) the payoffs of the belief-selves $\pi_{\hat{i}}(\cdot)$. If all three of these objects satisfy the requirements in Theorem 1, there exists a Dual-Selves Equilibrium in the game $\tilde{\mathcal{G}}_{OS}$. Finally, we have to argue that given $\mathbb{X}_{\hat{i}}$, $Q_i^{x_{\hat{i}}}$ and $\pi_{\hat{i}}(\cdot)$ the Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ is indeed a Cursed Equilibrium in game $\mathcal{G}_{OS}$.

Define $\mathbb{X}_{\hat{i}} = \Sigma_{-i}$, that is, the set of all combinations of distributional strategies of action-selves other than i. Moreover, define

$$Q_i^{x_{\hat{i}}}(y_i|s_i,x_i) = \mathcal{X} \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \left[\int_\Omega \mathbb{1}_{\{f_i(x_i,x_{-i},\omega)=y_i\}} g_i(s_{-i}|s_i) \times g_i(\omega|s_i) p^\Omega(d\omega)\right] x_{\hat{i}}(dx_{-i}, ds_{-i})$$
$$+ (1-\mathcal{X}) \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x,s) x_{\hat{i}}(dx_{-i}, ds_{-i}),$$

and $\pi_{\hat{i}}$ such that

$$\pi_{\hat{i}}(\sigma, x_{\hat{i}}) = -\sum_{y_i\in\mathbb{Y}_i} (Q_i^{\bar{\sigma}}(y_i|x_i,s_i) - Q_i^{x_{\hat{i}}}(y_i|x_i,s_i))^2.$$

By the same arguments as in *Step 1a* in the proof of Theorem 1, the belief-selves strategy sets Σ_{-i} are compact metric spaces in the weak topology. Furthermore, the set of strategies is clearly convex. We seek to show that for any sequence with $\left(\sigma^n, x_{\hat{i}}^n\right) \to \left(\sigma, x_{\hat{i}}\right)$ it also holds that $\pi_{\hat{i}}\left(\sigma^n, x_{\hat{i}}^n\right) \to \pi_{\hat{i}}\left(\sigma, x_{\hat{i}}\right)$.

$$\lim_{n\to\infty} \pi_{\hat{i}}\left(\sigma^n, x_{\hat{i}}^n\right) = \lim_{n\to\infty} -\sum_{y_i\in\mathbb{Y}_i} (Q_i^{\bar{\sigma}^n}(y_i|x_i,s_i) - Q_i^{x_{\hat{i}}^n}(y_i|x_i,s_i))^2$$
$$= -\sum_{y_i\in\mathbb{Y}_i} (\lim_{n\to\infty} Q_i^{\bar{\sigma}^n}(y_i|x_i,s_i) - \lim_{n\to\infty} Q_i^{x_{\hat{i}}^n}(y_i|x_i,s_i))^2.$$

As $Q_i^{\bar{\sigma}^n}(y_i|x_i,s_i)$ and $Q_i^{x_{\hat{i}}^n}(y_i|x_i,s_i)$ are integrals over continuous and bounded functions for all y_i over the set $\mathbb{X}\times\mathbb{S}$, we can apply Theorem 1.1 in Feinberg et al. (2014). Consequently, $\pi_{\hat{i}}\left(\sigma, x_{\hat{i}}\right)$ is continuous. Moreover, the expected payoff of action-self i is given by

$$\int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} \pi_i(x_i,y_i) Q_i^{x_{\hat{i}}}(y_i|s_i,x_i) d\sigma_i(x_i,s_i),$$

which is continuous in the weak topology by Feinberg et al. (2014). We conclude that by Theorem 1 a Dual-Selves Equilibrium exist.

In any Dual-Selves Equilibrium of game $\tilde{\mathcal{G}}_{OS}$, beliefs are chosen to maximize $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu(dx_{\hat{i}})$. The proposed $\pi_{\hat{i}}$, however, is nonpositive and the action of the belief-self that achieves $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu(dx_{\hat{i}}) = 0$ is in the set $\mathbb{X}_{\hat{i}}$ for all i. In fact, in equilibrium the beliefs satisfy $Q_i^{x_{\hat{i}}}(y_i|x_i,s_i) = Q_i^{\bar{\sigma}}(y_i|x_i,s_i)$ for all y_i, s_i, x_i and all players $i \in \{1,\dots,N\}$. Hence, the derived strategy profile constitutes a Cursed Equilibrium in game $\mathcal{G}_{OS}$. □

