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Article

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Journal of Derivatives and Quantitative Studies: Seonmul yeon'gu (JDQS)

Provided in Cooperation with:

Korea Derivatives Association (KDA)

Suggested Citation: Lee, Hyo-Chan; Park, Seyoung; Yoon, Jong Mun (2021) : Optimal investment with time-varying transition probabilities for regime switching, Journal of Derivatives and Quantitative Studies: Seonmul yeon'gu (JDQS), ISSN 2713-6647, Emerald, Leeds, Vol. 29, Iss. 2, pp. 102-115, <https://doi.org/10.1108/JDQS-12-2020-0032>

This Version is available at:

<https://hdl.handle.net/10419/337227>

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Optimal investment with time-varying transition probabilities for regime switching

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Abstract

Abstract – This study aims to generalize the following result of McDonald and Siegel (1986) on optimal investment: it is optimal for an investor to invest when project cash flows exceed a certain threshold. This study presents other results that refine or extend this one by integrating timing flexibility and changes in cash flows with time-varying transition probabilities for regime switching. This study emphasizes that optimal thresholds are either overvalued or undervalued in the absence of time-varying transition probabilities. Accordingly, the stochastic nature of transition probabilities has important implications to the search for optimal timing of investment.

Keywords Optimal investment, Time-varying transition probabilities, Regime switching, Cash flows, Option

Paper type Research paper

1. Introduction

An investor's optimal investment decision is to invest when project cash flows exceed a certain threshold (McDonald and Siegel, 1986). Here, the investment decision resembles an American-style call option, where there exists the flexibility of choosing optimal timing of option exercise. Based on this analogy, we extend the standard real option approach to a regime-switching environment with time-varying transition probabilities for regime switching and analyze the optimal investment decision [1].

Most real investment decisions have three important characteristics in varying degrees (Dixit and Pindyck, 1994). First, the investment is partially or completely *irreversible*, i.e.



investment cost incurred at investment time at least cannot be recovered. Second, the future cash flows from the investment are *uncertain*. Third, the timing of investment is *flexible*, i.e. an investor adjusts investment by an irreversible choice of investment time. In light of such investment characteristics, McDonald and Siegel (1986) also claim as follows:

The rule, “invest if the net present value of investing exceeds zero”, is only valid if the variance of the present value of future benefits and costs is zero or if the expected rate of growth of the present value is minus infinity; the value lost by following this suboptimal investment policy can be substantial.

This is precisely the direction we take here. We generalize this rule. Specifically, we generalize the McDonald and Siegel (1986) result to include regime switching (Guo *et al.*, 2005; Bensoussan *et al.*, 2012). By integrating timing flexibility and changes in cash flows with time-varying transition probabilities for regime switching, we obtain regime-dependent optimal policies for investment. We hope that some results of this paper will lend themselves to the study of implications for real investment activities.

To examine the impact of macroeconomic conditions on optimal investment in the simplest possible environment, we assume that only two regimes exist, an increasing “Bull” market (regime *B*) and a decreasing “bear” market (regime *b*). Importantly, we consider a Markov regime switching model with time-varying transition probabilities (Diebold *et al.*, 1994; Filardo, 1994). An investor would like to find the optimal timing of investment that gives the best profits in the long run taking into account that the transition probabilities between the two regimes vary over time. This is an optimal investment strategy that maximizes the expected profits of project cash flows that are governed by a two-state continuous-time Markov chain.

We present other results that refine or extend the result of McDonald and Siegel (1986) by integrating timing flexibility and changes in cash flows with time-varying transition probabilities for regime switching. We obtain regime-dependent optimal investment policies and reveal new insights into the implications of real investment activities when regimes can switch. We emphasize that optimal thresholds are either overvalued or undervalued in the absence of time-varying transition probabilities. Accordingly, the stochastic nature of transition probabilities has important implications to the search for optimal timing of investment.

2. Basic model

To examine the effect of macroeconomic conditions on optimal investment in the simplest possible environment, we assume that just two regimes, “Bull” (regime *B*) and “bear” (regime *b*), are possible. We also assume that regime *i* is observable and switches into regime *j* at the first jump time of a Poisson jump process with intensity $\lambda_i > 0$, for $i, j \in \{B, b\}$. The assumption of constant transition probabilities between two regimes (Guo *et al.*, 2005; Bensoussan *et al.*, 2012) is too restrictive to describe many empirical observations, such as asymmetric business-cycle features. To characterize the dynamics of regime switching better than the fixed-transition-probabilities version, we use a stochastic process for λ_i . Within the present model, the time T_i to leave regime *i* satisfies the following: for time $t \geq 0$,

$$\text{Probability of } \{T_i > t\} = 1 - \int_0^t \lambda_{i,s} ds, \quad i \in \{B, b\},$$

where

$$d\lambda_{i,s} = a_i \lambda_{i,s} ds + b_i \lambda_{i,s} dW_s^i, \quad \lambda_{i,0} = \lambda_i > 0, \quad a_i < 0, \quad b_i > 0. \quad (1)$$

Here, W_s^i for regime $i \in \{B, b\}$ are standard one-dimensional and two independent Brownian motions. T_i is stochastic, so our model incorporates time-varying probabilities of transitions between two regimes. When we set $a_i = -\lambda_i$ and $b_i = 0$, T_i follows an exponential distribution with intensity λ_i . This result implies the existence of probability $\lambda_i dt$ that regime i switches into regime j during an infinitesimal time interval dt . Further, the expected durations of regime switching are constant: the expected duration of regime i is $1/\lambda_i$ and the average fraction of time spent in regime i is $\lambda_j/(\lambda_i + \lambda_j)$. Therefore, the constant transition probabilities for regime switching can be regarded as a special case in our model.

