

Donker van Heel, Simon; Shephard, Neil G.

**Working Paper**

## Exponentially weighted estimands and the exponential family: Filtering, prediction and smoothing

Tinbergen Institute Discussion Paper, No. TI 2025-074/III

**Provided in Cooperation with:**

Tinbergen Institute, Amsterdam and Rotterdam

*Suggested Citation:* Donker van Heel, Simon; Shephard, Neil G. (2026) : Exponentially weighted estimands and the exponential family: Filtering, prediction and smoothing, Tinbergen Institute Discussion Paper, No. TI 2025-074/III, Tinbergen Institute, Amsterdam and Rotterdam

This Version is available at:

<https://hdl.handle.net/10419/339077>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

TI 2025-074/III  
Tinbergen Institute Discussion Paper

# Exponentially Weighted Estimands and the Exponential Family: Filtering, Prediction and Smoothing

**Revision: January 2026**

*Simon Donker van Heel<sup>1</sup>*  
*Neil Shephard<sup>2</sup>*

<sup>1</sup> Erasmus University Rotterdam, Tinbergen Institute

<sup>2</sup> Harvard University

Tinbergen Institute is the graduate school and research institute in economics of Erasmus University Rotterdam, the University of Amsterdam and Vrije Universiteit Amsterdam.

Contact: [discussionpapers@tinbergen.nl](mailto:discussionpapers@tinbergen.nl)

More TI discussion papers can be downloaded at <https://www.tinbergen.nl>

Tinbergen Institute has two locations:

Tinbergen Institute Amsterdam  
Gustav Mahlerplein 117  
1082 MS Amsterdam  
The Netherlands  
Tel.: +31(0)20 598 4580

Tinbergen Institute Rotterdam  
Burg. Oudlaan 50  
3062 PA Rotterdam  
The Netherlands  
Tel.: +31(0)10 408 8900

# Exponentially Weighted Estimands and the Exponential Family: Filtering, Prediction and Smoothing

Simon Donker van Heel<sup>b,c,\*</sup> and Neil Shephard<sup>a</sup>

January 9, 2026

<sup>a</sup>*Dept. of Economics and Dept. Statistics, Harvard University, Cambridge, MA 02138, USA*

<sup>b</sup>*Econometric Institute, Erasmus University Rotterdam, Rotterdam, The Netherlands*

<sup>c</sup>*Tinbergen Institute, The Netherlands*

## Abstract

We propose using a discounted version of a convex combination of the log-likelihood with the corresponding expected log-likelihood such that when they are maximized they yield a filter, predictor and smoother for time series. This paper then focuses on working out the implications of this in the case of the canonical exponential family. The results are simple exact filters, predictors and smoothers with linear recursions. A theory for these models is developed and the models are illustrated on simulated and real data.

**Keywords:** Exponential family; EWMA; Filtering; Likelihood; Time Series.

---

\*Corresponding author.

E-mail addresses: [donkervanheel@ese.eur.nl](mailto:donkervanheel@ese.eur.nl) (Simon Donker van Heel), [shephard@fas.harvard.edu](mailto:shephard@fas.harvard.edu) (Neil Shephard).

# 1 Introduction

We provide a simple way to carry out filtering, predicting and smoothing. It is based on a discounted version of a convex combination of a log-likelihood and the expected log-likelihood function. The discounting is carried out using an exponential function.

We focus on log-likelihoods based around the canonical exponential family. It has a similar structure to generalized linear regression models, but now for time series. Properties of the resulting procedures are established and illustrated by simulation and empirical work.

Section 2 develops the new filter, predictor and smoother. Section 3 looks at the canonical exponential family case. These are found by a convex numerical optimization.

Section 4 looks at a flexible special case of the canonical exponential family which results in analytic expressions for the filters, predictors and smoothers. They do not need any form of iteration or approximation. This special case covers a vast number of familiar models, e.g. based on distributions such as, for example, the Gaussian, multinomial, Poisson, Pareto, exponential, von Mises and Dirichlet.

Section 5 develops a quasi-likelihood based estimation strategy for the hyperparameters of this structure. This method is extended to allow for a two-step estimator, where an easy to use moment estimator reduces the dimension of the required numerical maximization.

Section 6 provides an empirical illustration based on a seven category Dirichlet model. The section uses the empirical results to provide simulation based evidence for the performance of the hyperparameter estimators. Section 7 details some conclusions.

## 2 Principle of weighting

Think of time series data  $y_{1:T} = \{y_1, \dots, y_T\}$  and a time series model for the corresponding random variables  $Y_{1:T} = \{Y_1, \dots, Y_T\}$ . Here  $T$  is the length of the time series. The properties of the time series model  $Y_{1:T}$  are determined by the abstract probability triple  $(\Omega, \mathcal{F}, P)$ .

**Definition 1** For the random variable  $Y_t$  write a frame

$$\log L_t(\theta; y_t), \quad \theta \in \Theta, y_t \in \mathcal{Y}_t, \quad t = 1, \dots, T,$$

for the specification of the filtering, smoothing and prediction estimands. The frame is called stable if it is of the form  $\log L(\theta; y_t)$ , i.e. not varying over time.

At no point do we regard this frame as true. It will index a class of filters, smoothers and predictors.

The form of the frame can vary over time, e.g. it could be based around the random variable  $Y_t \sim \text{Multinomial}(n_t, p)$ , where  $p$  is determined by  $\theta$  and  $n_t$  is regarded as non-stochastic (e.g. we have formed the probability mass function for  $Y_t$  by conditioning on  $n_t$ ) or  $Y_t \sim \text{Pois}(e^{\theta^T x_t})$  where  $x_t$  is a non-stochastic vector (or conditioned on).

At various points in the paper some assumptions are made about the form of the frame.<sup>4</sup> Here we list some of them.

**Assumption 1** The  $\mathbb{E}[\log L_t(\theta; Y_t)]$  exist for each  $t$  and  $\theta \in \Theta$ .

The expectation is under the probability measure for  $Y_t$ , not the probability model that would rationalize the frame.

**Assumption 2** The parameter space is  $\Theta = \mathbb{R}^d$ . For each  $t$  and all  $y \in \mathcal{Y}_t$ , the  $\log L_t(\theta; y)$  is concave and upper semicontinuous in  $\theta$ , taking values in  $\mathbb{R} \cup \{-\infty\}$ . The  $\mathbb{E}[\log L_t(\theta; Y_t)]$  is finite for all  $\theta \in \mathbb{R}^d$ , strictly concave in  $\theta$ , and satisfies the coercivity condition that  $\|\theta\| \rightarrow \infty$  implies that  $\mathbb{E}[\log L_t(\theta; Y_t)] \rightarrow -\infty$ .

---

<sup>4</sup>We follow the statistical convention of taking the log-likelihood as ignoring terms which only involve data not also hyperparameters. One way of implementing this, for a model with a probability density function, for example, is to take  $\log L_t(\theta; y_t) := \log f_t(y_t; \theta) - \log f_t(y_t; \theta_0)$  where  $\theta_0 \in \Theta$  is some arbitrary fixed value as  $\theta \in \Theta$  varies. As we need the expected log-likelihood to exist, removing these data terms lessens the requirements we need to prove the expected value exists.

The paper will be based around an exponential version of a weighting principle,<sup>2</sup>

The principle of weighting delivers three sequences of estimands through time: a filter, a predictor and a smoother. Notice that the estimands are random variables.

**Definition 2 (Exponentially weighted estimands)** *Let Assumptions 1-2 hold,  $\lambda \in [0, 1]$  be a discount parameter and  $\alpha \in [0, 1]$  an anchoring parameter. The exponentially weighted filter at time  $t$ , using data up to time  $t$  and the sequence of frames  $\{\log L_t(\theta; y_t)\}_{t=1}^T$ , is*

$$\tilde{\theta}_t = \arg \max_{\theta \in \Theta} Q_{\lambda,t}(\theta),$$

where

$$Q_{\lambda,t}(\theta) = \sum_{j=1}^t \lambda^{t-j} \{(1 - \alpha)\mathbb{E}[\log L_j(\theta; Y_j)] + \alpha \log L_j(\theta; Y_j)\}.$$

The time  $t$  exponentially weighted predictor based on data up to time  $(t - s)$ , for  $s \geq 1$ , is

$$\tilde{\theta}_{t|t-s} = \arg \max_{\theta \in \Theta} Q_{\lambda,t|t-s}(\theta),$$

where

$$Q_{\lambda,t|t-s}(\theta) = (1 - \alpha) \sum_{j=1}^t \lambda^{t-j} \mathbb{E}[\log L_j(\theta; Y_j)] + \alpha \sum_{j=1}^{t-s} \lambda^{t-j} \log L_j(\theta; Y_j).$$

The exponentially weighted smoother at time  $t$ , using data up to time  $T \geq t$ , is

$$\tilde{\theta}_{t|T} = \arg \max_{\theta \in \Theta} Q_{\lambda,t|T}(\theta),$$

where

$$Q_{\lambda,t|T}(\theta) = \sum_{j=1}^T \lambda^{|t-j|} \{(1 - \alpha)\mathbb{E}[\log L_j(\theta; Y_j)] + \alpha \log L_j(\theta; Y_j)\}.$$

---

<sup>2</sup>The extension to allow a prior density  $\pi(\theta)$  for  $\theta \in \Theta$ , yielding an exponentially weighted MAP (i.e. mode of a type of posterior), means just adding  $\log \pi(\theta)$  to our criterion function and then maximizing that with respect to  $\theta$  for each  $t$ . This is particularly important if  $\theta$  is high dimensional.

Note that Definition 2, Assumptions 1–2 and  $\alpha \in [0, 1]$  imply that  $Q_{\lambda,t}(\theta)$ ,  $Q_{\lambda,t|t-s}(\theta)$  and  $Q_{\lambda,t|T}$  admit unique maximizers  $\tilde{\theta}_t$ ,  $\tilde{\theta}_{t|t-s}$  and  $\tilde{\theta}_{t|T}$ , respectively.

As the likelihood function is invariant to reparameterization, like the maximum likelihood estimator (MLE), the exponentially weighted filter, predictor and smoother are invariant under one-to-one reparameterization, i.e. if  $\gamma = \gamma(\theta)$ , then  $\tilde{\gamma}_t = \gamma(\tilde{\theta}_t)$ . This will turn out to be a very powerful property in practice.

Intuitively,  $\lambda \in [0, 1]$  controls the rate at which older information is downweighted, while  $\alpha \in [0, 1]$  controls the weight placed on the anchor  $\mathbb{E}[\log L_j(\theta; Y_j)]$  terms.

**Remark 1** (a) When  $\alpha = 1$  the resulting exponentially weighted maximum likelihood estimator appears often in various literatures often under the label of local modeling or local regression, e.g. Chapter 8 of Fan and Yao (2005) and Fan et al. (1995). There the localization is typically carried out using a kernel weighting function — our use of the exponential function is a special case of this. The Gaussian likelihood for an unknown mean yields the celebrated EWMA of Brown (1956).

(b) Weighting likelihood contributions has appeared in some applied contexts, e.g. Dixon and Coles (1997) use a simplified form of exponentially weighted likelihoods over time to model football match outcomes, while Hu and Zidek (2002) and Blasques et al. (2016) weight over the cross-section.

(c) Luxenberg and Boyd (2024) look at exponentially weighted moving losses, replacing the likelihood with a loss function. Their work has some overlap with our approach, but focuses on penalization (which we do not), and does not consider the  $\mathbb{E}[\log L_j(\theta, Y_j)]$  terms.

(d) When  $\alpha < 1$  we do not know of a literature which uses this approach, including the works cited in parts (a)–(c). Of course, broadly, the expected log-likelihood can be thought as a form of shrinkage. Hence it relates to the vast literatures on empirical Bayes, ridge and Lasso regression, etc (e.g. Efron and Morris (1977), Efron (2012), Hoerl and Kennard (1970), Tibshirani (1996)).

(e) There is a large general Bayesian literature which raises the likelihood to a power less than one to obtain some “robustness” properties for the posterior. Some of the corresponding literature is discussed in [Holmes and Walker \(2017\)](#).

(f) When  $\alpha$  is close to one, the estimands are mostly determined by past observations; when  $\alpha$  is close to zero, the estimands are predominantly determined by the expected log-likelihoods  $\mathbb{E}[\log L_j(\theta, Y_j)]$ . The discount parameter  $\lambda \in [0, 1]$  determines how quickly past observations are downweighted, with larger values giving more weight to distant observations.

(g) In applications with naturally constrained parameters (e.g.  $\sigma > 0$ ), we work with a smooth one-to-one reparametrization such as a log or logit transform so that the resulting parameter vector  $\theta$  lives in  $\mathbb{R}^d$ . Since the exponentially weighted estimands are reparametrization invariant, this involves no loss of generality and allows us to take  $\Theta = \mathbb{R}^d$ . Together with the upper semicontinuity of the log-frame and the coercivity and strict concavity of the expected log-frame, this guarantees that all  $Q_{\lambda,t}$ ,  $Q_{\lambda,t|t-s}$  and  $Q_{\lambda,t|T}$  are coercive, concave and upper semicontinuous, and hence that their maximizers exist and are unique.

