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Mean Field Portfolio Games with Epstein-Zin Preferences

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Mean Field Portfolio Games with Epstein-Zin Preferences

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Mean Field Portfolio Games with Epstein-Zin Preferences

Guanxing Fu* and Ulrich Horst†

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Abstract

We study mean field portfolio games under Epstein-Zin preferences, which naturally encompass the classical time-additive power utility as a special case. In a general non-Markovian framework, we establish a uniqueness result by proving a one-to-one correspondence between Nash equilibria and the solutions to a class of BSDEs. A key ingredient in our approach is a necessary stochastic maximum principle tailored to Epstein-Zin utility and a nonlinear transformation. In the deterministic setting, we further derive an explicit closed-form solution for the equilibrium investment and consumption policies.

AMS Subject Classification: 93E20, 91B70, 60H30

Keywords: Epstein-Zin utility, mean field game, stochastic maximum principle

1 Introduction

In their seminal paper [10] Epstein and Zin developed a class of recursive preferences over intertemporal consumption lotteries that permit risk attitudes to be disentangled from the degree of intertemporal substitutability. Utility optimization problems with Epstein-Zin preferences have since been analyzed in various settings by many authors, including [7, 19, 27, 30, 31, 33].

In this paper we consider a class of mean-field portfolio games under Epstein-Zin utility with relative performance concerns in a general stochastic framework. Mean field games (MFGs) are a powerful tool to analyze strategic interactions in large populations when each individual player has only a small impact on the behavior of other players. Introduced independently by Huang, Malhamé and Caines [22] and Lasry and Lions [25], MFGs have been successfully applied to many economic and engineering problems ranging from optimal trading under market impact [13, 17, 16] to risk management [2], and from principal agent problems [9], to optimal exploitation of exhaustible resources [3, 18].

Mean-field portfolio games with relative performance criteria where all players trade a common stock were first analyzed by Espinosa and Touzi [11]. In a complete market setting, they established the existence of a unique Nash equilibrium for general utility functions; in an incomplete market settings with player-specific portfolio constraints, they proved the uniqueness under exponential utility. Frei and dos Reis [12] addressed the existence of equilibria in similar games. By leveraging the DPP from [11],

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they established a one-to-one correspondence between Nash equilibria and solutions to a multidimensional backward stochastic differential equation (BSDE) of quadratic growth and constructed a counterexample to show that equilibria in games with performance concerns may not exist. Lacker and Zariphopoulou [24] later considered a model where different players trade different stocks whose price dynamics is correlated through a common noise process. They identified a unique constant equilibrium in a time-homogeneous setting. Following up on these works, portfolio games with *time-additive utilities* have been extended my many authors to an array of different settings. dos Reis and Platonov [5] studied a portfolio game with forward utility. Fu and Zhou [14] extended the framework of [24] to a general non-Markovian setting. They established a one-to-one correspondence between Nash equilibria and BSDEs by extending the DPP in [11], thus obtaining existence and uniqueness of equilibria using BSDE theory. Hu and Zariphopoulou [20] analyzed equilibrium in an Itô-diffusion environment. Liang and Zhang [26] investigate a time-inconsistent portfolio game with exponential utility and non-exponential discounting. Models incorporating both investment and consumption are studied in [6, 15, 23].

To the best of our knowledge, portfolio games with Epstein-Zin preferences and performance concerns have only been studied in [4, 32]. We establish an existence of equilibrium result for such games with deterministic, though possibly time-varying model parameters, and a uniqueness of equilibrium result for general stochastic settings. Specifically, we prove that Nash equilibria in mean-field portfolio games with Epstein-Zin preferences are uniquely characterized in terms of a solution to a certain quadratic BSDE. Similar characterization results are often implicitly assumed - though rarely proved - in the literature. For instance, [4] proves the existence of a unique simple equilibrium assuming that optimizing the Epstein-Zin utility index is equivalent to optimizing the driver of some BSDE. This is not always the case, though; counterexample can easily be constructed. Our uniqueness result shows that the simple equilibrium obtained in [4] is indeed the unique bounded equilibrium.

The fact that any solution to a certain BSDE yields a Nash equilibrium is established using the martingale optimality principle (MOP). This approach is consistent with the methodology adopted in prior works such as [14, 15], and extends results in [21] beyond the benchmark case of time-additive utilities and results in [33] to models with mean-field interaction.

The fact that any Nash equilibrium must satisfy the BSDE arising from the MOP uses a novel stochastic maximum principle tailored to the Epstein-Zin setting. The key observation is that stochastic control problems involving Epstein-Zin preferences naturally lead to an FBSDE system as their state dynamics. Crucially, the driver of the BSDE component characterizing Epstein-Zin utility is not Lipschitz (see e.g. [8] for the maximum principle for recursive utilities with Lipschitz drivers) which prevents us from using established stochastic maximum principles for FBSDEs.

To connect the adjoint processes arising from the stochastic maximum principle with the desired BSDE characterization, we introduce a nonlinear transformation that maps the adjoint processes to a candidate process Y, which we expect to solve the targeted BSDE. Subsequently we demonstrate that the difference between the Epstein-Zin utility associated with the optimal investment–consumption strategy and a certain process involving the optimal wealth and the transformed process Y satisfies a linear BSDE with zero terminal condition. This crucial step confirms that the candidate process Y coincides with the solution to the BSDE arising from the MOP, thereby establishing a one-to-one relation between the Nash equilibrium and the solution to certain BSDE.

To complete the necessary part of our characterization result, we also need to make sure that the space of the equilibrium strategies and the solution space of the BSDE also correspond to each other. This is done with the help of the explicit expression for a class of linear BSDEs established in [1].

The remainder of the paper is organized as follows. Section 2 recalls both the single player benchmark model with Epstein-Zin utility along with its game-theoretic extensions. Section 3 establishes a one-to-one correspondence between the Nash equilibrium and the solution to a certain BSDE. The sufficient

condition—showing that a solution to the BSDE yields a Nash equilibrium—is presented in Section 3.1. The necessary condition—showing that any Nash equilibrium must correspond to a BSDE solution—is developed in Section 3.2. The uniqueness of the Nash equilibrium result is established in Section 4. In Section 5, we provide a closed-form expression for the equilibrium investment and consumption strategy in a deterministic setting.

2 The mean field game with Epstein-Zin utility

In this section we recall the benchmark model of a single player optimization problem with Epstein-Zin preferences along with a game-theoretic extension with performance concerns.

We fix a common time horizon $[0, T]$ for all investors and assume that all random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ that carries independent Brownian motions $W^0, W^1, \cdots, W^N$ for some $N \in \mathbb{N}$. The Brownian motions $W^1, \cdots, W^N$ capture *idiosyncratic noises* in an N-player model while the Brownian motion W^0 captures a *common noise* shared by all agents.

2.1 Single agent benchmark model

In a single agent benchmark model the agent can invest in a risk-free asset that pays interest at rates (r_t) and in a risky asset whose price process (S_t) follows the stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^1 + \sigma_t^0 dW_t^0, \qquad t \in [0, T].$$

Standing Assumption. The processes (r_t), (R_t), (σ_t) and (σ_t^0) are progressively measurable and bounded.

We denote by $\Pi = (\Pi_t)$ and $C = (C_t)$ a pair of progressively measurable processes that represent the investor's investment strategy and consumption plan, respectively. Specifically, Π_t denotes the amount of money invested in the risk stock and C_t denotes the amount of money consumed at time $t \in [0, T]$. The corresponding wealth process $X^{\Pi,C}$ then evolves according to the SDE

$$dX_t^{\Pi,C} = r_t X_t^{\Pi,C}\, dt + \Pi_t h_t\, dt + \Pi_t \sigma_t\, dW_t + \pi_t \sigma_t^0\, dW_t^0 - C_t\, dt, \qquad t \in [0, T].$$

In a model without mean-field interaction the dynamics of the investor's Epstein-Zin utility from consumption is given by

$$\tilde{V}_t^{\Pi,C} = \mathbb{E}\left[\int_t^T \tilde{f}(C_s, \tilde{V}_s^{\Pi,C})\, ds + \alpha U(X_T^{\Pi,C}) \middle| \mathcal{F}_t\right], \qquad t \in [0, T], \tag{2.1}$$

where it is assumed that all wealth is consumed at the terminal time, and where the *aggregator function* $f : [0, \infty) \times (-\infty, 0] \to \mathbb{R}$ is given by

$$f(c, v) = \frac{\delta c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\{(1-\gamma)v\}^{1-\frac{1}{\theta}} - \delta\widetilde{\theta} v, \quad \widetilde{\theta} := \frac{1-\gamma}{1-\frac{1}{\psi}}. \tag{2.2}$$

Here $\delta > 0$ represent the discount rate, $\gamma > 0$ specifies the relative risk aversion, $\psi > 0$ represents the elasticity of intertemporal substitution (EIS), $\alpha > 0$ is a rate of bequest, and the *bequest function* U is of power type, i.e.

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}. \tag{2.3}$$

Standing Assumption. We assume throughout this paper that[1]

$$\psi\gamma \geq 1, \quad \psi \geq 1. \tag{2.4}$$

2.2 Game theoretic extensions

The single agent model models with Epstein-Zin preferences has recently been extended to multi-player models and MFGs with relative performance concerns by [4]. Following their approach we consider an N-player model where all player share identical Epstein-Zin preferences, and where each player $i = 1, \cdots, N$ can trade a risk-free bond and a risky stock whose dynamics is given by

$$\frac{dS_t^i}{S_t^i} = R_t^i dt + \sigma_t^i dW_t^i + \sigma^{i0} dW_t^0, \qquad t \in [0, T]$$

The wealth dynamics is defined as in the single player case. Accounting for relative performance concerns we denote by

$$\bar{C}_t^i := \left(\prod_{j \neq i} C_t^j\right)^{\frac{1}{N-1}}, \quad \bar{X}_t^i := \left(\prod_{j \neq i} X_t^j\right)^{\frac{1}{N-1}}, \qquad t \in [0, T].$$

the ergodic average consumption and wealth, respectively, of player i's competitors and consider utility functionals of the form

$$V_t^{i,\Pi,C} = \mathbb{E}\left[\int_t^T f_i(C_s(\bar{C}_s^i)^{-\theta}, V_s^{\Pi,C})\, ds + \alpha_i U_i(X_T^{\Pi,C}\bar{X}_T^i)\middle| \mathcal{F}_t\right], \qquad t \in [0, T], \tag{2.5}$$

where f_i and U_i are defined as in (2.2) and (2.3), with δ, ψ and γ replaced by δ_i, ψ^i and γ^i, respectively.

Remark 2.1. We refer to [4] and references therein for a detailed economic motivation of the above preference functional. We emphasize, though, that our approach differs from [4] in one important aspect. In [4] the authors benchmark own consumption and wealth of each player against the ergodic averages $\left(\prod_{j=1}^N C^j\right)^{\frac{1}{N}}$ and $\left(\prod_{j=1}^N X^j\right)^{\frac{1}{N}}$. In particular, own consumption and wealth is included in the benchmark. We believe that our approach of only considering average consumption and wealth of competitors captures better the idea of relative performance concerns. For *time-additive utilities*, the two approaches result in equivalent optimization problems, as the problems can be transformed into each other by adjusting model parameters such as the risk aversion coefficient and the competition parameter; see [24, Remark 2.5] for details. For Epstein-Zin utilities, such parameter distortions lack economic intuition/justification and the two approaches may be no longer equivalent (except in the limit when the number of players tends to infinity).

To study the MFG version of the N-player model we fix an $\mathcal{F}^0$-progressively measurable stochastic process $\nu = (\nu_t)$ ("mean field externality") that captures the impact of aggregate intertemporal and terminal consumption on a representative player's utility. Specifically, in the MFG the representative investor's utility from consumption is then given by

$$V_t^{\Pi,C} = \mathbb{E}\left[\int_t^T f(C_s\nu_s^{-\theta}, V_s^{\Pi,C})\, ds + \alpha U(X_T^{\Pi,C}\nu_T^{-\theta})\middle| \mathcal{F}_t\right], \qquad t \in [0, T]. \tag{2.6}$$

In what follows an investment-consumption pair (Π, C) is called *admissible* if it satisfies the following properties (cf. [31]):

[1] Technically, our results also hold under the conditions $\psi\gamma \leq 1$ and $\gamma \leq 1$. However, the empirically relevant regime is $\psi > 1$ and $\gamma > 1$; see [33]. The choice in (2.4) allows for a unified treatment of Epstein-Zin and power utility.

- the wealth process $X^{\Pi,C}$ is a.s. strictly positive at all times and all terminal wealth is used for consumption, i.e. $X_T = C_T$;
- the consumption process C is a.s. strictly positive, and $\mathbb{E}\left[\int_0^T C_t^\beta \, dt + C_T^\beta\right] < \infty$ for all $\beta \in \mathbb{R}$;
- the investment process Π belongs to L^2.

The set of admissible investment-consumption pairs is denoted $\mathcal{A}$ and the representative investor's optimization problem is then given by

$$\sup_{(\Pi,C)\in\mathcal{A}} V_0^{\Pi,C} = \sup_{(\Pi,C)\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(C_s\nu_s^{-\theta}, V_s^{\Pi,C})\, ds + \alpha U(X_T^{\Pi,C}\nu_T^{-\theta})\right]. \tag{2.7}$$

Let $\big(C^*(\nu), \Pi^*(\nu)\big)$ be an optimal consumption-investment pair, given ν, with corresponding wealth process $X^*(\nu)$. In a mean-field equilibrium the expected optimal consumption and wealth processes coincides the "anticipated" externalities.

