

Jaakkola, Niko; Wagener, Florian Oskar Ottokar

**Working Paper**

## markov-perfect equilibria in differential games, with an application to climate policy

CESifo Working Paper, No. 10585

**Provided in Cooperation with:**

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*Suggested Citation:* Jaakkola, Niko; Wagener, Florian Oskar Ottokar (2025) : markov-perfect equilibria in differential games, with an application to climate policy, CESifo Working Paper, No. 10585, Munich Society for the Promotion of Economic Research - CESifo GmbH, Munich

This Version is available at:

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**CES ifo**

**10585  
2025**

Original Version: July 2023  
This Version: October 2025

# Working Papers

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**CES ifo**

Imprint:

**CESifo Working Papers**

ISSN 2364-1428 (digital)

Publisher and distributor: Munich Society for the Promotion  
of Economic Research - CESifo GmbH

Poschingerstr. 5, 81679 Munich, Germany  
Telephone +49 (0)89 2180-2740

Email [office@cesifo.de](mailto:office@cesifo.de)  
<https://www.cesifo.org>

Editor: Clemens Fuest

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# Markov-perfect equilibria in differential games, with an application to climate policy

Niko Jaakkola\* and Florian Wagener†

September 30, 2025

## Abstract

We introduce discontinuous Markovian strategies for differential games. The best response correspondence uniquely maps almost all profiles of opponents' strategies back to the strategy space. We thus make Markov-perfect equilibria (MPE) in a wide class of differential games well-behaved, resolving a long-standing open problem. We provide a readily applicable necessary and sufficient condition for best responses and MPE. We demonstrate our methods in a canonical model of non-cooperative mitigation of climate change. Our approach provides novel, economically important results: we obtain the entire set of symmetric Markov-perfect Nash equilibria, and demonstrate that the best equilibria can yield a major welfare improvement over the equilibrium which previous literature has focused on. International climate negotiations should be seen as being about coordination on good equilibria, rather than about bargaining over the limited surplus available in a dynamic prisoner's dilemma.

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\*University of Bologna, Piazza Scaravilli 2, 40126 Bologna, Italy; CESifo. Email: niko.jaakkola@unibo.it

†Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1402 EJ Amsterdam, The Netherlands; Tinbergen Institute; CESifo. Email: f.o.o.wagener@uva.nl. Tel: +31 20 525 4034 (corresponding author)

Climate change is widely recognised as a severe threat to economic and human wellbeing. Solving this problem has proven to be difficult: a stable climate is a global public good. Investment into such a public good, that is, refraining from burning fossil fuels, is costly, and all countries would prefer to free-ride on the efforts of others. Dynamic public good provision is typically seen to have the structure of a prisoner’s dilemma, with disappointing outcomes.

However, dynamic games often have multiple Nash equilibria, of which some Pareto-dominate others. In this paper, we construct the entire set of symmetric equilibria in Markovian strategies to a class of dynamic public good games and apply the result to a classic game of dynamic climate mitigation (van der Ploeg and de Zeeuw, 1992). There is a single stock of a public bad—carbon in the atmosphere causing harmful climate change—and a set of symmetric players benefitting individually from emitting more carbon. Players choose Markovian strategies: policy rules which condition emissions only on the current carbon stock. There are many equilibria; coordinating on a good one is of first-order importance for welfare. Indeed, the strategic incentives embedded into policy rules can support non-cooperative, Markov-perfect Nash equilibria (MPE) which get close to, or even achieve, the socially optimal welfare outcome. The welfare gains, relative to the equilibrium that has been the focus of the literature—for reasons of tractability—are substantial. We argue that climate negotiations should leverage self-enforcing strategic incentives to achieve socially beneficial outcomes.

It has been argued that equilibrium multiplicity is important in public goods games (Dockner and Long, 1993; Dutta and Radner, 2004).<sup>1</sup> However, we are the first to make this argument rigorously in a theoretically sound differential game framework. Our main innovation is to treat discontinuous Markovian strategies in a manner that makes differential games well-specified. We construct the entire, large set of symmetric MPE and discuss its welfare properties, showing that any continuous-value equilibrium Pareto-dominates the linear equilibrium, which has been the main focus of the literature. The best equilibria get remarkably close to the first-best.<sup>2</sup>

Our main contribution is thus methodological, and the main results two-fold: structural and constructive. Structurally, Theorem 1 shows that allowing discontinuous strategies makes the concept of Markov-perfect Nash equilibrium (MPE) well-founded: outcomes, and thus payoffs,

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<sup>1</sup>Dutta and Sundaram (1993) demonstrate, in a discrete-time renewable resource game, a “good” outcomes sustained by appropriate Markovian strategies. They do not discuss whether still further equilibria might exist.

<sup>2</sup>The continuous-time framework is essential to computing the entire set of (symmetric) equilibria. In discrete time, constructing the entire set—as opposed to presenting individual alternative equilibria—is to our knowledge difficult in a setting such as ours, because the optimal action depends on global properties of the value function.

can be computed for all strategy profiles, and unique best responses almost always exist. This is a novel contribution to the voluminous literature on differential games—which, despite the term, has not taken the “game” aspect sufficiently seriously (Bernhard, 2024). We put differential games on a solid game-theoretic foundation, solving, for the scalar state case, an open problem discussed in, for instance, the textbooks of Başar and Olsder (1982), Krasovskiĭ and Subbotin (1988), Fudenberg and Tirole (1991), and Dockner et al. (2000). We remark on this further below.

Constructively, Theorem 2 provides a readily applicable necessary and sufficient condition for characterising the best response to a profile of opponents’ strategies in a general class of public investment games. From this, MPE can readily be constructed; we give a general condition for constructing symmetric MPE, Theorem 3, and demonstrate its use in the climate application. Our methods apply to many other applications, without the need to resort to special functional specifications, such as the commonly-used linear-quadratic framework.

Our climate policy application is intended mainly as an example of how our methods can be used to derive novel results for even well-understood models. We use the seminal model of non-cooperative mitigation of climate change of van der Ploeg and de Zeeuw (1992), which uses a linear-quadratic framework so that a linear MPE exists. Dockner and Long (1993) proved the model also has non-linear equilibria, but many of these could not be extended to the entire state space (as did Tsutsui and Mino, 1990, for a different context). Rubio and Casino (2002) and Dockner and Wagener (2014) constructed even more “local MPE”. However, the validity of such “equilibria” undefined on the entire state space has been denied (Rowat, 2007; Bernhard, 2024). Rowat (2007) enumerated all the continuous, globally defined equilibria of this model. Recently, Schumacher et al. (2022), building on our approach, derive a complementary result (on which more below). In a renewable resource game, Sorger (1998) constructs a family of equilibria with a single discontinuity.<sup>3</sup>

We derive three main results for the application. First, we characterise the entire set of symmetric MPE. The set is large, but construction of any member is straightforward (Theorems 5 and 6). Most of the previously discovered equilibria are nested within the larger set of all symmetric equilibria, extending locally-defined equilibria into globally-defined, but discontinuous, MPE.<sup>4</sup>

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<sup>3</sup>Sorger (1998) is our closest precursor. We use a more general approach, and show that it is game-theoretically well-defined.

<sup>4</sup>In our parameterisation, only one continuous globally defined equilibrium exists.

Second, we show that the new equilibria we identify are important. Because we focus solely on symmetric equilibria, these can be Pareto-ranked. The linear equilibrium—a central focus of the extant literature—belongs to this set, but it is weakly Pareto-dominated by any other equilibrium with continuous value. That is, for any initial state, the linear equilibrium yields the lowest surplus available in any MPE, except for equilibria featuring bad coordination failures, which we discuss below.<sup>5</sup> The linear equilibrium involves adverse strategic incentives, as players try to exploit the fact that their opponents will reduce emission in response to a higher carbon stock. In a calibrated example, we show that choosing the best equilibrium can close between 50–100% of the welfare difference between the linear equilibrium and the first-best outcome.<sup>6</sup> From a welfare perspective the question of which equilibrium is played is therefore of first-order importance.

We can give intuition for why Pareto-dominant equilibria achieve high welfare. The best equilibria are similar to trigger strategies, but because we consider MPE, with strategies not conditioned on past actions, the trigger is defined in terms of the stock. In particular, in the constrained-efficient equilibrium steady state, the players respond differently if the state falls slightly below the steady state level, compared to the state rising slightly above the steady state level. A fall in the carbon stock below the steady state leads to a discontinuous increase in emission rates, so that any spare “capacity” is rapidly eaten up by the players, returning the economy rapidly to the steady state. If the carbon stock increases to above the steady state, this results in a small increase in emission rates, causing the economy to drift to higher carbon concentrations and into an eventual more highly polluted steady state. The threat of this adverse outcome ensures that players prefer to remain in the benign steady state.

Third, we show that there exist particularly bad equilibria involving extreme coordination failures; at high levels of atmospheric carbon, these can yield welfare outcomes even worse than the linear equilibrium. The coordination failure involves discontinuities in not just the actions, that is, the emission rates, but also in the value function. Such a discontinuity is only possible if no player can single-handedly move the economy from the high-carbon, low-value side of

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<sup>5</sup>In a working paper version of this paper, we show that the same property holds if the utility function is isoelastic instead of linear-quadratic, as long as the carbon demand elasticity is sufficiently small. The qualitative properties of the discontinuous equilibria change at the same elasticity threshold as those of the continuous equilibrium (Wirl, 2014).

<sup>6</sup>Schumacher et al. (2022) show that an equilibrium with a value function arbitrarily close to the value envelope, i.e. yielding the maximal value for any initial state achievable with any MPE, can be constructed. However, this requires that the number of discontinuities becomes arbitrarily high.

the discontinuity to the low-carbon, high-value side. Think, for example, of a large number of relatively small symmetric countries: if other countries are emitting heavily, then no single country can stop the rise of atmospheric carbon even if it was to stop emitting completely. In equilibrium it is then a best response to also emit heavily. All players, following the same logic, emit heavily only as an optimal response to a collective coordination failure.<sup>7</sup>

While the model is simple, it has some policy implications of import. First, and most critically, our results suggest that more attention should be paid to the possibility of multiple equilibria in the problem of noncooperative dynamic climate mitigation. Focusing on continuous strategies, as the previous literature has done, unavoidably frames the problem as a dynamic prisoners' dilemma, highlighting the noncooperative aspects of international climate policy which make the underlying environmental problem so difficult to solve. The presence of multiple equilibria suggests that climate negotiations may be able to coordinate on better ones. This could be conducive to a more collaborative atmosphere in global climate diplomacy.

Second, our results suggest what good climate agreements might look like. MPE are self-enforcing, and Dockner and Long (1993), among others, argued such equilibria could underpin self-enforcing climate agreements. Our model suggests that effective agreements may involve discontinuous, rather than gradual, decarbonisation paths, together with threats to stabilise at higher levels only, should a target be missed. These could do better than agreements in which emissions are gradually reduced on the way to a long-term carbon stabilisation target, and in which any overshooting of a target is undone by emission reductions. In such "gradual" agreements, emissions are dynamically strategic substitutes, and thus give bad strategic incentives. An agreement which involves countries responding to above-target concentrations by further increasing emissions can give an agreement more bite. Thus, setting discrete carbon concentration goals—aiming at a 2°C increase in global mean temperature, but should this not be met, setting the next target substantially higher, for example at 2.5°C—could make states more interested in sticking to the ambitious goal. To be clear, we are hesitant to make this claim forcefully due to the simplicity of the model, but our results warrant further investigation of strategic incentives embodied in dynamic climate treaties.

Before turning to the model, we discuss the technical complication which has not been addressed by the previous literature, why it has presented problems, and how we solve these. The

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<sup>7</sup>This is of course not a renegotiation-proof outcome. In fact, all of our equilibria feature some initial states from which the equilibrium trajectories are not renegotiation-proof were it possible to pause the evolution of the dynamics and play again. We discuss this further in Section 4.

complication relates to what kinds of policy functions should be allowed. In particular, best-response policy functions can generically be discontinuous in the state; but discontinuous policies can lead to nonexistence of classical solutions to the differential equation governing state evolution. This causes a difficulty: without a state trajectory, payoffs cannot be computed.

The literature has taken two approaches to this. The first is to allow only continuous Markovian strategies, ensuring existence and uniqueness of trajectories. Two problems ensue: first, it leads to non-existence of best responses, in the space of continuous functions, to many strategy profiles of other players, as the “natural” best response would be discontinuous.<sup>8</sup> Second, allowing only continuous strategies may rule out, a priori, equilibria which are technically unproblematic. In our application, this matters for welfare: any discontinuous strategy equilibrium with a continuous value function Pareto-dominates the unique continuous equilibrium.

The second approach (e.g., Sorger, 1998; Dockner et al., 2000; Rowat, 2007) allows discontinuous strategies in principle, but rules out strategy *profiles* leading to technical complications. A restriction to “admissible” strategy profiles implies that a player’s allowable set of strategies depends on the strategies simultaneously chosen by the other players: the model is then no longer a noncooperative game (in the sense of Nash, 1951).<sup>9</sup>

In the present paper, we extend the definition of a differential game to allow for strategies with discontinuities. Under our definition, state trajectories and payoffs can always be computed: they are so-called Filippov solutions to the ordinary differential equation governing the state dynamics (Filippov, 1988). We can thus allow strategy profiles to simply be product sets of the individual strategy spaces, as is standard in non-cooperative game theory. Our first main result states that each player has a unique, Markovian best response to almost every strategy profile of the other players (Theorem 1). These two features jointly solve the above, long-standing technical problem with MPE in differential games for the single state variable case, by guaranteeing that payoffs can always be computed and best responses exist almost always.<sup>10</sup>

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<sup>8</sup>The common focus on symmetric equilibria hides this problem. However, in some games, the continuity requirement implies the non-existence of any globally defined symmetric equilibria (Kossioris et al., 2008).

<sup>9</sup>This problem has recently been pointed out by Bernhard (2024). Klein and Rady (2011) discuss admissibility extensively. In their model, equilibria are sometimes sustained by the technical requirement of admissibility. Schumacher et al. (2022) provide a verification theorem—a sufficient condition—for a symmetric Markovian policy rule to constitute an MPE (essentially the same approach as in the previous literature). However, they do not cleanly specify an individual player’s problem, and their approach does not determine payoffs for “problematic” strategy profiles. Thus their problem cannot truly be said to be a non-cooperative game. Their approach also cannot answer questions about, for instance, the set of equilibria.

<sup>10</sup>We thus solve the critique of Bernhard (2024). A large fraction of the literature on differential games works in a single-state setting. Extension of our results to more dimensions is left for future work.

Our second main result, Theorem 2, gives a readily applicable necessary and sufficient condition for characterising the best response to a profile of opponents’ strategies. This condition supplements the maximum principle with local requirements that have to hold at a discontinuity. These additional requirements restrict which discontinuities are possible at a given state. Imposing a fixed point condition on the best response correspondence, a Nash equilibrium in Markov-perfect strategies can then be constructed. Theorem 3 formulates the resulting necessary and sufficient conditions for symmetric equilibrium profiles.

To obtain our results, we use the theory of viscosity solutions to Hamilton–Jacobi–Bellman equations (Bardi and Capuzzo-Dolcetta, 2008), building on and developing ideas by Barles et al. (2013, 2014).<sup>11</sup> Because our interests are economic and game-theoretic, we primarily focus on policies rather than values. In particular, our methods directly construct the Markovian policy function that is optimal in the face of discontinuous dynamics, where the discontinuities arise from the strategies of the other players; of course the results hold as well for discontinuities resulting from, for instance, regime shifts. Unlike Barles et al. (2013, 2014), we cannot assume controllability of the dynamics near discontinuities without placing undue restrictions on the admissible set of strategies of the other players. Finally, the application of our results to a fixed point of the best response correspondence, to construct conditions for MPE, is entirely novel.

In our definition of a differential game, we need to specify how the dynamics are resolved, and payoffs computed, in situations at which the dynamics involve discontinuities. In particular, we need to define meaningful payoffs for “chattering” or “relaxed” actions, where a player’s actions may switch infinitely often in time. However, while our specification must allow for such behaviour, our results show that such outcomes never occur in equilibrium. Indeed, all the MPE we identify have perfectly standard classical solutions to the state dynamics.

Our results apply to numerous differential games; our example suggests that our methods can yield novel insights into all of these. Our methods also relax the restrictions on parametric model specification needed for tractability, making it straightforward to study models with novel specifications. We discuss other possible extensions in Section 4.<sup>12</sup>

The paper proceeds as follows. Section 1 sets up the basic model. The main results, existence

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<sup>11</sup>We also rely on the theory of nonlinear dynamical systems to show that the best response is Markovian.

<sup>12</sup>Many learning models specify a learning process, sometimes requiring parametric restrictions, in which the dynamics only move in one direction between the times when new information arrives, specifically to avoid the problems which we tackle (Keller and Rady, 2015; Sun, 2024); or the state dynamics are restricted by imposing an admissibility requirement on strategies (Klein and Rady, 2011) or beliefs (Board and Meyer-ter Vehn, 2013; Hauser, 2024). Our methods obviate the need for such requirements, at least for the class of games we consider.

and characterisation of the best response, are given in Section 2. We show how these are used in an application to climate change in Section 3. Readers primarily interested in the transboundary pollution game can read the beginning of Section 1 and then skip to Section 3, referring to previous sections when necessary. Section 4 wraps up. Proofs of the main results are relegated to the Online Appendix.

## 1 The model

We consider MPE in a differential game between  $N$  players. Player  $i$ 's problem is to maximise

$$\int_0^{\infty} \exp(-\rho_i t) u_i(x(t), a_i(t)) dt \quad (1)$$

where  $u_i(x, a_i)$  is flow utility, a function of the current state variable  $x \in \mathcal{X} = [x_{\min}, x_{\max}] \subset \mathbb{R}$  and player  $i$ 's own current action  $a_i \in \mathcal{A}_i = [a_{\min,i}, a_{\max,i}] \subset \mathbb{R}$ . The discount rate  $\rho_i > 0$  is constant. The maximisation is conducted over trajectory–action pairs  $(x(t), a_i(t))$ , with  $x : [0, \infty) \rightarrow \mathcal{X}$  absolutely continuous and  $a_i : [0, \infty) \rightarrow \mathcal{A}_i$  measurable, satisfying

$$\dot{x}(t) = a_i(t) + g_i(x(t)), \quad x(0) = x_0, \quad (2)$$

for almost all  $t \geq 0$ . Here  $g_i(x) = \Phi_{-i}(x) + h(x)$  is the sum of the physical dynamics  $h(x)$  as well as the aggregate action  $\Phi_{-i}(x) \equiv \sum_{j \neq i} \phi_j(x)$  of the other players. The functions  $\phi_j(x)$  are Markovian policy rules: player  $j \neq i$  sets, at all times, its action according to  $a_j(t) = \phi_j(x(t))$ . Player  $i$ 's problem depends only on the aggregate action  $\Phi_{-i}(x)$ .<sup>13</sup>

Pollution build-up models are our main focus: the state  $x(t)$  is a pollution stock, utility  $u_i(x, a_i)$  is decreasing in the stock, that is  $\frac{\partial u_i}{\partial x} \leq 0$ , and actions  $a_i(t) \geq 0$  are emissions. The model is easily adapted to pollution abatement, accumulation of joint capital, and contributions to public goods; these adaptations entail only changes of signs.

Our primary interest is in MPE, which requires us to construct player  $i$ 's optimal policy, its best response to the other players' strategies, as a feedback (or Markovian) policy rule so that  $a_i(t) = \phi_i(x(t))$ .<sup>14</sup> An MPE is a strategy profile  $\phi = (\phi_1, \dots, \phi_N)$  such that each player's strategy is a best response to the other players' strategies, from any initial state. For

<sup>13</sup>In particular, player  $i$ 's best response is identical to any set of opponents' strategies which involve the same aggregate action.

<sup>14</sup>Our results of course also apply to a single-agent optimal control problem under discontinuous dynamics.

the notion of MPE to be well-defined, we have to be able to compute the payoffs for any strategy profile. Furthermore, the strategy space, the class of functions which the  $\phi_i$  are allowed to belong to, should be *strategically complete*: for every player  $i$  and every profile  $\phi_{-i} = (\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_N)$ , the optimal action schedule  $a_i^*(t)$  should be generated by a strategy  $\phi_i(x)$  from the strategy space.<sup>15</sup> That is, we request the strategy space to be sufficiently large to accommodate all best response action schedules. This is a consistency requirement, particular to differential games, connecting the action schedule space to the strategy space.

The literature has not addressed these desiderata: nearly each model we know of fails to satisfy at least one.<sup>16</sup> Our aim is to meet both. To do so, in the remainder of this section we allow strategies that are discontinuous in the state. In Section 2 we show that, after the extension, best replies almost always exist; they can be fully characterised; and consequently symmetric MPE can be characterised in a manner which allows us to construct the set of all equilibria, as demonstrated for an application in Section 3.

## 1.1 Strategies

In the remainder of this section, we consider the problem of a single player, and we therefore drop the indices  $i$  and  $-i$  until further notice. We want to construct a Markovian best response  $\phi(x)$  to an aggregate action  $\Phi(x)$ . The key question is the space of functions to which the strategy  $\phi(x)$  belongs: we discuss why a natural function space must admit discontinuous functions.

The Hamiltonian of player  $i$  is

$$H(x, p) = \max_{a \in A} [u(x, a) + p(a + g(x))] . \quad (3)$$

This maximisation problem is the static problem which player  $i$  must solve at each point in time; jumping slightly ahead, when player  $i$ 's value function  $V(x)$  is differentiable, the maximisation is conducted under  $p = V'(x)$ .

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<sup>15</sup>We focus on autonomous feedback strategies: strategies conditioned only on the state variable, not calendar date. We do not discuss open-loop strategies, that is, strategies which give actions as functions of time. It is well-known that open-loop strategies are not robust to unforeseen one-off shocks to the state: the resulting Nash equilibria lack strong time consistency, a concept similar to subgame perfection (Dockner et al., 2000).

<sup>16</sup>The locally-defined MPE of Dockner and Long (1993) do not allow computation of payoffs from certain initial states. Discontinuous strategies with an admissibility restriction on strategy profiles (Dockner et al., 2000) implies available strategies depend on other player's chosen strategies. And a requirement of continuity means the best response often does not exist in the space of continuous policies.

The first-order condition for an interior maximum to player  $i$ 's static problem is

$$\frac{\partial u}{\partial a}(x, a) + p = 0.$$

To consider only situations with unique maximisers, we assume  $\frac{\partial^2 u}{\partial a^2} < 0$  everywhere.<sup>17</sup> We can then solve for the optimal action in terms of state and costate as

$$a = a^*(x, p). \quad (4)$$

However, the dependence of the optimal policy  $\phi(x) = a^*(x, V'(x))$  on the state is neither smooth nor necessarily continuous. First, because the action space is compact, we have corner solutions to the maximisation in (3), with the action passing from interior values of  $\mathcal{A}$  to boundary, or corner, values when the costate passes a *switching point* (we italicise terms which we use later in the paper). This implies that the allowable strategy  $\phi(x)$  should admit kinks.

Second, it is well-known that if the dynamics are continuous and smooth but non-convex, there may exist a state  $\hat{x}$  such that the maximisation problem in Equation (3) has two distinct solutions, for values of the costate  $p$  equal to the left and right limits of  $V'(x)$  at  $\hat{x}$ . We call  $\hat{x}$  a Skiba, or *indifference*, state (Skiba, 1978; Wagener, 2003). At such a point, there are two optimal solutions—one moving towards lower values of the state, the other towards higher values, both yielding the same value at  $\hat{x}$ —so that the optimal Markovian policy  $\phi(x)$  is discontinuous. Such situations happen for a broad range of the opponents' aggregate action  $\Phi$ , even if that action is smooth. In other words, the optimal policy response to smooth and continuous Markovian strategies cannot, in general, be described with a continuous Markovian strategy.

The extant literature either assumes *a priori*, often implicitly, that strategies must be continuous, in which case best responses to other players' strategies do not exist if player  $i$ 's optimal policy features a Skiba point—and this occurs generically; or it imposes an admissibility or “feasibility” restriction which effectively conditions player  $i$ 's strategy set on strategies simultaneously chosen by the other players. Our framework does not need these highly restrictive assumptions.

We allow a strategy  $\phi(x)$  to have a finite number of discontinuities, or *jump points*.<sup>18</sup> Further, we require strategies to be piecewise *real analytic*, assuming that they can be expanded at each

<sup>17</sup>That is, we do not consider discontinuities due to flow utility not being strictly concave in the action.

<sup>18</sup>The mathematics for infinitely many jumps would be substantially more complicated, and we see policy functions with infinitely many jump points as having less practical relevance (but see Ivanov, 1990).

non-jump point in a power series with a positive radius of convergence. This implies strategies are infinitely differentiable outside of a jump point.<sup>19</sup> Finally, at jump points we require that strategies and their first order derivatives have finite one-sided limits when approaching the jumps. Formally, we introduce the strategy space  $\mathcal{S}$  as follows.

**Definition 1.** *The strategy space  $\mathcal{S}$  consists of functions  $\phi : \mathcal{X} \rightarrow \mathcal{A}$  that are real analytic, except at a finite number of points, and have one-sided derivatives everywhere.*

## 1.2 Dynamics and payoffs

Allowing discontinuous strategies requires a reinterpretation of the evolution equation (2). Supposing player  $i$  chooses a Markovian policy, the equation becomes  $\dot{x} = \phi(x) + g(x)$ . As long as the right hand side of this equation is a Lipschitz-continuous function of the state, that is, continuous and almost everywhere differentiable with a uniformly bounded derivative, the Picard-Lindelöf theorem ensures the existence and uniqueness of a state trajectory  $x(t)$  solving the equation. This is why the existing literature has often required the strategies  $\phi(x)$  to be Lipschitz-continuous: it guarantees payoffs can be computed. With a discontinuous right hand side, neither existence nor uniqueness of classical solutions can be guaranteed, and payoffs cannot be determined for all strategy profiles.

The main problem can be illustrated by the simple example

$$\dot{x}(t) = f(x(t)), \quad f(x) = \begin{cases} f_-, & x < 0, \\ f_+, & x \geq 0, \end{cases} \quad (5)$$

where the right hand side  $f$  has a jump discontinuity at  $x = 0$ . Consider the special situation that  $f_- > 0$  and  $f_+ < 0$ . Solutions to this equation increase at a constant rate as long as  $x(t)$  is negative, and decrease at a constant rate if  $x(t)$  is positive. Using the intuitive terminology of Barles et al. (2013), we call this a *push-push* situation, as the state is pushed towards  $x = 0$  by the dynamics. It is natural to expect that if the state arrives in  $x = 0$  at some point in time, it will continue to remain at the origin for all future times. But  $x(t) = 0$  is not a solution to the differential equation. In our context  $f_+$  and  $f_-$  would represent the limits of the dynamics arising

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<sup>19</sup>If the state–costate system of the optimisation has a steady state, and the function  $g$  is  $k$  times differentiable, then the best response policy function can only be guaranteed to be  $k - 1$  times differentiable. To map the best response back to the same function space, infinite differentiability is thus required. Real analyticity is a technical assumption and not particularly restrictive.

from all players, including player  $i$ , using a discontinuous Markovian policy. The solution to this technical problem matters: player  $i$ 's flow payoff at the steady state  $x(t) = 0$  depends on what is going on, in terms of player  $i$ 's action, at that point.