### 3 Canonical exponential family case

The rest of this paper will focus entirely on the case where the frame is based on a member of the canonical exponential family. This family includes most of the important probability models used in applied statistics. In our context, crucially, it allows either an analytic solution or an easy to compute solution for each  $t$ .

#### 3.1 Canonical exponential family

Start by recalling the definition of the canonical exponential family and three of its properties.

**Definition 3 (Canonical exponential family)** Assume a random variable  $Y \sim \text{CEF}(\theta, h, \psi)$  — that is its probability density function (or probability mass function for a discrete random

variable) can be written as a member of the minimal canonical exponential family:

$$f(y; \theta) = \exp\{\theta^T h(y) - \psi(\theta)\} / b(y), \quad \theta \in \Theta = \{\theta : \psi(\theta) < \infty\}, \quad y \in \mathcal{Y},$$

where  $\psi(\theta)$  is infinitely continuously differentiable and strictly convex with respect to  $\theta$  while  $b(y) > 0$  for all  $y \in \mathcal{Y}$ , (e.g. [McCullagh and Nelder \(1989\)](#) and [Efron \(2022\)](#)).

The cumulant function of the CEF is  $\log \mathbb{E}[e^{a^T h(Y)}; \theta] = \psi(\theta + a) - \psi(\theta)$ , so

$$\psi'(\theta) = \frac{\partial \psi(\theta)}{\partial \theta} = \mathbb{E}[h(Y); \theta] = \mu,$$

the finite expected value of  $h(Y)$  under the  $\text{CEF}(\theta, h, \psi)$  model and

$$\psi''(\theta) = \frac{\partial^2 \psi(\theta)}{\partial \theta \partial \theta^T} = V(h(Y); \theta),$$

the corresponding finite variance. The minimal CEF requires  $\psi(\theta)$  to be strictly convex so  $V(h(Y); \theta)$  is positive definite under the model. It is helpful to go from  $\mu = \mathbb{E}[h(Y)]$  to  $\theta$  through the inverse function  $\psi'^{-1}(\mu) = \theta$ . This inverse has a unique solution (as  $\psi$  is strictly convex), but in some cases it has to be solved numerically — a classic task in statistics as this is isomorphic to computing the MLE of  $\theta$  in the CEF. Famously, it can be carried out reliably and rapidly using a Newton-Raphson algorithm, which converges quadratically close to the solution. We state this formally as [Algorithm 1](#) for reference later.

**Algorithm 1 (Solving  $\theta = \psi'^{-1}(\mu)$ )** Starting from some point  $\theta^{(0)}$ , then iterate

$$\theta^{(i)} = \theta^{(i-1)} - \{\psi''(\theta^{(i-1)})\}^{-1} \{\psi'(\theta^{(i-1)}) - \mu\}, \quad i = 1, 2, \dots$$

until convergence, which is the solution  $\theta$ . Recall here  $\mu = \mathbb{E}[h(Y)]$ .

There is a substantial literature on time series models based around the exponential

family. Examples include Zeger and Qaqish (1988), Li (1994), Benjamin et al. (2003) and the review by Davis et al. (2021). The dynamic conditional score (DCS) filter and the generalized autoregressive score (GAS) filter of Harvey (2013) and Creal et al. (2013) can be applied in this context to deliver recursive filters. Our predictive version is closest to the generalized ARMA work of Benjamin et al. (2003), which is built around the exponential family and generalized linear models.

### 3.2 Filtering, prediction and smoothing

In what follows throughout we take the frame  $\log L_t(\theta; y) = \log f_t(y; \theta)$  as coming from  $\text{CEF}(\theta, h_t, \psi_t)$ . Then, for each  $t$ , (ignoring the implied  $b_t(y)$  term as it has no impact on  $\theta$  and dropping it means we do not need to make an assumption that  $\mathbb{E}[\log b_t(Y_t)]$  exists)

$$(1 - \alpha)\mathbb{E}[\log L_t(\theta; Y_t)] + \alpha \log L_t(\theta; Y_t) = \theta^\top \{(1 - \alpha)\mathbb{E}[h_t(Y_t)] + \alpha h_t(Y_t)\} - \psi_t(\theta).$$

In the case where the data is assumed to be strictly stationary, the frame is stable and  $\mathbb{E}[h(Y_t)]$  exists, then the right hand side of this expression simplifies to

$$\theta^\top \{(1 - \alpha)\mathbb{E}[h(Y_1)] + \alpha h(Y_t)\} - \psi(\theta).$$

To compactly write the exponentially weighted filter it is helpful to denote the exponentially weighted moving sums

$$x_{\lambda,t} = \sum_{j=1}^t \lambda^{t-j} \mathbb{E}[h_j(Y_j)], \quad h_{\lambda,t} = \sum_{j=1}^t \lambda^{t-j} h_j(Y_j),$$

as well as the double sided exponentially weighted moving sums

$$x_{\lambda,t|T} = \sum_{j=1}^T \lambda^{|t-j|} \mathbb{E}[h_j(Y_j)], \quad h_{\lambda,t|T} = \sum_{j=1}^T \lambda^{|t-j|} h_j(Y_j).$$

Each of these terms can be computed recursively, implying each entire series, e.g.  $h_{\lambda,1|T}, \dots, h_{\lambda,T|T}$ , can be computed in  $O(\sum_{t=1}^T \dim(h_t(y_t)))$  computations.

**Theorem 1** *Assume a minimal CEF frame  $\text{CEF}(\theta, h_t, \psi_t)$  for each  $t$ . Then the exponentially weighted filter  $(\tilde{\theta}_t)$ , predictor  $(\tilde{\theta}_{t|t-s}$  with  $s \geq 1$ ) and smoother  $(\tilde{\theta}_{t|T})$  solve:*

$$x_{\lambda,t} + \alpha(h_{\lambda,t} - x_{\lambda,t}) = \sum_{j=1}^t \lambda^{t-j} \psi'_j(\tilde{\theta}_t),$$

$$(1 - \alpha)x_{\lambda,t} + \alpha\lambda^s h_{\lambda,t-s} = (1 - \alpha) \sum_{j=1}^t \lambda^{t-j} \psi'_j(\tilde{\theta}_{t|t-s}) + \alpha \sum_{j=1}^{t-s} \lambda^{t-j} \psi'_j(\tilde{\theta}_{t|t-s}),$$

$$x_{\lambda,t|T} + \alpha(h_{\lambda,t|T} - x_{\lambda,t|T}) = \sum_{j=1}^T \lambda^{t-j} \psi'_j(\tilde{\theta}_{t|T}),$$

respectively. Then write the filtered, predicted and smoothed mean and variance as

$$\tilde{\mu}_t := \psi'_t(\tilde{\theta}_t), \quad \tilde{\mu}_{t|t-s} := \psi'_t(\tilde{\theta}_{t|t-s}), \quad \tilde{\mu}_{t|T} := \psi'_t(\tilde{\theta}_{t|T}),$$

and

$$\tilde{\Sigma}_t := \psi''_t(\tilde{\theta}_t), \quad \tilde{\Sigma}_{t|t-s} := \psi''_t(\tilde{\theta}_{t|t-s}), \quad \tilde{\Sigma}_{t|T} := \psi''_t(\tilde{\theta}_{t|T}).$$

**Proof.** Focus on the filtering case. Now

$$Q_{\lambda,t}(\theta) = \theta^T \{(1 - \alpha)x_{\lambda,t} + \alpha h_{\lambda,t}\} - \sum_{j=1}^t \lambda^{t-j} \psi_j(\theta).$$

Differentiate  $Q_{\lambda,t}(\theta)$  with respect to  $\theta$  and solve. This yields the stated result. The same argument applied to prediction and smoothing. The remaining results follow by invariance.

■

All the left hand side terms (e.g.  $x_{\lambda,t}$  and  $h_{\lambda,t}$ ) can be computed recursively. The right hand side, not so much, as they depend upon the value of  $\theta$  that the  $\psi_j$  function is evaluated

at. Typically  $\tilde{\theta}_t$ ,  $\tilde{\theta}_{t|t-s}$  and  $\tilde{\theta}_{t|T}$  have to be found by numerical root solving. Due to the strict convexity of  $\psi$  this is numerically straightforward, for each individual value of  $t$ , using Newton-Raphson, e.g. Algorithm [2](#) computes  $\tilde{\theta}_t$ , the exponentially weighted filter.

**Algorithm 2 (Computing  $\tilde{\theta}_t$ )** Starting from some point  $\tilde{\theta}_t^{(0)}$ , then iterate

$$\tilde{\theta}_t^{(i)} = \tilde{\theta}_t^{(i-1)} - \left\{ \sum_{j=1}^t \lambda^{t-j} \psi_j''(\tilde{\theta}_t^{(i-1)}) \right\}^{-1} \left\{ \sum_{j=1}^t \lambda^{t-j} \psi_j'(\tilde{\theta}_t^{(i-1)}) - \{x_{\lambda,t} + \alpha(h_{\lambda,t} - x_{\lambda,t})\} \right\}, \quad i = 1, 2, \dots$$

until convergence, which is the solution  $\tilde{\theta}_t$ .

A downside is that the numerical procedure has to be run separately for each  $t$ , so computing, for example, the time series  $\tilde{\theta}_1, \dots, \tilde{\theta}_T$  costs  $O(\{\sum_{t=1}^T \dim(h_t(y_t))^3\}^2)$  calculations.

### 3.3 A CEF( $\theta, h_t, \psi_t$ ) data generating process

We will use the following structure in our simulations later.

**Assumption 3 (Model based data generating process)** (a) Use a frame based on CEF( $\theta, h_t, \psi_t$ ), computing the sequence  $\tilde{\theta}_{t|t-1}$  for  $t = 1, 2, \dots, T$ , recursively through Definition [2](#). (b) Generate the data as

$$Y_t | Y_{1:t-1} \sim \text{CEF}(\tilde{\theta}_{t|t-1}, h_t, \psi_t), \quad t = 1, 2, \dots, T.$$

The stable frame version of this is where we set the frame to have  $h_t(y) = h(y)$  and  $\psi_t = \psi$  for  $t = 1, \dots, T$ .

The use of the frame to derive a filter, which is then used as an input into a data generating process echos the DCS/GAS models of [Harvey \(2013\)](#) and [Creal et al. \(2013\)](#).

Some examples of these types of simulations will be given shortly.

## 4 Special case with analytic solution: $\psi_t = n_t \psi$

The following special case has an analytic solution. We focus on it in the rest of this paper.

**Example 1** Use a minimal CEF( $\theta, h_t, n_t \psi$ ) frame for each  $t$  where  $n_t$  is a non-stochastic scalar. Then the exponentially weighted filter ( $\tilde{\theta}_t$ ), predictor ( $\tilde{\theta}_{t|t-s}$  with  $s \geq 1$ ) and smoother ( $\tilde{\theta}_{t|T}$ ) are:

$$\tilde{\theta}_t = \psi'^{-1}(\bar{m}_t), \quad \tilde{\theta}_{t|t-s} = \psi'^{-1}(\bar{m}_{t|t-s}), \quad \tilde{\theta}_{t|T} = \psi'^{-1}(\bar{m}_{t|T}),$$

respectively, where

$$\bar{m}_t = \frac{m_{\lambda,t}}{n_{\lambda,t}}, \quad \bar{m}_{t|t-s} = \frac{m_{\lambda,t|t-s}}{n_{\lambda,t|t-s}}, \quad \bar{m}_{t|T} = \frac{m_{\lambda,t|T}}{n_{\lambda,t|T}},$$

with

$$\begin{aligned} m_{\lambda,t} &= (1 - \alpha)x_{\lambda,t} + \alpha h_{\lambda,t}, & n_{\lambda,t} &= \sum_{j=1}^t \lambda^{t-j} n_j, \\ m_{\lambda,t|t-s} &= (1 - \alpha)x_{\lambda,t} + \alpha \lambda^s h_{\lambda,t-s}, & n_{\lambda,t|t-s} &= (1 - \alpha)n_{\lambda,t} + \alpha \lambda^s n_{\lambda,t-s}, \\ m_{\lambda,t|T} &= (1 - \alpha)x_{\lambda,t|T} + \alpha h_{\lambda,t|T}, & n_{\lambda,t|T} &= \sum_{j=1}^T \lambda^{|t-j|} n_j. \end{aligned}$$

Then

$$\tilde{\mu}_t = n_t \bar{m}_t, \quad \tilde{\mu}_{t|t-s} = n_t \bar{m}_{t|t-s}, \quad \tilde{\mu}_{t|T} = n_t \bar{m}_{t|T},$$

and

$$\tilde{\Sigma}_t = n_t \psi''(\tilde{\theta}_t), \quad \tilde{\Sigma}_{t|t-s} = n_t \psi''(\tilde{\theta}_{t|t-s}), \quad \tilde{\Sigma}_{t|T} = n_t \psi''(\tilde{\theta}_{t|T}).$$

This Example covers many interesting models.

