

Prummer, Anja; Nava, Francesco

Working Paper

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Discussion Paper, No. 2026/6

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Suggested Citation: Prummer, Anja; Nava, Francesco (2026) : Divisive by design shaping values in optimal mechanisms, Discussion Paper, No. 2026/6, Freie Universität Berlin, School of Business & Economics, Berlin,
<https://doi.org/10.17169/refubium-51440>

This Version is available at:

<https://hdl.handle.net/10419/338079>

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Divisive By Design

Shaping Values in Optimal Mechanisms

Anja Prummer
Francesco Nava

2., überarbeitete Auflage

School of Business & Economics

Discussion Paper

Economics

2026/6

DIVISIVE BY DESIGN: SHAPING VALUES IN OPTIMAL MECHANISMS*

Anja Prummer[†] Francesco Nava[‡]

February 23, 2026

Abstract

We study a principal who allocates a good to agents with private, independently distributed values through an optimal mechanism. The principal can strategically shape these value distributions by modifying the good's features, which affect agents' valuations. Our analysis reveals that optimal designs are frequently divisive – creating goods that appeal strongly to specific agents or agent types while being less valued by others. These divisive designs reduce information rents and increase total surplus, even though they reduce competition. Even when total surplus is constrained, some divisiveness in designs remains optimal.

Keywords: Value Design, Mechanism Design, Differentiation

JEL Codes: D82, D46, L15

*We thank Chris Bidner, Daniele Condorelli, Dan Kovenock, David McAdams, and Balazs Szentes for excellent comments. We are grateful to participants at various seminars and conferences for their insightful questions and comments. All remaining errors are ours. Both authors contributed equally.

[†]Freie Universität Berlin and Berlin School of Economics, anja.prummer@fu-berlin.de.

[‡]London School of Economics, f.nava@lse.ac.uk.

1 Introduction

A bureaucrat is tasked with allocating a plot of land to one of several interested parties. The interested parties include a company seeking a new production site, an entrepreneur who wants to build a water skiing facility, and a school superintendent in need of a plot for a new elementary school. The bureaucrat has the authority to impose rules and regulations on this plot, while facing some restrictions on what they can add. For example, they can specify certain setbacks, enforce environmental restrictions, or approve the construction of a highway to improve access to the plot, but perhaps not all of these measures jointly. The bureaucrat then allocates the land according to an optimal mechanism (Myerson (1981), Bulow and Roberts (1989)). The objective is to maximize transfers from the interested agents, either for the bureaucrat's benefit or for the community at large.¹ The question is: What are the optimal features that should be imposed on the plot of land?

We demonstrate that attributes that increase surplus while minimizing information rents are optimal, even if they reduce competition among bidders. Each feature influences the bidders' valuations in different ways. Some attributes may affect the valuation of one bidder without altering the others', while other features may impact the valuations of all bidders. We therefore consider a broad set of constraints, moment conditions, and distributional restrictions on how the features of a good shape valuations, and we characterize the optimal value distribution subject to these constraints.

Any combination of attributes that shapes a value design has three key implications: (i) it directly affects agents' valuations, in particular the maximum value, which we also refer to as surplus; (ii) it alters agents' private information, thereby influencing the incentives required for agents' values to be truthfully revealed; and (iii) it impacts the allocation probability, which in turn affects the transfers in the optimal mechanism. We find that attributes leading to more differentiated valuations help the principal screen and increase revenue, as they are associated with an increase in surplus and/or a reduction in information rents.

Value distributions are determined by the characteristics of the good. For example, building restrictions imposed on a plot of land can have various implications for value distributions. Environmental restrictions may increase the value of the land to both the

¹While these transfers may constitute bribes, they could also be perfectly legal contributions through PACs. Moreover, the transfers do not necessarily have to be monetary, but can take some other form. It only matters that the bureaucrat is able to assign a price to what is given by every bidder. In this sense, a school superintendent may be able to provide some favors, not necessarily money, that the bureaucrat values.

water ski entrepreneur and the company seeking a production site, as both entities value green technologies, and the restrictions allow them to credibly signal this commitment to their customers. The school, on the other hand, may be unaffected by these restrictions. Approving highway access may lower the value of the land for the school superintendent due to increased pollution, while increasing its value for the production site due to improved transportation access. The water skiing facility may not be significantly impacted, as access has improved, but water quality would need to be more closely monitored.

These examples illustrate that the connection between features and the resulting value distributions can be complex and context-specific. It matters which markets we consider, the type of buyers, and the characteristics that can be added to the good, to name but a few factors. Abstracting from these specifics, our goal is to find the optimal value designs, that is, the distributions the principal would like to achieve through the appropriate choice of attributes. To discipline our analysis, we consider settings where an increase in value for one (type of) agent is met by a decrease in value for another (type of) agent, as made precise in Section 3.² Consequently, we think of *divisive designs* as those that lead to (i) some agents valuing the good highly while others value it little, or (ii) some types of agents valuing the good substantially while other types do not.³

In general, the principal benefits from adding features to the good that make it uniformly more desirable, thereby increasing surplus. However, we assume that the principal cannot generate valuations exceeding some arbitrarily large value, \bar{v} .

If this were the only constraint, the principal would select features that ensure at least one agent values the good exactly at \bar{v} . Having even a single agent with such a value distribution maximizes surplus by generating the highest willingness to pay for the good. This design simultaneously minimizes information rents by making the value fully predictable. The principal can set a price of \bar{v} , thereby extracting the entire surplus. Even if all other agents have a valuation of zero, it would still be optimal for the principal to choose features that result in such a design.

The emergence of atoms, if possible, is generally a feature of optimal designs, as shown in Section 4.1. This follows because the principal's revenue in the optimal mechanism is a convex function of the agents' value distributions. Therefore, any solution to the design

²We show that value destruction cannot be optimal in Section 4.1, while increasing value is always beneficial. Therefore, we focus on the more interesting case where an increase is matched by a decrease in values.

³Note that both cases are observationally equivalent. Regardless of whether the change in values affects the types of an agent differentially or distinct agents, we would see agents with high and agents with low values.

problems we consider will be extreme, lying on the boundary of the constraint set. Put differently, value distributions that are convex combinations of other value designs are suboptimal.

A second insight is that the principal’s revenue in the optimal mechanism is monotonically increasing in first-order stochastic dominance, as previously established by [Devanur, Huang, and Psomas \(2016\)](#) and [Hart and Reny \(2015\)](#). This implies that the principal can never benefit from uniformly reducing values.

While these two features of optimal revenue help narrow the set within which our optimal value distributions lie, they do not fully determine the optimal value distribution. The optimal distribution is therefore determined by the constraints the principal faces.

No Spillovers Suppose that the principal can select features that influence the value of the good for one agent, while leaving the value for the other agents unchanged. We begin in [Section 4.2](#) by considering settings in which the principal cannot raise the expected value beyond some constant $k < \bar{v}$, but can otherwise shape the distribution of values arbitrarily. The principal will select features resulting in a binary distribution of values with mass at \bar{v} and zero, with mean k .

On average, the agent values the good at k , the upper bound. Furthermore, the principal selects features that lead to a binary distribution with positive mass assigned to one strictly positive value, in order to fully extract the gains from trade. The designed value distribution not only makes the agent’s value predictable for the designer conditional on trade, but also affects the probability of allocating the good to competing agents. The principal aims to keep the chances of the competing agents at a maximum, as this increases their willingness to pay and, thus, revenue. They achieve this by minimizing the odds that the agent with the binary distribution receives the good. This agent only receives the good if their realized value is strictly positive, and the chances of this occurring are minimized when the strictly positive value is as high as possible.

The optimality of such binary distributions extends to the case where all agents respond to the features of the good in the same way, and the principal faces a constraint on the mean of the designed distributions. Contrary to the widespread belief that higher variance increases agents’ information rents, this result demonstrates that this is not necessarily the case.⁴ For instance, when the increase in variance allows the principal to

⁴A folk wisdom in some of mechanism design literature appears to be that greater dispersion in bidder valuations generally increases information rents accruing to buyers at the expense of the seller’s revenue. For instance, [McAfee and McMillan \(1987\)](#) (p. 708) and [Milgrom \(2004\)](#) (p. 98) emphasize

better screen agents' valuations, the dispersion in values benefits the principal.

In sum, a principal prefers features that allow them to generate extreme distributions. These attributes are, with some probability, highly valued by agents, but they also carry the risk that the agents may no longer value the good. This value design is particularly valuable when agents appear a priori identical, and the introduction of a feature allows the principal to segment them. To illustrate this, consider again the bureaucrat facing different companies. The bureaucrat would like to attract a company that values environmental standards, but cannot observe this, and therefore imposes eco-friendly policies on the plot. For some companies, this aligns with their ethical standards and can make the brand more attractive to environmentally conscious consumers.⁵ For others, it is a negative, as their production standards do not meet these regulations. As a result, imposing such legislation can either increase or decrease the agents' values, and, at best, generate the binary distributions that are optimal in the absence of spillovers in valuations.

Spillovers While in the no-spillover case all agents may be affected by the choice of features in the same manner, in Section 4.3 we characterize optimal designs when the principal can select features of the good that redistribute values across agents. Making the good more appealing to some agents necessarily reduces its value to others.

We first consider a constraint on the sum of agents' expected values, mirroring the mean-bound constraint from the setting without spillovers. Under this restriction, the principal would ideally design a good that perfectly caters to a single agent, making their value both as high and as predictable as possible. This extreme specialization simultaneously maximizes surplus generation and eliminates information rents entirely, since the favored agent's willingness to pay becomes perfectly known.

However, the feasibility of such extreme value distributions is questionable in practice. We therefore complement this analysis with a distributional constraint that more realistically bounds the principal's design ability.

We consider a scenario where the principal can design any value distributions for the two agents, provided that their cumulative distributions sum to at least $H(v)$ pointwise.⁶ In such a setting, the optimal design exhibits maximal asymmetry: one agent's values

that the seller's inability to extract surplus stems from the need to leave rents to higher types to prevent mimicking, with these rents growing in the dispersion of values. Katzman, Reif, and Schwartz (2010) formalizes this intuition, showing that higher dispersion directly increases information rents for some class of distributions.

⁵For some evidence that these types of considerations matter to firms, see <https://scmguides.com/10-essential-considerations-before-building-a-new-production-facility/>

⁶ H is a non-decreasing function with image in $[0, 2]$.

are concentrated below the median of H , while the other agent's values lie entirely above the median. Maximal differentiation serves a dual purpose: it increases total surplus while simultaneously reducing information rents, as agents' valuations lie in a narrower set, thereby making them more predictable. This asymmetry decreases transfers from the disadvantaged agent, while it leads to an increase in transfers from the advantaged agent, which more than compensates. Therefore, maximally distinct value distributions are optimal.

This result has practical implications. When a principal faces competing entities and knows that a particular feature will increase one agent's valuation while decreasing another's, they should add the feature to maximize revenue.

Fixed Surplus With or without spillovers, surplus maximization is a key driver of divisive designs. Accordingly, in Section 5, we consider a fixed surplus distribution G to assess whether divisive designs remain optimal or whether more symmetric designs instead yield higher revenue.

If agents' value distributions are symmetric, it is optimal to allocate the good via a second-price auction with a reserve price. Given this optimal mechanism under symmetric designs, inducing asymmetries in value distributions reduces the principal's revenue. This does not imply that divisive designs are suboptimal; rather, any revenue gain must arise from tailoring the mechanism to the asymmetric distributions.

We show that is indeed possible to improve upon symmetric value distributions by inducing threshold designs. Threshold designs have the property, that for one agent, they generate a distribution with mass at the minimal value, while also assigning all the values above a given threshold. The other agent receives the values below the threshold. These designs limit information rents by reducing agents' scope for deviation, while at the same time maintaining competition between different agents.

Threshold designs outperform symmetric designs when exclusion under the symmetric designs is substantial. They also improve upon symmetric and maximally divisive designs when the surplus distribution is uniform. Thus, even when surplus is fixed, a degree of divisiveness remains optimal.

Applications While our primary example involves a bureaucrat or politician who allocates a good to one of several entities, our model can also provide insights in a variety of other settings. In the Supplementary Appendix, Section A, we discuss how our model relates to celebrity auctions as well as the art market.

More generally, our model can be viewed as a product design model, where the principal decides what kind of product to sell and how to sell it. The principal benefits from understanding what types of features agents value, motivating why market research that leads to better information about the customer base, what they value, and how they are affected by different attributes of the good can be extremely valuable.⁷ As described in [Bloch \(1995\)](#) p. 26, “Seeking the ‘ideal’ form for a product remains a significant goal for both designers and marketing management. The ideal form is a theoretical concept, [...], similar to a vanishing point in the distance.” [Bloch \(1995\)](#) continues by stating that the ideal form must be truly valued by consumers, evoke positive emotions, and be sympathetic to consumers’ tastes. “This ideal form must accomplish all of this, while simultaneously satisfying numerous design constraints.”—design constraints that depend on the specific setting. These are the considerations our model aims to formalize.

2 Related Literature

Our paper contributes to the literature on value and product design in allocation problems. Key insights in this area were pioneered by [Johnson and Myatt \(2006\)](#), who study environments where a price-setting monopolist influences market demand either by adding features to the product or through advertising. Their analysis focuses on reshaping demand through rotations and shifts, demonstrating that value distributions with minimal dispersion are optimal when the monopolist caters to a mass market, while dispersion is beneficial when the monopolist caters to different niches. While our approach is similar in spirit, our model differs fundamentally with implications for results. In the absence of niche-versus-mass market features, dispersion in values, both within and across buyers, generally increases revenue, due to higher surplus and lower information rents.⁸

The importance of increasing surplus becomes evident in light of [Cantillon \(2008\)](#). She shows that when surplus is fixed, revenue decreases with asymmetric value designs in

⁷This described approach is followed in the segmenting-targeting-positioning framework in marketing. A firm aims to divide consumers according to their characteristics, the segmentation. It then targets the market through an advertising campaign and finally positions the product in the market so as to raise consumers’ values. See for instance [Moutinho \(2000\)](#) for an application of this concept to tourism.

⁸While not considering product design, [Anderson and Renault \(2006\)](#) analyzes how attributes of a given product are advertised. These attributes or characteristics are taken as given, and the seller can choose to highlight some or all of them. In contrast, we allow for the seller to create these attributes which are immediately visible. Despite these modeling differences, [Anderson and Renault \(2006\)](#) finds that it is beneficial to advertise characteristics that screen out low-value customers, which allows for a price increase for high-value customers.

first- and second-price auctions without exclusion. We complement her result by showing that revenue can increase with asymmetric designs in the optimal mechanism, if there is exclusion.⁹

Value design has been previously explored in [Condorelli and Szentes \(2020, 2022\)](#). The former focuses on buyer-optimal value design with a single buyer, while the latter characterizes the optimal profit arising from different demand functions in a quantity competition setting. In contrast, we consider seller-optimal designs with multiple buyers and focus on optimal design strategies.

Our analysis technically relates to the literature on robust mechanism design under moment constraints, see for instance [Carrasco, Luz, Kos, Messner, Monteiro, and Moreira \(2018\)](#). They characterize the optimal mechanisms when the principal knows only moments of the buyer’s value distribution. In contrast, we consider a setting with multiple buyers, where the seller knows at the very least the distributions, if not the value if strictly positive. This difference in knowledge about the distributions also implies that in our setting the principal can extract full gains from trade under the mean constraint, while in [Carrasco et al. \(2018\)](#) revenue might be negative if only the first moment is known.

Finally, some of our results can be interpreted through the lens of information design.¹⁰ In these settings, the principal provides information about the product, as in [Bergemann and Pesendorfer \(2007\)](#), [Eso and Szentes \(2007\)](#), [Li and Shi \(2017\)](#), [Sorokin and Winter \(2018\)](#), and [Ganuja and Penalva \(2019\)](#). In contrast, we allow for the principal to design the product’s features, generating value for agents.

The fundamental difference is that information design changes only what agents know about their values, while value design changes the actual distribution of agents’ values. Providing information to agents about their values gives them additional private information and this means that the principal incurs screening costs. In value design, principal and agents know the distribution and the principal can select values that are trivially screenable – for instance, binary distributions with mass at one strictly positive value and zero – thereby eliminating information rents entirely.

This difference explains why our results diverge from [Bergemann and Pesendorfer \(2007\)](#), even when constraints technically match Bayes plausibility. In their setting, the

⁹In contrast to our focus on designing value distributions, [Deb and Pai \(2017\)](#) fix value designs and establish that symmetric mechanisms can heavily discriminate between agents when their value distributions differ.

¹⁰For instance, [Roesler and Szentes \(2017\)](#) and [Bobkova \(2019\)](#) study trade environments with a single buyer and seller, characterizing buyer-optimal information designs and the costs of information acquisition.

key trade-off is between the seller’s revenue and the bidders’ information rents. In our setting, this trade-off disappears: the optimal design simultaneously minimizes information rents and maximizes revenue. Consequently, even when the number of bidders exceeds two, information rents remain zero and every bidder has the same value design at the optimum (with the possible exception of the n -th bidder, where we obtain a multiplicity). In contrast, [Bergemann and Pesendorfer \(2007\)](#) finds that with three or more bidders, binary partitions are no longer optimal and the principal must incur incentive costs.

The case of a single bidder further illustrates the distinction. In [Bergemann and Pesendorfer \(2007\)](#), it is optimal to reveal no information: the principal sets the price equal to the expected value and extracts the entire surplus. In value design, revealing no information corresponds to placing all mass at the mean. But other designs are equally optimal, such as any binary distribution with mass at one strictly positive value and zero. With value design, uncertainty about the buyer’s value is irrelevant for revenue extraction when the distribution is appropriately designed.

In sum, although information design and value design are technically related, the economic insights differ substantially.

3 A Model of Constrained Value Design

We begin by introducing the mechanism design framework, which forms the foundation for the value design approach in our analysis. We then proceed to discuss the design of optimal distributions. For clarity, we focus on the case of two agents, but extensions to N agents are considered throughout.

The Environment and the Optimal Mechanism A principal is tasked with allocating a single unit of a good to one of two competing agents, A and B . The value of the good to agent $i \in \{A, B\}$ is denoted by $v_i \in [\underline{v}, \bar{v}]$, where $\bar{v} > \underline{v} \geq 0$ are exogenous upper and lower bounds on the value of the good. Each agent is privately informed of their own value, and the values are independently distributed. The cumulative distribution function of the value v_i is denoted by F_i , with support $V_i \subseteq [\underline{v}_i, \bar{v}_i]$, where \underline{v}_i and \bar{v}_i represent the smallest and largest values in V_i , respectively. The distributions F_A and F_B are commonly known to both the agents and the principal. The utility of an agent is given by the private value of the good, v_i , if the agent is awarded the good, minus any transfer t_i made to the principal. The principal derives no value from the good itself and seeks to maximize

revenue by selecting a mechanism to award the good, with revenue defined as the sum of the transfers, $t_A + t_B$.