Definition 2.2. An $\mathcal{F}^0$-progressively measurable stochastic process $\nu^* = (\nu_t^*)$ forms a mean-field equilibrium if a.s.

$$\widehat{\nu}_t^* = \mathbb{E}[\widehat{C}_t^*(\nu^*)|\mathcal{F}_t^0],\ 0 \le t < T, \quad \text{and} \quad \widehat{\nu}_T^* = \mathbb{E}[\widehat{X}_T^*(\nu^*)|\mathcal{F}_T^0]$$

where $\widehat{\xi} := \log \xi$ denotes the logarithm of a strictly positive random variable ξ.

In what follows we prove that at most one (in a certain class) mean-field equilibrium can exist under our standing assumptions. For the special case of deterministic model parameters we subsequently prove the existence (and hence uniqueness) of an equilibrium. Having solved the MFG, solving finite player game requires only minor adjustment of previously given arguments.

3 Characterization of mean-field equilibria

In this section we establish our characterization of equilibrium result. We prove that any mean-field equilibrium satisfies a certain BSDE and that, conversely, any solution to the said BSDE yields a mean-field equilibrium.

3.1 From BSDE to Nash equilibrium

To identify the FBSDE system and subsequently the BSDE that any mean-field equilibrium must satisfy we first solve the representative investor's optimization problem for any fixed externality using the martingale optimality principle. Subsequently, we solve the fixed-point problem to characterize MFG equilibria.

3.1.1 Optimization

To solve the representative investor's utility optimization problem for a given externality, we introduce the auxiliary process

$$R_t^{\Pi,C} = \alpha\frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t} + \int_0^t f\left(C_s\nu_s^{-\theta}, \alpha\frac{X_s^{1-\gamma}}{1-\gamma}e^{Y_s}\right) ds, \qquad t \in [0,T], \tag{3.1}$$

where Y denotes the first component of the (unique) solution to a BSDE of the form

$$-dY_t = g_t\,dt - Z_t\,dW_t - Z_t^0\,dW_t^0, \qquad Y_T = -\theta(1-\gamma)\log\nu_T, \tag{3.2}$$

where the driver g is to be determined such that the following conditions are satisfied:

- for each admissible investmet-consumption pair (Π, C) the process $R^{\Pi,C}$ is a supermartingale,
- for some admissible investmet-consumption pair (Π^*, C^*) the process R^{Π^*,C^*} is a martingale,
- the initial values $R_0^{\Pi,C}$ are independent of the choice of (Π, C).

The following remark justifies our approach. It shows that under the above assumptions, the pair (Π^*, C^*) is an optimal strategy.

Remark 3.1. From the definition of the utility processes $V^{\Pi,C}$ and $R^{\Pi,C}$, we know that

- the process $V_t^{\Pi,C} + \int_0^t f(C_s\nu_s^{-\theta}, V_s^{\Pi,C})\,ds$ is a local martingale
- the process $\alpha\frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t} + \int_0^t f(C_s\nu_s^{-\theta}, \alpha\frac{X_s^{1-\gamma}}{1-\gamma}e^{Y_s})\,ds$ is a local supermartingale,

and that

$$V_T^{\Pi,C} = \alpha\frac{(X_T^{\Pi,C})^{1-\gamma}}{1-\gamma}e^{Y_T}.$$

Thus, by the comparison principle in [33, proof of Proposition 2.2, Step 3], it holds that

$$\alpha\frac{(X_t^{\Pi,C})^{1-\gamma}}{1-\gamma}e^{Y_t} \geq V_t^{\Pi,C}, \qquad t\in[0,T].$$

Since the left-hand side of the above inequality is independent of (Π, C) at time $t = 0$ we see that

$$\alpha\frac{x^{1-\gamma}}{1-\gamma} \geq \sup_{\Pi,C} V_0^{\Pi,C}.$$

Since R^{Π^*,C^*} is assumed to be a martingale, the following equality holds for some process (Z, Z^0):

$$\alpha\frac{(X_t^{\Pi^*,C^*})^{1-\gamma}}{1-\gamma}e^{Y_t} + \int_0^t f\left(C_s^*\nu_s^{-\theta}, \alpha\frac{(X_s^{\Pi^*,C^*})^{1-\gamma}}{1-\gamma}e^{Y_s}\right)ds = \int_0^t Z_s\,dW_s + \int_0^t Z_s^0\,dW_s^0, \quad t\in[0,T].$$

It follows that the process

$$\alpha\frac{(X^{\Pi^*,C^*})^{1-\gamma}}{1-\gamma}e^{Y}$$

satisfies the recursive utility equation. As a result,

$$V_t^{\Pi^*,C^*} = \alpha\frac{(X_t^{\Pi^*,C^*})^{1-\gamma}}{1-\gamma}e^{Y_t}, \qquad t\in[0,T].$$

In particular, $V_0^{\Pi^*,C^*} = \alpha\frac{x^{1-\gamma}}{1-\gamma}e^{Y_0}$ from which we conclude that (Π^*, C^*) is an optimal strategy.

It remains to determine the driver g of the BSDE (3.2). By Itô's formula,

$$\begin{aligned}
&d\Big(\frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}\Big)\\
&= \frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}\left(-g_t + \frac{Z_t^2 + (Z_t^0)^2}{2}\right)dt + \frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}Z_t\,dW_t + \frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}Z_t^0\,dW_t^0\\
&\quad + e^{Y_t}\left\{r_tX_t^{1-\gamma} + \Pi_t h_t X_t^{-\gamma} - C_tX_t^{-\gamma} - \frac{\gamma}{2}X_t^{-1-\gamma}\Pi_t^2(\sigma_t^2 + (\sigma_t^0)^2)\right\}dt\\
&\quad + e^{Y_t}X_t^{-\gamma}\Pi_t\left(\sigma_t\,dW_t + \sigma_t^0\,dW_t^0\right)\\
&\quad + e^{Y_t}Z_tX_t^{-\gamma}\Pi_t\sigma_t\,dt + e^{Y_t}Z_t^0X_t^{-\gamma}\Pi_t\sigma_t^0\,dt,
\end{aligned}$$

from which we see that

$$
\begin{aligned}
&d\big(R_t^{\Pi,C}\big)\\
&= \alpha\frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}\left\{-g_t+\frac{Z_t^2+(Z_t^0)^2}{2}\right.\\
&\qquad\left.+(1-\gamma)\Big(r_t+\frac{\Pi_t}{X_t^*}h_t-\frac{C_t}{X_t}-\frac{\gamma}{2}\frac{\Pi_t^2}{X_t^2}(\sigma_t^2+(\sigma_t^0)^2)+Z_t\frac{\Pi_t}{X_t}\sigma_t+Z_t^0\frac{\Pi_t}{X_t}\sigma_t^0\Big)\right\}dt\\
&\quad+\alpha\left(\frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}Z_t+e^{Y_t}X_t^{-\gamma}\Pi_t\sigma_t\right)dW_t+\alpha\left(\frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}Z_t^0+e^{Y_t}X_t^{-\gamma}\Pi_t\sigma_t^0\right)dW_t^0\\
&\quad+\left\{\frac{\delta(C_s\nu_s^{-\theta})^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\left\{\alpha X_t^{1-\gamma}e^{Y_t}\right\}^{1-\frac{1}{\theta}}-\delta\widetilde{\theta}\alpha\frac{X_t^{1-\gamma}}{1-\gamma}e^{Y_t}\right\}dt.
\end{aligned}
$$

The driver of the above BSDE is strictly concave in the investment-consumption pair (Π_t, C_t). Setting its partial derivatives w.r.t. Π_t and C_t equal to zero, the poinwise maximizer is given by

$$
\begin{aligned}
\Pi^* &\equiv \frac{h+\sigma Z+\sigma^0 Z^0}{\gamma(\sigma^2+(\sigma^0)^2)}X := \pi^* X\\
C^* &\equiv \delta^\psi(\nu^{-\theta})^{\psi-1}(\alpha e^Y)^{-\frac{\psi}{\theta}}X := c^* X.
\end{aligned}
\tag{3.3}
$$

Taking these quantities back into the driver of $R^{\Pi,C}$ and letting

$$
\begin{aligned}
g := &\frac{Z^2+(Z^0)^2}{2}+(1-\gamma)r+\frac{1-\gamma}{2\gamma}\frac{(h+\sigma Z+\sigma^0 Z^0)^2}{\sigma^2+(\sigma^0)^2}\\
&+\frac{1-\gamma}{\psi-1}\delta^\psi(\nu^{-\theta})^{\psi-1}(\alpha e^Y)^{-\frac{\psi}{\theta}}-\delta\widetilde{\theta}
\end{aligned}
\tag{3.4}
$$

we see that the driver of the BSDE (3.1) vanishes for the pair (Π^*, C^*) and is non-positive for any admissible pair (Π, C). Thus, the process R^{Π^*,C^*} is a martingale while the process $R^{\Pi,C}$ is a supermartingale for any admissible strategy.

3.1.2 The fixed point condition

The fixed point condition reads

$$
\widehat{\nu}_t^* \equiv \mathbb{E}[\widehat{c}_t^*|\mathcal{F}_t^0]+\mathbb{E}[\widehat{X}_t^*|\mathcal{F}_t^0],\ 0\le t<T,\quad\text{and}\quad \widehat{\nu}_T^*\equiv\mathbb{E}[\widehat{X}_T^*|\mathcal{F}_T^0],
$$

where the process c^* is determined in (3.3). In particular, $c_T^*=1$ for any admissible consumption plane C^*. Moreover, for $0\le t<T$, in view of the optimality condition (3.3) the equilibrium consumption process satisfies

$$
c_t^* = \delta^\psi e^{-\theta(\psi-1)\log\nu_t^*}(\alpha e^{Y_t})^{-\frac{\psi}{\theta}} = \delta^\psi e^{-\theta(\psi-1)\mathbb{E}[\widehat{c}_t^*|\mathcal{F}_t^0]}e^{-\theta(\psi-1)\mathbb{E}[\widehat{X}_t^*|\mathcal{F}_t^0]}(\alpha e^{Y_t})^{-\frac{\psi}{\theta}},
\tag{3.5}
$$

which implies that the conditional expectation $\mathbb{E}[\widehat{c}_t^*|\mathcal{F}_t^0]$ satisfies the equation

$$
\mathbb{E}[\widehat{c}_t^*|\mathcal{F}_t^0] = \mathbb{E}[\psi\log\delta]-\mathbb{E}[\theta(\psi-1)]\mathbb{E}[\widehat{c}_t^*|\mathcal{F}_t^0]-\mathbb{E}[\theta(\psi-1)]\mathbb{E}[\widehat{X}_t^*|\mathcal{F}_t^0]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\log\alpha\right]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}Y_t\middle|\mathcal{F}_t^0\right].
$$

Thus,

$$
\mathbb{E}[\widehat{c}_t^*|\mathcal{F}_t^0] = \frac{1}{1+\mathbb{E}[\theta(\psi-1)]}\left\{\mathbb{E}[\psi\log\delta]-\mathbb{E}[\theta(\psi-1)]\mathbb{E}[\widehat{X}_t^*|\mathcal{F}_t^0]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\log\alpha\right]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}Y_t\middle|\mathcal{F}_t^0\right]\right\}
$$

and so

$$
\begin{aligned}
\widehat{\nu}_t^* &= \mathbb{E}[\widehat{X}_t^*|\mathcal{F}_t^0]+\mathbb{E}[\widehat{c}_t^*|\mathcal{F}_t^0]\\
&= \frac{1}{1+\mathbb{E}[\theta(\psi-1)]}\left\{\mathbb{E}[\psi\log\delta]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\log\alpha\right]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}Y_t\middle|\mathcal{F}_t^0\right]\right\}+\frac{1}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}[\widehat{X}_t^*|\mathcal{F}_t^0].
\end{aligned}
\tag{3.6}
$$

Define

$$\widetilde{Y} := Y + \theta(1-\gamma)\mathbb{E}[\widehat{X}^*|\mathcal{F}^0].$$

This process satisfies the backward equation with zero terminal condition

$$\begin{aligned}\widetilde{Y}_t = & -\theta(1-\gamma)\int_t^T \mathbb{E}\left[r_s + \pi_s^* h_s - c_s^* - \frac{1}{2}(\pi_s^*)^2(\sigma_s^2 + (\sigma_s^0)^2)\middle| \mathcal{F}_s^0\right] ds \\ & + \int_t^T \left\{\frac{Z_s^2 + (Z_s^0)^2}{2} + (1-\gamma)r_s + \frac{1-\gamma}{2\gamma}\frac{(h_s + \sigma_s Z_s + \sigma_s^0 Z_s^0)^2}{\sigma^2 + (\sigma^0)^2}\right. \\ & \left. + \frac{1-\gamma}{\psi-1}\delta^\psi(\nu_s^*)^{-\theta(\psi-1)}(\alpha e^{Y_s})^{-\frac{\psi}{\theta}} - \delta\widetilde{\theta}\right\} ds \\ & - \int_t^T Z_s\, dW_s - \int_t^T \left\{Z_s^0 + \theta(1-\gamma)\mathbb{E}[\pi_s^*\sigma_s^0|\mathcal{F}_s^0]\right\} dW_s^0.\end{aligned} \tag{3.7}$$

To bring this equation into standard BSDE format we set

$$\widetilde{Z} := Z, \qquad \widetilde{Z}^0 := Z^0 + \theta(1-\gamma)\mathbb{E}[\pi^*\sigma^0|\mathcal{F}^0] = Z^0 + \theta(1-\gamma)\mathbb{E}\left[\frac{h + \sigma Z + \sigma^0 Z^0}{\gamma(\sigma^2 + (\sigma^0)^2)}\sigma^0\middle|\mathcal{F}^0\right], \tag{3.8}$$

where the second equality is equivalent to

$$Z^0 = \widetilde{Z}^0 - \frac{\theta(1-\gamma)\mathbb{E}\left[\frac{\sigma^0(h+\sigma\widetilde{Z}+\sigma^0\widetilde{Z}^0)}{\gamma(\sigma^2+(\sigma^0)^2)}\middle|\mathcal{F}^0\right]}{1+\mathbb{E}\left[\frac{\theta(1-\gamma)(\sigma^0)^2}{\gamma(\sigma^2+(\sigma^0)^2)}\middle|\mathcal{F}^0\right]}. \tag{3.9}$$