**Remark 2** (a) In the special case of a stable frame CEF and strictly stationary process then  $x_{\lambda,t} = n_{\lambda,t} \mathbb{E}[h(Y_1)]$  and  $x_{\lambda,t|T} = n_{\lambda,t|T} \mathbb{E}[h(Y_1)]$ .

(b) For filtering, the  $\tilde{m}_{\lambda,t}$  is a convex combination of EWMA's of  $\mathbb{E}[h(Y_1)], \dots, \mathbb{E}[h(Y_t)]$  and of  $h(Y_1), \dots, h(Y_t)$ . For smoothing, the  $\tilde{m}_{\lambda,t|T}$  is a convex combination of the double sided EWMA's of  $\mathbb{E}[h(Y_1)], \dots, \mathbb{E}[h(Y_T)]$  and of  $h(Y_1), \dots, h(Y_T)$  for time  $t$ .

(c) If  $\lambda \in (0, 1)$ , and  $n_t = 1$  for all  $t$ , then  $n_{\lambda,t} \rightarrow 1/(1 - \lambda)$  as  $t \rightarrow \infty$ . If  $\lambda = 1$ , then  $n_{\lambda,t} = t$ , while if  $\lambda = 0$ , then  $n_{\lambda,t} = 1$ . Likewise  $n_{\lambda,t|T} \rightarrow (1 + \lambda)/(1 - \lambda)$  as  $t \rightarrow \infty$ . If  $\lambda = 1$ , then  $n_{\lambda,t|T} = T$ , while if  $\lambda = 0$ , then  $n_{\lambda,t|T} = 1$ .

(d) Famously the EWMA can be computed recursively. In our case it follows

$$n_{\lambda,t} = n_t + \lambda n_{\lambda,t-1}, \quad x_{\lambda,t} = \mathbb{E}[h(Y_t)] + \lambda x_{\lambda,t-1}, \quad h_{\lambda,t} = h(Y_t) + \lambda h_{\lambda,t-1},$$

initialized at  $n_{\lambda,0}, x_{\lambda,0}, h_{\lambda,0} := 0$ . Likewise, the double sided EWMA can be computed recursively

$$\begin{aligned} h_{\lambda,t|T} &= h_{\lambda,t} + \lambda(h_{\lambda,t+1|T} - \lambda h_{\lambda,t}), & h_{\lambda,T|T} &= h_{\lambda,T} \\ n_{\lambda,t|T} &= n_{\lambda,t} + \lambda(n_{\lambda,t+1|T} - \lambda n_{\lambda,t}), & n_{\lambda,T|T} &= n_{\lambda,T}, \end{aligned}$$

going backwards, using the output from the forward pass of the EWMA.

(e) The  $m_{\lambda,t}$  and  $m_{\lambda,t|t-1}$  can also be written recursively:

$$\begin{aligned} m_{\lambda,t} &= \mathbb{E}[h_t(Y_t)] + \alpha\{h_t(Y_t) - \mathbb{E}[h_t(Y_t)]\} + \lambda m_{\lambda,t-1}, \\ m_{\lambda,t|t-1} &= (1 - \alpha)\mathbb{E}[h_t(Y_t)] + \alpha\lambda h_{t-1}(Y_{t-1}) + \lambda m_{\lambda,t-1|t-2}. \end{aligned}$$

In the steady state stable frame case with  $n_t = 1$ , the  $\bar{m}_t = \tilde{\mu}_t$  and  $\bar{m}_{t|t-1} = \tilde{\mu}_{t|t-1}$  become

$$\begin{aligned} \tilde{\mu}_t &= (1 - \alpha)(1 - \lambda)\mathbb{E}[h(Y_1)] + \alpha(1 - \lambda)h(Y_t) + \lambda\tilde{\mu}_{t-1}, \\ \tilde{\mu}_{t|t-1} &= \frac{(1 - \alpha)(1 - \lambda)}{1 - \alpha(1 - \lambda)}\mathbb{E}[h(Y_1)] + \frac{\alpha\lambda(1 - \lambda)}{1 - \alpha(1 - \lambda)}h(Y_{t-1}) + \lambda\tilde{\mu}_{t-1|t-2}. \end{aligned}$$

The  $\tilde{\mu}_{t|t-1}$  relates to the generalized ARMA model of [Benjamin et al. \(2003\)](#) who model a link

function of the data as being linear in past conditional means and link functions of the data.

Define  $U_t := h(Y_t) - \tilde{\mu}_{t|t-1}$ , then

$$\begin{aligned}
h(Y_t) &= \tilde{\mu}_{t|t-1} + U_t \\
&= \frac{(1-\alpha)(1-\lambda)}{1-\alpha(1-\lambda)} \mathbb{E}[h(Y_1)] + \frac{\alpha\lambda(1-\lambda)}{1-\alpha(1-\lambda)} h(Y_{t-1}) + \lambda\tilde{\mu}_{t-1|t-2} + U_t \\
&= \frac{(1-\alpha)(1-\lambda)}{1-\alpha(1-\lambda)} \mathbb{E}[h(Y_1)] + \frac{\alpha\lambda(1-\lambda)}{1-\alpha(1-\lambda)} h(Y_{t-1}) + \lambda\{h(Y_{t-1}) - U_{t-1}\} + U_t \\
&= \frac{(1-\alpha)(1-\lambda)}{1-\alpha(1-\lambda)} \mathbb{E}[h(Y_1)] + \frac{\lambda}{1-\alpha(1-\lambda)} h(Y_{t-1}) + U_t - \lambda U_{t-1}.
\end{aligned}$$

If the data has the property that  $\mathbb{E}[h(Y_t)|Y_{1:t-1}] = \tilde{\mu}_{t|t-1}$  and  $\mathbb{E}[|Y_t|] < \infty$ , then  $\{U_t\}_{t=1}^T$  is a martingale difference (MD) sequence with respect to the filtration generated by the data. The steady state process is a vector ARMA(1,1)-MD process, with the autoregressive root being  $\lambda/(1-\alpha(1-\lambda)) \in [\lambda, 1)$ , assuming  $\lambda, \alpha \in [0, 1)^2$ , with the  $-\lambda$  moving average root. For example, if  $\lambda = 0.93$  then the AR(1) root is roughly 0.978 and 0.996 when  $\alpha = 0.7$  and  $\alpha = 0.95$ , respectively. Hence the process can have substantial memory, although individual autocorrelations can be modest due to near root cancellation. If additionally  $\mathbb{V}(h(Y_t))$  exists and is time invariant, then in steady state  $\{U_t\}$  is weak white noise (WN) and  $\{h(Y_t)\}$  is a vector ARMA(1,1)-WN covariance stationary process.

(f) For prediction,  $m_{\lambda,t|t-s}$  places non-negative weight  $(1-\alpha)\lambda^{t-j}$  on each  $\mathbb{E}[h_j(Y_j)]$  for  $j = 1, \dots, t$ , and non-negative weight  $\alpha\lambda^{t-j}$  on each  $h_j(Y_j)$  for  $j = 1, \dots, t-s$ . These weights sum to  $n_{\lambda,t|t-s}$ , so  $\bar{m}_{\lambda,t|t-s} = m_{\lambda,t|t-s}/n_{\lambda,t|t-s}$  has non-negative weights summing to one; likewise for  $\bar{m}_{\lambda,t|t}$  and  $\bar{m}_{\lambda,t|T}$ . Hence Jensen's inequality applies, e.g. for any convex  $\varphi$ ,

$$\varphi(\bar{m}_{\lambda,t|t-s}(h_{1:t})) \leq \bar{m}_{\lambda,t|t-s}(\varphi(h_1), \dots, \varphi(h_t)),$$

where  $\bar{m}_{\lambda,t|t-s}(h_{1:t})$  denotes the dependence on  $h_1(Y_1), \dots, h_t(Y_t)$ . The right-hand side applies the linear operator  $\bar{m}_{\lambda,t|t-s}$  to the component-wise transformed vector  $(\varphi(h_1), \dots, \varphi(h_t))$  with  $\varphi$  being any convex function. A simple version of this is where  $h(y) = y$  and  $\varphi(h(y)) = y^2$ .

## 4.1 Simulating ten CEF examples: design

We illustrate the exponentially weighted predictor on ten CEF examples. Given  $\tilde{\mu}_{t|t-1}$ , each yields a unique solution for  $\tilde{\theta}_{t|t-1} = (\psi')^{-1}(\tilde{\mu}_{t|t-1})$ . In cases 1–6 and 9, this can be solved analytically; in cases 7, 8, and 10, numerically via Algorithm 1. Tables 1–2 summarize the canonical parameterizations.

Table 1: Canonical exponential family parameterizations: univariate cases.

Distribution	$h(y)$	$\psi(\theta)$	$\psi'(\theta)$	$(\psi')^{-1}(\mu)$
1. Bernoulli, $y \in \{0, 1\}$	$y$	$\log(1 + e^\theta)$	$\frac{e^\theta}{1+e^\theta}$	$\log \frac{\mu}{1-\mu}$
2. Gaussian (known $\sigma$ ), $y \in \mathbb{R}$	$y$	$\frac{\theta^2}{2\sigma^2}$	$\frac{\theta}{\sigma^2}$	$\sigma^2 \mu$
3. Poisson, $y \in \{0, 1, 2, \dots\}$	$y$	$e^\theta$	$e^\theta$	$\log(\mu)$
4. Exponential, $y \in \mathbb{R}_{>0}$	$y$	$-\log(-\theta)$	$-\frac{1}{\theta}$	$-\frac{1}{\mu}$
5. Gaussian (zero mean), $y \in \mathbb{R}$	$y^2$	$-\frac{1}{2} \log(-2\theta)$	$-\frac{1}{2\theta}$	$-\frac{1}{2\mu}$
6. Pareto, $y \geq m > 0$	$\log(y)$	$-\log(-\theta) + \theta \log m$	$-\frac{1}{\theta} + \log m$	$-\frac{1}{\mu - \log m}$

Table 2: Canonical exponential family parameterizations: multivariate cases.

Distribution	$h(y)$	$\psi(\theta)$	$\psi'(\theta)$	$(\psi')^{-1}(\mu)$
7. Beta, $y \in [0, 1]$	$\begin{Bmatrix} \log(y) \\ \log(1-y) \end{Bmatrix}$	$\log B(\theta_1, \theta_2)$	$\begin{Bmatrix} \text{Dig}(\theta_1) - \text{Dig}(\theta_1 + \theta_2) \\ \text{Dig}(\theta_2) - \text{Dig}(\theta_1 + \theta_2) \end{Bmatrix}$	Alg. 1
8. Dirichlet, $y \in \Delta_{d-1}$	$\begin{Bmatrix} \log(y_1) \\ \vdots \\ \log(y_d) \end{Bmatrix}$	$\log B(\theta_1, \dots, \theta_d)$	$\begin{Bmatrix} \text{Dig}(\theta_1) - \text{Dig}(\sum_j \theta_j) \\ \vdots \\ \text{Dig}(\theta_d) - \text{Dig}(\sum_j \theta_j) \end{Bmatrix}$	Alg. 1
9. Gaussian, $y \in \mathbb{R}$	$\begin{Bmatrix} y \\ y^2 \end{Bmatrix}$	$-\frac{\theta_1^2}{4\theta_2} - \frac{1}{2} \log(-2\theta_2)$	$\begin{Bmatrix} -\frac{\theta_1}{2\theta_2} \\ \frac{\theta_1^2}{4\theta_2^2} - \frac{1}{2\theta_2} \end{Bmatrix}$	$\begin{Bmatrix} \frac{\mu_1}{\mu_2 - \mu_1^2} \\ -\frac{1}{2(\mu_2 - \mu_1^2)} \end{Bmatrix}$
10. von Mises, $y \in [0, 2\pi]$	$\begin{Bmatrix} \sin(y) \\ \cos(y) \end{Bmatrix}$	$\log I_0(\sqrt{\theta_1^2 + \theta_2^2})$	$\frac{I_1(\sqrt{\theta_1^2 + \theta_2^2})}{I_0(\sqrt{\theta_1^2 + \theta_2^2})} \begin{Bmatrix} \frac{\theta_1}{\sqrt{\theta_1^2 + \theta_2^2}} \\ \frac{\theta_2}{\sqrt{\theta_1^2 + \theta_2^2}} \end{Bmatrix}$	Alg. 1

Note:  $B(\cdot)$  denotes the beta function,  $\text{Dig}(\cdot)$  the digamma function,  $I_0$  the modified Bessel function of the 1st kind and  $I_1$  the modified Bessel function of the 1st kind of order 1, e.g. [Abramowitz and Stegun \(1970\)](#).

