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On the threshold representation of partial orderings

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ABSTRACT

In this paper, we provide a constructive proof for threshold representations of partial orderings in a finite setting and extend the construction to arbitrary sets. This contributes to a long-standing open problem posed in Nakamura (2002) and Aleskerov et al. (2007). We identify and discuss aspects of our proof that do not directly generalize to an infinite setting and, for each, provide intuitive sufficient conditions.

1. Introduction

It has long been recognized that an individual may be

“neither able to state which of two alternatives he prefers nor that they are equally desirable”.

[(Von Neumann and Morgenstern, 1944).]

This observation about incomparability is at the core of economic theory and has sparked an intense debate about the validity of the completeness axiom (see, among many, Von Neumann and Morgenstern (1944), Luce (1956), Aumann (1962), Bewley (2002), Ok (2002), Mandler (2005)). As a result, a vast literature on numerical representations of (possibly incomplete) preferences has investigated the sources of incomparability in relation to the structure of the preferences among comparable alternatives.

In this context, partial orderings have played a central role. From a descriptive standpoint, they capture economically significant relations such as stochastic-, vector-, and Pareto-dominance. From a normative standpoint, transitivity is one of the axioms of expected utility theory and is crucial to avoid money-pump arguments.

The earliest approach to numerical representations of incomplete preferences is known as the weak utility approach (Richter, 1966; Peleg, 1970; Sondermann, 1980; Alcantud and Rodriguez-Palmero, 1999). It relies on the existence of a real-valued function u such that $x > y$ implies $u(x) > u(y)$. This numerical representation “permits us to go from the partial ordering $>$ to the real-valued function” (Majumdar and Sen, 1976) but not vice versa. In other words, the utility function conveys less information than the partial order itself. This is an unsatisfactory

aspect of the weak utility approach when compared to representations of linear orders, which come without loss of information, i.e., $x > y$ if, and only if, $u(x) > u(y)$. Unfortunately, these representations are unattainable when the partial order is incomplete.

The most well-known proposal to tackle the informational problem is the multi-utility approach which is rooted in the notions of dominance we aforementioned, i.e., there is a set of utility functions \mathcal{U} and a Pareto aggregator such that $x > y$ if, and only if, $u(x) > u(y)$ for all $u \in \mathcal{U}$. Here, the source of incomparability stems from the inability of the decision-maker to find an alternative that dominates the other along all the dimensions he cares about. The multi-utility approach was conjectured by Von Neumann and Morgenstern (1944), analyzed choice-theoretically by Aizerman and Malishevski (1981), and has received extensive attention in recent years after the seminal paper of Ok (2002), see e.g. Dubra et al. (2004), Kaminski (2007), Evren and Ok (2011), Bosi and Herden (2012).

Less well-known is the threshold approach, in which incompleteness is modeled using a pair-dependent threshold, i.e., $x > y$ if, and only if, $u(x) - u(y) > \varepsilon(x, y)$. Here, the decision-maker is able to compare alternatives if, and only if, their utilities are sufficiently far apart, with a notion of “distance” that can change across comparisons. Describing incompleteness through a threshold dates back to Luce (1956), Fishburn (1970, 1973) – see Aleskerov et al. (2007) for a review – but its application to partial orders remains limited. The initial step towards this representation was made by Agaev and Aleskerov (1993), who observed that the triangle inequality for the threshold $\varepsilon(x, y)$ is sufficient to ensure transitivity. The necessity was established nearly

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ten years later by Nakamura (2002), who paved the way for further research:

“Our proof of the representation theorem is not constructive. It remains an open problem to give a constructive proof, which may also answer the question of whether arbitrary posets have quasi-metric² threshold representations”.

The absence of a representation for arbitrary sets is, indeed, a significant gap in the literature as stressed by Aleskerov et al. (2007):

“His [Nakamura’s] result relies on the Theorem of the Alternative... An important open problem is to know whether such a result holds for infinite sets”.

In this paper, we provide a constructive proof of Nakamura’s theorem and extend the representation to infinite sets.

2. Preliminaries

Binary relations and induced graphs

Let A be a nonempty set of alternatives and $>$ a binary relation on A . We say that $>$ is *irreflexive* if $\neg(x > x)$ for all $x \in A$; *asymmetric* if $x > y$ implies $\neg(y > x)$; *acyclic* if $x_1 > x_2 \dots x_{n-1} > x_n$ implies $\neg(x_n > x_1)$ for all $n \geq 2$; *transitive* if $x > y > z$, then $x > z$; *negatively transitive* if $\neg(x > y)$ and $\neg(y > z)$ implies $\neg(x > z)$; *connected* if $x \neq y$ implies $x > y$ or $y > x$. If $>$ is connected, asymmetric, and negatively transitive, we refer to it as *linear order*; if $>$ is asymmetric and negatively transitive, we refer to it as *weak order*; and if $>$ is irreflexive and transitive, we refer to it as *partial order*. Finally, for all $x, y \in A$, whenever $\neg(x > y)$ and $\neg(y > x)$, i.e. x, y are incomparable, we write $x \perp y$.

Given a binary relation $>$, we call *induced graph* the undirected graph $G = (A, E)$ where A is the set of vertices and E the set of edges defined as the set of unordered pair of elements in $>$ (if $x > y$, then $\{x, y\}$ is an edge).

A path p is a sequence z_0, z_1, \dots, z_n of (distinct) vertices such that for each z_i, z_{i+1} with $0 \leq i \leq n - 1$, either $z_i > z_{i+1}$ or $z_{i+1} > z_i$. For all $x, y \in A$, define by $\mathcal{P}_{x,y}$ the set of all paths between³ x and y and denote by $p_{x,y}$ (or p if the context is clear) the generic element of $\mathcal{P}_{x,y}$. If p is a path between x and y and there is no strictly contained path in p between x and y , then p is a *minimal path* between x and y . Moreover, let \mathcal{P} be the set of all paths in G .