By the revelation principle, the principal's problem can be solved by restricting attention to direct, incentive-compatible (IC), and individually rational (IR) mechanisms. A direct mechanism (x, t) consists of an allocation rule x and a transfer rule t ,

$$(x, t) : V_A \times V_B \rightarrow \Delta_-^2 \times \mathbb{R}^2,$$

where agents are asked to simultaneously report their types. The allocation rule $x(\mathbf{v}) = (x_A(\mathbf{v}), x_B(\mathbf{v}))$ specifies the probability with which agents A and B receive the good for any profile of reported values $\mathbf{v} \in V_A \times V_B$, and it belongs to the subsimplex Δ_-^2 , the set of allocation probabilities. The transfer rule $t(\mathbf{v}) = (t_A(\mathbf{v}), t_B(\mathbf{v}))$ specifies the transfer each agent must make to the principal for any profile of reported values \mathbf{v} . A direct mechanism is said to be IC and IR if, for any agent i and value v_i , we have that

$$v_i x_i(v_i) - t_i(v_i) \geq \max \{v_i x_i(z_i) - t_i(z_i), 0\} \text{ for any } z_i \in V_i.$$

This means that there exists a Bayesian Nash equilibrium in which all agents participate and report their types truthfully.¹¹ Denote the optimal revenue for any pair of distributions (F_A, F_B) as

$$R(F_A, F_B) = \max_{x, t} \mathbb{E}_F [t_A(\mathbf{v}) + t_B(\mathbf{v})]$$

$$\text{s. t. } v_i x_i(v_i) - t_i(v_i) \geq \max \{v_i x_i(z_i) - t_i(z_i), 0\} \text{ for any } z_i, v_i \in V_i, \quad i \in \{A, B\},$$

where \mathbb{E}_F denotes the expectation over the distribution of values $F = (F_A, F_B)$. Leveraging [Monteiro and Svaiter \(2010\)](#)'s generalization of the revenue equivalence theorem, we can express the revenue of an incentive compatible mechanism (x, t) as

$$R(x, t | F_A, F_B) = \mathbb{E}_F \left[x_A(\mathbf{v}) \hat{\psi}_A(v_A) + x_B(\mathbf{v}) \hat{\psi}_B(v_B) \right],$$

where $\hat{\psi}_i$ is defined as the *generalized virtual value* of agent i . For regular value distribu-

¹¹With a slight abuse of notation, we use the same operator to denote both the ex-post rules and the interim rules. The interim allocation probability of agent i with value v_i is $x_i(v_i) = \mathbb{E}_{F_{-i}} [x_i(\mathbf{v})] = \int_{\underline{v}_{-i}}^{\bar{v}_{-i}} x_i(\mathbf{v}) dF_{-i}(v_{-i})$. Similarly, the interim transfer is defined as $t_i(v_i) = \mathbb{E}_{F_{-i}} [t_i(\mathbf{v})] = \int_{\underline{v}_{-i}}^{\bar{v}_{-i}} t_i(\mathbf{v}) dF_{-i}(v_{-i})$. When cumulative distributions are non-differentiable or discontinuous on the support, we use Riemann-Stieltjes integrals to calculate the expectations implicitly.

tions, generalized virtual values coincide with *virtual values* and amount to

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

We refer to the difference between the value and the generalized virtual value as the *information rent*.

Finally, *realized surplus* is the highest realized value for the good among agents, $\max\{v_A, v_B\}$. For brevity, we call the expectation of this random variable *surplus*.

The Value Design Problem We turn to the value design problem at the core of the analysis. Any pair of distributions $F \equiv (F_A, F_B) \in \Delta[\underline{v}, \bar{v}]^2$ denotes a *value design*. The principal's value design problem is then

$$\max_{F_A, F_B} R(F_A, F_B) \quad \text{subject to} \quad (F_A, F_B) \in K,$$

where $K \subset \Delta[\underline{v}, \bar{v}]^2$ represents an arbitrary constraint on the set of value designs. We summarize the constraints we consider in Table 1 for the baseline case of two agents.

A key feature common to all constraints we consider is the trade-off between value increases and value reductions. In the case of no spillovers, value can only be redistributed across the different types of agents. Increasing the mass allocated to a higher value necessarily involves reallocating mass to lower values. In the presence of spillovers, increasing the value for one agent leads to a corresponding reduction for the other. In the case of distributional constraints, spillovers imply that a pointwise increase in value must be matched by a pointwise reduction, making the distributional constraints a natural extension of the moment condition.

Restriction Type	Moment Constraint	Distributional Constraint
No Spillovers	$\mathbb{E}_{F_i}[v] \leq k$	$\int_0^v G(t)dt \leq \int_0^v F_i(t)dt$
Spillovers	$\mathbb{E}_{F_A}[v] + \mathbb{E}_{F_B}[v] \leq k$	$H(v) \leq F_A(v) + F_B(v)$
Surplus Bound	–	$G(v) \leq F_A(v)F_B(v)$

Table 1: Design Constraints

Note: The parameter k is an arbitrary constant, H is a measure with mass 2, G is some cumulative distribution function. F_A and F_B are distributions chosen by the principal.

4 Optimal Value Design

We begin by characterizing general properties of optimal value distributions, before turning to the various constraints we consider. Results in the main text are proved in the Appendix. Corollaries and lemmas appearing in the Appendix are proved in Section F of the Supplementary Appendix. Proofs of remarks are omitted.

4.1 Optimal Revenue Properties

The first result establishes that the revenue of the optimal mechanism must be convex in the joint distribution of values $F_A(v_A)F_B(v_B)$. Intuitively, the result holds because optimal revenue is the value function of the classical mechanism design problem. Given that this problem is linear in F_A and F_B , its maximum must be convex.

Lemma 1 (Convexity of Revenue). *Optimal revenue is convex in the joint distribution of values. Specifically, for any $F, F', F'' \in \Delta[\underline{v}, \bar{v}]^2$ satisfying for some $a \in (0, 1)$*

$$F_A(v_A)F_B(v_B) \geq aF'_A(v_A)F'_B(v_B) + (1-a)F''_A(v_A)F''_B(v_B) \text{ for all } v \in [\underline{v}, \bar{v}]^2,$$

we have that $R(F) \leq aR(F') + (1-a)R(F'')$.

When $F_B = F'_B = F''_B$, the result shows that if F_A is an average of some other cumulative distributions, F'_A and F''_A , then it generates less revenue than $\max\{R(F'_A, F_B), R(F''_A, F_B)\}$. This insight implies that value designs, (F_A, F_B) , in the interior of the constraint set under consideration are necessarily suboptimal, while revenue-maximizing designs will lie on the boundary of the constraint set.¹²

The second insight establishes why it can never be optimal to select features that induce distributions that are first-order stochastically dominated, a result that dates back to [Devanur et al. \(2016\)](#) and [Hart and Reny \(2015\)](#). A value design F' first-order stochastically dominates value design $F \in \Delta[\underline{v}, \bar{v}]^2$, denoted by $F' \succ_1 F$, if

$$F'_i(v) \geq F_i(v) \text{ for all } v \in [\underline{v}, \bar{v}] \text{ and } i \in \{A, B\}.$$

Lemma 2 (FOSD ([Devanur, Huang, and Psomas, 2016](#); [Hart and Reny, 2015](#))). *For any $F, F' \in \Delta[\underline{v}, \bar{v}]^2$ such that $F' \succ_1 F$, we have that $R(F) \leq R(F')$.*

¹²Results in [Yang and Zentefis \(2024\)](#) demonstrate how to characterize extreme points within sets of monotone functions. Since revenue is convex, optimal designs will necessarily correspond to extreme points of the set being considered.

To gain intuition for this result, which plays a key role in our analysis, consider an allocation with two value distributions, F_A and F_B , and an arbitrary mechanism. For simplicity, assume that F_A is supported on a finite set of values (v^1, \dots, v^n) with respective probabilities (p^1, \dots, p^n) . Now, consider an alternative distribution supported on values $(v^1, \dots, v^i + \epsilon, \dots, v^n)$, where the only change is that the realization of v^i is increased by ϵ , while the remaining values and probabilities stay unchanged. This alternative distribution first-order stochastically dominates the original value distribution.

Next, consider the revenue from maintaining the same mechanism with one modification: Agent A 's transfer at value $v^i + \epsilon$ increases by $\epsilon \min_{k < i} (x_A(v^i) - x_A(v^k))$.¹³ The new mechanism remains incentive compatible for agent A under the new value distribution. When $v_A \neq v^i + \epsilon$, Agent A 's incentive constraints do not change, though deviations to $v^i + \epsilon$ become less advantageous. When $v_A = v^i + \epsilon$, agent A prefers reporting truthfully to reporting a smaller value, because the increased payment exactly offsets the additional gains from trade. Agent A also prefers truthtelling to reporting a higher value for ϵ small enough, since upward incentive constraints were slack in mechanism (x, t) by optimality. Further, interim payoffs are unchanged for the lowest type v^1 , preserving individual rationality.

Thus, the proposed mechanism is incentive compatible and individually rational while raising higher revenue for the principal. Since any first-order stochastic dominance shift can be decomposed into a sequence of such single-value changes, it cannot be optimal to select a distribution that is first-order stochastically dominated by some other feasible distribution.

Before proceeding to our constrained value design, observe that any design in which at least one agent values the good at \bar{v} with certainty is revenue-maximizing when the principal faces no design constraints, meaning $K = \Delta[\underline{v}, \bar{v}]^2$.

Remark 1. *In any unconstrained optimal value design, at least one agent has a value distribution satisfying $F_i(v) = \mathbb{1}(v \geq \bar{v})$ for all values $v \in [\underline{v}, \bar{v}]$.*

This result highlights the three forces that emerge throughout our analysis. First, the principal aims to increase values as this leads to higher transfers. Second, the principal prefers agents' values to be as predictable as possible, since this reduces information rents.

¹³Denote by (x, t) the optimal mechanism for (F_A, F_B) and by (\hat{x}, \hat{t}) the mechanism after the ϵ -increase. The allocation rules satisfy $\hat{x}(\mathbf{v}) = x(\mathbf{v})$ when $v_A \neq v^i$, and $\hat{x}(v^i + \epsilon, v_B) = x(v^i, v_B)$. The transfer for agent B remains unchanged. This also holds for agent A whenever $v_A \neq v^i + \epsilon$. For $v_A = v^i + \epsilon$, A 's transfer amounts to $\hat{t}_A(\mathbf{v}) = t_A(\mathbf{v}) + \epsilon \min_{k < i} (x_A(v^i) - x_A(v^k))$.

Third, it suffices to increase the value for *one* agent to \bar{v} , as the principal, knowing the value of that agent with certainty, is able to extract the full surplus, $\mathbb{E}[\max\{v_A, v_B\}]$, from them without ever selling to the other agent.

4.2 No Spillovers

We now turn to the analysis of the optimal distribution when the features of the good affect only agent A , while the valuation of agent B remains unchanged.

Moment Condition We begin by imposing a moment condition on the expected value of each agent, specifically that it cannot exceed some constant k . It immediately follows from Lemma 2 that this bound must hold with equality. If it did not, there would exist a value distribution in the constraint set that first-order stochastically dominates the current distribution, yielding higher revenue.

Lemma 1 and the implied convexity of revenue also suggests that optimal distributions must be atomic. We therefore define a class of binary distributions that take values at a point $h \in [k, \bar{v}]$ and \underline{v} , with a mean of k . Formally, for any $h \in [k, \bar{v}]$, the probability distribution P^h satisfies

$$P^h(v) = \begin{cases} \frac{h-k}{h-\underline{v}} & \text{if } v = \underline{v}, \\ \frac{k-v}{h-v} & \text{if } v = h. \end{cases} \quad (1)$$

$P^h(v)$ denotes the probability that the value equals v , while F^h is the cumulative distribution associated with P^h . The distribution that assigns all mass to the mean k is denoted by F^k . The distribution $F^{\bar{v}}$ places mass only at the maximal value \bar{v} and at the minimal value \underline{v} . We refer to this distribution as the *maximally spread* distribution and we establish that this distribution maximizes revenue given the restriction on the mean.

Proposition 1. *Let $\underline{v} = 0$. The value design $(F_A^*, F_B) = (F_A^{\bar{v}}, F_B)$ is optimal among all value designs (F_A, F_B) satisfying $\mathbb{E}_{F_A}[v] \leq k$. This design is uniquely optimal when $F_B \neq F^{\bar{v}}$. For $F_B = F^{\bar{v}}$, any design with $F_A^* = F^h$, $h \in [k, \bar{v}]$ is also optimal.*

Our result depends on the value distribution of agent B . If B possesses some arbitrary distribution, but not the maximally spread distribution, then for agent A the maximally spread distribution with mean k is the unique optimal distribution.

The optimal distribution maximizes surplus and reduces the information rent for agent A to zero. This occurs for any bimodal distribution with an atom at one strictly positive

value and mass at zero. The optimality of the maximal spread arises from the presence of the second agent, B .¹⁴ Agent B also competes for the good, and their transfer depends on the probability of receiving it. A higher allocation probability implies a higher transfer. The maximal spread for agent A maximizes agent B 's probability of receiving the good. To see this, note that agent B obtains the good if their virtual value is positive and exceeds that of agent A . The probability that B 's virtual value exceeds agent A 's increases with the probability that agent A has zero value. For a bimodal distribution with mean k and a single strictly positive atom at h , this probability equals $\frac{h-k}{h}$, which is increasing in h . Consequently, among all bimodal distributions with a fixed mean, the distribution that assigns mass to the maximal value and to zero places the greatest probability on zero. This makes the maximally spread distribution uniquely optimal.

The optimal mechanism for such a value design is simple: first, offer the good to agent A at price \bar{v} , and if A rejects, offer the good to agent B at some lower price.

If agent B 's distribution is maximally spread, then the optimal distribution for agent A is no longer unique. All distributions described in (1) with $h \geq k$ are now optimal. With these distributions, the principal can extract the entire surplus by allocating the good to agent B whenever B 's value equals the maximal value \bar{v} ; otherwise, the good is awarded to agent A .

Proposition 1 relies on the assumption that the minimal value \underline{v} equals zero, implying that the principal never allocates the good to agent A when his value is minimal. When the minimal value is positive, it may instead be optimal to allocate the good to agent A even when $v_A = \underline{v}$. If so, information rents accrued by A can become strictly positive. Whether it will be optimal to allocate the good to A in such circumstances will in general depend on B 's distribution. If, for instance, B 's distribution assigns all mass to the mean, then the principal will only allocate the good to A if their value is maximal, and the maximally divisive distribution remains optimal.

From a mechanism design perspective, the result is perhaps surprising since it demonstrates that an optimal design can maximize dispersion in values.¹⁵ But by maximizing overall dispersion, the principal simultaneously minimizes the variation in valuations (and consequently the information rents) for non-excluded types, who obtain the good with

¹⁴Without a second agent, any distribution that assigns mass to exactly one strictly positive value with expected value k is optimal.

¹⁵For instance, McAfee and McMillan (1987) (p. 708) states that “The larger is this variance, the larger on average is the difference between the highest and the second highest valuation, and so the larger is the economic rent to the winning bidder.”, a notion picked up again in Milgrom (2004) (p. 98) and formalized by Katzman et al. (2010).

positive probability, thereby increasing revenue.

If the principal can independently shape the value distribution of all agents, then bimodal distributions with mass at the maximal value \bar{v} and mass at $\underline{v} \geq 0$ remain optimal.

Proposition 2. *A value design, (F_A^*, F_B^*) , is optimal among all designs satisfying $\mathbb{E}_{F_i}[v] \leq k$ for all $i \in \{A, B\}$ if and only if for agents $i, j \in \{A, B\}$ with $j \neq i$, we have that $(F_i^*, F_j^*) = (F^{\bar{v}}, F^h)$, where $h = k$ if $\underline{v} > 0$ and $h \in [k, \bar{v}]$ if $\underline{v} = 0$.*

In contrast to Proposition 1, this result holds for any minimum value \underline{v} . As the principal, by creating the appropriate good, can now affect the distributions of all agents independently, they select a good that induces the maximal spread for one agent, while the second agent's distribution places all mass at the mean. This is optimal for any minimal value \underline{v} . If the minimal value equals zero, then any bimodal distribution with mean k , and mass at a single strictly positive value and zero, is also optimal for the second agent.

If the minimal value is positive, having two bimodal distributions is not optimal. To see this, suppose, to the contrary, that both agents possess the maximal spread. With probability $\frac{(\bar{v}-k)^2}{(\bar{v}-\underline{v})^2}$, both agents have the minimal value, and with the remaining probability, at least one agent has the maximal value. Surplus thus amounts to

$$\left(\frac{(\bar{v}-k)^2}{(\bar{v}-\underline{v})^2}\right)\underline{v} + \left(1 - \frac{(\bar{v}-k)^2}{(\bar{v}-\underline{v})^2}\right)\bar{v} = \bar{v} - \frac{(\bar{v}-k)^2}{(\bar{v}-\underline{v})}.$$

Consider instead a scenario in which an agent i 's distribution is the maximal spread, while the other agent j has all the mass at the mean. If so, then agent i receives the good if their valuation is maximal, and otherwise, agent j obtains the good. This results in the same surplus as if both agents had the maximal spread,

$$\left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)k + \left(1 - \left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)\right)\bar{v} = \bar{v} - \frac{(\bar{v}-k)^2}{(\bar{v}-\underline{v})}.$$

However, surplus does not necessarily equal revenue due to incentive compatibility. If both agents possess a bimodal distribution and $\underline{v} > 0$, it is not possible for the principal to extract the entire surplus. To see this, observe that if the principal was to sell to all high-value types at a price equal to \bar{v} , they would not be able to sell the good to low types, as this would undermine their incentive to report truthfully.¹⁶ Consequently, the principal cannot achieve a revenue that equals surplus by generating only bimodal distributions.

¹⁶If the principal allocates the good to the low type \underline{v} , all higher types of the agent must receive a

If agent j has all the mass at the mean, the principal can set a different price for this agent, namely k , and extract the average value from them when they do not trade with the high-value type of agent i . Therefore, revenue equals surplus when one distribution is the maximal spread and the other places all mass at the mean, in contrast to the case where both distributions are maximally spread.

If the minimum value equals zero, the optimal strategy is for one agent, say i , to have a maximally spread distribution, while the other agent, $j \neq i$, can have all mass at the mean k or any bimodal distribution with mass at one strictly positive value h and zero, with mean k . The principal can then sell the good in a manner similar to before – offering first the good to agent i at a price \bar{v} , and if rejected offering it to j at a price $h \in [k, \bar{v}]$, once again extracting the full surplus.

We consider the generalization of Proposition 2 to $n + 1$ agents. In any optimal design for these settings, n agents exhibit a maximally spread distribution. For the remaining agent the optimal distribution places all mass at the mean if $\underline{v} > 0$, and equals the bimodal distribution described in (1) with $h \geq k$ for $\underline{v} = 0$.

Interestingly, it is possible to extract the expected value from each agent if $\underline{v} = 0$ and \bar{v} diverges to infinity. In this case, revenue from the optimal design approaches $(n + 1)k$. As $\underline{v} = 0$, the probability that an agent has value \bar{v} amounts to $p = k/\bar{v}$. For a large maximal value \bar{v} , p is close to 0. As a result, the likelihood that more than one agent has a high value \bar{v} is negligible and we only take into account the probability that one agent possesses the high value \bar{v} . Then, revenue is bounded below by $(n + 1)p\bar{v} = (n + 1)k$.¹⁷

In general, it is not obvious that the principal can implement features that affect exactly one agent differentially relative to the other agents. Rather, the principal may only select features that affect all agents in the same manner. If this is the case, the principal benefits from having all agents possess the maximally spread value distribution. As long as one of the agents has a realized value of \bar{v} , the principal extracts the entire surplus by setting a price equal to \bar{v} . But the principal fails to sell the good when all agents have value \underline{v} , which happens with probability $P^{\bar{v}}(\underline{v})^{n+1}$. This probability converges to zero as $n \rightarrow \infty$. Therefore, for a sufficiently large number of potential buyers, the principal is only marginally hurt by selecting symmetric, but divisive designs.

Having agent-specific mean bounds, k_i for $i \in \{A, B\}$, does not change the results positive payoff by incentive compatibility. Otherwise, they would benefit from pretending to be type \underline{v} – trading less often at a lower price.

¹⁷For $\bar{v} = k/p$, revenue amounts to $\sum_{s=0}^n \binom{n+1}{s+1} p^s (1-p)^{n-s} k$ and converges to $(n + 1)k$ as $p \rightarrow 0$.

presented in Proposition 2. In the optimal design, one agent still possesses a maximally divisive design, while the optimal value distribution of the other agent depends on the minimum value.