For a heuristic explanation of how we resolve this issue, set  $x(0) = 0$ , and suppose that a decision maker sets  $\dot{x}(t)$  equal to either  $f_-$  or  $f_+$ . In particular, assume that if  $\dot{x}(t_0) = f_-$ , then this rate is maintained until  $x(t)$  attains the level  $x = \delta$ , at which point  $\dot{x}(t)$  is switched to  $f_+$ ; here  $\delta > 0$  is some tolerance parameter. Likewise, if  $\dot{x}(t_0) = f_+$ , this rate is kept until  $x(t)$  crosses the level  $x = -\delta$  and then it is switched to  $f_-$ .

The state  $x(t)$  is captured in an interval, bouncing between the boundary points  $x = -\delta$  and  $x = \delta$ . During a fraction  $\mu_-$  of the time, the state increases at a rate  $f_-$ ; during a complementary fraction  $\mu_+$ , it decreases at a rate  $f_+$ . These time fractions are inversely proportional to the absolute values of the rates, which implies

$$\mu_- = \frac{\frac{1}{|f_-|}}{\frac{1}{|f_-|} + \frac{1}{|f_+|}} = \frac{|f_+|}{|f_+| + |f_-|}, \quad \mu_+ = 1 - \mu_-. \quad (6)$$

For all tolerances  $\delta > 0$ , player  $i$ 's flow payoff is a weighted sum of the left action payoff, obtained when  $\dot{x}(t) > 0$ , and right action payoff, obtained when  $\dot{x}(t) < 0$ , with the weights given by the fraction of time the trajectory spends using these actions, respectively,  $\mu_-$  and  $\mu_+$ .

Taking the limit  $\delta \rightarrow 0$ , the state trajectory converges uniformly to the natural solution  $x(t) = 0$  for all  $t$ . This solution is formalised mathematically by replacing the differential equation (5) by a differential inclusion  $\dot{x}(t) \in F(x(t))$ . For this we introduce a set-valued function  $F$  such that  $F(x) = \{f(x)\}$  if  $x \neq 0$ , and we extend it to  $x = 0$  such that  $F(0)$  is convex and the graph of  $F$  is closed, that is,  $F(0) = [f_-, f_+]$ . The differential equation is then replaced by the differential inclusion for almost all  $t$ . This specification allows for the solution  $x(t) = 0$ . These generalised solutions are known as ‘‘chattering’’, ‘‘sliding’’ or Filippov solutions.<sup>20</sup>

If  $f_+ > 0$  and  $f_- < 0$  in (5), we have a *pull-pull* situation. The differential inclusion  $\dot{x}(t) \in F(x(t))$ , with initial condition  $x(0) = 0$ , has now multiple solutions:  $x(t) = f_+t$ ,  $x(t) = f_-t$ , and  $x(t) = 0$  for  $0 < t < t_0$  and  $x(t) = f_{\pm}(t - t_0)$  for  $t \geq t_0$  are all different solution trajectories. The first two are *regular* solutions, while the others are *singular*: the latter solutions are not stable under small variations of the initial state. Multiple regular solutions typically occur as

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<sup>20</sup>Because we work in one dimension, these solutions are sliding in a degenerate sense.

optimal trajectories originating at an indifference point, and so are perfectly standard.

In general, replacing differential equations with discontinuous right hand sides by differential inclusions that convexify the dynamics at discontinuities ensures the existence of solutions (Filippov, 1988). Uniqueness is not guaranteed, but this is less problematic in our context.

Extending differential inclusions to controlled systems is straightforward, as the controlled evolution equation (2) itself can be naturally reformulated as a differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0. \quad (7)$$

First, for all points where  $g(x)$  is continuous, we set  $F(x) = \mathcal{A} + g(x) = \{a + g(x) : a \in \mathcal{A}\}$ . That is, the set  $F(x)$  contains the rates of state increase available to player  $i$  at point  $x$ . We then extend  $F$  to any points of discontinuity by requiring that the graph of  $F$  is closed, that  $F(x)$  is convex for all  $x$ , and that  $F$  is upper semi-continuous. This uniquely determines  $F$ .

To be specific, let  $x$  be a point of discontinuity of  $g$ , and let  $g_-(x)$  and  $g_+(x)$  be the left-hand and right-hand limits of  $g(y)$  as  $y \uparrow x$  or  $y \downarrow x$  respectively. Then  $F(x)$  is the convex hull of the union of the sets  $F_-(x) = \mathcal{A} + g_-(x)$  and  $F_+(x) = \mathcal{A} + g_+(x)$ , and the elements  $\dot{x} \in F(x)$  can be represented, with  $a \in \mathcal{A}$  and  $0 \leq \mu \leq 1$ , as convex combinations <sup>21</sup>

$$\dot{x} = \mu(a + g_-(x)) + (1 - \mu)(a + g_+(x)) = a + \mu g_-(x) + (1 - \mu)g_+(x). \quad (8)$$

An (*admissible*) trajectory of the controlled system is an absolutely continuous function of the state  $x(t)$  for which (7) holds for almost all  $t \geq 0$ .

In the differential inclusion approach, trajectories are the primary objects. We need to determine the associated actions in order to evaluate the objective. By the Filippov selection theorem (Vinter, 2000, Theorem 2.3.13), if  $g$  has discontinuities  $\bar{x}_1, \dots, \bar{x}_J$ , there are measurable functions  $a(t), \mu_1(t), \dots, \mu_J(t)$ , such that for almost all  $t$

$$\dot{x}(t) = a(t) + g(x(t)), \quad (9)$$

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<sup>21</sup>This is different from Barles et al. (2013, 2014), where the actions  $a_-$  and  $a_+$  on both sides of a discontinuity take values in different spaces  $A_-$  and  $A_+$ .

if  $g$  is continuous at  $x(t)$ , and

$$\dot{x}(t) = a(t) + \mu_j(t)g_-(x(t)) + (1 - \mu_j(t))g_+(x(t)) \quad (10)$$

if  $x(t) = \bar{x}_j$ . The pair  $(x(t), a(t))$  is an (*admissible*) *trajectory–action pair* for the differential inclusion (7) if equations (9) and (10) hold for appropriately chosen weights.

To avoid technical complications, we consider differential games whose state space  $\mathcal{X}$  is a compact interval  $[x_{\min}, x_{\max}]$ ; we therefore have to specify what happens if the state leaves the state space. More precisely, we take it that the sum  $g(x) = \Phi(x) + h(x)$  of aggregate action and system response is only defined on the interior  $\overset{\circ}{\mathcal{X}}$  of  $\mathcal{X}$ . We extend the function  $g$  to all real numbers by setting  $g(x) = g_+(x_{\min})$  if  $x \leq x_{\min}$  and  $g(x) = g_-(x_{\max})$  if  $x \geq x_{\max}$ . This implies a corresponding extension of the set-valued function  $F$ . We redefine a trajectory to be admissible if  $x_0 \in \mathcal{X}$ , and if the differential inclusion (7) is satisfied almost everywhere. For an admissible trajectory, the *exit time* from  $\mathcal{X}$  is<sup>22</sup>

$$\Theta = \inf \{t \geq 0 : x(t) \notin \mathcal{X}\}.$$

### 1.3 Values

Having obtained the trajectories and the actions, we can evaluate the objective. At the exit time, the state leaves the state space, the game stops, and the player receives a boundary payoff  $\beta(x)$ , which is locally constant. That is, instead of (1), we consider the objective

$$U(x, a) = \int_0^\Theta \exp(-\rho t) u(x(t), a(t)) dt + \exp(-\rho \Theta) \beta(x(\Theta)).$$

The payoff of the game for the player is

$$V(x_0) = \sup U(x, a),$$

where the supremum is taken over all admissible trajectory–action pairs  $(x(t), a(t))$  with initial state  $x(0) = x_0$ . The supremum is actually attained, and can thus be replaced by a maximum,

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<sup>22</sup>Thus we properly consider a free end time problem, rather than an infinite horizon problem (see the critique by Bernhard, 2024). In many problems an appropriate specification of the state space eliminates any meaningful difference between the two; see our application in Section 3.

achieved by an *optimal trajectory-action pair* (see the Online Appendix). We note that  $V(x) = \beta(x_{\min})$  if  $x < x_{\min}$  and  $V(x) = \beta(x_{\max})$  if  $x > x_{\max}$ .

## 1.4 Feedback best response strategies

From this point onwards, we reinstate the indices  $i$  and  $-i$ . We want to understand whether the optimal trajectory-action pair of player  $i$ —a time trajectory—can be generated by a Markovian policy  $\phi_i(x) \in \mathcal{S}$ , for any initial state  $x_0$ , so that we can say that  $\phi_i$  is a *Markovian best response* to  $\phi_{-i} \in \mathcal{S}^{N-1}$ . Existence of Markovian best responses consistent with all optimal trajectory-action pairs is what we term strategic completeness. We first need to define exactly what a Markovian best response is; the difficulty clearly lies at the points at which  $\phi_{-i}$  is discontinuous. We consider a strategy profile  $\phi(x) \in \mathcal{S}^N$ . Because of the discontinuities, we again use a differential inclusion to describe the dynamics. We have to define how payoffs to a Markovian strategy are computed at points of discontinuity.

We set  $f^\phi(x) = \phi_i(x) + \Phi_{-i}(x) + h(x)$  and introduce a set-valued map by setting  $F^\phi(x) = \{f^\phi(x)\}$  for all points  $x$  where  $f^\phi(x)$  is continuous. As before, we extend  $F^\phi(x)$  to any point of discontinuity by requiring that the graph of  $F^\phi$  is closed, that  $F^\phi(x)$  is convex for all  $x$ , and that  $F^\phi$  is upper semi-continuous. For a point of discontinuity  $x$ , this implies that  $F^\phi(x)$  is the closed interval  $[f_-^\phi(x), f_+^\phi(x)]$  connecting  $f_-^\phi(x)$  to  $f_+^\phi(x)$ . Moreover, the set-valued map is extended past the boundaries of  $\mathcal{X}$  by setting it equal to its value at the nearest boundary point.

If  $x(t)$  is such that for almost all  $t$  we have

$$\dot{x}(t) \in F^\phi(x(t)), \quad x(0) = x_0, \quad (11)$$

then  $x(t)$  is an admissible Markovian trajectory when the player chooses the feedback strategy  $\phi_i(x)$  in response to  $\phi_{-i}(x)$ .

To compute payoffs, we need to specify utility. At points where  $f^\phi(x)$  is continuous, utility is

$$u_i^\phi(x) = u_i(x, \phi_i(x)).$$

If  $f^\phi(x)$  is not continuous at  $x$ , then  $\dot{x} \in F^\phi(x)$  can be written uniquely as a convex combination

$$\dot{x} = \mu(\phi_{i,-}(x) + g_{i,-}(x)) + (1 - \mu)(\phi_{i,+}(x) + g_{i,+}(x)) \quad (12)$$

with  $0 \leq \mu \leq 1$  again being the time fraction the solution spends on the left hand side of the discontinuity. Player  $i$ 's utility is then

$$u_i^\phi(x) = \mu u_i(x, \phi_{i,-}(x)) + (1 - \mu) u_i(x, \phi_{i,+}(x)). \quad (13)$$

With this choice, player  $i$ 's payoff is

$$V_i^\phi(x_0) = \sup \left[ \int_0^\Theta \exp(-\rho_i t) u_i^\phi(x(t)) dt + \exp(-\rho_i \Theta) \beta_i(x(\Theta)) \right], \quad (14)$$

where the supremum is taken over all solutions of (11), including singular solutions.

We emphasise that, under the differential inclusion approach, state and action trajectories, and thus values, can be computed for all strategy profiles  $\phi$ , so that individual strategies can be freely chosen from  $\mathcal{S}_i$ , independently of the other players' strategies, in line with the notion of noncooperative games (Nash, 1951; Bernhard, 2024).

If, for a strategy profile  $\phi$ , player  $i$ 's payoff coincides with the payoff achieved by the optimal trajectory-action pair in response to  $\phi_{-i}$ , i.e.  $V_i^\phi(x_0) = V_i(x_0)$  for all  $x_0$ , we say that  $\phi_i$  is a *Markovian best response* to  $\phi_{-i}$ .

The key difference between equations (10) and (12) is that the latter requires a control to be a convex combination of the left and right limiting Markovian control. Then player  $i$ 's payoff is invariant to pointwise deviations in  $\phi_i$ , and a Markovian strategy  $\phi_i(x)$  encodes all the information required to compute payoffs, as the weight  $\mu$  can be computed from the limits  $\phi_{i,-}(\bar{x})$ ,  $\phi_{i,+}(\bar{x})$  for any  $\bar{x}$ . We need to show that even under these restrictions Markovian best responses always exists; this is the first of our main results, to which we turn now.<sup>23</sup>

## 2 Results

This section states the three main results of our article. The first asserts the well-definedness of the best-response correspondence for almost all strategy profiles. We first make precise our

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<sup>23</sup>To see that this is not obvious and instead depends on further restrictions of the model, consider a symmetric game with  $N = 3$ , discount rate  $\rho = 1$ , utility  $u(x, a_i) = a_i^2 - x^2$ , dynamics  $\dot{x} = \sum_{i=1}^N a_i$ , state space  $\mathcal{X} = \mathbb{R}$  and action space  $\mathcal{A} = [-1/2, 1]$ . This game has a symmetric equilibrium  $\phi$  with steady state  $\bar{x} = 2/3$ , such that  $\phi_i(x) = 1$  for  $x < \bar{x}$  and  $\phi_i(x) = -1/2$  for  $x > \bar{x}$ . However, the optimal trajectory-action pairs imply players setting  $a_i(t) = 1$ ,  $\mu_i(t) = 1$  for all  $t$  such that  $x(t) = \bar{x}$ , which gives a higher payoff than any Markovian policy. This example violates our assumption of  $u(x, a_i)$  being monotonic in  $x$ , as well as Assumption 2 below.

meaning of “almost all” by defining the concept of a “shy” set. Shy sets are analogues of measure zero sets in infinite-dimensional spaces, such as function spaces.

Let  $S$  be a subset of a complete metric linear space  $\mathcal{V}$ . The set  $S$  is *nowhere dense*, that is, “topologically small”, if its complement contains an open and dense set; it is *shy*, or “measure-theoretically small”, if there exists a Borel set  $B$  containing  $S$  and a measure on  $\mathcal{V}$  that takes a finite value on some compact set, such that  $\text{meas}(B - v) = 0$  for all  $v \in \mathcal{V}$ : here  $B - v = \{b - v : b \in B\}$ . Note that if  $\mathcal{V}$  is finite dimensional, a set is shy if and only if its Lebesgue measure is 0 (Hunt et al., 1992).

A *collection of jump points* is a finite discrete set  $\mathcal{J} = \{\bar{x}_1, \dots, \bar{x}_J\}$  with  $\bar{x}_1 = x_{\min}$  and  $\bar{x}_J = x_{\max}$ . We denote by  $\mathcal{F}_{\mathcal{J}}$  the set of functions that are real analytic on  $\mathcal{X} \setminus \mathcal{J}$ , such that their derivatives have finite one-sided limits everywhere. The strategy space  $\mathcal{S}_{\mathcal{J},i}$  of player  $i$  consists of those functions  $\phi_i \in \mathcal{F}_{\mathcal{J}}$  such that  $\phi_i(x) \in \mathcal{A}_i$  for all  $x \in \mathcal{X}$ . The strategy space  $\mathcal{S}_{\mathcal{J},-i} \subset \mathcal{F}_{\mathcal{J}}^{N-1}$  of all players except player  $i$  is defined similarly. The strategy spaces  $\mathcal{S}_i$  and  $\mathcal{S}_{-i}$  are the union of, respectively, the sets  $\mathcal{S}_{\mathcal{J},i}$  and  $\mathcal{S}_{\mathcal{J},-i}$  over all collections  $\mathcal{J}$ . The spaces  $\mathcal{F}_{\mathcal{J},i}$  and  $\mathcal{F}_{\mathcal{J},-i}$  can be given the structure of a complete metric linear space, see e.g. Krantz and Parks (2002, Section 2.6).

We make three assumptions; the first two have already been motivated in Section 1.

**Assumption 1.** *The state evolution is*

$$\dot{x}(t) = \sum_{i=1}^n a_i(t) + h(x(t)), \quad x(0) = x_0,$$

with  $h(x)$  a real analytic function.

**Assumption 2.** *Utilities  $u_i(x, a_i)$  are real analytic and satisfy  $\frac{\partial u_i}{\partial x} \leq 0$  and  $\frac{\partial^2 u_i}{\partial a_i^2} < 0$ .*

We add the following assumption on the boundary payoffs.

**Assumption 3.** *The boundary payoffs satisfy the inequalities  $\beta_i(x_{\min}) \geq u_i(x_{\min}, a_i)/\rho$  and  $\beta_i(x_{\max}) \leq u_i(x_{\max}, a_i)/\rho$  for all  $a_i \in \mathcal{A}_i$ .*

The third assumption, together with the fact that the  $u_i(x, a_i)$  are decreasing in  $x$ , is used to show that the value functions  $V_i(x)$  are decreasing in  $x$ : the state is a public bad. The boundary payoffs only matter in situations where a player can take the state out of the state space, and so end the game; see Theorem 2 below. Often models can be set up in such a way that exit is not possible, as in the application in Section 3.

**Theorem 1.** *Given a collection  $\mathcal{J}$  of jump points, for every  $i$  there is a shy and nowhere dense set  $\mathcal{E}_{\mathcal{J},-i} \subset \mathcal{S}_{\mathcal{J},-i}$  such that the Markovian best response mapping  $\mathcal{B}_i : \mathcal{S}_{\mathcal{J},-i} \setminus \mathcal{E}_{\mathcal{J},-i} \rightarrow \mathcal{S}_i$  is well-defined: for every strategy profile  $\phi_{-i} \in \mathcal{S}_{\mathcal{J},-i} \setminus \mathcal{E}_{\mathcal{J},-i}$ , player  $i$  has exactly one Markovian best response  $\phi_i \in \mathcal{S}_i$ .*

The proof, detailed in Online Appendices B–E, is structured as follows. First, direct arguments are used to show a number of properties of the value function  $V_i$  of player  $i$ : the value is decreasing and continuous everywhere except for a finite number of points. Then  $V_i$  is shown to be the unique viscosity solution to the Hamilton–Jacobi–Bellman equation of player  $i$  within the class of functions that possess these properties. To reach this result, we extend results of Barles et al. (2013, 2014) on optimal control problems with discontinuous dynamics to the situation that the dynamics are not controllable at a jump point.

Having characterised the value function, we build a regularity theory for  $V_i$ . This is a new contribution. We begin by showing that  $V_i$  is differentiable on a dense set. The points of differentiability yield initial points of trajectories of the state–costate equations. As the state–costate system is piecewise real analytic, this implies that  $V_i$  is real analytic on the projection of these trajectories to the state space. The trajectories end on steady states of the state–costate equations, of which there are potentially infinitely many. The case that the steady states accumulate on a jump point of  $\phi_{-i}$  is shown to be shy. Steady states which have an accumulation point which is not a jump point of  $\phi_{-i}$  have to be on the centre manifold of the accumulation point. A result of Aulbach (1986) implies that all points on this centre manifold are necessarily steady states, and that the centre manifold is real analytic. The value function is also real analytic on the projection of the centre manifolds on the state space. Apart from the shy cases, we end up with finitely many intervals of steady states and finitely many non-constant state trajectories, along which the value function is real analytic; consequently the best response belongs to  $\mathcal{S}_i$ . Uniqueness of the best response is clear from the construction, and the result is proved.

The theorem shows that our specifications of differential game and Markovian strategy space are fit for purpose: they ensure almost full strategic completeness, in the sense that each player will have a unique best response, in the same strategy space, to any profile of the other players’ strategies in the complement of the exceptional shy set  $\mathcal{E}_{\mathcal{J},-i}$ , and that the best response generates optimal action schedules for all initial states.

Explicit sufficient conditions can be formulated for a strategy profile  $\phi_{-i}(x)$  to be not in the shy

set  $\mathcal{E}_{\mathcal{J},-i}$ : these are of evident importance for applying Theorem 1. One such condition is given in Online Appendix E in Proposition E.10.

As the best response is piecewise real analytic, it can be characterised by classical conditions in the regions of analyticity, and by compatibility conditions at the interfaces. The second main result of this article, Theorem 2, formulates these. Its proof is given in Online Appendix F.

To state the theorem, we introduce the set

$$\mathcal{A}_{i,0} = \{a_i \in \mathcal{A}_i : a_i + \mu g_{i,-}(x) + (1 - \mu)g_{i,+}(x) = 0 \text{ for some } 0 \leq \mu \leq 1\} \quad (15)$$

of *stabilising actions*: if for  $x = x(t)$  we have  $a_i = \phi_{i,-}(x) = \phi_{i,+}(x) \in \mathcal{A}_{i,0}$ , then  $0 \in F^\phi(x)$ . At a boundary of an interval  $(x_1, x_2)$ , where  $x_1 < x_2$ , we introduce the outward pointing normal  $n(x)$  by setting  $n(x_1) = -1$  and  $n(x_2) = 1$ . The set-valued function  $F(x)$  *points into*  $(x_1, x_2)$  at a boundary point  $x \in \{x_1, x_2\}$ , if  $n(x)\dot{x} \leq 0$  for all  $\dot{x} \in F(x)$ . If  $x_1 = x_2$ , the condition is void.

**Theorem 2.** *Let  $\mathcal{J}$  be a collection of jump points, and assume  $\phi_{-i} \in \mathcal{D}_{\mathcal{J},-i} \setminus \mathcal{E}_{\mathcal{J},-i}$ .*

*Then  $\phi_i = \mathcal{B}_i(\phi_{-i})$  if and only if the following hold.*

(i) *Maximum principle: If  $x \in \mathcal{X} \setminus \mathcal{J}$  and  $V_i^\phi$  is differentiable at  $x$ , then*

$$\phi_i(x) = a_i^*(x, (V_i^\phi)'(x)). \quad (16)$$

(ii) *Monotonicity:  $V_i^\phi$  is decreasing.*

(iii) *Value discontinuities: If  $V_i^\phi$  is not continuous at  $x$ , then  $x \in \mathcal{J}$ ,  $f_-^\phi(x)$  points into  $(x_{\min}, x)$  and  $F_+(x)$  into  $(x, x_{\max})$ .*

(iv) *Value at interfaces: If  $x \in \mathcal{J}$ , then  $V_i^\phi(x) \geq \max_{a_i \in \mathcal{A}_{i,0}} u_i(x, a_i) / \rho_i$ .*

(v) *Regularity at push–push steady states: If  $x \in \mathcal{J}$  is such that  $f_-^\phi(x) > 0 > f_+^\phi(x)$ , then  $V_i^\phi(x)$  is differentiable at  $x$ .*

The conditions can be interpreted. The Hamilton–Jacobi–Bellman equation of player  $i$ 's value function reads as

$$\rho_i V_i(x) = u_i(x, a_i^*(x, V_i'(x))) + V_i'(x)(a_i^*(x, V_i'(x)) + g_i(x)), \quad (17)$$

while the value  $V_i^\phi$ , obtained when the profile  $\phi$  is played, satisfies

$$\rho V_i^\phi(x) = u_i^\phi(x) + (V_i^\phi)'(x) (\phi_i(x) + g_i(x)) \quad (18)$$

at points where  $\phi(x)$  is differentiable. If Condition (i) holds, then (18) implies that the value function  $V_i^\phi$  satisfies the Hamilton–Jacobi–Bellman equation (17) at all points where  $V_i^\phi$  is differentiable, that is, almost everywhere. This is the classical condition. By itself, it is insufficient to conclude that  $V_i^\phi = V_i$  and that  $\phi$  is the best response. In order to draw this conclusion, we have to show, first, that both  $V_i$  and  $V_i^\phi$  are viscosity solutions of (17), and, second, that viscosity solutions of (17) are unique. The function of Conditions (ii)–(v) is to ensure that  $V_i^\phi$  is a viscosity solution of (17).

Condition (ii) follows from the fact that the stock is a public bad, and says that there are no strategic incentives so perverse as to make the stock locally a good for player  $i$ .

The remaining conditions place restrictions on the best response at discontinuities of the other players' strategies. Condition (iii) says that a discontinuity in value is only possible at points where at least one of the other players' strategies is discontinuous, and which are such that player  $i$  is unable to control the dynamics back to the region of low stock if they ever end up on the high side of the discontinuity. Value discontinuities can occur at the boundary if it is impossible to exit the state interval. Condition (ii) implies that the value can only have downward (not upward) discontinuities.

Condition (iv) ensures that the value at an interface point is at least the value that can be obtained by stabilising the dynamics at that point.

Finally, Condition (v) follows from the fact that value is continuous at push-push states. If a player's best response is to be pushed strictly towards such a state, they end up at the same point whether approaching from the left or the right, and close to that point the continuity of the payoffs implies that the marginal value of the stock does not depend significantly on the direction of approach.

Our third result focuses on symmetric games, where all players have the same utility and the same discount rate, and on symmetric equilibria of these games. We leave the extension to non-symmetric equilibria to future work. Because of the symmetry, we drop the index  $i$ . The resulting ambiguity about the meaning of  $\phi$  is resolved as follows: it will stand for the strategy of a generic player in ordinary expressions, and for the equilibrium profile when used as a

superscript, as in  $f^\phi$ .

The result gives a differential equation that has to be satisfied by equilibrium strategies in the interior  $\mathring{A}$  of the action space  $\mathcal{A}$ . We call  $\phi(x)$  *interior* if  $\phi(x) \in \mathring{A}$ . We introduce the *game Hamiltonian* in state–action variables

$$G(x, a) = u(x, a) - \frac{\partial u}{\partial a}(x, a)(Na + h(x)).$$

If  $\phi(x)$  is a Nash equilibrium strategy, then  $G(x, \phi(x))$  gives the value of  $H(x, (V^\phi)'(x))$ , and hence of  $\rho V^\phi(x)$ , in the situation that  $\Phi(x) = (N - 1)\phi(x)$  and  $\phi(x)$  is interior.

**Theorem 3.** *Let  $\mathcal{J}$  be a collection of jump points, and assume  $\phi \in \mathcal{S}_{\mathcal{J}} \setminus \mathcal{E}_{\mathcal{J}}$ . Then  $\phi$  is a symmetric Nash equilibrium if and only if conditions (ii)–(iv) of Theorem 2 hold, together with:*

(i) *Maximum principle: If  $\phi(x)$  is interior, then*

$$\frac{d}{dx}G(x, \phi(x)) = -\rho \frac{\partial u}{\partial a}(x, \phi(x)). \quad (19)$$

*Moreover for non-interior  $\phi(x)$  we have that if  $\phi(x) = a_{\min}$ , then  $(V^\phi)'(x) \leq -\frac{\partial u}{\partial a}(x, a_{\min})$ , and if  $\phi(x) = a_{\max}$ , then  $(V^\phi)'(x) \geq -\frac{\partial u}{\partial a}(x, a_{\max})$ .*

(v) *No push–push steady states: There is no point  $x \in \mathcal{X}$  such that  $f_-^\phi(x) > 0 > f_+^\phi(x)$ .*

The differential equation (19) specifies a family of candidate Markov policies, to which the interior pieces of any piecewise-defined Markovian strategy must belong. The equation only depends on the fundamentals of the problem, and it can be solved without needing to know anything about the equilibrium.<sup>24</sup> This greatly simplifies the construction of any equilibrium.<sup>25</sup> Conditions (iii)–(v) then impose restrictions on how local pieces are connected together.