For case 2, the resulting predictor  $\tilde{\mu}_{t|t-1}$  is the stationary version of the **ARMA**(1,1) process; we compare the filter with the Kalman filter in Section [4.3](#). For case 5, note that  $\mu = \mathbb{E}[Y^2] = V(Y)$ , and the resulting predictor  $\tilde{\mu}_{t|t-1}$  is the stationary version of the **GARCH**(1,1) model of [Engle \(1982\)](#) and [Bollerslev \(1986\)](#). For case 9,  $\tilde{\sigma}_{t|t-1}^2 = \tilde{\mu}_{2,t|t-1} - \tilde{\mu}_{1,t|t-1}^2 \geq 0$  by invariance and Jensen's inequality. For the cases 7, 8, and 10, given  $\mu$ , the corresponding  $\theta = (\psi')^{-1}(\mu)$ , which is unique, can be found numerically via Algorithm [1](#).

Following Assumption [3](#), we simulate each process using  $Y_t|Y_{1:t-1} \sim \text{CEF}(\tilde{\theta}_{t|t-1}, h, \psi)$  with

$$\tilde{\theta}_{t|t-1} = (\psi')^{-1}(\tilde{m}_{t|t-1}) = (\psi')^{-1}\left(\frac{m_{\lambda,t|t-1}}{n_{\lambda,t}}\right),$$

where  $m_{\lambda,t|t-1} = \mathbb{E}[h(Y_1)] + \lambda m_{\lambda,t-1}$ , recalling Example [1](#). We impose stability by centering at  $\mathbb{E}[h(Y_1)] = \mathbb{E}[Y_1] = 0.5$  (Bernoulli),  $\mathbb{E}[h(Y_1)] = \mathbb{E}[Y_1] = 0$  (Gaussian with known  $\sigma^2 = 1$ ),  $\mathbb{E}[h(Y_1)] = \mathbb{E}[Y_1] = 1$  (Poisson, Exponential),  $\mathbb{E}[h(Y_1)] = \mathbb{E}[Y_1^2] = 1$  (Gaussian with zero mean), and  $\mathbb{E}[h(Y_1)] = \mathbb{E}[\log(Y_1)] = 1/3$  (Pareto with scale  $m = 1$ , and shape  $\theta_0 = -3$ ). For the bivariate cases, we take  $\mathbb{E}[h(Y_1)] = \psi'(\theta_0)$  with initial values  $\theta_0 = (2, 5)'$  for Beta (corresponding to **Beta**(2, 5)),  $\theta_0 = (0, -1/2)'$  for Gaussian (corresponding to mean 0, variance 1), and  $\theta_0 = (0, -2)'$  for von Mises (corresponding to mean direction  $\pi$ , concentration 2).

## 4.2 Simulating ten CEF examples: results

Figures [1](#) and [2](#) show simulated time series  $Y_t$  and conditional predictors for  $t = 5, \dots, T = 2000$  with  $\lambda = 0.93$ ,  $\alpha \in \{0.70, 0.95\}$ , and  $\tilde{\theta}_{1|0} = (\psi')^{-1}(\mathbb{E}[h(Y_1)])$ . Figure [1](#) displays the conditional mean  $\mathbb{E}[\widetilde{Y}_t|Y_{1:t-1}] = \psi'(\tilde{\theta}_{t|t-1})$  for Bernoulli, Gaussian (known  $\sigma^2 = 1$ ), and Poisson. Integer-valued observations are jittered with  $\text{Unif}(-0.1, 0.1)$  noise for visualization. Figure [2](#) shows the exponential distribution with conditional mean  $\mathbb{E}[\widetilde{Y}_t|Y_{1:t-1}] = \psi'(\tilde{\theta}_{t|t-1})$ , the Gaussian (zero mean) with conditional standard deviation  $\tilde{\sigma}_{t|t-1} = \sqrt{\psi'(\tilde{\theta}_{t|t-1})}$ , and the Pareto with conditional mean  $\mathbb{E}[\widetilde{Y}_t|Y_{1:t-1}] = \frac{\tilde{\theta}_{t|t-1}}{\tilde{\theta}_{t|t-1}+1}$  when  $\tilde{\theta}_{t|t-1} < -1$  and  $\infty$  otherwise.

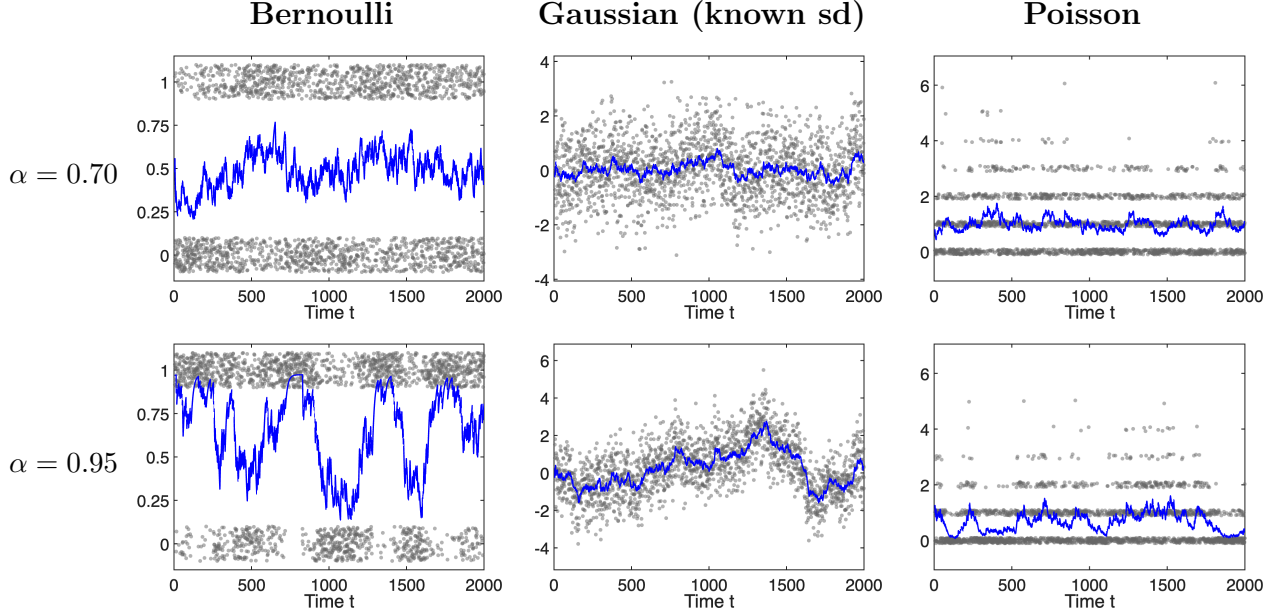


Figure 1: Simulation from  $Y_t|Y_{1:t-1} \sim \text{CEF}(\tilde{\theta}_{t|t-1}, h, \psi)$  for time  $t = 5, \dots, T = 2000$  with discount parameter  $\lambda = 0.93$ . Top: anchoring parameter  $\alpha = 0.7$ ; bottom:  $\alpha = 0.95$ . Columns: evolution of  $\mathbb{E}[Y_t|Y_{1:t-1}] = \psi'(\tilde{\theta}_{t|t-1})$  in blue with observations  $Y_t$  as gray circles.

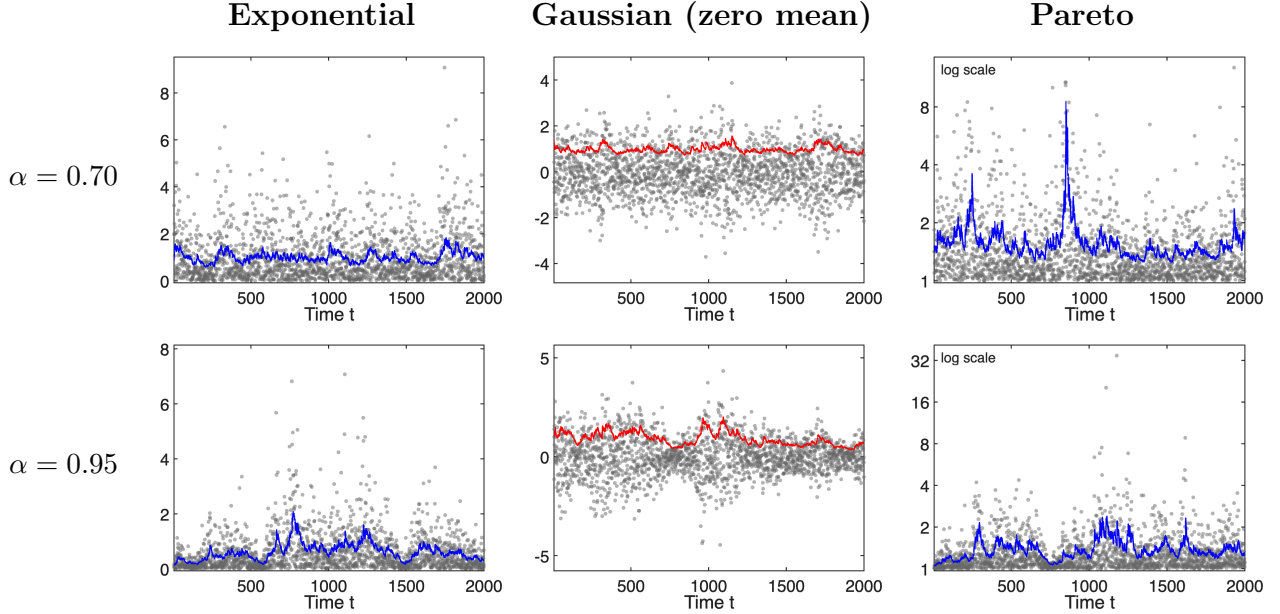


Figure 2: Simulation from  $Y_t|Y_{1:t-1} \sim \text{CEF}(\tilde{\theta}_{t|t-1}, h, \psi)$  for time  $t = 5, \dots, T = 2000$  with discount parameter  $\lambda = 0.93$ . Top: anchoring parameter  $\alpha = 0.7$ ; bottom:  $\alpha = 0.95$ . Gray circles show observations  $Y_t$ . Blue lines show the conditional expectation:  $\mathbb{E}[Y_t|Y_{1:t-1}] = \psi'(\tilde{\theta}_{t|t-1})$  for the exponential, and  $\mathbb{E}[Y_t|Y_{1:t-1}] = \frac{\tilde{\theta}_{t|t-1}}{\tilde{\theta}_{t|t-1} + 1}$  for the Pareto (y-axis on log scale) when  $\tilde{\theta}_{t|t-1} < -1$  and  $\infty$  otherwise. The red line for the Gaussian (zero mean) shows the conditional standard deviation  $\tilde{\sigma}_{t|t-1} = \sqrt{\psi'(\tilde{\theta}_{t|t-1})}$ .

Figure 3 shows simulation results for the three bivariate cases: Beta, Gaussian with time-varying mean and variance, and von Mises. Each column displays the evolution of the conditional expectation  $\mathbb{E}[\widetilde{Y}_t | \widetilde{Y}_{1:t-1}]$  with simulated observations. For the Beta distribution, the conditional mean is  $\widetilde{\theta}_{1,t|t-1} / (\widetilde{\theta}_{1,t|t-1} + \widetilde{\theta}_{2,t|t-1})$ . For Gaussian, it is  $-\widetilde{\theta}_{1,t|t-1} / (2\widetilde{\theta}_{2,t|t-1})$  and the conditional standard deviation  $\widetilde{\sigma}_{t|t-1} = \sqrt{-1 / (2\widetilde{\theta}_{2,t|t-1})}$  is shown as a red line. For von Mises, we present the mean direction  $\widetilde{\mu}_{t|t-1} = \text{atan2}(\widetilde{\theta}_{1,t|t-1}, \widetilde{\theta}_{2,t|t-1})$ , modulo  $2\pi$  to ensure  $\widetilde{\mu}_{t|t-1} \in [0, 2\pi]$ , where  $\text{atan2}(y, x)$  is the four-quadrant arctangent of  $(x, y)$ . For the anchoring parameter  $\alpha = 0.95$ , the predictors oscillate more around the centering values than for  $\alpha = 0.70$ , due to weaker mean reversion.