Corollary 1 (to Proposition 2). *If agents face different constraints, $\mathbb{E}_{F_i}[v] \leq k_i$, a value design (F_A^*, F_B^*) is optimal if: for some agent $i \in \{A, B\}$, F_i^* has mean k_i and support $V_i^* = [\underline{v}, \bar{v}]$, while for the other agent $j \neq i$, F_j^* has mean k_j and support $V_j^* = \{k_j\}$ if $\underline{v} > 0$, and support $V_j^* = \{0, h\}$ for some $h \in [k_j, \bar{v}]$ otherwise.*

The magnitude of the mean does not determine which agent receives a given distribution. Since surplus is invariant to which agent attains the maximal spread, revenue is likewise unaffected.¹⁸ Therefore, a priori differences in average valuations do not determine agents' final valuations.

Second-Order Stochastic Dominance It is worth noting that Proposition 1 also establishes that $F^{\bar{v}}$ is the optimal design relative to any distribution that is second-order stochastically dominated by an arbitrary distribution G with mean $\mathbb{E}_G[v] = k$. This follows from two insights. First, any distribution F_A that is second-order stochastically dominated by G satisfies

$$\int_0^v G(t)dt \leq \int_0^v F_A(t)dt \quad \text{for all } v \in [\underline{v}, \bar{v}], \quad (2)$$

and, therefore, has a lower mean than G . Therefore, F_A satisfies our earlier moment condition $\mathbb{E}_{F_A}[v] \leq k$. Second, $F^{\bar{v}}$ is second-order stochastically dominated by G . Thus, $F^{\bar{v}}$ is also the optimal distribution among all F_A satisfying the constraint in (2).

Corollary 2 (to Proposition 1). *Let $\underline{v} = 0$. The value design $(F_A^*, F_B) = (F^{\bar{v}}, F_B)$ is optimal among all value designs (F_A, F_B) satisfying (2). It is the unique optimal design if $F_B \neq F^{\bar{v}}$.*

The result hinges on the assumption that $\underline{v} = 0$. To see why we require this assumption, let $F_B = F^{\bar{v}}$. We know from Proposition 2 that in this case, for $\underline{v} > 0$, placing all the mass at the mean k for agent A would be optimal. This highlights that if the lowest value \underline{v} is strictly positive, then it may be optimal to choose a distribution that second-order stochastically dominates the given distribution G . In general, for a strictly positive

¹⁸See the Supplementary Appendix F for further details.

minimal value $\underline{v} > 0$, the optimal distribution for A depends on B 's and cannot readily be pinned down.

However, for $\underline{v} = 0$ and $F_B = F^{\bar{v}}$, any distribution F^h that is bimodal and assigns mass to one strictly positive value and additionally, is second-order stochastically dominated by G is optimal. If B 's distribution does not correspond to the maximal spread $F_B \neq F^{\bar{v}}$, while the minimal value \underline{v} equals zero, the maximal spread becomes once again the unique optimum for agent A .

In sum, in the absence of spillovers, the principal will aim to create divisive, bimodal value designs in which agents either highly value the good or value it little, if at all.

4.3 Spillovers

While in the no-spillover case all agents may be affected by the choice of features in the same manner, we now turn to settings in which an attribute affects agents' values differently. As in the no-spillover case, if the principal could design a good that is more appealing to all agents, they would do so. We therefore focus on scenarios in which designs reallocate value across agents—that is, making the good more appealing to one agent necessarily makes it less appealing to others.

Moment Condition We begin by considering a constraint that restricts only the sum of means of the value distributions, mirroring the mean bound in the no spillover case. We first assume that the principal can achieve any value design (F_A, F_B) such that

$$\mathbb{E}_{F_A}[v] + \mathbb{E}_{F_B}[v] \leq k, \tag{3}$$

for some constant $k \in [2\underline{v}, 2\bar{v}]$. Since the value for each agent is bounded below by \underline{v} , k must exceed $2\underline{v}$. If the mean k were greater than $2\bar{v}$, then it would always be possible for the principal to allocate mass one to \bar{v} for both agents and extract the full surplus. For $k \in [2\underline{v}, 2\bar{v}]$, in any optimal design, the principal leaves one agent with the smallest possible value while allocating all remaining value to the other agent. To state the result, let $F_{k-\underline{v}}^h$ denote a binary distribution with atoms at h and \underline{v} and with mean $k - \underline{v}$.¹⁹

Proposition 3. *The value design (F_A^*, F_B^*) is optimal among all designs satisfying (3):*

¹⁹The probability distribution associated with $F_{k-\underline{v}}^h$ corresponds to the probability distribution defined in (1), except for the mean being $k - \underline{v}$, instead of k .

- (1) For $k \in [2\underline{v}, \underline{v} + \bar{v}]$ if and only if $F_i^*(v) = F_{k-\underline{v}}^h(v)$ with $h \in [k - \underline{v}, \bar{v}]$ for some agent $i \in \{A, B\}$, while $F_j^*(v) = \mathbb{1}(v \geq \underline{v})$ for all values $v \in [\underline{v}, \bar{v}]$ for agent $j \neq i$;
- (2) For $k \in [\underline{v} + \bar{v}, 2\bar{v}]$ if and only if $F_i^*(v) = \mathbb{1}(v \geq \bar{v})$ for all values $v \in [\underline{v}, \bar{v}]$ for some agent $i \in \{A, B\}$.

If $k \geq \underline{v} + \bar{v}$, the principal assigns the maximal value \bar{v} to agent i with certainty and sells the good to i at price \bar{v} . The remaining value can be assigned to agent j arbitrarily, as agent j never receives the good.

If $k < \underline{v} + \bar{v}$, it is not feasible to assign the maximal value with certainty to one agent. The principal then assigns agent j the minimum value \underline{v} with certainty. For agent i , any bimodal distribution with mean $k - \underline{v}$ and support \underline{v}, h , where $h \in [k - \underline{v}, \bar{v}]$, is optimal. Full surplus extraction can still be achieved by a mechanism that sets the price for agent i equal to h . If agent i does not have valuation h , the principal offers the good to agent j at price \underline{v} . This extracts the entire surplus, as the expected revenue equals

$$\frac{k - \underline{v}}{h - \underline{v}}h + \left(1 - \frac{k - \underline{v}}{h - \underline{v}}\right)\underline{v} = k.$$

By the same logic as in Proposition 2, any other distribution where revenue matches surplus entails either a loss in surplus or a strictly positive information rent, making the class of binary distributions optimal.

Ultimately, the two agents value the goods very differently. This insight extends to an $n + 1$ -agent setting, where as much value as possible is allocated to a single agent. The optimality of this extremely divisive value design raises questions about whether divisive designs remain optimal under more restrictive constraints.

Distributional Constraints We therefore consider pointwise distributional constraints and begin with additive constraints. To define this class of constraints, consider an arbitrary measure defined on the interval $[\underline{v}, \bar{v}]$ with mass two on that interval. Let $H(v)$ denote the associated cumulative distribution.²⁰ We ask how the principal would design the two value distributions (F_A, F_B) if their sum had to be bounded above pointwise by some cumulative measure H , i.e., if

$$F_A(v) + F_B(v) \geq H(v) \quad \text{for all } v \in [\underline{v}, \bar{v}]. \quad (4)$$

²⁰Formally, let \mathcal{H} denote the measure and define $H(v) = \mathcal{H}([\underline{v}, v])$.

We find that the principal maximizes revenue by creating two maximally different value distributions. To state the result, it is useful to define the median v^M for the measure associated with H , as the smallest value \hat{v} such that $\int_{\underline{v}}^{\hat{v}} dH(v) \geq 1$.

Proposition 4. *Value design (F_A^*, F_B^*) is optimal among all designs satisfying (4) if and only if for some agent $i \in \{A, B\}$ and $j \neq i$,*

$$\begin{aligned} F_i^*(v) &= H(v) - 1 && \text{if } v \in [v^M, \bar{v}], \\ F_j^*(v) &= \min\{H(v), 1\} && \text{if } v \in [\underline{v}, v^M]. \end{aligned}$$

Distributions where one agent's values lie below the median of H and the other agent's values lie above the median of H yield higher revenue than any other design fulfilling the constraint. Such value designs fulfill at once the dual objectives of the principal – namely, surplus maximization and information rent minimization. Put differently, the two objectives are aligned at the optimal design, (F_A^*, F_B^*) . These distributions maximize surplus, $\mathbb{E}[\max\{v_A, v_B\}]$. Due to the convexity of the maximum operator, it is optimal to generate the highest possible expected value for one agent (again, one agent with a high value is all that the principal needs). Additionally, the design (F_A^*, F_B^*) minimizes agents' information rents by narrowing the support of both agents' distributions.

To establish the optimality of maximally divisive designs, we first leverage Lemma 2 to show that the constraint must bind at every point because revenue is increasing in first-order stochastic dominance – it is never optimal to destroy surplus. Therefore, we can restrict attention to the setting where $F_A(v) + F_B(v) = H(v)$.

Note that we are not placing any restrictions on the properties of $H(v)$. It may be non-differentiable, contain atoms, or display gaps in the support. We begin by considering the subsets of the support of H where H is differentiable, assuming these exist. We show that on these subsets, F_A and F_B must have disjoint supports. This follows from Lemma 1. Suppose, to the contrary, that there exist values v (with positive measure) for which the density $h(v) = \frac{dH(v)}{dv}$ is split between the two agents. Then, we can show that this split density is a convex combination of two other designs fulfilling (4) in which densities are more extreme. To see this, partition the set of values for which density is split into two subsets with equal measure, W' and W'' . For the subset W' , remove some density ϵ for each value from A and allocate it to B . For the other subset, W'' , remove the same density for each value from B and allocate it to A , such that both are still well-defined cumulative

distributions. This leads to F'_A and F'_B . Now construct another two distributions, F''_A and F''_B , by increasing the density for A in the set W' by ϵ , and decreasing it in the set W'' . Consequently, the density of B must decrease for each value in the set W' , while it increases for all the values in W'' . We can then write F'_A and F''_A as

$$\begin{aligned} F'_A(v) &= F_A(v) - \epsilon(v - \hat{v}), \\ F''_A(v) &= F_A(v) + \epsilon(v - \hat{v}), \end{aligned}$$

where $v \in W'$ and $\hat{v} = \inf(W')$ is the smallest value in the closure of W' . Then, $F_A(v) = \frac{1}{2}F'_A(v) + \frac{1}{2}F''_A(v)$. As the same argument extends to F_B , F'_B , and F''_B , and every value in W' and W'' , we have established that F_A and F_B are a convex combination of some other value designs. Consequently, by Lemma 1, F_A and F_B cannot be optimal, meaning that any value design that splits densities does not maximize revenue. Therefore, at the solution of this problem, the value distributions of the two agents must have supports that are disjoint.

We then establish that, among all value designs with disjoint supports, revenue is maximized by the most divisive design (even when the measure H is atomic). Consider a design with disjoint support where the support for B 's value distribution is $[\underline{v}, w_1] \cup [w_2, w_3]$, while that for A is $[w_1, w_2] \cup [w_3, \bar{v}]$. For such a design, the principal must ensure the incentive compatibility of the mechanism, which requires, for instance, deterring the agents from pretending to have a low value. Thus, the principal must prevent agent B from reporting \underline{v} as their value, even when their realized value v_B is well above the median. Similarly, the principal must deter agent A from reporting w_1 as their value, even when their realized value v_A is close to \bar{v} . However, deterring such large changes in values will require considerable compensation to the agents in the form of information rents.

As a result, the principal benefits from shrinking the supports: giving A all values above the median and giving B all values below the median. In this resulting design, fewer deviations need to be deterred, since A 's largest deviation will be to $v^M > w_1$, and because B will no longer have values above the median. In such a design, the principal will pay less to elicit any given value. Nevertheless, A 's revenue contribution will increase, as they will pay all transfers associated with values above the median (which tend to be larger), while B 's revenue contribution will decrease.

The optimality of maximal value differentiation generalizes to $n+1$ agents. In this case, only the distribution of the two most favored agents is pinned down. One agent receives

all the highest values with measure one, while another agent attains the remaining highest values with measure one. Only these two agents, with the highest values, receive the good with positive probability. Therefore, the value designs for all other agents are arbitrary, subject to them fulfilling the constraint.

5 The Role of Surplus

All of the optimal divisive designs we considered so far were also surplus maximizing within the class of designs considered. This raises two questions: Can optimal designs still be divisive if this does not lead to surplus gains? And can it ever be optimal to create designs that reduce surplus?

5.1 The Surplus-Information Trade-Off

We begin with the second question and show that a decrease in surplus can increase revenue – as long as the reduction in information rents outweighs this surplus loss. This is summarized in Corollary 3.

Corollary 3 (to Proposition 1). *Let $\underline{v} = 0$. Consider any value design (F_A, F_B) in which $\mathbb{E}_{F_A}[v] = \mathbb{E}_{F_B}[v] = k$, $F_B \neq F^{\bar{v}}$, and $\mathbb{E}_{F_A}[\max\{\psi_A(v), 0\}] < m < k$. If so, value design (F_A, F_B) raises less revenue than the value design $(F_m^{\bar{v}}, F_B)$.*

This result builds on Proposition 1 and establishes that the principal always benefits from reducing the mean of one agent’s distribution if they can make their value sufficiently predictable by selecting the binary distribution $F_m^{\bar{v}}$, with mass at zero and \bar{v} and mean m .²¹

With the binary distribution $F_m^{\bar{v}}$, the principal can extract agent A ’s expected value m .²² By construction, the expected value exceeds the expected virtual value that the principal can extract for any other distribution F_A . This result relies on the minimum value \underline{v} being zero, and on F_B being some distribution other than the maximal spread, $F^{\bar{v}}$. If F_B is the maximal spread, multiple distributions F_A are optimal, as described in Proposition 1. Whether it is then optimal to reduce surplus depends on the given F_A . If, for instance, F_A is such that it only allocates mass at the mean, then no improvement can be achieved.²³

²¹For the associated probability distribution, see expression (1), now with mean m instead of k .

²²This can be achieved through the same mechanism as outlined in the discussion after Proposition 1.

²³This result is morally close to [Condorelli and Szentes \(2020\)](#), who show that when the buyer selects

5.2 Fixed Surplus Distribution

We turn to the question whether divisive designs remain optimal if the distribution of surplus is constrained. Namely, we fix a cumulative distribution function G and ask what the optimal design is if the resulting surplus distribution $F_A F_B$ has to be first-order stochastically dominated by G , formally

$$F_A(v)F_B(v) \geq G(v) \quad \text{for all } v \in [\underline{v}, \bar{v}]. \quad (5)$$

As shown in Section C of the Supplementary Appendix, Lemma 2 immediately implies that optimal designs for this constraint always generate full surplus meaning that constraint (5) must hold with equality. Therefore, the surplus generating motive of optimal designs is shut down. The remaining trade-off for the designer is one between reducing information rents by making agents more distinct and enhancing competition by making them more similar. It turns out that some degree of divisiveness tends to balance these countervailing forces best.

To make this point, we begin by comparing the revenue under the symmetric (or equivalently, minimally divisive) design with asymmetric ones. In any symmetric mechanism, the minimally divisive design raises more revenue than any asymmetric design. However, when the principal optimally selects the mechanism, we establish that asymmetric designs improve upon minimally divisive ones when there is sufficient exclusion. While this shows that some divisiveness can be beneficial, maximal divisiveness is no longer optimal when the minimal value equals zero.

In order to characterize the optimal revenue for symmetric designs, define the *minimally divisive design* F^- as the one satisfying $(F_A^-(v), F_B^-(v)) = (\sqrt{G(v)}, \sqrt{G(v)})$ for all $v \in [\underline{v}, \bar{v}]$. For this design, the optimal mechanism reduces to a second price auction with a reserve price r^- . The allocation rule associated with mechanism $(x(r^-), t(r^-))$ satisfies $x_i(v_i, v_j | r^-) = \mathbb{1}(v_i > \max\{v_j, r^-\})$ for any $i \neq j$, meaning an agent i receives the good if their valuation exceeds the reserve price r^- as well as agent j 's valuation. Agent i then pays agent j 's valuation or the reserve price, whichever is higher. The reserve price in the optimal auction can be found by solving $\hat{\psi}^-(r^-) = 0$, whenever there is exclusion.

To calculate the revenue in this auction for arbitrary designs, we therefore require

 their distribution, they are also willing to reduce its expected value. However, in that context the buyer does so in order to have a less predictable value when not excluded, protecting the information rents of these types. Consequently, their optimal design is never binary.

the distribution of the second highest value. Given that we only have two agents the second highest value equals the minimum value. For a design F satisfying $F_A F_B = G$, the minimum is distributed according to J , where

$$J(v) = 1 - (1 - F_A(v))(1 - F_B(v)) = F_A(v) + F_B(v) - G(v).$$

Let $J^-(v) = 2\sqrt{G(v)} - G(v)$ denote the distribution of the minimum value in the minimally divisive design.

The next result establishes that it is not possible to increase revenue through value design when the surplus is fixed and the mechanism is a symmetric auction with some reserve price r , not necessarily the optimal one.

Proposition 5. *The distribution of the minimum value in the minimally divisive design, F^- , first-order stochastically dominates the distribution of the minimum in any other design satisfying $F_A F_B = G$, with $J^-(v) < J(v)$ for all $v \in [\underline{v}, \bar{v}]$.*

Therefore, in any standard auction $(x(r), t(r))$ with reserve price r , revenue from the minimally divisive design exceeds revenue from any other design satisfying $F_A F_B = G$.

The result implies that naively designing value distributions while holding the mechanism constant can lead to a reduction in revenue. This contrasts with our insights from constraint (4). There, even with a symmetric mechanism, the maximally divisive design was revenue maximizing. As we cannot find distributions that outperform the minimally divisive ones in a fixed symmetric mechanism, asymmetric distributions can increase revenue only when coupled with an adjustment in the allocation rule.

Moreover, every design satisfying the constraint yields the same surplus. This means there is a large set of designs that are a priori candidates to be an optimum, making it challenging to identify the revenue-maximizing design and the corresponding mechanism. We therefore select a specific class of designs, threshold designs, and show that such designs when associated with their optimal mechanism outperform the minimally divisive design. A w -threshold design $F^{[w]}$ satisfies for some $w \in [\underline{v}, \bar{v}]$ and some $i \neq j$:

$$(F_i^{[w]}(v), F_j^{[w]}(v)) = \begin{cases} (G(v)/G(w), G(w)) & \text{if } v < w \\ (1, G(v)) & \text{if } v \geq w \end{cases}$$

A threshold design assigns to agent i all possible low values (below threshold w), and to agent j value \underline{v} with probability $G(w)$ and all possible high values (above w), while

ensuring the constraint $F_A F_B = G$ is satisfied. Intuitively, it captures an intermediate level of divisiveness. Agent j receives the high values of G and mass at \underline{v} , which generates a high level of dispersion in values. At the same time, low values are concentrated on agent i . This concentration in combination with allocating the high values to agent j (with the exception of the mass at \underline{v}) generates divisiveness in values across agents.

The dispersion and differentiation of these threshold designs offer two key advantages. First, they restrict possible deviations: agent i can only deviate to values below w , while agent j can only deviate to values above w and \underline{v} ; the latter is a deviation that is irrelevant to the principal. This restriction reduces information rents. Second, they maintain competition between high types of agent i and low types of agent j , reducing the cost of truthful revelation.