To represent regime-dependent cash flows, we consider a stream of profit after taking a project as the following project cash flows X_t modeled by a geometric Brownian motion equipped with a two-state Markov regime switching model:

$$dX_t = \mu_i X_t dt + \sigma_i X_t dW_t, \quad X_0 = x > 0, \quad (2)$$

where W_t is a standard one-dimensional Brownian motion, $i \in \{B, b\}$ is determined by a two-state continuous time Markov chain, $\mu_i > 0$ is the expected rate of returns on project cash flows, and $\sigma_i > 0$ is the standard deviation of returns on cash flows, i.e. the volatility of project cash flows. Further, we assume that $\mu_B > \mu_b$ and $\sigma_b > \sigma_B$. The project cash flows and the regime switching have asymmetric correlations in that $dW_t \cdot dW_t^i = \rho_i dt$ for $\rho_i \in [-1, 1]$.

The presence of time-varying probabilities of transitions between two regimes directly affects the expected rate of returns on cash flows. The project cash flows X_t can be rewritten as (Appendix):

$$dX_t = (\mu_i + \rho_i b_i \sigma_i) X_t dt + \sigma_i X_t dW_t^{i,Q}$$

where $W_t^{i,Q}$ is a standard one-dimensional Brownian motion defined under the Q probability measure [2]. The economic intuition is straightforward. An investor encounters undiversifiable risk stemming from time-variation of regime intensity λ_i , which is well captured by the term involving Brownian motions W_s^i in equation (1). Thus, the investor requires additional risk premia that are included in the expected rate of returns on cash flows. The risk premia can be positive or negative according to the sign of the correlations ρ_i . A negative correlation implies that investing in project cash flows could be a partial hedging tool for the unhedgeable risk of stochastic regime switching, and hence decreases the expected returns on cash flows.

An investor would like to find the optimal timing of investment that gives the best profits in the long run, taking into account that the transition probabilities between two regimes vary over time. This is an optimal investment strategy that maximizes the expected profits of project cash flows which are governed by a two-state continuous time Markov chain. More specifically, the optimal investment problem can be formulated as

$$\max_{\tau} E[e^{-\delta \tau} (X_{\tau} - c)],$$

where $\delta > 0$ is the appropriate discount rate, τ is optimal timing of investment, and $c > 0$ is an investment cost. The investor owns an embedded option to invest that can only be exercised once, i.e. she can undertake the investment irreversibly at some optimally chosen

time τ . The optimal timing of investment should be determined endogenously, so the investment time is stochastic from today's perspective. The investor receives $X_\tau - c$ when she invests at some endogenously chosen time τ . She pays the investment cost c only at τ . We assume that this cost can be financed from the investor's own wealth.

Following McDonald and Siegel (1986), the investor's objective is to find a certain threshold of project cash flows over which it is optimal for the investor to undertake an irreversible investment. Most importantly, the endogenously determined threshold is not only regime dependent, but also significantly affected by the presence of time-varying transition probabilities for regime switching. Technically, we solve the optimal investment problem by using the variational inequality approach (Bensoussan and Lions, 1982; Øksendal, 2007). More specifically, in regime $i = \{B, b\}$ we define $\phi_i(x)$ as the expected discounted payoff from the irreversible investment project undertaken at time τ and it is given by:

$$\phi_i(x) = E[e^{-\delta\tau}(X_\tau - c)]$$

In regime i , the investor's objective is to find an optimal stopping time τ to maximize the expected discounted payoff ϕ_i :

$$\Phi_i(x) = \max_{\tau \geq 0} \phi_i(x)$$

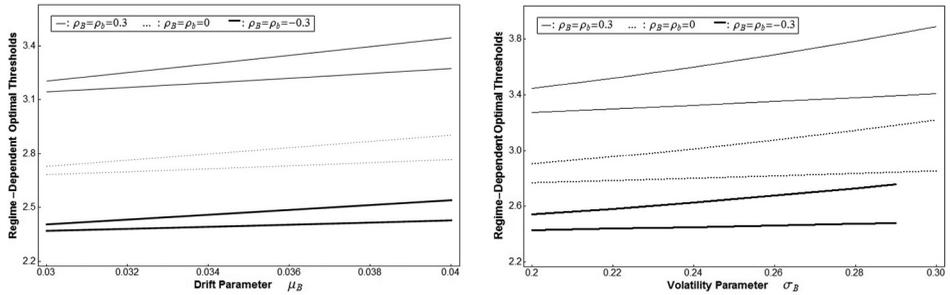
3. Numerical results

We present graphical illustration and additional detail in discussion of our optimal investment strategy. To do this, we exploit regime-dependent optimal investment policies that are significantly affected by the presence of time-varying probabilities between two regimes (Appendix).

We set the baseline parameter values for numerical analysis as follows (Guo *et al.*, 2005). We set appropriate discount rate $\delta = 0.06$ [3]. The drift and volatility parameters of project cash flows during regime B are $\mu_B = 0.04$ and $\sigma_B = 0.2$, and during regime b are $\mu_b = 0.01$ and $\sigma_b = 0.3$. Our parameter choice for the drift and volatility reflects the fact that while volatility is higher in the bear market regime than in the Bull market regime, drift is higher in the Bull market regime than in the bear market regime (Ang and Bekaert, 2002). δ should be larger than the growth rates μ_i of project cash flows; otherwise, an investor never pays to invest, because the value of the investment opportunity is infinite (McDonald and Siegel, 1986). The parameters governing the persistence of regime chosen as follows: $\lambda_B = 0.15$ and $\lambda_b = 0.1$ [4]. Regarding the investment cost, we set $c = 1$ (Dixit and Pindyck, 1994) [5]. Finally, we set for the comparison purpose the volatility parameter b_i of regime intensity λ_i to 0, i.e. we consider constant probabilities of transitions between two regimes. We will set b_i to 0.1 in Figure 1 for consideration of time-varying regime transition probabilities.