Moreover, using the definition of $\widetilde{Y}$, the equalities in (3.5) and (3.6) imply that on $[0,T)$

$$\begin{aligned}c^* &= \delta^\psi \alpha^{-\frac{\psi}{\theta}} \exp\left(-\theta(\psi-1)\widehat{\nu} - \frac{\psi}{\widetilde{\theta}}Y\right) \\ &= \delta^\psi \alpha^{-\frac{\psi}{\theta}} \exp\left(-\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\left\{\mathbb{E}[\psi\log\delta] - \mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\log\alpha\right]\right\} + \frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\widetilde{Y}\middle|\mathcal{F}^0\right] - \frac{\psi}{\widetilde{\theta}}\widetilde{Y}\right) \\ &= \exp\left(-\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\left\{\mathbb{E}[\psi\log\delta] - \mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\log\alpha\right] - \mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\widetilde{Y}\middle|\mathcal{F}^0\right]\right\} + \psi\log\delta - \frac{\psi}{\widetilde{\theta}}\log\alpha - \frac{\psi}{\widetilde{\theta}}\widetilde{Y}\right).\end{aligned} \tag{3.10}$$

Taking the first equality in (3.8) and (3.9) into the first equality in (3.3), we get the equilibrium investment process

$$\pi^* = \frac{h + \sigma\widetilde{Z} + \sigma^0\widetilde{Z}^0}{\gamma(\sigma^2 + (\sigma^0)^2)} - \frac{\sigma^0\theta(1-\gamma)\mathbb{E}\left[\frac{\sigma^0(h+\sigma\widetilde{Z}+\sigma^0\widetilde{Z}^0)}{\gamma(\sigma^2+(\sigma^0)^2)}\middle|\mathcal{F}^0\right]}{\gamma(\sigma^2 + (\sigma^0)^2)\left(1+\mathbb{E}\left[\frac{\theta(1-\gamma)(\sigma^0)^2}{\gamma(\sigma^2+(\sigma^0)^2)}\middle|\mathcal{F}^0\right]\right)} \tag{3.11}$$

Taking the first equality in (3.8) and (3.9) into (3.7), we get

$$\begin{aligned}\widetilde{Y}_t = & -\theta(1-\gamma)\int_t^T \mathbb{E}[r_s|\mathcal{F}_s^0]\, ds + \int_t^T \left\{(1-\gamma)r_s - \delta\widetilde{\theta}\right\} ds + \theta(1-\gamma)\int_t^T \mathbb{E}[c_s^*|\mathcal{F}_s^0]\, ds \\ & + \int_t^T \frac{1-\gamma}{\psi-1}c_s^*\, ds + \int_t^T \mathcal{J}_{\widetilde{Z},\widetilde{Z}^0}(s)\, ds - \int_t^T \widetilde{Z}_s\, dW_s - \int_t^T \widetilde{Z}_s^0\, dW_s^0,\end{aligned} \tag{3.12}$$

where c^* is given by (3.10), and $\mathcal{J}_{\widetilde{Z},\widetilde{Z}^0}$ collects all terms containing $\widetilde{Z}$ and $\widetilde{Z}^0$, which is the same as [15, Appendix B], with γ there replaced by $1-\gamma$.

The following theorem summarizes the findings of this subsection.

Theorem 3.2. *If the mean field BSDE* (3.12) *admits a solution* $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0) \in L^\infty \times H^2_{BMO} \times H^2_{BMO}$*, then the mean-field portfolio game admits an equilibrium consumption rate* $c^* \in L^\infty$ *and an equilibrium investment rate* $\pi^* \in H^2_{BMO}$ *given by* (3.10) *and* (3.11)*, respectively. Let* X^* *be the corresponding wealth process it holds that* $(c^*X^*, \pi^*X^*) \in \mathcal{A}$*.*

Remark 3.3. Theorem 3.2 only considers the consumption and investment in equilibrium. The best response can also be studied if the dynamics of ν is assumed to follow some SDE.

3.2 From Nash equilibrium to BSDE

So far we derived a BSDE whose solution yields a mean-field equilibrium to our portfolio game. We proceed to prove that *any* Nash equilibrium satisfies the previously obtained BSDE.

In what follows we denote f_1 and f_2 the derivative of f w.r.t. the first and the second argument, respectively. In particular,

$$f_2(C,V) = \frac{\delta C^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\left(1-\frac{1}{\widetilde{\theta}}\right)(1-\gamma)\big((1-\gamma)V\big)^{-\frac{1}{\theta}} - \delta\widetilde{\theta} \tag{3.13}$$

We start with the following necessary maximum principle for Epstein-Zin preferences.

Proposition 3.4. *Let ν by an $\mathcal{F}_0$-progressively measurable process such that $\mathbb{E}\left[\int_0^T \nu_t^\beta\, ds + \nu_T^\beta\right] < \infty$ for each $\beta \in \mathbb{R}$. Let $(\Pi^*, C^*) = (\Pi^*(\nu), C^*(\nu)) \in \mathcal{A}$ be an optimal control of (2.7) and (X^*, V^*) be the corresponding state process. Define the following (well posed) system of adjoint equations:*

$$\begin{cases} -dp_t = r_t p_t\, dt - q_t\, dW_t - q_t^0\, dW_t^0 := r_t p_t\, dt - \widetilde{q}_t^\top\, d\widetilde{W}_t, \quad p_T = -\alpha U'(X_T^*\nu_T^{-\theta})\nu_T^{-\theta} Q_T \\ dQ_t = f_2(C_t^*\nu_t^{-\theta}, V_t^*)Q_t\, dt, \quad Q_0 = -1. \end{cases} \tag{3.14}$$

Then, it holds that

$$ph + q\sigma + q\sigma^0 = 0, \qquad p + f_1(C^*\nu^{-\theta}, V^*)\nu^{-\theta}Q = 0. \tag{3.15}$$

Proof. **Step 1.** *A priori estimates.* We start with a series of a priori estimates.

- We first consider the utility process. For any admissible pair (Π, C) it follows from [31, Theorem 4.6] that
$$Y^\gamma \le V^{\Pi,C} \le U_\gamma \circ U_{\frac{1}{\psi}}^{-1}(Y^{\frac{1}{\psi}}), \tag{3.16}$$
where (following the notation in [31])
$$Y_t^\varrho := e^{\delta t}\mathbb{E}\left[\int_t^T \delta e^{-\delta s} U_\varrho(C_s\nu_s^{-\theta})\, ds + e^{-\delta T}U_\varrho(C_T\nu_T^{-\theta})\,\middle|\,\mathcal{F}_t\right] \quad \text{and} \quad U_\varrho(C) := \alpha\frac{C^{1-\varrho}}{1-\varrho}.$$
Our assumptions on the consumption plan C guarantee that the lower bound of the utility process $V^{\Pi,C}$ belongs to $\bigcap_{\beta>0} S^\beta$. Since
$$U_\gamma \circ U_{\frac{1}{\psi}}^{-1}(x) = \frac{1}{1-\gamma}\left\{\left(1-\frac{1}{\psi}\right)x\right\}^{\frac{1-\gamma}{1-\frac{1}{\psi}}},$$
the upper bound also belongs to $\bigcap_{\beta>0} S^\beta$. Moreover, it follows from (3.16) that $(1-\gamma)V^{\Pi,C} \ge 0$. Hence,
$$f_2(C,V) \le -\delta\widetilde{\theta}.$$
- We proceed to bound the adjoint variables. By (3.13), the process Q satisfies
$$0 > Q_t = -\exp\left(\int_0^t f_2(C_s^*, V_s^*)\, ds\right) \ge -e^{-\delta\widetilde{\theta}t}.$$

Moreover, the processes p and q satisfy

$$0 < p_t = \mathbb{E}\left[-e^{\int_t^T r_s\,ds}\alpha(X_T^*)^{-\gamma}\nu_T^{-\theta(1-\gamma)}Q_T\middle|\mathcal{F}_t\right] \le \mathbb{E}\left[e^{\int_t^T r_s\,ds-\delta\widetilde{\theta}T}\alpha(X_T^*)^{-\gamma}\nu_T^{-\theta(1-\gamma)}\middle|\mathcal{F}_t\right].$$

Since $X_T^* = C_T^*$ and because C_T^* is integrable we deduce that $p \in \bigcap_{\beta>0} S^\beta$. Since

$$\int_t^T \widetilde{q}_s^\top\, d\widetilde{W}_s = \int_t^T r_s p_s\,ds + p_T - p_t,$$

this implies that

$$\mathbb{E}\left[\left(\int_0^T q_s^2\,ds\right)^{\frac{\beta}{2}} + \left(\int_0^T (q_s^0)^2\,ds\right)^{\frac{\beta}{2}}\right] < \infty, \quad \text{for all } \beta > 0.$$

Step 2. *A variational system.* Let us fix an admissible strategy (Π, C) and consider the variational system

$$\begin{cases} d\mathcal{X}_t = \Big\{r_t\mathcal{X}_t + (\Pi_t - \Pi_t^*)h_t - (C_t - C_t^*)\Big\}\,dt + (\Pi_t - \Pi_t^*)\sigma_t\,dW_t + (\Pi_t - \Pi_t^*)\sigma_t^0\,dW_t^0, \\ -d\mathcal{V}_t = \Big\{f_1(C_t^*\nu_t^{-\theta}, V_t^*)(C_t - C_t^*)\nu_t^{-\theta} + f_2(C_t^*\nu_t^{-\theta}, V_t^*)\mathcal{V}_t\Big\}\,dt - \mathcal{Z}_t\,dW_t - \mathcal{Z}_t^0\,dW_t^0, \\ \mathcal{X}_0 = 0, \quad \mathcal{V}_T = \alpha U'(X_T^*\nu_T^{-\theta})\mathcal{X}_T\nu_T^{-\theta}. \end{cases} \tag{3.17}$$

By [29, Proposition 6.2.1],

$$\mathcal{V}_t = \mathbb{E}\left[\mathcal{V}_T e^{\int_t^T f_2(C_s^*\nu_s^{-\theta}, V_s^*)\,ds} + \int_t^T f_1(C_s^*\nu_s^{-\theta}, V_s^*)(C_s - C_s^*)\nu_s^{-\theta}e^{\int_t^s f_2(C_s^*\nu_s^{-\theta}, V_s^*)\,dr}\,ds\middle|\mathcal{F}_t\right],$$

which implies that $\mathcal{V} \in \bigcap_{\beta>0} S^\beta$. Hence,

$$\mathbb{E}\left[\left(\int_0^T \mathcal{Z}_s^2\,ds\right)^{\beta} + \left(\int_0^T (\mathcal{Z}_s^0)^2\,ds\right)^{\beta}\right] < \infty, \quad \text{for all } \beta > 0.$$

To identify the initial value $\mathcal{V}_0$ we apply an integration by parts to obtain that

$$\begin{aligned} &\mathcal{V}_T Q_T \\ &= -\mathcal{V}_0 + \int_0^T \mathcal{V}_t f_2(C_t^*\nu_t^{-\theta}, V_t^*)Q_t\,dt - \int_0^T Q_t\Big\{f_1(C_t^*\nu_t^{-\theta}, V_t^*)(C_t - C_t^*)\nu_t^{-\theta} + f_2(C_t^*\nu_t^{-\theta}, V_t^*)\mathcal{V}_t\Big\}\,dt \\ &\quad + \int_0^T Q_t\mathcal{Z}_t\,dW_t + \int_0^T Q_s\mathcal{Z}_s^0\,dW_s^0 \\ &= -\mathcal{V}_0 - \int_0^T Q_t f_1(C_t^*\nu_t^{-\theta}, V_t^*)(C_t - C_t^*)\nu_t^{-\theta}\,dt + \int_0^T Q_t\mathcal{Z}_t\,dW_t + \int_0^T Q_t\mathcal{Z}_t^0\,dW_t^0 \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &\mathcal{X}_T p_T \\ &= \int_0^T \mathcal{X}_t q_t\,dW_t + \int_0^T \mathcal{X}_t q_t^0\,dW_t^0 + \int_0^T p_t\Big\{h_t(\Pi_t - \Pi_t^*) - (C_t - C_t^*)\Big\}\,dt \\ &\quad + \int_0^T \sigma_t p_t(\Pi_t - \Pi_t^*)\,dW_t + \int_0^T \sigma_t^0 p_t(\Pi_t - \Pi_t^*)\,dW_t^0 + \int_0^T \sigma_t q_t(\Pi_t - \Pi_t^*)\,dt + \int_0^T \sigma_t^0 q_t^0(\Pi_t - \Pi_t^*)\,dt \\ &= \int_0^T \Big\{p_t h_t(\Pi_t - \Pi_t^*) - p_t(C_t - C_t^*) + \sigma_t q_t(\Pi_t - \Pi_t^*) + \sigma_t^0 q_t^0(\Pi_t - \Pi_t^*)\Big\}\,dt \\ &\quad + \int_0^T \mathcal{X}_t q_t\,dW_t + \int_0^T \mathcal{X}_t q_t^0\,dW_t^0 + \int_0^T \sigma_t p_t(\Pi_t - \Pi_t^*)\,dW_t + \int_0^T \sigma_t p_t^0(\Pi_t - \Pi_t^*)\,dW_t^0. \end{aligned} \tag{3.19}$$

Summing up (3.18) and (3.19) shows that

$$\begin{aligned}\mathcal{V}_0 = &\int_0^T (p_t h_t + q_t\sigma_t + q_t^0\sigma_t^0)(\Pi_t - \Pi_t^*)\,dt - \int_0^T \left(p_t + Q_t f_1(C_t^*\nu_t^{-\theta}, V_t^*)\nu_t^{-\theta}\right)(C_t - C_t^*)\,dt\\ &+ \int_0^T \left\{Q_t\mathcal{Z}_t + \mathcal{X}_t q_t + \sigma_t p_t(\Pi_t - \Pi_t^*)\right\} dW_t + \int_0^T \left\{Q_t\mathcal{Z}_t^0 + \mathcal{X}_t q_t^0 + \sigma_t^0 p_t(\Pi_t - \Pi_t^*)\right\} dW_t.\end{aligned}$$

In view of Step 1, the stochastic process

$$\int_0^{\cdot} \left\{Q_t\mathcal{Z}_t + \mathcal{X}_t q_t + \sigma_t p_t(\Pi_t - \Pi_t^*)\right\} dW_t + \int_0^{\cdot} \left\{Q_t\mathcal{Z}_t^0 + \mathcal{X}_t q_t^0 + \sigma_t^0 p_t(\Pi_t - \Pi_t^*)\right\} dW_t^0$$

is a true martingale. Thus,

$$\mathcal{V}_0 = \mathbb{E}\left[\int_0^T (p_t h_t + q_t\sigma_t + q_t^0\sigma_t^0)(\Pi_t - \Pi_t^*)\,dt - \int_0^T \left(p_t + Q_t f_1(C_t^*\nu_t^{-\theta}, V_t^*)\nu_t^{-\theta}\right)(C_t - C_t^*)\,dt\right]. \quad (3.20)$$

Step 3: *A perturbation result.* We now prove perturbation results for state processes from which we then conclude our maximum principle.