Note that even though our specification allows for situations in which utilities are computed as a convex combination of utilities of two different actions, when actions switch “infinitely often”, Theorem 3 states that this outcome never arises in equilibrium. All symmetric MPE (indeed, all MPE) have well-defined payoffs also under classical solutions to the state dynamics.

<sup>24</sup>A similar equation will not hold for non-interior points: knowledge of the value  $V^\phi(x)$  is needed to determine the behaviour at interface points, and this depends on the entire equilibrium.

<sup>25</sup>Equation (19) is not new *per se*; similar equations have been used in applied work since at least Tsutsui and Mino (1990), and generalisations have been presented in various contexts (see e.g. Rincón-Zapatero et al., 1998). Note that in a discrete-time framework the Bellman equation cannot be solved without knowledge of the full value function, which depends on the equilibrium. In continuous time, the Hamilton–Jacobi–Bellman equation can be used locally to construct continuous candidate solutions. This benefit has been highlighted also in the recent literature on continuous-time macroeconomics (e.g. Achdou et al., 2014; Brunnermeier and Sannikov, 2016).

We emphasise that Theorems 1 and 2 are of a new kind. All the conditions of Theorems 2 and 3 are moreover local, in terms of the state. They are used to construct the family of candidate Markov policies, using Equation (19), and to deduce global restrictions that rule out candidate strategies. We turn to an application to make this more concrete for the symmetric case.

### 3 Application

Our results can be used to find all symmetric Markov-perfect Nash equilibria, with finitely many jump points, of a differential game. We consider the canonical transboundary stock pollution game of van der Ploeg and de Zeeuw (1992). This model is characterised by a discount rate  $\rho$  and a quadratic flow utility

$$u(x, a) = \alpha a - \beta a^2/2 - \gamma x^2/2,$$

which are the same for all players. Here,  $a$  is the emission rate of a stock pollutant, and  $x$  is the pollution stock. Emissions generate benefits, the stock damages. The natural decay rate is proportional to the stock,  $h(x) = -\delta x$ , so that the dynamics of the pollution stock are

$$\dot{x}(t) = \sum_{j=1}^N a_j(t) - \delta x(t).$$

This linear–quadratic specification ensures the existence of a piecewise linear MPE that is defined on the whole state space.<sup>26</sup> Dockner and Long (1993) showed that the game also admits locally defined nonlinear equilibria of the type discussed, in a different context, by Tsutsui and Mino (1990).<sup>27</sup> However, as these locally-defined equilibria are not supported on the entire state space, the outcome is not defined in regions of the state space which could be reached with alternative strategies. It is then not clear in what sense the purported equilibrium strategies can be said to be best responses (Rowat, 2007; Bernhard, 2024).

In illustrations, we use a stylised calibration in terms of atmospheric CO<sub>2</sub> pollution to show that this simple model produces quantitatively important results. We assume the players to be symmetric “major powers”. Table 1 lists the calibration assumptions.

<sup>26</sup>We often use “equilibrium strategies” and “MPE” synonymously.

<sup>27</sup>Rubio and Casino (2002) show a further class of equilibria. Our methods subsume many of the “local MPE” in the literature, showing how they are extended to globally defined discontinuous strategies.

Parameter	Quantity	Value	Unit
$N$	Number of players	5	
$\rho$	Discount rate	0.015	1/y
$\delta$	Natural decay rate	0.001	1/y
$x_0$	Initial carbon stock	0.5	TtC
	Optimal steady state carbon stock	1.0	TtC
$\alpha/\beta$	Business-as-usual emission rate	0.01	TtC/y
	Social cost of carbon today, first-best	400	T\$/TtC

Table 1: Calibration assumptions

The unit abbreviations are ‘y’ for year, ‘TtC’ for  $10^{12}$  metric tonnes of carbon, and ‘T\$’ for  $10^{12}$  US\$. The low natural decay rate reflects the long persistence of CO<sub>2</sub> in the atmosphere; the optimal steady state refers to the steady state reached under full cooperation.<sup>28</sup> The business-as-usual emission rate is the optimal pollution level if pollution damages are not taken into account, that is, if  $\gamma = 0$ : given that the model is stationary, its value reflects emissions from fossil fuel use as of 2024. The calibrated values of the parameters are shown in Table 2. Using these parameter values, we construct natural units of the model, given in Table 3.

Parameter	Value	Unit
$\alpha$	678	T\$/TtC
$\beta$	$339 \cdot 10^3$	T\$ y/TtC <sup>2</sup>
$\gamma$	1.953	T\$/y TtC <sup>2</sup>

Table 2: Calibration results

Expressing all quantities in multiples of these units leads to a model where  $\alpha = \beta = \gamma = 1$ ,  $\rho = 6.25$  and  $\delta = 0.417$ . As the natural time scale of the model is about four centuries, the natural discount rate  $\rho$  is not close to zero. We formulate our analytic results for  $N$  players,

Quantity	Natural unit	Expression
Time	417 y	$\sqrt{\beta/\gamma}$
Pollutant	0.833 TtC	$\alpha/\sqrt{\beta\gamma}$
Value	565 T\$	$\alpha^2/\sqrt{\beta\gamma}$

Table 3: Natural units

setting  $\alpha = \beta = \gamma = 1$ ,  $\mathcal{A} = [0, 1]$  and  $\mathcal{X} = [0, N/\delta]$ : the actions range between 0 and the bliss point, and the state space covers all states that are reachable from the initial state  $x(0) = 0$ .

<sup>28</sup>We calibrate current and long-term-optimal carbon *stocks* to *cumulative emissions*. In reality, cumulative emissions are closely related to the degree of climate change, because of offsetting changes to the carbon-temperature relationship and the rate of decay of atmospheric carbon (Matthews et al., 2009). Our simple model should not be read too literally, but as an illustrative example.

### 3.1 Constructing the Markov-perfect Nash equilibria

We use Theorem 3 to determine all piecewise continuously differentiable, symmetric Markov-perfect Nash equilibria, that is, symmetric policy rules  $a = \phi(x)$  with finitely many discontinuities that give the emission flow as a function of the carbon stock. The starting point of the analysis is the differential equation (19). The pieces of piecewise strategies must be chosen from its solutions; thus our first task is to construct these.

In the transboundary pollution game, the Hamiltonian of an individual player is

$$H(x, p) = \max_a \left[ a - a^2/2 - x^2/2 + p (a + \Phi(x) - \delta x) \right].$$

The optimal action satisfies

$$a^*(x, p) = \begin{cases} 0 & \text{for } p \leq -1, \\ 1 - p & \text{for } -1 < p < 0, \\ 1 & \text{for } p \geq 0. \end{cases}$$

For an interior action, using symmetry and  $p = 1 - a^*(x, p)$ , the game Hamiltonian is given by

$$G(x, a) = a - a^2/2 - x^2/2 - (1 - a)(Na - \delta x).$$

We write Equation (19), which characterises equilibrium strategies away from jumps, as

$$\phi'(x) = \frac{(\rho + \delta)(\phi(x) - 1) + x}{(2N - 1)\phi(x) - (N - 1) - \delta x}. \quad (20)$$

Any piecewise continuously differentiable symmetric Nash equilibrium  $\phi$  has to satisfy the differential equation (20) locally, by Theorem 3(i). As the right hand side is real analytic if the denominator does not vanish, it follows that the solution  $\phi$  is also real analytic. In particular, a piecewise continuously differentiable equilibrium is necessarily piecewise real analytic.

Figure 1 shows a number of local solutions  $a = \phi(x)$  of Equation (20), drawn as solid curves; the figure focuses on the most interesting part of the state space. Candidate pieces of an equilibrium policy rule must coincide with such a solution. The dotted line indicates the points where the denominator of the right hand side of (20) vanishes. As almost all points on this *line of*

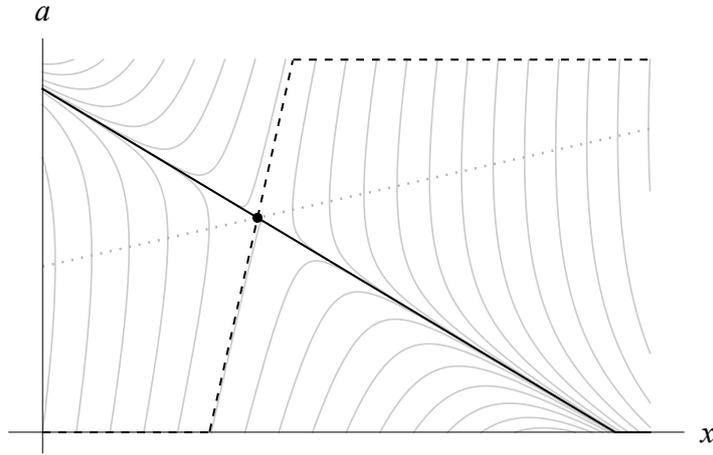


Figure 1: Local solutions of Equation (20). Indicated are the central singularity (black dot), the piecewise linear strategies  $a = \phi_1(x)$  (solid) and  $a = \phi_2(x)$  (dashed), as well as the line of singularities (dotted).

*singularities* involve solutions with a vertical tangent, they cannot belong to any strategy in the strategy space  $\mathcal{S}$ , which requires at least one-sided differentiability.

The exception is the central singularity, where both numerator and denominator of (20) vanish: through this point two solutions pass that are both linear functions in a neighbourhood of the point. The graph of the decreasing solution, indicated by a solid curve, intersects the lower bound  $a = 0$ : we extend this solution by setting its value equal to 0 for all states above the intersection point. We thus obtain a piecewise linear function  $\phi_1(x)$ , defined on the whole state space, which is the ‘linear’ Nash equilibrium strategy examined by van der Ploeg and de Zeeuw (1992).<sup>29</sup> We extend the increasing solution similarly to a piecewise linear function  $\phi_2(x)$ , assuming the values of the parameters  $\rho$ ,  $\delta$  and  $N$  to be such that the increasing solution intersects the lines  $a = 0$  and  $a = 1$  at states in the interior of the state space: this is the situation of our calibration.

With the family of solutions to Equation (19) at hand, we now start ruling out candidate equilibria by using conditions (ii)–(v) of Theorem 3. The task is to understand which policy rules  $\phi(x)$ , constructed from pieces of candidate solutions illustrated in Figure 1 and pieced together using the conditions, can be extended so that they cover the entire state domain.

**Continuous equilibria.** We first show that  $\phi_1(x)$  is the unique continuous-strategy equilibrium. Assume that  $\phi(x)$  is a continuous Nash equilibrium strategy not equal to either  $\phi_1(x)$  or  $\phi_2(x)$ . Then either  $\phi(x) \geq \max\{\phi_1(x), \phi_2(x)\}$  everywhere, or  $\phi(x) \leq \min\{\phi_1(x), \phi_2(x)\}$  everywhere:

<sup>29</sup>We remind the reader that we now consider only symmetric strategies  $\phi(x)$ , and the subscripts on  $\phi_1$ ,  $\phi_2$  indicate particular candidate strategies (and not player identity).

otherwise the graph of  $\phi(x)$  intersects the line of singularities not at the central singularity, and cannot be continued as the graph of a continuous function defined on the whole state space.

We infer that  $\phi(x)$  is either not emitting for small values of the state, or it is maximally emitting for large values of the state. That is, there is either an interval of positive length including  $x = 0$  such that  $\phi(x) = 0$  on that interval, or an interval of positive length including  $x_{\max} = N/\delta$  such that  $\phi(x) = 1$ . We show that neither situation can obtain for an equilibrium strategy.

For assume that  $\phi(x)$  is an equilibrium strategy that is not emitting for small values of the state. Clearly  $V^\phi(0) = 0$ , and, on an interval containing 0, Equation (18) reads as

$$\rho V^\phi(x) = -x^2/2 - \delta x(V^\phi)'(x).$$

Solving this differential equation, we obtain  $V^\phi(x) = -x^2/(4\delta + 2\rho)$ . However, Equation (16) requires that  $(V^\phi)'(x) \leq -1$  close to  $x = 0$ , which is not the case.<sup>30</sup>

The situation that  $\phi(x)$  is maximally emitting for large values of the state is even simpler. As  $V^\phi(x)$  is decreasing by Theorem 3(ii), we have  $(V^\phi)'(x) \leq 0$  close to  $x_{\max}$ . But (16) implies that  $(V^\phi)'(x) \geq 0$  close to  $x_{\max}$ . Hence  $V^\phi(x)$  has to be constant there. Equation (18) however implies that  $\rho V^\phi(x) = (1 - x^2)/2$ , which contradicts the constancy of  $V^\phi(x)$ . The argument shows more than claimed: it implies that an equilibrium strategy cannot be maximally emitting on any interval of positive length.

Hence there cannot be a continuous equilibrium strategy other than  $\phi_1(x)$  or  $\phi_2(x)$ . Actually, the latter strategy is also disqualified, as it is both not emitting for small values and maximally emitting for large values. We therefore have obtained the following result.

**Proposition 3.1.** *The piecewise-linear strategy  $\phi_1$  is the unique continuous Nash equilibrium strategy of the transboundary pollution game.*

**Discontinuous equilibria with continuous value.** Having dealt with continuous equilibrium strategies, we use the family of interior candidate strategies, together with the local conditions imposed by Theorem 3, to construct global restrictions on discontinuous equilibrium strategies  $\phi(x)$  when the value  $V^\phi(x)$  is still continuous. To do this, we construct three regions and prove that no MPE strategy piece can pass through any of these. We then show that any point not in

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<sup>30</sup>In optimal control, such strategies would violate the transversality condition.

these regions can be part of an MPE strategy, and show how a globally-defined strategy can be constructed from such a point.

An easily verified observation is central to how candidate strategies can be pieced together:

**Proposition 3.2.** *The function  $G(x, a)$  is convex in  $a$ , takes its minimum at the line of singularities, and satisfies  $\rho V^\phi(x) = G(x, \phi(x))$  if  $\phi(x)$  is interior and continuous at  $x$ .*

We begin with translating Condition (v) of Theorem 3, which excludes the possibility of strong push–push steady states (see Section 1.2) in equilibrium, into geometric terms.

Figure 2 is a modification of Figure 1. The dashed line indicates the steady state locus

$$Na - \delta x = 0.$$

At this locus total emissions are balanced by natural decay. Where the strategy involves emissions above this locus, the stock grows over time; where below, the stock diminishes. A strong push-push steady state is thus a situation connecting two candidate equilibrium strategy pieces such that the left piece is strictly above the locus, and the right piece is strictly below.

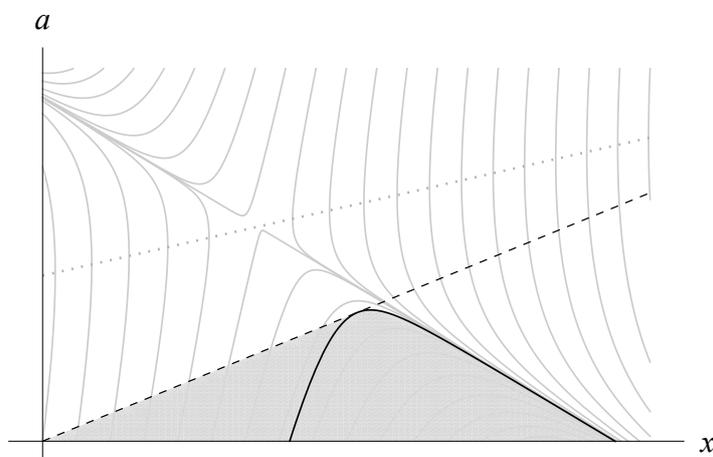


Figure 2: First forbidden region. Indicated are the region NE (gray), the steady state locus (dashed), the solution  $a = \psi_1(x)$  tangent to it (solid), and the line of singularities (dotted).

Let  $\psi_1(x)$  be the unique solution of Equation (20) whose graph is tangent to the steady state locus.<sup>31</sup> The shaded area indicates the union of the set of points under the steady state locus up to the tangency, and the points under the graph of  $\psi_1(x)$  (both strictly). This union will be called NE for “no emissions”, as solutions of (20) in this region without strong push-push points necessarily do not emit for small values of the state. Formally:

<sup>31</sup>This solution was considered in some detail by Dockner and Long (1993).

**Proposition 3.3.** *The graph of a piecewise differentiable equilibrium strategy with continuous value function cannot intersect NE.*

*Proof.* Arguing by contradiction, we assume that  $\phi(x)$  is a piecewise differentiable equilibrium strategy with continuous value and that  $x_0$  is a state such that  $(x_0, \phi(x_0)) \in \text{NE}$ .

The region NE has the property that if  $x_1 < x_0$  and  $\phi$  is continuous in the interval  $x_1 < x \leq x_0$ , then  $(x_1, \phi_+(x_1))$  is also in NE.<sup>32</sup> If  $\phi(x)$  is continuous for all states up to and including  $x_0$ , then necessarily  $\phi(x)$  is not emitting at small values of the state. But this has already been shown to be impossible for an equilibrium strategy.

Hence  $\phi(x)$  has to have jump points below  $x_0$ . Let  $\bar{x}$  indicate the largest of these. For all equilibrium strategies with continuous values, we have  $\rho V^\phi(x) = G(x, \phi(x))$  if  $\phi(x)$  is interior, see Proposition 3.2. The left and right hand limits of  $\phi$  at  $\bar{x}$  are moreover distinct at  $\bar{x}$ , that is  $\phi_-(\bar{x}) \neq \phi_+(\bar{x})$ , while by Condition (iii) of Theorem 3 their values are equal, so that

$$G(x, \phi_-(x)) = G(x, \phi_+(x)).$$

The game Hamiltonian  $G(\bar{x}, a)$  is convex in  $a$  and takes its minimum at the line of singularities, which is above the steady state locus; the point  $(\bar{x}, \phi_+(\bar{x}))$  is in NE, hence under the steady state locus and therefore under the line of singularities. Proposition 3.2 implies that  $(\bar{x}, \phi_-(\bar{x}))$  is above the line of singularities, hence above the steady state locus. We conclude that  $\bar{x}$  is a strong push-push steady state, which according to Theorem 3(v) yields a contradiction.  $\square$

The impossibility of strong push-push steady states has further implications: in Figure 3, we again indicate the steady state locus, but we added a second locus, the *conjugate* steady state locus, also indicated by a dashed line, of points  $(x, a^\dagger)$  which are such that if  $(x, a)$  is on the steady state locus, then  $(x, a^\dagger)$  is above the line of singularities and

$$G(x, a^\dagger) = G(x, a).$$

Let  $\psi_2(x)$  be the unique solution of Equation (20) that is tangent to the conjugate steady state locus. The shaded region is the union of the set of points above the conjugate steady state locus from the tangency onwards, and the points above the graph of  $\psi_2(x)$ . This region is called ME

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<sup>32</sup>Recall that  $\phi_+$  refers to the right-hand limit.

for “maximal emissions”, as solutions of (20) in this region without strong push-push points necessarily emit maximally for large values of the state. Formally:

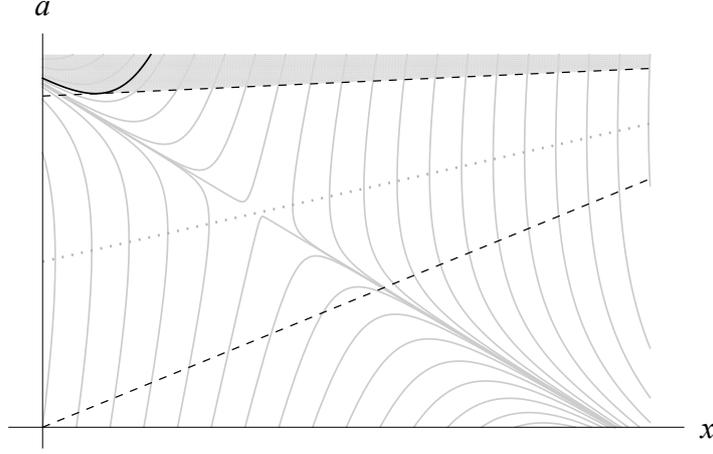


Figure 3: Second forbidden region. Indicated are the region ME (gray), the steady state locus and its conjugate (both dashed), the solution  $a = \psi_2(x)$  tangent to the conjugate steady state locus (solid), and the line of singularities (dotted).

**Proposition 3.4.** *The graph of a piecewise differentiable equilibrium strategy with continuous value function cannot intersect ME.*

The proof is analogous to the proof of Proposition 3.3 and is omitted.

The third region that cannot be entered by equilibrium strategies is illustrated in Figure 4. It is

$$\text{NC} = \left\{ (x, a) \in \mathcal{X} \times \mathring{A} : G(x, a) < G(x, \phi_1(x)) \right\}.$$

The region is a subset of the interior, bounded by the graph of the unique continuous equilibrium strategy  $\phi_1(x)$  and its conjugate  $\phi_1^\dagger(x)$ , which is defined by the requirement that  $G(x, \phi_1(x)) = G(x, \phi_1^\dagger(x))$ . We shall show that solutions of (20) in this region cannot be continued to globally defined continuous-value equilibrium strategies; hence NC is the “no continuation” region.

**Proposition 3.5.** *The graph of a piecewise differentiable equilibrium strategy with continuous value function cannot intersect NC.*

*Proof.* By contradiction: assume  $\phi(x)$  is a globally defined equilibrium strategy whose graph is in NC for some state  $x_0$ . Introducing  $\Delta(x) = G(x, \phi(x)) - G(x, \phi_1(x))$ , this is equivalent to assuming  $\Delta(x_0) < 0$ . Let  $(x_s, a_s)$  denote the central singularity. As  $a_s = \phi_1(x_s)$  minimises  $G(x_s, a)$ , it follows that  $\Delta(x_s) \geq 0$ .

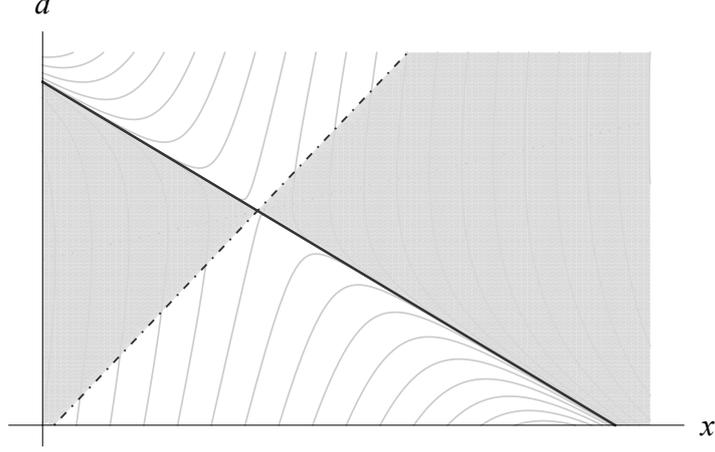


Figure 4: Third forbidden region. Indicated are the region NC (gray), the strategy  $a = \phi_1(x)$  (solid), and its conjugate  $a = \phi_1^\dagger(x)$  (dash-dotted).

We shall show that  $\Delta(x)$  has to decrease as  $x$  moves from  $x_0$  towards  $x_s$ . Equation (19) implies

$$\frac{d}{dx}\Delta(x) = \rho(\phi(x) - \phi_1(x)). \quad (21)$$

If  $\phi(x)$  jumps, the value of  $G$ , and hence of  $\Delta$ , does not change: hence  $\Delta(x)$  is continuous and piecewise differentiable in  $x$ , and Equation (21) holds everywhere except at finitely many points. It follows that  $\Delta$  is decreasing to the left of the central singularity, where  $\phi(x) < \phi_1(x)$ , and increasing to the right of it. In particular  $0 \leq \Delta(x_s) < \Delta(x_0) < 0$ , which is a contradiction.  $\square$

Reformulating Proposition 3.5 gives one of our central results.<sup>33</sup>

**Theorem 4.** *The piecewise linear Markov-perfect equilibrium strategy  $\phi_1(x)$  achieves the lowest possible payoff compared to all other continuous value equilibrium strategies.*

Thus the linear feedback MPE, a focus of the literature since Starr and Ho (1969), is Pareto-dominated by all other MPE associated with a continuous value function. Any candidate equilibrium piece yielding a lower value is in the no-continuation region, and extending it toward the central singularity of Equation (20) only makes the associated value diverge further from the value given by the linear equilibrium. At the central singularity, the linear strategy gives the lowest possible value of all candidate equilibrium pieces; no lower-valued equilibrium can extend across this point.

<sup>33</sup>Compare this to Wirl (1996), who considers a situation where there are multiple continuous MPEs and who shows that the linear equilibrium achieves the lowest payoff of all continuous MPEs.

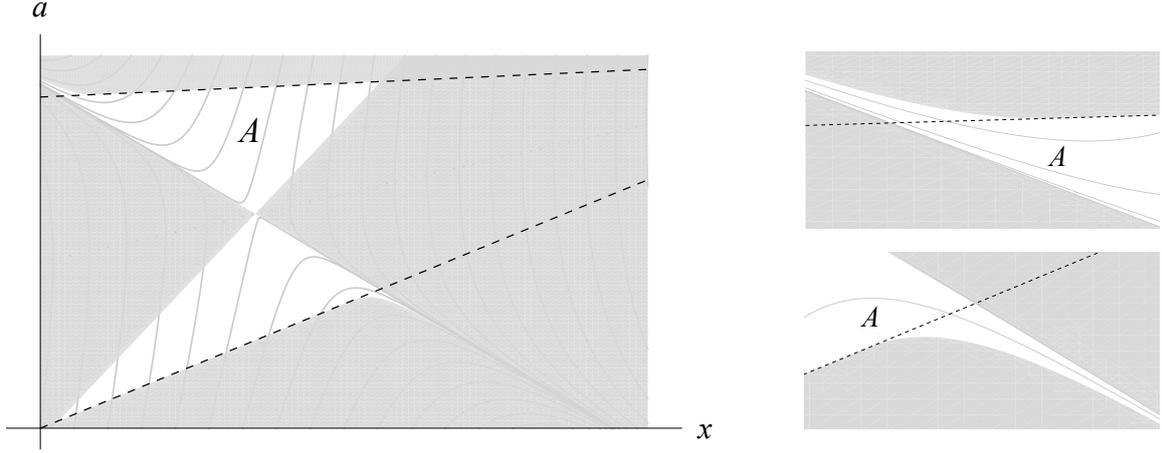


Figure 5: Left: the allowed region. Right: detailed view below the steady state locus (bottom) and above the conjugate steady state locus (top). Indicated are the region  $A$  (white), and the steady state locus and its conjugate (both dashed).

The *allowed* region  $A = (\mathcal{X} \times \mathcal{A}) \setminus (\text{NE} \cup \text{ME} \cup \text{NC})$  (Figure 5) is the complement of the union of the forbidden regions. Our discussion up to this point can be summarised as follows: the graph of a continuous-value equilibrium strategy is necessarily contained in the allowed region.

The next result characterises all equilibrium strategies with finitely many jump points.

**Theorem 5.** *A piecewise differentiable function  $\phi(x)$  with finitely many jump points is a continuous value Markov-perfect equilibrium strategy of the transboundary pollution game if and only if the following hold.*

- (i) *Its graph is contained in the allowed region  $A$ .*
- (ii) *At points where  $\phi(x)$  is differentiable and interior, Equation (20) holds.*
- (iii) *The function  $G(x, \phi(x))$  is continuous.*

The proof of this theorem is given in Appendix A.