3.1 Constant Transition probabilities

Table 1 reports the comparative statistics for regime-dependent optimal investment policies (\bar{x}_B, \bar{x}_b) . The investment opportunity set that consists of the drift and volatility of project cash flows is not constant, and regimes B and b have fundamental parameters. This consideration of a stochastic investment opportunity set has a *first-order* effect on optimal investment strategies. Specifically, for the baseline parameter values, optimal timing of



Notes: This figure illustrates regime-dependent optimal thresholds according to the changes of drift and volatility parameters. To investigate the effects of time-varying probabilities of transitions between two regimes, we allow for the case where the volatility parameter b_i ($i \in \{B, b\}$) of regime intensity λ_i is not zero. We set the volatility parameter b_i to 0.1. Then we consider two symmetric correlations: $\rho_B = \rho_b = 0.3$ and $\rho_B = \rho_b = -0.3$. The baseline parameter values are fixed as follows: discount rate $\delta = 0.06$, drift and volatility parameters of project cash flows in regime B are $\mu_B = 0.04$ and $\sigma_B = 0.2$, drift and volatility parameters in regime b are $\mu_b = 0.01$ and $\sigma_b = 0.3$, parameters governing the persistence of regime are $\lambda_B = 0.15$ and $\lambda_b = 0.1$, investment cost $c = 1$

Figure 1.
Regime-dependent
optimal thresholds
(symmetric
correlations)

investment in regime B is faster than optimal timing in regime b , i.e. optimal thresholds \bar{x}_B in regime B are consistently smaller than those \bar{x}_b in regime b (Table 1).

An investor's irreversible investment decision can be regarded as an American-style call option, where the investor has the flexibility of choosing the time of irreversible investment. The standard option pricing model shows that option value increases as market volatility increases. Also, the real option value for investment increases as project volatility increases, in the sense that increase in uncertainty causes increase in the value of an investor's investment opportunities (Dixit and Pindyck, 1994). As Dixit and Pindyck (1994) states:

Thus investment is highly sensitive to volatility in project values, irrespective of investors' or managers' risk preferences, and irrespective of the extent to which the riskiness of project values is correlated with the market.

Our results support the positive convexity effect of volatility on option value. The table confirms that regime-dependent optimal investment policies (\bar{x}_B, \bar{x}_b) increase with the volatility of any regime. Most importantly, the possibility of a regime switching means that the investor's optimal strategy depends on the existence of the other regime. Specifically, the optimal strategy of investment timing is characterized by a different threshold in each regime. Due to the possibility of regime shift, each threshold incorporates the threshold in the other regime.

Changes in volatility in a given regime have major effects on investment thresholds of any state. For the baseline parameters, the thresholds are given a pair of $(\bar{x}_B, \bar{x}_b) = (2.79, 2.97)$. When the volatility of regime B is increased from 20% to 25%, the thresholds of any state also are raised by the amount $(0.04, 0.18)$, so $(\bar{x}_B, \bar{x}_b) = (2.83, 3.15)$. Similarly, when the volatility of regime b is increased from 30% to 35%, the thresholds of any state also are raised by the amount $(0.35, 0.31)$, so $(\bar{x}_B, \bar{x}_b) = (3.14, 3.28)$. In this sense, we show monotonically increasing relations of optimal thresholds as the volatility of project cash flows increases.