Let $p = (z_0, z_1, \dots, z_{n+1})$ for some integer $n > 0$ be a path. We say that p is an *alternating path* if for all $i = 0, \dots, n - 1$ the following conditions are met: (i) $z_i > z_{i+1}$ implies $z_{i+1} < z_{i+2}$; and (ii) $z_i < z_{i+1}$ implies $z_{i+1} > z_{i+2}$. Moreover, we define $\mathcal{M}_{x,y}$ as the set of all alternating paths between x and y .

Alternating paths will be the backbone of our argument, and in Lemma 1 we show that they can be categorized into four disjoint groups.

Lemma 1. *Let $>$ be an asymmetric binary relation on A , and G be the induced graph. If $p = (z_0, z_1, \dots, z_{n+1})$ for some integer $n > 0$ is an alternating path, then p satisfies one and only one of the following conditions:*

(up-up) n is odd and $z_0 > z_1 < z_2 > \dots < z_{n+1}$;

(down-down) n is odd and $z_0 < z_1 > z_2 < \dots > z_{n+1}$;

(down-up) n is even and $z_0 < z_1 > z_2 < \dots < z_{n+1}$;

(up-down) n is even and $z_0 > z_1 < z_2 > \dots > z_{n+1}$.

² Following Monjardet (1981), a quasi-metric is a function ϵ such that $\epsilon(x, y) \geq 0$, $\epsilon(x, y) = \epsilon(y, x)$, and $\epsilon(x, y) + \epsilon(y, z) \geq \epsilon(x, z)$. In other words, there may exist two distinct elements x, y such that $\epsilon(x, y) = 0$.

³ We say ‘between’ and not ‘from’ and ‘to’, because G is an undirected graph. Moreover, we identify a path of only two endpoints with the set of its endpoints (a path p between x and y consisting of only x and y is identified with $\{x, y\}$).

Representable binary relations and induced weighted graphs

We say that $>$ has a Threshold (T) representation, or shortly, $>$ is *semimetric representable* if there is a function $u : A \rightarrow \mathbb{R}$ and a semi-metric⁴ $\epsilon : A \times A \rightarrow \mathbb{R}$ such that

$$x > y \iff u(x) - u(y) > \epsilon(x, y)$$

If ϵ is a metric, we say $>$ is *metric representable*.

The following lemma shows that acyclicity is necessary and sufficient for semimetric representability. The statement imposes that A is at most a countable set, while the more general result for arbitrary sets, which requires density⁵ on top of acyclicity, can be found as Theorem 6.16 in Aleskerov et al. (2007).

Lemma 2 (Theorem 6.15, Aleskerov et al., 2007). *Let A be at most a countable set. The following conditions are equivalent:*

(i) $>$ is semimetric representable;

(ii) $>$ is irreflexive and acyclic;

(iii) there is a function $u : A \rightarrow \mathbb{R}$ such that $x > y \implies u(x) > u(y)$;

(iv) there is a bounded function $u : A \rightarrow \mathbb{R}$ such that $x > y \implies u(x) > u(y)$.

If condition (iii) holds, we say that u is a representation of the binary relation $>$.

The induced graph G together with the weight function $w : E \rightarrow \mathbb{R}$ defined as $w((x, y)) = |u(x) - u(y)|$ is called an *induced weighted graph*. For all paths $p \in \mathcal{P}$, extend the function w to \mathcal{P} additively, that is, by defining $w(p) = \sum_{e \in p} w(e)$ and call $w(p)$ the *weight* of p .

Properties of the induced weighted graphs of partial orders

Transitivity has important implications for the properties of the induced weighted graphs and the relationships between paths and alternating paths. We summarize these properties in two lemmas. The proofs are rather intuitive, hence, we relegate them to the Appendix.

First, Lemma 3 shows that transitivity implies that every path can be reduced to an alternating path that is fully included in the original path and has the same weight. Consequently, for our purposes, it will suffice to focus on alternating paths.

Lemma 3. *Let $>$ be a partial order on a set A , u a representation of $>$, and G an induced weighted graph. For all $x, y \in A$ and $p \in \mathcal{P}_{x,y}$ there is an alternating path $p' \in \mathcal{M}_{x,y}$ such that all points in p' are in p and $w(p) = w(p')$.*

Secondly, Lemma 4 shows that the weight of non-minimal paths is always strictly greater than the difference in utility between the two endpoints. This result ensures that utility differences are less or equal than the minimum distance between two points in the induced weighted graphs.

Lemma 4. *Let $>$ be a partial order on a set A , u a representation of $>$, and G an induced weighted graph. For all $x, y \in A$ and $p \in \mathcal{P}_{x,y}$ with $p \neq \{x, y\}$ we have that $|u(x) - u(y)| < w(p)$.*

3. A constructive proof: the finite case

We now present a construction for the two components of the threshold representation: (i) the (weak) utility function, and (ii) the metric. First, notice that the existence of a (weak) utility function is guaranteed by Lemma 2 and, therefore, we concentrate on the construction of the metric.

⁴ A *semimetric* is a function ϵ such that $\epsilon(x, y) \geq 0$, $\epsilon(x, y) = \epsilon(y, x)$, and $\epsilon(x, y) = 0 \iff x = y$. In other words, it may violate the triangle inequality.

⁵ Let $>^T$ be the transitive closure of $>$. We say that $B \subseteq A$ is dense in A for $>^T$, if for all $x, y \in A$, $x >^T y$ implies $x >^T z$ or $z >^T y$ for some $z \in B$. Then A satisfies density if there is a countable set B that is dense in A for $>^T$.

We begin by showing that the function describing the minimum weight among (weighted) paths for a given pair of alternatives is a metric (following the nomenclature in Monjardet (1981) and Lemma 5, henceforth, we refer to this metric as the *minimum weighted path metric*). For all $x, y \in A$, if $x \neq y$ and $\mathcal{P}_{x,y} \neq \emptyset$, let $\delta(x, y) = \min_{p \in \mathcal{P}_{x,y}} w(p)$. If $x = y$, set $\delta(x, x) = 0$ and if $\mathcal{P}_{x,y} = \emptyset$ set $\delta(x, y) = \delta$, with $\delta > \max\{\max_{p \in \mathcal{P}} w(p), \max_{x,y \in A} |u(x) - u(y)|\}$.