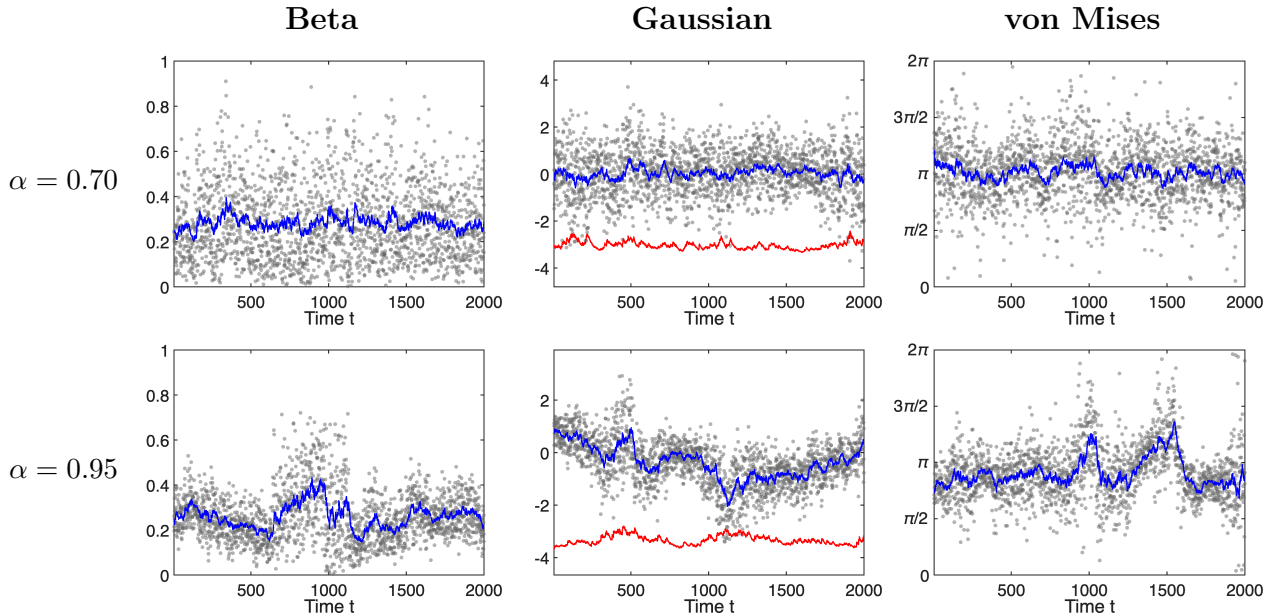


Figure 3: Simulation from  $Y_t | Y_{1:t-1} \sim \text{CEF}(\widetilde{\theta}_{t|t-1}, h, \psi)$  for time  $t = 5, \dots, T = 2000$  with discount parameter  $\lambda = 0.93$ . Top: anchoring parameter  $\alpha = 0.7$ ; bottom:  $\alpha = 0.95$ . Columns: evolution of the conditional expectation  $\mathbb{E}[\widetilde{Y}_t | \widetilde{Y}_{1:t-1}]$  (blue line) with observations  $Y_t$  (gray circles) for the Beta, Gaussian (time-varying mean & variance), & von Mises distributions. For the Gaussian distribution, the red line shows the conditional standard deviation  $\widetilde{\sigma}_{t|t-1}$ , shifted down by 4 units for visualization.

### 4.3 Gaussian case: comparison to Kalman filter

The Gaussian example with known standard deviation, case 2, links strongly to the Kalman filter for the univariate Gaussian local level model

$$Y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} N(0, \sigma_\varepsilon^2),$$

$$\mu_{t+1} = \mu_t + \eta_t, \quad \eta_t \stackrel{iid}{\sim} N(0, q\sigma_\varepsilon^2), \quad (\varepsilon_{1:T} \perp\!\!\!\perp \eta_{2:T}) | \mu_1, \quad \mu_1 \sim N(a_{1|0}, \sigma_\varepsilon^2 P_1).$$

The Kalman filter implies  $a_{t|t-1} = \mathbb{E}[\mu_t | Y_{1:t-1}] = \mathbb{E}[\mu_{t-1} | Y_{1:t-1}]$  is given by (e.g. [Harvey \(1989\)](#) and [Durbin and Koopman \(2012\)](#))

$$a_{t+1|t} = a_{t|t-1} + K_t(Y_t - a_{t|t-1}), \quad K_t = \frac{P_t}{P_t + 1}, \quad P_{t+1} = K_t + q,$$

where  $K_t$  is called the Kalman gain and  $\sigma_\varepsilon^2 P_t = \text{V}(\mu_t | Y_{1:t-1})$ . If  $q > 0$ , in steady state,  $K_t$  and  $P_t$  converge to  $K = \frac{P}{P+1}$  and  $P = K + q$ . Then  $K^2 + qK - q = 0$ , implying  $K = \frac{-q + \sqrt{q^2 + 4q}}{2}$ .

In steady state, the one-step predictor is

$$a_{t+1|t} = (1 - K)a_{t|t-1} + KY_t = K \sum_{j=0}^{\infty} (1 - K)^j Y_{t-j},$$

and EWMA with discount hyperparameter

$$\lambda = 1 - K = \frac{2 + q - \sqrt{(2 + q)^2 - 4}}{2}.$$

The discount rate monotonically declines as  $q$  increases, going from 1 to 0.

Our predictor in steady state, setting  $\mathbb{E}[Y_1] = 0$  and  $\alpha = 1$ , implies

$$\tilde{\mu}_{t+1|t} = \lambda \tilde{\mu}_{t|t-1} + (1 - \lambda)Y_t.$$

A more subtle comparison is out of steady state. For the Kalman filter, take  $P_1 = \infty$ , then  $K_1 = 1$ . In our predictor,

$$\tilde{\mu}_{t+1|t} = \left(1 - \frac{1}{n_{\lambda,t}}\right) \tilde{\mu}_{t|t-1} + \frac{1}{n_{\lambda,t}} Y_t.$$

We note that if  $\lambda > 0$ , then  $n_{\lambda,t} = \frac{1-\lambda^t}{1-\lambda}$ . Hence it is interesting to plot  $K_t \times n_{\lambda,t}$  against  $t$ , when  $P_{1|0} = \infty$ , the so called diffuse initial conditions for a Kalman filter (e.g. Chapter 1 of [Durbin and Koopman \(2012\)](#)). If this product is less than one, then our recursion gives more weight on  $Y_t$  than the Kalman filter, and subsequently, less on the past.

Then  $K_t n_{\lambda,t} = 1$  if  $t = 1$ . It is also 1 for every value of  $t$  when  $q = 0$  for then  $n_{\lambda,t} = t$ . 1 is also the limit as  $t \rightarrow \infty$  for  $q > 0$  (the steady state case). However the product is not 1 for every  $t$  and  $q > 0$ .

The easiest way to see differences is to think of  $q$  as small, then, for example,  $\lambda \approx 1 - q^{1/2}$ ,  $n_{\lambda,2} \approx 2 - q^{1/2}$ ,  $K_2 \approx \frac{1}{2} - q$ , and  $K_2 n_{\lambda,2} \approx 1 - q^{1/2}/2$ . Figure [4](#) plots the product  $K_t \times n_{\lambda,t}$  for  $q \in \{0.001, 0.1, 0.3, 1, 2, 10\}$  against  $t$ . The worst case is when  $q$  is tiny, as it takes quite a large  $t$  for the effect to disappear (as we move to the steady state).

## 5 Hyperparameter estimation

The analysis in this Section will be based on a working model

$$Y_t | Y_{1:t-1} \sim \text{CEF}(\tilde{\theta}_{t|t-1}; h_t, \psi_t), \quad t = 1, \dots, T,$$

where  $\tilde{\theta}_{t|t-1}$  comes from the exponentially weighted estimands. Later we will explicitly note the dependence of  $\tilde{\theta}_{t|t-1}$  on some hyperparameters, denoted  $\omega$ .

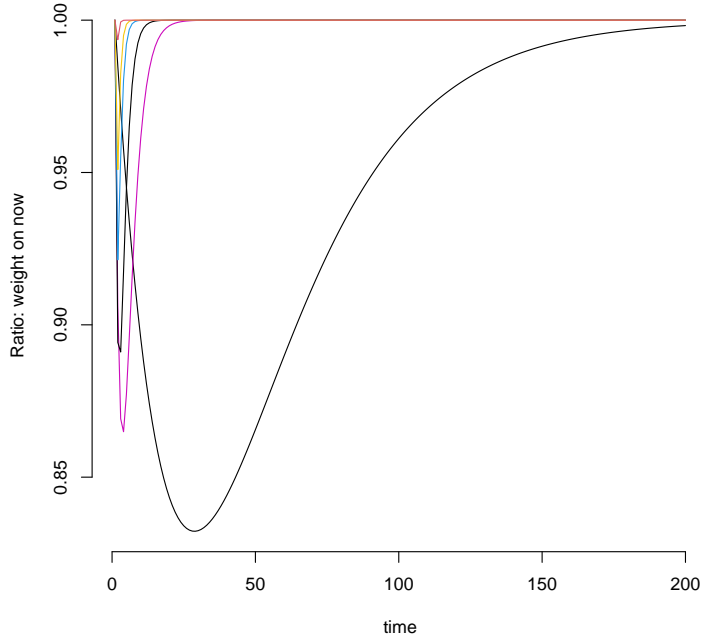


Figure 4: The product  $K_t n_{\lambda,t}$  against  $t$  for  $q \in \{0.001, 0.1, 0.3, 1, 2, 10\}$ . This is the ratio of the Kalman filter's weight on  $Y_t$  (under diffuse initialization) to the EWMA weight on  $Y_t$ , showing the impact of initial conditions. Values below 1 mean the Kalman filter places less weight on the current observation than the EWMA. All curves converge to 1 in steady state. Deviations from 1 are largest, and convergence to steady-state is slowest, for small  $q$ .

Under the working model, write

$$\tilde{\mu}_{t|t-1} := \psi'_t(\tilde{\theta}_{t|t-1}), \quad \tilde{\Sigma}_{t|t-1} := \psi''_t(\tilde{\theta}_{t|t-1}),$$

the model's conditional mean and variance, respectively.

The working model yields a working log-likelihood via the prediction decomposition, ignoring constants,

$$l_t = \sum_{j=1}^t \Delta l_j, \quad \Delta l_t = h_t(y_t)^T \tilde{\theta}_{t|t-1} - \psi_t(\tilde{\theta}_{t|t-1}).$$

To make sense of this working log-likelihood function we need some regularity assumptions.

**Assumption 4** Assume for a sequence  $\{\tilde{\theta}_{t|t-1}\}_{t=1}^T$  there exists constants  $c_{1:2}, d_{1:2}$  such that

for all  $t$ :

(a) the  $\mathbb{E}[|h(Y_t)^\top \tilde{\theta}_{t|t-1}|] < c_1$  and  $\mathbb{E}[|\psi_t(\tilde{\theta}_{t|t-1})|] < d_1$ .

(b) the  $\mathbb{V}[h(Y_t)^\top \tilde{\theta}_{t|t-1}] < c_2$  and  $\mathbb{V}[\psi_t(\tilde{\theta}_{t|t-1})] < d_2$ .

## 5.1 A quasi-likelihood interpretation

As the working log-likelihood

$$l_t, \quad t = 1, \dots, T,$$

comes from the CEF  $(\tilde{\theta}_{t|t-1}, h_t, \psi_t)$ , under weak regularity conditions, it will also be a quasi-likelihood — so long as the predictive mean is modeled correctly. This property does not depend upon the particular features of  $\tilde{\theta}_{t|t-1}$ . We demonstrate this for the canonical exponential family time series case. It then applies as a special case to when  $\tilde{\theta}_{t|t-1}$  is formed from the weighted predictor.

Now make some assumptions about the data.

**Assumption 5** *Under the data, assume that:*

(a)  $\mathbb{E}[|h(Y_t)|] < \infty$  and denote  $\mathbb{E}[h_t(Y_t)|Y_{1:t-1}] := \mu_{t|t-1}$ , where  $t = 1, \dots, T$ .

(b)  $\mathbb{V}[h_t(Y_t)]$  exists, the  $\Sigma_{t|t-1} := \mathbb{V}[h_t(Y_t)|Y_{1:t-1}]$ , for  $t = 1, \dots, T$  and there exists a positive definite matrix  $C$  such that  $\Sigma_{t|t-1} - C$  is positive semidefinite for every  $t$ .

Using strict convexity of  $\psi_t$  and Assumption [5\(a\)](#) there exists a unique  $\theta_{t|t-1}$  such that  $\mu_{t|t-1} = \psi'_t(\theta_{t|t-1})$ .

**Definition 4** *Assume the sequence  $\{\theta_{t|t-1}\}_{t=1}^T$  obeys Assumption [4\(a\)](#) and  $\theta_{t|t-1} = (\psi')^{-1}(\mu_{t|t-1})$  from Assumption [5\(a\)](#). Then define the oracle log-likelihood:*

$$l_t^* = \sum_{j=1}^t \Delta l_j^*, \quad \Delta l_t^* = h_t(y_t)^\top \theta_{t|t-1} - \psi_t(\theta_{t|t-1}), \quad t = 1, \dots, T.$$

Under Assumptions [4\(a\)](#) and [5\(a\)](#), define the two sequences

$$M_t = \sum_{j=1}^t \{l_j - \mathbb{E}[l_j^* | Y_{1:j-1}]\}, \quad t = 1, 2, \dots, T,$$

$$C_t = \sum_{j=1}^t \{\mathbb{E}[l_j | Y_{1:j-1}] - \mathbb{E}[l_j^* | Y_{1:j-1}]\}.$$

then  $\{M_t\}_{t=1}^T$  is a supermartingale with respect to the data's natural filtration. The  $\{C_t\}_{t=1}^T$  is a previsible drift with increments

$$c_t = \psi'(\theta_{t|t-1})^\top \{\tilde{\theta}_{t|t-1} - \theta_{t|t-1}\} - [\psi\{\tilde{\theta}_{t|t-1}\} - \psi\{\theta_{t|t-1}\}] \leq 0,$$

by convexity of  $\psi$  from Definition [3](#) and then using p.69 of [Boyd and Vandenberghe \(2004\)](#). The equality is only obtained iff  $\tilde{\theta}_{t|t-1} = \theta_{t|t-1}$ , that is only if the working model has the correct predictive mean. Thus under Assumptions [4\(a\)](#) and [5\(a\)](#) the log-likelihood process  $\{l_t\}_{t=1}^T$  can be view as a quasi-likelihood process.