Theorem 5 fully characterises all continuous-value MPE with finitely many jump points. It has long been known that Equation (20) is necessary for an MPE. Our result adds the allowed region  $A$  to the necessary conditions, and shows that these three conditions are also sufficient.

The intuition underlying the welfare implications can be understood by introducing the effective discount factor for time  $t$  of a player as

$$D(t) \equiv \exp\left(-(\rho + \delta)t + \int_0^t (N - 1)\phi'(x(s)) ds\right).$$

This discount factor takes into account the effect of a marginal change in the carbon stock on other players future emissions. For integrating an individual player's costate equation along the equilibrium path to the next jump time  $t_1$ , or to  $t_1 = \infty$  if there is no jump, we obtain for the player's equilibrium cost of carbon  $-V'(x(t))$  that

$$-V'(x_0) = \int_0^{t_1} D(t)x(t) dt + \lim_{t \uparrow t_1} D(t)(-V'(x(t))).$$

This is the present value of marginal damages up to time  $t_1$ , discounted by the effective discount factor  $D(t)$ , plus the discounted marginal damage value at  $t_1$ .

The linear equilibrium has no jumps, hence  $t_1 = \infty$ . It is apparent from Figure 5 that the linear equilibrium strategy is the most accommodating strategy; i.e.,  $\phi'(x) \geq \phi'_1(x)$  for all  $\phi$ . In other words, in the linear equilibrium, emissions are everywhere dynamic strategic substitutes: a marginal increase in carbon concentrations induces other players to lower their emissions, resulting in lower effective cost of carbon for the emitter. This is expressed through the lower effective discount factor. As all players reason in this way, collective pollution is high and eventually leads to high carbon stocks, harming welfare. The remaining non-linear equilibria involve less accommodation to carbon stocks than the linear equilibrium, so that their effective discount factor is always larger than its linear counterpart.

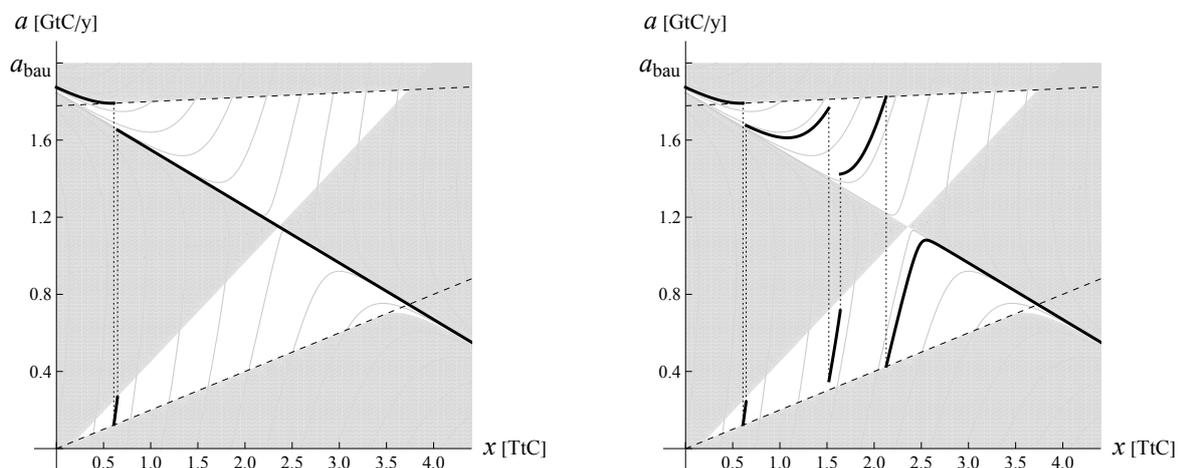


Figure 6: Two Markov-perfect Nash equilibria that are not piecewise linear. Indicated are the allowed region (white), an equilibrium strategy (solid, with dotted jumps), and the steady state locus and its conjugate (both dashed).

Figure 6 displays two MPE, that is, two alternative strategies which are mutual best responses to themselves. The left panel displays a strategy which features high emissions for values of

the carbon stock up to the critical level of 0.6 TtC. In this region, the equilibrium is Pareto-dominant, and carbon stocks grow rapidly. When the critical state is reached, emissions are reduced discontinuously and the state is stabilised at the critical level. If for some reason the stock grows above this steady state, the strategy prescribes all players to start increasing their emissions, and the economy starts drifting to higher carbon stocks. Upon reaching a second critical level of 0.65 TtC, emissions are increased discontinuously by all players, and the economy starts moving more rapidly towards a second steady state, with a carbon stock of 3.7 TtC. It is this threat of a substantial and rapid increase of the stock which ensures that the players are happy to stabilise at a lower level of the stock. The right panel displays a second equilibrium which behaves similarly at low carbon stocks, but in which there are two steady states at higher carbon stocks, and several discontinuities.

The low-carbon steady-state is thus semi-stable. To the right,  $\phi'(x)$  is large and positive, so that the effective discount factor is large: if a player emits an additional unit, other players would increase their emissions rapidly, resulting in much higher costs for the original emitter: the equilibrium cost of carbon for the player is high. This reaction of the other players acts as a trigger response to deviations of the first player, which allows the equilibrium to stabilise such a small carbon stock with low emissions.<sup>34</sup>

Along the equilibrium trajectory approaching this steady state from below, the players understand that stabilisation at a small carbon stock is about to happen, and that the main effect of additional emissions is to bring forward the moment when emissions are discontinuously cut. The additional costs of the emission are to a large degree compensated by the fact that the low emission regime has been brought forward: hence the marginal cost of the additional amount of stock is low. After the emission cut, the beneficial effect of additional emissions is gone, and the trigger reaction described above of the other players sets in, all resulting in a discontinuous change of the player's cost of carbon.<sup>35</sup> Note that while low steady states can be supported by MPE, these steady states are reached too fast, with suboptimally high emission rates, relative to the efficient outcome.

**Equilibria with value discontinuities.** Finally we turn to equilibrium strategies whose value function  $V^\phi(x)$  is not continuous. By Theorem 3(iii), a value discontinuity must be located at

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<sup>34</sup>Sorger (1998) constructs a family of MPE to a renewable resource game which almost stabilise the efficient steady state in a similar manner.

<sup>35</sup>The stable equilibrium considered by Dockner and Long (1993) is also in the equilibrium set, extended to a global strategy. It does not have attractive welfare properties, as the steady-state carbon stock is quite high.

a point  $\bar{x}$  at which the strategy is discontinuous, and the left hand limit  $\phi(\bar{x})$  of the equilibrium strategy is below the steady state locus, while the right hand limit satisfies

$$a + (N - 1)\phi_+(\bar{x}) - \delta\bar{x} \geq 0$$

for all  $a \in \mathcal{A}$ , so that no player can individually take the state to the benign side of the discontinuity. This implies  $\phi_+(\bar{x}) \geq \delta\bar{x}/(N - 1)$ . If  $\bar{x}$  is the smallest state at which there is a value discontinuity, then it is necessarily to the right of the unique tangency point of a solution of Equation (19) to the steady state locus. This gives a lower bound on the location of value discontinuities; in particular, any value discontinuity must be located in the right half of region NC, at a high carbon stock, above the steady-state of the linear equilibrium.

The right hand limit  $\phi_+(\bar{x})$  is either above or below the line of singularities. If it is above that line, there has to be a second jump, this time with continuous value, to avoid the graph of  $\phi(x)$  entering the forbidden region ME of eventually maximally emitting strategies. To the right of this jump, the graph of  $\phi$  is extended in the same manner as before. If  $\phi_+(\bar{x})$  is below the line of singularities, its graph can either be extended continuously for all larger states, or it may have continuous value jumps to points above the steady state locus, or discontinuous value jumps from points below this locus. See Figure 7.

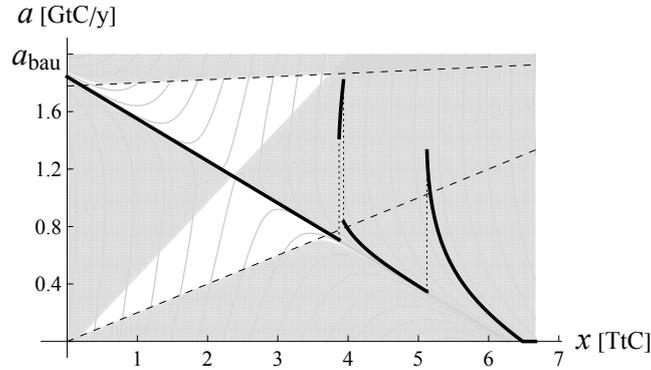


Figure 7: An equilibrium strategy with two value discontinuities. Legend as in Figure 6.

These equilibria are miscoordination traps, which sustain bad long-run outcomes, but from which no player can escape by unilateral action.

**Equilibrium construction.** Theorem 5 implies the following result.

**Theorem 6.** *Each point of the allowed region  $\mathcal{A}$  is on the graph of an MPE strategy.*

*Proof.* The local solution of Equation (19) that passes through a given point can be continued for larger values of the state until either it intersects the conjugate steady state locus, from whence it jumps to the steady state locus, or it intersects the boundary of NC, from which it jumps to the linear strategy  $\phi_1(x)$ , or it intersects the line  $a = 0$ , from which it is continued at that value. Continuing the solution for smaller values of the state is similar, only that then intersecting the steady state locus instead of its conjugate triggers a jump. Strategies constructed in this manner will have at most two jumps; however, more jumps can be inserted in the interior of, respecting value matching and Proposition 3.2<sup>36</sup> □

Computation of particular equilibria is straightforward, requiring only the integration of Equation (20) between any points of discontinuity, and a continuous-value connection at these points.

**Properties of Pareto-dominant equilibria.** Theorem 6 allows us to construct the value envelope  $\bar{V}(x)$ , which for every state  $x$  gives the maximal value that can be supported by a Nash equilibrium. The solution to (19) tangent to the conjugate steady state locus is  $\psi_2(x)$ ; for states below this tangency, we have that  $\bar{V}(x) = V^{\psi_2}(x)$ . Similarly, as the solution tangent to the steady state locus is  $\psi_1(x)$ , for states above that tangency we have  $\bar{V}(x) = V^{\psi_1}(x)$ . Between the two tangency points,  $\bar{V}(x)$  is the value  $G(x, a)$  for points  $(x, a)$  on the steady state locus. We recover the result of Schumacher et al. (2022).

**Theorem 7.** *There is no single Nash equilibrium strategy  $\phi$  such that  $\bar{V}(x) = V^\phi(x)$  for all  $x$ .*

*Proof.* The steady state locus between the two tangency states is not a local solution of (19). □

The welfare properties of MPE are illustrated by comparing them with the business-as-usual solution, which we take as the emission path of a decision maker that completely ignores costs arising from pollution, and the fully cooperative solution. In Figure 8(a) we give the added value, relative to the business-as-usual solution, of the cooperative solution, the linear equilibrium, which is the lowest value a continuous-value MPE can achieve, and the value envelope, which gives for every state the highest MPE value that can be achieved when the dynamics commence at that state. Figure 8(b) gives the corresponding relative values, where business-as-usual corresponds to 0% and the cooperative solution to 100%. The linear equilibrium consistently achieves a bit more than half of the possible efficiency gain, whereas the value envelope reaches between 80% and 100% of the first best solution.

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<sup>36</sup>Discontinuous actions do not need to satisfy a “smooth pasting” principle: for any player, the point of discontinuity is not endogenously chosen, but determined by the other players’ strategies, taken as given.

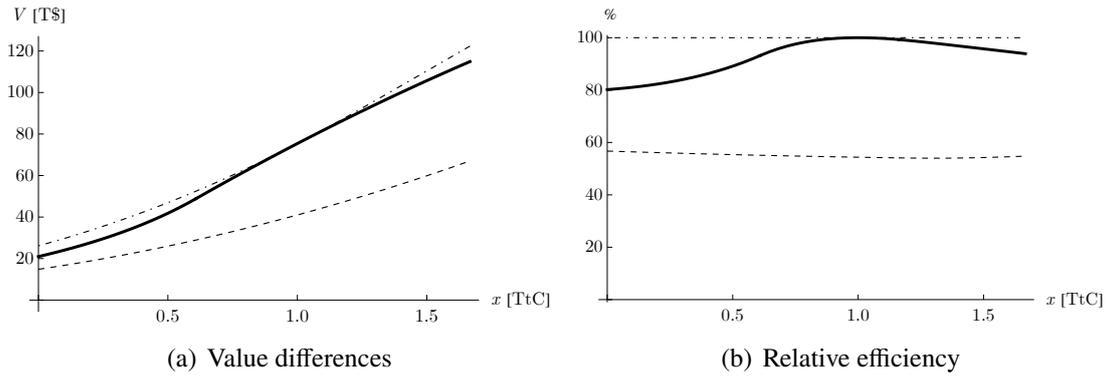


Figure 8: Added value and relative efficiency, both with respect to business-as-usual, for different equilibrium strategies: cooperative (dash-dotted), best Nash (solid), linear Nash (dashed)

Stock evolutions and emissions are shown for two equilibria, the piecewise linear one and one that is Pareto optimal for the initial state  $x_0 = 0.5$  TtC. The strategies are illustrated in Figure 9(a): the Pareto optimal strategy prescribes higher emissions until the pollution stock reaches 0.61 TtC, which happens in 13 years time, by which time emissions are cut to steady state level emissions of 0.12 GtC/y.<sup>37</sup>

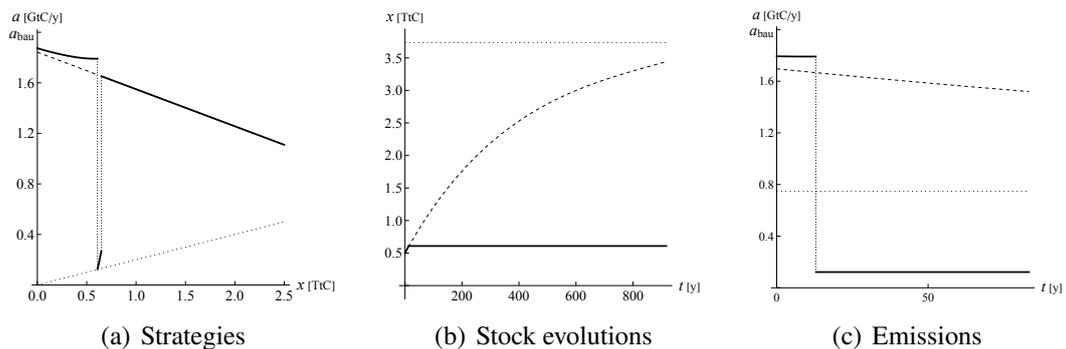


Figure 9: Strategies, stock evolutions and emissions for the equilibrium that is Pareto-dominant at initial state  $x = 0.5$  TtC (solid), the linear equilibrium (dashed), and the steady state curve (dotted, panel 9(a)) and the steady state level of stock and emissions of the linear strategy (dotted, panels 9(b) and 9(c)).

Our results highlight that equilibrium multiplicity has important welfare implications. In particular, ensuring that a good equilibrium is played can lead to large welfare improvements. Climate negotiations can be seen as a forum for equilibrium choice (Dockner and Long, 1993). Coordination on a benign equilibrium may be more important than negotiating hard over who gets what fraction of the meagre surplus available to the players of a dynamic prisoner's dilemma.

<sup>37</sup>Such a radical emission reduction could be challenging because of adjustment costs, reflecting unmodelled capital stocks. This makes an extension to more dimensions of particular interest.

## 4 Conclusion

We have shown how discontinuous strategies put differential games on a solid conceptual footing: best responses almost always exist and are unique. We have also given a necessary and sufficient condition for constructing best responses. Illustrating its use in a model of non-cooperative mitigation of climate change, we have constructed a large class of symmetric equilibria, all but one of them featuring discontinuous strategies. In other applications, with non-convex natural dynamics, well-defined continuous equilibria do not exist at all; our formulation allows the study of MPE in such games also (Kossioris et al., 2008; Dockner and Wagener, 2014).

The importance of multiple equilibria in our application highlights the importance of equilibrium selection. Dockner and Long (1993) argued that climate negotiations could help states coordinate on a relatively benign equilibrium.<sup>38</sup> Our substantive message is twofold. First, in a non-cooperative world, gradual stabilisation of emissions gives poor strategic incentives and leads to bad outcomes; in our simple model, equilibria with gradual stabilisation have poor welfare properties. Second, more efficient long-run steady states are sustained in equilibrium by trigger-like (but Markovian) asymmetric responses, in which upward deviations from the benign steady state are punished by the players choosing collectively to slowly drift to a steady state with a higher carbon concentration. The efficient steady state is reached in finite time, with a discontinuous reduction in carbon emissions, suggesting that agreeing on radical climate policies might be better than agreeing on gradual reductions, specifically because radical policies provide better strategic incentives (similar to Dutta and Sundaram, 1993).<sup>39</sup> Such equilibria can be indexed by a “stabilisation target”, which triggers a “net zero” emission policy when reached. Thus, strategic leverage can be used to develop agreements with more effective incentives.

Our methodological results open up many research directions. The present model is deterministic; adding noise to the state dynamics, or Poisson shocks, are interesting potential extensions. We expect that our results can be extended to multiple state variables, although this is not a trivial task. Renegotiation-proofness in our equilibria (or the lack of it) is a further interesting property warranting further study (the contribution of Schumacher et al., 2022, was also motivated by this same observation). Lastly, in our application we have considered symmetric

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<sup>38</sup>As we have stressed, however, their treatment of multiple equilibria was problematic exactly for the reasons we address in the present paper. See also Dutta and Radner (2004).

<sup>39</sup>Van Der Ploeg and Venables (2022) give further reasons for radical climate policies, based on strategic complementarities and multiple steady states.

equilibria; our results also allow, as well as motivate, the study of asymmetric equilibria.

Finally, given that our methods have found several novel and important features of a well-studied model, we expect these methods to yield new insights in the numerous applications of differential games which exist in the literature. While we have focused on a deterministic public good differential game, we expect our methods can also facilitate the analysis of various types of games of incomplete information, or with stochastic elements, in continuous time.

## Acknowledgements

For helpful comments, we thank Buz Brock, Guilherme Carmona, Larry Karp, Matt Mitchell, Rick van der Ploeg, Colin Rowat, Gerhard Sorger, Daniel Spiro, Georges Zaccour and Aart de Zeeuw, as well as seminar audiences at Aalto University, Colorado State University, GERAD, ifo Institute, the Swedish University of Agricultural Sciences, and the Universities of Amsterdam, Bologna, Oslo, Uppsala, and Utrecht.

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## A Proof of Theorem 5

*Proof.* The necessity of the conditions has been discussed in the run-up to the statement of Theorem 5. To show sufficiency, assume that  $\phi(x)$  satisfies the conditions.

Let  $x_1$  be the state at which the graph of the linear equilibrium  $\phi_1(x)$  intersects the steady state locus. By Condition (i), below the steady state locus and to the right of  $x = x_1$ , we have  $\psi_1(x) \leq \phi(x) \leq \phi_1(x)$ . Condition (iii) implies that  $\phi(x)$  is continuous, as there are no points in  $A$  above the steady state locus and to the right of  $x = x_1$ . In the interval  $[a, b]$ , where  $a$  and  $b$  are such that  $\psi_1(a) = 0$  and  $\phi_1(b) = 0$ , we have  $\phi'(x) < 0$  by Condition (ii). Hence there is a state  $x_b \in [a, b]$  such that  $\phi(x)$  is interior below  $x_b$  and  $\phi(x) = 0$  above  $x_b$ .

Introduce the function

$$W(x) = G(x, \phi(x))/\rho = u(x, \phi(x))/\rho - \frac{\partial u}{\partial a}(x, \phi(x))(N\phi(x) - \delta x)/\rho. \quad (22)$$

We shall show that  $W(x) = V^\phi(x)$  whenever  $\phi(x)$  is interior. Equation (19) implies that

$$W'(x) = \frac{d}{dx}G(x, \phi(x))/\rho = -\frac{\partial u}{\partial a}(x, \phi(x)).$$

This allows us to rewrite Equation (22) as

$$\rho W(x) = u(x, \phi(x)) + W'(x)(N\phi(x) - \delta x).$$

This equation is the same as Equation (18), which is satisfied by the value function  $V^\phi$ . Proposition A.1 implies that solutions to this equation are unique on all orbits of  $\dot{x} = N\phi(x) - \delta x$  that converge to a stable or a non-degenerate semi-stable steady state. As all trajectories of Equation (19) either intersect the steady state locus transversally or are non-degenerately, that is, quadratically, tangent to it, this is satisfied, and we conclude that  $W(x) = V^\phi(x)$  and  $\rho V^\phi(x) = G(x, \phi(x))$  for  $x \leq x_b$ , hence Condition (i) of Theorem 3 holds for interior  $\phi(x)$ .

We show Condition (i) also for non-interior  $\phi(x)$ . We have to show that for  $x > x_b$  we have

$$(V^\phi)'(x) \leq -\frac{\partial u}{\partial a}(x, 0) = -1.$$

The fact that  $x_b$  is the boundary point of the set of states where  $\phi(x)$  is interior implies that

$(V^\phi)'(x_b) = -1$  and  $(V^\phi)''(x_b) \leq 0$ . Arguing by contradiction, assume there is  $x > x_b$  such that  $(V^\phi)'(x) > -1$ . Then there is  $x_b < x_1 < x$  such that  $(V^\phi)'(x_1) = -1$  and  $(V^\phi)''(x_1) \geq 0$ .

On the interval  $x > x_b$ , Equation (18), which determines  $V^\phi$ , takes the form

$$\rho V^\phi(x) = -\frac{1}{2}x^2 - \delta x (V^\phi)'(x).$$

Taking the derivative with respect to the state and rearranging yields

$$(V^\phi)''(x) = -(\rho + \delta)(V^\phi(x))' / (\delta x) - 1/\delta.$$

We find that

$$0 \leq (V^\phi)''(x_1) = (\rho + \delta)/(\delta x_1) - 1/\delta < (\rho + \delta)/(\delta x_b) - 1/\delta = (V^\phi)''(x_b) \leq 0,$$

which constitutes a contradiction. We conclude that Condition (i) holds.

The previous argument also shows that  $V^\phi$  is decreasing for non-interior  $\phi(x)$ . Using Equation (19), we find for interior  $\phi(x)$  that

$$(V^\phi)'(x) = \frac{1}{\rho} \frac{d}{dx} G(x, \phi(x)) = \phi(x) - 1 < 0,$$

which shows that  $V^\phi$  is decreasing everywhere: this shows Condition (ii) of Theorem 2.

Formally, we put  $\beta(x_{\min}) = \infty$  and  $\beta(x_{\max}) = -\infty$ . The state space has been constructed such that the dynamics always point into it at the boundary, which takes care the value discontinuity at the boundary points: Condition (iii) holds there. Other value discontinuities are ruled out by the third assumption of the result to be proved, verifying Condition (iii) everywhere.

We verify Condition (iv) in the situation that  $\phi_+(x) > \phi_-(x)$ , the proof of the other situation being entirely similar. We write

$$\mathcal{A}_0 = \{a \in [0, 1] : a - \delta x + (N - 1)\phi = 0 \text{ for some } \phi \in [\phi_-(x), \phi_+(x)]\},$$

and we introduce  $\bar{a}_- = \delta x - (N - 1)\phi_-(x)$  and  $\bar{a}_+ = \delta x - (N - 1)\phi_+(x)$ . Then

$$\max_{a \in \mathcal{A}_0} u(x, a) = \begin{cases} -\infty & \text{if } \bar{a}_- < 0 \text{ or } \bar{a}_+ > 1 \\ u(x, \bar{a}_-) & \text{if } 0 \leq \bar{a}_- \leq 1, \\ u(x, 1) & \text{if } \bar{a}_+ \leq 1 < \bar{a}_-. \end{cases}$$

If  $\bar{a}_- < 0$  or  $\bar{a}_+ > 1$ , there is nothing to prove.

Assume first that  $0 \leq \bar{a}_- \leq 1$ . If  $\phi_-(x)$  is interior, we have

$$\begin{aligned} \rho V^\phi(x) - \max_{a \in \mathcal{A}_0} u(x, a) &= G(x, \phi_-(x)) - u(x, \bar{a}_-) \\ &= u(x, \phi_-(x)) - \frac{\partial u}{\partial a}(x, \phi_-(x))(N\phi_-(x) - \delta x) - u(x, \bar{a}_-). \end{aligned}$$

Noting that  $N\phi_-(x) - \delta x = \phi_-(x) - \bar{a}_-$ , and using the fact that  $u(x, a)$  is concave in  $a$ , we obtain

$$\rho V^\phi(x) - \max_{a \in \mathcal{A}_0} u(x, a) = u(x, \phi_-(x)) + \frac{\partial u}{\partial a}(x, \phi_-(x))(\bar{a}_- - \phi_-(x)) - u(x, \bar{a}_-) \geq 0.$$

If  $\phi_-(x)$  is not interior, it is zero and  $\bar{a}_- = \delta x$ . Above we have shown that then  $(V^\phi)'(x) \leq -1 = \frac{\partial u}{\partial a}(x, 0)$ . Again using concavity of  $u(x, a)$  in  $a$ , it follows that

$$\begin{aligned} \rho V^\phi(x) - \max_{a \in \mathcal{A}_0} u(x, a) &= u(x, 0) + (V^\phi)'(x)(-\delta x) - u(x, \bar{a}_-) \\ &\geq u(x, 0) + \frac{\partial u}{\partial a}(x, 0)(\bar{a}_- - 0) - u(x, \bar{a}_-) \geq 0 \end{aligned}$$

We proceed by investigating the situation that  $\bar{a}_- > 1$ . If  $\phi_-(x)$  is interior, we have

$$\begin{aligned} \rho V^\phi(x) - \max_{a \in \mathcal{A}_0} u(x, a) &= \rho V^\phi(x) - u(x, 1) = G(x, \phi_-(x)) - u(x, 1) \\ &\geq u(x, \phi_-(x)) - \frac{\partial u}{\partial a}(x, \phi_-(x))(N\phi_-(x) - \delta x) - u(x, \phi_-(x)) - \frac{\partial u}{\partial a}(x, \phi_-(x))(1 - \phi_-(x)) \\ &= -\frac{\partial u}{\partial a}(x, \phi_-(x))((N - 1)\phi_-(x) + 1 - \delta x) \geq 0. \end{aligned}$$

At the first inequality, we have again used concavity; for the second inequality, we have used that  $\frac{\partial u}{\partial a} \geq 0$  and  $(N - 1)\phi_-(x) + 1 - \delta x < 0$ , which is a consequence of  $\bar{a}_- > 1$ .

In the non-interior case we have  $\phi_1(x) = 0$  and  $\bar{a}_- = \delta x > 1$ . Then

$$\begin{aligned} \rho V^\phi(x) - \max_{a \in \mathcal{A}_0} u(x, a) &= u(x, 0) + (V^\phi)'(x)(-\delta x) - u(x, 1) \\ &\geq u(x, 0) + \frac{\partial u}{\partial a}(x, 0)(1 - 0) - u(x, 1) \geq 0. \end{aligned}$$

This concludes the verification of Condition (iv).

To verify Condition (v) of Theorem 3, the impossibility of push-push steady states, we remark that such a steady state would entail a jump from the part of the allowed region  $A$  above the steady state locus to the part below it. As the latter region is to the right of the unique tangency  $(x_1, a_1)$  of  $\psi_1(x)$  to the steady state locus, and the former to the left of the unique tangency  $(x_2, a_2)$  of  $\psi_2(x)$  to the conjugate steady state locus, it is sufficient to show that  $x_1 > x_2$ .