To compare the benefits of the threshold design with those of a symmetric distribution, we require the minimally divisive design to be *weakly regular*. Weak regularity holds if G is differentiable and the virtual value $\psi^-(v)$ of \sqrt{G} is increasing when non-negative, with

$$\psi^-(v) = v - \frac{2(\sqrt{G(v)} - G(v))}{g(v)}.$$

This condition relaxes the usual regularity assumption by requiring virtual values to only be increasing for non-excluded values.²⁴ Weak regularity suffices to ensure that allocating the good to the highest non-negative virtual value is always optimal.

We now establish why threshold designs are particularly valuable, when the minimally divisive design leads to significant exclusion.

Proposition 6. *If for some G , the minimally divisive design is weakly regular, and for some h ,*

$$h(G(r^-) - G(h)) > \int_{r^-}^{\bar{v}} (v - r^-) dJ^-(v), \quad (6)$$

then there exist threshold designs raising more revenue than the minimally divisive one.

For the minimally divisive design, revenue in the optimal mechanism has two components. First, the designer receives r^- whenever the highest value exceeds the reserve price r^- , which happens with probability $(1 - G(r^-))$. Additionally, the designer receives the difference between the smallest of the two values and the reserve price whenever the

²⁴Formally, a distribution F is weakly regular if it is differentiable and if for any $v \in [\underline{v}, \bar{v}]$ such that $\psi^-(v) \geq 0$ and any $v' > v$, we have that $\psi^-(v') > \psi^-(v)$.

smallest value exceeds the reserve price – the right-hand side of (6). The latter captures the gains from having agents compete when both of their values are high, relative to posting a price for each agent.

Now consider a r^- -threshold design in which agent j possesses an atom at the minimal value \underline{v} and the values above the threshold. Agent j is first offered the good at price r^- and buys it if $v_j \geq r^-$. If j declines, the good is offered to i at price $h < r^-$, who accepts if $v_i \geq h$. The revenue from agent j simply amounts to $r^-(1 - G(r^-))$, while revenue from i equals $h(G(r^-) - G(h))$ – the left-hand side of (6).²⁵ Thus, when exclusion is significant in the minimally divisive design ($G(r^-)$ is large), (6) holds and threshold designs can outperform the minimally divisive one. This follows because the threshold design allows the principal to segment the market into high and low value agents.

We further highlight in Section C of the Supplementary Appendix that threshold designs outperform proportional designs if the surplus distribution G is uniform. A proportional design satisfies for some $\gamma \in [0, 1]$, $(F_A(v_A), F_B(v_B)) = (G(v_A)^\gamma, G(v_B)^{1-\gamma})$. This has been the approach taken by Cantillon (2008). She shows that in this case symmetric distributions generate higher revenue relative to asymmetric ones in the class of proportional designs. However, symmetric distributions are not necessarily optimal if the mechanism can also be adjusted. Moreover, when the distribution of surplus is ternary, with mass at $\underline{v} = 0$, and two strictly positive values, threshold designs are uniquely optimal as shown in Section C of the Supplementary Appendix.

While some divisiveness, in the form of threshold designs, can improve revenue relative to symmetric designs, maximally divisive designs are often not optimal. In fact, these designs are revenue minimizing whenever $\underline{v} = 0$.

Proposition 7. *If $\underline{v} = 0$, then any maximally divisive design F^+ is revenue-minimizing among designs satisfying $F_A F_B = G$.*

When $\underline{v} = 0$, in the maximally divisive design, one agent has zero value with certainty. Thus, the problem reduces to a single buyer setting in which posted prices are optimal. Yet, with a fixed surplus distribution, any posted price generates identical revenue regardless of the design chosen, since the probability of trade depends only on G via $\Pr(\max\{v_A, v_B\} \geq v) = 1 - F_A(v)F_B(v) = 1 - G(v)$, which is invariant to the design.

²⁵The revenue from agent j equals the probability that j 's valuation exceeds r^- times the reserve price. Revenue from i is the price h times the probability that i 's value exceeds h , multiplied with the probability that j 's value is below r^- , $h(G(r^-) - G(h)) = hG(r^-)(1 - G(h)/G(r^-))$.

However, when both agents' values can be strictly positive and agents have private information about their value, posted prices cannot be optimal. In this case, the principal prefers that agents compete for the good, as this leads to additional transfers. Therefore, for any other design fulfilling the constraint, the principal would select a different mechanism, and obtain a revenue that exceeds that of a posted price – thereby making maximal differentiation revenue-minimizing.

In the Supplementary Appendix, we show why this insight does not necessarily carry over when $\underline{v} > 0$. In fact, when the distribution of surplus G is binary and values are strictly positive, maximal differentiation is the unique optimal design. The principal can extract the full surplus by offering the good to the agent with distribution G at price \bar{v} . If the agent does not purchase the good, then the principal awards the good to the other agent at price \underline{v} . In contrast, any other design requires that both agents can have value \bar{v} . But then full surplus extraction is not possible, as the principal now faces agents with two different, strictly positive values.

In sum, threshold designs, which introduce some asymmetries both within and between different agents, increase revenue relative to symmetric designs whenever there is sufficient exclusion. However, the features of the optimal design will depend on the exact surplus distribution and on whether the minimal value \underline{v} is strictly positive or not, as made precise in Supplementary Appendix C.

6 Conclusion

Our results establish that the divisiveness of a good is generally a valuable feature. Divisiveness raises the value of a good for some agents, while diminishing it for others. This leads to an increase in surplus and simultaneously, a reduction in information rents, which both increase revenue and thus benefit the principal.

When surplus is fixed, then our results are more nuanced in terms of divisiveness. Divisiveness can still be optimal, but so can other designs, potentially even the minimally divisive one. This hints at the importance of surplus generation. However, it is not necessarily the case that a reduction in surplus is always suboptimal – it can be beneficial if it is exceeded by a reduction in information rents.

Our analysis has abstracted from design costs and correlations in values. In the Supplementary Appendix Sections D and E, we show how to extend our insights to these

settings, and explain why correlation in values would simplify our analysis.

Finally, when it comes to product design, our optimal distributions can serve as guidance that can be applied in any context. A product designer should aim for attributes that make goods divisible.

A Proofs

Proof of Lemmas 1 and 2: Convexity and FOSD We begin by deriving a key revenue optimality property of ex-post optimal mechanisms. We then exploit this optimality property to show that revenue is monotone in FOSD order and convex in the joint distribution of values.

Optimality Fix a value design $F \in \Delta(V_A) \times \Delta(V_B)$. Consider any direct, IC, IR mechanism $(x, t) : V_A \times V_B \rightarrow \Delta_-^2 \times \mathbb{R}^2$, with $x_i(\mathbf{v})$ and $t_i(\mathbf{v})$ denoting respectively i 's likelihood of winning the good and i 's payment conditional on reports $\mathbf{v} \in V_A \times V_B$. Denote the ex-ante revenue of such a mechanism when the value design is F by

$$\bar{R}(x, t|F) = \int_V [t_A(\mathbf{v}) + t_B(\mathbf{v})] dF(\mathbf{v}).$$

Recall that the revenue in the optimal direct, IC, IR mechanism is given by

$$R(F) = \max_{x, t} \bar{R}(x, t|F) \text{ subject to } \int_{V_{-i}} x_i(\mathbf{v}) v_i - t_i(\mathbf{v}) dF_{-i}(v_{-i}) \geq \max \left\{ \int_{V_{-i}} x_i(v'_i, v_{-i}) v_i - t_i(v'_i, v_{-i}) dF_{-i}(v_{-i}), 0 \right\},$$

for all $v'_i, v_i \in V_i$ and all $i \in \{A, B\}$. Denote by $(x(F), t(F))$ an optimal mechanism for the selected design F , and note that

$$R(F) = \bar{R}(x(F), t(F)|F).$$

Among optimal mechanisms, select $(x(F), t(F))$ to be dominant strategy incentive compatible. Such a mechanism always exists by Theorem 2 in [Manelli and Vincent \(2010\)](#) which establishes the equivalence between Bayesian IC and dominant strategy IC allocations in IPV settings. Because the mechanism is dominant strategy incentive compatible $x_i(v_i, v_{-i})$ is non-decreasing in v_i for all v_{-i} and all i . Moreover, players' payments are determined by Vickrey's rule

$$t_i(\mathbf{v}|F) = x_i(\mathbf{v}|F) v_i - \int_{\underline{v}_i}^{v_i} x_i(s, v_{-i}|F) ds - (x_i(\underline{v}_i, v_{-i}|F) \underline{v}_i - t_i(\underline{v}_i, v_{-i}|F)).$$

Because in an optimal mechanism the Bayesian individual rationality constraint binds, without loss further select a mechanism in which $x_i(\underline{v}_i, v_{-i}|F) \underline{v}_i = t_i(\underline{v}_i, v_{-i}|F)$ for all v_{-i}

and all i . Because mechanism $(x(F), t(F))$ is dominant strategy incentive compatible and individually rational, it remains incentive compatible and individually rational even when a different design, say $F' \in \Delta(V_A) \times \Delta(V_B)$, is selected.²⁶ This is because in dominant strategy incentive compatible and individually rational mechanisms, agents' incentives are unaffected by the distribution of values. Further, optimality of a mechanism implies the following key property

$$R(F') \geq \bar{R}(x(F), t(F)|F'), \quad (7)$$

with the inequality being strict whenever the distributions differ for non-excluded types.

FOSD Next, consider any design $F' \in \Delta(V_A) \times \Delta(V_B)$ that first-order stochastically dominates F , meaning that

$$F'_i(v_i) \leq F_i(v_i) \text{ for all } v_i \in V_i \text{ and all } i \in \{A, B\}. \quad (8)$$

Observe that by the assumption in expression (8), the joint cumulative distribution $F'_A(v_A)F'_B(v_B)$ first-order stochastically dominates the joint distribution $F_A(v_A)F_B(v_B)$. The monotonicity of revenue in the FOSD order then follows by collecting these insights as

$$\begin{aligned} R(F') &\geq \bar{R}(x(F), t(F)|F') = \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)] dF'(\mathbf{v}) \\ &\geq \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)] dF(\mathbf{v}) = R(F). \end{aligned}$$

The first inequality holds by (7). The second inequality follows because the aggregate transfer $t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)$ is non-decreasing in \mathbf{v} . This follows because the aggregate transfer in the Vickrey rule increases in the aggregate allocation probability, $x_A(\mathbf{v}|F) + x_B(\mathbf{v}|F)$, and because aggregate allocation probability cannot decrease in \mathbf{v} in an optimal mechanism. If the aggregate allocation probability ever decreased on the joint support $V_A \times V_B$, revenue could be increased by selecting an allocation rule that never decreases on the joint support and that is incentive compatible.

²⁶When the support of F differs from that of F' , we can extend the allocation rule, $x(F)$, to cover the entire support of F' while retaining ex-post incentive compatibility and individual rationality by adding constant portions to it.

Convexity Finally, consider any two designs $F', F'' \in \Delta(V_A) \times \Delta(V_B)$ and any number $a \in (0, 1)$ such that

$$aF'_A(v_A)F'_B(v_B) + (1 - a)F''_A(v_A)F''_B(v_B) \leq F_A(v_A)F_B(v_B) \text{ for all } \mathbf{v} \in V_A \times V_B. \quad (9)$$

As before, because the Vickrey mechanism $(x(F), t(F))$ remains ex-post incentive compatible and individually rational even when F' and F'' are selected as value designs, we have that

$$R(F') \geq \bar{R}(x(F), t(F)|F') \text{ and } R(F'') \geq \bar{R}(x(F), t(F)|F''). \quad (10)$$

Convexity then follows by collecting these insights because

$$\begin{aligned} aR(F') + (1 - a)R(F'') &\geq a\bar{R}(x(F), t(F)|F') + (1 - a)\bar{R}(x(F), t(F)|F'') \\ &= \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)](adF'(\mathbf{v}) + (1 - a)dF''(\mathbf{v})) \\ &\geq \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)]dF(\mathbf{v}) = R(F), \end{aligned}$$

where the first inequality holds by averaging the inequalities in (10), while the second inequality follows by (9) as the aggregate transfer $t_A(\mathbf{v}) + t_B(\mathbf{v})$ increases in values. ■

Proof of Proposition 1: One-Agent Mean Bound We compare revenue under $F_A^* = F^{\bar{v}}$ to revenue when setting some other distribution F_A such that $\mathbb{E}_{F_A}[v] = k$. The change from F_A to F_A^* has two effects: (i) it affects the virtual valuation of agent A ; and (ii) it affects the optimal allocation rule for the good. We first establish that revenue under F_A^* is higher than revenue under F_A , even when the allocation rule is suitably fixed. This establishes that F_A^* is an optimal design as optimizing the mechanism could only increase revenue further. We conclude by establishing uniqueness for $F_B \neq F^{\bar{v}}$.

By Theorem 5 in [Monteiro and Svaiter \(2010\)](#), we know that for an arbitrary value design (F_A, F_B) , optimal revenue can always be written as

$$R(F_A, F_B) = \int_{V_A} \psi_A(v)x_A(v)dF_A(v) + \int_{V_B} \psi_B(v)x_B(v)dF_B(v).$$

where ψ_i is a non-decreasing function denoting the *generalized virtual value* of agent i . Further, as shown in Section B of the Supplementary Appendix, the generalized virtual

value for the binary distribution $F_A^* = F^{\bar{v}}$ amounts to

$$\psi_A^*(v) = \begin{cases} \bar{v} & \text{if } v = \bar{v} \\ 0 & \text{if } v = \underline{v} = 0 \end{cases}.$$

Thus, optimal revenue under distribution F_A^* is given by

$$R(F_A^*, F_B) = \int_{V_A^*} \psi_A^*(v) x_A^*(v) dF_A^*(v) + \int_{V_B} \psi_B(v) x_B^*(v) dF_B(v). \quad (11)$$

Next we express the allocation rule, x , that is optimal for value distributions (F_A, F_B) in the quantile space. To do so, recall that for an arbitrary cumulative distribution, F , the inverse is defined as $v_i(q) = \inf\{v \mid F_i(v) \geq q\}$ for all $q \in [0, 1]$. The quantile representation of allocation rule, x , is then

$$y(q_A, q_B) = x(v_A(q_A), v_B(q_B)) \text{ for all } (q_A, q_B) \in [0, 1]^2.$$

Now consider an allocation rule x^+ that under design (F_A^*, F_B) , leads to the same allocation rule y in the quantile space. Formally, define such an allocation rule as follows

$$x^+(v_A, v_B) = y(F_A^*(v_A), F_B(v_B)) \text{ for all } (v_A, v_B) \in V_A^* \times V_B.$$

By construction, allocation rule x^+ under value design (F_A^*, F_B) is feasible, since it coincides in the quantile space with allocation rule x under design (F_A, F_B) – given that $x^+(v_A^*(q_A), v_B(q_B)) = y(q_A, q_B)$. Further, by construction interim allocation rules for x^+ under design (F_A^*, F_B) and for x under design (F_A, F_B) must coincide in the quantile space, and amount to $y_i(q_i) = \int_0^1 y_i(q_i, q_j) dq_j$ for $i \neq j$.

Consider the Vickrey transfer, t^+ , associated with such an allocation rule. Next, we establish that

$$\bar{R}(x^+, t^+ | F_A^*, F_B) \geq R(F_A, F_B).$$

Showing this immediately delivers the result, because revenue would further increase when the mechanism is optimally tailored to the value design (F_A^*, F_B) . Because interim allocations are fixed in the quantile space and virtual values are unchanged for B in either

the value or the quantile space, we have that

$$\int_{V_B} \psi_B(v) x_B^+(v) dF_A(v) = \int_{V_B} \psi_B(v) x_B(v) dF_A(v).$$

To prove the result, it therefore suffices to show that

$$\int_{V_A^*} \psi_A^*(v) x_A^+(v) dF_A^*(v) \geq \int_{V_A} \psi_A(v) x_A(v) dF_A(v).$$

To show this, first observe that the expected virtual value of agent A satisfies

$$\int_{V_A^*} \psi_A^*(v) dF_A^*(v) = \int_{V_A^*} v dF_A^*(v) = k = \int_{V_A} v dF_A(v) \geq \int_{V_A} \max\{\psi_A(v), 0\} dF_A(v),$$

where the inequality follows by individual rationality because the expected transfer from agent A to the designer, when their value distribution is F_A , cannot exceed the expected value of the good. In the quantile space, the latter is equivalent to

$$\int_{[0,1]} \psi_A^*(v_A^*(q)) dq = k \geq \int_{[0,1]} \max\{\psi_A(v_A(q)), 0\} dq. \quad (12)$$

Since the good is only ever allocated to an agent with the highest non-negative generalized virtual value (see Section 4.1 in [Monteiro and Svaiter \(2010\)](#)), we have that $\psi_A(v_A(q)) \in [0, \bar{v}]$ whenever $y_A(q) > 0$. Further $\psi_A^*(v_A^*(q))$ is increasing and satisfies $\psi_A^*(v_A^*(q)) \in \{0, \bar{v}\}$. Because the interim allocation rule y_A is non-decreasing by incentive compatibility, we have that

$$\begin{aligned} \int_{[0,1]} \psi_A^*(v_A^*(q)) dq &\geq \int_{[0,1]} \max\{\psi_A(v_A(q)), 0\} dq \\ \Rightarrow \int_{[0,1]} \psi_A^*(v_A^*(q)) y_A(q) dq &\geq \int_{[0,1]} \psi_A(v_A(q)) y_A(q) dq, \end{aligned}$$

where the implication follows because the allocation rule $y_A(q)$ places more weight on quantiles at which $\psi_A^*(v_A^*(q)) = \bar{v} > \psi_A(v_A(q))$ and less weight on quantiles at which $\psi_A^*(v_A^*(q)) = 0 < \psi_A(v_A(q))$ relative to a scenario in which all quantiles receive the same weight. Because the interim allocation rule was fixed in the quantile space, we can rewrite the final condition in the value space to obtain the desired conclusion, namely that

$$\int_{V_A^*} \psi_A^*(v) x_A^+(v) dF_A^*(v) \geq \int_{V_A} \psi_A(v) x_A(v) dF_A(v).$$

The latter establishes that it is never optimal to select a distribution for A with an expected virtual value which is strictly smaller than the expected value. There are other distributions than the maximal spread that allow for the expected value to be equal to the expected virtual value. But, all these distributions can attach positive probability to at most one strictly positive value. Suppose to the contrary, the principal selected a value distribution in which the agent had more than one strictly positive value. In such instances, the only way for the principal to extract the full surplus would require awarding the good with certainty to A whenever agent A has positive value. But then the principal would not be able to extract the full surplus from agent A , since they could only sell to A at the smallest of their positive values, yielding the contradiction.

To show that F_A^* is the unique optimum when $F_B \neq F^{\bar{v}}$, consider any other binary distribution $F_A = F^h$ with mean k and a single atom on positive values at h . Let $\underline{q} = 1 - (k/h)$ denote the probability of having value zero. Revenue can be written as

$$\begin{aligned} R(F_A, F_B) &= \underline{q} \int_{V_B} \max\{\psi_B(v_B), 0\} dF_B(v_B) + (1 - \underline{q}) \int_{V_B} \max\{\psi_B(v_B), h\} dF_B(v_B) \\ &= \underline{q} \int_{V_B} \max\{\psi_B(v_B), 0\} dF_B(v_B) + \int_{V_B} \max\{(1 - \underline{q})\psi_B(v_B), k\} dF_B(v_B), \end{aligned} \quad (13)$$

where the first equality is derived in Section B of the Supplementary Appendix, and follows because the optimal mechanism allocates the good to B when $v_A = 0$ and B 's generalized virtual valuation is positive, or $v_A = h$ and B 's generalized virtual valuation exceeds h .