Lemma 5. *Let \succ be a partial order on a finite set A , u a representation of \succ , and G an induced weighted graph. Then $\delta : A^2 \rightarrow \mathbb{R}$ is a metric.*

Proof. Clearly for all $x, y \in A$ we have $\delta(x, y) \geq 0$ and if $x = y$, then $\delta(x, y) = 0$. Suppose $x \neq y$. If $\mathcal{P}_{x,y} = \emptyset$, then $\delta(x, y) = \delta > \max_{p \in \mathcal{P}} w(p) \geq 0$. Assume $\mathcal{P}_{x,y} \neq \emptyset$ and let $p \in \mathcal{P}_{x,y}$ such that $\delta(x, y) = w(p)$. Let e be an edge in p , obtain $w(e) > 0$, and so $\delta(x, y) = w(p) \geq w(e) > 0$. To prove the triangular inequality, let $x, y, z \in A$. If either $\mathcal{P}_{x,y} = \emptyset$ or $\mathcal{P}_{y,z} = \emptyset$, then we are done.

Assume both the sets are nonempty to obtain $\mathcal{P}_{(x,z)} \neq \emptyset$. Let $p \in \mathcal{P}_{(x,z)}$ such that $w(p) = \delta(x, z)$. If the path p contains an edge with y as endpoint, then we can split the path p into a path p' between x and y and a path p'' between y and z . We show that $\delta(x, y) = w(p')$ and $\delta(y, z) = w(p'')$. By contradiction, assume there is a path q' between x and y such that $w(q') < w(p')$. Consider the path q defined by q' and p'' to obtain $w(q) = w(q') + w(p'') < w(p') + w(p'') = \delta(x, z)$, a contradiction. A similar reasoning shows that $\delta(y, z) = w(p'')$.

If the path p does not contain an edge with y as endpoint we obtain that for any path q that has y as endpoint of an edge, $\delta(x, z) = w(p) \leq w(q)$. Let q be the path formed by considering a shortest path q' between x and y and a shortest path q'' between y and z . Since $\delta(x, y) + \delta(x, z) = w(q') + w(q'') = w(q) \geq w(p) = \delta(x, z)$, we obtain the thesis. \square

3.1. The representation

Endowed with a (weak) utility function (Lemma 2) and a metric (Lemma 5), we can now prove that partial orders are *metric representable*.

Denote by $l(p)$ the number of edges in the path p (for instance, if $p \in \mathcal{P}_{x,y}$ only contains x and y as endpoints, then $l(p) = 1$). In Lemma 6, we show that when two alternatives are ordered the minimum weighted metric takes the simple form of utility difference.

Lemma 6. *For all $x, y \in A$ the following conditions are equivalent:*

- (i) $l(x, y) = 1$;
- (ii) $x \succ y$ or $y \succ x$;
- (iii) $\delta(x, y) = |u(x) - u(y)|$.

Proof. (i) \implies (ii) By definition.

(ii) \implies (iii) By hypothesis we have $\mathcal{P}_{x,y} \neq \emptyset$, since $\{x, y\} \in \mathcal{P}_{x,y}$. Therefore there is a path p such that $w(p) = \delta(x, y)$. By Lemma 4, if $p \neq \{x, y\}$, then $w(\{x, y\}) = |u(x) - u(y)| < w(p) = \delta(x, y)$, a contradiction. Therefore $p = \{x, y\}$ and $\delta(x, y) = |u(x) - u(y)|$.

(iii) \implies (i) Since $\delta > |u(x) - u(y)|$, we obtain that $\mathcal{P}_{x,y} \neq \emptyset$. Let $p \in \mathcal{P}_{x,y}$ such that $w(p) = \delta(x, y) = |u(x) - u(y)|$. By Lemma 4, if $p \neq \{x, y\}$, then $w(\{x, y\}) = |u(x) - u(y)| < w(p) = \delta(x, y)$. We conclude that $p = \{x, y\}$ and $l(x, y) = 1$. \square

An immediate corollary of Lemma 6 yields the representation for all ordered alternatives.

Corollary 1. *For all $\gamma \in [0, 1]$, if $x \succ y$, then $u(x) - u(y) > \gamma \delta(x, y)$.*

For unordered alternatives, instead, we can only leverage previous results to obtain the representation for each pair of alternatives separately.

Corollary 2. *For all $x, y \in A$, there exists a $\gamma_{x,y} \in]0, 1[$ such that if $x \perp y$ then $|u(x) - u(y)| \leq \gamma_{x,y} \delta(x, y)$.*

Proof. Suppose $x, y \in A$. If $\mathcal{P}_{x,y} = \emptyset$, then we are done since $\delta > |u(x) - u(y)|$. If $\mathcal{P}_{x,y} \neq \emptyset$, by Lemmas 3 and 4, there is $p \in \mathcal{M}_{x,y}$ such that $|u(x) - u(y)| < w(p) = \delta(x, y)$. Hence there is a $\gamma_{x,y}$ such that $|u(x) - u(y)| \leq \gamma_{x,y} \delta(x, y)$. \square

Corollary 2 identifies pair-dependent or local scaling factors for which the representation holds. These factors are related to the minimum weighted path metric as shown in the following definition.

Definition 3 (Local Scaling Factor). For all $x, y \in A$:

$$0 < \gamma_{x,y} = \frac{|u(x) - u(y)|}{\delta(x, y)} = \frac{|u(x) - u(y)|}{\min_{p \in \mathcal{P}_{x,y}} w(p)} < 1$$

To complete the proof, we need to show that the representation holds globally, i.e. our rescaling of the metric δ must hold for all unordered alternatives together. We do so using a natural sufficient condition, i.e., if, for all $x, y \in A$, $\gamma_{x,y} = \gamma$ then $\gamma\delta$ is a metric.

Theorem 1.

\succ is irreflexive and transitive $\iff \succ$ is metric representable.