We denote for any admissible pair (Π, C) and any $\rho > 0$ by (X^ρ, V^ρ) be the state process corresponding to the control

$$(\Pi^* + \rho(\Pi - \Pi^*), C^* + \rho(C - C^*)).$$

- We first establish the convergence of the processes $X^\rho - X^*$ and $\frac{X^\rho - X^*}{\rho} - \mathcal{X}$ where $\mathcal{X}$ was defined in (3.17). The dynamics of the process $X^\rho - X^*$ follows the linear SDE

$$\begin{aligned}d(X_t^\rho - X_t^*) &= r_t(X_t^\rho - X_t^*)\,dt + \rho h_t(\Pi_t - \Pi_t^*)\,dt + \rho\sigma_t(\Pi_t - \Pi_t^*)\,dW_t\\ &\quad + \rho\sigma_t^0(\Pi_t - \Pi_t^*)\,dW_t^0 - \rho(C_t - C_t^*)\,dt\\ X_0^\rho - X_0^* &= 0\end{aligned}$$

  and for each $\beta > 0$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|X_t^\rho - X_t^*|^\beta\right]\\ &\le C\rho^\beta\mathbb{E}\left[\left(\int_0^T |\Pi_s - \Pi_s^*| + |C_s - C_s^*|\,ds\right)^\beta\right] + C\rho^\beta\mathbb{E}\left[\left(\int_0^T (\Pi_s - \Pi_s^*)^2\,ds\right)^{\frac{\beta}{2}}\right]\\ &\to 0, \quad \text{as } \rho \to 0.\end{aligned}$$

  Moreover, uniqueness of solutions to the above SDE yields that a.s.

$$\frac{X^\rho - X^*}{\rho} - \mathcal{X} = 0.$$

- Next, we consider the convergence of the processes $V^\rho - V^*$ and $\frac{V^\rho - V^*}{\rho} - \mathcal{V}$ where $\mathcal{V}$ was also defined in (3.17). The BSDE for the process $V^\rho - V^*$ can be linearized as

$$\begin{cases}\begin{aligned}-d(V_t^\rho - V_t^*) &= (C_t^\rho - C_t^*)\nu_t^{-\theta}\int_0^1 f_1(C_t^*\nu_t^{-\theta} + \lambda(C_t^\rho - C_t^*)\nu_t^{-\theta}, V_t^\rho)\,d\lambda\\ &\quad + (V_t^\rho - V_t^*)\int_0^1 f_2(C_t^*\nu_t^{-\theta}, V_t^* + \lambda(V_t^\rho - V_t^*))\,d\lambda\\ &\quad - (Z_t^\rho - Z_t^*)\,dW_t - (Z_t^{\rho,0} - Z_t^{0,*})\,dW_t^0\\ &:= \left\{A_t(V_t^\rho - V_t^*) + B_t\right\}dt - (Z_t^\rho - Z_t^*)\,dW_t - (Z_t^{\rho,0} - Z_t^{0,*})\,dW_t^0,\\ V_T^\rho - V_T^* &= \alpha(X_T^\rho - X_T^*)\mu_T^{-\theta}\int_0^1 U'(X_T^*\mu_T^{-\theta} + \lambda(X_T^\rho - X_T^*)\mu_T^{-\theta})\,d\lambda.\end{aligned}\end{cases}$$

We note that

$$B \in \bigcap_{\beta>0} S^\beta \quad \text{and} \quad V_T^\rho - V_T^* \in \bigcap_{\beta>0} S^\beta.$$

The above BSDE admits a unique solution if we truncate B and replace B by $B \wedge n$ for some $n \in \mathbb{N}$; the corresponding solution is denoted by $(\Delta V^n, \Delta Z^n)$. Standard comparison arguments show that ΔV^n is increasing in n; we denote the pointwise limit by ΔV. Since

$$\Delta V_t^n = \mathbb{E}\left[e^{\int_t^T A_s\, ds}(V_T^\rho - V_T^*) + \int_t^T e^{\int_t^s A_r\, dr}(B_s \wedge n)\, ds \middle| \mathcal{F}_t \right]$$

it follows by monotone convergence that

$$\Delta V_t = \mathbb{E}\left[e^{\int_t^T A_s\, ds}(V_T^\rho - V_T^*) + \int_t^T e^{\int_t^s A_r\, dr} B_s\, ds \middle| \mathcal{F}_t \right],$$

and so it follows from the previous result that

$$\lim_{\rho\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\Delta V_t|^\beta \right] = 0 \quad \text{for any } \beta > 0.$$

We now apply a similar argument to the process $\frac{V^\rho - V^*}{\rho} - \mathcal{V}$. The BSDE for this process can be linearized as

$$\begin{cases} -d\left(\dfrac{V_t^\rho - V_t^*}{\rho} - \mathcal{V}_t\right) = \left\{ \left(\dfrac{V_t^\rho - V_t^*}{\rho} - \mathcal{V}_t\right) \displaystyle\int_0^1 f_2(C_t^* \nu_t^{-\theta}, V_t^* + \lambda(V_t^\rho - V_t^*))\, d\lambda \right. \\ \qquad\qquad + \mathcal{V}_t \displaystyle\int_0^1 \left\{ f_2(C_t^* \nu_t^{-\theta}, V_t^* + \lambda(V_t^\rho - V_t^*)) - f_2(C_t^* \nu_t^{-\theta}, V_t^*) \right\} d\lambda \\ \qquad\qquad \left. + (C_t - C_t^*)\nu_t^{-\theta} \displaystyle\int_0^1 \left\{ f_1(C_t^* \nu_t^{-\theta} + \lambda(C_t^\rho - C_t^*)\nu_t^{-\theta}, V_t^*)\, d\lambda - f_1(C_t^* \nu_t^{-\theta}, V_t^*) \right\} d\lambda \right\} \\ \qquad\qquad - \left(\dfrac{Z_t^\rho - Z_t^*}{\rho} - \mathcal{Z}_t\right) dW_t - \left(\dfrac{Z_t^\rho - Z_t^*}{\rho} - \mathcal{Z}_t^0\right) dW_t^0 \\ \qquad := \left\{ \left(\dfrac{V_t^\rho - V_t^*}{\rho} - \mathcal{V}_t\right) I_1(t) + I_2(t) + I_3(t) \right\} dt \\ \qquad\qquad - \left(\dfrac{Z_t^\rho - Z_t^*}{\rho} - \mathcal{Z}_t\right) dW_t - \left(\dfrac{Z_t^{0,\rho} - Z_t^{0,*}}{\rho} - \mathcal{Z}_t^0\right) dW_t^0, \\ \dfrac{V_T^\rho - V_T^*}{\rho} - \mathcal{V}_T = \alpha \mathcal{X}_T \mu_T^{-\theta} \displaystyle\int_0^1 \left\{ U'(X_T^* \mu_T^{-\theta} + \lambda(X_T^\rho - X_T^*)\mu_T^{-\theta}) - U'(X_T^* \mu_T^{-\theta}) \right\} d\lambda. \end{cases}$$

Truncating the processes I_2 and I_3 by n the resulting BSDE admits the unique solution

$$(\Delta V)_t^{\rho,n} = \mathbb{E}\left[\left(\frac{V_T^\rho - V_T^*}{\rho} - \mathcal{V}_T\right) e^{\int_t^T I_1(r)\, dr} + \int_t^T e^{\int_t^s I_1(r)\, dr}(I_2(s) \wedge n + I_3(s) \wedge n)\, ds \middle| \mathcal{F}_t \right].$$

A standard comparison principle along with the monotone convergence theorem shows that the sequence $(\Delta V)_t^{\rho,n}$ as $n \to \infty$ converges pointwise to the process

$$(\Delta V)_t^{\rho} = \mathbb{E}\left[\left(\frac{V_T^\rho - V_T^*}{\rho} - \mathcal{V}_T\right) e^{\int_t^T I_1(r)\, dr} + \int_t^T e^{\int_t^s I_1(r)\, dr}(I_2(s) + I_3(s))\, ds \middle| \mathcal{F}_t \right].$$

The previously established convergence of $X^\rho - X^*$ and $V^\rho - V^*$ yields that

$$\lim_{\rho\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |(\Delta V)_t^\rho|^\beta \right] = 0 \quad \text{for each } \beta > 0.$$

Step 4. *Conclusion.* In view of (3.20) and the previously established convergence of the processes $\frac{V^\rho - V^*}{\rho} - \mathcal{V}$, it holds for any admissible (Π, C) that

$$\begin{aligned} 0 \;&\geq \frac{V_0^\rho - V_0^*}{\rho} \to \mathcal{V}_0 \\ &= \mathbb{E}\left[\int_0^T (p_t h_t + q_t \sigma_t + q_t^0 \sigma_t^0)(\Pi_t - \Pi_t^*)\, dt - \int_0^T \left(p_t + Q_t f_1(C_t^*, V_t^*)\nu_t^{-\theta}\right)(C_t - C_t^*)\, dt\right]. \end{aligned} \tag{3.21}$$

□

Remark 3.5. The inequality (3.21) should be compared with the utility gradient method in [31, Theorem 3.4], where no adjoint variables were introduced:

$$\begin{aligned} &V_0(C) - V_0(C^*) \\ &\leq \mathbb{E}\left[\int_0^T e^{\int_t^s f_2(C_s^*\nu_s^{-\theta}, V_s^*)\, ds} f_1(C^*\nu_s^{-\theta}, V_s^*)(C_s - C_s^*)\nu_s^{-\theta}\, ds + \alpha e^{\int_0^T f_2(C_s^*\nu_s^{-\theta}.V_s^*)\, ds} U'(C_T^*\nu_T^{-\theta})\nu_T^{-\theta}(C_T - C_T^*)\right], \end{aligned}$$

where $V_0(C)$ is the utility index corresponding to the consumption plan C. To obtain our BSDE (3.4) from any given optimal strategy (or equilibrium strategy), a version of utility gradient inequality in terms of adjoint variables in the spirit of (3.21) is required.

We are now ready to show that any optimal control satisfies the BSDE with the driver (3.4). As a result, any equilibrium strategy must satisfy mean field BSDE (3.12) determined in Section 3.1.2.

Theorem 3.6. *Let $(\Pi^*, C^*) \in \mathcal{A}$ be an optimal control and (X^*, V^*) be the corresponding state process. Then,*

$$\Pi^* \equiv \frac{h + \sigma Z + \sigma^0 Z^0}{\gamma(\sigma^2 + (\sigma^0)^2)} X^* := \pi^* X^*, \quad C^* \equiv \delta^\psi (\nu^{-\theta})^{\psi-1} (\alpha e^Y)^{-\frac{\psi}{\theta}} X^* := c^* X^*, \tag{3.22}$$

where (Y, Z, Z^0) satisfies the BSDE with the driver (3.4).

Proof. The proof consists of three steps. First, we define a stochastic process Y in terms of the optimal wealth process X^* and the adjoint processes p and Q which - together with some process (Z, Z^0) to be defined - satisfies a BSDE. We also relate Π^* with the (Z, Z^0).

We then verify that the optimal utility process satisfies $V^* = \alpha \frac{(X^*)^{1-\gamma}}{1-\gamma} e^Y$. In a final step, we verify that the optimal consumption process C^* can be expressed in terms of Y, and (Y, Z, Z^0) satisfies the same BSDE as in Section 3.1.1.