Let  $A(x)$  and  $B(x)$  be functions such that their graphs equal, respectively, the steady state locus and the conjugate steady state locus. That is,  $A(x) = \delta x/N$  and  $B(x) > A(x)$  satisfies  $G(x, A(x)) = G(x, B(x))$  for all  $x$ .

We claim that if  $\phi(x)$  is a local solution of (19) that intersects the steady state locus at  $x_2$ , then  $\phi'(x_2) > A'(x_2)$ . Indeed, we have

$$\begin{aligned} \frac{\partial G}{\partial x}(x_2, A(x_2)) + \frac{\partial G}{\partial a}(x_2, A(x_2))A'(x_2) &= \frac{d}{dx}G(x, A(x)) \Big|_{x=x_2} = \frac{d}{dx}G(x, B(x)) \Big|_{x=x_2} \\ &= \frac{\partial G}{\partial x}(x_2, B(x_2)) + \frac{\partial G}{\partial a}(x_2, B(x_2))B'(x_2). \end{aligned}$$

We have  $\psi_2(x_2) = a_2 = B(x_2) > A(x_2) = \phi(x_2)$  and  $\psi_2'(x_2) = B'(x_2)$ . This implies

$$\begin{aligned} \frac{\partial G}{\partial x}(x_2, B(x_2)) + \frac{\partial G}{\partial a}(x_2, B(x_2))B'(x_2) &= \frac{\partial G}{\partial x}(x_2, \psi(x_2)) + \frac{\partial G}{\partial a}(x_2, \psi(x_2))\psi'(x_2) \\ &= \frac{d}{dx}G(x, \psi(x)) \Big|_{x=x_2} = -\frac{\partial u}{\partial a}(x_2, \psi(x_2)) > -\frac{\partial u}{\partial a}(x_2, \phi(x_2)) = \frac{d}{dx}G(x, \phi(x)) \Big|_{x=x_2} \\ &= \frac{\partial G}{\partial x}(x_2, \phi(x_2)) + \frac{\partial G}{\partial a}(x_2, \phi(x_2))\phi'(x_2) = \frac{\partial G}{\partial x}(x_2, A(x_2)) + \frac{\partial G}{\partial a}(x_2, A(x_2))\phi'(x_2). \end{aligned}$$

Here we have used Equation (19) in the third and fourth equality, and the inequality is a consequence of strict concavity of  $u(x, a)$  with respect to  $a$ . We conclude that

$$\frac{\partial G}{\partial a}(x_2, A(x_2))(A'(x_2) - \phi'(x_2)) > 0,$$

and as  $\frac{\partial G}{\partial a}(x, a) < 0$  along the steady state locus, the claim has been demonstrated.

Assume now that  $x_1 \leq x_2$ . As  $\psi_1(x)$  touches the steady state locus at  $(x_1, a_1)$  from below, it follows that all solutions intersecting the steady state locus at a point  $(x, A(x))$  such that  $x > x_1$  have a slope less than  $A'(x)$ . This contradicts the slope  $\phi'(x_2)$  being larger than  $A'(x_2)$ . We conclude that  $x_2 < x_1$ , and that Condition (v) is satisfied.  $\square$

The following auxiliary result, used above, solves Hamilton–Jacobi–Bellman equations for the situation that utility and dynamics do not depend on an action variable. The proof is standard.

**Proposition A.1.** *Let  $g(x)$  and  $v(x)$  be real analytic, and let  $\bar{x}$  be such that  $g(\bar{x}) = 0$ . Consider for  $\rho > 0$  the differential equation*

$$\rho W(x) - v(x) - W'(x)g(x) = 0 \quad a.e. \quad (23)$$

- (i) *All solutions  $W(x)$  of (23) are continuous close to  $\bar{x}$  and satisfy  $W(\bar{x}) = v(\bar{x})/\rho$ .*
- (ii) *All solutions are real analytic in  $I \setminus \{\bar{x}\}$  for some open interval  $I$  containing  $\bar{x}$ , and  $W'(x)$  tends to a finite or infinite limit as  $x \rightarrow \bar{x}$ .*
- (iii) *If  $g'(\bar{x}) < 0$ , the equation has a unique and real analytic solution.*
- (iv) *If  $g'(\bar{x}) = 0$  and  $g''(\bar{x}) \neq 0$ , the equation has a unique solution for  $x < \bar{x}$  if  $g''(\bar{x}) > 0$  and for  $x > \bar{x}$  if  $g''(\bar{x}) < 0$ .*

## Online appendix

We assume throughout the Online appendix that Assumptions 1, 2, and 3 hold, without explicitly mentioning the fact. Also the following notational conventions are used.

Time is always nonnegative:  $t \geq 0$ . If  $S \subset \mathbb{R}$  is measurable,  $|S|$  denotes the Lebesgue measure of  $S$ . For bounded functions  $f : S \rightarrow \mathbb{R}$ , we set  $\|f\|_\infty = \sup_{z \in S} |f(z)|$ . If  $f : S \rightarrow \mathbb{R}$  and  $g : S \rightarrow \mathbb{R}$  are real-valued functions,  $f \leq g$  is taken to mean that  $f(s) \leq g(s)$  for all  $s \in S$ .

We set  $g(x) = \Phi(x) + h(x)$ . At points  $x$  where  $g$  is real analytic, we introduce

$$f(x, a) = a + g(x).$$

A point where  $g$  is not real analytic is called a jump point, even if  $g$  is continuous there; the set of jump points is denoted  $\mathcal{J}$ . Given a jump point  $\bar{x} \in \mathcal{J}$ , we write  $f_-(x, a)$  and  $f_+(x, a)$  for  $f(x, a)$  if, respectively,  $x < \bar{x}$  and  $x > \bar{x}$ . By hypothesis on  $\mathcal{S}$ , the functions  $f_-$  and  $f_+$ , together with their state derivatives, can be extended continuously to  $\bar{x}$ . The values of  $f_-$  and  $f_+$  and their state derivatives at  $(\bar{x}, a)$  are defined as these extensions.

## B Singular and regular value functions

A sufficient condition for uniqueness of solutions to the Hamilton–Jacobi–Bellman equation is equality of value and regular value function (Barles et al., 2013, 2014). The object of this section is to show this equality in our context.

We first note two existence results.

**Proposition B.1.** *For every initial state  $x_0$ , there is a trajectory–action pair  $(x^*, a^*)$  such that  $x^*(0) = x_0$  and  $V(x_0) = U(x^*, a^*)$ . For every initial state  $x_0$  and action schedule  $a$ , there is a trajectory  $x$  such that  $(x, a)$  is a trajectory–action pair.*

*Proof.* The first statement follows from Assumption 2, see for instance Cesari (1983). For the second statement, see (Filippov, 1988, Chapter 2, §7, Theorem 5) □

A given action schedule gives a better payoff if the initial state is lower. This implies that the value function is decreasing.

**Proposition B.2.** *Let  $(x, a)$  be a trajectory–action pair with initial state  $x_0$  such that  $U(x, a) = V(x_0)$ . For every  $y_0 \leq x_0$ , there is a trajectory–action pair  $(y, a)$  with initial state  $y_0$  such that  $U(y, a) \geq U(x, a)$ .*

*Proof.* Let  $(z, a)$  be a trajectory–action pair with initial state  $y_0$ . Set  $y(t) = \min\{x(t), z(t)\}$  for all  $t$ . Then  $(y, a)$  is a trajectory–action pair with initial state  $y_0$  and  $y \leq x$ . Assumptions 2 and 3 imply  $U(z, a) \geq U(x, a)$ .  $\square$

**Proposition B.3.** *The value function  $V$  is decreasing and  $\beta(x_{\min}) \geq V(x) \geq \beta(x_{\max})$ .*

*Proof.* Take initial states  $x_0, y_0$  such that  $y_0 \leq x_0$  and let  $(x, a)$  be a trajectory–action pair such that  $U(x, a) = V(x_0)$ . By Proposition B.2, there is a trajectory–action pair  $(y, a)$  with initial state  $y_0$  such that  $U(y, a) \geq U(x, a)$ . We conclude that  $V(y_0) \geq U(y, a) \geq U(x, a) = V(x_0)$ . The second statement immediately follows from Assumption 3.  $\square$

**Proposition B.4.** *The value function  $V$  and the regular value function  $V^{\text{reg}}$  are equal.*

*Proof.* Assume the proposition is false. Then there is a state  $x_0$  such that  $V(x_0) > V^{\text{reg}}(x_0)$ . Let the trajectory–action pair  $(x, a)$  have initial state  $x_0$  and be such that  $U(x, a) = V(x_0)$ .

As  $U(x, a) = V(x_0) > V^{\text{reg}}(x_0)$ , the trajectory–action pair  $(x, a)$  is non-regular: hence there is a jump point  $\bar{x} \in \mathcal{J}$  and a time interval  $I = (t_1, t_2)$  of positive length such that for  $t \in I$  we have  $x(t) = \bar{x}$ ,  $f_-(\bar{x}, a(t)) < 0$  and  $f_+(\bar{x}, a(t)) > 0$ .

We construct a trajectory  $y$  by setting  $y(t) = x(t)$  for  $t \in [0, t_1]$ , letting  $y$  solve  $\dot{y} = f_-(y, a)$  for  $t \in I$  with initial state  $y(t_1) = x(t_1)$ , and continuing  $y$  for larger values of  $t$  such that  $(y, a)$  is a trajectory–action pair. We set  $z(t) = \min\{x(t), y(t)\}$ . Then  $(z, a)$  is a trajectory–action pair and  $z \leq x$ . As  $V(x_0) \geq U(z, a) \geq U(x, a) = V(x_0)$ , the trajectory  $z$  is optimal as well.

Jaakkola and Wagener (2025) and Assumption 2 imply that  $z$  is monotone. As it is locally decreasing, for  $t > t_1$  close to  $t_1$ , it is decreasing. Hence it cannot return to the jump point  $\bar{x}$ . It may stay at a second jump point  $\bar{x}^{(2)}$  for  $t \in I^{(2)} = (t_1^{(2)}, t_2^{(2)})$  which is such that  $t \in I^{(2)}$  we have  $x(t) = \bar{x}^{(2)}$ ,  $f_-(\bar{x}^{(2)}, a(t)) < 0$  and  $f_+(\bar{x}^{(2)}, a(t)) > 0$ . As before, we construct a trajectory  $z^{(2)}$  such that  $z^{(2)} \leq z$  and  $z^{(2)}(t) < z(t)$  if  $t \in (t_1^{(2)}, t_2^{(2)})$ , while  $U(z^{(2)}, a) = U(z, a) = V(x_0)$ .

As there are at most  $J - 2$  interior jump points, repeating the construction at most  $J - 2$  times yields an optimal regular trajectory, a contradiction.  $\square$

## C Controllability

The strategy profile chosen by the other players may imply that the dynamics facing the responding player are not controllable at an interface. This prevents us from applying the results of Barles et al. (2013) directly. In the following, we investigate the consequences of non-controllability for the properties of the value function.

**Definition 2.** *The dynamics are left (right) controllable at a point  $\bar{x} \in \mathcal{J}$  if there is an open interval  $I$  containing 0 such that  $I \subset f_-(\bar{x}, \mathcal{A})$  ( $I \subset f_+(\bar{x}, \mathcal{A})$ ). If the dynamics are both left and right controllable at  $\bar{x}$ , they are controllable.*

For a jump point  $\bar{x}$ , we introduce the compact sets  $\mathcal{X}_- = \{x \in \mathcal{X} : x \leq \bar{x}\}$  and  $\mathcal{X}_+ = \{x \in \mathcal{X} : x \geq \bar{x}\}$ . We also introduce the state-constrained value function  $V_-^{\text{sc}}$  and  $V_+^{\text{sc}}$  of the optimisation problems for which the state restrictions  $x(t) \in \mathcal{X}_-$  or  $x(t) \in \mathcal{X}_+$  are, respectively, required to hold for all  $t$ , and where the respective dynamics left and right of  $\bar{x}$  are given by  $f_-$  and  $f_+$ .

Additionally, we introduce  $V_j^{\text{sc}}(\bar{x})$  for the value function of the optimisation problem with the state restriction  $x(t) = \bar{x}$  for all  $t$ , where only action schedules that satisfy  $\mu(t)f_-(\bar{x}, a(t)) + (1 - \mu(t))f_+(\bar{x}, a(t)) = 0$  for some  $\mu(t) \in [0, 1]$  are admitted. Finally for  $k \in \{-, \mathcal{J}, +\}$  we set  $V_k^{\text{sc}}(\bar{x}) = -\infty$  if there is no trajectory starting at  $\bar{x}$  that satisfies the particular state constraint.

The following result establishes necessary and sufficient conditions for the continuity of the value function at interfaces. To formulate it, we need to introduce the notions of one-sided semi-repellers and semi-attractors.

**Definition 3.** *A point  $\bar{x} \in \mathcal{J}$  is a left semi-repeller if  $f_-(\bar{x}, a) \leq 0$  for all  $a \in \mathcal{A}$ , and a right semi-repeller if  $f_+(\bar{x}, a) \geq 0$  for all  $a \in \mathcal{A}$ .*

*Likewise,  $\bar{x} \in \mathcal{J}$  is a left semi-attractor if  $f_-(\bar{x}, a) \geq 0$  for all  $a \in \mathcal{A}$ , and a right semi-attractor if  $f_+(\bar{x}, a) \leq 0$  for all  $a \in \mathcal{A}$ .*

**Proposition C.1.** *Let  $\bar{x} \in \mathcal{J}$ . Then the following hold.*

- (i)  $V(\bar{x}) = \max\{V_-^{\text{sc}}(\bar{x}), V_j^{\text{sc}}(\bar{x}), V_+^{\text{sc}}(\bar{x})\}$ .
- (ii) *If the dynamics are left (right) controllable at  $\bar{x}$ , then  $V$  is left (right) Lipschitz continuous in a left (right) neighbourhood of  $\bar{x}$ .*
- (iii) *At a right semi-attractor  $\bar{x}$ ,  $V$  is right continuous and  $V(\bar{x}) = \max\{V_-^{\text{sc}}(\bar{x}), V_j^{\text{sc}}(\bar{x})\}$ .*
- (iv) *At a left semi-attractor  $\bar{x}$ ,  $V$  is left continuous and  $V(\bar{x}) = \max\{V_j^{\text{sc}}(\bar{x}), V_+^{\text{sc}}(\bar{x})\}$ .*

(v) At a left semi-repeller  $\bar{x}$ ,  $V$  is left continuous and  $V(\bar{x}) = V_-^{\text{sc}}(\bar{x})$ .

(vi) The value function  $V$  is not continuous at  $\bar{x}$  if and only if either  $\bar{x} \in \hat{\mathcal{X}}$  is a right semi-repeller and  $V_+^{\text{sc}}(\bar{x}) < \max\{V_-^{\text{sc}}(\bar{x}), V_j^{\text{sc}}(\bar{x})\}$ , or  $\bar{x} = x_{\min}$  is a right semi-repeller and  $V_+^{\text{sc}}(\bar{x}) < \beta(\bar{x})$ , or  $\bar{x} = x_{\max}$  and  $V_-^{\text{sc}}(\bar{x}) > \beta(\bar{x})$ .

The assumption  $\frac{\partial u}{\partial x} \leq 0$  causes the asymmetry in the results. For the proof see Appendix G.1.

## D Viscosity solutions

This section proves that the value function of the optimisation problem is, in an appropriate sense, the unique viscosity solution of a Hamilton–Jacobi–Bellman (HJB) equation.

We introduce a number of relevant concepts. For  $x \in \mathcal{X} \setminus \mathcal{J}$  and  $p \in \mathbb{R}$ , the *Hamilton function* is

$$H(x, p) = \max_{a \in \mathcal{A}} [u(x, a) + pf(x, a)]. \quad (24)$$

Assumptions 1 and 2 imply that for fixed  $(x, p)$ , the function  $u(x, a) + pf(x, a)$  has a unique maximiser  $a = a^*(x, p)$  in  $\mathcal{A}$ . If  $a^*(x, p) \in \mathring{\mathcal{A}}$ , then  $\frac{\partial u}{\partial a}(x, a^*(x, p)) + p \frac{\partial f}{\partial a}(x, a^*(x, p)) = 0$ .

For  $x \in \mathcal{J}$ , we define  $H_-(x, p)$  and  $H_+(x, p)$  by respectively substituting  $f_-$  and  $f_+$  for  $f$  in (24). We introduce Hamilton functions  $H_-(x, p) = \rho\beta(x)$  for  $x \leq x_{\min}$  and  $H_+(x, p) = \rho\beta(x)$  for  $x \geq x_{\max}$ . For  $x \in \mathcal{J}$  we define the Hamilton function as

$$H(x, p) = \max\{H_-(x, p), H_+(x, p)\}.$$

For interface points  $x \in \mathcal{J}$ , we also introduce the *interface Hamilton function*

$$H^{\mathcal{J}}(x) = \max_{a \in \mathcal{A}_0} u(x, a),$$

where  $\mathcal{A}_0$  is the set of stabilising actions introduced in (15). We set  $H^{\mathcal{J}}(x) = -\infty$  if the set  $\mathcal{A}_0$  is empty.

Let  $S \subset \mathbb{R}^m$ . The *upper semi-continuous envelope*  $W^*$  of a function  $W : S \rightarrow \mathbb{R}$  is

$$W^*(x) = \limsup_{\delta \downarrow 0} \{W(z) : z \in S, |z - x| \leq \delta\} \quad \text{for all } x \in \bar{S}.$$

The *lower semi-continuous envelope*  $W_*$  of  $W$  is defined analogously, with inf replacing sup. We have for instance that  $H^*(x, p) = H(x, p)$  everywhere, while  $H_*(x, p)$  differs from  $H(x, p)$  only at  $x \in \mathcal{J}$ , in which case

$$H_*(x, p) = \min\{H_-(x, p), H_+(x, p)\}.$$

The *superdifferential*  $D^+W(x)$  of a bounded upper semicontinuous function  $W : S \rightarrow \mathbb{R}$  at a point  $x \in S$  is the set

$$D^+W(x) = \left\{ p \in \mathbb{R} : \limsup_{\substack{z \rightarrow x \\ z \in S}} \frac{W(z) - W(x) - p(z - x)}{|x - z|} \leq 0 \right\}.$$

The *subdifferential*  $D^-W(x)$  of a bounded lower semicontinuous function  $W$  on  $S$  at  $x$  is defined similarly, with sup replaced by inf and  $\leq$  by  $\geq$  (see e.g. Bardi and Capuzzo-Dolcetta, 2008, Chapter V). We have that  $p \in D^+W(x)$  if and only if there is a continuously differentiable function  $\psi$  such that  $\psi'(x) = p$  and  $W - \psi$  restricted to  $\mathcal{X}$  has a local maximum at  $x$ . An analogous characterisation exists for subdifferentials.

**Definition 4.** A function  $W : \mathcal{X} \rightarrow \mathbb{R}$  is a viscosity supersolution of the equation

$$\rho W(x) - H(x, W'(x)) = 0 \tag{25}$$

if for all  $x \in \mathcal{X}$  and all  $p \in D^-W_*(x)$  we have

$$\rho W_*(x) - H_*(x, p) \geq 0, \tag{26}$$

and if for all  $x \in \mathcal{J}$ , we have

$$\rho W(x) - H^{\mathcal{J}}(x) \geq 0. \tag{27}$$

The function  $W$  is a viscosity subsolution of (25) if for all  $x \in \mathcal{X}$  and all  $p \in D^+W^*(x)$

$$\rho W^*(x) - H^*(x, p) \leq 0. \tag{28}$$

Finally,  $W$  is a viscosity solution of (25) if it is both a supersolution and a subsolution.

Note that there is no condition analogous to (27) for subsolutions.

**Theorem D.1.** *The value function  $V$  is a viscosity solution of equation (25).*

*Proof.* To show that  $V$  is a supersolution at a point  $x_0$ , take a continuously differentiable function  $\psi$  such that  $V_*(x_0) = \psi(x_0)$  and  $V(x) \geq \psi(x)$  everywhere. Then  $p = \psi'(x_0) \in D^-V_*(x_0)$ .

Take  $a(t) = \bar{a}$  constant, and let  $(x, a)$  be a trajectory–action pair with initial state  $x_0$ . Consider first the possibility that  $x(t)$  exits from  $\mathcal{X}$  immediately for some  $\bar{a}$ . If  $x_0 = x_{\min}$ , then  $f_-(x_0, a) = f_+(x_0, a) < 0$  for some  $a$ , the function  $V$  is continuous at  $x_0$  by Proposition C.1(vi), and  $V(x_0) = \beta(x_0)$ . If  $x_0 = x_{\max}$ , then  $V_*(x_0) = \beta(x_0)$  follows directly. In both cases (26) holds.

If  $x(t)$  does not exit immediately, then for  $h > 0$  small, the state  $x(t)$  is in the interior of  $\mathcal{X}$  for  $0 < t < h$ . The dynamic programming principle implies

$$\psi(x_0) = V_*(x_0) \geq \int_0^h u(x(t), a(t)) \exp(-\rho t) dt + \psi(x(h)) \exp(-\rho h).$$

Subtracting  $\psi(x_0)$  left and right, dividing by  $h$ , and letting  $h$  tend to 0 yields

$$\begin{aligned} \rho V_*(x_0) &\geq u(x_0, \bar{a}) + \left. \frac{d}{dt} \psi(x(t)) \right|_{t=0} = u(x_0, \bar{a}) + p(\mu f_-(x_0, \bar{a}) + (1 - \mu) f_+(x_0, \bar{a})) \quad (29) \\ &= \mu [u(x_0, \bar{a}) + p f_-(x_0, \bar{a})] + (1 - \mu) [u(x_0, \bar{a}) + p f_+(x_0, \bar{a})]. \end{aligned}$$

We first note that if  $x_0$  is an interface point and  $\bar{a} \in \mathcal{A}_0$ , then  $\mu f_- + (1 - \mu) f_+ = 0$  and  $\rho V_*(x_0) \geq u(x_0, \bar{a})$  for all  $\bar{a} \in \mathcal{A}_0$ , which implies (27).

If  $x_0$  is a value discontinuity, then Proposition C.1(vi) implies that  $x_0$  is a right semi-repeller, and that for every  $\bar{a}$  the trajectory  $x(t)$  can be chosen to be increasing: hence

$$\rho V_*(x_0) \geq u(x_0, \bar{a}) + f_+(x_0, \bar{a}). \quad (30)$$

holds for every  $\bar{a}$ .

Hence we may assume henceforth that  $V$  is continuous at  $x_0$ , and that  $V_*(x_0) = V(x_0)$ . If  $x_0$  is not an interface point, then  $f_- = f_+ = f$  and for every  $\bar{a}$  the inequality (29) reduces to (30) with  $f_+$  replaced by  $f$ .

If  $f_-(x_0, a) \leq f_+(x_0, a)$  for  $a = \bar{a}$ , then this inequality is true for all  $a$ . In this situation, as  $p \leq 0$ , Equation (29) implies (30) with  $f$  replaced by  $f_+$  for all  $\bar{a}$ . Similarly, if  $f_-(x_0, \bar{a}) \geq f_+(x_0, \bar{a})$ ,

then (29) implies (30) with  $f$  replaced by  $f_-$  for all  $\bar{a}$ .

In all of these situations, maximising (30) over  $\bar{a}$  yields either  $\rho V_*(x_0) \geq H_-(x_0, p)$  or  $\rho V_*(x_0) \geq H_+(x_0, p)$ , and hence (26) holds. This proves that  $V$  is a supersolution.

The proof that  $V$  is a subsolution is analogous to the corresponding part of the proof of Theorem 2.5 of Barles et al. (2013), where it is shown, in a minimisation context, that the value function is a supersolution. This proof does neither depend on the controllability assumption made in the statement of that theorem, nor on the continuity of the value function.  $\square$

Proposition C.1 motivates the following definition.

**Definition 5.** *The class  $\mathcal{G}$  consists of functions  $W : \mathcal{X} \rightarrow \mathbb{R}$  such that*

- (i)  $W$  is decreasing;
- (ii)  $W$  is left continuous everywhere, except at  $x_{\min}$ , where it is right continuous;
- (iii)  $W$  is continuous on  $\mathcal{X} \setminus \mathcal{J}$ ;
- (iv) if  $W$  is not continuous at  $\bar{x} \in \mathcal{J}$ , then this point is either  $x_{\max}$  or a right semi-repeller.

The following result is central.

**Theorem D.2.** *The value function  $V$  is the unique viscosity solution of equation (25) in  $\mathcal{G}$ .*

This is a direct corollary of the following comparison result.

**Proposition D.1.** *Let  $v \in \mathcal{G}$  and  $w \in \mathcal{G}$  be respectively a supersolution and a subsolution of (25). Then  $v(x) \geq w(x)$  for all  $x \in \mathcal{X}$  where  $v$  and  $w$  are continuous.*

To prove this proposition, we need a number of technical results. The first relates to a subsolution version of condition (27). This subsolution version either holds, or an alternative property must be true. The proof is a straightforward adaptation of Barles et al. (2013, Theorem 3.3).

**Proposition D.2.** *Let  $w \in \mathcal{G}$  be a subsolution of (25) and  $\bar{x} \in \mathcal{J} \setminus \{x_{\min}, x_{\max}\}$ .*

*Then either of the following two statements holds.*

- A.  $\rho w(\bar{x}) - H^{\mathcal{J}}(\bar{x}) \leq 0$
- B. (i) *If  $w$  is continuous at  $\bar{x}$ , then there is a constant  $\eta > 0$ , an index  $\sigma \in \{-, +\}$ , and a sequence  $x_{0,k} \rightarrow \bar{x}$  such that for each  $k$  there is a trajectory–action pair  $(x_k, a_k)$  with  $x_k(0) = x_{0,k}$ ,  $x_k(t) \in \mathcal{X}_\sigma$  for all  $t \in [0, \eta]$  and*

$$w(x_{0,k}) \leq \int_0^\eta u(x_k(t), a_k(t)) \exp(-\rho t) dt + w(x_k(\eta)) \exp(-\rho \eta) \quad (31)$$

(ii) If  $w$  is not continuous at  $\bar{x}$ , the previous statement holds with  $\sigma = -$ .

The next result, proved in Section G.2, settles the continuous case of Proposition D.1.

**Proposition D.3.** *Let  $v \in \mathcal{G}$  be a continuous viscosity supersolution, and  $w \in \mathcal{G}$  a continuous viscosity subsolution of the HJB equation (25). Then  $v \geq w$  in  $\mathcal{X}$ .*

To prove the discontinuous case, we need a technical result. Let  $\hat{x}_\ell$ , for  $1 \leq \ell < L$ , be the right semi-repellers of the dynamics; then  $\hat{x}_\ell \in \mathcal{J}$ . Introduce the intervals  $\mathcal{X}^{(\ell)} = [\hat{x}_{L-\ell}, x_{\max}]$  as well as extensions  $v_\ell, w_\ell$  to  $\mathcal{X}^{(\ell)}$  of the respective restrictions of  $v$  and  $w$  to  $\mathcal{X}^{(\ell)} \setminus \{\hat{x}_{L-\ell}\}$  such that  $v_\ell$  and  $w_\ell$  are right continuous at the left boundary point  $\hat{x}_{L-\ell}$  of  $\mathcal{X}^{(\ell)}$ : as  $v, w \in \mathcal{G}$  these extensions are well-defined and unique.