C.2 | Existence of PNE

Next, we turn to the concept of PNE. In analogy to Kőszegi and Rabin (2006) and Dato et al. (2017), we define the utility of a loss-averse player as follows:

$$U_i(\sigma_i, \hat{\sigma}_i, \sigma_{-i}) = \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} \pi_i(x_i,y_i) Q_i^{\sigma}(y_i|s_i,x_i) d\sigma_i(x_i,s_i)$$
$$+ \eta \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} Q_i^{\sigma}(y_i|s_i,x_i) \left[\int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{\tilde{y}_i\in\mathbb{Y}_i} Q_i^{\sigma}(\tilde{y}_i|\tilde{s}_i,\tilde{x}_i) h\left[\pi_i(x_i,y_i)\right.\right.$$
$$\left.\left. - \pi_i(\tilde{x}_i,\tilde{y}_i)\right] \hat{\sigma}_i(d\tilde{x}_i, d\tilde{s}_i)\right] \sigma_i(dx_i, ds_i)$$
$$= \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} \left[\pi_i(x_i,y_i) + \eta \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{\tilde{y}_i\in\mathbb{Y}_i} Q_i^{\sigma}(\tilde{y}_i|\tilde{s}_i,\tilde{x}_i) h\left[\pi_i(x_i,y_i)\right.\right.$$
$$\left.\left. - \pi_i(\tilde{x}_i,\tilde{y}_i)\right] \hat{\sigma}_i(d\tilde{x}_i, d\tilde{s}_i)\right] \cdot Q_i^{\sigma}(y_i|s_i,x_i) d\sigma_i\left(x_i,s_i\right),$$

where $\hat{\sigma}_i$ denotes player i's belief about the own strategy, and $h[\cdot] : \mathbb{R} \to \mathbb{R}$ is a continuous gain–loss function. According to Dato et al. (2017), a PNE of the game is defined as follows:

Definition 4. A strategy profile σ is a PNE of game $\mathcal{G}_{OS}$ if, for all players $i \in \{1, \dots, N\}$, we have $U_i(\sigma_i, \sigma_i, \sigma_{-i}) \geq U_i(\tilde{\sigma}_i, \sigma_i, \sigma_{-i})$ for all $\tilde{\sigma}_i \in \Sigma_i$.

This objective function directly depends on the players' beliefs about their own strategy in equilibrium, because their reference point is shaped by their expectations about their own future choices. Consequently, the standard assumptions of Nash Equilibrium do not apply in game $\mathcal{G}_{OS}$. However, in Dual-Selves games, we allow the belief-selves' actions to influence the payoff of the corresponding action-self directly. We can, therefore, apply Theorem 1 to extend the existence result in Dato et al. (2017) to nonfinite games.

Corollary 3. *Assume that Assumption 1 holds. Then there exists a PNE in game $\mathcal{G}_{OS}$.*

Proof. The application of Theorem 1 requires the following steps. First, we have to define a game $\tilde{\mathcal{G}}_{OS}$, which reflects the structure of game $\mathcal{G}_{OS}$ but features players with dual-selves. As laid out in Section 4.1, this requires us to define (i) the space $\mathbb{X}_{\hat{i}}$, that specifies the action sets of the belief-selves, (ii) how these actions relate to the consequences that the action-selves anticipate $Q_i^{x_{\hat{i}}}$, and (iii) the objectives of the belief-selves $\pi_{\hat{i}}(\cdot)$. If all three of these objects satisfy the requirements in Theorem 1, there exists a Dual-Selves Equilibrium in the game $\tilde{\mathcal{G}}_{OS}$. Finally, we have to argue that given $\mathbb{X}_{\hat{i}}$, $Q_i^{x_{\hat{i}}}$ and $\pi_{\hat{i}}(\cdot)$ the Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ is indeed a PNE in game $\mathcal{G}_{OS}$.

Define $\mathbb{X}_{\hat{i}}$ to be equal to Σ, where Σ is the set of distributional strategies. Moreover, define $Q_i^{x_{\hat{i}}}$ such that

$$Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) = \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x, s) x_{\hat{i}}(dx_{-i}, ds_{-i})$$

and $\pi_{\hat{i}}$ as follows:

$$\pi_{\hat{i}}(\sigma, x_{\hat{i}}) = -\sum_{y\in\mathbb{Y}_i} (Q_i^{x_{\hat{i}}}(y|x_i, s_i) - Q_i^{\sigma}(y|x_i, s_i))^2.$$

The expected payoff of the action-self given the belief $x_{\hat{i}}$ is then given by

$$\begin{aligned} U_i\left(\sigma_i, x_{\hat{i}}\right) &= \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} \pi_i(x_i, y_i) Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) d\sigma_i(x_i, s_i) \\ &+ \eta \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) \Bigg[\int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{\tilde{y}_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}(\tilde{y}_i|\tilde{s}_i, \tilde{x}_i) h[\pi_i(x_i, y_i) \\ &\qquad - \pi_i(\tilde{x}_i, \tilde{y}_i)] x_{\hat{i}}(d\tilde{x}_i, d\tilde{s}_i) \Bigg] \sigma_i(dx_i, ds_i). \end{aligned}$$

Note that we do not need and therefore have dropped $x_{\hat{i}}$ as an argument of the payoff function of action-self i. By the same arguments as in *Step 1a* from the proof of Theorem 1, the belief-selves strategy sets Σ are compact metric spaces in the weak topology. Furthermore, the set of strategies is clearly convex.

The continuity of $Q_i^{x_{\hat{i}}}$ on $\mathbb{X}_{\hat{i}} \times \mathbb{S}_i \times \mathbb{X}_i$ and the continuity of $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}}) h(dx_{\hat{i}})$ on $\mathcal{M}(\mathbb{X}_{\hat{i}}) \times \Sigma$ follow from the same arguments as in Section C.1.