Step 1. We start by defining a stochastic process Y in terms of the adjoint processes p and Q defined in (3.14) through

$$p = \alpha U'(X^*) e^Y (-Q).$$

That is,

$$Y = \log p - \log \alpha - \log U'(X^*) - \log(-Q).$$

By Itô's formula,

$$\begin{aligned} &dY_t \\ &= \left\{ -\frac{U''(X_t^*)}{U'(X_t^*)} X_t^* (r_t + \pi_t^* h_t - c_t^*) - \frac{U'''(X_t^*)U'(X_t^*) - (U''(X_t^*))^2}{2(U'(X_t^*))^2} \left\{ (X_t^*\pi_t^*\sigma_t)^2 + (X_t^*\pi_t^*\sigma_t^0)^2 \right\} \right. \\ &\quad \left. - r_t - \frac{q_t^2 + (q_t^0)^2}{2p_t^2} - f_2(c_t^* X_t^* \nu_t^{-\theta}, V_t^*) \right\} dt \\ &\quad + \left\{ \frac{q_t}{p_t} - \frac{U''(X_t^*)}{U'(X_t^*)} \pi_t^* X_t^* \sigma_t \right\} dW_t + \left\{ \frac{q_t^0}{p_t} - \frac{U''(X_t^*)}{U'(X_t^*)} \pi_t^* X_t^* \sigma_t^0 \right\} dW_t^0. \end{aligned} \tag{3.23}$$

Let

$$Z := \frac{q}{p} - \frac{U''(X^*)}{U'(X^*)}\pi^* X^* \sigma, \quad Z^0 := \frac{q^0}{p} - \frac{U''(X^*)}{U'(X^*)}\pi^* X^* \sigma^0,$$

so that

$$q = p\left\{\frac{U''(X^*)}{U'(X^*)}\pi^* X^* \sigma + Z\right\}, \quad q^0 = p\left\{\frac{U''(X^*)}{U'(X^*)}\pi^* X^* \sigma^0 + Z^0\right\}. \tag{3.24}$$

From the first equality in (3.15) we conclude that

$$ph + p\left\{\frac{U''(X^*)}{U'(X^*)}\pi^* X^* \sigma + Z\right\}\sigma + p\left\{\frac{U''(X^*)}{U'(X^*)}\pi^* X^* \sigma^0 + Z^0\right\}\sigma^0 = 0,$$

which implies that

$$\pi^* = -\frac{h + \sigma Z + \sigma^0 Z^0}{\frac{U''(X^*)}{U'(X^*)} X^*(\sigma^2 + (\sigma^0)^2)}.$$

Since U is a power function,

$$\pi^* = \frac{h + \sigma Z + \sigma^0 Z^0}{\gamma(\sigma^2 + (\sigma^0)^2)}. \tag{3.25}$$

We emphasize that (3.25) coincides with (3.22) only if (Y, Z, Z^0) follows the BSDE with the driver (3.4).

Step 2. By the second equality in (3.15), we have that

$$p + f_1(c^* X^* \nu^{-\theta}, V^*) Q \nu^{-\theta} = 0$$

and so

$$\alpha U'(X^*) e^Y (-Q) + \delta (c^* X^* \nu^{-\theta})^{-\frac{1}{\psi}} \left\{(1-\gamma)V^*\right\}^{1-\frac{1}{\theta}} Q\nu^{-\theta} = 0.$$

This implies that

$$c^* = (X^*)^{\psi\gamma - 1} e^{-\psi Y} \left(\frac{\delta}{\alpha}\right)^{\psi} \{(1-\gamma)V^*\}^{\psi(1-\frac{1}{\theta})} \nu^{-\theta(\psi-1)} \tag{3.26}$$

and that

$$(c^* X^* \nu^{-\theta})^{1-\frac{1}{\psi}} = \left(\frac{\delta}{\alpha}\right)^{\psi(1-\frac{1}{\psi})} \{(1-\gamma)V^*\}^{\psi(1-\frac{1}{\theta})(1-\frac{1}{\psi})} (X^*)^{\psi\gamma(1-\frac{1}{\psi})} e^{-\psi(1-\frac{1}{\psi})Y} \nu^{-\theta(\psi-1)}. \tag{3.27}$$

We proceed by comparing the BSDEs for $\alpha\frac{(X^*)^{1-\gamma}}{1-\gamma}e^Y$ and for V^*. Since $Y_T = -\theta(1-\gamma)\log\nu_T$, the processes V^* and $\alpha\frac{(X^*)^{1-\gamma}}{1-\gamma}e^Y$ share the same terminal value. Moreover, by Itô's formula,

$$\begin{aligned}
\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t} = \frac{x^{1-\gamma}}{1-\gamma}e^{Y_0} &+ \int_0^t \frac{(X_s^*)^{1-\gamma}}{1-\gamma}e^{Y_s}\Big\{-(1-\gamma)r_s + \gamma h_s \pi_s^* - \frac{\gamma}{2}(\pi_s^*)^2(\sigma_s^2 + (\sigma_s^0)^2) \\
&\qquad - \frac{1}{2}(-\gamma\pi_s^*\sigma_s + Z_s)^2 - \frac{1}{2}(-\gamma\pi_s^*\sigma_s^0 + Z_s^0)^2 + \frac{1}{2}Z_s^2 + \frac{1}{2}(Z_s^0)^2 \\
&\qquad -\gamma c_s^* - \frac{\delta(c_s^* X_s^* \nu_s^{-\theta})^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}(1-\gamma)(1-\frac{1}{\widetilde{\theta}})\left\{(1-\gamma)V_s^*\right\}^{-\frac{1}{\theta}} + \delta\widetilde{\theta}\Big\}\, ds \\
&+ \int_0^t e^{Y_s}\left\{(X_s^*)^{1-\gamma}(r_s + \pi_s^* h_s) - (X_s^*)^{1-\gamma}c_s^* - \frac{\gamma}{2}(X_s^*)^{1-\gamma}(\pi_s^*)^2(\sigma_s^2 + (\sigma_s^0)^2)\right\} ds \\
&+ \int_0^t (X_s^*)^{1-\gamma}\pi_s^*\sigma_s e^{Y_s} Z_s\, ds + \int_0^t (X_s^*)^{1-\gamma}\pi_s^*\sigma_s^0 e^{Y_s} Z_s^0\, ds \\
&+ \int_0^t \frac{(X_s^*)^{1-\gamma}}{1-\gamma}e^{Y_s}\widetilde{Z}_s^\top\, d\widetilde{W}_s + \int_0^t e^{Y_s}(X_s^*)^{1-\gamma}\pi_s^*\widetilde{\sigma}_s^\top\, d\widetilde{W}_s.
\end{aligned}$$

Collecting all terms with π^* and/or (Z, Z^0) in the driver of $\frac{(X^*)^{1-\gamma}}{1-\gamma}e^Y$, we get

$$\begin{aligned}
&\frac{(X^*)^{1-\gamma}}{1-\gamma}e^Y\left\{\gamma h\pi^* - \frac{\gamma}{2}(\pi^*)^2(\sigma^2+(\sigma_s^0)^2) - \frac{1}{2}(-\gamma\pi^*\sigma+Z)^2 - \frac{1}{2}(-\gamma\pi^*\sigma^0+Z^0)^2 + \frac{1}{2}Z^2 + \frac{1}{2}(Z^0)^2\right\}\\
&+e^Y\left\{(X^*)^{1-\gamma}\pi^*h - \frac{\gamma}{2}(X^*)^{1-\gamma}(\pi^*)^2(\sigma^2+(\sigma^0)^2)\right\} + (X^*)^{1-\gamma}\pi^*\sigma e^Y Z + (X^*)^{1-\gamma}\pi^*\sigma^0 e^Y Z^0\\
=\ &\frac{(X^*)^{1-\gamma}}{1-\gamma}e^Y\left\{\gamma h\pi^* - \frac{\gamma}{2}(\pi^*)^2(\sigma^2+(\sigma^0)^2) - \frac{1}{2}(-\gamma\pi^*\sigma+Z)^2 - \frac{1}{2}(-\gamma\pi^*\sigma^0+Z^0)^2 + \frac{1}{2}Z^2 + \frac{1}{2}(Z^0)^2\right.\\
&\qquad\left.+(1-\gamma)\pi^*h - \frac{(1-\gamma)\gamma}{2}(\pi^*)^2(\sigma^2+(\sigma^0)^2) + (1-\gamma)\pi^*\sigma Z + (1-\gamma)\pi^*\sigma^0 Z^0\right\}\\
=\ &0,
\end{aligned}$$

where we used the representation (3.25) obtained in Step 1. Thus,

$$\begin{aligned}
\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t} = \frac{x^{1-\gamma}}{1-\gamma}e^{Y_0} &+ \int_0^t \frac{(X_s^*)^{1-\gamma}}{1-\gamma}e^{Y_s}\left\{-c_s^* - \frac{\delta(c_s^*X_s^*\nu_s^{-\theta})^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}(1-\gamma)(1-\frac{1}{\widetilde{\theta}})\left\{(1-\gamma)V_s^*\right\}^{-\frac{1}{\widetilde{\theta}}} + \delta\widetilde{\theta}\right\}ds\\
&+\int_0^t \frac{(X_s^*)^{1-\gamma}}{1-\gamma}e^{Y_s}(\widetilde{Z}_s^\top + (1-\gamma)\pi_s^*\widetilde{\sigma}_s^\top)\,d\widetilde{W}_s.
\end{aligned}$$

By (3.26) and (3.27), the dynamics of $\frac{(X^*)^{1-\gamma}}{1-\gamma}e^Y$ can be rewritten as

$$\begin{aligned}
&\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t}\\
=\ &\frac{x^{1-\gamma}}{1-\gamma}e^{Y_0} + \int_0^t\left\{-\frac{1}{1-\gamma}\left(\frac{\delta}{\alpha}\right)^\psi (X_s^*)^{\psi\gamma-\gamma}e^{(1-\psi)Y_s}\left\{(1-\gamma)V_s^*\right\}^{\psi(1-\frac{1}{\theta})}\nu_s^{-\theta(\psi-1)}\right.\\
&-\left(\frac{1}{1-\frac{1}{\psi}}-\frac{1}{1-\gamma}\right)\frac{(X_s^*)^{1-\gamma}}{1-\gamma}e^{Y_s}\delta(1-\gamma)\left(\frac{\delta}{\alpha}\right)^{\psi-1}\left\{(1-\gamma)V_s^*\right\}^{\psi(1-\frac{1}{\theta})(1-\frac{1}{\psi})-\frac{1}{\theta}}(X_s^*)^{\gamma(\psi-1)}e^{(1-\psi)Y_s}\nu^{-\theta(\psi-1)}\\
&\left.+\delta\widetilde{\theta}\frac{(X_s^*)^{1-\gamma}}{1-\gamma}e^{Y_s}\right\}ds + \int_0^t\frac{(X_s^*)^{1-\gamma}}{1-\gamma}e^{Y_s}(\widetilde{Z}_s^\top + (1-\gamma)\pi_s^*\widetilde{\sigma}_s^\top)\,d\widetilde{W}_s.
\end{aligned}$$

Taking (3.27) into account, the dynamics of V^* can be rewritten as

$$\begin{aligned}
-dV_t^* &= \left\{\frac{\delta(c_t^*X_t^*\nu_t^{-\theta})^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\{(1-\gamma)V_t^*\}^{1-\frac{1}{\widetilde{\theta}}} - \delta\widetilde{\theta}V_t^*\right\}dt - Z_t^*\,dW_t - Z_t^{0,*}\,dW_t^0\\
&= \left\{\frac{\delta}{1-\frac{1}{\psi}}\left(\frac{\delta}{\alpha}\right)^{\psi-1}\{(1-\gamma)V_t^*\}^{\psi(1-\frac{1}{\theta})}(X_t^*)^{\gamma(\psi-1)}e^{(1-\psi)Y_t}\nu_t^{-\theta(\psi-1)} - \delta\widetilde{\theta}V_t^*\right\}dt - Z_t^*\,dW_t - Z_t^{0,*}\,dW_t^0
\end{aligned}$$

As a result,

$$\begin{aligned}
&d\left(\alpha\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t} - V_t^*\right)\\
=\ &\alpha\left(\frac{1}{1-\frac{1}{\psi}}-\frac{1}{1-\gamma}\right)\left(\frac{\delta}{\alpha}\right)^\psi\{(1-\gamma)V_t^*\}^{\psi(1-\frac{1}{\theta})}(X_t^*)^{\gamma(\psi-1)}e^{(1-\psi)Y_t}\nu_t^{-\theta(\psi-1)}\,dt\\
&-\alpha\left(\frac{1}{1-\frac{1}{\psi}}-\frac{1}{1-\gamma}\right)\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t}\delta(1-\gamma)\left(\frac{\delta}{\alpha}\right)^{\psi-1}\{(1-\gamma)V_t^*\}^{\psi(1-\frac{1}{\theta})-1}(X_t^*)^{\gamma(\psi-1)}e^{(1-\psi)Y_t}\nu_t^{-\theta(\psi-1)}\,dt\\
&+\delta\widetilde{\theta}\left(\alpha\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t} - V_t^*\right)dt + Z_t'\,dW_t + Z_t'^0\,dW_t^0\\
=\ &\alpha(\widetilde{\theta}-1)\left(\frac{\delta}{\alpha}\right)^\psi\{(1-\gamma)V_t^*\}^{\psi(1-\frac{1}{\theta})-1}(X_t^*)^{\gamma(\psi-1)}e^{(1-\psi)Y_t}\nu_t^{-\theta(\psi-1)}\left(V_t^* - \alpha\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t}\right)dt
\end{aligned}$$

$$+\delta\widetilde{\theta}\left(\alpha\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t}-V_t^*\right)\,dt+Z_t'\,dW_t+Z_t'^0\,dW_t^0$$
$$:=\mathcal{D}_t\left(\alpha\frac{(X_t^*)^{1-\gamma}}{1-\gamma}e^{Y_t}-V_t^*\right)\,dt+Z_t'\,dW_t+Z_t'^0\,dW_t^0,$$

for some positive process $\mathcal{D}$. Thus, the difference

$$\alpha\frac{(X^*)^{1-\gamma}}{1-\gamma}e^{Y}-V^*$$

satisfies a linear BSDE with monotone driver and zero terminal condition. Such BSDEs admit a unique solution, namely zero; see [28, Theorem 4.1] and [29, Proposition 6.2.1].