Expression (13) is differentiable in \underline{q} . Differentiating (13) with respect to \underline{q} , shows that revenue increases in \underline{q} , meaning that the optimal distribution will set \underline{q} to be as large as possible, or equivalently h to be as large as possible. Letting $V_+(w) = \{v_B \in [0, \bar{v}] | \psi_B(v_B) \geq w\}$ for $w \geq 0$ denote the set of values for which B 's virtual valuation weakly exceeds w , we find that

$$\frac{\partial R(F_A, F_B)}{\partial \underline{q}} = \int_{V_+(0)} \psi_B(v_B) dF_B(v_B) - \int_{V_+(h)} \psi_B(v_B) dF_B(v_B) \geq 0,$$

where the inequality holds, since $V_+(h) \subseteq V_+(0)$. Moreover, the inequality is strict whenever $\psi_B(v_B) \in (0, h)$ for some $v_B \in V_B$. Thus, it is optimal to maximize \underline{q} which is accomplished by setting $h = \bar{v} \geq \bar{v}_B$. This establishes the uniqueness result when $F_B \neq F^{\bar{v}}$. To prove the optimality of designs F^h for $h \in [k, \bar{v}]$ when $F_B = F^{\bar{v}}$, it suffices to show that

any design F^h yields the same revenue as F_A^* , which is immediate and thus omitted. ■

Proof of Proposition 2: Two-Agent Mean Bound The proof proceeds in three steps. First, we demonstrate that the surplus is convex in the agents' realized values, and we leverage this convexity to derive value designs that maximize surplus. Second, we argue that the value designs specified in the proposition simultaneously maximize both surplus and revenue. Third, we establish that no alternative value design can achieve optimality. We conclude by arguing why nothing changes when mean bounds are player specific, $k_i \neq k$.

Convexity in Values Consider any design (F_A, F_B) fulfilling the mean bound. Surplus in this design amounts to

$$S(F_A, F_B) = \int_{\underline{v}_A}^{\bar{v}_A} \int_{\underline{v}_B}^{\bar{v}_B} \max\{v_A, v_B\} dF_B(v_B) dF_A(v_A),$$

where Riemann-Stieltjes integrals are used to calculate the expectations implicitly. Conditional on any realized value of $v_A \in [\underline{v}_B, \bar{v}_B]$, surplus amounts to

$$\begin{aligned} \hat{S}(v_A) &= \int_{\underline{v}_B}^{\bar{v}_B} \max\{v_A, v_B\} dF_B(v_B) = \int_{\underline{v}_B}^{v_A} v_A dF_B(v_B) + \int_{v_A}^{\bar{v}_B} v_B dF_B(v_B) \\ &= v_A F_B(v_A) + [v_B F_B(v_B)]_{v_A}^{\bar{v}_B} - \int_{v_A}^{\bar{v}_B} F_B(v_B) dv_B \\ &= \bar{v}_B - \int_{v_A}^{\bar{v}_B} F_B(v_B) dv_B, \end{aligned}$$

where the second equality is algebraic, the third follows from calculations and integration by parts, and the fourth equality simply factors redundant terms.

For $v_A \geq \bar{v}_B$, conditional surplus amounts to $\hat{S}(v_A) = v_A$ since $\max\{v_A, \bar{v}_B\} = v_A$. For $v_A \leq \underline{v}_B$, conditional surplus amounts to $\hat{S}(v_A) = \underline{v}_B$ since $\max\{v_A, \underline{v}_B\} = \underline{v}_B$. Moreover, $\hat{S}(v_A)$ is differentiable in $v_A \in [\underline{v}_A, \bar{v}_A]$ and it is increasing since for $v_A \in [\underline{v}_B, \bar{v}_B]$,

$$\hat{S}'(v_A) = -\frac{d}{dv_A} \int_{v_A}^{\bar{v}_B} F_B(v_B) dv_B = F_B(v_A)$$

by the Leibniz rule. The derivative equals one, $\hat{S}'(v_A) = 1$, for $v_A > \bar{v}_B$, while it is equal to zero, $\hat{S}'(v_A) = 0$, for $v_A < \underline{v}_B$.

Finally, $\hat{S}(v_A)$ is convex since $\hat{S}(v_A) - \hat{S}(v'_A) \geq (v_A - v'_A) \hat{S}'(v'_A)$ for any values $v_A, v'_A \in$

$[\underline{v}_A, \bar{v}_A]$. To show this, assume without loss that $v_A \geq v'_A$. Observe that when $v'_A > \bar{v}_B$,

$$\hat{S}(v_A) - \hat{S}(v'_A) = v_A - v'_A = (v_A - v'_A)\hat{S}'(v'_A),$$

given that A 's values always exceed B 's value. When $\bar{v}_B \geq v_A \geq v'_A \geq \underline{v}_B$,

$$\begin{aligned} \hat{S}(v_A) - \hat{S}(v'_A) &= \int_{v'_A}^{v_A} F_B(v_B) dv_B \geq \int_{v'_A}^{v_A} F_B(v'_A) dv_B \\ &= (v_A - v'_A)F_B(v'_A) = (v_A - v'_A)\hat{S}'(v'_A), \end{aligned} \tag{14}$$

where the inequality follows because F_B is increasing. When $v_A > \bar{v}_B \geq v'_A \geq \underline{v}_B$,

$$\begin{aligned} \hat{S}(v_A) - \hat{S}(v'_A) &= v_A - \bar{v}_B + \hat{S}(\bar{v}_B) - \hat{S}(v'_A) \geq v_A - \bar{v}_B + (\bar{v}_B - v'_A)\hat{S}'(v'_A) \\ &\geq (v_A - v'_A)\hat{S}'(v'_A), \end{aligned}$$

where the equality follows because $\hat{S}(v_A) = v_A$ for $v_A > \bar{v}_B$ and $\hat{S}(\bar{v}_B) = \bar{v}_B$, the first inequality follows from (14), and the second inequality follows from $\hat{S}'(v'_A) \leq 1$ and $v_A > \bar{v}_B$. When $v_A > \bar{v}_B$ and $\underline{v}_B > v'_A$,

$$\hat{S}(v_A) - \hat{S}(v'_A) = v_A - \underline{v}_B \geq 0 = (v_A - v'_A)\hat{S}'(v'_A),$$

where the inequality follows as $v_A > \bar{v}_B$ and the final equality follows as $\hat{S}'(v'_A) = 0$.

When $\bar{v}_B \geq v_A \geq \underline{v}_B > v'_A$,

$$\hat{S}(v_A) - \hat{S}(v'_A) \geq 0 = (v_A - v'_A)\hat{S}'(v'_A),$$

where the inequality follows because conditional surplus is increasing and the equality as $\hat{S}'(v'_A) = 0$. Finally, for $\underline{v}_B > v_A$, we have that

$$\hat{S}(v_A) - \hat{S}(v'_A) = (v_A - v'_A)\hat{S}'(v'_A) = 0,$$

given that B 's values always exceed A 's. Thus, surplus conditional on v_A , $\hat{S}(v_A)$, is globally convex in v_A given that \hat{S} is differentiable and $\hat{S}(v_A) - \hat{S}(v'_A) \geq (v_A - v'_A)\hat{S}'(v'_A)$.

Surplus Maximizing Designs The surplus maximizing design F_A given F_B must solve

$$\max_{F_A} \int_{\underline{v}}^{\bar{v}} \hat{S}(v_A) dF_A(v_A) \text{ subject to } \int_{\underline{v}}^{\bar{v}} v_A dF_A(v_A) \leq k.$$

Because conditional surplus is convex and increasing in v_A while the constraint is linear, surplus maximization requires maximally spreading the distribution of v_A while keeping the mean at k – meaning that setting $F_A = F^{\bar{v}}$ maximizes surplus for any F_B . Thus, the convexity of conditional surplus in realized values implies that for any distribution of values F_j , surplus will be maximized by any design (F_A, F_B) satisfying $F_i = F^{\bar{v}}$ for some $i \in \{A, B\}$ and $\int_{\underline{v}}^{\bar{v}} v_j dF_j(v_j) = k$ for $j \neq i$. In any such design, surplus amounts to

$$\begin{aligned} S^* = S(F_A, F_B) &= P^{\bar{v}}(\bar{v}) \int_{\underline{v}_j}^{\bar{v}_j} \max\{\bar{v}, v_j\} dF_j(v_j) + P^{\bar{v}}(\underline{v}) \int_{\underline{v}_j}^{\bar{v}_j} \max\{\underline{v}, v_j\} dF_j(v_j) \\ &= P^{\bar{v}}(\bar{v}) \int_{\underline{v}_j}^{\bar{v}_j} \bar{v} dF_j(v_j) + P^{\bar{v}}(\underline{v}) \int_{\underline{v}_j}^{\bar{v}_j} v_j dF_j(v_j) = \frac{k - \underline{v}}{\bar{v} - \underline{v}} \bar{v} + \frac{\bar{v} - k}{\bar{v} - \underline{v}} k, \end{aligned}$$

where the first equality follows from the definition of surplus, the second equality follows because $v_j \in [\underline{v}, \bar{v}]$, and the third as $\int_{\underline{v}_j}^{\bar{v}_j} v_j dF_j(v_j) = k$.

Revenue Optimality For any value design, surplus must bound revenue from above. This follows because inefficient optimal allocations may result in less surplus being produced for particular value realizations, and because the utility of any player i is non-negative by individual rationality, $U_i(F_A, F_B) = \int_{\mathbf{V}} [v_i x_i(\mathbf{v}|F) - t_i(\mathbf{v}|F)] dF(\mathbf{v}) \geq 0$. Therefore,

$$S(F_A, F_B) \geq R(F_A, F_B) + U_A(F_A, F_B) + U_B(F_A, F_B) \geq R(F_A, F_B),$$

and revenue is no larger than surplus in any value design. But for any of the proposed optimal design in the statement of the result, we have that $R(F_A, F_B) = S^*$, and thus these designs must indeed be optimal.

To see this, first, suppose $\underline{v} > 0$. Consider the surplus maximizing design satisfying $(F_i^*, F_j^*) = (F^{\bar{v}}, F^k)$ for some $i \neq j$, and the following IC mechanism

$$x_i(v) = \mathbb{1}(v_i = \bar{v}), \quad x_j(v) = 1 - x_i(v), \quad t_i(v) = x_i(v)\bar{v}, \quad t_j(v) = x_j(v)k.$$

Revenue in such a mechanism coincides with surplus and amounts to

$$R(F_A, F_B) = P^{\bar{v}}(\bar{v})\bar{v} + P^{\underline{v}}(\underline{v})k = S^*.$$

Next, suppose $\underline{v} = 0$. Consider any surplus maximizing design satisfying $(F_i^*, F_j^*) = (F^{\bar{v}}, F^h)$ for some $i \neq j$, and the following IC mechanism

$$x_i(v) = \mathbb{1}(v_i = \bar{v}), \quad x_j(v) = \mathbb{1}(v_j = h, v_i \neq \bar{v}), \quad t_i(v) = x_i(v)\bar{v}, \quad t_j(v) = x_j(v)h.$$

Revenue in such a mechanism coincides with surplus and amounts to

$$R(F_A, F_B) = P^{\bar{v}}(\bar{v})\bar{v} + P^{\underline{v}}(\underline{v})P^h(h)h = P^{\bar{v}}(\bar{v})\bar{v} + P^{\underline{v}}(\underline{v})k = S^*.$$

where we have used $P^h(h)h = k$ by assumption.

No other surplus-maximizing design can be optimal though, because the only way to ensure that revenue equals S^* entails guaranteeing that the generalized virtual values are equal to values, $\psi_i(v_i) = v_i$, whenever trade occurs. But for the latter to hold for agent j , their value distribution F_j can have either a single atom at the mean k , or only two atoms with one atom at 0 and the other above the mean k – which is only possible if $\underline{v} = 0$. In any other surplus-maximizing design, the designer needs to pay information rents to some type of agent j to get them to reveal their type truthfully and thus cannot extract the full surplus. ■

Proof of Proposition 3: Reallocating Value, Fixed Mean The proof establishes the result for $k \in [2\underline{v}, \underline{v} + \bar{v}]$, since the other cases are immediate. To begin, we characterize the optimal designs when $\mathbb{E}_{F_i}[v] = k_i$ for some $i \in \{A, B\}$ and $\mathbb{E}_{F_j}[v] = k - k_i$ for $j \neq i$. We will then let the designer choose $k_i \in [\underline{v}, k - \underline{v}]$ to maximize revenue.

As in the first part of the proof of Proposition 2, the surplus-maximizing design F_i given F_j must solve

$$S^+(k_i) = \max_{F_i} \int_{\underline{v}}^{\bar{v}} \hat{S}(v_i) dF_i(v_i) \text{ subject to } \int_{\underline{v}}^{\bar{v}} v_i dF_i(v_i) = k_i.$$

As before, because \hat{S} is convex and increasing in v_i while the constraint is linear, there exists a surplus-maximizing value design F_i which is maximally spread and has mean k_i . In particular, surplus will be maximized by designs (F_A, F_B) satisfying $F_i(v) = (\bar{v} - k_i)/(\bar{v} - \underline{v})$

for all $v < \bar{v}$ and some $i \in \{A, B\}$ and $\int_{\underline{v}}^{\bar{v}} v_j dF_j(v_j) = k - k_i$ for $j \neq i$. In these designs, surplus amounts to

$$S^+(k_i) = \frac{k_i - \underline{v}}{\bar{v} - \underline{v}} \bar{v} + \frac{\bar{v} - k_i}{\bar{v} - \underline{v}} (k - k_i).$$

Surplus S^+ is convex in k_i , decreasing for $k_i < k/2$, and increasing thereafter. Thus when selecting k_i to maximize surplus, optimality requires k_i to either be equal to \underline{v} or to $k - \underline{v}$. For both of these values of k_i , surplus is maximized and $S^+(\underline{v}) = S^+(k - \underline{v}) = k - \underline{v}$.

This completes the proof as for any of the designs in the statement of the result $R(F_A^*, F_B^*) = k - \underline{v}$, given that the principal either trades with j at price h or sells to i at price \underline{v} . ■

Proof of Proposition 4: Reallocating Value, Fixed Distribution We want to show that the maximally divisive design

$$\begin{aligned} F_B^*(v) &= \min\{H(v), 1\} && \text{if } v \in [\underline{v}, v^M) \\ F_A^*(v) &= H(v) - 1 && \text{if } v \in [v^M, \bar{v}], \end{aligned}$$

is optimal. To do so, we first argue that the constraint in (4) must bind almost surely. We then show that in an optimal design F , the supports of the two value distributions must have disjoint interiors. We conclude by establishing that the maximally divisive design $(H - 1, \min\{H, 1\})$ leads to higher revenue than any other value design with disjoint interiors.

Binding Constraint To see why the constraint must bind almost surely in some optimal design, posit that some optimal design F satisfies $F_A(v) + F_B(v) > H(v)$ on some set W with positive measure. If so, there exists a value design $F' \succ_1 F$ that satisfies constraint (4) and first-order stochastically dominates the original design F —implying that $R(F') \geq R(F)$.

Disjoint Interiors Next, we show that because revenue is convex in the joint distribution of values, and the set of value designs meeting the constraint $F_A + F_B = H$ is convex, optimal value distributions for the two agents must have essentially disjoint supports. For an arbitrary set V , denote its interior by $\overset{\circ}{V}$. Formally, this part of the proof establishes that in any optimal design the intersection between the interior of the two supports must

be empty,

$$\mathring{V}_A \cap \mathring{V}_B = \emptyset.$$

To prove this, proceed by contradiction and assume that there exists an optimal design $F = (F_A, F_B)$ that satisfies $F_A + F_B = H$ and $W = \mathring{V}_A \cap \mathring{V}_B \neq \emptyset$. If so, for any $v \in W$, we have that $F_A(v), F_B(v) \in (0, 1)$, since v belongs to the interior of both supports. Furthermore, for any $v, v' \in W$ such that $v' > v$, we have that $F_i(v') - F_i(v) > 0$, because cumulative distributions strictly increase in the interior of the support.

This step of the proof hinges on redistributing measure between F_A and F_B on the set W . To ensure that the resulting new measures are proper distributions, the measure redistributed must be capped. To do so, construct a function $\varepsilon : W \rightarrow \mathbb{R}_+$ that satisfies for any $v, v' \in W$ such that $v' > v$

$$\varepsilon(v') - \varepsilon(v) \in (-\min_i\{F_i(v') - F_i(v)\}, \min_i\{F_i(v') - F_i(v)\}). \quad (15)$$

Condition (15) requires that any changes in the function ε are smaller in absolute value than changes in either of the cumulative distributions. It is always possible to find a function ε fulfilling (15) because $F_i(v') > F_i(v)$ in the interior of the support.

Now construct two alternative designs F' and F'' as follows. Let

$$F'_A(v) = \begin{cases} F_A(v) + \varepsilon(v) & \text{if } v \in W \\ F_A(v) & \text{if } v \in V \setminus W \end{cases} \quad \text{and} \quad F''_A(v) = \begin{cases} F_A(v) - \varepsilon(v) & \text{if } v \in W \\ F_A(v) & \text{if } v \in V \setminus W \end{cases},$$

and set $F'_B = H - F'_A$ and $F''_B = H - F''_A$. Condition (15) ensures that all cumulative distributions remain increasing in both value designs F' and F'' , because the changes in ε are small in absolute value relative to the changes in F_i . Further we have that

$$F_A(v_A)F_B(v_B) \geq F_A(v_A)F_B(v_B) - \varepsilon(v_A)\varepsilon(v_B) = \frac{F'_A(v_A)F'_B(v_B)}{2} + \frac{F''_A(v_A)F''_B(v_B)}{2}.$$

The convexity of revenue then delivers a contradiction, because F cannot be optimal since

$$R(F) < R(F')/2 + R(F'')/2 < \max\{R(F'), R(F'')\},$$

where the inequalities are strict by selecting $\varepsilon(v)$ to be strictly positive on W .

Convex Support For any optimal value design, the next lemma shows that because supports, V_A and V_B , have disjoint interiors, they must also be convex.

Lemma 3. *If a value design is optimal subject to (4), it has convex supports V_A and V_B .*

This lemma is proven in the Supplementary Appendix. Thus, the maximally divisive design is the unique value design satisfying $F_A + F_B = H$ with disjoint convex supports, establishing its optimality. ■

Proof of Proposition 5: Auctions and Surplus Bound To prove the result on second-order statistics, fix an arbitrary value $v \in [\underline{v}, \bar{v}]$, and consider the following relaxed problem (which does not account for the restriction requiring cumulative distributions to increase),

$$\min_{F_A(v), F_B(v)} F_A(v) + F_B(v) - G(v) \text{ s.t. } F_A(v)F_B(v) \geq G(v).$$

The objective function of this problem is linear, and the constraint set is quasi-convex. Taking first-order conditions implies that at the minimum

$$F_A(v) = F_B(v) = \sqrt{G(v)}.$$

Because $\sqrt{G(v)} \in [0, 1]$ and $\sqrt{G(v)}$ is increasing, for all $v \in [\underline{v}, \bar{v}]$, this procedure delivers a proper value design $(F_A^-, F_B^-) = (\sqrt{G}, \sqrt{G})$. Further by construction, we have that for this value design

$$J^-(v) = 2\sqrt{G(v)} - G(v) < F_A(v) + F_B(v) - G(v) = J(v).$$

Thus, the distribution of the second-order statistic J^- associated with the minimally divisive design (F_A^-, F_B^-) first-order stochastically dominates the distribution of the second-order statistic J associated with any other design (F_A, F_B) satisfying $F_A F_B = G$.