Next, we turn to the continuity of $U_i(\sigma_i, x_{\hat{i}})$ in the weak topology. For any sequence $\lim_{n\to\infty}(\sigma_i^n, x_{\hat{i}}^n) \to (\sigma_i, x_{\hat{i}})$, we get

$$\begin{aligned} \lim_{n\to\infty} U_i\left(\sigma_i^n, x_{\hat{i}}^n\right) &= \lim_{n\to\infty} \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} \pi_i(x_i, y_i) Q_i^{x_{\hat{i}}^n}(y_i|s_i, x_i) d\sigma_i^n(x_i, s_i) \\ &+ \lim_{n\to\infty} \eta \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}^n}(y_i|s_i, x_i) \left[\int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{\tilde{y}_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}^n}(\tilde{y}_i|\tilde{s}_i, \tilde{x}_i) h\left[\pi_i(x_i, y_i) - \pi_i(\tilde{x}_i, \tilde{y}_i)\right] x_{\hat{i}}^n(d\tilde{x}_i, d\tilde{s}_i) \right] \sigma_i^n(dx_i, ds_i). \end{aligned}$$

Making use of the insights from above and Assumption 1, we can rewrite Q_i^{σ} and $Q_i^{x_{\hat{i}}}$ such that

$$Q_i^{\sigma}(y_i|s_i, x_i) = \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x, s) \prod_{j\neq i} \sigma_j\left(dx_j, ds_j\right)$$

and

$$Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) = \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x, s) \prod_{j\neq i} x_{\hat{i},j}\left(dx_j, ds_j\right),$$

where $x_{\hat{i},j}$ denotes self $\hat{i}$'s belief about the strategy of action-self j. The continuity and boundedness of $\Pr(y_i|x,s)$ implies that both probability distributions are continuous and bounded. Applying Feinberg et al. (2014), we get

$$\lim_{n\to\infty} U_i\left(\sigma_i^n, x_{\hat{i}}^n\right) = \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} \pi_i(x_i,y_i) Q_i^{x_{\hat{i}}}(y_i|s_i,x_i)d\sigma_i(x_i,s_i)$$

$$+ \lim_{n\to\infty} \eta \int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{y_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}(y_i|s_i,x_i)\left[\int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{\tilde{y}_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}(\tilde{y}_i|\tilde{s}_i,\tilde{x}_i)h[\pi_i(x_i,y_i) - \pi_i(\tilde{x}_i,\tilde{y}_i)]x_{\hat{i}}^n(d\tilde{x}_i,d\tilde{s}_i)\right]\sigma_i^n(dx_i,ds_i).$$

To exchange limits, we have to show that the function in the integral is bounded and continuous on $\mathbb{X}_i \times \mathbb{S}_i$. For every sequence $(x_i^n, s_i^n) \to (x_i, s_i)$, we have

$$\lim_{n\to\infty} \sum_{y_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}\left(y_i|s_i^n,x_i^n\right)\left[\int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{\tilde{y}_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}(\tilde{y}_i|\tilde{s}_i,\tilde{x}_i)h\left[\pi_i\left(x_i^n,y_i\right) - \pi_i(\tilde{x}_i,\tilde{y}_i)\right]x_{\hat{i}}(d\tilde{x}_i,d\tilde{s}_i)\right]$$

$$= \sum_{y_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}(y_i|s_i,x_i)\left[\int_{\mathbb{X}_i\times\mathbb{S}_i} \sum_{\tilde{y}_i\in\mathbb{Y}_i} Q_i^{x_{\hat{i}}}(\tilde{y}_i|\tilde{s}_i,\tilde{x}_i)h[\pi_i(x_i,y_i) - \pi_i(\tilde{x}_i,\tilde{y}_i)]x_{\hat{i}}(d\tilde{x}_i,d\tilde{s}_i)\right]$$

since the sum and multiplication of continuous and bounded functions is continuous and bounded. As a consequence, we get

$$\lim_{n\to\infty} U_i\left(\sigma_i^n, x_{\hat{i}}^n\right) = U_i\left(\sigma_i, x_{\hat{i}}\right).$$

Therefore, a Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ exists. In any Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ beliefs are chosen to maximize $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu(dx_{\hat{i}})$. The proposed $\pi_{\hat{i}}$, however, is nonpositive and the belief that achieves $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu(dx_{\hat{i}}) = 0$ is in the set $\mathbb{X}_{\hat{i}}$ for all i. In fact, in equilibrium the beliefs satisfy $Q_i^{x_{\hat{i}}}(y|x_i,s_i) = Q_i^{\sigma}(y|x_i,s_i)$ for all s_i, x_i and all action-selves $i \in \{1,\dots,N\}$. Hence, the derived strategy profile constitutes a PNE in game $\mathcal{G}_{OS}$. □