Parameters	The riskiness of project cash flows					
	Volatility of regime B			Volatility of regime b		
	2(0%)	25(0%)	30(0%)	30(0%)	35(0%)	40(0%)
Base parameter	(2.79, 2.97)	(2.83, 3.15)	(2.88, 3.38)	(2.79, 2.97)	(3.14, 3.28)	(3.53, 3.62)
$\mu_B = 0.04$	-0.01 (2.69, 2.75)	(2.74, 2.91)	(2.79, 3.12)	(2.69, 2.75)	(3.03, 3.02)	(3.40, 3.32)
	-0.005 (2.74, 2.86)	(2.78, 3.03)	(2.84, 3.25)	(2.74, 2.86)	(3.08, 3.15)	(3.46, 3.47)
	+0.005 (2.84, 3.08)	(2.89, 3.28)	(2.94, 3.51)	(2.84, 3.08)	(3.21, 3.41)	(3.61, 3.77)
	+0.01 (2.90, 3.20)	(2.94, 3.40)	(2.99, 3.64)	(2.90, 3.20)	(3.28, 3.54)	(3.69, 3.93)
$\mu_b = 0.01$	-0.003 (2.68, 2.87)	(2.72, 3.05)	(2.76, 3.25)	(2.68, 2.87)	(3.01, 3.16)	(3.38, 3.49)
	-0.0015 (2.73, 2.92)	(2.78, 3.10)	(2.83, 3.32)	(2.73, 2.92)	(3.08, 3.22)	(3.46, 3.55)
	+0.0015 (2.85, 3.03)	(2.89, 3.21)	(2.95, 3.44)	(2.85, 3.03)	(3.21, 3.34)	(3.61, 3.69)
	+0.003 (2.91, 3.08)	(2.96, 3.27)	(3.01, 3.50)	(2.91, 3.08)	(3.28, 3.41)	(3.69, 3.76)
$\delta = 0.06$	-0.005 (2.97, 3.20)	(3.02, 3.40)	(3.08, 3.65)	(2.97, 3.20)	(3.37, 3.55)	(3.80, 3.93)
	-0.025 (2.87, 3.08)	(2.92, 3.27)	(2.98, 3.50)	(2.87, 3.08)	(3.25, 3.40)	(3.66, 3.77)
	+0.025 (2.71, 2.88)	(2.75, 3.05)	(2.80, 3.27)	(2.71, 2.88)	(3.05, 3.17)	(3.42, 3.49)
	+0.005 (2.64, 2.79)	(2.68, 2.96)	(2.73, 3.17)	(2.64, 2.79)	(2.96, 3.07)	(3.32, 3.37)
$c = 1$	-0.5 (1.39, 1.49)	(1.42, 1.58)	(1.44, 1.69)	(1.39, 1.49)	(1.57, 1.64)	(1.76, 1.81)
	-0.25 (2.09, 2.23)	(2.12, 2.36)	(2.16, 2.53)	(2.09, 2.23)	(2.36, 2.46)	(2.65, 2.71)
	+0.25 (3.48, 3.71)	(3.54, 3.94)	(3.61, 4.22)	(3.48, 3.71)	(3.93, 4.10)	(4.41, 4.52)
	+0.5 (4.18, 4.46)	(4.25, 4.73)	(4.33, 5.07)	(4.18, 4.46)	(4.71, 4.92)	(5.29, 5.43)
$\lambda_B = 0.15$	-0.02 (2.80, 3.01)	(2.85, 3.22)	(2.90, 3.47)	(2.80, 3.01)	(3.15, 3.31)	(3.54, 3.64)
	-0.01 (2.79, 2.99)	(2.84, 3.18)	(2.89, 3.42)	(2.79, 2.99)	(3.14, 3.29)	(3.53, 3.63)
	+0.01 (2.78, 2.95)	(2.83, 3.13)	(2.88, 3.34)	(2.78, 2.95)	(3.14, 3.26)	(3.53, 3.61)
	+0.02 (2.78, 2.94)	(2.82, 3.10)	(2.87, 3.31)	(2.78, 2.94)	(3.13, 3.25)	(3.53, 3.60)
$\lambda_b = 0.1$	-0.01 (2.77, 2.95)	(2.82, 3.12)	(2.86, 3.33)	(2.77, 2.95)	(3.13, 3.26)	(3.52, 3.61)
	-0.005 (2.78, 2.96)	(2.82, 3.14)	(2.87, 3.36)	(2.78, 2.96)	(3.14, 3.27)	(3.53, 3.61)
	+0.005 (2.79, 2.98)	(2.84, 3.17)	(2.90, 3.40)	(2.79, 2.98)	(3.15, 3.29)	(3.53, 3.62)
	+0.01 (2.80, 2.99)	(2.85, 3.18)	(2.91, 3.42)	(2.80, 2.99)	(3.15, 3.29)	(3.54, 3.63)

Notes: The baseline parameter values are fixed as follows: discount rate $\delta = 0.06$, drift and volatility parameters of project cash flows in regime B are $\mu_B = 0.04$ and $\sigma_B = 0.2$, drift and volatility parameters in regime b are $\mu_b = 0.01$ and $\sigma_b = 0.3$, parameters governing the persistence of regime are $\lambda_B = 0.15$ and $\lambda_b = 0.1$, investment cost $c = 1$

Table 1. Comparative statistics for (\bar{x}_B, \bar{x}_b)

Concerning the effects of varying drift rates of any regime, we show monotonically increasing relations of optimal thresholds as the drift of project cash flows increases. The rationale behind this result is that the expected appreciation in the value of an irreversible investment option rises as the expected rate of growth of project cash flows increases. As a result, waiting becomes cheaper than investing immediately. Further, changes in drift in a given regime have a large impact on investment thresholds of any state. For the baseline parameters, the thresholds are given as: $(\bar{x}_B, \bar{x}_b) = (2.79, 2.97)$. If the drift of regime B is raised by the amount 0.01; $\mu_B = 0.05$, then thresholds of any state also are increased from $(\bar{x}_B, \bar{x}_b) = (2.79, 2.97)$ to $(\bar{x}_B, \bar{x}_b) = (2.90, 3.20)$. Similarly, if the drift of regime b is raised by the amount 0.003; $\mu_b = 0.013$, then thresholds of any state also are increased from $(\bar{x}_B, \bar{x}_b) = (2.79, 2.97)$ to $(\bar{x}_B, \bar{x}_b) = (2.91, 3.08)$.

Optimal thresholds increase as discount rate δ decreases or as investment cost c increases. As expected, a decrease in δ increases the present value of expected project cash flows after investment decision, and hence increases the value of investment option, i.e. increases the opportunity cost of investing now. Accordingly, investment can be delayed as δ decreases. Of course, an increase in c raises the value of an investor's investment option, and drives increase in the thresholds.

The functional dependence between regime-dependent optimal thresholds depends on the durations of regimes. Interestingly, the effects of varying the duration of a regime are asymmetric. A decrease of the persistence of regime B , i.e. an increase in λ_B reduces the opportunity cost of irreversible investment in regime B and hence lowers optimal threshold \bar{x}_B in regime B . One might wonder why investment in regime B can be hastened, because relatively frequent transition from regime B to regime b is expected. However, as persistence of regime B decreases, the expected appreciation in the value of investment option decreases, whereas the riskiness of project cash flows increases. To improve our understanding of the expected appreciation and the riskiness, we introduce regime-adjusted drift:

$$\tilde{\mu} = \frac{\lambda_b}{\lambda_B + \lambda_b} \mu_B + \frac{\lambda_B}{\lambda_B + \lambda_b} \mu_b$$

and volatility as

$$\tilde{\sigma} = \frac{\lambda_b}{\lambda_B + \lambda_b} \sigma_B + \frac{\lambda_B}{\lambda_B + \lambda_b} \sigma_b$$

of project cash flows. When the parameter λ_B that governs the persistence of regime B is raised by 0.02 to $\lambda_B = 0.17$, then $\tilde{\mu}$ decreases from 0.022 to 0.021, whereas $\tilde{\sigma}$ increases from 0.26 to 0.263. Accordingly, adjustments in the persistence of regime represent a trade-off between these two quantities, so the optimal thresholds depend crucially on $\tilde{\mu}$ and $\tilde{\sigma}$. Combining all the effects, optimal thresholds can decrease when regime B is of shorter duration than regime b . In contrast, optimal thresholds can increase when regime B is of longer duration than regime b ; this instance is well captured when λ_B decreases or λ_b increases.