To prove the result on standard auctions, consider any standard auction with reserve price r —meaning that

$$x_i(\mathbf{v}|r) = \mathbb{1}(v_i > \max\{v_j, r\}).$$

Select any incentive compatible payment rule for such an auction, for instance $t_i(\mathbf{v}|r) = \max\{v_j, r\} \mathbb{1}(v_i > \max\{v_j, r\})$. By revenue equivalence, the revenue in this auction is

maximized by the minimally divisive design, since

$$\begin{aligned}\bar{R}(x(r), t(r)|F_A, F_B) &= r(J(r) - G(r)) + \int_r^{\bar{v}} v dJ(v) = \int_r^{\bar{v}} \max\{v, r\} dJ(v) - rG(r) \\ &\leq \int_r^{\bar{v}} \max\{v, r\} dJ^-(v) - rG(r) = \bar{R}(x(r), t(r)|F_A^-, F_B^-),\end{aligned}$$

where the inequality follows since $J^- \succeq_1 J$ and $\max\{r, v\}$ is increasing in v . ■

Proof of Proposition 6: The Benefits of Divisive Designs If the minimally divisive design is weakly regular, its revenue amounts to

$$R(F^-) = 2r^- \left(\sqrt{G(r^-)} - G(r^-) \right) + \int_{r^-}^{\bar{v}} v dJ^-(v) = (1 - G(r^-))r^- + \int_{r^-}^{\bar{v}} (v - r^-) dJ^-(v).$$

Fix some number $h < r^-$ and a threshold design with $w = r^-$, namely $F^{[r^-]}$, in which B has low values. Then consider a mechanism (x, t) satisfying

$$\begin{aligned}x_A(v_A, v_B) &= \mathbb{1}(v_A \geq r^-) \text{ and } x_B(v_A, v_B) = \mathbb{1}(v_B \geq h \text{ and } v_A = \underline{v}), \\ t_A(v_A, v_B) &= \mathbb{1}(v_A \geq r^-)r^- \text{ and } t_B(v_B, v_A) = \mathbb{1}(v_B \geq h \text{ and } v_A = \underline{v})h.\end{aligned}$$

In this mechanism, the good is sold to A at price r^- when their value exceeds r^- , and is offered to B at price h otherwise. The revenue from this mechanism for threshold design $F^{[r^-]}$ amounts to

$$\bar{R}(x, t|F^{[r^-]}) = (1 - G(r^-))r^- + (G(r^-) - G(h))h.$$

The result then follows since

$$R(F^{[r^-]}) \geq \bar{R}(x, t|F^{[r^-]}) > r^-(1 - G(r^-)) + \int_{r^-}^{\bar{v}} (v - r^-) dJ^-(v) = R(F^-),$$

given the assumptions invoked on G . ■

Proof of Proposition 7: Maximal Divisiveness and Surplus Bounds To prove the result, let $\underline{v} = 0$ and consider the maximally divisive value design $F^+ = (G, 1)$. By the optimality of posted prices in single-buyer settings, the optimal mechanism for value design F^+ has a simple structure. It requires selling to bidder A at some fixed price r^+ , whenever A is willing to purchase at r^+ , while not selling the good otherwise—for

instance, when G is regular, $r^+ = \psi_G^{-1}(0)$. The same revenue though can also be obtained with any other value design (\hat{F}_A, \hat{F}_B) satisfying $\hat{F}_A(v)\hat{F}_B(v) = G(v)$ for all v . This can be accomplished by selecting a mechanism (x, t) in which the principal sets again r^+ as the price of the good and sells to one of the agents willing to pay said amount. Revenue from such a mechanism, (x, t) , coincides with the optimal revenue for the maximally divisive design F^+ , since

$$R(F^+) = r^+(1 - G(r^+)) = r^+ \Pr(\max\{v_A, v_B\} \geq r^+) = \bar{R}(x, t|\hat{F}) \leq R(\hat{F}),$$

where the inequality follows as posted price mechanisms need not be optimal and are generically suboptimal when both agents can have positive value for the good. ■

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DIVISIVE BY DESIGN: SUPPLEMENTARY

APPENDIX

Anja Prummer* Francesco Nava†

February 23, 2026

Contents

A Applications	2
B Designing Mechanisms with General Distributions	3
C Fixed Surplus Constraint: Additional Results	8
D Correlated Private Values	12
E Cost of Value Designs	14
F Proofs: Corollaries and Appendix Results	15

*Freie Universität Berlin and Berlin School of Economics, anja.prummer@fu-berlin.de.

†London School of Economics, f.nava@lse.ac.uk.

A Applications

Consider a celebrity auction. In these auctions, celebrities offer fans the opportunity to spend time with them for a specific activity.¹ The activity, which serves as a feature of the good, can either increase, decrease, or have no effect on the value of certain agents. To make this example more concrete, imagine a famous tennis player participating in such an auction. The tennis player could offer a casual tennis match, an ambitious training session, or, alternatively, a dinner with the auction winner. The type of good—whether a celebrity plus casual tennis, a celebrity plus intense training session, or a celebrity plus dinner—can significantly influence valuations and has the potential to help screen bidders’ preferences. Some bidders may prefer an intense training session, others may simply want to meet their favorite tennis player and play a casual round, while some might favor the dinner to have the opportunity for more personal interaction. Our results suggest that the activity chosen should be divisive – catering strongly to the preferences of specific bidders, thereby increasing surplus and facilitating better screening.

Similarly, our model applies to the art market. An artist creating an object with the goal of selling it should aim to incorporate features that increase surplus while minimizing information rents. The artist can select the art form (e.g., sculpture or painting), the materials used, and the size of the object. Each of these decisions has the potential to influence bidders’ preferences and, in turn, their bids (Bocart, Gertsberg, and Pownall, 2022). For example, choosing the color blue in painting has been associated with higher revenues (Ma, Noussair, and Renneboog, 2022). Similarly, larger paintings generate more income, even after accounting for the increased material costs.² The artist’s goal is then to create a piece that increases the value for buyers and, additionally, better screens their valuations. This helps explain why art often polarizes, with some customers valuing it substantially while others may feel alienated.

¹For example, this can be booked through <https://www.charitybuzz.com/>.

²See, for instance, https://www.gallerytoday.com/blog/122_what-determines-the-price-of-a-painting-does-size-matter.html, which argues that this is due to larger paintings being seen as more prestigious.

B Designing Mechanisms with General Distributions

Our analysis in the main paper relies on optimal value designs that frequently feature atomic distributions with mass points at specific values. In this section, we demonstrate how to apply classical revenue equivalence insights when agents' value distributions, F_A and F_B , may contain atoms or singular continuous parts. A more general treatment of similar topics appears in [Monteiro and Svaiter \(2010\)](#). In this section we focus on how to derive the specific revenue expressions appearing in the proofs of Propositions 1, 2 and 4.

We use Riemann-Stieltjes integrals to keep the notation compact. This means that the differential accounts for jumps at points of discontinuity of F and continuity points at which the density may not exist. To gain intuition, note that when the measure has no singular continuous part, for a cumulative distribution F with support V and a function $u : V \rightarrow \mathbb{R}$, we have

$$\int_V u(\mathbf{v}) dF(\mathbf{v}) = \int_S u(\mathbf{v}) \frac{\partial F(\mathbf{v})}{\partial \mathbf{v}} d\mathbf{v} + \sum_{\mathbf{v} \in D} u(\mathbf{v}) [F_+(\mathbf{v}) - F_-(\mathbf{v})],$$

where $D \subseteq V$ denotes the set of discontinuity points of F , $S \subseteq V$ denotes the set of points where F is differentiable, and $F_+(\mathbf{v})$ and $F_-(\mathbf{v})$ denote the right and left limits of F at \mathbf{v} , respectively.

To prove the desired result, it is convenient to proceed in the quantile space. For any $q \in [0, 1]$, let $v_i(q) = \inf\{v_i \in [\underline{v}, \bar{v}] | F_i(v_i) \geq q\}$ denote the value associated to the q -th quantile in agent i 's distribution. Observe that by construction $v_i(q)$ is left-continuous since cumulative distributions are right-continuous.

Since the revelation principle holds irrespective of distributional assumptions, we begin by considering in the quantile space a direct mechanism $(y, p) : [0, 1]^2 \rightarrow \Delta(A, B, 0) \times \mathbb{R}^2$. Define the interim allocation probability and the expected transfer of agent i reporting quantile q'_i , when others are sincere, as follows

$$y_i(q'_i) = \int_0^1 y_i(q'_i, q_{-i}) dq_{-i},$$

$$p_i(q'_i) = \int_0^1 p_i(q'_i, q_{-i}) dq_{-i}.$$

With this notation, the expected payoff of type q_i claiming to be q'_i is given by

$$v_i(q_i)y_i(q'_i) - p_i(q'_i).$$

As usual, a mechanism (y, p) is (Bayesian) incentive compatible when truth-telling is optimal for all agents i all quantiles q_i , meaning that

$$v_i(q_i)y_i(q_i) - p_i(q_i) = \max_{q'_i \in [0,1]} [v_i(q_i)y_i(q'_i) - p_i(q'_i)] \equiv u_i(q_i).$$

Even when value distributions are not well-behaved, incentive compatibility can be restated as requiring that the interim probability of winning the object increases in the agent's type, as the following lemma shows.

Lemma: A direct mechanism is incentive compatible for agent i if and only if $y_i(q_i)$ is non-decreasing and $du_i(q_i) = y_i(q_i)dv_i(q_i)$ for all $q_i \in [0, 1]$.

Proof (necessity): If incentive compatibility holds for agent i , for any two values $q_i, q'_i \in [0, 1]$, we have that

$$u_i(q'_i) \geq v_i(q'_i)y_i(q_i) - p_i(q_i) = u_i(q_i) + (v_i(q'_i) - v_i(q_i))y_i(q_i).$$

The same inequality holds when switching q_i and q'_i , so we obtain

$$(v_i(q'_i) - v_i(q_i))y_i(q'_i) \geq u_i(q'_i) - u_i(q_i) \geq (v_i(q'_i) - v_i(q_i))y_i(q_i). \quad (\text{B.1})$$

Condition (B.1) also implies that

$$(v_i(q'_i) - v_i(q_i))(y_i(q'_i) - y_i(q_i)) \geq 0,$$

which is equivalent to $y_i(q_i)$ being non-decreasing. Dividing both sides of (1) by $(q'_i - q_i)$ and taking the limit as $q'_i \rightarrow q_i$ yields

$$\lim_{q'_i \rightarrow q_i} \frac{u_i(q'_i) - u_i(q_i)}{q'_i - q_i} = \frac{du_i(q_i)}{dq_i} = \frac{dv_i(q_i)}{dq_i}y_i(q_i),$$

or equivalently $du_i(q_i) = y_i(q_i)dv_i(q_i)$.

Proof (sufficiency): If $du_i(q_i) = y_i(q_i)dv_i(q_i)$, the expected payoff by fundamental the-

orem of calculus satisfies

$$u_i(q_i) - u_i(0) = \int_0^{q_i} y_i(q) dv_i(q).$$

Thus, for any q_i and any q'_i , we have

$$u_i(q'_i) - u_i(q_i) = \int_0^{q'_i} y_i(q) dv_i(q) - \int_0^{q_i} y_i(q) dv_i(q) \geq (v_i(q'_i) - v_i(q_i))y_i(q_i),$$

where the inequality holds because $y_i(q_i)$ is non-decreasing. This inequality is equivalent to incentive compatibility. ■

We can use this restatement of incentive compatibility to derive the classical revenue equivalence result for this setting.

Revenue Equivalence Theorem: In any incentive compatible, direct mechanism (y, p) , the interim transfer of agent i with quantile q_i is given by:

$$p_i(q_i) = p_i(0) - v_i(0)y_i(0) + v_i(q_i)y_i(q_i) - \int_0^{q_i} y_i(q) dv_i(q).$$

Proof: Since $u_i(q_i) = v_i(q_i)y_i(q_i) - p_i(q_i)$ and $u_i(0) = v_i(0)y_i(0) - p_i(0)$, incentive compatibility implies that

$$v_i(q_i)y_i(q_i) - p_i(q_i) = v_i(0)y_i(0) - p_i(0) + \int_0^{q_i} y_i(q) dv_i(q).$$

Thus, interim transfers in any two incentive compatible mechanisms with the same allocation rule must coincide up to a constant. ■

Revenue equivalence implies that even when value distributions are discontinuous, the designer's problem coincides with the maximization of virtual surplus. To see this, recall that a mechanism is individually rational if

$$u_i(q_i) \geq 0 \quad \text{for all } q_i.$$

When incentive compatibility holds, individual rationality (IR) simplifies to

$$u_i(0) \geq 0 \quad \text{or} \quad p_i(0) \leq v_i(0)y_i(0).$$

By revenue equivalence, we also know that in an incentive-compatible and direct mechanism, the expected payment of agent i is given by

$$\mathbb{E}[p_i(q_i)] = -u_i(0) + \int_0^1 v_i(q_i) y_i(q_i) dq_i - \int_0^1 \int_0^{q_i} y_i(q) dv_i(q) dq_i. \quad (\text{B.2})$$

Exploiting Riemann-Stieltjes integrals, we can change the order of integration to obtain

$$\int_0^1 \int_0^{q_i} y_i(q) dv_i(q) dq_i = \int_0^1 \int_q^1 dq_i y_i(q_i) dv_i(q) = \int_0^1 (1 - q_i) y_i(q_i) dv_i(q_i).$$

By replacing this expression in condition (B.2) and collecting terms, we obtain that

$$\mathbb{E}[p_i(q_i)] = -u_i(0) + \int_0^1 y_i(q) d[(q_i - 1)v_i(q_i)].$$

With this notation, the principal's revenue from (y, p) can as usual be written as

$$\hat{R}(y, p | F_A, F_B) = \sum_{i \in N} \left[-u_i(0) + \int_0^1 y_i(q_i) d[(q_i - 1)v_i(q_i)] \right]. \quad (\text{B.3})$$

Next consider $(q_i - 1)v_i(q_i)$ and $y_i(q_i)$ as the two parts of the integral. Observe that the two parts have bounded variation and no common jumps, since $(q_i - 1)v_i(q_i)$ is left-continuous by construction, while $y_i(q_i)$ can be selected to be right-continuous without loss.³ Thus, even if F has atoms or singular continuous parts, we can apply standard integration by parts to obtain that

$$\hat{R}(y, p | F_A, F_B) = \sum_{i \in \{A, B\}} \left[v_i(0) y_i(0) + \int_0^1 v_i(q_i) (1 - q_i) dy_i(q_i) \right].$$

As with non-atomic distributions, the optimal mechanism can be found by maximizing this expression for revenue subject to incentive compatibility and individual rationality, which require $y_i(q_i)$ to be non-decreasing and $u_i(0) \geq 0$ for all agents i . Formally, a feasible interim allocation rule is optimal for suitable transfers if and only if it solves

$$\max_y \hat{R}(y, p | F_A, F_B) \quad \text{subject to} \quad (v_i(q'_i) - v_i(q_i))(y_i(q'_i) - y_i(q_i)) \geq 0 \quad \text{for all } q'_i, q_i \in [0, 1],$$

and $u_i(0) = 0$ since individual rationality binds at the optimum.

³Allocation probabilities on gaps are irrelevant for revenue as long as incentive compatibility holds.

Implication Proposition 4: To map back to the value space, define the corresponding direct mechanism $(x(\mathbf{v}), t(\mathbf{v})) = (y(F_A(v_A), F_B(v_B)), p(F_A(v_A), F_B(v_B)))$. Observe that because F_i is increasing and y_i is left-continuous x_i must be right-continuous. With this notation and requiring individual rationality to bind at the lowest value $u_i(0) = 0$, equation (B.3) can be rewritten in the value space as

$$R(x, t|F_A, F_B) = \sum_{i \in \{A, B\}} \left[\underline{v}_i x_i(\underline{v}_i) + \int_{V_i} v(1 - F_i(v)) dx_i(v) \right]. \quad (4)$$

This is the revenue expression found in expression (F.4) of the proof of Lemma 3.

Implication Proposition 1: When F_i is a binary distribution with only values \underline{v} and h in the support and $F_i(\underline{v}) = \underline{q}$, we can further simplify expression (B.3) since

$$\int_0^1 y_i(q_i) d[(q_i - 1)v_i(q_i)] = \int_0^{\underline{q}} \underline{v} y_i(q_i) dq_i + \int_{\underline{q}}^1 h y_i(q_i) dq_i + y_i(\underline{q})(\underline{q} - 1)(h - \underline{v}).$$

When $\underline{v} = 0$, this further reduces to

$$\int_0^1 y_i(q_i) d[(q_i - 1)v_i(q_i)] = \int_{\underline{q}}^1 h y_i(q_i) dq_i + y_i(\underline{q})(\underline{q} - 1)h$$

Define $\hat{\psi}_i(v_i) = h\mathbb{1}(v_i = h)$. Because in any optimal mechanism $y_i(\underline{q}) = 0$ (the good is never allocated when the value equals zero), the previous expression can be rewritten in the value space as

$$\int_0^1 y_i(q_i) d[(q_i - 1)v_i(q_i)] = \int_{V_i} \hat{\psi}_i(v_i) x_i(v_i) dF_i(v_i),$$

which amounts to expression (11) in the proof of Proposition 1. Here $\hat{\psi}_i$ represents the generalized virtual value of agent i .

Implication Proposition 2: Since the good is only ever allocated to an agent with the highest non-negative generalized virtual value (see Section 4.1 in Monteiro and Svaiter (2010)), we further obtain expression (13) in the proof of Proposition 2, namely

$$R(F_i, F_j) = \underline{q} \int_{V_j} \max\{\psi_j(v_j), 0\} dF_j(v_j) + (1 - \underline{q}) \int_{V_j} \max\{\psi_j(v_j), h\} dF_j(v_j).$$

C Fixed Surplus Constraint: Additional Results

Binding Constraint We begin by showing that in any optimal design the surplus constraint must bind for all non-excluded values. A value $v \in [\underline{v}, \bar{v}]$ is non-excluded by the optimal mechanism if $x_A(v|F^*) + x_B(v|F^*) > 0$.

Proposition A1. *Value design (F_A^*, F_B^*) is optimal among all designs satisfying (5) only if $F_A^*(v)F_B^*(v) = G(v)$ for all non-excluded values $v \in [\underline{v}, \bar{v}]$.*

The result follows because if the constraint was slack at some non-excluded value v in the support of a design F , one could construct a design F' that first-order stochastically dominates F while still satisfying the constraint. Revenue would increase strictly whenever the constraint is slack for some non-excluded values, precluding the optimality of F .

Binary and Ternary Distributions While we have demonstrated that asymmetric designs when accompanied by an appropriately asymmetric mechanism can improve upon symmetric designs, establishing optimality is challenging. The optimal design depends on whether the minimal value \underline{v} equals or exceeds zero as well as the specific surplus distribution G . If the minimal value exceeds zero, the maximally divisive design, defined as $(F_i^+(v), F_j^+(v)) = (G(v), 1)$ for all $v \in [0, \bar{v}]$ and $j \neq i$, can be optimal. However, this is not the case if the minimum value equals zero. The optimal design in this case depends on the surplus distribution function, G . For ternary distributions, assigning mass to exactly three values, with minimal value $\underline{v} = 0$, w -threshold designs are always optimal. We summarize these results in Proposition A2. For convenience, let V_G be the support of G .

Proposition A2. *The unique revenue maximizing design is:*

- *the maximally divisive design F^+ , when $\underline{v} > 0$ and $V_G = \{\underline{v}, \bar{v}\}$;*
- *the w -threshold design $F^{[w]}$, when $\underline{v} = 0$ and $V_G = \{\underline{v}, w, \bar{v}\}$.*

If $\underline{v} > 0$ and the surplus distribution G equals a binary distribution with mass at \underline{v} and \bar{v} , the maximally divisive design is optimal. The principal can extract the full surplus by offering the good to the agent with distribution G at price \bar{v} . If the agent does not purchase the good, then the principal awards the good to the other agent at price \underline{v} . In contrast, any other design requires that both agents can have value \bar{v} . But then full surplus extraction is not possible, as the principal now faces agents with two different,

strictly positive values.⁴ This contrasts sharply, with the sub-optimality of maximally divisive design established in the main text for the case in which $\underline{v} = 0$.