If the magnitude of  $\tilde{\theta}_{t|t-1} - \theta_{t|t-1}$  is small, then

$$c_t \approx -\frac{1}{2}(\tilde{\theta}_{t|t-1} - \theta_{t|t-1})^\top \psi''(\theta_{t|t-1})(\tilde{\theta}_{t|t-1} - \theta_{t|t-1}).$$

More broadly, under Assumptions [4\(a\)](#) and [5\(a\)](#), the  $\{M_t - C_t\}_{t=1}^T$  is a martingale. Further, under Assumptions [4\(b\)](#) and Assumption [5\(b\)](#) the angle bracket of the log-likelihood process

$$\langle l \rangle_t := \sum_{j=1}^t V(l_j | Y_{1:j-1}) = \sum_{j=1}^t \tilde{\theta}_{j|j-1}^\top \Sigma_{j|j-1} \tilde{\theta}_{j|j-1} = \langle M - C \rangle_t, \quad t = 1, \dots, T$$

the angle-bracket process of the  $\{M_t - C_t\}_{t=1}^T$  process. For all cases where  $\langle M - C \rangle_T \rightarrow \infty$  then  $(M_T - C_T) / \langle M - C \rangle_T \rightarrow 0$  almost surely, by the martingale strong law of large numbers.

**Remark 3** (a) This quasi-likelihood interpretation is not surprising, it falls in line with the storied history of, for example, generalized linear models, e.g. [Wedderburn \(1974\)](#) and [McCullagh and Nelder \(1989\)](#). More broadly quasi-likelihoods have played a vast role in modern statistics with the estimation theory going back at least to [Cox \(1961\)](#), [Huber \(1967\)](#), [Gallant \(1987\)](#) and [White \(1982\)](#).

(b) The scaled  $\{C_t(\omega)\}_{t=1}^T$  process measures how the scaled working log-likelihood  $l_T$  minus the scaled oracle  $l_T^*$  drifts downwards as  $T$  increases.

## 5.2 Maximum likelihood estimation

Now turn to the maximum likelihood estimator of the hyperparameters based on the above exponential family quasi-likelihood, where the predictor comes from exponential weighting and uses a CEF frame.

**Assumption 6** Based on a  $\text{CEF}(\theta, h_t, \psi_t)$  frame, write  $\tilde{\mu}_{t|t-1}(\omega)$  as the predictor using some finite dimensional hyperparameter vector  $\omega \in \Omega$ . The hyperparameter vector may include any static parameters  $\phi \in \Phi$  from the model density. Assume there exists a  $\omega^*$  such that

$$\hat{\mu}_{t|t-1}(\omega^*) = \tilde{\mu}_{t|t-1}, \quad \text{for all } t = 1, \dots, T,$$

in the data and that  $\tilde{\mu}_{t|t-1}(\omega^*) \neq \tilde{\mu}_{t|t-1}$  for all  $\omega \in (\Omega \setminus \omega^*)$ . Then refer to  $\omega^*$  as the pseudo-true or oracle value.

The leading version of this is given in [Example 2](#).

**Example 2** For a strictly stationary stochastic process  $\{Y_t\}_{t \geq 1}$ , assume that  $\mathbb{E}[h(Y_1)]$  exists and lies in  $\mathcal{H} := h(\mathcal{Y}) \subseteq \mathbb{R}^k$ . For the exponentially weighted estimands under the stable frame minimal  $\text{CEF}(\theta, h, \psi)$ , in steady state,

$$\tilde{\mu}_{t|t-1}(\omega) = \frac{(1-\alpha)(1-\lambda)}{1-\alpha(1-\lambda)} \mathbb{E}[h(Y_1)] + \frac{\alpha\lambda(1-\lambda)}{1-\alpha(1-\lambda)} h(Y_{t-1}) + \lambda \tilde{\mu}_{t-1|t-2},$$

following Remark 2(e). Here the hyperparameters are

$$\omega := (\mathbb{E}[h(Y_1)]^\top, \alpha, \lambda)^\top \in \Omega^* = \mathcal{H} \times (0, 1) \times (0, 1).$$

The  $\tilde{\mu}_{t|t-1}(\omega)$  is infinitely differentiable with respect to  $\omega$ , linear in transforms of the past data  $h(y_1), \dots, h(y_{t-1})$ ,  $\lambda$  and  $\mathbb{E}[h(Y_1)]$ , but nonlinear in  $\lambda$ . More generally, if the model density contains additional static parameters  $\phi$  (e.g., a static variance parameter  $\sigma^2$  in the Gaussian case that is not modeled by the exponentially weighted estimand), these can be included in the hyperparameter vector

$$\omega := (\mathbb{E}[h(Y_1)]^\top, \alpha, \lambda, \phi^\top)^\top \in \Omega^* = \mathcal{H} \times (0, 1) \times (0, 1) \times \Phi.$$

Such static parameters enter the  $l_t(\omega)$  but do not affect the structure of the predictor  $\tilde{\mu}_{t|t-1}$ .

The resulting maximum likelihood estimate formed from the quasi-likelihood is:

$$\hat{\omega}_{\text{MLE}} = \arg \max_{\omega \in \Omega} l_T(\omega),$$

noting explicitly how the quasi-likelihood is impacted by the choice of  $\omega$  through  $\{\tilde{\theta}_{t|t-1}(\omega)\}_{t=1}^T$ .

Under Assumption 6, the corresponding  $\omega^*$  is the pseudo-true value of  $\omega$  for this quasi-likelihood function.

Throughout we will posit that  $\tilde{\theta}_{t|t-1}(\omega)$  is infinitely differentiable for all  $\omega \in \Omega^*$  and write

$$\tilde{\mu}'_{t|t-1}(\omega) = \frac{\partial \tilde{\mu}_{t|t-1}(\omega)^\top}{\partial \omega}.$$

The corresponding score up to time  $1 \leq t \leq T$  is

$$\begin{aligned} S_t(\omega; y_{1:t}) &:= \frac{\partial l_t(\omega)}{\partial \omega} \\ &= \sum_{j=1}^t s_j(\omega), \quad s_t(\omega) := \frac{\partial l_t(\omega)}{\partial \omega} = \frac{\partial \tilde{\theta}_{t|t-1}(\omega)^\top}{\partial \omega} \frac{\partial l_t(\omega)}{\partial \tilde{\theta}_{t|t-1}(\omega)}. \end{aligned}$$

Assume that for all  $\omega \in \Omega$  that  $\tilde{\Sigma}_{t|t-1}(\omega) = \psi''(\tilde{\theta}_{t|t-1}(\omega))$  is invertible. Then

$$s_t(\omega) = \tilde{\theta}'_{t|t-1}(\omega) \{h(y_t) - \tilde{\mu}_{t|t-1}(\omega)\}, \quad \text{where} \quad \tilde{\theta}'_{t|t-1}(\omega) := \frac{\partial \tilde{\theta}_{t|t-1}(\omega)^\top}{\partial \omega} = \tilde{\mu}'_{t|t-1}(\omega) \tilde{\Sigma}_{t|t-1}^{-1}(\omega),$$

recalling  $\tilde{\Sigma}_{t|t-1}(\omega) := \psi''(\tilde{\theta}_{t|t-1}(\omega))$ . The Hessian is

$$H_t(\omega) = - \sum_{j=1}^t l''_j(\omega), \quad \text{where} \quad l''_t(\omega) = \frac{\partial^2 l_t(\omega)}{\partial \omega \partial \omega^\top}.$$

**Remark 4** (a) Let  $s_t := s_t(\omega^*; Y_{1:t})$ , and  $S_t = \sum_{j=1}^t s_j$ . Then  $\{S_t\}_{t=1}^T$  is a martingale sequence with respect to the data's natural filtration so long as  $\mathbb{E}[|s_t|] < \infty$ . Under the [Brown \(1971\)](#) martingale central limit theorem the

$$\langle S, S \rangle_T^{-1/2} S_T \xrightarrow{D} N(0, I),$$

noting that  $\langle S, S \rangle_T = \sum_{t=1}^T \tilde{\theta}'_{t|t-1} \Sigma_{t|t-1} (\tilde{\theta}'_{t|t-1})^\top$ , where  $\tilde{\theta}'_{t|t-1} := \tilde{\theta}'_{t|t-1}(\omega^*)$  and  $\Sigma_{t|t-1} := \Sigma_{t|t-1}(\omega^*)$ .

(b) By a multivariate mean value expansion  $0 = S_T - \bar{H}_T(\hat{\omega}_{\text{MLE}} - \omega^*)$ , where  $\bar{H}_T = \int_0^1 H_T(\omega^* + u(\hat{\omega}_{\text{MLE}} - \omega^*)) du$ . Hence if  $\langle S, S \rangle_T$  is invertible then

$$\{\langle S, S \rangle_T^{-1/2} \bar{H}_T\}(\hat{\omega}_{\text{MLE}} - \omega^*) = \langle S, S \rangle_T^{-1/2} S_T \xrightarrow{D} N(0, I).$$

If  $\bar{H}_T$  is invertible, then  $W_T = \bar{H}_T^{-1} \langle S, S \rangle_T \bar{H}_T^{-1}$ , is an infeasible approximation to the variance-

covariance matrix of the estimator.

(c) In practice we use the estimator  $\hat{W}_T = H_T^{-1}(\hat{\omega}_{\text{MLE}})[\hat{S}, \hat{S}]_T H_T^{-1}(\hat{\omega}_{\text{MLE}})$ , where  $[\hat{S}, \hat{S}]_T = \sum_{t=1}^T \hat{s}_t \hat{s}_t^\top$ , with  $s_t = s_t(\hat{\omega}_{\text{MLE}}; Y_{1:t})$ , as a feasible approximation to the variance-covariance matrix of the estimator. Hence this is a sandwich matrix of the tradition we see in quasi-likelihood estimation, going back to at least [Cox \(1961\)](#) and [Huber \(1967\)](#).

### 5.3 Two step alternative to the MLE

An alternative to the MLE is a two step procedure (e.g. [Newey and McFadden \(1994\)](#), [Engle and Mezrich \(1996\)](#), [Francq et al. \(2013\)](#)), which has the following structure.

**Definition 5 (2-step estimator)** For a strictly stationary stochastic process  $\{Y_t\}_{t \geq 1}$ , assume  $\mathbb{E}[h(Y_1)] \in \mathcal{H}$  exists. For the exponentially weighted estimands under the stable frame minimal CEF  $(\theta, h, \psi)$ , let  $\phi$  denote any additional static parameters from the model density.

Write

$$\hat{\omega}_{2\text{Step}} = \begin{pmatrix} \widehat{\mathbb{E}[h(Y_1)]}_{2\text{Step}} \\ \hat{\alpha}_{2\text{Step}} \\ \hat{\lambda}_{2\text{Step}} \\ \hat{\phi}_{2\text{Step}} \end{pmatrix}.$$

Then compute:

1. The method of moments estimator:

$$\widehat{\mathbb{E}[h(Y_1)]}_{2\text{Step}} = \frac{1}{T} \sum_{t=1}^T h(Y_t).$$

2. The likelihood based estimator:

$$\{\hat{\alpha}_{2\text{Step}}, \hat{\lambda}_{2\text{Step}}, \hat{\phi}_{2\text{Step}}\} = \arg \max_{\{\alpha, \lambda, \phi\} \in (0,1)^2 \times \Phi} \sum_{t=1}^T l_t(\widehat{\mathbb{E}[h(Y_1)]}_{2\text{Step}}, \alpha, \lambda, \phi),$$

Here the numerical optimization of the likelihood is only  $2 + \dim(\phi)$  dimensional. This estimation strategy can be very attractive when  $\mathbb{E}[h(Y_1)]$  is high dimensional.