C.3 | Existence of Psychological Equilibrium

Next, consider the concept of psychological games. Take game $\mathcal{G}_{OS}$ in which each player only has one-self. In analogy to Geanakoplos et al. (1989), define the beliefs of these players as follows. A first-order belief is a probability measure over the product of the strategy sets of all other players. Hence, the set of first-order beliefs of player i is given by $B_i^1 = \mathcal{M}(\Sigma_{-i})$. The sets of higher-order beliefs are then inductively defined as

$$\begin{aligned} B_i^k &= \mathcal{M}(\Sigma_{-i} \times B_{-i}^1 \times \cdots \times B_{-i}^k) & & \\ B_{-i}^k &= \times_{j\neq i} B_j^k & B^k &= \times_i B_i^k \\ B_i &= \times_{k\in\{1,\cdots,K\}} B_i^k & B &= \times_i B_i. \end{aligned}$$

For simplicity, we assume that the payoff-relevant beliefs are of finite order K. Player i's payoff if she holds belief $b_i \in B_i$, she plays x_i and outcome y_i is realized is then $\pi_i(x_i, y_i, b_i) : \mathbb{X}_i \times \mathbb{Y}_i \times B_i \to \mathbb{R}$ and expected payoff is given by

$$U_i(b_i,\sigma) = \int_{\mathbb{X}_i\times\mathbb{S}_i} Q_i^{\sigma}(y_i|s_i,x_i)\pi_i(x_i,y_i,b_i)\sigma_i(dx_i,ds_i).$$

We make the following assumption on the game $\mathcal{G}_{OS}$ to transfer the notion of a Psychological Equilibrium to our setting.

Assumption 4. $\pi_i(x_i, y_i, b_i)$ is continuous on $\mathbb{X}_i \times \mathbb{Y}_i \times B_i$.

Geanakoplos et al. (1989) require that beliefs about other players' actions are correct in Equilibrium. If σ is an Equilibrium profile of strategies, all players therefore expect their opponents to play σ_{-i}, and that each opponent j expects that all other players play σ^{-j} and so on. Denote the corresponding belief system by $\beta(\sigma) = (\beta_1(\sigma), \dots, \beta_n(\sigma)) \in B$. According to Geanakoplos et al. (1989), a Psychological Equilibrium is then defined as follows:

Definition 5. A profile $(b, \sigma) \in B \times \Sigma$ is a Psychological Equilibrium of game $\mathcal{G}_{OS}$ if:

i. $b = \beta(\sigma)$;
ii. for all players $i \in \{1,\dots,N\}$, we have $U_i(b_i, (\sigma_i, \sigma_{-i})) \geq U_i(b_i, (\tilde{\sigma}_i, \sigma_{-i}))$ for all $\tilde{\sigma}_i \in \Sigma_i$.

The following corollary extends the result in Geanakoplos et al. (1989) to nonfinite games.

Corollary 4. *Suppose that Assumptions 1, 2, and 4 hold, then there exist a Psychological Equilibrium in $\mathcal{G}_{OS}$.*

Proof. The application of Theorem 1 requires the following steps. First, we have to define a game $\tilde{\mathcal{G}}_{OS}$, which reflects the structure of game $\mathcal{G}_{OS}$ but features players with dual-selves. As laid out in Section 4.1, this requires us to define (i) the space $\mathbb{X}_{\hat{i}}$, that specifies the action sets of the belief-selves, (ii) how these actions relate to the consequences that the action-selves anticipate $Q_i^{x_{\hat{i}}}$, and (iii) the objectives of the belief-selves $\pi_{\hat{i}}(\cdot)$. If all three of these objects satisfy the requirements in Theorem 1, then there exists a Dual-Selves Equilibrium in the game $\tilde{\mathcal{G}}_{OS}$. Finally, we have to argue that given $\mathbb{X}_{\hat{i}}$, $Q_i^{x_{\hat{i}}}$ and $\pi_{\hat{i}}(\cdot)$ the Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ is indeed a Psychological Equilibrium in game $\mathcal{G}_{OS}$. Define $\mathbb{X}_{\hat{i}}$ to be equal to B_i and $\pi_{\hat{i}}$ as follows:

$$\pi_{\hat{i}}(\sigma, x_{\hat{i}}) = -d(x_{\hat{i}}, \beta(\sigma)),$$

where $d(\cdot,\cdot)$ is a bounded metric on B, which exists and is continuous because B_i is a metric space.[16]

The action spaces are assumed to be compact, and hence they and their Cartesian-product are also complete (see De la Fuente 2000, Theorem 8.16, 96). By Theorem 3.2 in Parthasarathy (1967), Σ is then a tight set of probability measures. Also, since it is a set of measures over separable and complete metric spaces, it is complete itself by Theorem 6.5 in Parthasarathy (1967) and hence closed in the weak topology. With the same argument, the sets B_i^k are closed because they are the Cartesian product of sets of measures over complete and compact metric spaces. By Prohorov's Theorem, it follows that the strategy sets Σ, and the belief sets B_i are compact metric spaces in the weak topology. Furthermore, the set of strategies is clearly convex.