Step 3. Taking the identity $V^*=\alpha\frac{(X^*)^{1-\gamma}}{1-\gamma}e^{Y}$ from Step 2 into (3.26) we get

$$c^*=\delta^{\psi}(\alpha Y)^{-\frac{\psi}{\theta}}\nu^{-\theta(\psi-1)}. \tag{3.28}$$

Taking (3.24) and (3.28) into the driver of (3.23), one can verify that (Y,Z,Z^0) satisfies the BSDE in Section 3.1.1. □

By Theorem 3.2 and Theorem 3.6 we have the following one-to-one correspondence between the equilibrium investment and consumption rates and the solution to the BSDE (3.12).

Theorem 3.7. *Equilibrium investment and consumption rates of mean field portfolio games with Epstein-Zin utility that satisfy the conditions*

$$(\pi^*,c^*)\in H^2_{BMO}\times L^\infty,\quad (\pi^*X^*,c^*X^*)\in\mathcal{A},\quad \mathbb{E}\left[p_T|\mathcal{F}_t\right]=\mathcal{E}\left(\int_0^t\vartheta_s^\top\,d\widetilde{W}_s\right)\ \text{for some}\ \vartheta\in H^2_{BMO} \tag{3.29}$$

admit a one-to-one correspondence to solutions $(\widetilde{Y},\widetilde{Z},\widetilde{Z}^0)\in L^\infty\times H^2_{BMO}\times H^2_{BMO}$ *to the mean field BSDE* (3.12). *The relation is given by*

$$\pi^*=\frac{h+\sigma\widetilde{Z}+\sigma^0\widetilde{Z}^0}{\gamma(\sigma^2+(\sigma^0)^2)}-\frac{\sigma^0\theta(1-\gamma)\mathbb{E}\left[\frac{\sigma^0(h+\sigma\widetilde{Z}+\sigma^0\widetilde{Z}^0)}{\gamma(\sigma^2+(\sigma^0)^2)}\middle|\mathcal{F}^0\right]}{\gamma(\sigma^2+(\sigma^0)^2)\left(1+\mathbb{E}\left[\frac{\theta(1-\gamma)(\sigma^0)^2}{\gamma(\sigma^2+(\sigma^0)^2)}\middle|\mathcal{F}^0\right]\right)} \tag{3.30}$$

and

$$\begin{cases} c^*=\exp\left(-\dfrac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\left\{\mathbb{E}[\psi\log\delta]-\mathbb{E}\left[\dfrac{\psi}{\widetilde{\theta}}\log\alpha\right]-\mathbb{E}\left[\dfrac{\psi}{\widetilde{\theta}}\widetilde{Y}\middle|\mathcal{F}^0\right]\right\}\right. \\ \qquad\qquad\left.+\psi\log\delta-\dfrac{\psi}{\widetilde{\theta}}\log\alpha-\dfrac{\psi}{\widetilde{\theta}}\widetilde{Y}\right)\quad \text{on } [0,T), \\ c_T^*=1. \end{cases} \tag{3.31}$$

Proof. We proceed in three steps.

Step 1. By Theorem 3.6, any equilibrium rate (π^*,c^*) such that $(\pi^*X^*,c^*X^*)\in\mathcal{A}$ must be characterized by the solution to the following mean field FBSDE system

$$\begin{cases} dX_t^*=X_t^*\Big((r_t+\pi_t^*h_t)\,dt+\pi_t^*\sigma_t\,dW_t+\pi_t^*\sigma_t^0\,dW_t^0\Big)-c_t^*X_t^*\,dt \\ -dY_t=\left\{\dfrac{Z_t^2+(Z_t^0)^2}{2}+(1-\gamma)r_t+\dfrac{1-\gamma}{2\gamma}\dfrac{(h_t+\sigma_tZ_t+\sigma_t^0Z_t^0)^2}{\sigma_t^2+(\sigma_t^0)^2}\right. \\ \qquad\left.+\dfrac{1-\gamma}{\psi-1}\delta^{\psi}(\nu_t^*)^{-\theta(\psi-1)}(\alpha e^{Y_t})^{-\frac{\psi}{\theta}}-\delta\widetilde{\theta}\right\}\,dt-Z_t\,dW_t-Z_t^0\,dW_t^0 \\ X_0=x,\quad Y_T=-\theta(1-\gamma)\mathbb{E}[\widehat{X}_T^*|\mathcal{F}_T^0], \end{cases} \tag{3.32}$$

where

$$\pi^* = \frac{h + \sigma Z + \sigma^0 Z^0}{\gamma(\sigma^2 + (\sigma^0)^2)}, \qquad c^* = \delta^\psi (\nu^*)^{-\theta(\psi-1)} (\alpha e^Y)^{-\frac{\psi}{\theta}},$$

and

$$\widehat{\nu}^* = \frac{1}{1 + \mathbb{E}[\theta(\psi-1)]} \left\{ \mathbb{E}[\psi \log \delta] - \mathbb{E}\left[\frac{\psi}{\widetilde{\theta}} \log \alpha\right] - \mathbb{E}\left[\frac{\psi}{\widetilde{\theta}} Y \middle| \mathcal{F}^0\right] \right\} + \frac{1}{1 + \mathbb{E}[\theta(\psi-1)]} \mathbb{E}[\widehat{X}^* | \mathcal{F}^0].$$

By the same argument as in the proof of Theorem 3.2, (π^*, c^*) satisfies (3.30) and (3.31), with $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0)$ satisfies (3.12).

The other direction has been established by Theorem 3.2. In the next two steps, we will verify that the equilibrium space and the solution space also have a one-to-one correspondence.

Step 2. If the triple $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0) \in L^\infty \times H^2_{BMO} \times H^2_{BMO}$ is a solution to the mean field BSDE (3.12), then the strategy (π^*, c^*) defined in (3.30) and (3.31) satisfies $(\pi^*, c^*) \in H^2_{BMO} \times L^\infty$.

It remains to verify that the adjoint variable p defined through (π^*, c^*) and (3.14) satisfies the third condition in (3.29). To do so, we apply the explicit expression for adjoint equation in [1, Section 1.4.1]. In view of Proposition 3.4 and Theorem 3.6 we have the following correspondence between adjoint variables and the optimal control:

$$(Z, Z^0) = \left(\frac{q}{p} + \gamma \sigma \pi^*, \frac{q^0}{p} + \gamma \sigma^0 \pi^* \right), \tag{3.33}$$

where we recall (Z, Z^0) is the (Z, Z^0)-component of (3.32), which has a one-to-one correspondence with the $(\widetilde{Z}, \widetilde{Z}^0)$-component of (3.12). It implies that $(Z, Z^0) \in H^2_{BMO} \times H^2_{BMO}$.

By [1, Equation (51) and Equation (52)],

$$\left(\frac{q_t}{p_t}, \frac{q^0_t}{p_t} \right) = \vartheta_t, \tag{3.34}$$

where ϑ satisfies

$$\mathbb{E}[p_T | \mathcal{F}_t] = \mathcal{E}\left(\int_0^t \vartheta_s^\top \, d\widetilde{W}_s \right).$$

Since Z, Z^0 and π^* belong to H^2_{BMO}, it follows from (3.33) that the processes $\frac{q}{p}$ and $\frac{q^0}{p}$ also belong to H^2_{BMO}. The same holds for the process ϑ, due to (3.34). Thus, the third condition in (3.29) holds.

Step 3. If (π^*, c^*) is an equilibrium strategy satisfying the all conditions in (3.29), then it follows from Step 1 that it can be characterized by the mean field BSDE (3.12) via (3.30) and (3.31). In particular, the relation (3.31) implies that

$$\widetilde{Y} = -\frac{\widetilde{\theta}}{\psi} \log c^* - \frac{\theta \widetilde{\theta}(\psi - 1)}{\psi} \mathbb{E}[\log c^* | \mathcal{F}^0] + \widetilde{\theta} \log \delta - \log \alpha,$$

which belongs to L^∞ since $c^* \in L^\infty$ by (3.29). Using the relations (3.33) and (3.34) again, we have Z and Z^0 belong H^2_{BMO} by (3.29). Thus, $(\widetilde{Z}, \widetilde{Z}^0) \in H^2_{BMO} \times H^2_{BMO}$. □

Remark 3.8. The third condition in (3.29) is consistent with the reverse Hölder inequality in [11, 12, 14, 15], where time additive utility (exponential or power utility) is considered. In general, this condition is necessasry to establish the one-to-one correspondence between the equilibrium investment and the Z-component of solution to some (F)BSDE in the BMO space. However, this condition can be dropped if either the following two conditions holds:

- If there is only common noise, i.e. $\sigma^0 = 0$, then (3.30) yields a one-to-one correspondence between π^* and Z^0, which implies that $\pi^* \in H^2_{BMO}$ is equivalent to $Z^0 \in H^2_{BMO}$, even without the third condition in (3.29). This also implies that $\pi^* \in L^\infty$ and $Z^0 \in L^\infty$.
- If all coefficients are deterministic, then $\widetilde{Z} = \widetilde{Z}^0 = 0$ and $\pi^* \in H^2_{BMO}$ trivially holds, if it exists.

4 Uniqueness of mean-field equilibrium strategy in $H^2_{BMO} \times L^\infty$

By Theorem 3.7, to establish our uniqueness of equilibrium result, it is sufficient and necessary to show that the BSDE (3.12) admits at most one solution.

Theorem 4.1. *For each $R > 0$, there exists at most one equilibrium investment and consumption rates $(\pi^*, c^*) \in H^2_{R,BMO} \times L^\infty$ satisfying (3.29), when the competition parameter θ is small enough. Here, $H^2_{R,BMO}$ is the R-ball of H^2_{BMO}.*

Proof. Let $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0)$ and $(\widetilde{Y}', \widetilde{Z}', \widetilde{Z}^{0'})$ be solutions to the BSDE (3.12) in $L^\infty \times H^2_{BMO} \times H^2_{BMO}$ and let

$$\Delta Y := \widetilde{Y} - \widetilde{Y}', \quad \Delta Z := \widetilde{Z} - \widetilde{Z}', \quad \Delta Z^0 := \widetilde{Z}^0 - \widetilde{Z}^{0'}.$$

It follows that

$$\Delta Y_t = \theta(1-\gamma) \int_t^T \mathbb{E}[\Delta c_s^* | \mathcal{F}_s^0]\, ds + \int_t^T \frac{1-\gamma}{\psi - 1} \Delta c_s^*\, ds + \int_t^T \Delta \mathcal{J}_s\, ds - \int_t^T \Delta Z_s\, dW_s - \int_t^T \Delta Z_s^0\, dW_s^0,$$

where

$$\Delta c^* = \frac{c^* - c^{*'}}{\Delta Y} \Delta Y$$

with $\frac{c^* - c^{*'}}{\Delta Y}$ being bounded since $\widetilde{Y}, \widetilde{Y}' \in L^\infty$ and where the process $\Delta \mathcal{J}$ can be expressed in terms of the functions f^a and f^{ab} defined in [15, Appendix B] as

$$\begin{aligned}
\Delta \mathcal{J} = & -\theta\gamma \mathbb{E}\left[f^{\sigma^h} \Delta Z + f^{\sigma^0 h} \Delta Z^0 \middle| \mathcal{F}^0 \right] + \theta\gamma \mathbb{E}\left[\theta\gamma f^{\sigma^0 h} \middle| \mathcal{F}^0 \right] \frac{\mathbb{E}\left[f^{\sigma^0 \sigma} \Delta Z + f^{\sigma^0 \sigma^0} \Delta Z^0 \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}[\theta\gamma f^{\sigma^0 \sigma^0} | \mathcal{F}^0]} \\
& + \theta\gamma \mathbb{E}\left[\frac{1}{2}(1-\gamma) \widetilde{\sigma}^\top \widetilde{\sigma} \left\{ 2f^h + f^\sigma(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) - \frac{\theta\gamma f^{\sigma^0} \mathbb{E}\left[2f^{\sigma^0 h} + f^{\sigma^0 \sigma}(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0 \sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) | \mathcal{F}^0 \right]}{1 + \mathbb{E}[\theta\gamma f^{\sigma^0 \sigma^0} | \mathcal{F}^0]} \right\} \right. \\
& \qquad \left. \times \left\{ f^\sigma \Delta Z + f^{\sigma^0} \Delta Z^0 - \frac{\theta\gamma f^{\sigma^0} \mathbb{E}\left[f^{\sigma^0 \sigma} \Delta Z + f^{\sigma^0 \sigma^0} \Delta Z^0 \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}\left[\theta\gamma f^{\sigma^0 \sigma^0} \middle| \mathcal{F}^0 \right]} \right\} \middle| \mathcal{F}^0 \right] \\
& - \frac{1}{2} \left\{ \widetilde{Z}^0 + \widetilde{Z}^{0'} - \frac{\theta\gamma \mathbb{E}\left[2f^{\sigma^0 h} + f^{\sigma^0 \sigma}(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0 \sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}\left[\theta\gamma f^{\sigma^0 \sigma^0} \middle| \mathcal{F}^0 \right]} \right\} \frac{\theta\gamma \mathbb{E}\left[f^{\sigma^0 \sigma} \Delta Z + f^{\sigma^0 \sigma^0} \Delta Z^0 \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}\left[\theta\gamma f^{\sigma^0 \sigma^0} \middle| \mathcal{F}^0 \right]} \\
& - \frac{\gamma(1-\gamma) \widetilde{\sigma}^\top \widetilde{\sigma}}{2} \left\{ 2f^h + f^\sigma(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) - \frac{\theta\gamma f^{\sigma^0} \mathbb{E}\left[2f^{\sigma^0 h} + f^{\sigma^0 \sigma}(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0 \sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}\left[\theta\gamma f^{\sigma^0 \sigma^0} \middle| \mathcal{F}^0 \right]} \right\} \\
& \qquad \times \frac{\theta\gamma f^{\sigma^0} \mathbb{E}\left[f^{\sigma^0 \sigma} \Delta Z + f^{\sigma^0 \sigma^0} \Delta Z^0 \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}\left[\theta\gamma f^{\sigma^0 \sigma^0} \middle| \mathcal{F}^0 \right]} \\
& + \frac{\widetilde{Z} + \widetilde{Z}'}{2} \Delta Z \\
& + \frac{\gamma(1-\gamma) \widetilde{\sigma}^\top \widetilde{\sigma}}{2} \left\{ 2f^h + f^\sigma(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) - \frac{\theta\gamma f^{\sigma^0} \mathbb{E}\left[2f^{\sigma^0 h} + f^{\sigma^0 \sigma}(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0 \sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}\left[\theta\gamma f^{\sigma^0 \sigma^0} \middle| \mathcal{F}^0 \right]} \right\} f^\sigma \Delta Z \\
& + \frac{\gamma(1-\gamma) \widetilde{\sigma}^\top \widetilde{\sigma}}{2} \left\{ 2f^h + f^\sigma(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) - \frac{\theta\gamma f^{\sigma^0} \mathbb{E}\left[2f^{\sigma^0 h} + f^{\sigma^0 \sigma}(\widetilde{Z} + \widetilde{Z}') + f^{\sigma^0 \sigma^0}(\widetilde{Z}^0 + \widetilde{Z}^{0'}) \middle| \mathcal{F}^0 \right]}{1 + \mathbb{E}\left[\theta\gamma f^{\sigma^0 \sigma^0} \middle| \mathcal{F}^0 \right]} \right\} f^{\sigma^0} \Delta Z^0
\end{aligned}$$