The two step procedure can be viewed as a method of moments estimator (Pearson, 1894), based around a random function

$$S_t(\omega; y_{1:t}) := \sum_{j=1}^t s_j(\omega; y_{1:j}), \quad s_t(\omega; y_{1:t}) := \begin{pmatrix} h(Y_t) - \mathbb{E}[h(Y_1)] \\ \partial l_t(\omega)/\partial \alpha \\ \partial l_t(\omega)/\partial \lambda \\ \partial l_t(\omega)/\partial \phi \end{pmatrix}$$

for  $t = 1, \dots, T$ , then  $\hat{\omega}_{2\text{Step}}$  is a method of moments estimator which solves  $S_T(\hat{\omega}_{2\text{Step}}; Y_{1:T}) = 0$ , while  $\mathbb{E}[s_t(\omega^*; Y_{1:t})] = 0$  for each  $t = 1, \dots, T$ . The corresponding Hessian is

$$H_T(\omega) := - \sum_{t=1}^T \frac{\partial s_t(\omega)}{\partial \omega^T} = \sum_{t=1}^T \begin{pmatrix} I & 0 & 0 \\ -\partial^2 l_t(\omega)/\partial \alpha \partial \omega^T \\ -\partial^2 l_t(\omega)/\partial \lambda \partial \omega^T \\ -\partial^2 l_t(\omega)/\partial \phi \partial \omega^T \end{pmatrix}.$$

**Remark 5** (a) Write  $s_t = s_t(\omega^*; Y_{1:t})$  and  $S_t = S_t(\omega^*; Y_{1:t})$ . The last three elements of  $\{s_t\}$  are martingale differences. The remaining elements,  $h(Y_t) - \mathbb{E}[h(Y_1)]$ , have an unconditional zero mean but possibly substantial time series memory. Write out a corresponding CLT as

$$V_T^{-1/2} S_T(\omega^*) \xrightarrow{D} N(0, I), \quad \text{where } V_T = V(S_T(\omega^*)),$$

which needs the memory in  $\{s_t\}$  to be controlled. The martingale difference elements have limited memory, the problem is the time series of  $\{h(Y_t) - \mathbb{E}[h(Y_1)]\}$ . A basic way of generating a CLT for these type of objects is to assume the  $h(Y_t)$  series is strictly stationary and exhibits  $m$ -dependence (e.g. Janson (2021)).

(b) By a multivariate mean value expansion  $0 = S_T - \bar{H}_T(\hat{\omega}_{2\text{Step}} - \omega^*)$ , where  $\bar{H}_T = \int_0^1 H_T(\omega^* + u(\hat{\omega}_{2\text{Step}} - \omega^*)) du$ . Thus

$$V_T^{-1/2} \bar{H}_T(\hat{\omega}_{2\text{Step}} - \omega^*) = V_T^{-1/2} S_T \xrightarrow{D} N(0, I).$$

So the infeasible variance matrix for the two step estimator is  $W_T = \bar{H}_T^{-1} V_T \bar{H}_T^{-1}$ .

(c) In practice we use the estimator of the covariance matrix  $\hat{W}_T = H_T^{-1}(\hat{\omega}_{2\text{Step}}) \hat{V}_T H_T^{-1}(\hat{\omega}_{2\text{Step}})^T$ , where  $\hat{V}_T$  approximates  $V(\hat{S}_T)$ . The latter can be estimated using  $T$  times a long-run variance of the time series  $\hat{s}_1, \dots, \hat{s}_T$ , where  $\hat{s}_t = s_t(\hat{\omega}_{2\text{Step}}; Y_{1:t})$ .

## 6 Empirical example

### 6.1 Dirichlet based frame: household financial situation

We apply the exponentially weighted predictor and smoother to monthly data on household financial situation expectations from the University of Michigan Survey of Consumers, covering January 1978 to September 2025 ( $T = 573$  observations). The survey asks respondents two questions: first, whether they are better off or worse off financially than a year ago, and second, whether they expect to be better off or worse off a year from now. The combination of responses yields seven mutually exclusive categories:

- (i) Continuous Increase (better off now and expect to be better off),
- (ii) Intermittent Increase (one period better, one period same),
- (iii) Remain Unchanged (same in both periods),
- (iv) Intermittent Decline (one period worse, one period same),
- (v) Continuous Decline (worse off now and expect to be worse off),
- (vi) Mixed Change (improvement followed by decline or vice versa),
- (vii) Don't Know/No answer.

For each month, only the sample proportions (percentages) falling into each category are reported, not the underlying individual responses or sample sizes.<sup>3</sup>

We model the observed proportions  $y_t = (y_{1,t}, \dots, y_{7,t})^T$  as realizations from a time-varying seven category Dirichlet distribution  $Y_t \mid Y_{1:t-1} \sim \text{Dirichlet}(\tilde{\theta}_{t|t-1})$ , with  $\tilde{\theta}_{t|t-1} = (\tilde{\theta}_{1,t|t-1}, \dots, \tilde{\theta}_{7,t|t-1})^T$ , with concentration parameters  $\tilde{\theta}_{j,t|t-1} > 0$  for all  $j \in \{1, \dots, 7\}$  and all  $t$ . Each observation is treated as a single draw of a proportion vector. The predictor is based on a Dirichlet frame.

The exponentially weighted predictor and smoother are computed via the recursions in Example 1 using the Dirichlet distribution’s CEF parameterization given in Section 4.1.

Following the two-step quasi-likelihood procedure from Definition 5, we estimate the centering parameter  $\mathbb{E}[h(Y_1)]$  by the sample mean  $\mathbb{E}[\widehat{h(Y_1)}] = T^{-1} \sum_{t=1}^T (\log y_{1,t}, \dots, \log y_{7,t})^T$ . Next, we fix  $\mathbb{E}[\widehat{h(Y_1)}]$  and estimate  $(\alpha, \lambda)$  as maximizing  $\sum_{t=1}^T \log f(y_t; \tilde{\theta}_{t|t-1}(\alpha, \lambda))$ .

For the household financial situation data with seven categories, we obtain the parameter estimates  $(\mathbb{E}[\widehat{h(Y_1)}]^T, \hat{\alpha}, \hat{\lambda}) \approx (-1.76, -1.41, -1.78, -1.77, -2.73, -2.23, -3.53, 0.95, 0.65)$ . The high anchoring estimate  $\hat{\alpha} = 0.95$  indicates little long-run effect, suggesting that shifts in household financial expectations, once they occur, tend to persist a long time. The discount parameter  $\hat{\lambda} = 0.65$  implies a half-life of approximately 1.6 months for the exponential weights. Following Remark 2(e), the steady state process is a vector ARMA(1,1)-MD process, with autoregressive root  $\hat{\lambda}/\{1 - \hat{\alpha}(1 - \hat{\lambda})\} \approx 0.974$ , and moving average root  $-\hat{\lambda} \approx -0.65$ . Hence, the process exhibits substantial memory, with substantial root cancellation.

Figure 5 presents the estimation results. Each panel shows the evolution of one expectation category over the 47-year period. The gray circles represent observed monthly proportions  $y_{k,t}$ . The light colored lines show the exponentially weighted predictor, the dark colored lines the exponentially weighted smoother. Both track major shifts in household financial expectations. There are the sharp declines in both “Continuous Increase” and “In-

---

<sup>3</sup>While monthly sample sizes are available, our framework would require within-month i.i.d. responses, which is unlikely given the clustered sampling design. We leave them out of our analysis.

“Intermittent Increase” during the 2008 financial crisis, and, more strikingly, since around 2018. This deterioration in optimism about household finances is mirrored by corresponding increases in negative expectations: “Intermittent Decline” and “Continuous Decline” roughly track their inverse patterns. The categories “Remain Unchanged” and “Don’t Know/No Answer” stay relatively stable at 15-20% and 3%, respectively, throughout the sample.

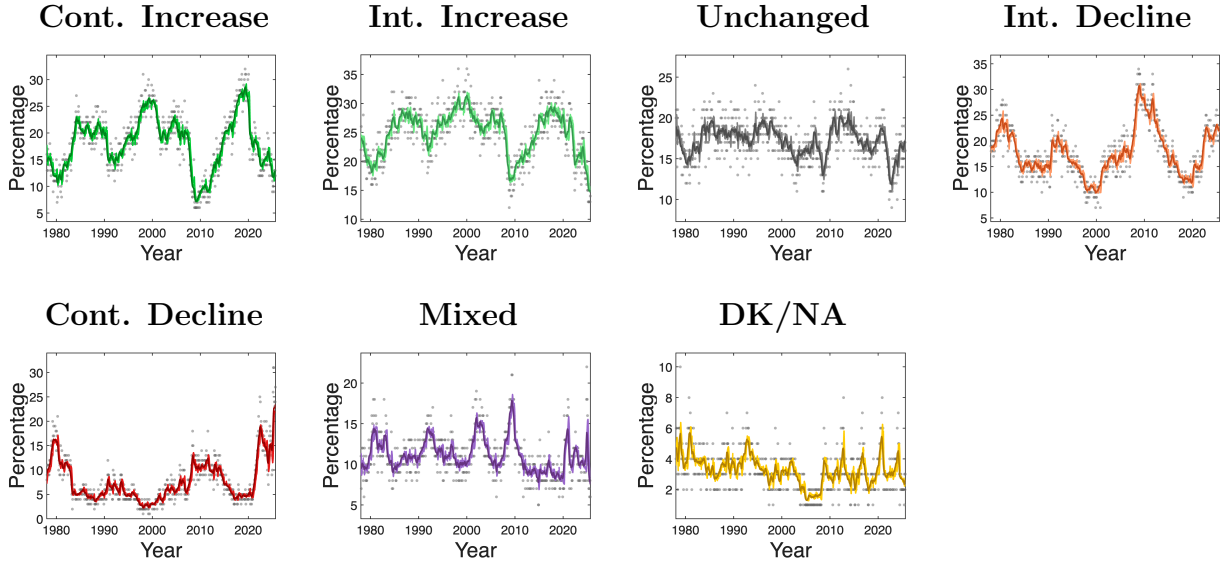


Figure 5: Household financial situation: Uni. Michigan Survey of Consumers (Jan 1978 to Sept 2025). Categories combine current versus past financial situation, & expectations for the year ahead (Cont. = Continuous, Int. = Intermittent, DK = Don’t Know, NA = No Answer): Continuous Increase means better off now & expecting better, Intermittent Increase indicates improvement in one period only. Each panel shows, for one response category, the observed monthly proportions  $y_{k,t}$  (gray circles), the predictor  $y_{k,t|t-1} = \tilde{\theta}_{k,t|t-1} / \sum_{j=1}^7 \tilde{\theta}_{j,t|t-1}$  (light line) & the smoother  $y_{k,t|T} = \tilde{\theta}_{k,t|T} / \sum_{j=1}^7 \tilde{\theta}_{j,t|T}$  (dark line). Estimates:  $(\hat{\alpha}, \hat{\lambda}) = (0.95, 0.65)$ .

## 6.2 Monte Carlo assessment of estimator precision

To investigate the finite-sample precision of the quasi-likelihood estimator from Definition 5, we conduct a Monte Carlo study using the estimated model from Section 6.1 as the data-generating process (DGP). We generate time-varying proportion vectors following Assumption 3, that is,  $Y_t | Y_{1:t-1} \sim \text{Dirichlet}(\theta_{t|t-1})$ , where  $\theta_{t|t-1} = (\theta_{1,t|t-1}, \dots, \theta_{7,t|t-1})^\top$ , where  $\theta_{t|t-1}$  evolves according to the exponentially weighted predictor (Theorem 1), denot-

ing  $(\pi^*, \alpha^*, \lambda^*)$  for the hyperparameters in the DGP. We initialize each prediction path with  $\tilde{\theta}_{1|0} = (\psi')^{-1}(\pi^*)$  where  $\pi^* = \mathbb{E}[\widehat{h}(y_1)]$  is the estimated centering parameter from Section 6.1.

To investigate how the anchoring  $\alpha^*$  affects hyperparameter estimation, we consider four scenarios:

- (i)  $(\pi^*, \alpha^*, \lambda^*) = (\widehat{\pi}, 0.01, 0.65)$  (strong anchoring),
- (ii)  $(\pi^*, \alpha^*, \lambda^*) = (\widehat{\pi}, 0.30, 0.65)$  (moderate anchoring),
- (iii)  $(\pi^*, \alpha^*, \lambda^*) = (\widehat{\pi}, 0.60, 0.65)$  (weak anchoring),
- (iv)  $(\pi^*, \alpha^*, \lambda^*) = (\widehat{\pi}, 0.95, 0.65)$  (no anchoring, matching estimated values),

where  $\pi^* = (-1.76, -1.41, -1.78, -1.77, -2.73, -2.23, -3.53)'$  and  $\lambda^* = 0.65$  is fixed across all scenarios to match the estimates from the household data.

Figure 6 displays the quasi log-likelihood surfaces and confidence regions for the estimated  $(\alpha, \lambda)$ . Confidence regions contract systematically as  $T$  increases, with maximum quasi-likelihood estimates (red stars) concentrating near the DGP values (blue circles). The pattern of contraction depends strongly on the anchoring hyperparameter  $\alpha^*$ . When  $\alpha^* = 0.01$  (first row), the confidence interval for  $\lambda$  remains wide even at  $T = 10,000$ . This aligns with Remark 1(e): when  $\alpha$  is close to zero, the frame is nearly stationary, and  $\lambda$  has minimal impact on the filter and predictor, making it difficult to estimate from the data. Contrastingly, when  $\alpha^* \in \{0.60, 0.95\}$  (third and fourth rows), confidence intervals for both hyperparameters contract rapidly even at moderate sample sizes.