As  $\hat{x}_\ell$  is a right semi-repeller, we can restrict the optimisation problem to  $\mathcal{X}^{(\ell)}$ :  $v_\ell$  and  $w_\ell$  are then the proper restrictions of  $v$  and  $w$ , while remaining supersolution and subsolution:

**Lemma D.1.** *Assume that  $v_\ell$  and  $w_\ell$  are, for  $1 \leq \ell < L$ , respectively a supersolution and a subsolution of (25) on the set  $\mathcal{X}^{(\ell)} \setminus \{\hat{x}_{L-\ell}\}$ . Then  $v_\ell$  and  $w_\ell$  are also, respectively, a supersolution and a subsolution of (25) on  $\mathcal{X}^{(\ell)}$ .*

The proof is a viscosity test function argument, given in Section G.3.

*Proof of Proposition D.1.* The proof proceeds by induction. The induction hypothesis is that the inequality  $v_\ell(x) \geq w_\ell(x)$  holds for all  $x \in \mathcal{X}^{(\ell)}$  where  $v_\ell$  and  $w_\ell$  are continuous. Lemma D.1 implies that  $v_\ell$  and  $w_\ell$  are respectively a supersolution and a subsolution of (25) on  $\mathcal{X}^{(\ell)}$ .

The functions  $v_1$  and  $w_1$  are continuous on  $\mathcal{X}^{(1)}$ . Proposition D.3 then implies the induction hypothesis for  $\ell = 1$ .

Assuming that the hypothesis is true for  $\ell \geq 1$ , consider  $\Delta_{\ell+1} = w_{\ell+1} - v_{\ell+1}$ . By hypothesis,  $\Delta_{\ell+1} \leq 0$  on  $\mathcal{X}^{(\ell)} \setminus \{\hat{x}_{L-\ell}\}$ . If  $\Delta_{\ell+1} \leq 0$  on  $\mathcal{X}^{(\ell+1)}$ , there is nothing left to prove. If not, then  $\Delta_{\ell+1}$  takes a positive maximum  $M = \Delta_{\ell+1}(\bar{x}) > 0$  at a point  $\bar{x} \in [\hat{x}_{L-(\ell+1)}, \hat{x}_{L-\ell}]$ , as the interval is compact and  $\Delta_{\ell+1}$  restricted to this interval is continuous.

If  $\bar{x} \neq \hat{x}_{L-\ell}$ , the same arguments used in the proof of Proposition D.3 can be used to derive a contradiction to the statement that  $M > 0$ . Hence we may assume that  $\bar{x} = \hat{x}_{L-\ell}$ . As this is an interface point, Proposition D.2, on which Proposition D.3 is based, applies.

If  $v_{\ell+1}$  and  $w_{\ell+1}$  are continuous at  $\bar{x}$ , the argument of Proposition D.3 for interface points again produces a contradiction.

If  $v_{\ell+1}$  is discontinuous at  $\bar{x}$ , but  $w_{\ell+1}$  is not, we have that  $M = w_{\ell+1}(\bar{x}) - v_{\ell+1}(\bar{x}) > 0$  and, since  $\Delta_{\ell+1} \leq 0$  if  $x > \bar{x}$ , also that  $\lim_{x \downarrow \bar{x}} (w_{\ell+1}(x) - v_{\ell+1}(x)) \leq 0$ , which implies that  $v_{\ell+1}(\bar{x}) < \lim_{x \downarrow \bar{x}} v_{\ell+1}(\bar{x})$ . But then  $v_{\ell+1}$  cannot be an element of  $\mathcal{G}$ .

Finally, if  $w_{\ell+1}$  is discontinuous at  $\bar{x}$ , the argument of Proposition D.3 for interface points holds again, as in Alternative B the sequence elements  $x_k$  satisfy  $x_k \leq \bar{x}$  for all  $k$ .

We conclude that the induction hypothesis also holds for  $\ell + 1$ , and therefore for all  $1 \leq \ell < L$ .

This completes the proof.  $\square$

*Proof of Theorem D.2.* By Theorem D.1,  $V$  is a viscosity solution to (25), and according to Proposition C.1, it is in  $\mathcal{G}$ .

Assume  $W \in \mathcal{G}$  is another viscosity solution to (25). As  $V$  is a supersolution and  $W$  is a subsolution, by Proposition D.1 we have that  $V \geq W$  at all points in  $\mathcal{X}$  where  $V$  and  $W$  are continuous. Interchanging  $V$  and  $W$  yields also that  $V \leq W$  at all points of continuity. As  $V$  and  $W$  are either both left or both right continuous everywhere, this shows that  $V = W$  for all  $x \in \mathcal{X}$ .  $\square$

## E Existence of the best response map

Let  $\mathcal{J}$  be a given collection of jump points and  $\mathcal{S}_{\mathcal{J},-i}$  the space of profiles  $\phi_{-i}$  with jump points in  $\mathcal{J}$ . This section shows that the *best response map*  $\phi_i = \mathcal{B}_i(\phi_{-i})$  is well-defined on  $\mathcal{S}_{\mathcal{J},-i}$  in the complement of a set  $\mathcal{E}_{\mathcal{J},-i}$ , and it shows that the latter set is shy, that is, small in a topological as well as a measure-theoretical sense. The strategy  $\phi_i$  is the Markovian best response of player  $i$  to the strategy profile  $\phi_{-i}$  of the other players: given the profile  $\phi_{-i}$ , the strategy  $\phi_i$  is the feedback strategy such that for every initial state there is a trajectory–action pair  $(x(t), a(t))$ , with  $a(t)$  induced by  $\phi_i(x(t))$ , maximising the total payoff.

The construction starts with a general profile  $\phi_{-i} \in \mathcal{S}_{-i}$ . First the value function of player  $i$  is shown to be differentiable on a dense set of points. Then the regularity is improved: using the state–costate equations, the value function is shown to be real analytic on non-constant optimal state orbits. Compact intervals not containing interface points intersect only finitely many of these orbits. Restricted to such a compact interval, the set of points at which the value function

is not real analytic is shown to be discrete, hence finite. Consequently the associated strategy can fail to be in  $\mathcal{S}_i$  only if the points of non-analyticity accumulate on an interface point: the final step of the proof shows that the set of profiles  $\phi_{-i}$  for which this occurs is shy.

## E.1 Notations

We introduce a number of auxiliary quantities: the functions

$$p_{\min}(x) = -\frac{\partial u}{\partial a}(x, a_{\min}), \quad p_{\max}(x) = -\frac{\partial u}{\partial a}(x, a_{\max});$$

the corner sets  $\mathcal{P}_{\min} = \{(x, p) : p \leq p_{\min}(x)\}$ ,  $\mathcal{P}_{\max} = \{(x, p) : p \geq p_{\max}(x)\}$ ; and the interior set  $\mathcal{P}_{\text{int}} = \{(x, p) : p_{\min}(x) < p < p_{\max}(x)\}$ . With these definitions, the maximiser  $a = a^*(x, p)$  equals  $a_{\min}$  if  $(x, p) \in \mathcal{P}_{\min}$ ,  $a_{\max}$  if  $(x, p) \in \mathcal{P}_{\max}$ , and it takes a value in  $\mathring{\mathcal{A}}$  if  $(x, p) \in \mathcal{P}_{\text{int}}$ .

The boundaries of the sets  $\mathcal{P}_{\min}$ ,  $\mathcal{P}_{\max}$  and  $\mathcal{P}_{\text{int}}$  are the *switching manifolds*

$$\mathcal{S}_{\min} = \{(x, p) : p = p_{\min}(x)\}, \quad \mathcal{S}_{\max} = \{(x, p) : p = p_{\max}(x)\}, \quad \mathcal{S} = \mathcal{S}_{\min} \cup \mathcal{S}_{\max}.$$

## E.2 Differentiability of the value function in a dense set

As the value function  $V$  is in the class  $\mathcal{G}$ , it is continuous on  $\mathcal{X} \setminus \mathcal{J}$ . We partition this set into three sets  $\mathcal{X}_L(0)$ ,  $\mathcal{X}_M$  and  $\mathcal{X}_N$ , based on whether  $H(x, p)$  has a single minimiser with respect to  $p$  or not, and, if not, whether  $H(x, p) = \rho V(x)$  is locally solvable for  $p$  or not.

**Definition 6.** Let  $c \geq 0$  and  $k \in \{0, 1, 2, \dots\}$ . Introduce the sets

- (i)  $\mathcal{D}_k = \{x \in \mathcal{X} \setminus \mathcal{J} : V \text{ is } k \text{ times differentiable at } x\}$ ;
- (ii)  $\mathcal{X}_L(c) = \left\{x \in \mathcal{X} \setminus \mathcal{J} : \frac{\partial H}{\partial p} < -c \text{ if } (x, p) \in \mathcal{P}_{\min} \text{ and } \frac{\partial H}{\partial p} > c \text{ if } (x, p) \in \mathcal{P}_{\max}\right\}$ ;
- (iii)  $\mathcal{X}_M = \left\{x \in \mathcal{X} \setminus \mathcal{J} : \frac{\partial H}{\partial p}(x, p) \neq 0 \text{ for all } p\right\}$ ;
- (iv)  $\mathcal{X}_N = \left\{x \in \mathcal{X} \setminus \mathcal{J} : \frac{\partial H}{\partial p}(x, p) = 0 \text{ for more than one value of } p\right\}$ ;

As  $\frac{\partial H}{\partial p}(x, p) = 0$  for precisely one value of  $p$  if and only if  $x \in \mathcal{X}_L(0)$ , the sets  $\mathcal{X}_L(0)$ ,  $\mathcal{X}_M$  and  $\mathcal{X}_N$  are mutually disjoint and satisfy  $\mathcal{X}_L(0) \cup \mathcal{X}_M \cup \mathcal{X}_N = \mathcal{X} \setminus \mathcal{J}$ .

**Proposition E.1.** The value function  $V$  is differentiable almost everywhere on  $\mathcal{X}_L(0)$ , and it is real analytic on  $\mathcal{X}_M \cup \mathring{\mathcal{X}}_N$ .

*Proof.* Take  $c > 0$ ,  $x \in \mathcal{X}_L(c)$  and  $p \in D^-V(x)$ . The value function  $V$  is continuous in  $\mathcal{X} \setminus \mathcal{J}$  and a supersolution; hence we have for every  $a \in \mathcal{A}$  that

$$\rho V(x) \geq H(x, p) \geq u(x, a) + pf(x, a).$$

As  $\|V\|_\infty \leq \|u\|_\infty/\rho$ , this implies

$$pf(x, a) \leq \rho V(x) - u(x, a) \leq 2\|u\|_\infty.$$

Taking  $a = a_{\max}$ , and using that  $f(x, a_{\max}) = \frac{\partial H}{\partial p}(x, p_{\max}(x)) > c$ , we find  $p < 2\|u\|_\infty/c$ . Similarly, for  $a = a_{\min}$ , we have  $f(x, a_{\min}) = \frac{\partial H}{\partial p}(x, p_{\min}(x)) < -c$  and  $p > -2\|u\|_\infty/c$ . We conclude that if  $p \in D^-V(x)$ , then  $|p| < 2\|u\|_\infty/c$ .

By remark II.5.16 of Bardi and Capuzzo-Dolcetta (2008) the function  $V$  is Lipschitz continuous on  $\mathcal{X}_L(c)$  with Lipschitz constant  $2\|u\|_\infty/c$ ; Rademacher's theorem (Clarke et al., 1998, Chapter 3, Corollary 4.19) subsequently ensures almost everywhere differentiability of  $V$  on  $\mathcal{X}_L(c)$ , and hence on  $\mathcal{X}_L(0) = \bigcup_{c>0} \mathcal{X}_L(c)$ , proving the first part of the statement.

Next, consider  $x \in \mathcal{X}_M$ . By the implicit function theorem the solution  $p = \kappa(x, w)$  of  $w = H(x, p)$  is locally real analytic.

Assume that  $\frac{\partial H}{\partial p}(x, \kappa(x, w)) > 0$ , the other situation being similar. As  $x \in \mathcal{X}_M$ , this implies that  $\frac{\partial H}{\partial p}(x, p) > 0$  for all  $p$ , and  $H(x, p)$  is increasing in  $p$ . If  $p \in D^+V(x)$ , the subsolution property of  $V$  implies

$$H(x, \kappa(x, \rho V(x))) = \rho V(x) \leq H(x, p).$$

We obtain that  $\kappa(x, \rho V(x)) \leq p$ , that is,  $\kappa(x, \rho V(x)) - V'(x) \leq 0$  in the viscosity sense. Similarly, the supersolution property implies for  $p \in D^-V(x)$  that  $\kappa(x, \rho V(x)) - V'(x) \geq 0$  in the viscosity sense. Again using Bardi and Capuzzo-Dolcetta (2008, Remark II.5.16), it follows that  $V(x)$  is a classical solution of  $V'(x) = \kappa(x, \rho V(x))$ . As  $\kappa$  is real analytic, so is  $V$ . We conclude that  $V$  is a real analytic solution of  $\rho V(x) = H(x, V'(x))$  in  $\mathcal{X}_M$ .

Finally, for  $x \in \mathring{\mathcal{X}}_N$ ,  $\rho V(x) = u(x, a_{\min})$  or  $\rho V(x) = u(x, a_{\max})$ , and  $V$  is again real analytic. This proves the second part of the proposition.  $\square$

As a corollary of this result, we obtain

**Proposition E.2.** *The set  $\mathcal{D}_1$  is dense in  $\mathcal{X}$ .*

*Proof.* If this were not the case, there is a boundary point  $\bar{x}$  of  $\mathcal{X}_N$  such that  $V$  is not differentiable for any point in an open interval  $I$  with positive length containing  $\bar{x}$ . As  $\bar{x}$  is a boundary point, the interval  $I$  contains points that are not in  $\mathcal{X}_N$ . Hence the intersection  $I \cap (\mathcal{X}_L(0) \cup \mathcal{X}_M)$  is nonempty. But this intersection is open, and therefore it contains a positive measure subset of points where  $V$  is differentiable, which is a contradiction.  $\square$

### E.3 Canonical trajectories

To extend the domain of differentiability of the value function, we show that differentiability is carried forward along optimal orbits by the costate dynamics. This is closely related to the Pontryagin Maximum Principle, the difference being that we here have initial values rather than terminal values for the costate equation. The result is an adaptation of Cannarsa and Frankowska (1991, Theorem 3.3) to the present context.

**Proposition E.3.** *Let  $(x^*, a^*)$  be an optimal trajectory–action pair with initial point  $x_0 \in \mathcal{D}_1$  and let  $T \geq 0$  be such that  $x^*(t) \in \mathcal{X} \setminus \mathcal{J}$  for all  $0 < t < T$ .*

*Let moreover  $p^*$  be the solution of*

$$\dot{p}(t) = \rho p(t) - \frac{\partial u}{\partial x}(x^*(t), a^*(t)) - p(t) \frac{\partial f}{\partial x}(x^*(t), a^*(t)), \quad p(0) = V'(x_0). \quad (32)$$

*Then for every  $0 < t < T$  we have  $x^*(t) \in \mathcal{D}_1$ ,  $V'(x^*(t)) = p^*(t)$ , and  $a^*(t) = a^*(x^*(t), p^*(t))$ .*

Proposition E.3 can be expressed in the more familiar form that, except at jump points, an optimal trajectory necessarily satisfies the *canonical equations*

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t)), \quad \dot{p}(t) = \rho p(t) - \frac{\partial H}{\partial x}(x(t), p(t)),$$

with  $x(0) = x_0$ ,  $p(0) = V'(x_0)$ . This motivates the following definition:

**Definition 7.** *The canonical vector field  $X : \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}^2$  is given as  $X(x, p) = (\frac{\partial H}{\partial p}(x, p), \rho p - \frac{\partial H}{\partial x}(x, p))$ . A canonical trajectory is a trajectory of  $X$ . An optimal canonical trajectory is a canonical trajectory  $(x, p)$  such that  $x$  is an optimal trajectory.*

The next proposition, whose proof is left to the reader, states familiar properties of non-constant optimal trajectories: only at their initial states or at jump points the value function may be non-differentiable; they are monotone; and they correspond to optimal canonical trajectories, which moreover either converge to steady states of  $X$ , or reach a jump point in finite time.

**Proposition E.4.** *Let  $(x^*, a^*)$  be a non-constant optimal trajectory–action pair, and let  $T = \inf \{t > 0 : x^*(t) \notin \mathcal{J}\}$ . Then*

- (i)  $x^*(t) \in \mathcal{D}_1$  for all  $0 < t < T$ ;
- (ii) there is a unique initial point  $(x_0, p_0)$  such that the canonical trajectory  $(x, p)$  through this point satisfies  $x^*(t) = x(t)$  and  $V'(x^*(t)) = p(t)$  for all  $0 < t < T$ ;
- (iii) either  $\dot{x}(t) > 0$  for all  $0 \leq t < T$ , or  $\dot{x}(t) < 0$  for all  $0 \leq t < T$ ;
- (iv) If  $T = \infty$ , there is a steady state  $(\bar{x}, \bar{p})$  such that  $(x(t), p(t)) \rightarrow (\bar{x}, \bar{p})$  as  $t \rightarrow \infty$ .

## E.4 Markovian best responses

This section constructs the piecewise real analytic Markovian best response strategy.

**Definition 8.** *An optimal orbit is a maximal interval  $I \subset \mathcal{X} \setminus \mathcal{J}$  such that there are  $T \in [0, \infty]$  and  $T_0 \in \{-\infty, 0\}$ , a time interval  $\mathcal{T} = [0, T)$  or  $\mathcal{T} = (T_0, T)$ , and a state trajectory  $x : \mathcal{T} \rightarrow I$  with the property that for every  $x_0 \in I$  there is  $\tau \in \mathcal{T}$  such that  $x_\tau(t) = x(\tau + t)$  satisfies  $x_\tau(0) = x_0$  and  $x_\tau$  is an optimal trajectory. If the interval  $I$  consists of a single point, it is an optimal steady state; if  $I$  has positive length, it is an optimal non-constant orbit.*

The intervals  $[0, T)$  and  $(-\infty, T)$  are needed to cater for, respectively, optimal orbits starting at indifference points and optimal orbits tending to steady states as time tends to  $-\infty$ .

The next two results, proved in Section G.4, give the structure of the set of optimal non-constant orbits: there are at most countably many of these orbits, and they can only accumulate on the set  $\mathcal{J}$  of jump points. Moreover, restricted to the interior of a non-constant orbit, a best response exists and is real analytic.

**Proposition E.5.** *A compact interval  $C \subset \mathcal{X} \setminus \mathcal{J}$  intersects finitely many optimal non-constant orbits.*

For the remainder of the section, we introduce for an interval  $I \subset \mathcal{X} \setminus \mathcal{J}$  the notation  $\mathcal{S}(I)$  for the strategy space of player  $i$ 's strategies that are defined on  $I$ . We also introduce a switching point as the crossing of a switching manifold by a trajectory.

**Definition 9.** A canonical trajectory  $(x, p)$  passes at time  $t_0 \geq 0$  through a switching point  $(x(t_0), p(t_0)) \in \mathcal{S}$ , if in every neighbourhood of  $t_0$  there are times  $t_1, t_2$  with  $(x(t_1), p(t_1)) \in \mathcal{P}_{int}$  and  $(x(t_2), p(t_2)) \in \mathcal{P}_{min} \cup \mathcal{P}_{max}$ . Switching points of optimal state trajectories are defined as switching points of their associated canonical trajectories.

**Proposition E.6.** Let  $C \subset \mathcal{X} \setminus \mathcal{J}$  be a compact interval and  $I$  a non-constant optimal orbit. Then  $V$  is real analytic, except at finitely many points, on the interior of  $C \cap I$ , and the function  $\phi(x) = a^*(x, V'(x))$  is in  $\mathcal{S}(C)$ .

By piecing together the results for constant and non-constant optimal orbits, we obtain the uniqueness of Markovian best responses.

**Proposition E.7.** Let  $C \subset \mathcal{X} \setminus \mathcal{J}$  be a compact interval. Then  $V$  is piecewise real analytic on  $C$ , and the function given as  $\phi(x) = a^*(x, V'(x))$  for all  $x \in \mathcal{D}_1 \cap C$  can be extended to a unique strategy  $\phi \in \mathcal{S}(C)$ . If a Markovian best response to  $\Phi$  exists, its restriction to  $C$  equals  $\phi$ . Any point of non-analyticity  $\bar{x} \notin \mathcal{J}$  of  $\phi$  is contained in the closure of some non-constant orbit.

*Proof.* By Proposition E.5, the interval  $C$  intersects only finitely many non-constant optimal orbits  $I_1, \dots, I_m$ . The complement of the intersection consists of finitely many nonintersecting compact intervals  $N_1, \dots, N_n$  that contain only optimal steady states.

If  $x \in \mathring{N}_k$ , then  $V(x) = u(x, \bar{a}(x))/\rho$ , where  $a = \bar{a}(x)$  solves  $a + \Phi(x) + g(x) = 0$ . The function  $\bar{a}(x)$ , and hence  $V(x)$  and  $\phi(x) = a^*(x, V'(x)) = \bar{a}(x)$ , are real analytic in this set. Hence points of non-analyticity are in the closure of some non-constant optimal orbit.

Proposition E.6 implies that  $V$  is real analytic on the interior of  $C_k = C \cap I_k$  and  $\phi \in \mathcal{S}(C_k)$  for each  $k$ . We therefore conclude that  $V$  is piecewise real analytic on  $C$  and  $\phi \in \mathcal{S}(C)$ .

It is clear that  $\phi$  is uniquely determined: it remains to show that it has the best response property. Let  $V$  be differentiable at a point  $x_0 \in \mathcal{X} \setminus \mathcal{J}$ , and let  $(x^*, a^*)$  be the optimal trajectory–action pair with initial point  $x_0$ . Then Proposition E.3 implies the best response property

$$a^*(t) = a^*(x^*(t), V'(x^*(t))) = \phi(x^*(t))$$

for all  $t \geq 0$ . □

The next result shows that at a strong push-push point, the value function is differentiable and the best response  $\phi(x)$  is continuous.

**Proposition E.8.** Let  $\bar{x} \in \mathcal{J}$ , and let  $I$  be an open interval containing  $\bar{x}$  such that  $V$  is differentiable on  $I \setminus \{\bar{x}\}$ . Set  $f^\phi(x) = \phi(x) + \Phi(x) + h(x)$ . If  $f_-^\phi(\bar{x}) > 0 > f_+^\phi(\bar{x})$ , then  $V$  is continuously differentiable at  $\bar{x}$ .

*Proof.* We note that as  $\bar{x}$  is not a right repeller,  $V$  is continuous at  $\bar{x}$ .

Let  $\mu \in (0, 1)$  be such that  $\mu f_-^\phi(\bar{x}) + (1 - \mu)f_+^\phi(\bar{x}) = 0$ , and set  $p_- = V'_-(\bar{x})$  and  $p_+ = V'_+(\bar{x})$ . Then  $\rho V(\bar{x}) = \mu u(\bar{x}, \phi_-(\bar{x})) + (1 - \mu)u(\bar{x}, \phi_+(\bar{x}))$ , by Equation (13). We also have  $\rho V(\bar{x}) = H_-(\bar{x}, p_-) = H_+(\bar{x}, p_+)$ . Combining these equalities, we see that

$$\begin{aligned} 0 &= \mu H_-(\bar{x}, p_-) + (1 - \mu)H_+(\bar{x}, p_+) - ((1 - \mu)u(\bar{x}, \phi_+(\bar{x})) + \mu u(\bar{x}, \phi_-(\bar{x}))) \\ &= \mu p_- f_-^\phi(\bar{x}) + (1 - \mu)p_+ f_+^\phi(\bar{x}) = \mu(p_- - p_+)f_-^\phi(\bar{x}). \end{aligned}$$

As  $\mu \neq 0$  and  $f_-^\phi(\bar{x}) \neq 0$ , we infer that  $p_- = p_+$ ,  $V(x)$  is continuously differentiable at  $\bar{x}$ , and  $\phi(x) = a^*(x, V'(x))$  is continuous at  $\bar{x}$ .  $\square$

We need a technical result to dispose of a boundary case.

**Proposition E.9.** A non-constant optimal canonical trajectory  $(x, p)$  such that  $p(t_0) = 0$  for some  $t_0 \geq 0$  does not pass through switching points.

*Proof.* As  $V'(x(t)) = p(t)$  for  $t > 0$  and  $V$  is decreasing, we have  $p(t) \leq 0$  for all  $t$ . Solving (32) for  $p(t_0) = 0$  yields

$$p(t) = - \int_{t_0}^t \frac{\partial u}{\partial x}(x(s), a^*(x(s), p(s))) \exp \left( \int_s^t \left( \rho - \frac{\partial f}{\partial x}(x(r), a^*(x(r), p(r))) \right) dr \right) ds.$$

Assumption 2 implies  $p(t) \geq 0$  for  $t \geq t_0$ . Hence  $p(t) = 0$  for all  $t \geq t_0$ , and, by real analyticity, for all  $t \geq 0$ .

Consequently, for all  $t \geq 0$ , the value function  $V(x(t))$  is constant, and, by the Hamilton–Jacobi equation,  $V(x(t)) = u(x(t), a^*(x(t), p(t)))/\rho$ . If the trajectory passes through switching points, there is a time interval  $I$  of nonzero length such that  $(x(t), p(t)) \in \mathcal{P}_{\min} \cup \mathcal{P}_{\max}$ . Assume for definiteness that  $(x(t), p(t)) \in \mathcal{P}_{\max}$  if  $t \in I$ . Then  $a^*(x(t), p(t)) = a_{\max}$  and  $V(x(t)) = u(x(t), a_{\max})/\rho = V(x(0))$  for all  $t \in I$ . By real analyticity, it follows that, first,  $V(x(t)) = u(x(t), a_{\max})/\rho$ , second,  $a^*(x(t), p(t)) = a_{\max}$ , and third,  $(x(t), p(t)) \in \mathcal{P}_{\max}$ , for all  $t \geq 0$ . We conclude that the orbit does not pass through any switching points.  $\square$

For  $z \in \mathcal{S}$ , let  $n(z)$  be the unit normal vector to  $\mathcal{S}$  pointing out of  $\mathcal{P}_{\text{int}}$  at  $z$ , and let

$$\omega_b(x) = n(x, p_b(x)) \cdot X(x, p_b(x)) \quad (33)$$

for  $b \in \{\min, \max\}$ . If  $\omega_b(x) = 0$ , then the canonical vector field is tangent to  $\mathcal{S}_b$ , and we call  $(x, p_b(x))$  a *switching tangency*.

*Proof of Theorem 1.* Take  $\phi_{-i} \in \mathcal{S}_{\mathcal{J}, -i}$ . Let  $\phi_i : \mathcal{X} \setminus \mathcal{J} \rightarrow \mathbb{R}$  be the unique candidate best response to  $\phi_{-i}$ , given by Proposition E.7, which is defined on compact subsets of  $\mathcal{X} \setminus \mathcal{J}$ . It remains to show that  $\phi_i \notin \mathcal{S}_i$  can only occur for a shy set  $\mathcal{E}_{\mathcal{J}, -i}$  of profiles  $\phi_{-i}$ .