Proof (\implies). If $\succ = \emptyset$, then for all distinct $x, y \in A$, let $\epsilon(x, y) = 1 = u(x) - u(y)$ and $\epsilon(x, y) = 0$ if $x = y$. The thesis is immediate. Next, assume $\succ \neq \emptyset$ and let u be a representation of \succ . By Lemma 5 and Corollaries 1, 2, fix $\gamma = \max_{x,y: l(x,y) > 1} \gamma_{x,y}$ and $\epsilon(x, y) = \gamma\delta(x, y)$ to obtain that (u, ϵ) is a metric threshold representation

(\impliedby) This is implied by Lemma 2 and Agaev and Aleskerov (1993). \square

3.1.1. From local to global: some algorithmic properties of the representation

Corollaries 1, 2 and the successive Theorem 1 deal with the passage from a local representation, i.e. one that holds for each pair in isolation, to a global representation, i.e. one that holds for all pairs. This step implies going from the local scaling factors $\gamma_{x,y}$ to the unique global scaling factor γ via the following maximization problem

$$\max_{x,y: l(x,y) > 1} \gamma_{x,y}$$

which can be equivalently stated as finding “minimum-weighted-paths”. Several algorithms with varying degrees of time complexity have been proposed, and their analysis is beyond the scope of this paper. Nonetheless, we provide two results that briefly tackle the issue of time complexity by substantially reducing the set of relevant paths.

First, as a corollary of Lemma 3, we show that calculating minimal weighted paths between any two points can be achieved by focusing solely on alternating paths.

Corollary 4. *For all $x, y \in A$,*

$$\min_{p \in \mathcal{P}_{x,y}} w(p) = \min_{p' \in \mathcal{M}_{x,y}} w(p')$$

Second, we show that within the set of alternating paths, only those with minimal distance are relevant. This is shown to be true regardless of their type (see Lemma 1 for the taxonomy of alternating paths), though differences in the meaning of “minimal path” exist.

Before stating our result, we introduce further notation. First, for all $x, y \in A$, $p \in \mathcal{P}_{x,y}$, and w, t vertices of the path p denote by $p \upharpoonright_{w,t}$ the path between w and t such that each vertex of $p \upharpoonright_{w,t}$ is a vertex of p and call it the *path restricted to w and t* or the *restriction of p to w and t* . Second, for all $x, y \in A$ and $p \in \mathcal{P}_{x,y}$ denote the ratio between the utility difference and the weighted path as $\gamma_{x,y}^p = \frac{|u(x) - u(y)|}{w(p)}$. Importantly, note that these ratios are the foundation of our constructive proof.

Lemma 7. *Let \succ be a partial order on a set A , u a representation of \succ , and G an induced weighted graph. For all $x, y \in A$ and $p = (x, z_1, \dots, z_n, y) \in \mathcal{M}_{x,y}$ with integer $n \geq 2$ we have that*

- (i) If p is (up-up) or (down-down), then $\gamma_{x,y}^p < \max\{\gamma_{x,z_n}^{p_{x,z_n}}, \gamma_{z_1,y}^{p_{z_1,y}}\}$;
- (ii) If $p = (x, z_1, \dots, z_n, y)$ is (down-up) with $u(x) \geq u(y)$, then $\gamma_{x,y}^p \leq \min\{\gamma_{x,z_n}^{p_{x,z_n}}, \gamma_{z_1,y}^{p_{z_1,y}}\}$;
- (iii) If p is (up,down) with $u(x) \geq u(y)$ and $n = 4$, then $\gamma_{x,y}^p \leq \max\{\gamma_{x,z_3}^{p_{x,z_3}}, \gamma_{z_2,y}^{p_{z_2,y}}\}$;
- (iv) If p is (up,down) with $u(x) \geq u(y)$ and $n > 4$, then $\gamma_{x,y}^p \leq \max\{\gamma_{x,z_3}^{p_{x,z_3}}, \gamma_{z_4,y}^{p_{z_4,y}}\}$.

4. Extending the representation to infinite sets

Nakamura (2002) conjectured that providing a constructive proof of his theorem would also answer the representation problem for arbitrary sets. In this section, we show that, with some adjustments, our representation can indeed be extended to arbitrary sets.

The adjustments regard two parts of our original proof: the construction of the minimum weighted path metric δ (see Lemma 5) and the definition of ϵ from δ using the scaling factor γ (see Corollary 2 and Theorem 1).

We note that, as mentioned prior to Lemma 2, in this section, we require the set to satisfy density (for a definition, see footnote 4). This condition is equivalent to the existence of a representation (Theorem 6.16 in Aleskerov et al. (2007)). Further, to avoid marginal cases, throughout this section we assume that $\succ \neq \emptyset$ and that there are at least two distinct $x, y \in A$ such that $0 < \gamma_{x,y} < 1$.

4.1. Re-defining the minimum weighted path distance

In Lemma 5, we defined the distance between two points as the minimum length of all paths between them. In the finite case, when two points are not connected, we set their distance to be bigger than the maximum of all possible weighted distances between connected elements. However, in the infinite case, these distances might diverge, making this strategy unfeasible.

A mirroring issue arises when dealing with sequences of weighted paths whose distances converge to zero. Indeed, for two distinct points having the same utility, the distance between elements converging to these points is zero, violating the definition of a metric.

To tackle these issues, we first propose a modified definition for δ that fails to meet the requirements of a metric; then we provide sufficient conditions that will bridge the gap to our original construction.

Definition 5. For all $x, y \in A$, define

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \infty & \text{if } \mathcal{P}_{x,y} = \emptyset, \\ \inf_{p \in \mathcal{P}_{x,y}} w(p) & \text{if } \mathcal{P}_{x,y} \neq \emptyset. \end{cases}$$

On the one hand, δ deals with the issue of infinitely distant points by setting their distance to infinity, i.e. δ is an extended metric.⁶ This approach stems from the very intuition related to the finite case, as it allows for two disconnected points to be “as far as possible”. On the other hand, the use of the infimum, in place of the minimum as in Definition 3, implies that two distinct points that converge to points with the same utility may have distance equal to zero, implying that δ is not a metric but a quasi-metric.⁷

4.1.1. From an extended quasi-metric to a metric

First, infinite distances in Definition 5 are necessary to describe the distance between points that are not connected in a potentially

⁶ The operations with ∞ are standard: for any real number x , $x + \infty = \infty$ and $\frac{x}{\infty} = 0$; if $x \neq 0$, then $x\infty = \infty$.