If $\underline{v} = 0$ and the surplus distribution G is ternary, then threshold designs are optimal. In this case, each agent has exactly one strictly positive valuation by setting $F_A^{[w]}(0) = p/q$, $F_A^{[w]}(w) = 1$, while $F_B^{[w]}(0) = F_B^{[w]}(w) = q$, $F_B^{[w]}(\bar{v}) = 1$. Given these distributions, the principal first offers the good to agent B at price \bar{v} . If B declines, the good is then offered to agent A at price w . Consequently, the optimal mechanism allows the principal to fully extract surplus.

Threshold Design Uniform Example To illustrate the profitability of threshold designs, we work through a concrete example where values are uniformly distributed, so $G(v) = v$. This setting offers clean analytical solutions while capturing the essential economic forces at play.

Consider first the symmetric case where both agents have identical value distributions: $F_A^-(v) = F_B^-(v) = \sqrt{v}$. The virtual values in this design are:

$$\psi_i^-(v) = 3v - 2\sqrt{v}.$$

These virtual values turn positive when $v > 4/9$, establishing the optimal reserve price. Since virtual values are strictly increasing for $v \geq 4/9$ (with derivative $3 - 1/\sqrt{v} \geq 3/2 > 0$), the optimal mechanism allocates the object to the highest-value bidder above this threshold. This yields revenue of:

$$R(F^-) = \int_{4/9}^1 (3v - 2\sqrt{v}) dv = 43/162 \approx 0.265.$$

At the other extreme, consider a maximally divisive design where only agent A values the good: $F_A^+(v) = v$ and $F_B^+(v) = 1$. This reduces the problem to a simple monopoly pricing problem with a single buyer. The optimal mechanism becomes a posted price of $1/2$, generating revenue of exactly $R(F^+) = 1/4 = 0.25$.

Notably, maximal divisiveness performs *worse* than the symmetric design. This may appear to be a striking reversal of the typical advantage of asymmetric designs we see in

⁴Either the principal has to provide information rents to at least one agent, or they can choose to not allocate the good to agents with low value. Alternatively, they could pool them and set a price equal to \underline{v} . Regardless, they cannot extract the entire surplus.

other settings. But the optimal design is not a symmetric one; instead it is a threshold design aimed at extracting surplus from types that would otherwise be excluded.

Now consider a w -threshold design for $w \in [0, 1]$. This design assigns to agent A a uniform distribution on $[0, w]$: $F_A^{[w]}(v) = \min\{v/w, 1\}$, and to agent B value 0 with probability $G(w)$ and uniform distribution on $[w, 1]$: $F_B^{[w]}(v) = \max\{w, v\}$. This creates a distinct segmentation: agent A is assigned to low values while agent B ends up with the high values and an atom at 0. The virtual values become

$$\psi_A^w(v) = 2v - w \text{ for } v \leq w \quad \text{and} \quad \psi_B^w(v) = 2v - 1 \text{ for } v \geq w.$$

Since these are increasing, the optimal mechanism allocates to the agent with the highest non-negative virtual value. This generates revenue:

$$R(F^{[w]}) = w \int_0^w \max\{2v_A - w, 0\} d\frac{v_A}{w} + \int_0^w \int_w^1 \max\{2v_A - w, 2v_B - 1, 0\} dv_B d\frac{v_A}{w},$$

where the two terms amount respectively to the case in which B has value 0 and the case in which they value above w .

The analysis splits into two cases based on whether w exceeds $1/2$, as depicted in Figure 1.

Case $w > 1/2$: In this scenario, there is never exclusion. After working through the integration regions where different agents have higher virtual values, revenue becomes

$$R(F^{[w]}) = \frac{w^2}{4} + \frac{(1-w)(25w^2 - 2w + 1)}{24w}.$$

Case $w \leq 1/2$: In this scenario, it is possible that the good is not allocated to either agent. Revenue calculations are further complicated by exclusion, but yield

$$R(F^{[w]}) = \frac{w^2}{4} + \frac{6 + 3w - 5w^2}{24}.$$

Figure 2 shows that revenue peaks at $w^* \approx 0.648$, achieving approximately 0.336—a substantial improvement over both benchmarks. Revenue in this w^* -threshold design is 27% higher than in the symmetric design and 34% higher than in the maximally divisive design.

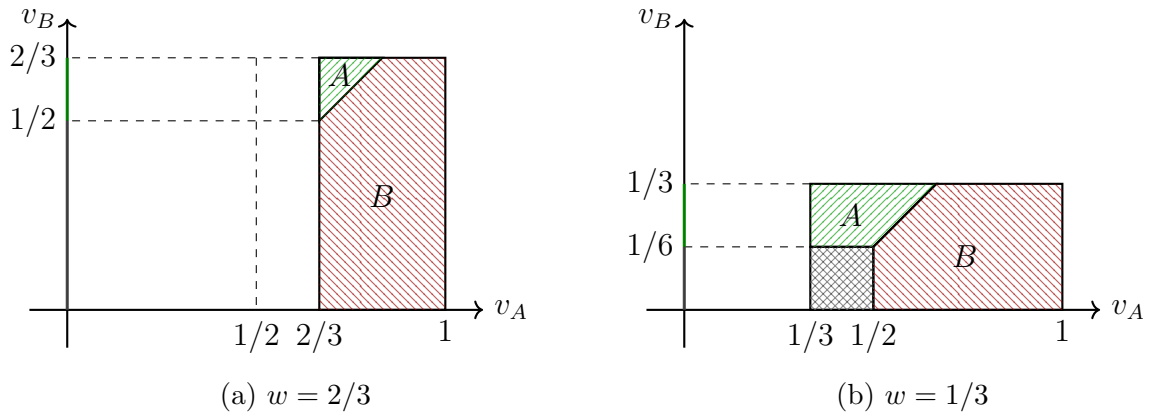


Figure 1: Integration regions for w -threshold designs when $G(v) = v$. Panels (a)–(b) depict integration regions for different values of w : in green north-east lines, value profiles for which A wins the good; in red north-west lines, value profiles for which B wins the good; and in gray crosshatch, value profiles for which the good is not allocated.

The threshold design maintains competitive pressure between agents (unlike the posted price solution) while creating specialized market segments that reduce information rents. Agent A faces competition only from low-value realizations of agent B , while agent B competes primarily in the high-value range where the surplus gains justify the information rents.

This example demonstrates that when surplus is constrained, optimal designs remain divisive but require nuanced market segmentation rather than extreme differentiation or symmetry.

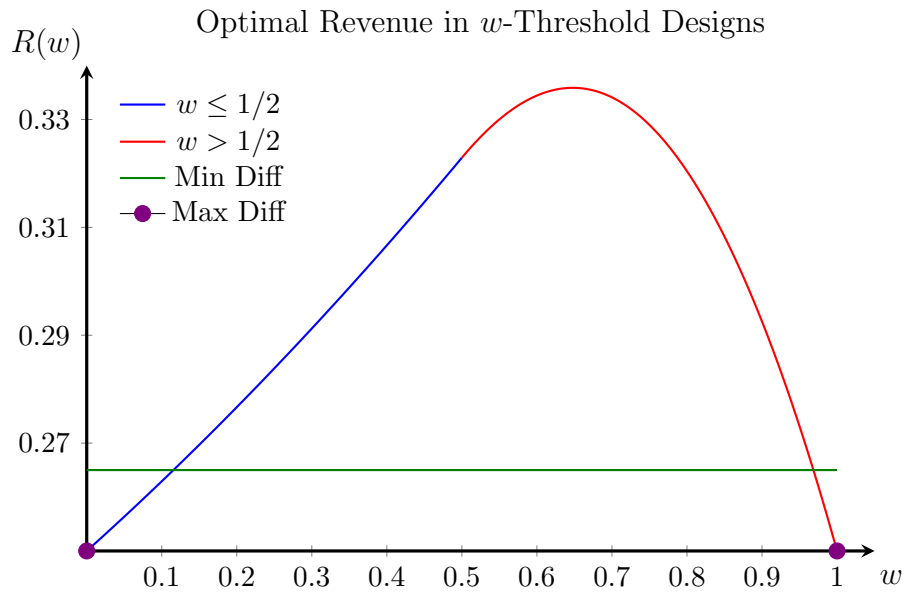


Figure 2: The plot depicts optimal revenue as a function of w , as well as optimal revenue in both the maximally and minimally divisive designs.

D Correlated Private Values

Maintaining the independence of value distributions across design problems is a natural assumption if one believes that agents should not be able to infer anything about their competitors' realized values based on their own. This assumption is not only plausible but also makes the problem more challenging, as it requires accounting for agents' information rents, rather than focusing solely on surplus design. However, many of our results extend to settings where signals are not independent, and bidders have some information about the realized values of their competitors.

Consider a correlated information setting, as discussed by [Cr mer and McLean \(1985\)](#), where revenue generically coincides with surplus, $\mathbb{E}[\max\{v_A, v_B\}]$. To provide some insight while staying close to the core of our analysis, we allow the principal to design marginal value distributions for both bidders, but not the correlation structure between their values, which is fixed. Additionally, we assume that bidders know their own value and have some information about their competitors' values. This approach mirrors our baseline analysis, where the principal designs the marginal value distributions for both agents but cannot influence the independence of their signals.

To fix the correlation structure, consider any copula $W : [0, 1]^2 \rightarrow [0, 1]$, where $W(F_A(v_A), F_B(v_B)) = \Pr(V_A \leq v_A, V_B \leq v_B)$ identifies the joint probability of an event given the marginal distributions F_A and F_B . By Sklar's Theorem, copulas can be used to capture arbitrary correlation structures—such as independence, perfect positive correlation (concordance), and perfect negative correlation (discordance). Assume that W determines the underlying correlation of tastes across agents and cannot be influenced by the principal, as was the case in the original setup where $W(F_A(v_A), F_B(v_B)) = F_A(v_A)F_B(v_B)$. Instead, let the principal design the two marginal value distributions, $F_A(v_A)$ and $F_B(v_B)$. This approach is suitable for settings in which the features of the goods can be determined, but the correlation in tastes across agents must be taken as given due to factors beyond the principal's control.

No Spillovers When designing marginal distributions subject to the mean-bound constraint $\mathbb{E}_{F_i}[v_i] \leq k$ for all i , extreme bimodal designs remain optimal. This follows from the fact that variance increases the expected surplus, $\mathbb{E}[\max\{v_A, v_B\}]$, due to the convexity of the maximum operator. As an example, consider the case where values are discordant,

such that $W(F_A, F_B) = \min\{F_A, F_B\}$ for any pair of probabilities (F_A, F_B) . Additionally, assume $\bar{v} = 2$ and $k = 1$. In this case, surplus, or equivalently revenue, is maximized by a value design (F_A, F_B) in which both agents value the good at $v_i = 2$ or $v_i = 0$ with equal probability. This design corresponds to the maximally divisive design described in Proposition 2. Under such a maximally spread two-atom distribution, surplus is exactly equal to 2, since at least one agent must value the good at 2 due to the discordance. Thus, revenue also equals 2, and the principal secures this surplus by awarding the good with certainty to the agent with the realized value equal to 2. No other value design can lead to higher surplus, as values never exceed 2. Therefore, designs in which both distributions are maximally spread are optimal when values are negatively correlated.

With concordance $W(F_A, F_B) = \max\{0, F_A + F_B - 1\}$, the design in which one agent always values the good at the mean, $k = 1$, while the other agent's value is maximally spread, as described in Proposition 2, yields a surplus of 1.5. This exceeds the surplus from the design where both agents are maximally spread, which results in a surplus of 1. Similarly, if both agents have all mass at the mean, the surplus equals 1. Consequently, with positively correlated values, the principal is better off when one agent has all mass at the mean, while the other agent's value is maximally spread, in line with our finding when values are independent.

Spillovers When designing marginal distributions subject to the linear constraint, $F_A + F_B = H$, maximally divisive designs remain optimal. This follows because such designs increase surplus, $\mathbb{E}[\max\{v_A, v_B\}]$, by minimizing the chance of having two agents with high values.

As an example, consider the concordant copula discussed above, and assume that $H(v) = v$ for all $v \in [0, 2]$. In this case, surplus in the maximally divisive design, $(F_A, F_B) = (H - 1, H)$, coincides with the expected value, $\mathbb{E}_{H-1}[v] = 1.5$, of the high-value agent A . Thus, revenue also amounts to 1.5. This revenue can be achieved by never allocating the good to agent B , but always awarding it to agent A . To extract full surplus, the transfer of the high-value agent A is set to the value associated with the quantile reported by agent B , i.e., $t_A(\mathbf{v}) = \inf\{v \mid H(v) - 1 \geq H(v_B)\}$, while B never pays anything. In such a setting, revenue under the minimally divisive design, where both agents draw values from $H/2$, simply amounts to the expected value of one of the two agents, $\mathbb{E}_{H/2}[v] = 1$. This highlights that the optimality of maximally divisive value

designs, as discussed in Proposition 4, does not depend on the independence of value distributions, but holds in more general settings. However, other designs may also be optimal for specific correlation structures.

In the same example, when the copula is discordant, the minimally divisive design yields the same surplus as the maximally divisive one. Further, discordance generally increases surplus since differentiation in values increases gains from trade.

E Cost of Value Designs

It is natural in many contexts to assume that attributes can be added for free, as it is far from obvious whether and how much the addition of features costs. For completeness, we explore how our results change if we allow for costs, with $C(F)$ representing the cost of a particular value design F .

A natural starting point is to consider costs that increase with the “quality” of the design, measured by first-order stochastic dominance. Under such cost structures, the principal would first select a distribution according to costs, which subsequently serves as the boundary distribution that cannot be exceeded. This transforms our constrained optimization problems into cost-benefit trade-offs, where the principal balances the revenue gains from better designs against their implementation costs.

We characterize the types of cost functions that yield the same designs as those derived for our various constraints, demonstrating the robustness of our main insights.

No Spillovers If an attribute affects each agent individually, but there are no spillovers, then this corresponds to a setting with separable cost functions, i.e., $C(F) = c(F_A) + c(F_B)$. Designs that are optimal for $\mathbb{E}_{F_i}[v] \leq k$ would naturally arise if the separable cost functions depend solely on the mean of the value distribution chosen for each agent. Formally, this would be represented as $C(F) = c(\mathbb{E}_{F_A}[v]) + c(\mathbb{E}_{F_B}[v])$ for some increasing function $c : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}_+$. In such environments, the designer would first select the mean based on cost considerations, and then choose the shape of the optimal design by maximizing revenue subject to the constraint $\mathbb{E}_{F_i}[v] \leq k$. Thus, even in costly design settings, the principal would still choose features that result in bimodal value designs.

Spillovers With spillovers, a cost that depends on the sum of average values for the good will still yield the result that it is optimal to assign all value to one agent, in the manner described in Proposition 3.

For the additive constraint, $H(v) = F_A(v) + F_B(v)$, cost functions must satisfy a weak linearity property to ensure the optimality of the maximal split distribution. This requires that any two value designs with the same sum incur the same cost. Formally, this would be represented as $C(F) = C(F')$ if $F_A + F_B = F'_A + F'_B$. This assumption is satisfied by common design cost functions, such as entropy, and would hold for integrable cost functions of the form:

$$\int_0^{\bar{v}} c(v) d[F_A(v) + F_B(v)]$$

for some function $c : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}_+$.

Finally, designs that are optimal for the multiplicative constraint, $G(v) = F_A(v)F_B(v)$, would also be optimal in costly design settings where design costs depend solely on the distribution of surplus and increase according to first-order stochastic dominance.

F Proofs: Corollaries and Appendix Results

Proof of Corollary 1: Asymmetric Mean Bound If agents faced different mean bounds, $k_A \neq k_B$, the logic of Proposition 2 continues to apply. To see this, suppose that both agents possess the maximal spread binary distribution subject to their respective mean constraint. With probability $\frac{(\bar{v}-k_A)(\bar{v}-k_B)}{(\bar{v}-\underline{v})^2}$, both agents would have the minimal value, and with the remaining probability, at least one agent would have the maximal value. Surplus would therefore amount to

$$\left(\frac{(\bar{v}-k_A)(\bar{v}-k_B)}{(\bar{v}-\underline{v})^2} \right) \underline{v} + \left(1 - \frac{(\bar{v}-k_A)(\bar{v}-k_B)}{(\bar{v}-\underline{v})^2} \right) \bar{v} = \bar{v} - \frac{(\bar{v}-k_A)(\bar{v}-k_B)}{(\bar{v}-\underline{v})}.$$

Consider instead a scenario in which an agent i 's distribution was maximally spread, while the other agent j had all the mass at the mean. If so, then agent i would receive the good if their valuation is maximal, and otherwise, agent j would obtain the good. This would

result in the same surplus as if both agents had the maximal spread,

$$\left(\frac{\bar{v} - k_i}{\bar{v} - \underline{v}}\right) k_j + \left(1 - \left(\frac{\bar{v} - k_i}{\bar{v} - \underline{v}}\right)\right) \bar{v} = \bar{v} - \frac{(\bar{v} - k_A)(\bar{v} - k_B)}{(\bar{v} - \underline{v})}.$$

Yet if $\underline{v} > 0$, surplus cannot be fully extracted in the first scenario, since trading with some agent at a price smaller or equal to \underline{v} precludes full surplus extraction from the same agent when their type is \bar{v} . ■

Proof of Corollary 2: Second-Order Stochastic Dominance We show that F^* is second-order stochastically dominated by G . The proof then follows because any distribution that is second-order stochastically dominated by G must have a mean no higher than k , and because F^* was optimal among all distributions with mean no higher than k .

If $G = F^*$, then F^* is second-order stochastically dominated by G trivially. So, suppose that $G \neq F^*$. If so, for some $\hat{v} < \bar{v}$ in the support of G and all $v \in [0, \hat{v})$,

$$\Delta(v) = G(v) - F^*(v) < 0,$$

because F^* maximizes the probability that the value is equal to 0 among all distributions with mean equal to k . Moreover $\Delta(v)$ is non-decreasing for any $v < \bar{v}$, because G is non-decreasing and F^* is constant for $v < \bar{v}$. Finally observe that at \bar{v} , Riemann-Stieltjes integration by parts yields

$$\int_0^{\bar{v}} \Delta(s) ds = [s(G(s) - F^*(s))]_0^{\bar{v}} - \int_0^{\bar{v}} s d(G(s) - F^*(s)) = \mathbb{E}_{F^*}[v] - \mathbb{E}_G[v] = 0,$$

where the final equality holds because both distributions have mean k . But if so, F^* is second-order stochastically dominated by G , since $\int_0^v \Delta(s) ds = \int_0^v G(s) - F^*(s) ds < 0$ for all $v < \bar{v}$. ■

Proof of Corollary 3: Profitable Surplus Reduction The observation follows directly from the proof of Proposition 1, but noting that equation (12) must now hold by the assumptions invoked on F_A and m . ■

Proof of Lemma 3: Convex Supports We restrict attention to value designs F in which the designed distributions have supports $\hat{V}_B = [\underline{v}, w_1] \cup [w_2, w_3]$ and $\hat{V}_A \subseteq [w_1, w_2] \cup$

$[w_3, \bar{v}]$, for $\underline{v} < w_1 < w_2 < w_3 \leq \bar{v}$, and establish that neither yields higher revenue than the maximally divisive design. Since further splitting of supports cannot improve on designs already shown to be suboptimal, this establishes convexity. Note that by the definition of the median value, $w_1 < v^M < w_3$.