If  $\phi_i$  has infinitely many points of non-analyticity, by Propositions E.6 and E.7 they have to be either endpoints of non-constant optimal orbits or switching points. By Proposition G.1 every compact subinterval of  $\mathcal{X} \setminus \mathcal{J}$  contains only finitely many of these: hence points of non-analyticity have to accumulate on the set  $\mathcal{J}$ . Switching points have to accumulate on switching tangencies. As between two endpoints of a non-constant orbit there is a steady state, at an accumulation point of endpoints the canonical vector field has to vanish. We show that the set  $\mathcal{E}_{\mathcal{J}, -i} \subset \mathcal{S}_{\mathcal{J}, -i}$  of profiles  $\phi_{-i}$  such that  $\phi_i$  has one of these properties is shy.

We introduce the function  $w(x) = \prod_{k=1}^J (x - \bar{x}_k)$ , and the component functions  $\psi_j^{(1)}(x) = 1$  and  $\psi_j^{(2)}(x) = w(x)$ . Denote by  $\mathcal{L}$  the two-dimensional space spanned by the vector-valued functions  $\psi_{-i}^{(1)}(x)$  and  $\psi_{-i}^{(2)}(x)$ , and equip  $\mathcal{L}$  with the natural Lebesgue measure. This measure takes finite values on compact subsets of  $\mathcal{L}$ . Finally, for  $\phi_{-i} \in \mathcal{S}_{\mathcal{J}, -i}$ , we declare  $X_{\phi_{-i}}$  to be the canonical vector field on  $\mathcal{X} \times \mathbb{R}$  associated to  $\phi_{-i}$ .

Let  $\mathcal{M}_{0,-} = \{\phi_{-i} : X_{\phi_{-i}, -}(x, p) = 0 \text{ for some } x \in \mathcal{J}, p \in \mathbb{R}\}$ , and let  $\mathcal{M}_{0,+}$  be defined analogously. If the best response  $\phi_i$  to  $\phi_{-i}$  has an accumulation point of non-switching non-analytic points, then  $\phi_{-i}$  is in  $\mathcal{M}_{0,-} \cup \mathcal{M}_{0,+}$ . Similarly, let  $b \in \{\min, \max\}$ ,  $\sigma \in \{-, +\}$ , and  $n_b(x)$  the normal vector to  $\mathcal{S}_b$  at  $(x, p_b(x))$  pointing out of  $\mathcal{P}_{\text{int}}$ . If for  $\phi_{-i}$  a point  $\bar{x} \in \mathcal{J}$  is an accumulation point of switching points, it is a switching tangency and, by Proposition E.9, we have  $p_{b,\sigma}(\bar{x}) \neq 0$ . That is,  $\phi_{-i} \in \mathcal{M}_{b,\sigma}$  where

$$\mathcal{M}_{b,\sigma} = \{\phi_{-i} : \omega_{b,\sigma}(\bar{x}; \phi_{-i}) = 0 \text{ and } p_{b,\sigma}(\bar{x}; \phi_{-i}) \neq 0 \text{ for some } \bar{x} \in \mathcal{J}\}.$$

Summarising, the set of profiles  $\mathcal{E}_{\mathcal{J}, -i}$  whose best response have infinitely many points of non-analyticity satisfies  $\mathcal{E}_{\mathcal{J}, -i} \subset \mathcal{M} \equiv \bigcup_{\sigma \in \{-, +\}} \mathcal{M}_{0,\sigma} \cup \mathcal{M}_{\min,\sigma} \cup \mathcal{M}_{\max,\sigma}$ .

We prove shyness of  $\mathcal{M}_{\max,+}$ , the arguments for the other sets being similar. We have to show that the intersection  $(\mathcal{M}_{\max,+} - \phi_{-i}) \cap \mathcal{L}$  has measure zero in  $\mathcal{L}$  for every  $\phi_{-i} \in \mathcal{F}_j^{N-1}$ . Elements of this intersection satisfy  $\hat{\phi}_{-i} - \phi_{-i} = \lambda_1 \psi_{-i}^{(1)} + \lambda_2 \psi_{-i}^{(2)}$  for  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $\hat{\phi}_{-i} \in \mathcal{M}_{\max,+}$ .

Introducing  $\hat{\Phi}_+(x) = \sum_{j \neq i} \hat{\phi}_{j,+}(x)$ , the condition  $\hat{\phi}_{-i} \in \mathcal{M}_{\max,+}$  translates to

$$0 = \omega_{\max,+}(\bar{x}; \hat{\phi}_{-i}) = n_{\max,1,+}(\bar{x})(a^*(\bar{x}, p_{\max,+}(\bar{x})) + \hat{\Phi}_+(\bar{x}) + h(\bar{x})) \quad (34)$$

$$+ n_{\max,2,+}(\bar{x}) \left( p_{\max,+}(\bar{x})(\rho - \hat{\Phi}'_+(\bar{x}) - h'(\bar{x})) - \frac{\partial u}{\partial x}(\bar{x}, a^*(\bar{x}, p_{\max,+}(\bar{x}))) \right)$$

for some  $\bar{x} \in \mathcal{J}$ . We have  $(\psi_j^{(1)})'(\bar{x}) = 0$ ,  $\psi_j^{(2)}(\bar{x}) = 0$  and  $(\psi_j^{(2)})'(\bar{x}) = w'(\bar{x}) \neq 0$ . Recalling that  $N$  is the number of players, Equation (34) reduces to

$$0 = n_{\max,+}(\bar{x}) \cdot X_{\phi_{-i}} + (N - 1) (n_{\max,1,+}(\bar{x})\lambda_1 - n_{\max,2,+}(\bar{x})p_{\max,+}(\bar{x})w'(\bar{x})\lambda_2). \quad (35)$$

As  $n_{\max,2,+}(\bar{x})$ ,  $p_{\max,+}(\bar{x})$  and  $w'(\bar{x})$  are nonzero, Equation (35) defines a line in  $\mathcal{L}$ . Hence there are at most  $J - 1$  lines in  $\mathcal{L}$ , one for every  $\bar{x}_j \in \mathcal{J}$  except  $\bar{x}_J$ , covering the set  $(\mathcal{M}_{\max,+} - \phi_{-i}) \cap \mathcal{L}$ ; this shows that the intersection has measure zero. We conclude that  $\mathcal{M}_{\max,+}$  is shy.

Unions and subsets of shy sets are shy (Hunt et al., 1992). Hence  $\mathcal{M}$  and  $\mathcal{E}_{\mathcal{J},-i} \subset \mathcal{M}$  are shy. The complement of a shy set is dense. As  $\mathcal{M}$  is closed, it follows that its complement is open, and that  $\mathcal{M}$ , and therefore  $\mathcal{E}_{\mathcal{J},-i}$ , are nowhere dense.

Take  $\phi_{-i} \in \mathcal{S}_{\mathcal{J},-i} \setminus \mathcal{M}$ . Then the strategy profile  $\phi_{-i}$  has only finitely many points of non-analyticity in  $\mathcal{X} \setminus \mathcal{J}$ . It remains to show that  $\phi_i(x)$  and  $\phi'_i(x)$  have finite limits as  $x$  tends to points in  $\mathcal{J}$  either from the left or from the right.

Let  $I \subset \mathcal{X} \setminus \mathcal{J}$  be a maximal open interval on which  $\phi_i$  is real analytic such that at least one of its boundary points  $\bar{x}$  is in  $\mathcal{J}$ : assume that it is the lower boundary point, the other case being similar. Set  $\bar{p} = \lim_{x \downarrow \bar{x}} V'(x)$ . If  $(x, V'(x))$  is contained in  $\mathcal{P}_b$  for all  $x \in I$  and some  $b \in \{\min, \max\}$ , the best response  $\phi_i(x)$  equals  $a_b$  and can be extended to a continuously differentiable function on the closure of  $I$ .

If  $(x, V'(x)) \in \mathcal{P}_{\text{int}}$  for all  $x \in I$  close to  $\bar{x}$ , and if  $\bar{z} = (\bar{x}, \bar{p})$  is not a steady state, we have, since the first component of  $X_{\phi_{-i},+}(x, p)$  does not vanish at  $\bar{x}$ , that there is a continuously differentiable extension of the canonical trajectory through  $\bar{z}$  to a neighbourhood of  $\bar{z}$ , giving rise to a continuously differentiable extension of  $\phi_i$  to  $\bar{x}$ . If  $\bar{z}$  is a steady state, the graph  $(x, V'(x))$  is tangent to a—necessarily one-dimensional—eigenspace of  $DX_{\phi_{-i}}(\bar{z})$  at  $\bar{z}$  and can

be extended as a continuously differentiable function as well. This shows that  $\phi_i \in \mathcal{S}_i$ .  $\square$

**Proposition E.10.** *If the profile  $\phi_{-i}$  is such that the vector field  $X_{\phi_{-i}}$  has only finitely many steady states and switching tangencies, then there is a unique best response  $\phi_i \in \mathcal{S}_i$  to  $\phi_{-i}$ .*

## F Characterisation of the best response map

This section provides the proof of Theorem 2. In this section the index  $i$  is dropped:  $\phi$  stands for  $\phi_i$ ,  $V$  for  $V_i$ , etc. We have as before that  $g(x) = \Phi(x) + h(x) = \sum_{j \neq i} \phi_j(x) + h(x)$ . We introduce  $H^\phi(x, p) = u(x, \phi(x)) + p(\phi(x) + g(x))$  and  $H_\sigma^\phi(x, p) = u(x, \phi_\sigma(x)) + p(\phi_\sigma(x) + g_\sigma(x))$  for  $\sigma \in \{-, +\}$ .

The value  $V^\phi$  accruing to player  $i$  when playing  $\phi$  has the following properties, proved in Appendix G.5.

**Proposition F.1.** *Assume  $\phi \in \mathcal{S}_j$ . Then there is a finite subset  $\mathcal{Q} \subset \mathcal{X} \setminus \mathcal{J}$  such that the following hold.*

- (i) *The function  $V^\phi$  is continuous in  $\mathcal{X} \setminus \mathcal{J}$  and real analytic in  $\mathcal{X} \setminus (\mathcal{J} \cup \mathcal{Q})$ .*
- (ii) *If  $x \in \mathcal{X} \setminus (\mathcal{J} \cup \mathcal{Q})$ , then  $\rho V^\phi(x) = H^\phi(x, (V^\phi)'(x))$ , while if  $x \in \mathcal{Q}$ , then there is  $\sigma \in \{-, +\}$  such that  $f_\sigma^\phi(x) = 0$  and  $\rho V^\phi(x) = u_\sigma^\phi(x)$ .*
- (iii) *If  $x \in \mathcal{X}$  and  $z \uparrow x$  ( $z \downarrow x$ ), there is a value  $p_-, (p_+) \in [-\infty, 0]$ , such that  $(V^\phi)'(z) \rightarrow p_-$  ( $(V^\phi)'(z) \rightarrow p_+$ ).*

*Proof of Theorem 2.* In the first part of the proof Conditions (i)–(v) of Theorem 2 are shown to imply that  $V^\phi$  is in the class  $\mathcal{G}$  and that it is a viscosity solution of (25). Using Theorem D.2 we conclude that  $V^\phi = V$  and that  $\phi = \phi_i$  is a best response to  $\phi_{-i}$ .

The second part shows if  $\phi$  is a best response, Conditions (i)–(v) hold true.

### F.1 Notations

For a given  $x$ , we write  $p_- = V'_-(x)$  and  $p_+ = V'_+(x)$ . Proposition F.1(iii) ensures that these limits exist everywhere, if we allow them to take the values  $-\infty$ .

## F.2 Sufficiency

Assume that Conditions (i)–(v) hold. Proposition F.1(i), as well as Conditions (ii) and (iii) imply that  $V^\phi \in \mathcal{G}$ . We have to show that it is a viscosity solution of (25).

*Subdifferentials and superdifferentials.* If  $x$  is not a boundary point of  $\mathcal{X}$ ,  $V^\phi$  is continuous at  $x$  and  $p_- < p_+$ , then  $D^-V^\phi(x) = [p_-, p_+]$  and  $D^+V^\phi(x) = \emptyset$ ; similarly, if  $p_+ < p_-$ , then  $D^-V^\phi(x) = \emptyset$  and  $D^+V^\phi(x) = [p_+, p_-]$ ; finally if  $p_- = p_+ = p$ , then  $D^-V^\phi(x) = D^+V^\phi(x) = \{p\}$ . The last situation occurs if and only if  $V^\phi$  is differentiable at  $x$ .

If  $V^\phi$  is not continuous at  $x$ , Condition (ii) implies that

$$D^-(V^\phi)_*(x) = (-\infty, p_+] \quad \text{and} \quad D^+(V^\phi)^*(x) = (-\infty, p_-].$$

*Non-interface points.* If  $x \in \mathcal{X} \setminus \mathcal{J}$ , by Proposition F.1(i) the function  $V^\phi(x)$  is continuous at  $x$ .

If  $x \notin \mathcal{Q}$ , where  $\mathcal{Q}$  is given by Proposition (F.1), then  $V^\phi$  is differentiable at  $x$  and  $p = (V^\phi)'(x)$ . Condition (i) and Proposition F.1(ii) imply

$$H(x, p) = u(x, a^*(x, p)) + pf(x, a^*(x, p)) = u(x, \phi(x)) + pf(x, \phi(x)) = H^\phi(x, p). \quad (36)$$

Hence  $\rho V^\phi(x) = H(x, p)$  and  $V^\phi(x)$  is a viscosity solution at  $x$ .

If  $x \in \mathcal{Q}$ , then  $V^\phi$  is not differentiable at  $x$  and  $p_- \neq p_+$ . By taking left and right limits in (36), we find that  $\rho V^\phi(x) = H(x, p_-) = H(x, p_+)$ . Convexity of  $H(x, p)$  in  $p$  implies that if  $f_-^\phi(x) = \frac{\partial H}{\partial p}(x, p_-)$  and  $f_+^\phi(x) = \frac{\partial H}{\partial p}(x, p_+)$  are both nonzero, they cannot have the same sign.

If  $p = p_-$  minimises  $H(x, p)$ , then so does  $p = p_+$  and  $\frac{\partial H}{\partial p}(x, p) = f(x, \phi(x)) = 0$  for all  $p \in [p_-, p_+]$ . It follows that  $\rho V^\phi(x) = u(x, \phi(x)) = H(x, p)$  for all  $p$  in both  $D^-V^\phi(x)$  and  $D^+V^\phi(x)$  — one of them is empty — and  $V^\phi(x)$  is a viscosity solution at  $x$ .

If  $p = p_-$  does not minimise  $H(x, p)$ , then  $\frac{\partial H}{\partial p}(x, p_-) = f(x, \phi_-(x)) < 0 < f(x, \phi_+(x)) = \frac{\partial H}{\partial p}(x, p_+)$ , the other possibility being excluded by Condition (v). Convexity of  $H(x, p)$  in  $p$  then implies that  $p_- < p_+$  and for all  $p \in [p_-, p_+] = D^-V^\phi(x)$  we have, again by convexity, that  $\rho V^\phi(x) \geq H(x, p)$ . Hence  $V^\phi$  is a supersolution at  $x$ ; it is also a subsolution as  $D^+V^\phi(x) = \emptyset$ .

*Jump points at which  $V^\phi$  is continuous.* Consider next  $x \in \mathcal{J}$  with  $V^\phi$  continuous at  $x$ .

If  $D^-V^\phi(x) = [p_-, p_+]$  is nonempty, we have to show that

$$\rho V^\phi(x) \geq \min\{H_-(x, p), H_+(x, p)\}$$

for all  $p \in D^-V^\phi(x)$ . By continuity,  $\rho V^\phi(x) = H_-(x, p_-) = H_+(x, p_+)$ . Arguing by contradiction, assume there is a point  $\hat{p} \in (p_-, p_+)$  such that

$$H_-(x, p_-) = \rho V^\phi(x) < H_-(x, \hat{p}) \quad \text{and} \quad H_+(x, p_+) = \rho V^\phi(x) < H_+(x, \hat{p}).$$

Convexity in  $p$  implies  $\hat{f}_- \equiv \frac{\partial H_-}{\partial p}(x, \hat{p}) \geq (H_-(x, \hat{p}) - H_-(x, p_-))/(\hat{p} - p_-) > 0$  and similarly  $\hat{f}_+ \equiv \frac{\partial H_+}{\partial p}(x, \hat{p}) < 0$ . Hence there is  $0 < \mu < 1$  such that  $\mu \hat{f}_- + (1 - \mu) \hat{f}_+ = 0$ . Then

$$\begin{aligned} \rho V^\phi(x) &\geq u(x, a^*(x, \hat{p})) = u(x, a^*(x, \hat{p})) + \hat{p}(\mu \hat{f}_- + (1 - \mu) \hat{f}_+) \\ &= \mu H_-(x, \hat{p}) + (1 - \mu) H_+(x, \hat{p}) \geq \min\{H_-(x, \hat{p}), H_+(x, \hat{p})\} > \rho V^\phi(x), \end{aligned}$$

by Condition (iv), which is a contradiction, proving (26) in this situation. Equation (27) is an immediate consequence of Condition (iv).

Next, assume that  $D^+V^\phi(x)$  is nonempty. We have to show that

$$\rho V^\phi(x) \leq \max\{H_-(x, p), H_+(x, p)\} \tag{37}$$

for all  $p \in D^+V^\phi(x) = [p_+, p_-]$ . If (37) fails to hold for some  $\hat{p} \in (p_+, p_-)$ , then

$$H_-(x, p_-) = \rho V^\phi(x) > H_-(x, \hat{p}) \quad \text{and} \quad H_+(x, p_+) = \rho V^\phi(x) > H_+(x, \hat{p}).$$

In this situation convexity yields that  $f_-^\phi(x) = \frac{\partial H_-}{\partial p}(x, p_-) > 0$  and  $f_+^\phi(x) = \frac{\partial H_+}{\partial p}(x, p_+) < 0$ . By condition (v) the function  $V^\phi$  is differentiable at  $x$ , and we have therefore that  $p_- = p_+$  and  $\rho V^\phi(x) = H_-(x, p) = H_+(x, p)$  for all  $p \in D^+V^\phi(x)$ . Hence (28) holds.

*Jump points at which  $V^\phi$  is not continuous.* To show (26) at  $x \in \mathcal{J}$  if  $V^\phi$  is not continuous there, assume the existence of  $\hat{p} \in D^-(V^\phi)_*(x) = (-\infty, p_+]$  for which  $\rho(V^\phi)_*(x) < \min\{H_-(x, \hat{p}), H_+(x, \hat{p})\}$ . Then  $\hat{p} < p_+$  and  $H_+(x, p_+) = \rho(V^\phi)_*(x) < H_+(x, \hat{p})$ . Convexity implies that  $\frac{\partial H_+}{\partial p}(x, \hat{p}) < 0$ . This however contradicts Condition (iii), which states that  $f_+(x, a^*(x, p)) = \frac{\partial H_+}{\partial p}(x, p) \geq 0$  for all  $p$ .

Turning to (28), assume there is  $\hat{p} \in D^+(V^\phi)^* = (-\infty, p_-]$  for which

$$\rho(V^\phi)^*(x) > \max\{H_-(x, \hat{p}), H_+(x, \hat{p})\}.$$

Then  $\hat{p} < p_-$  and  $H_-(x, p_-) = \rho(V^\phi)^*(x) > H_-(x, \hat{p})$ . Convexity implies here that  $f_-^\phi(x) = \frac{\partial H_-}{\partial p}(x, p_-) > 0$ , again contradicting Condition (iii).

This concludes the proof of the sufficiency part.

### F.3 Necessity

To prove the necessity of Conditions (i)–(v), assume that  $\phi$  is the Markovian best response to  $\phi_{-i}$ . Then  $V^\phi = V$ , and  $V^\phi$  is the unique viscosity solution of (25).

*Maximum principle.* If  $x \in \mathcal{X} \setminus \mathcal{J}$  and  $V$  is differentiable at  $x$ , then  $D^-V(x) = D^+V(x) = \{V'(x)\}$ , and (26) and (28) imply that  $\rho V(x) = H(x, V'(x))$ . Moreover, since  $V = V^\phi$ , we also have that  $V'(x) = (V^\phi)'(x) \equiv p$  and  $H(x, p) = H^\phi(x, p)$ , which is equivalent to

$$u(x, \phi(x)) + p(\phi(x) + g(x)) = \max_a [u(x, a) + p(a + g(x))],$$

and therefore implies Condition (i).

*Monotonicity.* Condition (ii) follows from Proposition B.3.

*Value discontinuities.* To show Condition (iii), note that, since  $V \in \mathcal{G}$ , if  $V$  fails to be continuous at  $x$ , then  $x \in \mathcal{J} \setminus \{x_{\max}\}$  and  $f_+(x, a) \geq 0$  for all  $a \in \mathcal{A}$ , or  $x = x_{\max}$  and the condition on  $F_+(x)$  is void. It remains to show that  $f_-^\phi(x) = \frac{\partial H_-}{\partial p}(x, p_-) \leq 0$  if  $x \in \mathcal{J} \setminus \{x_{\min}\}$ .

Assume therefore that  $f_-^\phi(x) > 0$ . We have  $V^*(x) = \max\{V_-^{\text{sc}}(x), V_j^{\text{sc}}(x)\}$  by Proposition C.1(vi). If  $V^*(x) = V_j^{\text{sc}}(x)$ , then  $f_+^\phi(x) = 0$  and  $V_j^{\text{sc}}(x) = u(x, \phi_+(x)) \leq V_+^{\text{sc}}(x)$ , implying that  $V$  is continuous at  $x$ .

So assume that  $V^*(x) = V_-^{\text{sc}}(x)$ . As  $V_-^{\text{sc}}(x)$  is a constrained viscosity solution on  $\mathcal{X}_-$  (see Bardi and Capuzzo-Dolcetta, 2008, Definition IV.5.6), for all  $p \in D^+V^*(x) = (-\infty, p_-]$  we have that  $H_-(x, p_-) = \rho V^*(x) \leq H_-(x, p)$ , and consequently  $\frac{\partial H_-}{\partial p}(x, p_-) \leq 0$ , which had to be proved.

*Value at interface steady states.* Condition (iv) is a direct consequence of (27).

*Strong push–push steady state.* Condition (v) is a consequence of Proposition E.8.

This completes the proof of Theorem 2. □

## G Proofs

### G.1 Proof of Proposition C.1

*Proof of Proposition C.1.* Theorem 2 of Jaakkola and Wagener (2025) implies that all optimal trajectories are monotone. Hence an optimal trajectory–action pair starting at  $\bar{x}$  is either decreasing, constant, or increasing. This implies statement (i).

The proof of (ii) is standard, see e.g. Bardi and Capuzzo-Dolcetta (2008, Proposition III.2.3).

For (iii), let  $\bar{x}$  be a right semi-attractor. As  $V(x)$  is decreasing, it is right upper semi-continuous.

Proposition IV.3.4 of Bardi and Capuzzo-Dolcetta (2008) implies that the value function is also lower semi-continuous at  $\bar{x}$ . This establishes right continuity.

Finally, if there is a trajectory that starts at  $\bar{x}$  and remains in  $\mathcal{X}_+$  for all  $t \geq 0$ , it must be equal to  $x(t) = \bar{x}$ . Hence  $V_+^{\text{sc}}(\bar{x}) \leq V_j^{\text{sc}}(\bar{x})$ , which shows the second part of the statement.

To show (iv), assume  $\bar{x}$  to be a left semi-attractor. As  $V$  is decreasing, it is left lower semi-continuous at  $\bar{x}$ .

If  $f_-(\bar{x}, a_{\min}) > 0$ , then there are  $m > 0$  and  $\delta > 0$  such that  $f_-(z, a) > m$  for all  $z \in [\bar{x} - \delta, \bar{x}]$  and all  $a \in \mathcal{A}$ . Fix  $x \in [\bar{x} - \delta, \bar{x}]$ , and let  $(x, a)$  be a trajectory–action pair such that  $x(0) = x_0$  and  $V(x_0) = U(x, a)$ . Then there is  $0 < \tau < |x_0 - \bar{x}|/m$  such that  $x(t) < \bar{x}$  for all  $0 < t < \tau$  and  $x(\tau) = \bar{x}$ . This implies

$$\begin{aligned} V(x_0) - V(\bar{x}) &= \left( \int_0^\tau u(x(t), a(t)) \exp(-\rho t) dt + (\exp(-\rho\tau) - 1)V(\bar{x}) \right) \\ &\leq 2\|u\|_\infty \tau = \frac{2\|u\|_\infty}{m} |x_0 - \bar{x}|, \end{aligned}$$

which shows that  $V$  is left upper semi-continuous at  $\bar{x}$ .

If  $f_-(\bar{x}, a_{\min}) = 0$ , take  $\varepsilon > 0$  and  $\delta > 0$ , and let  $T$  be the unique solution of the equation  $\exp(-\rho T)\|u\|_\infty/\rho = \varepsilon/2$ ; this solution is positive if  $\varepsilon > 0$  is sufficiently small. Let  $L > 1$  be such that  $|f_-(x, a) - f_-(\bar{x}, a)| \leq L|x - \bar{x}|$  and  $|u(x, a) - u(\bar{x}, a_{\min})| \leq L(|x - \bar{x}| + (a - a_{\min}))$  for all  $x$  and  $a$ , and let  $\delta_1 = \exp(-LT)\delta$ .

Take  $x_0 \in (\bar{x} - \delta_1, \bar{x})$ , and let  $x$  be any state trajectory with  $x(0) = x_0$ . Set  $\tau = \inf\{t \geq 0 : x(t) \in \mathcal{X}_+\}$ , and let  $\theta = \min\{\tau, T\}$ . Using the Gronwall inequality, we have that  $-\delta < -\exp(Lt)|x_0 - \bar{x}| \leq x(t) - \bar{x} \leq 0$  for all  $0 \leq t \leq \theta$ .

Let now  $(x, a)$  be a trajectory–action pair such that  $x(0) = x_0$  and  $V(x_0) = U(x, a)$ . To obtain an estimate for the payoff on the time interval  $[0, \theta]$ , we split it in a part  $\mathcal{T}_1$  where the state moves quickly to the right, which restricts the amount of time it can spend in this set, and a part  $\mathcal{T}_2 \cup \mathcal{T}_3$  where it moves slowly to the right, or not at all, restricting the value of  $a(t)$  from above, and hence the payoff.

Take  $\eta > 0$  and form the partition  $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$  of the interval  $[0, \theta]$ , where  $\mathcal{T}_1 = \{t : \dot{x}(t) > \eta\}$ ,  $\mathcal{T}_2 = \{t : 0 \leq \dot{x}(t) \leq \eta\}$ ,  $\mathcal{T}_3 = \{t : \dot{x}(t) < 0\}$ , and  $\mathcal{T}_4 = \{t : x \text{ is not differentiable at } t\}$ . Note that  $\mathcal{T}_4$  is a set of measure zero, and that some of the sets may be empty.