⁷ For example, consider an up-up alternating path $p_n = \{x, z_n, y\}$ such that $u(z_n) \rightarrow u(x) = u(y)$ and $x, y > z_n$, e.g. $u(x) = u(y) = 1$ and $u(z_n) = 1 - \frac{1}{n}$.

infinite graph. It follows that a sufficient condition to restore finiteness of the metric is to assume that the induced graph G is connected as this ensures that, even if the set of all weighted distances is not bounded, δ is well-defined.

Second, the attentive reader may have noticed that, in the finite case, strengthening Nakamura’s original statement from quasi-metrics to metrics was inconsequential. Given the reasoning above, this is no more the case. In Definition 5, we go back to Nakamura’s statement by allowing δ to be a quasi-metric — while keeping the partial order arbitrary other than requiring a representation. Now, we impose a condition on the partial order that suffices to guarantee that δ is a metric and, in doing so, clarifies when the difference between quasi-metric and metric becomes consequential.

Definition 6 (Sequential Indiscernible). For all distinct $x, y \in A$, we say x, y are sequentially indiscernible if there are $x_n, y_n \in A$ such that $u(x_n), u(y_n) \rightarrow u(x) = u(y)$, $\neg(x \perp x_n)$, and $\neg(y \perp y_n)$.

This property captures when two distinct alternatives are indistinguishable by looking at their behavior at nearby sequences that converge to their utilities. To appreciate the economic intuition, we briefly discuss, from a revealed preference perspective, the two ingredients of the definition: (i) $u(x) = u(y)$, and (ii) $\neg(x \perp x_n)$, and $\neg(y \perp y_n)$. The first ingredient, $u(x) = u(y)$, is intuitive as if the utilities differ, an intermediate alternative z , i.e. $u(x) > u(z) > u(y)$, could be used to distinguish them. The second ingredient is more interesting and rules out ambiguity coming from the incomparability relation.⁸ For example, if x, y are both better, worse, or in between the sequences x_n, y_n , then they can be seen as the same point on a line despite being distinct. If the condition fails (either $x \perp x_n$, or $y \perp y_n$, or both), instead, ambiguity remains as the local comparisons between x, x_n and y, y_n are missing. In other words, in the absence of incomparability, i.e. the threshold collapses to zero, it may be that $x \perp x_n$ and $\neg(y \perp y_n)$ implying that x, y can be told apart.

Lemma 8. For all distinct $x, y \in A$, if they are not sequentially indiscernible, then $\delta(x, y) \neq 0$.

Proof. Toward a contradiction suppose there are $x, y \in A$ such that $\delta(x, y) = 0$. We obtain that there are paths p_n such that $w(p_n) \rightarrow 0$. Since $w(p_n) \geq |u(x) - u(y)|$, we obtain $u(x) = u(y)$. Moreover, since $w(p_n) \rightarrow 0$, by considering the second point x_n and second to last point y_n of each p_n we get, respectively, $u(x_n) \rightarrow u(x)$ and $u(y_n) \rightarrow u(y)$ with $\neg(x_n \perp x)$ and $\neg(y_n \perp y)$, a contradiction by hypothesis. \square

Remark 1. Note that, Lemma 8 holds vacuously whenever the utility function has no ties. This observation gives away another sufficient condition that constraints the representation while leaving the (representable) partial order arbitrary. This is arguably a weak property given that partial orders are a generalization of linear orders for which the utility representation is such that for all pairs of distinct alternatives $u(x) \neq u(y)$. Hence, any utility function defined over the linear extension of \succ would satisfy this property.

The following proposition includes the discussions of this section showing under which conditions we obtain a quasi-metric, extended metric, or a metric.

Proposition 1. The following conditions hold.

- (i) δ is an extended quasi metric;
- (ii) If G is connected, then δ is a quasi-metric;

⁸ This argument is connected to the literature that studies the relationship between indifference and incomparability, e.g. Mandler (2005), Eliaz and Ok (2006).

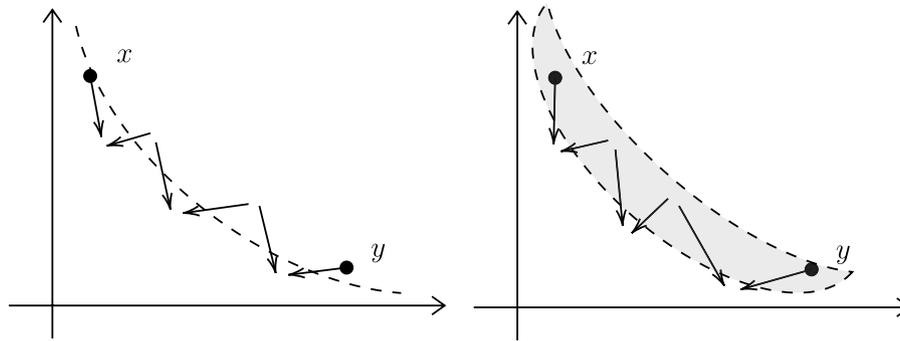


Fig. 1. Visualization of sequential comparability.

- (iii) If for all $x, y \in A$, the pair x, y is not sequentially indiscernible, then δ is an extended metric;
- (iv) If G is connected and all pairs of points are not sequentially indiscernible, then δ is a metric.

4.2. Defining the threshold ϵ from the metric δ

The second step in extending the representation is to define ϵ . As aforementioned, in the finite case, a global rescaling of δ was possible through means of the ratio between the difference in utility between two points and their distance in the induced graph which maximum was less than 1 (see Theorem 1 and the discussion the preceded it). This is a necessary condition for the representation as it follows from Corollary 1 that if the rescaling factor is equal to one then the partial order is not representable. In the infinite case, this global rescaling does not follow through as two pathological issues, one local and the other global, may disallow us from bounding the ratio strictly below one.