Since the metric is continuous and bounded on $B_i \times B_i$ also $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu(dx_{\hat{i}})$ is continuous on $\mathcal{M}(\mathbb{X}_{\hat{i}}) \times \Sigma$. Moreover, the expected payoff functions are continuous since $Q_i^{\sigma}(y_i|s_i, x_i)\pi_i(x_i, y_i, b_i)$ is continuous and bounded (for a proof, see the proofs of Corollaries 3 and 5). Therefore, a Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ exists. In any Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ beliefs are chosen to maximize $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu(dx_{\hat{i}})$. The proposed $\pi_{\hat{i}}$, however, is nonpositive and the belief that achieves $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu(dx_{\hat{i}}) = 0$ is in the set $\mathbb{X}_{\hat{i}}$ for all i. In fact, in equilibrium the belief needs to satisfy $b_i = \beta_i(\sigma)$ for all action-selves $i \in \{1, \dots, N\}$. Hence, the derived strategy profile has to constitute a Psychological Equilibrium. □

C.4 | Berk–Nash Equilibrium

To transfer the notion of a Berk–Nash Equilibrium introduced by Esponda and Pouzo (2016) to our setting, consider the game $\mathcal{G}_{OS}$ in which every player has only one-self. As the optimality criterion that determines how beliefs about primitives and opponents' strategies are chosen, Esponda and Pouzo (2016) make use of the weighted Kullback–Leibler divergence (wKLD). Transferred to our notation, the wKLD for action-self i is given by

$$K_i(\sigma, x_{\hat{i}}) = \int_{s_i \in \mathbb{S}_i} \int_{x_i \in \mathbb{X}_i} E^{Q_i^{\sigma}(\cdot|s_i, x_i)} \left[\ln \frac{Q_i^{\sigma}(y_i|s_i, x_i)}{Q_i^{x_{\hat{i}}}(y_i|s_i, x_i)} \right] \sigma_i(dx_i|s_i) \cdot p^{S_i}(ds_i), \tag{C.1}$$

where $x_{\hat{i}} \in \mathbb{X}_{\hat{i}}$ reflects the model that player i has formed about the game. Essentially, this model summarizes what player i beliefs about the structure of the game and the strategies that her opponents will play (see Esponda and Pouzo 2016). We define the set of parameter values $\mathbb{X}_{\hat{i}}(\sigma) \subset \mathbb{X}_{\hat{i}}$ to be the set of all beliefs such that the wKLD is minimized:

$$\mathbb{X}_{\hat{i}}(\sigma) \equiv \arg\min_{x_{\hat{i}} \in \mathbb{X}_{\hat{i}}} K_i(\sigma, x_{\hat{i}}).$$

The interpretation is that $\mathbb{X}_{\hat{i}}(\sigma)$ is the set of beliefs inducing probability distributions over outcomes that best match the objective distribution over outcomes. A Berk–Nash Equilibrium requires each player to choose a strategy that is optimal given her beliefs. A strategy σ_i for player i is optimal given $\mu_i \in \mathcal{M}(\mathbb{X}_{\hat{i}})$ if $\sigma_i(x_i, s_i) > 0$ implies that

$$x_i \in \arg\max_{\bar{x}_i \in \mathbb{X}_i} E^{\bar{Q}_i^{\mu_i}(\cdot|s_i, \bar{x}_i)}[\pi_i(\bar{x}_i, y_i)], \tag{C.2}$$

where $\bar{Q}_i^{\mu_i}(\cdot|s_i, x_i) = \int_{\mathbb{X}_{\hat{i}}} Q_i^{x_{\hat{i}}}(\cdot|s_i, x_i)\mu_i(dx_{\hat{i}})$ is the distribution over consequences of player i, conditional on $(s_i, x_i) \in \mathbb{S}_i \times \mathbb{X}_i$, induced by μ_i. According to Esponda and Pouzo (2016), a Berk–Nash Equilibrium is defined as follows.

Definition 6. A strategy profile σ is a Berk–Nash Equilibrium of game $\mathcal{G}_{OS}$ if, for all players $i \in I$, there exists $\mu_i \in \mathcal{M}(\mathbb{X}_{\hat{i}})$ such that (i) σ_i is optimal given μ_i, and (ii) $\mu_i \in \mathcal{M}(\mathbb{X}_{\hat{i}}(\sigma))$, that is, if $\hat{x}_i$ is in the support of μ_i, then $\hat{x}_i \in \arg\min_{x_{\hat{i}} \in \mathbb{X}_{\hat{i}}} K_i(\sigma, x_{\hat{i}})$

Similar to Esponda and Pouzo (2016), we maintain the following assumption about feasible beliefs.

Assumption 5. For all $i \in I$ and for all $\sigma_{-i} \in \Sigma_{-i}$, $y_i \in \mathbb{Y}_i$ and $(x_{\hat{i}}^n, s_i^n, x_i^n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(x_{\hat{i}}^n, s_i^n, x_i^n) = (x_{\hat{i}}, s_i, x_i)$ there exists $\epsilon > 0$ such that if $Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) < \epsilon$, there exist $n' \in \mathbb{N}$ and $m \in \mathbb{N}$ with $m < \infty$ such that $Q^{\sigma}\left(y_i|s_i^{n''}, x_i^{n''}\right)^m \leq Q_i^{x_{\hat{i}}^{n''}}\left(y_i|s_i^{n''}, x_i^{n''}\right)$ for all $n'' > n'$.