$$+\frac{1}{2}\left\{\widetilde{Z}^0+\widetilde{Z}^{0\prime}-\frac{\theta\gamma\mathbb{E}\left[2f^{\sigma^0 h}+f^{\sigma^0\sigma}(\widetilde{Z}+\widetilde{Z}')+f^{\sigma^0\sigma^0}(\widetilde{Z}^0+\widetilde{Z}^{0\prime})\middle|\mathcal{F}^0\right]}{1+\mathbb{E}\left[\theta\gamma f^{\sigma^0\sigma^0}\middle|\mathcal{F}^0\right]}\right\}\Delta Z^0$$
$$:=\theta\Delta\widetilde{\mathcal{J}}+\mathcal{C}\Delta Z+\mathcal{C}^0\Delta Z^0.$$

All terms in the definition of $\Delta\mathcal{J}$ that do not congtain θ are linear terms of $(\Delta Z,\Delta Z^0)$ and can hence be dropped by a change of measure:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}=\mathcal{E}\left\{\int_0^{\cdot}\mathcal{C}_s\,dW_s+\int_0^{\cdot}\mathcal{C}_s^0\,dW_s^0\right\}.$$

Since $\widetilde{Z}$, $\widetilde{Z}^0$, $\widetilde{Z}'$ and $\widetilde{Z}^{0\prime}$ are in H^2_{BMO}, $(W^{\mathbb{Q}},W^{0,\mathbb{Q}})$ is a $\mathbb{Q}$-Brownian motion, where

$$\begin{cases}W_{\cdot}^{\mathbb{Q}}=W_{\cdot}-\displaystyle\int_0^{\cdot}\mathcal{C}_s\,ds,\\ W_{\cdot}^{0,\mathbb{Q}}=W_{\cdot}^0-\displaystyle\int_0^{\cdot}\mathcal{C}_s^0\,ds.\end{cases}$$

As a result,

$$\Delta Y_t=\theta(1-\gamma)\int_t^T\mathbb{E}[\Delta c_s^*|\mathcal{F}_s^0]\,ds+\int_t^T\frac{1-\gamma}{\psi-1}\Delta c_s^*\,ds+\theta\int_t^T\Delta\widetilde{\mathcal{J}}_s\,ds-\int_t^T\Delta Z_s\,dW_s^{\mathbb{Q}}-\int_t^T\Delta Z_s^0\,dW_s^{0,\mathbb{Q}}.$$

Itô's formula and standard estimate imply that

$$\begin{aligned}&\operatorname*{ess\,sup}_{\omega,t\le s\le T}(\Delta Y_s)^2+\|\Delta Z\|^2_{BMO,\mathbb{Q}}+\|\Delta Z^0\|^2_{BMO,\mathbb{Q}}\\ &\le\ C\int_t^T\operatorname*{ess\,sup}_{\omega,t\le r\le s}(\Delta Y_r)^2\,ds+\theta C\|\Delta Z\|^2_{BMO,\mathbb{Q}}+\theta C\|\Delta Z^0\|^2_{BMO,\mathbb{Q}},\end{aligned}$$

where C depends on R. It implies $\Delta Y=\Delta Z=\Delta Z^0=0$ by Grönwall's inequality and letting θ be small enough. □

5 Wellposedness under deterministic parameters

This section proves the existence of an equilibrium in closed form for models with deterministic parameters;[2] both the MFG and the N-player game are considered. Moreover, we verify that this equilibrium is the unique equilibrium in the space $L^\infty\times L^\infty$, without additional integrability assumptions. As special case we show that the simple equilibrium obtained in [4] is indeed the unique equilibrium in their setting.

5.1 The MFG

We start with the analysis of the MFG; similar results for N-player games require only minor modifications of the arguments that follow.

Uniqueness in $L^\infty\times L^\infty$. To prove the uniqueness of equilibria in the space of bounded processes we denote by $(\widetilde{Y},\widetilde{Z},\widetilde{Z}^0)$ and $(\widetilde{Y}',\widetilde{Z}',\widetilde{Z}^{0\prime})$ two solutions to the BSDE (3.12) in $L^\infty\times L^\infty\times L^\infty$ and set

$$\Delta Y:=\widetilde{Y}-\widetilde{Y}',\quad \Delta Z:=\widetilde{Z}-\widetilde{Z}',\quad \Delta Z^0:=\widetilde{Z}^0-\widetilde{Z}^{0\prime}.$$

[2] They may depend on an initial distribution capturing initial heterogeneity.

The triple $(\Delta Y, \Delta Z, \Delta Z^0)$ satisfies the BSDE

$$\begin{aligned}\Delta Y_t = &\int_t^T \left(A_{1,s}\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\Delta Y_s \middle| \mathcal{F}_s^0 \right] + A_{2,s}\Delta Y_s \right) ds + \int_t^T \left(A_{3,s}\mathbb{E}[A_{4,s}\Delta Z_s|\mathcal{F}_s^0] + A_{5,s}\mathbb{E}[A_{6,s}\Delta Z_s^0|\mathcal{F}_s^0] \right) ds \\ &+ \int_t^T \left(A_{7,s}\Delta Z_s + A_{8,s}\Delta Z_s^0 \right) ds - \int_t^T \Delta Z_s \, dW_s - \int_t^T \Delta Z_s^0 \, dW_s^0,\end{aligned}$$

where the coefficients $A_1, ..., A_8$ all belong to L^∞ since $(\widetilde{Y}, \widetilde{Z}, \widetilde{Z}^0)$ and $(\widetilde{Y}', \widetilde{Z}', \widetilde{Z}^{0'})$ are assumed to belong to $L^\infty \times L^\infty \times L^\infty$. Standard estimates show that

$$\begin{aligned}&\mathbb{E}\left[(\Delta Y_t)^2\right] + \mathbb{E}\left[\int_t^T (\Delta Z_s)^2 + (\Delta Z_s^0)^2 \, ds \right] \\ &= 2\mathbb{E}\left[\int_t^T \left(A_{1,s}\Delta Y_s \mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\Delta Y_s \middle| \mathcal{F}_s^0 \right] + A_{2,s}(\Delta Y_s)^2 + A_{3,s}\Delta Y_s \mathbb{E}\left[A_{4,s}\Delta Z_s|\mathcal{F}_s^0 \right] + A_5 \Delta Y_s \mathbb{E}[A_{6,s}\Delta Z_s^0|\mathcal{F}_s^0] \right) ds \right] \\ &\quad + 2\mathbb{E}\left[\int_t^T \Delta Y_s \left(A_{7,s}\Delta Z_s + A_{8,s}\Delta Z_s^0 \right) ds \right] \\ &\leq C\mathbb{E}\left[\int_t^T (\Delta Y_s)^2 \, ds \right] + \frac{1}{2}\mathbb{E}\left[\int_t^T (\Delta Z_s)^2 + (\Delta Z_s^0)^2 \, ds \right],\end{aligned}$$

which, by Grönwall's inequality, implies that $\Delta Y = \Delta Z = \Delta Z^0$.

We note that $(\pi, c) \in L^\infty \times L^\infty$ implies the first two conditions in (3.29) automatically hold. Moreover, recalling the second point in Remark 3.8, we know that there exists at most one equilibrium investment rate and consumption rate in $L^\infty \times L^\infty$.

Closed form solution. We proceed to give the unique equilibrium in closed form. If the model parameters are deterministic, then $\widetilde{Z} = \widetilde{Z}^0 = 0$, which implies that

$$Z = 0, \qquad Z^0 = -\frac{\theta(1-\gamma)\mathbb{E}\left[\frac{\sigma^0 h}{\gamma(\sigma^2+(\sigma^0)^2)}\right]}{1+\mathbb{E}\left[\frac{\theta(1-\gamma)(\sigma^0)^2}{\gamma(\sigma^2+(\sigma^0)^2)}\right]},$$

and

$$\pi^* = \frac{h}{\gamma(\sigma^2+(\sigma^0)^2)} - \frac{\sigma^0}{\gamma(\sigma^2+(\sigma^0)^2)} \times \frac{\theta(1-\gamma)\mathbb{E}\left[\frac{\sigma^0 h}{\gamma(\sigma^2+(\sigma^0)^2)}\right]}{1+\mathbb{E}\left[\frac{\theta(1-\gamma)(\sigma^0)^2}{\gamma(\sigma^2+(\sigma^0)^2)}\right]}.$$

Taking the above equalities into the driver of $\widetilde{Y}$, we get that

$$\begin{aligned}\widetilde{Y}_t = &-\theta(1-\gamma)\int_t^T \mathbb{E}\left[r_s + \pi_s^* h_s - c_s^* - \frac{1}{2}(\pi_s^*)^2(\sigma_s^2 + (\sigma_s^0)^2) \right] ds \\ &+ \int_t^T \left\{ \frac{(Z_s^0)^2}{2} + (1-\gamma)r_s + \frac{1-\gamma}{2\gamma}\frac{(h_s+\sigma_s^0 Z_s^0)^2}{\sigma_s^2+(\sigma_s^0)^2} + \frac{1-\gamma}{\psi-1}c_s^* - \delta\widetilde{\theta} \right\} ds \\ = &\int_t^T \left\{ \theta(1-\gamma)\mathbb{E}[c_s^*] + \frac{1-\gamma}{\psi-1}c_s^* \right\} ds \qquad (5.1) \\ &-\theta(1-\gamma)\int_t^T \mathbb{E}\left[r_s + \pi_s^* h_s - \frac{1}{2}(\pi_s^*)^2(\sigma_s^2+(\sigma_s^0)^2) \right] ds \\ &+ \int_t^T \left\{ \frac{(Z_s^0)^2}{2} + (1-\gamma)r_s + \frac{1-\gamma}{2\gamma}\frac{(h_s+\sigma^0 Z^0)^2}{\sigma^2+(\sigma^0)^2} - \delta\widetilde{\theta} \right\} ds.\end{aligned}$$

We now show that the above equation can be reduced to a system of Riccati equations. To this end, we set

$$A := \frac{(Z^0)^2}{2} + (1-\gamma)r + \frac{1-\gamma}{2\gamma}\frac{(h+\sigma^0 Z^0)^2}{\sigma^2+(\sigma^0)^2} - \delta\widetilde{\theta} - \theta(1-\gamma)\mathbb{E}\left[r + \pi^* h - \frac{1}{2}(\pi^*)^2(\sigma^2+(\sigma^0)^2) \right]. \quad (5.2)$$

Moreover, from (3.31) we get that

$$\begin{aligned} c^* &= \exp\left(-\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\left\{\mathbb{E}[\psi\log\delta]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\log\alpha\right]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}\widetilde{Y}\middle|\mathcal{F}^0\right]\right\}+\psi\log\delta-\frac{\psi}{\widetilde{\theta}}\log\alpha-\frac{\psi}{\widetilde{\theta}}\widetilde{Y}\right) \\ &= \exp\left(\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}\left[\widehat{Y}\middle|\mathcal{F}^0\right]-\widehat{Y}\right), \end{aligned} \tag{5.3}$$

where

$$\widehat{Y} := -\psi\log\delta+\frac{\psi}{\widetilde{\theta}}\log\alpha+\frac{\psi}{\widetilde{\theta}}\widetilde{Y}.$$

Expressing the optimal consumption plan c^* in terms of $\widehat{Y}$ as shown in (5.3) and recalling the equation (5.1) we see that

$$\begin{aligned} \widehat{Y}' &= -\frac{\psi}{\widetilde{\theta}}\theta(1-\gamma)\mathbb{E}\left[\exp\left(\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}[\widehat{Y}]-\widehat{Y}\right)\right]-\frac{\psi}{\widetilde{\theta}}\frac{1-\gamma}{\psi-1}\exp\left(\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}[\widehat{Y}]-\widehat{Y}\right)-\frac{\psi}{\widetilde{\theta}}A \\ &= -\theta(\psi-1)\mathbb{E}\left[\exp\left(\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}[\widehat{Y}]-\widehat{Y}\right)\right]-\exp\left(\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}[\widehat{Y}]-\widehat{Y}\right)-\frac{\psi}{\widetilde{\theta}}A. \end{aligned}$$