4.2.1. Local issues

Starting with the local issue – recall, we use the term “local” because it regards pairs in isolation, i.e. we analyze $\gamma_{x,y}$ – given two incomparable points $x, y \in A$ ($x \perp y$), suppose there is a sequence of paths $\{p_n\}$ such that $w(p_n) \rightarrow |u(x) - u(y)|$, then $\delta(x, y) = |u(x) - u(y)|$. This in turn implies that the $\max_{x,y:l(x,y)>1} \gamma_{x,y} = 1$. Similarly to the previous section, we provide a sufficient condition to avoid this issue.

Definition 7 (Local Sequential Comparability). We say that $x, y \in A$ with $u(x) \neq u(y)$ are locally sequentially comparable if the following condition holds:

$$\exists \{p_n\} \subseteq \mathcal{M}_{x,y} : w(p_n) \rightarrow |u(x) - u(y)| \implies \neg(x \perp y).$$

If every $x, y \in A$ are sequentially comparable, we say that the pair (\succ, u) is locally sequentially comparable.

Intuitively, Definition 7 says that if the alternating paths converge to a direct comparison, then the partial order should reflect it. In economic terms, consider two consumption bundles x, y as in Fig. 1. In the left panel, y lays above the indifference curve that passes through x , hence $u(y) > u(x)$. An alternating path is a sequence of comparable alternatives that lie above and below the indifference curve. As $w(p)$ approaches $|u(x) - u(y)|$, the path approaches the indifference curve with a slight deviation as y lays above it. Sequential comparability says that since every point in the path is comparable, the DM is able to evaluate trade-offs that are arbitrarily close to those involving x, y , hence x, y should be comparable. The right panel illustrates the contrapositive of the definition. The grey area represents the set of bundles incomparable to x and contains y . In this case, one cannot draw an alternating path between x, y that approaches their utility difference as eventually incomparable alternatives would appear.

Then note that, local sequential comparability considers the case of pairs with distinct utilities. This detail allows us, in the statement

of our main result (Theorem 2), to address the gap between metric and quasi-metric, i.e. between ours and Nakamura’s result, solely by imposing that alternatives are not sequentially indiscernible. As such, we distinguish the role of local sequential comparability as a more general necessary condition for our result. The following Corollary is immediate.

Corollary 8. If the pair (\succ, u) is locally sequentially comparable, then for all $x, y \in A$ with $x \perp y$ we have $\gamma_{x,y} < 1$.

4.2.2. Global issues

Global issues arise when we rescale δ considering all alternatives together. As aforementioned, in the finite case, it was sufficient to find the maximum of the local scaling, i.e. $\gamma_{x,y}$, after checking that it was “proper” (i.e., a coefficient less than 1). In the infinite case, however, the supremum of all ratios $\gamma_{x,y}$ might lead to a trivial scaling, namely $\gamma = \sup_{l(x,y)>1} \gamma_{x,y} = 1$. In this case, we can obtain a sequence of local scaling factors (γ_{x_n,y_n}) that converges to one. Indeed, if no proper global scaling is possible (i.e., $\gamma = 1$), then we can find a sequence of pairs of points, each with an alternating path, of ratio converging to 1. In turn, this may happen because of a local issue, which is already tackled in the previous subsection, or because of a strictly increasing sequence of ratios. The following lemma formalizes this intuition in the general case where γ may not be 1.

Lemma 9. For all $n \in \mathbb{N}$, there are $x_n, y_n \in A$, such that $x_n \perp y_n$, and $p_n \in \mathcal{M}_{x_n,y_n}$ such that all paths p_n are either up-up, down-down, up-down, or down-up of minimal length⁹ and $\gamma_{x_n,y_n}^{p_n} \rightarrow \gamma$.

To tackle this issue, we propose the following sufficient condition, called “global sequential comparability”, and its immediate corollary. This property deals with a sequence of local comparisons that taken separately satisfy local sequential comparability but together fall into the global pathology described above.

Definition 9 (Global Sequential Comparability). The pair (\succ, u) satisfies global sequential comparability if there is no sequence of distinct pairs (x_n, y_n) and paths of minimal length (p_n) such that $(\gamma_{x_n,y_n}^{p_n})$ is a strictly increasing sequence converging to 1.

Lemma 10. If (\succ, u) is locally and globally sequentially comparable, then $\gamma < 1$.

We finally obtain the following theorem.

Theorem 2. Let \succ be a partial order and u a representation such that (\succ, u) is locally and globally sequentially comparable. The following conditions hold.

⁹ By paths of minimal length, we refer to the minimum amount of point in an alternating path needed to define that type of path. So three points for up-up and down-down, and 4 points for up-down and down-up.

- (i) $>$ admits an extended quasi-metric representation;
- (ii) if G is connected, then $>$ admits a quasi-metric representation;
- (iii) if all pairs of points are not sequentially indiscernible, $>$ admits an extended metric representation;
- (iv) if G is connected and all pairs of points are not sequentially indiscernible, then $>$ admits a metric representation.

Proof. By Lemma 10, we have $\gamma < 1$. For all $x, y \in A$, let $\epsilon(x, y) = \gamma \delta(x, y)$. It is immediate to check that ϵ satisfies the required conditions in each of the four cases. \square

Theorem 2 generalizes Nakamura’s result to infinite sets under two mild assumptions: local and global sequential comparability. The remainder of the theorem deals with conditions that allows us to propose a representation with a metric (as in the finite case) instead of an extended quasi-metric. Therefore, in developing this result, we have taken care to specify each issue that prevent the extension from finite to infinite sets separately, and for each propose a solution. In doing so, we believe to have clarified the remarkable flexibility of our construction as well as its limitations.

CRedit authorship contribution statement

Daniele Caliari: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Daive Carpentiere:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization.