3.2 Time-varying transition probabilities

To investigate the effects of time-varying probabilities of transitions between two regimes, we allow for the case where the volatility parameter b_i $i \in \{B, b\}$ of regime intensity λ_i is not zero. We set the volatility parameter b_i to 0.1. Then we consider two symmetric correlations: $\rho_B = \rho_b = 0.3$ and $\rho_B = \rho_b = -0.3$. We will also consider asymmetric correlations supporting the existence of substantial heterogeneity across magnitudes of conditional correlations. The correlations between portfolios (or expected cash flows) tend to show a sharp increase which has been observed in the period of global financial crisis (2007–2009), which has turned out to contribute to contagion effects with the networks of interconnections that link individuals, financial institutions and firms with one another (Kim and Park, 2020) [6].

The presence of time-varying probabilities between two regimes has a significant effect on the regime-dependent optimal thresholds (Figure 1). We confirm that the optimal thresholds drift up as the drift and volatility parameters of project cash flows increase (Table 1 and Figure 1). One intriguing observation is that the optimal thresholds depend crucially on the current level of symmetric correlations ρ_i ($i \in \{B, b\}$). That is, the optimal thresholds drift upward as the current level of correlations increases. The economic intuition is that investing in project cash flows that have negative correlations with the regime switching could be a partial hedging tool against the unhedgeable or undiversifiable risk of stochastic regime switching, i.e. in such a case an investor requires negative risk premia. Then the negative correlation decreases the expected returns on cash flows, i.e. reduces the level of μ_i ($i \in \{B, b\}$). As a result, a negative and sizable correlation induces an investor to undertake the investment early. We emphasize that optimal thresholds from the

comparative statistics (Table 1) are either overvalued or undervalued in the absence of time-varying transition probabilities. Accordingly, the stochastic nature of transition probabilities has important functions in the search for optimal timing of investment.

To contemplate the effects of asymmetric correlations on the optimal investment, we average the correlations across regime B and regime b by using the average fraction of time spent in each regime [7]:

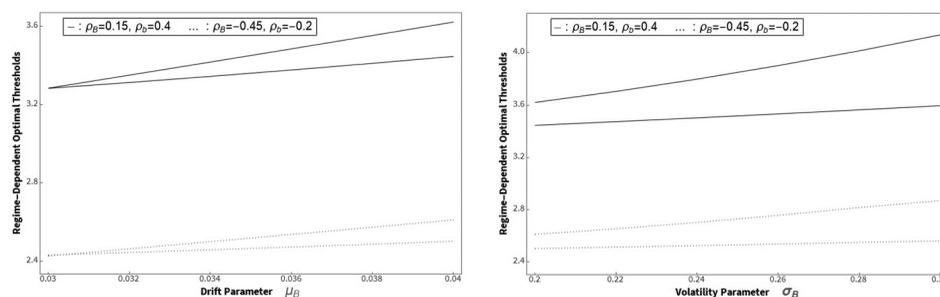
$$\tilde{\rho} = \frac{\lambda_b}{\lambda_B + \lambda_b} \rho_B + \frac{\lambda_B}{\lambda_B + \lambda_b} \rho_b,$$

which is a regime-adjusted correlation just as regime-adjusted drift $\tilde{\mu}$ and volatility $\tilde{\sigma}$. We then obtain asymmetric values of ρ_B and ρ_b by setting the regime-adjusted correlation to 0.3 and -0.3 , having the two values of symmetric correlations used so far as a benchmark. We consider two asymmetric correlations: $\rho_B = 0.15, \rho_b = 0.4$ and $\rho_B = -0.45, \rho_b = -0.2$ [8].

In case of asymmetric correlations, the qualitative features of regime-dependent optimal thresholds with symmetric correlations remain unchanged (Figure 2), i.e. the optimal thresholds rise as the drift and volatility parameters of project cash flows increase.

4. Conclusion

In this paper, we develop an analytically tractable investment model by generalizing McDonald and Siegel (1986)'s investment framework to include time-varying transition probabilities for regime switching. Consistent with the canonical investment framework, we also discover the existence of a certain threshold of cash flows over which it is optimal for an investor to invest in a new project. However, using this threshold for investment without taking into account time-varying transition probabilities represents an over-simplified situation, i.e. optimal investment thresholds are either overestimated or underestimated in the absence of time-varying transition probabilities. Therefore, neglecting the stochastic



Notes: This figure illustrates regime-dependent optimal thresholds according to the changes of drift and volatility parameters. To investigate the effects of time-varying probabilities of transitions between two regimes, we allow for the case where the volatility parameter b_i ($i \in \{B, b\}$) of regime intensity λ_i is not zero. We set the volatility parameter b_i to 0.1. Then we consider two asymmetric correlations: $\rho_B = 0.15, \rho_b = 0.4$ and $\rho_B = -0.45, \rho_b = -0.2$. The baseline parameter values are fixed as follows: discount rate $\delta = 0.06$, drift and volatility parameters of project cash flows in regime B are $\mu_B = 0.04$ and $\sigma_B = 0.2$, drift and volatility parameters in regime b are $\mu_b = 0.01$ and $\sigma_b = 0.3$, parameters governing the persistence of regime are $\lambda_B = 0.15$ and $\lambda_b = 0.1$, investment cost $c = 1$

Figure 2. Regime-dependent optimal thresholds (asymmetric correlations)

nature of transition probabilities can be costly to the investor who aims to make the optimal investment decision.