Taking expectations we get that

$$\mathbb{E}[\widehat{Y}]' = -(\mathbb{E}\left[\theta(\psi-1)\right]+1)\mathbb{E}\left[\exp\left(\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}[\widehat{Y}]-\widehat{Y}\right)\right]-\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}A\right]$$

from which we conclude that

$$\frac{\theta(\psi-1)}{\mathbb{E}\left[\theta(\psi-1)\right]+1}\mathbb{E}[\widehat{Y}]'-\widehat{Y}' = \exp\left(\frac{\theta(\psi-1)}{1+\mathbb{E}[\theta(\psi-1)]}\mathbb{E}[\widehat{Y}]-\widehat{Y}\right)-\frac{\theta(\psi-1)}{\mathbb{E}\left[\theta(\psi-1)\right]+1}\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}A\right]+\frac{\psi}{\widetilde{\theta}}A.$$

Let us now put

$$\mathring{Y} := \frac{\theta(\psi-1)}{\mathbb{E}\left[\theta(\psi-1)\right]+1}\mathbb{E}[\widehat{Y}]-\widehat{Y} \quad \text{and} \quad \breve{Y} = \exp(\mathring{Y}).$$

Then,

$$\mathring{Y}' = e^{\mathring{Y}}-\frac{\theta(\psi-1)}{\mathbb{E}\left[\theta(\psi-1)\right]+1}\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}A\right]+\frac{\psi}{\widetilde{\theta}}A.$$

and so the function $\breve{Y}$ satisfies the Riccati equation

$$\breve{Y}' = \breve{Y}^2+\left(-\frac{\theta(\psi-1)}{\mathbb{E}\left[\theta(\psi-1)\right]+1}\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}A\right]+\frac{\psi}{\widetilde{\theta}}A\right)\breve{Y}$$

with the terminal condition

$$\breve{Y}_T = \exp\left(\frac{\theta(\psi-1)}{\mathbb{E}[\theta(\psi-1)]+1}\mathbb{E}\left[-\psi\log\delta+\frac{\psi}{\widetilde{\theta}}\log\alpha\right]-\left(-\psi\log\delta+\frac{\psi}{\widetilde{\theta}}\log\alpha\right)\right) := D. \tag{5.4}$$

The unique solution of the above Riccati equation is given by

$$\breve{Y}_t = D\left\{\exp\left(\int_t^T B_r\,dr\right)+D\int_t^T\exp\left(\int_t^s B_r\,dr\right)\,ds\right\}$$

where

$$B = \left(-\frac{\theta(\psi-1)}{\mathbb{E}\left[\theta(\psi-1)\right]+1}\mathbb{E}\left[\frac{\psi}{\widetilde{\theta}}A\right]+\frac{\psi}{\widetilde{\theta}}A\right). \tag{5.5}$$

Thus, we have shown the following result, which includes [4, Theorem 2.6] as a special case.

Theorem 5.1. *If the model parameters h, σ and σ^0 are deterministic, then the unique equilibrium investment and consumption plan are given as follows:*

$$\pi_t^* = \frac{h_t}{\gamma(\sigma_t^2 + (\sigma_t^0)^2)} - \frac{\sigma_t^0}{\gamma(\sigma_t^2 + (\sigma_t^0)^2)} \times \frac{\theta(1-\gamma)\mathbb{E}\left[\frac{\sigma_t^0 h_t}{\gamma(\sigma_t^2+(\sigma_t^0)^2)}\right]}{1+\mathbb{E}\left[\frac{\theta(1-\gamma)(\sigma_t^0)^2}{\gamma(\sigma_t^2+(\sigma_t^0)^2)}\right]}, \quad t \in [0,T]$$

and

$$c_t^* = D\left\{\exp\left(\int_t^T B_r\, dr\right) + D\int_t^T \exp\left(\int_t^s B_r\, dr\right) ds\right\}, \quad t \in [0,T],$$

where A, D and B are given by (5.2), (5.4) *and* (5.5). *In particular, the simple equilibrium strategy obtained in [4] is unique in $L^\infty \times L^\infty$ in their setting.*

5.2 The N-player game

Having solved the MFG, solving N-player games requires only minor modifications of previously given arguments. In the N-player game we set

$$\nu = \left(\prod_{j\neq i} C^j\right)^{\frac{1}{N-1}}.$$

A lengthy yet relatively straightforward computation yields the equilibrium investment strategy

$$\pi_t^{i,*} = \frac{h_t^i}{\gamma^i(\sigma^i)^2 + \left(\gamma^i - \frac{\theta^i(1-\gamma^i)}{N-1}\right)(\sigma_t^{i0})^2} - \frac{\theta^i(1-\gamma^i)\sigma_t^{i0}}{\gamma^i(\sigma_t^i)^2 + \left(\gamma^i - \frac{\theta^i(1-\gamma^i)}{N-1}\right)(\sigma_t^{i0})^2}\frac{\phi_t^N}{1+\psi_t^N}, \quad t \in [0,T]$$

where

$$\phi^N := \frac{1}{N-1}\sum_{j=1}^N \frac{h^j\sigma^{j0}}{\gamma^j(\sigma^j)^2 + \left(\gamma^j - \frac{\theta^j(1-\gamma^j)}{N-1}\right)(\sigma^{j0})^2}$$

and

$$\psi^N := \frac{1}{N-1}\sum_{j=1}^N \frac{\theta^j(1-\gamma^j)(\sigma^{j0})^2}{\gamma^j(\sigma^j)^2 + \left(\gamma^j - \frac{\theta^j(1-\gamma^j)}{N-1}\right)(\sigma^{j0})^2},$$

and the equilibrium consumption plan

$$c^{i,*} = D^i\left\{\exp\left(\int_t^T B_r^i\, dr\right) + D^i\int_t^T \exp\left(\int_t^s B_r^i\, dr\right) ds\right\},$$

where the coefficients B^i and D^i are given by

$$B^i = b^i A^i - \frac{a^i}{1+\frac{1}{N-1}\sum_{i=1}^N a^i}\frac{1}{N-1}\sum_{i=1}^N b^i A^i,$$

$$b^i = \frac{\frac{\psi^i-1}{1-\gamma^i}}{1-\frac{\theta^i(\psi^i-1)}{N-1}}, \quad a^i = \frac{\theta^i(\psi^i-1)}{1-\frac{\theta^i(\psi^i-1)}{N-1}},$$

with

$$\begin{aligned} A^i = & -\theta^i(1-\gamma^i)\frac{1}{N-1}\sum_{j\neq i}\left\{r^j + \pi^{j,*}h^j - \frac{1}{2}\left((\sigma^j)^2 + (\sigma^{j0})^2\right)(\pi^{j,*})^2\right\} + \frac{(Z^{i0})^2}{2} + \frac{1}{2}\sum_{j\neq i}(Z^{ij})^2 \\ & + (1-\gamma^i)r^i + \frac{1-\gamma^i}{2\gamma^i}\frac{(h^i+\sigma^{i0}Z^{i0})^2}{(\sigma^i)^2+(\sigma^{i0})^2} - \delta^i\widetilde{\theta}^i, \end{aligned}$$

$$Z^{i0} = -\frac{\frac{\theta^i(1-\gamma^i)}{N-1}}{1-\frac{\theta^i(1-\gamma^i)(\sigma^{i0})^2}{(N-1)\gamma^i((\sigma^j)^2+(\sigma^{j0})^2)}}\sum_{j\neq i}\frac{\sigma^{j0}h^j}{\gamma^j\{(\sigma^j)^2+(\sigma^{j0})^2\}}$$
$$+\frac{\frac{\theta^i(1-\gamma^i)}{N-1}}{1-\frac{\theta^i(1-\gamma^i)(\sigma^{i0})^2}{(N-1)\gamma^i((\sigma^j)^2+(\sigma^{j0})^2)}}\cdot\frac{1}{1+\psi^N}\sum_{i=1}^{N}\frac{\frac{\theta^i(1-\gamma^i)(\sigma^{i0})^2}{(N-1)\gamma^i((\sigma^j)^2+(\sigma^{j0})^2)}}{1-\frac{\theta^i(1-\gamma^i)(\sigma^{i0})^2}{(N-1)\gamma^i((\sigma^j)^2+(\sigma^{j0})^2)}}\sum_{j\neq i}\frac{\sigma^{j0}h^j}{\gamma^j\{(\sigma^j)^2+(\sigma^{j0})^2\}},$$

and

$$Z^{ij} = -\frac{\theta^i\gamma^i}{N-1}\frac{\sigma^{i0}Z^{i0}+h^i}{\gamma^i((\sigma^i)^2+(\sigma^{i0})^2)}.$$

References

[1] Abel Cadenillas and Ioannis Karatzas. The stochastic maximum principle for linear, convex optimal control with random coefficients. 33:590–624, 1995.

[2] R. Carmona, J. Fouque, S. Mousavi, and L. Sun. Systemic risk and stochastic games with delay. *arXiv:1607.06373*, 2016.

[3] P. Chan and R. Sircar. Bertrand and cournot mean field games. *Applied Mathematics & Optimization*, 71(3):533–569, 2015.

[4] J. Dianetti, F. Riedel, and L. Stanca. Optimal consumption and investment under relative performance criteria with Epstein-Zin utility. *arXiv:2402.07698*, 2024.

[5] G. dos Reis and V. Platonov. Forward utilities and mean-field games under relative performance concerns. *Particle Systems and Partial Differential Equations*, VI, VII, VIII, 2021.

[6] G. dos Reis and V. Platonov. Forward utility and market adjustments in relative investment-consumption games of many players. *SIAM Journal on Financial Mathematics*, 13(3):844–876, 2022.

[7] D. Duffie and L. G. Epstein. Stochastic differential utility. *Econometrica*, 60(2):353–394, 1992.

[8] N. El Karoui, S. Peng, and M. C. Quenez. A dynamic maximum principle for the optimization of recursive utilities under constraints. *The Annals of Applied Probability*, 11(3):664–693, 2001.

[9] R. Elie, T. Mastrolia, and Dylan Possamaï. A tale of a principal and many many agents. *arXiv:1608.05226*, 2016.

[10] L. G. Epstein and Stanley E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57(4):937–969, 1989.

[11] G. Espinosa and N. Touzi. Optimal investment under relative performance concerns. *Mathematical Finance*, 25(2):221–257, 2015.

[12] C. Frei and G. dos Reis. A financial market with interacting investors: does an equilibrium exist? *Mathematics and Financial Economics*, 4:161–182, 2011.

[13] G. Fu, P. Graewe, U. Horst, and A. Popier. A mean field game of optimal portfolio liquidation. *to appear in Mathematics of Operations Research*, 2021.

[14] G. Fu and C. Zhou. Mean field portfolio games. *to appear in Finance and Stochastics*, 2022.

[15] Guanxing Fu. Mean field portfolio games with consumption. *Mathematics and Financial Economics*, 17(1):79–99, 2023.

[16] Guanxing Fu, Paul P. Hager, and Ulrich Horst. Mean-field liquidation games with market drop-out. *arXiv:2303.05783*, 2023.

[17] Guanxing Fu, U. Horst, and Xia. Portfolio liquidation games with self-exciting order flow. *Mathematical Finance*, 30(4):1020–1065, 2022.

[18] P. Graewe, U. Horst, and R. Sircar. A maximum principle approach to a deterministic mean field game of control with absorption. *SIAM Journal on Control and Optimization*, 60(5):3173–3190, 2022.

[19] Martin Herdegen, David Hobson, and Joseph Jerome. The infinite-horizon investment–consumption problem for epstein–zin stochastic differential utility. ii: Existence, uniqueness and verification for $\vartheta \in (0, 1)$. *Finance and Stochastics*, 27(1):159–188.

[20] R. Hu and T. Zariphopoulou. N-player and mean-field games in Itô-diffusion markets with competitive or homophilous interaction. *Stochastic Analysis, Filtering, and Stochastic Optimization: A Commemorative Volume to Honor Mark HA Davis's Contributions*, 2022.

[21] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.

[22] M. Huang, R. Malhamé, and P. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information and Systems*, 6(3):221–252, 2006.

[23] D. Lacker and A. Soret. Many-player games of optimal consumption and investment under relative performance criteria. *Mathematics and Financial Economics*, 14(2):263–281, 2020.

[24] D. Lacker and T. Zariphopoulou. Mean field and n-agent games for optimal investment under relative performance criteria. *Mathematical Finance*, 29(4):1003–1038, 2019.

[25] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, 2007.

[26] Z. Liang and K. Zhang. Time-inconsistent mean field and \({n}\)-agent games under relative performance criteria. *SIAM J. Financial Math.*, 15(4):1047–1082, 2024.

[27] Anis Matoussi and Hao Xing. Convex duality for epstein–zin stochastic differential utility. *Mathematical Finance*, 28(4):991–1019, 2018.

[28] Étienne Pardoux. BSDEs, weak convergence and homogenization of semilinear PDEs. In *Nonlinear analysis, differential equations and control. Proceedings of the NATO Advanced Study Institute and séminaire de mathématiques supérieures, Montréal, Canada, July 27–August 7, 1998*, pages 503–549. Dordrecht: Kluwer Academic Publishers, 1999.

[29] H. Pham. *Continuous-time stochastic control and optimization with financial applications*. Springer, 2008.

[30] Mark Schroder and Costis Skiadas. Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory*, 89(1):68–126, 1999.

[31] T. Seiferling and F.M. Seifried. Epstein-Zin stochastic differential utility: existence, uniqueness, concavity, and utility gradient. *SSRN*, 2015.

[32] J. Wang. A mean field game of Merton's portfolio problem under relative performance criteria. Master's thesis, Humboldt Universität zu Berlin, 2023.

[33] H. Xing. Consumption–investment optimization with Epstein–Zin utility in incomplete markets. *Finance and Stochastics*, 21:227–262, 2017.