Appendix. Proofs omitted from the main text

Proof of Lemma 3

Suppose $x, y \in A$ and $p \in \mathcal{P}_{x,y}$. If p is an alternating path, we are done. Assume $p = (z_0, \dots, z_{n+1})$ is not an alternating path and obtain that there is an i such that there are z_i, z_{i+1}, z_{i+2} with $z_i > z_{i+1} > z_{i+2}$ or $z_i < z_{i+1} < z_{i+2}$. By transitivity of $>$ we obtain, respectively, $z_i > z_{i+2}$ or $z_i < z_{i+2}$. Moreover, note that

$$w(p) = \sum_{j=0}^n |u(z_{j+1}) - u(z_j)| = \sum_{j=0, j \neq i, i+1}^n |u(z_{j+1}) - u(z_j)| + |u(z_{i+1}) - u(z_i)| + |u(z_{i+2}) - u(z_{i+1})|.$$

In both cases we obtain

$$w(p) = \sum_{j=0, j \neq i, i+1}^n |u(z_{j+1}) - u(z_j)| + |u(z_{i+1}) - u(z_i)| + |u(z_{i+2}) - u(z_{i+1})| = \sum_{j=0, j \neq i, i+1}^n |u(z_{j+1}) - u(z_j)| + |u(z_{i+2}) - u(z_i)| = w(p'),$$

where $p' = (z_0, \dots, z_i, z_{i+2}, \dots, z_{n+1})$. If p' is an alternating path, we are done, otherwise iterate the process to obtain the thesis. To note that the process must end, observe that the path p' contains fewer points than the path p . Hence, every time we get a path that is not an alternating path we may remove a point. Since there is a finite amount of points in a path, this process must end. \square

Proof of Lemma 4

Suppose that $p \neq \{x, y\}$. By Lemma 3, there is $p' \in \mathcal{M}_{x,y}$ such that $w(p) = w(p')$. Hence we show that $|u(x) - u(y)| < w(p')$. If $u(x) - u(y) = 0$, then we are done. Without loss of generality, assume $u(x) > u(y)$. To do so, by Lemma 1, we have to check four cases.

(up-up) Suppose $p' = (x, z_0, \dots, z_{2m+1}, y)$ with $m \in \mathbb{N}$ and $x > z_0 < \dots > z_{2m+1} < y$. Since $u(y) > u(z_{2m+1})$, we obtain

$$|u(x) - u(y)| = u(x) - u(y) < u(x) - u(z_{2m+1}) + u(y) - u(z_{2m+1}) + \sum_{i=1}^m u(z_{2i}) - u(z_{2i-1}) = w(p').$$

(down-down) Suppose $p' = (x, z_0, \dots, z_{2m+1}, y)$ with $m \in \mathbb{N}$ and $x < z_0 > \dots < z_{2m+1} > y$. Since $u(z_1) > u(x)$, we obtain

$$|u(x) - u(y)| = u(x) - u(y) < u(z_1) - u(y) + u(z_1) - u(x) + \sum_{i=1}^m u(z_{2i+1}) - u(z_{2i}) = w(p').$$

(up-down) Suppose $p' = (x, z_0, \dots, z_{2m}, y)$ with $m \in \mathbb{N} \setminus \{0\}$ and $x > z_0 < \dots < z_{2m+1} > y$. We obtain

$$|u(x) - u(y)| = u(x) - u(y) < u(x) - u(y) + \sum_{i=1}^m u(z_{2i}) - u(z_{2i-1}) = w(p').$$

(down-up) Suppose $p' = (x, z_0, \dots, z_{2m}, y)$ with $m \in \mathbb{N} \setminus \{0\}$ and $x < z_0 > \dots > z_{2m+1} < y$. Since $u(z_1) > u(x)$ and $u(y) > u(z_{2m})$, we obtain

$$|u(x) - u(y)| < u(z_1) - u(z_{2m}) + u(z_1) - u(x) + u(y) - u(z_{2m}) + \sum_{i=1}^{m-1} u(z_{2i+1}) - u(z_{2i}) = w(p'). \quad \square$$

Proof of Lemma 7

(i) Suppose $x, y \in A$ and $p = (x, z_1, \dots, z_n, y) \in \mathcal{M}_{x,y}$. If $u(x) = u(y)$, then we are done in all cases.

Without loss of generality, suppose $u(x) > u(y)$. Starting with (i), assume p is either (up-up) or (down-down). If p is (up-up), then consider $p \upharpoonright_{x, z_n}$. Since $u(y) > u(z_n)$, we obtain that $u(x) - u(y) < u(x) - u(z_n)$. Moreover, clearly, $w(p \upharpoonright_{x, z_n}) \leq w(p)$. Hence $\gamma_{x,y}^p < \gamma_{x, z_n}^{p \upharpoonright_{x, z_n}}$.

If p is (down-down), then consider $p \upharpoonright_{z_1, y}$. Since $u(z_1) > u(x)$, we obtain that $u(x) - u(y) < u(z_1) - u(y)$. Moreover, clearly, $w(p \upharpoonright_{z_1, y}) \leq w(p)$. Hence $\gamma_{x,y}^p < \gamma_{z_1, y}^{p \upharpoonright_{z_1, y}}$.

(ii) Can be argued in a similar way.

(iii) To simplify the notation, for all $j = 1, \dots, 4$, let $a_{2j} = u(x_{2j}) - u(x_{2j-1})$. Toward a contradiction, suppose $\gamma_{x,y}^p > \max\{\gamma_{x, z_3}^{p \upharpoonright_{x, z_3}}, \gamma_{z_2, y}^{p \upharpoonright_{z_2, y}}\}$. We obtain that

$$\frac{u(x) - u(z_3)}{w(p \upharpoonright_{x, z_3})} = \frac{u(x) - u(z_3)}{u(x) - u(z_3) + 2a_2} < \frac{u(x) - u(y)}{u(x) - u(y) + 2(a_2 + a_4)} = \frac{u(x) - u(y)}{w(p)}$$

and

$$\frac{u(z_2) - u(y)}{w(p \upharpoonright_{z_2, y})} = \frac{u(z_2) - u(y)}{u(z_2) - u(y) + 2a_4} < \frac{u(x) - u(y)}{u(x) - u(y) + 2(a_2 + a_4)} = \frac{u(x) - u(y)}{w(p)}.$$

Hence

$$(u(x) - u(z_3))2(a_2 + a_4) < (u(x) - u(y))2a_2 \quad \text{and} \quad (u(z_2) - u(y))2(a_2 + a_4) < (u(x) - u(y))2a_4.$$

Summing the two inequalities we obtain

$$(a_2 + a_4)(u(x) - u(y) + u(z_2) - u(z_3)) < (u(x) - u(y))(a_2 + a_4),$$

a contradiction, since $(u(z_2) - u(z_3))(a_2 + a_4) > 0$.