This assumption restricts misspecified beliefs that attach zero probability to some outcomes given other players' strategies in two ways. First, it requires that no misspecification renders a consequence impossible if it, in fact, occurs with positive probability. Second, we also impose a similar restriction on consequences in an ϵ-environment of subjectively impossible consequences: When approaching a subjective model that renders a consequence impossible, the subjective probability must not converge zero at a much faster rate than the objective probability. While the latter is trivially fulfilled in any discrete game, the first part reduces the set of covered misspecifications compared to Esponda and Pouzo (2016). For instance, it rules out the belief that other action-selves only follow pure strategies in some discrete games. Nevertheless, a broad class of models fulfills Assumption 5. For example, all combinations of games and misspecified beliefs in which players attach positive subjective probabilities to every consequence fulfill this assumption.

Corollary 5. *Suppose that Assumptions 1, 2, and 5 hold, then a Berk–Nash Equilibrium exists.*

Proof. Again, the application of Theorem 1 requires us to define a game $\tilde{\mathcal{G}}_{OS}$, which reflects the structure of game $\mathcal{G}_{OS}$ but features players with dual-selves. As laid out in Section 4.1, this requires us to define $\pi_{\hat{i}}$ additionally to the set of actions of the belief-selves which is given by the set of potential models $\mathbb{X}_{\hat{i}}$. Take $\pi_{\hat{i}}\left(\sigma, x_{\hat{i}}\right) = -K_i\left(\sigma, x_{\hat{i}}\right)$. Then from Definition 6 we can infer that a Dual-Selves Equilibrium in game $\tilde{\mathcal{G}}_{OS}$ constitutes a Berk–Nash Equilibrium in $\mathcal{G}_{OS}$. Therefore, it suffices to show that we can apply Theorem 1. We can do so if $K_i\left(\sigma, x_{\hat{i}}\right)$ is continuous.

Therefore, we seek to show that for any sequence with $\left(\sigma^n, \mu_{\hat{i}}^n\right) \to \left(\sigma, \mu_{\hat{i}}\right)$ it also holds that $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma^n, x_{\hat{i}})\mu_{\hat{i}}^n(dx_{\hat{i}}) \to \int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu_{\hat{i}}(dx_{\hat{i}})$. We get

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma^n, x_{\hat{i}})\mu_{\hat{i}}^n(dx_{\hat{i}}) = \lim_{n\to\infty} -\int_{\mathbb{X}_{\hat{i}}} K_i\left(\sigma^n, x_{\hat{i}}\right)\mu_{\hat{i}}^n\left(dx_{\hat{i}}\right)\\
&= \lim_{n\to\infty} -\int_{\mathbb{X}_{\hat{i}}\times\mathbb{S}_i\times\mathbb{X}_i} E^{Q_i^{\sigma^n}(\cdot|s_i,x_i)}\left[\ln\frac{Q_i^{\sigma^n}(y_i|s_i,x_i)}{Q_i^{x_{\hat{i}}}(y_i|s_i,x_i)}\right]\sigma_i^n(dx_i|s_i)\cdot p^{S_i}(ds_i)\mu_{\hat{i}}^n\left(dx_{\hat{i}}\right)\\
&= \lim_{n\to\infty} \int_{\mathbb{X}_{\hat{i}}\times\mathbb{S}_i\times\mathbb{X}_i} -\sum_{y_i\in\mathbb{Y}_i} Q_i^{\sigma^n}(y_i|s_i,x_i)\ln Q_i^{\sigma^n}(y_i|s_i,x_i)\sigma_i^n(dx_i,ds_i)\mu_{\hat{i}}^n\left(dx_{\hat{i}}\right) \qquad \text{(C.3)}
\end{aligned}
$$

$$
+\lim_{n\to\infty} \int_{\mathbb{X}_{\hat{i}}\times\mathbb{S}_i\times\mathbb{X}_i} \sum_{y_i\in\mathbb{Y}_i} Q_i^{\sigma^n}(y_i|s_i,x_i)\ln Q_i^{x_{\hat{i}}}(y_i|s_i,x_i)\sigma_i^n(dx_i,ds_i)\mu_{\hat{i}}^n\left(dx_{\hat{i}}\right). \qquad \text{(C.4)}
$$

First, consider the term in (C.3). To show continuity on $\mathbb{X}_i \times \mathbb{S}_i$, we first start by arguing that the integral and the limit in (C.3) can be exchanged. For this purpose, we show that $\sum_{y_i\in\mathbb{Y}_i} -Q_i^{\sigma^n}(y_i|s_i,x_i)\ln Q_i^{\sigma^n}(y_i|s_i,x_i)$ is bounded and continuous on $\mathbb{X}_i \times \mathbb{S}_i$. For any sequence (x_i^n, s_i^n) with $\lim_{n\to\infty}(x_i^n, s_i^n) = (x_i, s_i)$, we get

$$
\lim_{n\to\infty} Q_i^{\sigma}\left(y_i|s_i^n, x_i^n\right) = \lim_{n\to\infty} \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr\left(y_i|(x_i^n, x_{-i}), (s_i^n, s_{-i})\right)\prod_{j\neq i}\sigma_j\left(dx_j, ds_j\right).
$$