Notes

1. The real option approach has had a lot of applications in many areas of quantitative management with interest in operations management and finance/economics. Recently, Kang and Han (2020) have applied the real option theory to a military service system.
2. Q is the probability measure that has the following relationship with the physical probability measure P :

$$dQ = e^{b_i W_i - \frac{1}{2} b_i^2 t} dP.$$

Please refer to [Appendix](#) for more technical details and derivations.

3. In this paper, a time discount rate of δ is assumed to be constant, rather than regime dependent. Our paper focus is on the effects of regime-dependent changes in project cash flows with time-varying regime transition probabilities, thereby abstracting away complex issues of other parameters. In this paper, the drift and volatility of the project cash flows do change across regimes ([Ang and Bekaert, 2002](#)), maintaining the discount rate constant. We fully acknowledge that the time discount rate could be regime dependent, which may significantly affect the optimal policies. Such changes in the discount rate (or the preference parameters), however, are known to give rise to considerable challenges in solving optimal investment problems because the principle of dynamic programming is no longer applicable. For the purpose of simplicity and tractability as in a large body of literature on regime switching ([Hackbarth et al., 2006](#); [Jang et al., 2007](#); [Park, 2018](#); [Kim et al., 2021](#)), we decide to use a constant as the discount rate. To address any concerns that the discount rate is merely constant, we vary it in Table 1 and partially observe the effects of changes in the discount rate on the optimal strategies.
4. The expected durations of a Bull regime and a bear regime are $1/\lambda_B$ and $1/\lambda_b$, respectively. As a result, the bear market regime lasts longer than the Bull market regime. To reflect the empirical reality that the Bull market regime typically lasts longer than the bear market regime, λ_b could be higher than λ_B , which has been considered in Table 1.
5. We focus on the effects of regime-dependent project cash flows in the simplest possible economic setup, thereby neglecting the effects of time-varying investment cost. The investment cost could be a constant ratio to the expected cash flow at the time of optimal investment or some other size of investment. Even though the investment cost is constant, the optimal investment time determined in this paper is still stochastic from today's perspective. The constant investment cost is chosen purely for analytical convenience following the literature on investment ([Miao and Wang, 2007](#)).
6. The correlation fluctuations have been considered in investment models ([Buraschi et al., 2010](#)). More recently, [Kim et al. \(2021\)](#) have incorporated in their investment model sector-level investment with correlation fluctuations by namely dynamic and asymmetric correlations between industry and market portfolios. In addition to the time-varying and regime-dependent investment opportunity, we consider in our investment model asymmetric correlations between portfolios (or expected cash flows).
7. With regime intensities, λ_B and λ_b , the regime B lasts on average $1/\lambda_B$ and the regime b lasts on average $1/\lambda_b$. The average fraction of time spent in the regime B and the regime b is then given by $\lambda_b/(\lambda_B + \lambda_b)$ and $\lambda_B/(\lambda_B + \lambda_b)$, respectively.
8. In any case of asymmetric correlations, the chosen parameter values reflect the fact that the correlation in the regime b is inclined to be larger than that in the regime B , contributing to contagion effects at times of economic recessions.

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Appendix

In this appendix, we derive optimal timing of investment selected by an investor. We obtain regime-dependent optimal investment policies that are significantly affected by the presence of time-varying probabilities of transitions between two regimes. The optimal investment is characterized by two regions: a continuation region in which the investor's optimal choice is to delay an irreversible investment project; and a stopping region in which she should undertake the investment.

Prior to moving on the technical details behind the optimal investment strategies with time-varying regime transition probabilities in Section 3. Numerical Results, we recall some notations used in Section 2. The Basic Model.

The time T_i to leave regime $i \in \{B, b\}$ satisfies the following: for time $t \geq 0$:

$$\text{Probability of } \{T_i > t\} = 1 - \int_0^t \lambda_{i,s} ds$$

where

$$d\lambda_{i,s} = a_i \lambda_{i,s} ds + b_i \lambda_{i,s} dW_s^i, \quad \lambda_{i,0} = \lambda_i > 0, \quad a_i < 0, \quad b_i > 0$$

Here, W_s^i for regime $i \in \{B, b\}$ are standard one-dimensional and two independent Brownian motions.

We consider a stream of profit after taking a project as the following project cash flows X_t :

$$dX_t = \mu_i X_t dt + \sigma_i X_t dW_t, \quad X_0 = x > 0$$

where W_t is a standard one-dimensional Brownian motion, i is determined by a two-state continuous time Markov chain with B identifying the ‘‘Bull’’ state and b identifying the ‘‘bear’’ one, $\mu_i > 0$ is the expected returns on project cash flows, and $\sigma_i > 0$ is the standard deviation of returns on cash flows, i.e. the volatility of project cash flows. The project cash flows and the regime switching have the asymmetric correlations in the sense that $dW_t \cdot dW_t^i = \rho_i dt$ for $\rho_i \in [-1, 1]$.

The optimal investment problem can be formulated as the following:

$$\max_{\tau} E[e^{-\delta \tau} (X_{\tau} - c)]$$

where $\delta > 0$ is the appropriate discount rate, τ is optimal timing of investment, and $c > 0$ is an investment cost.

Theorem. It is optimal for an investor to invest when project cash flows exceed regime-dependent thresholds \bar{x}_i in regime $i (i \in \{B, b\})$.