(iv) Toward a contradiction, suppose $\gamma_{x,y}^p > \max\{\gamma_{x, z_3}^{p \upharpoonright_{x, z_3}}, \gamma_{z_4, y}^{p \upharpoonright_{z_4, y}}\}$. We obtain that

$$\frac{u(x) - u(z_3)}{w(p \upharpoonright_{x, z_3})} = \frac{u(x) - u(z_3)}{u(x) - u(z_3) + 2a_2} < \frac{u(x) - u(y)}{u(x) - u(y) + 2(\sum_{j=1}^{\frac{n}{2}} a_{2j})} = \frac{u(x) - u(y)}{w(p)}$$

and

$$\frac{u(z_4) - u(y)}{w(p \upharpoonright_{z_4, y})} = \frac{u(z_4) - u(y)}{u(z_4) - u(y) + 2(\sum_{j=3}^{\frac{n}{2}} a_{2j})} < \frac{u(x) - u(y)}{u(x) - u(y) + 2(\sum_{j=1}^{\frac{n}{2}} a_{2j})} = \frac{u(x) - u(y)}{w(p)}.$$

Hence

$$(u(x) - u(z_3))2(\sum_{j=1}^{\frac{n}{2}} a_{2j}) < (u(x) - u(y))2a_2 \quad \text{and} \quad (u(z_4) - u(y))2(\sum_{j=1}^{\frac{n}{2}} a_{2j}) < (u(x) - u(y))2(\sum_{j=3}^{\frac{n}{2}} a_{2j}).$$

Summing the two inequalities we obtain

$$\left(\sum_{j=1}^{\frac{n}{2}} a_{2j}\right)(u(x) - u(y) + u(z_4) - u(z_3)) < (u(x) - u(y))(a_2 + \left(\sum_{j=3}^{\frac{n}{2}} a_{2j}\right)),$$

a contradiction as in case (iii). \square

Proof of Proposition 1

(i) The statement can be proved by showing the following: for all x, y, z , if $\mathcal{P}_{x,y}, \mathcal{P}_{y,z}$ and $\mathcal{P}_{x,z}$ are nonempty, then $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

For all $p \in \mathcal{P}_{x,y}$, and $p' \in \mathcal{P}_{y,z}$, denote by $p + p'$ the path formed by joining the paths p and p' . Note that for all $p \in \mathcal{P}_{x,y}$, and $p' \in \mathcal{P}_{y,z}$,

$$\inf_{q \in \mathcal{P}_{x,z}} w(q) \leq w(p + p') \leq w(p) + w(p').$$

Since both p and p' are arbitrary, we obtain that

$$\delta(x, z) = \inf_{q \in \mathcal{P}_{x,z}} w(q) \leq \inf_{p \in \mathcal{P}_{x,y}} w(p) + \inf_{p' \in \mathcal{P}_{y,z}} w(p') = \delta(x, y) + \delta(y, z).$$

(ii) Immediate by Definition 5.

(iii) Follows by Lemma 8.

(iv) Follows by Conditions (iii) and (iv). \square

Proof of Lemma 9

First we prove the following result.

Lemma 11. For all $n \in \mathbb{N}$ let $B_n \subseteq \mathbb{R}^+ \setminus \{0\}$, $a_n \in \mathbb{R}$ such that $0 < a_n < \inf B_n$. Let for all $n \in \mathbb{N}$, $\gamma_n = \frac{a_n}{\inf B_n}$, and $\gamma = \sup_n \gamma_n$. There is a sequence $\{ \frac{a_n}{b_n} : b_n \in B_n \}$ such that $\frac{a_n}{b_n} \rightarrow \gamma$.

Proof. Let (γ_n) be a sequence converging to γ . Moreover, for all $n \in \mathbb{N}$, there is a sequence $(b_n^m) \subseteq B_n$ such that $b_n^m \rightarrow \inf B_n$. Hence, for all $n \in \mathbb{N}$ and for all $\delta_n > 0$ there is $\eta_n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m \geq \eta_n$ we have $\inf B_n \leq b_n^m < \inf B_n + \delta_n$. Therefore, $\gamma_n \geq \frac{a_n}{b_n^m} > \frac{a_n}{\inf B_n + \delta_n}$. Fix $\delta_n = \frac{a_n}{n}$ and note that

$$\lim_n \frac{a_n}{\inf B_n} \cdot \frac{\inf B_n + \frac{a_n}{n}}{a_n} = \lim_n 1 + \gamma_n \cdot \frac{1}{n} = 1.$$

We conclude that the sequence (γ_n) and $(\frac{a_n}{\inf B_n + \frac{a_n}{n}})$ have the same limit.

Hence, fixing $b_n^m = b_n^{\eta_n}$, we obtain that

$$\gamma_n \geq \frac{a_n}{b_n^{\eta_n}} > \frac{a_n}{\inf B_n + \frac{a_n}{n}}.$$

And so

$$\frac{a_n}{b_n^{\eta_n}} \rightarrow \gamma. \quad \square$$

By Lemmas 7 and 11, find $x_n, y_n \in A$ and $p_n \in \mathcal{M}_{x_n, y_n}$ of minimal length such that $\frac{|u(x_n) - u(y_n)|}{w(p_n)} \rightarrow \gamma$. For all $n \in \mathbb{N}$, p_n is either up-up, down-down, up-down, or down-up. Hence there is a subsequence (p_{n_k}) such that $\frac{|u(x_{n_k}) - u(y_{n_k})|}{w(p_{n_k})} \rightarrow \gamma$ and such that all p_{n_k} are either up-up, down-down, up-down, or down-up. \square

Data availability

No data was used for the research described in the article.

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