Since $\Pr\left(y_i|(x_i^n, x_{-i}), (s_i^n, s_{-i})\right)$ is bounded and continuous on $\mathbb{X}\times\mathbb{S}$ $\forall\, y_i$ and i, we can apply Theorem 1.1 in Feinberg et al. (2014) and get

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr\left(y_i|(x_i^n, x_{-i}), (s_i^n, s_{-i})\right)\prod_{j\neq i}\sigma_j\left(dx_j, ds_j\right)\\
&= \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|(x_i, x_{-i}), (s_i, s_{-i}))\prod_{j\neq i}\sigma_j\left(dx_j, ds_j\right) = Q_i^{\sigma}(y_i|s_i, x_i),
\end{aligned}
$$

which implies continuity of $Q_i^{\sigma}(y_i|s_i,x_i)$. It is also bounded as $Q_i^{\sigma}(y_i|s_i,x_i) \in [0,1]$ holds. Hence, the function $\sum_{y_i\in\mathbb{Y}_i} -Q_i^{\sigma^n}(y_i|s_i,x_i)\ln Q_i^{\sigma^n}(y_i|s_i,x_i)$ is bounded and continuous. By Theorem 1.1 in Feinberg et al. (2014) and the continuity of $Q_i^{\sigma}(y_i|s_i,x_i)$ on $\mathbb{X}_i\times\mathbb{S}_i$, we can thus exchange the limit and the integral in (C.3). In addition, we have to show that $\sigma^n \to \sigma$ implies $Q_i^{\sigma^n}(y_i|s_i,x_i) \to Q_i^{\sigma}(y_i|s_i,x_i)$. Since $\mathbb{X}_i$ and $\mathbb{X}_{-i}$ are compact metric spaces they are complete, by De la Fuente (2000, Theorem 8.16, 96), and separable. Furthermore, $\mathbb{S}_i$ and $\mathbb{S}_{-i}$ are complete separable metric spaces such that $\mathbb{X}_i\times\mathbb{S}_i$ and $\mathbb{X}_{-i}\times\mathbb{S}_{-i}$ are complete and separable. Hence, we can apply Theorem 2.8 from Billingsley (1999) and deduce that $\sigma^n \to \sigma$ implies $\sigma_{-i}^n \to \sigma_{-i}$. Using this insight, we can apply Theorem 1.1 in Feinberg et al. (2014) because $\Pr(y_i|x,s)$ is a bounded and continuous function on the space $\mathbb{X}_{-i}\times\mathbb{S}_{-i}$ $\forall\, y_i$ and i:

$$
\lim_{n\to\infty} \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x,s)\prod_{j\neq i}\sigma_j^n\left(dx_j, ds_j\right) = \int_{\mathbb{X}_{-i}\times\mathbb{S}_{-i}} \Pr(y_i|x,s)\prod_{j\neq i}\sigma_j\left(dx_j, ds_j\right),
$$

which is equivalent to $\lim_{n\to\infty} Q_i^{\sigma^n}(y_i|s_i,x_i) = Q_i^{\sigma}(y_i|s_i,x_i)$. Hence,

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{X}_{\hat{i}}\times\mathbb{S}_i\times\mathbb{X}_i} -\sum_{y_i\in\mathbb{Y}_i} Q_i^{\sigma^n}(y_i|s_i,x_i)\ln Q_i^{\sigma^n}(y_i|s_i,x_i)\sigma_i^n(dx_i,ds_i)\mu_{\hat{i}}^n\left(dx_{\hat{i}}\right)\\
&= -\int_{\mathbb{X}_{\hat{i}}\times\mathbb{S}_i\times\mathbb{X}_i} \sum_{y_i\in\mathbb{Y}_i} Q_i^{\sigma}(y_i|s_i,x_i)\ln Q_i^{\sigma}(y_i|s_i,x_i)\sigma_i(dx_i,ds_i)\mu_{\hat{i}}\left(dx_{\hat{i}}\right).
\end{aligned}
$$

Now, consider the term in (C.4). Similar to before, we will argue that

$$
Q_i^{\sigma}(y_i|s_i,x_i)\ln Q_i^{x_{\hat{i}}}(y_i|s_i,x_i) \equiv \rho_{\hat{i}}(x_{\hat{i}}, s_i, x_i)
$$

is bounded as well as continuous on $\mathbb{X}_{\hat{i}}\times\mathbb{X}_i\times\mathbb{S}_i$. We start by proving continuity of $\rho_{\hat{i}}(\cdot)$. Consider any $(x_{\hat{i}}, s_i, x_i)$ with $Q_i^{x_{\hat{i}}}(y_i|s_i,x_i) > 0$. Since $Q_i^{x_{\hat{i}}}(y_i|s_i,x_i)$ is continuous and strictly larger than zero also $lnQ_i^{x_{\hat{i}}}(y_i|s_i,x_i)$ is continuous at $(x_{\hat{i}}, s_i, x_i)$. As $Q_i^{\sigma}(y_i|s_i,x_i)$ is continuous, $\rho_{\hat{i}}(\cdot)$ is continuous at $(x_{\hat{i}}, s_i, x_i)$.