Clearly

$$x(\theta) - x_0 = \int_0^\theta \dot{x}(t) dt = \int_{\mathcal{T}_1} \dot{x}(t) dt + \int_{\mathcal{T}_2} \dot{x}(t) dt + \int_{\mathcal{T}_3} \dot{x}(t) dt.$$

Since  $-\exp(L\theta)|x_0 - \bar{x}| \leq x(\theta) - \bar{x} \leq 0$ , the measure  $|\mathcal{T}_1|$  of the first partitioning set satisfies

$$\begin{aligned} \eta|\mathcal{T}_1| &\leq \int_{\mathcal{T}_1} \dot{x}(t) dt \leq \int_{\mathcal{T}_1} \dot{x}(t) dt + \int_{\mathcal{T}_2} \dot{x}(t) dt = x(\theta) - x_0 - \int_{\mathcal{T}_3} \dot{x}(t) dt \\ &\leq |x(\theta) - x_0| - \int_{\mathcal{T}_3} \dot{x}(t) dt \leq |x(\theta) - \bar{x}| + |\bar{x} - x_0| - \int_{\mathcal{T}_3} f_-(x(t), a_{\min}) dt \\ &\leq (1 + \exp(L\theta))|x_0 - \bar{x}| + \int_{\mathcal{T}_3} L|x(t) - \bar{x}| dt \\ &\leq (1 + \exp(L\theta) + L\theta \exp(L\theta)) |x_0 - \bar{x}| \equiv C_1|x_0 - \bar{x}|. \end{aligned}$$

Consequently, the integral of the discounted flow payoff evaluated over  $\mathcal{T}_1$  is bounded by

$$\int_{\mathcal{T}_1} u(x(t), a(t)) \exp(-\rho t) dt \leq \|u\|_\infty |\mathcal{T}_1| \leq \frac{C_1 \|u\|_\infty}{\eta} |x_0 - \bar{x}|.$$

To estimate the payoff evaluated over  $\mathcal{T}_2 \cup \mathcal{T}_3$ , we need an upper bound on  $a(t)$ . We have for  $t \in \mathcal{T}_2 \cup \mathcal{T}_3$  that

$$\eta \geq \dot{x}(t) = f_-(x(t), a(t)) \geq -L|x(t) - \bar{x}| + a(t) - a_{\min};$$

here we used  $f(\bar{x}, a_{\min}) = 0$ . Hence  $a(t) - a_{\min} \leq \eta + L|x(t) - \bar{x}|$  and, since  $L > 1$ ,

$$|u(x(t), a(t)) - u(\bar{x}, a_{\min})| \leq 2L^2|x(t) - \bar{x}| + L\eta.$$

This implies

$$\begin{aligned} \int_{\mathcal{J}_2 \cup \mathcal{J}_3} u(x(t), a(t)) \exp(-\rho t) dt &\leq \int_0^\theta \left( u(\bar{x}, a_{\min}) + 2L^2|x(t) - \bar{x}| + L\eta \right) \exp(-\rho t) dt \\ &\leq (1 - \exp(-\rho\theta))u(\bar{x}, a_{\min})/\rho + C_2|x_0 - \bar{x}| + LT\eta, \end{aligned}$$

where  $C_2 = 2L^2 \exp(LT)$ .

Combining these estimates yields

$$\begin{aligned} V(x) &\leq \int_{\mathcal{J}_1} + \int_{\mathcal{J}_2} + \int_{\mathcal{J}_3} u(x(t), a(t)) dt + \exp(-\rho\theta)V(x(\theta)) \\ &\leq (1 - \exp(-\rho\theta))u(\bar{x}, a_{\min})/\rho + \exp(-\rho\theta)V(x(\theta)) \\ &\quad + \frac{C_1\|u\|_\infty}{\eta}|x - \bar{x}| + C_2|x_0 - \bar{x}| + LT\eta. \end{aligned}$$

Choose  $\eta = \varepsilon/(6LT)$  and  $|x_0 - \bar{x}| < \min\{\delta_1, \varepsilon^2/(36C_1LT\|u\|_\infty), \varepsilon/(6C_2)\}$ , and recall that  $V_-^{\text{sc}}(\bar{x}) = u(\bar{x}, a_{\min})/\rho \leq V(\bar{x})$ : all this implies

$$V(x_0) \leq (1 - \exp(-\rho\theta))V(\bar{x}) + \exp(-\rho\theta)V(x(\theta)) + \varepsilon/2.$$

If  $\theta = T$ , then  $\exp(-\rho\theta)V(x(\theta)) \leq \varepsilon/2$  and  $V(x_0) \leq V(\bar{x}) + \varepsilon$ , showing that  $V$  is left upper semi-continuous at  $\bar{x}$ . If  $\theta = \tau$ , then  $V(x(\theta)) = V(\bar{x})$  and  $V(x_0) \leq V(\bar{x}) + \varepsilon/2$ , again showing that  $V$  is left upper semi-continuous at  $\bar{x}$ . As lower semi-continuity is assured, it follows that  $V$  is left continuous at  $\bar{x}$ .

As  $\bar{x}$  is a left semi-attractor, a trajectory starting at  $\bar{x}$  and remaining in  $\mathcal{X}_-$  for all  $t \geq 0$  must satisfy  $x(t) = \bar{x}$  for all  $t$ : therefore  $V_-^{\text{sc}}(\bar{x}) \leq V_j^{\text{sc}}(\bar{x})$ .

Proceeding to (v), let  $\bar{x}$  be a left semi-repeller, and let  $a = \bar{a}$  maximise  $u(\bar{x}, a)$ . Let moreover  $(x, a)$  be a trajectory–action pair such that  $x(0) = \bar{x}$ ,  $a(t) = \bar{a}$  and  $x(t) \in \mathcal{X}_-$  for all  $t$ . Then  $V_-^{\text{sc}}(\bar{x}) \geq U(x, a) \geq u(\bar{x}, \bar{a})/\rho \geq V_j^{\text{sc}}(\bar{x})$ . Moreover, for any trajectory–action pair  $(\tilde{x}, \tilde{a})$  such that  $\tilde{x}(0) = \bar{x}$  and  $\tilde{x}(t) \in \mathcal{X}_+$  for all  $t$ , we have

$$u(\bar{x}, \bar{a}) - u(\tilde{x}(t), \tilde{a}(t)) \geq u(\bar{x}, \bar{a}) - u(\bar{x}, \tilde{a}(t)) + u(\bar{x}, \tilde{a}(t)) - u(\tilde{x}(t), \tilde{a}(t)) \geq 0,$$

as  $\bar{a}$  maximises  $u(\bar{x}, \cdot)$  and as  $u(x, a)$  is decreasing in  $x$ . This implies  $V_-^{\text{sc}}(\bar{x}) \geq u(\bar{x}, \bar{a})/\rho \geq V_+^{\text{sc}}(\bar{x})$ . We conclude that  $V(\bar{x}) = V_-^{\text{sc}}(\bar{x})$ , and hence that  $V$  is left continuous at  $\bar{x}$ .

Finally, we prove (vi) The sufficiency of the condition is clear. To show necessity, we combine (ii), (iv) and (v) to infer that  $V$  is always left-continuous, except at  $x_{\min}$ . Statements (ii) and (iii) imply that it can only fail to be continuous at non-boundary points if  $\bar{x}$  is a right semi-repeller and  $V_+^{\text{sc}}(\bar{x}) < V(\bar{x})$ . It fails to be continuous at the left boundary  $\bar{x} = x_{\min}$  if  $\bar{x}$  is a right semi-repeller and  $V_+^{\text{sc}}(\bar{x}) < \beta(\bar{x})$ , and at the right boundary  $\bar{x} = x_{\max}$  if  $V_-^{\text{sc}}(\bar{x}) > \beta(\bar{x})$ .  $\square$

## G.2 Proof of Proposition D.3

We formulate superoptimality and suboptimality principles at interfaces.

**Lemma G.1.** *Let  $\mathcal{J} = \{\bar{x}_1, \dots, \bar{x}_J\}$  be such that  $\bar{x}_{j-1} < \bar{x}_j$  for all  $1 < j \leq J$ ,  $v$  a supersolution and  $w$  a subsolution of (25), such that  $v, w \in \mathcal{G}$ . Take  $1 < j \leq J$  and  $\xi_1 \in (\bar{x}_{j-1}, \bar{x}_j)$ , and denote by  $\tau_- = \inf\{t \geq 0 : x(t) \notin (\xi_1, \bar{x}_j)\}$  the exit time from  $(\xi_1, \bar{x}_j)$ . Then for all  $t \geq 0$  and all  $x_0 \in (\xi_1, \bar{x}_j)$ , we have for  $\theta_- = \min\{t, \tau_-\}$  that*

$$v(x_0) \geq \sup_{(x,a)} \left( \int_0^{\theta_-} u(x(s), a(s)) \exp(-\rho s) ds + \exp(-\rho\theta_-)v(x(\theta_-)) \right) \quad (38)$$

and

$$w(x_0) \leq \sup_{(x,a)} \left( \int_0^{\theta_-} u(x(s), a(s)) \exp(-\rho s) ds + \exp(-\rho\theta_-)w(x(\theta_-)) \right), \quad (39)$$

where the suprema are taken over trajectory–action pairs  $(x, a)$  that satisfy  $x(0) = x_0$ .

Take  $1 \leq j < J$  and  $\xi_2 \in (\bar{x}_j, \bar{x}_{j+1})$ , and set  $\tau_+ = \inf\{t \geq 0 : x(t) \notin (\bar{x}_j, \xi_2)\}$ . If  $v$  or  $w$  are, respectively, continuous at  $\bar{x}_j$ , then for all  $t \geq 0$  and  $x_0 \in (\bar{x}_j, \xi_2)$  the inequalities (38) or (39) also hold, respectively, with  $\theta_-$  replaced by  $\theta_+ = \min\{t, \tau_+\}$ .

*Proof.* As the hypotheses imply that  $v$  and  $w$  restricted to  $[\xi_1, \bar{x}_j]$  and  $[\bar{x}_j, \xi_2]$  are continuous, equation (38) is implied by Bardi and Capuzzo-Dolcetta (2008, Remark III.2.34), and (39) by Bardi and Capuzzo-Dolcetta (2008, Remark IV.3.16).  $\square$

*Proof of Proposition D.3.* The continuous function  $w - v$  takes on the compact set  $\mathcal{X}$  a maximum  $M$  at a point  $x_m$ . Assume that  $M > 0$ , as otherwise the lemma is proved.

If  $x_m$  is neither an interface point nor a boundary point of  $\mathcal{X}$ , the proof uses the classical “doubling of variables” technique, (see Bardi and Capuzzo-Dolcetta, 2008, Theorem II.3.1) to derive a contradiction.

If  $x_m \in \partial\mathcal{X}$ , say  $x_m = \bar{x}_0$ , the case  $x_m = \bar{x}_J$  being similar, then (26) and (28) imply either  $w(x_m) \leq \beta(x_m) \leq v(x_m)$ , contradicting  $M > 0$ , or that one of the following holds:  $\rho w(x_m) - H_+(x_m, p) \leq 0$  for all  $p \in D^+w(x_m)$ , or  $\rho v(x_m) - H_+(x_m, p) \geq 0$  for all  $p \in D^-w(x_m)$ . The argumentation proceeds then as in the proof of Bardi and Capuzzo-Dolcetta (2008, Theorem V.4.16).

Hence we only have to consider the situation that  $x_m$  is an interface point. According to Proposition D.2, one of two alternatives can obtain. If Alternative A is true, then we have  $\rho w(x_m) \leq H^J(x_m) \leq \rho v(x_m)$ , as the second inequality is implied by (27). This implies that  $w(x_m) - v(x_m) = M \leq 0$ , a contradiction.

Hence Alternative B holds. Let  $\{x_{0,k}\}$  and  $\{(x_k(t), a_k(t))\}$  be the sequences provided by that alternative. From (38) we obtain that for  $k$  sufficiently large

$$v(x_{k,0}) \geq \int_0^\eta u(x_k(t), a_k(t)) \exp(-\rho t) dt + v(x_k(\eta)) \exp(-\rho\eta). \quad (40)$$

Combining (31) and (40) yields

$$w(x_{0,k}) - v(x_{0,k}) \leq (w(x_k(\eta)) - v(x_k(\eta))) \exp(-\rho\eta) \leq M \exp(-\rho\eta).$$

Taking the limit  $k \rightarrow \infty$  yields then  $M \leq M \exp(-\rho\eta) < M$ , again a contradiction. We conclude that necessarily  $M \leq 0$ .  $\square$

### G.3 Proof of Lemma D.1

*Proof.* We give the proof for the subsolution case; the supersolution case is similar.

For the duration of the proof, we write  $\bar{x} = \hat{x}_{L-\ell}$ ,  $\mathcal{X} = \mathcal{X}^{(\ell)} = [\bar{x}, x_{\max}]$ , and  $w = w_\ell$ . By hypothesis, the subsolution property holds for all  $x \in \mathcal{X} \setminus \{\bar{x}\}$ . Assuming that the statement of the proposition is false, there is a  $C^1$  function  $\psi$  such that first,  $\psi(\bar{x}) = w(\bar{x})$ , second,  $w(x) - \psi(x)$  restricted to  $\mathcal{X}$  is maximal at  $\bar{x}$ , and third,

$$\rho w(\bar{x}) - H_+(\bar{x}, \psi'(\bar{x})) > 0. \quad (41)$$

Introduce  $\Delta(y) = w(\bar{x} + y) - \psi(\bar{x} + y) - y^2$ . Then  $\Delta$  is continuous for  $y \geq 0$ , maximal at  $y = 0$ , and  $\Delta(0) = 0$ . Continuity implies that for every  $n > 0$  there is  $\xi_n > 0$  such that  $\Delta(\xi_n) > -1/n$ .

On the other hand, if  $y \geq 2/\sqrt{n}$ , then  $\Delta(y) \leq -y^2 \leq -4/n$ . It follows that  $0 < \xi_n < 2/\sqrt{n}$ .

Set  $\varepsilon_n = \xi_n/n$ . The function

$$\Delta(y) - \varepsilon_n/y = w(\bar{x} + y) - (\psi(\bar{x} + y) + y^2 + \varepsilon_n/y)$$

satisfies  $\Delta(\xi_n) - \varepsilon_n/\xi_n \geq -2/n$  and  $\Delta(y) - \varepsilon_n/y \leq -4/n$  if  $y \geq 2/\sqrt{n}$ . Hence it takes its maximum at a point  $0 < y_n < 2/\sqrt{n}$ , and, setting  $x_n = \bar{x} + y_n$ , we have

$$p_n \equiv \psi'(x_n) + 2y_n - \varepsilon_n/y_n^2 \in D^+w(x_n).$$

As  $y_n$  maximises  $\Delta(y) - \varepsilon/y$ , we have, first, that  $\Delta(y_n) - \varepsilon_n/y_n \geq \Delta(\xi_n) - \varepsilon_n/\xi_n \geq -2/n$ , and, second, that  $0 < \varepsilon_n/y_n \leq \Delta(y_n) + 2/n \leq 2/n$ . Consequently, if  $n \rightarrow \infty$  we obtain

$$p_n y_n = \psi'(x_n) y_n + 2y_n^2 - \varepsilon_n/y_n \rightarrow 0. \quad (42)$$

As the point  $\bar{x}$  is a right semi-repeller, we have that  $\frac{\partial H_+}{\partial p}(\bar{x}, p) = f_+(\bar{x}, p) \geq 0$  for all  $p$ .

Writing  $a_n = a^*(x_n, p_n)$ , by the mean value theorem there are  $0 < \theta_n^{(1)}, \theta_n^{(2)} < 1$  such that

$$\begin{aligned} H_+(x_n, p_n) &= u(\bar{x}, a_n) + \frac{\partial u}{\partial x}(\bar{x} + \theta_n^{(1)} y_n, a_n) y_n + p_n \left( f_+(\bar{x}, a_n) + \frac{\partial f_+}{\partial x}(\bar{x} + \theta_n^{(2)} y_n, a_n) y_n \right) \\ &= H_+(\bar{x}, p_n) + r_n \leq H_+(\bar{x}, \psi'(x_n) + 2y_n) + r_n, \end{aligned}$$

where we have set  $r_n = \frac{\partial u}{\partial x}(\bar{x} + \theta_n^{(1)} y_n, a_n) y_n + p_n \frac{\partial f_+}{\partial x}(\bar{x} + \theta_n^{(2)} y_n, a_n) y_n$ , and where we have used that  $p_n \leq \psi'(x_n) + 2y_n$  as well as the fact that  $H_+(\bar{x}, p)$  is nondecreasing in  $p$ . As  $\frac{\partial u}{\partial x}$  and  $\frac{\partial f_+}{\partial x}$  are bounded, equation (42) implies that  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Since  $w$  is a subsolution, we have  $\rho w(x_n) \leq H_+(x_n, p_n) \leq H_+(\bar{x}, \psi'(x_n) + 2y_n) + r_n$ . Taking the limit  $n \rightarrow \infty$  then yields  $\rho w(\bar{x}) \leq H_+(\bar{x}, \psi'(\bar{x}))$ , contradicting (41).  $\square$

## G.4 Proofs of Propositions E.5 and E.6

We note an implication of real analyticity. Recall that  $\omega_b$  is defined in (33).

**Lemma G.2.** *Let  $b \in \{\min, \max\}$  and let  $C \subset \mathcal{X} \setminus \mathcal{J}$  be a compact set.*

(i) *If the cardinality of  $\{x \in C : f(x, a_b) = 0\}$  is infinite, then  $f(x, a_b) = 0$  for all  $x \in C$ .*

(ii) If the cardinality of  $\{x \in C : \omega_b(x) = 0\}$  is infinite, then  $\mathcal{S}_b$  restricted to  $C$  is invariant under  $X$ .

*Proof.* Both assertions follow from the fact that a real analytic function whose zeros have an accumulation point vanishes identically.  $\square$

The following result provides a condition for unicity and analyticity of the centre manifold.

**Theorem G.1** (Aulbach (1986)). *Let  $\bar{\zeta}$  be a steady state of a real analytic vector field  $Y : N \rightarrow \mathbb{R}^2$ , where  $N$  is a neighbourhood of  $\bar{\zeta}$  in  $\mathbb{R}^2$ . Let  $\lambda_1 = 0$  and  $\lambda_2 > 0$  be the eigenvalues of  $DY(\bar{\zeta})$ , and let  $E_1$  and  $E_2$  be the corresponding eigenspaces.*

*If every neighbourhood of  $\bar{\zeta}$  contains a fixed point of  $Y$  different from  $\bar{\zeta}$ , then there is a disk  $D \subset N$  of positive radius, centred at  $\bar{\zeta}$ , and a unique analytic local centre–stable manifold  $W^{\text{cs}} \subset D$ , tangent to  $E_1$ , such that all points on  $W^{\text{cs}}$  are steady states of  $Y$ .*

*Proof of Proposition E.5.* Assume the statement is false. Then for every  $n > 0$  there are initial points  $\tilde{x}_{n,0}$  located in different non-constant optimal orbits. We may assume that they are ordered in an infinite increasing or decreasing sequence: we prove the result for the increasing case, the other being similar. By Proposition E.4.(ii) there are unique  $\tilde{p}_{n,0}$  such that the non-constant trajectory starting at  $\tilde{x}_{n,0}$  is the state component of the optimal canonical trajectory  $(\tilde{x}_n(t), \tilde{p}_n(t))$  starting at  $(\tilde{x}_{n,0}, \tilde{p}_{n,0})$ , and such that  $\tilde{p}_n = V'(\tilde{x}_n(t))$  for  $t > 0$ . Proposition E.4.(iv) implies that this canonical trajectory converges to a point  $(\bar{x}_n, \bar{p}_n)$  with  $\bar{x}_n \in (\tilde{x}_{n,0}, \tilde{x}_{n+1,0})$ .

If infinitely many of these limit points are in  $\mathcal{P}_{\text{max}}$ , then the cardinality of  $\{x : f(x, a_{\text{max}}) = 0\}$  is infinite and  $f(x, a_{\text{max}}) = 0$  for all  $x \in C$ . But then  $\mathcal{P}_{\text{max}}$ , and hence  $\mathcal{S}_{\text{max}}$ , are invariant under  $X$  and all the trajectories converging to the limit points are also in  $\mathcal{P}_{\text{max}}$ , and hence constant. The reasoning for  $\mathcal{P}_{\text{min}}$  is analogous.

Hence only finitely many limit points are not in  $\mathcal{P}_{\text{int}}$ . An infinite subsequence of limit points in  $\mathcal{P}_{\text{int}}$  converges to a steady state  $(\bar{x}, \bar{p}) \in \overline{\mathcal{P}_{\text{int}}}$ . By Theorem G.1, the invariant centre–stable manifold of  $(\bar{x}, \bar{p})$  consists of steady states and contains infinitely many trajectories  $(\tilde{x}_n(t), \tilde{p}_n(t))$ , again contradicting the fact that these trajectories are non-constant.  $\square$

**Proposition G.1.** *A compact set  $C \subset \mathcal{X} \setminus \mathcal{J}$  contains only finitely many switching points of any optimal orbit.*

*Proof.* Between any two switching points on  $\mathcal{S}_{\max}$  there is a point where  $X$  is tangent to  $\mathcal{S}_{\max}$ . If there are infinitely many switching points on  $\mathcal{S}_{\max}$ , the cardinality of tangency points is infinite and  $\mathcal{S}_{\max}$  is invariant. But this contradicts the existence of switching points. The reasoning for  $\mathcal{S}_{\min}$  is analogous.  $\square$

The proof of the following technical result about canonical trajectories is left to the reader.

**Lemma G.3.** *Let  $(x(t), p(t))$  be a canonical trajectory such that  $x(t) \rightarrow \bar{x}_{\pm}$  as  $t \rightarrow \pm\infty$ . Then  $p(t) \rightarrow \bar{p}_{\pm}$  as  $t \rightarrow \pm\infty$ , with  $\bar{p}_{\pm} \in [-\infty, 0]$ .*

*Proof of Proposition E.6.* Let  $\mathcal{T}$  and  $x : \mathcal{T} \rightarrow I$  be as in Definition 8, and let  $C_I = C \cap I$ . If  $x_0 \in \mathring{C}_I$ , then  $x_0 = x(t_0)$  for some  $t_0 \in \mathring{\mathcal{T}}$ . By Proposition E.4,  $V$  is differentiable in a neighbourhood of  $x_0$  and there is a canonical trajectory  $(x(t), p(t))$  defined on  $\mathcal{T}$  such that  $p(t) = V'(x(t))$  for all  $t \in \mathring{\mathcal{T}}$ . Set  $p_0 = p(t_0)$ . If  $(x_0, p_0)$  is not a switching point, then  $(x(t), p(t))$  is real analytic for  $t$  close to  $t_0$ , since it is locally the trajectory of a real analytic vector field  $X$ . Hence we can solve  $x = x(t)$  as  $t = x^{-1}(x)$  around  $x_0$ , and obtain that  $V'(x) = p(x^{-1}(x))$  is real analytic at  $x_0$ .

By Proposition G.1, there are finitely many switching points of  $x$  in  $C_I$ . We have to show that one-sided derivatives of  $\phi$  exist at the switching points and the end points of  $C_I$ . The argument for the switching points is straightforward and left to the reader. If  $x_0 = x(t_0)$  is an end point of  $C_I$ , but  $t_0 \in \mathring{\mathcal{T}}$ , then  $\phi$  is real analytic at  $x_0$  by the same argument as given above.

We are left with  $t_0$  being either  $T_0$  or  $T$ . We give the argument for the first situation, leaving the case  $t_0 = T$  to the reader. Proposition E.4.(ii) and Lemma G.3 imply that  $(x(t), p(t))$  converges either to  $(\bar{x}, -\infty)$  or to a finite limit  $(x_0, p_0)$  as  $t \downarrow T_0$ . In the first case we have  $\phi(x(t)) = a^*(x(t), p(t)) = a_{\min}$  for  $t$  in a neighbourhood of  $T_0$ , and  $x$  in a neighbourhood of  $x_0$ , and it is clear that  $\phi$  can be differentially extended.

In the second case, if the point  $(x_0, p_0)$  is not a steady state of the canonical vector field, then  $T_0 = 0$ . If  $\dot{x}(0) \neq 0$ , then  $(x(t), p(t))$  is real analytic in a neighbourhood of  $t = 0$  and  $V'(x)$  and  $\phi(x)$  can be analytically extended to an open interval containing  $x_0$ . So assume  $\dot{x}(0) = 0$ . The point  $x_0$  is boundary point of a second non-constant optimal orbit  $I_1$ . Let  $x_1 : \mathcal{T}_1 \rightarrow I_1$  be as in Definition 8, and let  $(x_1(t), p_1(t))$  be the associated optimal orbit, with initial state  $(x_0, p_1(0)) \neq (x_0, p_0)$ . As  $\dot{x}(0) = \frac{\partial H}{\partial p}(x_0, p_0) = 0$ , it follows that  $\rho V(x_0) = H(x_0, p_1(0)) > H(x_0, p_0) = \rho V(x_0)$ , a contradiction. We conclude that  $\dot{x}(0) = 0$  is not possible.

We are left with the situation that  $T_0 = -\infty$  and  $\bar{z} = (\bar{x}, \bar{p})$  is a steady state of the canonical vector field with  $\bar{x} \in C$ . Lemma G.1 implies that there is a time  $t_3$  such that  $z(t) = (x(t), p(t))$  does not pass through a switching point for  $t \in (-\infty, t_3)$ . If  $z(t) \in \mathcal{P}_{\max}$  for  $t \in (-\infty, t_3)$ , then  $\phi(x(t)) = a_{\max}$  for those values of  $t$ , and we conclude as above; similarly for  $z(t) \in \mathcal{P}_{\min}$ . If  $z(t) \in \mathcal{P}_{\text{int}}$  for all  $t \in (-\infty, t_3)$ , the trajectory is tangent to an eigenspace  $E$  of  $DX_I(\bar{z})$ , where  $X_I$  is the real analytic extension to an open neighbourhood of the restriction of  $X$  to  $\overline{\mathcal{P}_{\text{int}}}$ . The fact that  $\frac{\partial^2 H}{\partial p^2}(\bar{z}) > 0$  implies that none of the eigenspaces of  $DX_I(\bar{z})$  is vertical.

As all eigenspaces are one-dimensional non-vertical lines, it follows that the second derivative  $V''(x(t)) = X_2(z(t))/X_1(z(t))$  converges, as  $t \rightarrow \bar{t}$ , to the inclination  $\bar{w}$  of  $E$  with respect to the horizontal axis. Consequently  $\phi'(x(t)) = \frac{\partial a^*}{\partial x}(x(t), p(t)) + \frac{\partial a^*}{\partial p}(x(t), p(t))V''(x(t))$  converges to  $\frac{\partial a^*}{\partial x}(\bar{z}) + \frac{\partial a^*}{\partial p}(\bar{z})\bar{w}$ .  $\square$

## G.5 Proof of Proposition F.1

*Proof of Proposition F.1.* The value function  $V^\phi(x)$  satisfies Equation (18). As  $f^\phi$  is real analytic on the interval  $I_j = (\bar{x}_j, \bar{x}_{j+1})$ , either  $f^\phi(x) = 0$  for all  $x \in I_j$ , and  $\mathcal{Q}_j = \emptyset$ , or the set  $\mathcal{Q}_j = \{x \in I_j : f^\phi(x) = 0\}$  is finite. In the first case  $V^\phi(x) = u^\phi(x)/\rho$  is real analytic on  $I_j$ . In the second, Equation (18) and Proposition A.1 imply that  $V^\phi$  is continuous on  $I_j$  and real analytic whenever  $f^\phi(x) \neq 0$ . By setting  $\mathcal{Q} = \cup_j \mathcal{Q}_j$ , Properties (i) and (ii) hold true.

Property (iii) is a direct consequence of Proposition A.1(ii).  $\square$