Proof. The investor’s objective function is rewritten by an integral form. Specifically, the assumption of the time T_i to leave regime i in which regime intensity λ_i is formulated by a geometric Brownian motion yields:

$$\begin{aligned} \Phi_i(x) &= \max_{\tau \geq 0} E[e^{-\delta \tau} (X_{\tau} - c)] \\ &= \max_{\tau \geq 0} E \left[\int_0^{\tau} \lambda_{i,s} e^{-\delta \tau} \Phi_j(X_s) ds + \left(1 - \int_0^{\tau} \lambda_{i,s} ds \right) e^{-\delta \tau} (X_{\tau} - c) \right] \\ &= \max_{\tau \geq 0} E^Q \left[\int_0^{\tau} e^{-(\lambda_i + \delta)t} \lambda_i \Phi_j(X_t) dt + e^{-(\lambda_i + \delta)\tau} (X_{\tau} - c) \right], \end{aligned}$$

where Q is the probability measure that has the following relationship with the physical probability measure P :

$$dQ = e^{b_i W_t^i - \frac{1}{2} b_i^2 t} dP.$$

Here, we set the drift parameter a_i of regime intensity λ_i to $-\lambda_{i0} = -\lambda_i$. By the Girsanov's theorem:

$$W_t^{i,Q} \equiv W_t - \rho_i b_i t$$

is a Brownian motion under the Q probability measure. Then the cash flows process (2) is restated as:

$$\begin{aligned} dX_t &= \mu_i X_t dt + \sigma_i X_t \left(dW_t^{i,Q} + \rho_i b_i dt \right) \\ &= (\mu_i + \rho_i b_i \sigma_i) X_t dt + \sigma_i X_t dW_t^{i,Q} \end{aligned}$$

Notice that the presence of time-varying transition probabilities increases or decreases the expected rate of returns on project cash flows, according to the sign of correlations ρ_i and the magnitude of the volatility parameters b_i and σ_i of regime intensity λ_i and cash flows X_t , respectively.

According to the connection between the optimal stopping problem and variational inequality (Bensoussan and Lions, 1982; Øksendal, 2007), we obtain the following variational inequality: for any $x > 0$ and $i, j \in \{B, b\}$:

$$\left\{ \begin{array}{l} -\frac{1}{2}x^2\sigma_i^2\Phi_i''(x) - (\mu_i + \rho_i b_i \sigma_i)x\Phi_i'(x) + \delta\Phi_i(x) - \lambda_i(\Phi_j(x) - \Phi_i(x)) \geq 0, \\ \Phi_i(x) \geq x - c, \\ \left[-\frac{1}{2}x^2\sigma_i^2\Phi_i''(x) - (\mu_i + \rho_i b_i \sigma_i)x\Phi_i'(x) + \delta\Phi_i(x) - \lambda_i(\Phi_j(x) - \Phi_i(x)) \right] [\Phi_i(x) - (x - c)] = 0 \end{array} \right. \quad (3)$$

The strict inequality and equality in the first inequality in equation (3) hold in the stopping region and the continuation region, respectively. Further, the strict inequality in the second inequality in equation (3) represents the case where the investor's objective function Φ prior to investment is strictly larger than the payoff from the irreversible investment project. Accordingly, in this case the investor is in the continuation region and optimally delays the investment decision. When her objective function Φ approaches the payoff from the investment, the investor is in the stopping region and hence she undertakes the investment project. Notice that the third equality in equation (3) is required because of the fact that for any $x > 0$, one of the first two inequalities in equation (3) must hold.

We show that the optimal investment problem is solved by finding regime-dependent stopping boundaries i.e. free boundaries \bar{x}_i , which should be determined by value-matching and smooth-pasting conditions. The problem is formulated as follows: for $i, j \in \{B, b\}$:

$$\left\{ \begin{array}{l} -\frac{1}{2}x^2\sigma_i^2\Phi_i''(x) - (\mu_i + \rho_i b_i \sigma_i)x\Phi_i'(x) + \delta\Phi_i(x) - \lambda_i(\Phi_j(x) - \Phi_i(x)) = 0, \quad 0 < x < \bar{x}_i \\ \Phi_i(x) = x - c, \quad x \geq \bar{x}_i \end{array} \right. \quad (4)$$

As the expected discounted payoff ϕ_i is strictly increasing in x and the fact that $\mu_B > \mu_b$ and $\sigma_b > \sigma_B$, it is straightforward to show that $\bar{x}_b > \bar{x}_B$. Then the problem (4) can be restated as the following system of differential equations:

$$\left\{ \begin{array}{l} -\frac{1}{2}x^2\sigma_i^2\Phi_i''(x) - (\mu_i + \rho_i b_i \sigma_i)x\Phi_i'(x) + \delta\Phi_i(x) - \lambda_i(\Phi_j(x) - \Phi_i(x)) = 0, \quad 0 < x < \bar{x}_i, \\ \Phi_b(\bar{x}_b) = \bar{x}_b - c, \quad \Phi_b'(\bar{x}_b) = 1, \\ \Phi_B(\bar{x}_B) = \bar{x}_B - c, \quad \Phi_B'(\bar{x}_B) = 1. \end{array} \right. \quad (5)$$

The value-matching condition imposes an equality at \bar{x}_i between the objective functions Φ_i and the payoff from the irreversible investment project. Moreover, the functions Φ_i should be piecewise C^2 because project cash flows follow a (piecewise) continuous Borel-bounded function (Karatzas and Shreve, 1991). The functions Φ_i satisfies the continuity and smoothness and hence the smooth-pasting condition imposes an equality at \bar{x}_i between the first derivatives of Φ_i and $x - c$ taken with respect to x .