## 7 Conclusions

The exponentially weighted estimands combined with the CEF delivers simple exact filters, predictors and smoothers. They are in the same statistical spirit as generalized linear models (e.g. McCullagh and Nelder (1989)) and are relatively easy to fit to data using a quasi-likelihood approach either all at once or using a two step approach.

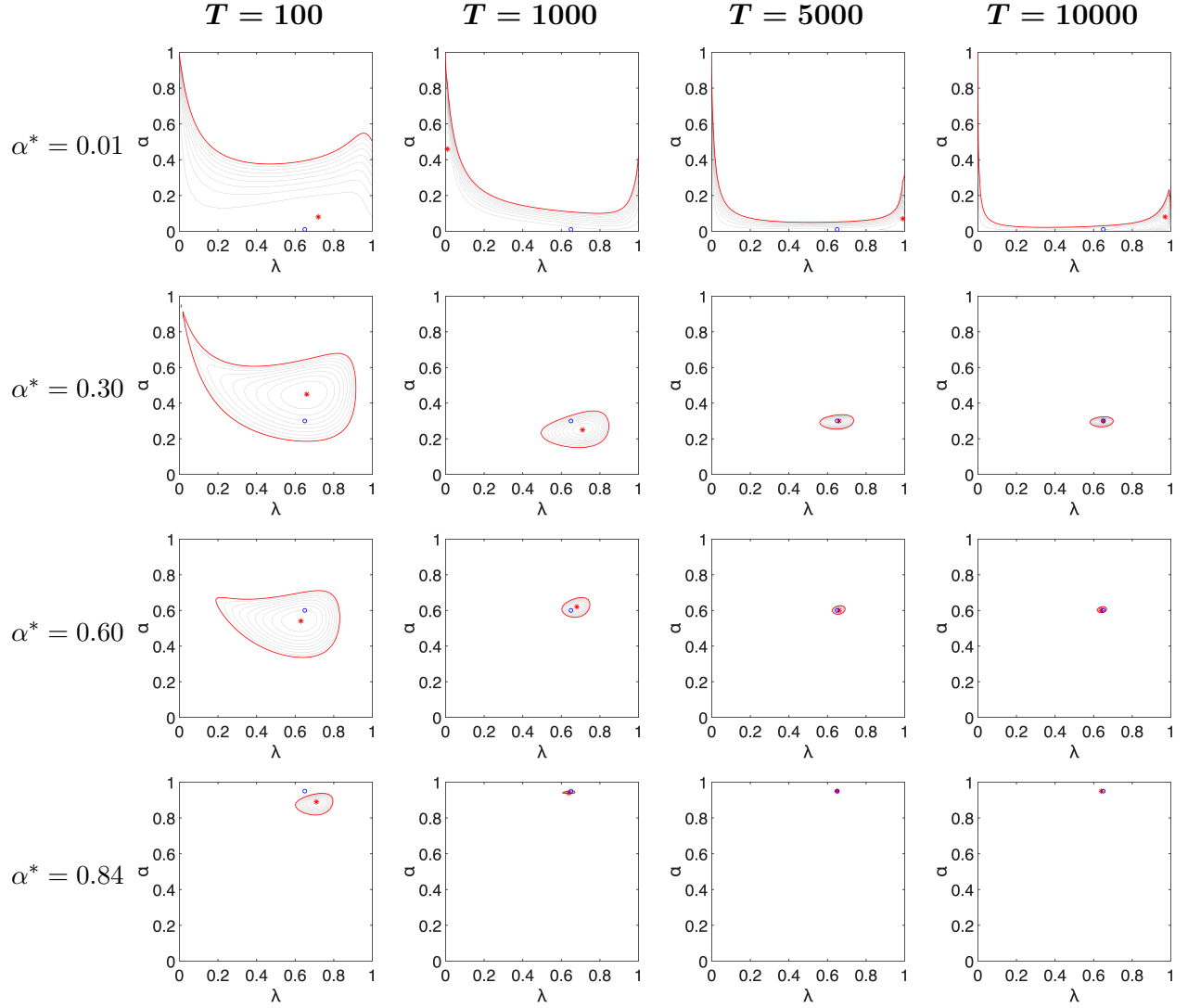


Figure 6: Monte Carlo: quasi-likelihood estimator precision for household financial expectations. Rows: anchoring hyperparameters: (i)  $\alpha^* = 0.01$ , (ii)  $\alpha^* = 0.30$ , (iii)  $\alpha^* = 0.60$ , and (iv)  $\alpha^* = 0.95$  (matching estimated household data). All scenarios use the estimated centering parameter  $\pi^* = \widehat{\mathbb{E}[h(y_1)]} = (-1.76, -1.41, -1.78, -1.77, -2.73, -2.23, -3.53)'$  and discount parameter  $\lambda^* = 0.65$  from Section 6.1. Display: quasi log-likelihood surface  $\ell(\alpha, \lambda)$  with the maximum quasi-likelihood estimate (red star) and DGP value (blue circle). The red contour traces the 99% confidence region. Columns:  $T \in \{100, 1000, 5000, 10000\}$ .

The exponentially weighted estimands framework can be used on non-exponentially family models and even for general loss functions. In such cases the optimization has to be carried out numerically or through a sequential approximation. This then relates to work on stochastic gradient descent (e.g. [Robbins and Munro \(1951\)](#), [Toulis and Airolidi \(2015, 2017\)](#), [Toulis et al. \(2016\)](#), [Toulis et al. \(2021\)](#)), [Lange et al. \(2024\)](#), [Donker van Heel et al. \(2025\)](#), as well as the DCS/GAS filter of [Harvey \(2013\)](#) and [Creal et al. \(2013\)](#).

## Acknowledgments

We are grateful for the comments and encouragement from Dick van Dijk, Siem Jan Koopman, Rutger-Jan Lange and Bram van Os. Donker van Heel conducted this research during a visiting research appointment at Harvard University’s Department of Statistics.

## Disclosure Statement

Donker van Heel thanks the Fulbright Program for financial support under their scholarship program. The authors report there are no competing interests to declare.

## References

- Abramowitz, M. and I. A. Stegun (1970). *Handbook of Mathematical Functions*. New York: Dover Publications Inc.
- Benjamin, M. A., R. A. Rigby, and D. M. Stasinopoulos (2003). Generalized autoregressive moving average model. *Journal of the American Statistical Association* 98, 214–223.
- Blasques, F., S. J. Koopman, M. Mallee, and Z. Zhang (2016). Weighted maximum likelihood for dynamic factor analysis and forecasting with mixed frequency data. *Journal of Econometrics* 193(2), 405–417.
- Bollerslev, T. (1986). Generalised autoregressive conditional heteroskedasticity. *Journal of Econometrics* 51, 307–327.
- Boyd, S. and L. Vandenberghe (2004). *Convex Optimization*. Cambridge University Press.
- Brown, B. M. (1971). Martingale central limit theorems. *Annals of Mathematical Statistics* 49, 59–66.

- Brown, R. G. (1956). *Exponential smoothing for predicting demand*. Cambridge, Massachusetts: Author D Little, Inc.
- Cox, D. R. (1961). Tests of separate families of hypotheses. *Proceedings of the Berkeley Symposium 4*, 105–123.
- Creal, D., S. J. Koopman, and A. Lucas (2013). Generalized autoregressive score models with applications. *Journal of Applied Econometrics 28*, 777–795.
- Davis, R. A., K. Fokianos, S. H. Holan, H. Joe, J. Livsey, R. Lund, V. Pipiras, and N. Ravishanker (2021). Count time series: A methodological review. *Journal of the American Statistical Association 116*(535), 1533–1547.
- Dixon, M. J. and S. G. Coles (1997). Modelling association football scores and inefficiencies in the football betting market. *Journal of the Royal Statistical Society: Series C (Applied Statistics) 46*(2), 265–280.
- Donker van Heel, S., R.-J. Lange, B. van Os, and D. J. C. van Dijk (2025). Stability and performance guarantees for misspecified multivariate score-driven filters. *Preprint*. <https://ssrn.com/abstract=5127937>.
- Durbin, J. and S. J. Koopman (2012). *Time Series Analysis by State Space Methods* (2 ed.). Oxford: Oxford University Press.
- Efron, B. (2012). *Large-Scale Inference: Empirical Bayes Methods for Estimation, Testing, and Prediction*. Cambridge University Press.
- Efron, B. (2022). *Exponential Families in Theory and Practice*. Cambridge University Press.
- Efron, B. and C. Morris (1977). Stein’s paradox in statistics. *Scientific American 236*, 119–127.
- Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of the United Kingdom inflation. *Econometrica 50*, 987–1007.
- Engle, R. F. and J. Mezrich (1996). GARCH for groups. *Risk*, 36–40.
- Fan, J., N. E. Heckman, and M. P. Wand (1995). Local polynomial kernel regression for generalized linear models and quasi-likelihood functions. *Journal of the American Statistical Association 90*, 141–150.
- Fan, J. and Q. Yao (2005). *Nonlinear Time Series*. New York: Springer.
- Francq, C., L. Horvath, and J.-M. Zakoïan (2013). Merits and drawbacks of variance targeting in garch models. *Journal of Financial Econometrics 9*, 619–656.
- Gallant, A. R. (1987). *Nonlinear Statistical Models*. New York: John Wiley.
- Harvey, A. C. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge: Cambridge University Press.
- Harvey, A. C. (2013). *Dynamic models for volatility and heavy tails: With applications to financial and economic time series*. Cambridge University Press.
- Hoerl, A. E. and R. W. Kennard (1970). Ridge regression: Biased estimation for nonorthogonal problems. *Technometrics 12*, 55–67.

- Holmes, C. C. and S. G. Walker (2017). Assigning a value to a power likelihood in a general bayesian model. *Biometrika* 104, 497–503.
- Hu, F. and J. V. Zidek (2002). The weighted likelihood. *Canadian Journal of Statistics* 30(3), 347–371.
- Huber, P. J. (1967). The behavior of maximum likelihood estimates under nonstandard conditions. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, pp. 221–233. University of California Press.
- Janson, S. (2021). A central limit theorem for m-dependent variables. *Preprint*. <https://arxiv.org/abs/2108.12263>.
- Lange, R.-J., B. van Os, and D. J. van Dijk (2024). Implicit score-driven filters for time-varying parameter models. *Preprint*. <https://ssrn.com/abstract=4227958>.
- Li, W. K. (1994). Time series models based on generalized linear models: Some further results. *Biometrics* 50, 506–511.
- Luxenberg, E. and S. Boyd (2024). Exponentially weighted moving models. *Preprint*. <https://arxiv.org/abs/2404.08136>.
- McCullagh, P. and J. A. Nelder (1989). *Generalized Linear Models* (2 ed.). London: Chapman & Hall.
- Newey, W. K. and D. McFadden (1994). Large sample estimation and hypothesis testing. In R. F. Engle and D. McFadden (Eds.), *The Handbook of Econometrics, Volume 4*, pp. 2111–2245. North-Holland.
- Pearson, K. (1894). Contributions to the mathematical theory of evolution. *Philosophical Transactions of the Royal Society, Series A* 185, 71–110.
- Robbins, H. and S. Munro (1951). A stochastic approximation method. *The Annals of Mathematical Statistics* 22, 400–407.
- Tibshirani, R. (1996). Regression shrinkage and selection via the Lasso. *Journal of the Royal Statistical Society, Series B* 58, 267–288.
- Toulis, P. and E. M. Airoldi (2015). Scalable estimation strategies based on stochastic approximations: Classical results and new insights. *Statistics and Computing* 25, 781–795.
- Toulis, P. and E. M. Airoldi (2017). Asymptotic and finite-sample properties of estimators based on stochastic gradients. *Annals of Statistics* 45(4), 1694–1727.
- Toulis, P., T. Horel, and E. M. Airoldi (2021). The proximal Robbins–Monro method. *Journal of the Royal Statistical Society Series B: Statistical Methodology* 83(1), 188–212.
- Toulis, P., D. Tran, and E. M. Airoldi (2016). Towards stability and optimality in stochastic gradient descent. In *Artificial Intelligence and Statistics*, pp. 1290–1298. PMLR.
- Wedderburn, R. W. M. (1974). Quasi-likelihood functions, generalized linear models and the Gauss-Newton methods. *Biometrika* 61, 439–47.
- White, H. (1982). Maximum likelihood estimation of misspecified models. *Econometrica* 50, 1–25.
- Zeger, S. L. and B. Qaqish (1988). Markov regression models for time series, a quasi likelihood approach. *Biometrics* 44, 1019–1032.