Next, consider any $(x_{\hat{i}}, s_i, x_i)$ with $Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) = 0$. As a preliminary step, we show that also $\rho_{\hat{i}}(y_i|s_i, x_i) = 0$ holds. Take any sequence $\left(x_{\hat{i}}^n, s_i^n, x_i^n\right)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}(x_{\hat{i}}^n, s_i^n, x_i^n) = (x_{\hat{i}}, s_i, x_i)$. Since for all $n'' > n'$ we have $0 \leq Q_i^{\sigma}\left(y_i|s_i^{n''}, x_i^{n''}\right)^m \leq Q_i^{x_{\hat{i}}^{n''}}\left(y_i|s_i^{n''}, x_i^{n''}\right)$ such that $0 \leq \lim_{n\to\infty} Q_i^{\sigma}\left(y_i|s_i^n, x_i^n\right)^m \leq \lim_{n\to\infty} Q_i^{x_{\hat{i}}^n}\left(y_i|s_i^n, x_i^n\right)$, and therefore $0 \leq \lim_{n\to\infty} Q_i^{\sigma}(y_i|s_i^n, x_i^n)^m \leq 0$. As in Esponda and Pouzo (2016), we define $ln(0) = -\infty$ and $0 \cdot \infty = 0$, which implies $\rho_{\hat{i}}(x_{\hat{i}}, s_i, x_i) = 0$.

Next, we show that $\rho_{\hat{i}}(x_{\hat{i}}, s_i, x_i)$ is bounded. Obviously, $\rho_{\hat{i}}(x_{\hat{i}}, s_i, x_i)$ is bounded above by zero. Suppose first that $Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) \geq \epsilon$. Note that $Q_i^{\sigma}(y_i|s_i, x_i) \leq 1$ and $\ln Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) \leq 0$. Hence,

$$0 \geq \rho_{\hat{i}}\left(x_{\hat{i}}, s_i, x_i\right) = Q_i^{\sigma}(y_i|s_i, x_i) \cdot \ln Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) \geq \ln Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) \geq 1 - \frac{1}{\epsilon}.$$

Consider $Q_i^{x_{\hat{i}}}(y_i|s_i, x_i) < \epsilon$, now. Since $\rho_{\hat{i}}(x_{\hat{i}}, s_i, x_i)$ is continuous at any $(x_{\hat{i}}, s_i, x_i)$ we can write

$$0 \geq \rho_{\hat{i}}\left(x_{\hat{i}}, s_i, x_i\right) = \lim_{n\to\infty} \rho_{\hat{i}}(x_{\hat{i}}^n, s_i^n, x_i^n) \geq \lim_{n\to\infty} m \cdot Q_i^{\sigma}\left(y_i|s_i^n, x_i^n\right) \cdot \ln Q_i^{\sigma}\left(y_i|s_i^n, x_i^n\right)$$

$$\Leftrightarrow 0 \geq \rho_{\hat{i}}\left(x_{\hat{i}}, s_i, x_i\right) \geq m \cdot Q_i^{\sigma}(y_i|s_i, x_i) \cdot \ln Q_i^{\sigma}(y_i|s_i, x_i) \geq m\left(Q_i^{\sigma}(y_i|s_i, x_i) - 1\right).$$

Since $m < \infty$, $\rho_{\hat{i}}(\cdot)$ is bounded. By Theorem 1.1 in Feinberg et al. (2014), we can write

$$\lim_{n\to\infty} \int_{\mathbb{X}_{\hat{i}}\times\mathbb{S}_i\times\mathbb{X}_i} - \sum_{y_i\in\mathbb{Y}_i} Q_i^{\sigma^n}(y_i|s_i, x_i) \ln Q_i^{x_{\hat{i}}}(y_i|s_i, x_i)\sigma_i^n(dx_i, ds_i)\mu_{\hat{i}}^n\left(dx_{\hat{i}}\right)$$

$$= \int_{\mathbb{X}_{\hat{i}}\times\mathbb{S}_i\times\mathbb{X}_i} - \sum_{y_i\in\mathbb{Y}_i} Q_i^{\sigma}(y_i|s_i, x_i) \ln Q_i^{x_{\hat{i}}}(y_i|s_i, x_i)\sigma_i(dx_i, ds_i)\mu_{\hat{i}}\left(dx_{\hat{i}}\right),$$

which also makes use of the fact that $\sigma^n \to \sigma$ implies $Q_i^{\sigma^n}(y_i|s_i, x_i) \to Q_i^{\sigma}(y_i|s_i, x_i)$. Putting the limits of (C.3) and (C.4) together yields

$$\lim_{n\to\infty} \int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma^n, x_{\hat{i}})\mu_{\hat{i}}^n(dx_{\hat{i}}) = \int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu_{\hat{i}}(dx_{\hat{i}}),$$

that is, $\int_{\mathbb{X}_{\hat{i}}} \pi_{\hat{i}}(\sigma, x_{\hat{i}})\mu_{\hat{i}}(dx_{\hat{i}})$ is continuous in the weak topology. □