The system (5) can be solved up to closed-form expressions for endogenously chosen timing of investment which is characterized by regime-dependent stopping boundaries \bar{x}_i over which it is optimal for an investor to undertake an investment in regime i . More specifically, we obtain the following explicit solutions for the objective functions Φ_b and Φ_B :

$$\begin{aligned} \Phi_b(x) &= A_b x^{\beta_b} + A_B x^{\beta_B}, \quad 0 < x < \bar{x}_B, \\ \Phi_B(x) &= B_b x^{\beta_b} + B_B x^{\beta_B}, \quad 0 < x < \bar{x}_B, \\ \Phi_B(x) &= Cx^{\alpha_b} + C^* x^{\alpha_b^*} + \frac{\lambda_b(x - c)}{\lambda_b + \delta - (\mu_b + \rho_b b_b \sigma_b)}, \quad \bar{x}_B < x < \bar{x}_b \end{aligned} \quad (6)$$

where $\alpha_i > 1$ and $\alpha_i^* < 0$ are the two roots to the following characteristic equation:

$$CE_i(n) = -\frac{1}{2}\sigma_i^2 n(n-1) - (\mu_i + \rho_i b_i \sigma_i)n + \delta + \lambda_i = 0, \quad i \in \{B, b\},$$

and β_b and β_B ($1 < \beta_b < \alpha_i < \beta_B$) are the two roots to the following equation:

$$CE_b(n)CE_B(n) = \lambda_b \lambda_B.$$

Using the explicit solutions proposed by equation (6) and piecewise C^2 condition of Φ_i with value-matching and smooth-pasting conditions given in equation (5), we derive:

$$\left\{ \begin{array}{l} A_b(\bar{x}_B)^{\beta_b} + A_B(\bar{x}_B)^{\beta_B} = \bar{x}_B - c, \\ \beta_b A_b(\bar{x}_B)^{\beta_b-1} + \beta_B A_B(\bar{x}_B)^{\beta_B-1} = 1, \\ C(\bar{x}_b)^{\alpha_b} + C^*(\bar{x}_b)^{\alpha_b^*} + \frac{\lambda_b(\bar{x}_b - c)}{\lambda_b + \delta - (\mu_b + \rho_b b_b \sigma_b)} = \bar{x}_b - c, \\ \alpha_B C(\bar{x}_b)^{\alpha_B-1} + \alpha_B^* C^*(\bar{x}_b)^{\alpha_B^*-1} + \frac{\lambda_b}{\lambda_b + \delta - (\mu_b + \rho_b b_b \sigma_b)} = 1, \\ B_b(\bar{x}_B)^{\beta_b} + B_B(\bar{x}_B)^{\beta_B} = C(\bar{x}_B)^{\alpha_b} + C^*(\bar{x}_B)^{\alpha_b^*} + \frac{\lambda_b(\bar{x}_B - c)}{\lambda_b + \delta - (\mu_b + \rho_b b_b \sigma_b)}, \\ \beta_b B_b(\bar{x}_B)^{\beta_b-1} + \beta_B B_B(\bar{x}_B)^{\beta_B-1} = \alpha_B C(\bar{x}_B)^{\alpha_B-1} + \alpha_B^* C^*(\bar{x}_B)^{\alpha_B^*-1} + \frac{\lambda_b}{\lambda_b + \delta - (\mu_b + \rho_b b_b \sigma_b)}. \end{array} \right. \quad (7)$$

Furthermore, for $0 < x < \bar{x}_B$ substituting the explicit solutions in [equation \(6\)](#) into the first equation in [equation \(5\)](#) yields the following relationships between A_i and B_i :

$$B_b = \frac{\lambda_B}{CE_B(\beta_b)} A_b \quad \text{and} \quad B_B = \frac{\lambda_B}{CE_B(\beta_B)} A_B. \quad (8)$$

Note that we have six unknown constants (A_b or B_b, A_B or $B_B, C, C^*, \bar{x}_b, \bar{x}_B$) and six equations associated with the constants in [equation \(7\)](#). After a little simple calculation gives the following expressions for A_b, A_B, C , and C^* in terms of \bar{x}_b and \bar{x}_B :

$$\begin{aligned} A_b &= \frac{(\beta_B - 1)\bar{x}_B - \beta_B C}{(\beta_B - \beta_b)(\bar{x}_B)^{\beta_b}}, & A_B &= \frac{(\beta_b - 1)\bar{x}_B - \beta_b C}{(\beta_b - \beta_B)(\bar{x}_B)^{\beta_B}}, \\ C &= \frac{(\delta - (\mu_b + \rho_b b_b \sigma_b))}{(\alpha_B^* - \alpha_B)(\lambda_b + \delta - (\mu_b + \rho_b b_b \sigma_b))(\bar{x}_b)^{\alpha_B}} \left((\alpha_B^* - 1)\bar{x}_b - \alpha_B^* C \right), & (9) \\ C^* &= \frac{(\delta - (\mu_b + \rho_b b_b \sigma_b))}{(\alpha_B - \alpha_B^*)(\lambda_b + \delta - (\mu_b + \rho_b b_b \sigma_b))(\bar{x}_b)^{\alpha_B^*}} \left((\alpha_B - 1)\bar{x}_b - \alpha_B C \right). \end{aligned}$$

According to the relationships given in [equation \(9\)](#), if two free boundaries \bar{x}_b and \bar{x}_B are determined, then the four constants of (A_b, A_B, C, C^*) also are automatically determined. Using the relationships (8) between A_i and B_i and substituting the four constants in [equation \(9\)](#) into the last two equations in [equation \(7\)](#), we can determine \bar{x}_b and \bar{x}_B , numerically. **Q.E.D.**

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