As shown in Section B of the Supplementary Appendix, optimal revenue can always be written as

$$R(F) = \sum_{i \in \{A, B\}} \left[\underline{v}_i x_i(\underline{v}_i) + \int_{V_i} v(1 - F_i(v)) dx_i(v) \right]. \quad (\text{F.4})$$

As the supports of the two distributions are disjoint by the previous argument, we can redefine variables so that

$$\begin{aligned} (\hat{F}_A(v), d\hat{x}_A(v)) &= \begin{cases} (F_A(v), dx_A(v)) & \text{if } v \geq v^M \text{ and } v \in V_A \\ (F_B(v), dx_B(v)) & \text{if } v \geq v^M \text{ and } v \in V_B \\ (0, 0) & \text{if } v < v^M \end{cases} , \\ (\hat{F}_B(v), d\hat{x}_B(v)) &= \begin{cases} (F_A(v), dx_A(v)) & \text{if } v < v^M \text{ and } v \in V_A \\ (F_B(v), dx_B(v)) & \text{if } v < v^M \text{ and } v \in V_B \\ (0, 0) & \text{if } v \geq v^M \end{cases} . \end{aligned}$$

With this labeling, revenue can be rewritten as

$$R(F) = \underline{v}x_B(\underline{v}) + w_1x_A(w_1) + \int_{\underline{v}}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) + \int_{v^M}^{\bar{v}} v(1 - \hat{F}_A(v))d\hat{x}_A(v).$$

For some non-negative number M , consider the alternative interim allocation rule

$$\begin{aligned} \hat{x}_B(v) &= x_B(\underline{v}) + \int_{\underline{v}}^v d\hat{x}_B(v) \text{ for } v \in [\underline{v}, v^M], \\ \hat{x}_A(v) &= M + \int_{v^M}^v d\hat{x}_A(v) \text{ for } v \in [v^M, \bar{v}]. \end{aligned}$$

This allocation rule is incentive compatible because all differentials are positive by incentive compatibility of the original allocation rule. Further it is interim feasible for the maximally divisive design $F^* = (F_A^*, F_B^*)$ as we establish in a separate part of the proof below. Thus to prove that the maximally divisive design raises more revenue than $R(F)$,

it suffices to show that

$$\begin{aligned} & \underline{v}\hat{x}_B(\underline{v}) + v^M\hat{x}_A(v^M) + \int_{\underline{v}}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) + \int_{v^M}^{\bar{v}} v(1 - F_A^*(v))d\hat{x}_A(v) \\ & > \underline{v}x_B(\underline{v}) + w_1x_A(w_1) + \int_{\underline{v}}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) + \int_{v^M}^{\bar{v}} v(1 - \hat{F}_A(v))d\hat{x}_A(v). \end{aligned} \quad (\text{F.5})$$

Showing that (F.5) holds would establish the result by the usual logic because $R(F^*)$, the revenue accruing to the principal when the design is F^* and the optimal mechanism is selected, must exceed the left-hand side of (F.5) which amounts to revenue under some other incentive compatible mechanism.

We consider three distinct cases: (i) $w_2 \leq v^M$, $w_1 > \underline{v}$, $w_3 < \bar{v}$; (ii) $w_2 > v^M$, $w_1 > \underline{v}$, $w_3 < \bar{v}$; (iii) $w_3 = \bar{v}$.

Case (i): In this scenario, inequality (F.5) can be amended to

$$\begin{aligned} & v^M M + \int_{w_1}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) + \int_{v^M}^{w_3} v(1 - F_A^*(v))d\hat{x}_A(v) \\ & > w_1x_A(w_1) + \int_{w_1}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) + \int_{v^M}^{w_3} v(1 - \hat{F}_A(v))d\hat{x}_A(v). \end{aligned} \quad (\text{F.6})$$

This follows because the distributions are identical below w_1 and above w_3 and $v^M \in V_B$. Further, we have that $\hat{F}_A(v) = F_B(v) = H(v) - H(w_2) + H(w_1)$ and $F_A^*(v) = H(v) - 1$ for all $v \in [v^M, w_3]$, and thus

$$\begin{aligned} & \int_{v^M}^{w_3} v(1 - F_A^*(v))d\hat{x}_A(v) - \int_{v^M}^{w_3} v(1 - \hat{F}_A(v))d\hat{x}_A(v) = \int_{v^M}^{w_3} v(\hat{F}_A(v) - F_A^*(v))d\hat{x}_A(v) \\ & = (1 - H(w_2) + H(w_1)) \int_{v^M}^{w_3} v d\hat{x}_A(v). \end{aligned}$$

We also have that $\hat{F}_B(v) = F_B(v) = H(v) - H(w_2) + H(w_1)$ for $v \in [w_2, v^M]$, $\hat{F}_B(v) = F_A(v) = H(v) - H(w_1)$ for $v \in [w_1, w_2]$, while $F_B^*(v) = H(v)$ for $v \in [w_1, v^M]$, and thus

$$\begin{aligned} & \int_{w_1}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) - \int_{w_1}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) = \int_{w_1}^{v^M} v(\hat{F}_B(v) - F_B^*(v))d\hat{x}_B(v) \\ & = \int_{w_2}^{v^M} v(F_B(v) - H(v))d\hat{x}_B(v) + \int_{w_1}^{w_2} v(F_A(v) - H(v))d\hat{x}_B(v) \\ & = -(H(w_2) - H(w_1)) \int_{w_2}^{v^M} v d\hat{x}_B(v) - H(w_1) \int_{w_1}^{w_2} v d\hat{x}_B(v). \end{aligned}$$

In light of the last two identities, condition (F.6) can be rewritten as

$$\begin{aligned} & v^M M - w_1 x_A(w_1) + (1 - H(w_2) + H(w_1)) \int_{v^M}^{w_3} v d\hat{x}_A(v) \\ & - (H(w_2) - H(w_1)) \int_{w_2}^{v^M} v d\hat{x}_B(v) - H(w_1) \int_{w_1}^{w_2} v d\hat{x}_B(v) > 0. \end{aligned}$$

As $H(w_1), H(w_2) - H(w_1) < 1$, it suffices to show that

$$v^M M - w_1 x_A(w_1) - \int_{w_1}^{v^M} v d\hat{x}_B(v) > 0.$$

Integrating by parts, this is equivalent to

$$\begin{aligned} & v^M M - w_1 x_A(w_1) + \int_{w_1}^{v^M} \hat{x}_B(v) dv > [\hat{x}_B(v)v]_{w_1}^{v^M} \\ & = v^M [x_B(v^M) - x_B(w_2) + x_A(w_2) - x_A(w_1) + x_B(w_1)] - w_1 x_B(w_1). \end{aligned} \tag{F.7}$$

Further because allocation rules are increasing, we have that $\int_{w_1}^{v^M} \hat{x}_B(v) dv > (v^M - w_1)x_B(w_1)$. Consequently equation (F.7) holds as long as

$$v^M (M - x_B(v^M) - x_B(w_2) + x_A(w_2)) + (v^M - w_1)x_A(w_1) > 0,$$

which holds for $M = x_B(v^M) - x_B(w_2) + x_A(w_2)$ —thus establishing the result.

Case (ii): As in case (i), we need to establish inequality (F.5). As in case (i), we can establish that

$$\int_{w_2}^{w_3} v(1 - F_A^*(v)) d\hat{x}_A(v) - \int_{w_2}^{w_3} v(1 - \hat{F}_A(v)) d\hat{x}_A(v) = (1 - H(w_2) + H(w_1)) \int_{w_2}^{w_3} v d\hat{x}_A(v),$$

because $\hat{F}_A(v) = F_B(v) = H(v) - H(w_2) + H(w_1)$ and $F_A^*(v) = H(v) - 1$ for $v \in [w_2, w_3]$.

Moreover, we have that

$$\int_{v^M}^{w_2} v(1 - F_A^*(v)) d\hat{x}_A(v) - \int_{v^M}^{w_2} v(1 - \hat{F}_A(v)) d\hat{x}_A(v) = (1 - H(w_1)) \int_{v^M}^{w_2} v d\hat{x}_A(v),$$

because $\hat{F}_A(v) = F_A(v) = H(v) - H(w_1)$ and $F_A^*(v) = H(v) - 1$ for $v \in [v^M, w_2]$. Finally,

we have that

$$\int_{w_1}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) - \int_{w_1}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) = -H(w_1) \int_{w_1}^{w_2} vd\hat{x}_B(v),$$

because $\hat{F}_B(v) = F_A(v) = H(v) - H(w_1)$ and $F_B^*(v) = H(v)$ for $v \in [w_1, v^M]$. In light of the last identities, condition (F.5) can be rewritten as

$$\begin{aligned} & v^M M - w_1 x_A(w_1) + (1 - H(w_2) + H(w_1)) \int_{w_2}^{w_3} vd\hat{x}_A(v) \\ & + (1 - H(w_1)) \int_{v^M}^{w_2} vd\hat{x}_A(v) - H(w_1) \int_{w_1}^{v^M} vd\hat{x}_B(v) > 0. \end{aligned}$$

As $H(w_1), H(w_2) - H(w_1) < 1$, this again reduces to

$$v^M M - w_1 x_A(w_1) - \int_{w_1}^{v^M} vd\hat{x}_B(v) > 0,$$

which from integration by parts, again reduces to

$$v^M M - w_1 x_A(w_1) + \int_{w_1}^{v^M} \hat{x}_B(v)dv > v^M(x_A(v^M) - x_A(w_1) + x_B(w_1)) - w_1 x_B(w_1). \quad (\text{F.8})$$

Because $\int_{w_1}^{v^M} \hat{x}_B(v)dv > (v^M - w_1)x_B(w_1)$, equation (F.8) holds as long as

$$v^M(M - x_A(v^M)) + (v^M - w_1)x_A(w_1) > 0,$$

which holds for $M = x_A(v^M)$ —thus establishing the result.

Case (iii): In this scenario, inequality (F.5) can be amended to

$$\begin{aligned} & v^M M + \int_{w_1}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) + \int_{v^M}^{w_2} v(1 - F_A^*(v))d\hat{x}_A(v) \\ & > w_1 x_A(w_1) + \int_{w_1}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) + \int_{v^M}^{w_2} v(1 - \hat{F}_A(v))d\hat{x}_A(v). \end{aligned} \quad (\text{F.9})$$

As before, we have that

$$\int_{v^M}^{w_2} v(1 - F_A^*(v))d\hat{x}_A(v) - \int_{v^M}^{w_2} v(1 - \hat{F}_A(v))d\hat{x}_A(v) = (1 - H(w_1)) \int_{v^M}^{w_2} vd\hat{x}_A(v),$$

because $\hat{F}_A(v) = F_A(v) = H(v) - H(w_1)$ and $F_A^*(v) = H(v) - 1$ for $v \in [v^M, w_2]$. Finally, we have that

$$\int_{w_1}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) - \int_{w_1}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) = -H(w_1) \int_{w_1}^{w_2} v d\hat{x}_B(v),$$

because $\hat{F}_B(v) = F_A(v) = H(v) - H(w_1)$ and $F_B^*(v) = H(v)$ for $v \in [w_1, v^M]$. But then, condition (F.9) can be rewritten as

$$v^M M - w_1 x_A(w_1) + (1 - H(w_1)) \int_{v^M}^{w_2} v d\hat{x}_A(v) - H(w_1) \int_{w_1}^{v^M} v d\hat{x}_B(v) > 0. \quad (\text{F.10})$$

which can be shown to hold following the same steps as in case (ii).

Interim Feasibility: To conclude, we establish that the proposed allocations are interim feasible. Define V_i^* as the support of distribution F_i^* . Theorem 1 in [Border \(1991\)](#) implies that our earlier interim allocation rules (\hat{x}_A, \hat{x}_B) are feasible for the maximally divisive design F^* if and only if for any $(v_A, v_B) \in V_A^* \times V_B^*$

$$\begin{aligned} \int_{\underline{v}}^{v_B} \hat{x}_B(v) dH(v) + \int_{v^M}^{v_A} \hat{x}_A(v) dH(v) &\leq 1 - (1 - F_B^*(v_B))(1 - F_A^*(v_A)) \\ &= 1 - (1 - H(v_B))(2 - H(v_A)). \end{aligned} \quad (\text{F.11})$$

To show that this condition holds, denote the difference between the left- and right-hand sides of (F.11) by

$$\Gamma(v_A, v_B) = \int_{\underline{v}}^{v_B} \hat{x}_B(v) dH(v) + \int_{v^M}^{v_A} \hat{x}_A(v) dH(v) - 1 + (1 - H(v_B))(2 - H(v_A)).$$

We want to show that $\Gamma(v_A, v_B) \leq 0$ for all $(v_A, v_B) \in V_A^* \times V_B^*$. Observe that $\Gamma(\bar{v}, v^M) \leq 0$ must hold, since

$$\Gamma(\bar{v}, v^M) = \int_{\underline{v}}^{v^M} \hat{x}_B(v) dH(v) + \int_{v^M}^{\bar{v}} \hat{x}_A(v) dH(v) - 1 = \int_{\underline{v}}^{\bar{v}} x_A(v) + x_B(v) dH(v) - 1 \leq 0,$$

where the second equality holds by definition of the new allocation rule and the disjoint support assumption, and where the inequality holds because the aggregate probability of trade in the original mechanism had to be smaller than 1. Because $\Gamma(\bar{v}, v^M) \leq 0$, we also have that $\Gamma(\bar{v}, v_B) \leq \Gamma(\bar{v}, v^M) \leq 0$ for all $v_B \leq v^M$. Further, we have that for $v_A = v^M$

and all $v_B \leq v^M$,

$$\Gamma(v^M, v_B) = \int_{\underline{v}}^{v_B} \hat{x}_B(v) dH(v) - H(v_B) \leq 0,$$

where the inequality holds as B can receive the good with probability no higher than 1. To conclude, observe that Γ is single-dip in v_A . To see this, note that

$$\frac{d\Gamma(v_A, v_B)}{dv_A} = dH(v_A)(\hat{x}_A(v_A) - 1 + H(v_B)).$$

But $d\Gamma(v_A, v_B)/dv_A$ can be negative only when $\hat{x}_A(v_A) < 1 - H(v_B)$ and must remain positive once it becomes larger than 0, given that \hat{x}_A is increasing by incentive compatibility. Because of this, (F.11) must hold, since for all $(v_A, v_B) \in V_A^* \times V_B^*$,

$$\Gamma(v_A, v_B) \leq \max\{\Gamma(v^M, v_B), \Gamma(\bar{v}, v_B)\} \leq 0,$$

where the first inequality holds since $\Gamma(v_A, v_B)$ is single-dip in v_A by the previous argument, and the second one holds since $\Gamma(v^M, v_B) \leq 0$ and $\Gamma(\bar{v}, v_B) \leq 0$. ■

Proof of Proposition A1: No Value Destruction with Surplus Bound We establish the result by contradiction. Suppose by contradiction that there exists an optimal design (F_A^*, F_B^*) , in which $F_A^*(w)F_B^*(w) > G(w)$ for some non-excluded value w . If so, an alternative value design (F'_A, F'_B) exists that first-order stochastically dominates the original design, $F' \succ_1 F^*$, and that still satisfies the constraint

$$F_A^*(w)F_B^*(w) > F'_A(w)F'_B(w) \geq G(w).$$

Because revenue is increasing in first-order stochastic dominance, the optimal revenue under design F^* is no larger than the optimal revenue under design F' . Further, revenue strictly increases when value w is not excluded and the mechanism is fixed to $(x(F^*), t(F^*))$, because higher transfers will be paid more often in the F' design. ■

Proof of Proposition A2: Surplus Bounds and Optimality When the surplus distribution is binary ($G(v) = p$ for $v \in [\underline{v}, \bar{v})$ and $G(\bar{v}) = 1$), the allocation rule and the transfers in the optimal mechanism for the maximally divisive, $(F_i^+, F_j^+) = (G, 1)$,

amount to

$$\begin{aligned} x_i(v_i, v_j|F^+) &= \mathbb{1}(v_i = \bar{v}) \text{ and } x_j(v_j, v_i|F^+) = \mathbb{1}(v_i = \underline{v}), \\ t_i(v_i, v_j|F^+) &= x_i(v_i, v_j|F^+)\bar{v} \text{ and } t_j(v_j, v_i|F^+) = x_j(v_j, v_i|F^+)\underline{v}. \end{aligned}$$

These rules award the good to i in case of high-value at price \bar{v} , and sell the good to j at price \underline{v} otherwise. This mechanism is incentive compatible since interim allocation rules are increasing. It is individually rational, as ex-post payoffs are non-negative. Further, it is optimal because revenue coincides with surplus $R(F^+) = S(F^+)$. When $\underline{v} > 0$, this design is uniquely optimal. To see this, consider any other value design \hat{F} in which $\hat{F}_A \hat{F}_B = G$. If so, $\hat{V}_i = \hat{V}_j = \{\underline{v}, \bar{v}\}$. By construction, the designs \hat{F} and F^+ generate the same surplus, $S(\hat{F}) = S(F^+)$. To show that $R(\hat{F}) < R(F^+)$, we argue that some agents must obtain a positive information rent in the optimal mechanism associated with value design \hat{F} . By incentive compatibility, if an agent reports the high value truthfully, it must be that

$$U_i(\bar{v}|\hat{F}) = x_i(\bar{v}|\hat{F})\bar{v} - t_i(\bar{v}|\hat{F}) \geq x_i(\underline{v}|\hat{F})\bar{v} - t_i(\underline{v}|\hat{F}).$$

If $x_A(\underline{v}) = x_B(\underline{v}) = 0$, the result follows because the mechanism does not generate full surplus, and thus $R(\hat{F}) < S(\hat{F})$. If instead $x_i(\underline{v}) > 0$ for at least some player i , by individual rationality, we further have that

$$U_i(\bar{v}|\hat{F}) \geq x_i(\underline{v})\bar{v} - t_i(\underline{v}) > x_i(\underline{v})\underline{v} - t_i(\underline{v}) = U_i(\underline{v}|\hat{F}) \geq 0.$$

If so, agent i 's ex-ante utility is strictly positive, $U_i(\hat{F}) = \hat{F}_i(\bar{v})U_i(\bar{v}|\hat{F}) + \hat{F}_i(\underline{v})U_i(\underline{v}|\hat{F}) > 0$. Thus, we have that

$$R(\hat{F}) \leq S(\hat{F}) - U_A(\hat{F}) - U_B(\hat{F}) < S(F^+) = R(F^+).$$

This establishes that divisive designs maximize revenue.

To prove the second part of the result, consider any ternary distribution, $G(v) = p$ for $v \in [0, w)$, $G(v) = q > p$ for $v \in [w, \bar{v})$, and $G(\bar{v}) = 1$, and value design $F^{[w]}$. In such a value design, $F_i^{[w]}(0) = F_i^{[w]}(w) = q$, while $F_j^{[w]}(0) = p/q$, $F_j^{[w]}(w) = 1$. The optimal

mechanism associated with this design amounts to

$$x_i(v_i, v_j | F^{[w]}) = \mathbb{1}(v_i = \bar{v}) \text{ and } x_j(v_j, v_i | F^{[w]}) = \mathbb{1}(v_j = w \text{ and } v_i = 0),$$

$$t_i(v_i, v_j | F^{[w]}) = x_i(v_i, v_j | F^{[w]})\bar{v} \text{ and } t_j(v_j, v_i | F^{[w]}) = x_j(v_j, v_i | F^{[w]})w.$$

These rules award the good to i in case of high-value at price \bar{v} , and otherwise offer to sell the good to j at price w . This mechanism is incentive compatible since interim allocation rules are increasing. It is individually rational, as ex-post payoffs are non-negative. Further, it is optimal because revenue coincides with surplus $R(F^{[w]}) = S(F^{[w]})$. It is uniquely optimal because any other design for which the constraint binds would lead to an agent having two positive values and accruing some information rent. ■

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