

Alfaro, Rodrigo; Inzunza, Alejandra

**Article**

## Modeling S&P500 returns with GARCH models

Latin American Journal of Central Banking (LAJCB)

**Provided in Cooperation with:**

Center for Latin American Monetary Studies, Mexico City

*Suggested Citation:* Alfaro, Rodrigo; Inzunza, Alejandra (2023) : Modeling S&P500 returns with GARCH models, Latin American Journal of Central Banking (LAJCB), ISSN 2666-1438, Elsevier, Amsterdam, Vol. 4, Iss. 3, pp. 1-11,  
<https://doi.org/10.1016/j.latcb.2023.100096>

This Version is available at:

<https://hdl.handle.net/10419/336377>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*



<https://creativecommons.org/licenses/by-nc-nd/4.0/>

Contents lists available at [ScienceDirect](https://www.sciencedirect.com)

# Latin American Journal of Central Banking

journal homepage: [www.elsevier.com/locate/latcb](http://www.elsevier.com/locate/latcb)

## Modeling S&P500 returns with GARCH models<sup>☆</sup>

Rodrigo Alfaro\*, Alejandra Inzunza

Central Bank of Chile, Agustinas 1180, Santiago, Chile



### ARTICLE INFO

#### JEL classification:

G13  
C52

#### Keywords:

GARCH option pricing models  
VIX  
Tail-risk statistics

### ABSTRACT

This paper provides several estimates of the GARCH models' parameters for the S&P500 index, based on returns and CBOE VIX. Using a daily sample collected from 2007 to 2022, we can conclude that adding the VIX information improves the estimates of the long-term volatility. By providing an external validation of the model using an option-based index reported by the Federal Reserve of Minneapolis, we are able to propose a calibrate model to track the tail-risk of this stock index.

### 1. Introduction

GARCH models were first introduced by Engle (1982) and then widely used in financial econometrics to model asset prices empirically (Singleton, 2002) as they are a powerful tool for characterizing the distribution of asset prices, which oftentimes requires Monte Carlo simulations to obtain relevant statistics, such as tail-risk measures. Duan (1995) established that a GARCH model can also be used to value options by introducing the so-called locally risk-neutral valuation relationship (LRNVR), which needs a simple algebraic manipulation to change the probability measure. For instance, if the returns can be modeled with a standard GARCH in mean, that characterizes the physical measure, then option prices must be evaluated using a non-linear GARCH model that characterizes the risk-neutral measure. The extension to other GARCH models is straightforward as long as the disturbances are normally distributed. Approximation formulae have already been published for several cases (eg., Duan et al., 1999; Heston and Nandi, 2000; Hao and Zhang, 2013; Kannianen et al., 2014). Thus, under both the physical and risk-neutral measures, the shape of these density functions depends on the GARCH model's parameters, as well as the initial level of conditional variance. Since conditional variance is also estimated with the model, the GARCH model's parameters are crucial for proper estimation of density and probability functions.

Hao and Zhang (2013) and Kannianen et al. (2014) pointed out that the VIX, a well-known volatility index generated by the Chicago Board Option Exchange (CBOE) using option prices, can be obtained in closed form from several GARCH models. Therefore, they both recommended including it as an additional variable to estimate GARCH's parameters but reached opposite conclusions. On one hand, Hao and Zhang (2013) argued that the joint estimation generates "unreasonable parameters", and that the implied VIX from the GARCH model does not align with the observed one in "various statistical aspects". On the other hand, Kannianen et al. (2014) claimed that joint estimates "improve option pricing performance" when compared with other methods, such as direct calibration. Although Zhang and Zhang (2020) tried to close that discrepancy by modifying the LRNVR, the data does not support their approach.

In this paper, we utilize several empirical strategies for estimating the parameters of two GARCH models, considering the daily data of the S&P500 returns and VIX from January 2007 to December 2022, and combining both series with different weights. This expands on previous papers that considered equal weights for the returns and the VIX. Additionally, we discuss alternative ways

<sup>☆</sup> We thank the helpful comments of three anonymous referees. The opinions expressed herein are those of the authors and do not necessarily represent the views of the Central Bank of Chile or its Board members.

\* Corresponding author.

E-mail addresses: [ralfaro@bcentral.cl](mailto:ralfaro@bcentral.cl) (R. Alfaro), [ainzunza@bcentral.cl](mailto:ainzunza@bcentral.cl) (A. Inzunza).

to introduce VIX information, including defining the error term with the VIX squared, level of the VIX, and logarithm of the VIX. Based on these methods, we conclude the following two points. First, the parameter associated with the risk-premium increases from 9% (only-returns) to 18% - 27% (depending on the function) when the VIX information is included and a similar weight to the returns is used. Point estimates are higher when only the VIX information is included. These findings are in line with [Hao and Zhang \(2013\)](#), who used a sample collected between January 1990 and August 2009, but extend to a more recent sample and use different functions to incorporate the VIX information (ie., level, squares, logarithms). This does not imply that the model with VIX information generates a higher tail-risk, because other parameters of the model are also involved in such measures. However, it demonstrates that unreasonable values are obtained when the VIX information is weighted at 100%.

Second, the estimate of the unconditional volatility (long-term volatility) decreases when the VIX information is included. This result is robust when we consider the level of VIX, VIX squared, or logarithm of VIX. Thus, our point estimate that only uses returns implies 21% long-term volatility (annual terms), whereas the point estimate that uses the logarithm of VIX (or the level of VIX) stands at around 16%, and 18% when the VIX-squared is considered. These findings differ from [Hao and Zhang \(2013\)](#) and [Kannianen et al. \(2014\)](#), who found that the unconditional volatility increases when the VIX information is included. Interestingly, estimates of this parameter seem to be “reasonable values” even when the VIX information is weighted at 100%. A zoom-in shows that reduction occurs rapidly when the VIX information is included and is already recognizable at 5% weight.

To study the adequacy of our GARCH models, we use a tail-risk indicator provided by the Federal Reserve of Minneapolis (FRM), which approximates a Large Decrease Probability (LDP) in the stock market index over the next 6 months (20% decline). By simulating LDP using our GARCH models, we find that the Threshold GARCH (TGARCH) provides a better fit for the data when we consider the VIX information in logarithm. However, there is room for improvement in the model, which is something that we will leave for future developments.

The article is organized as follows. In [Section 2](#), we introduce the GARCH models used in this research, along with the LRNVR, the closed-form expression for the VIX, and the Monte Carlo approach for other option-based indexes (eg., LDP). In [Section 3](#), we provide estimates of the parameters of the GARCH and TGARCH models based on a daily sample collected from 2007 to 2022, compare between the VIX and the model-implied VIX and conduct an exercise using the LDP series. In [Section 4](#), we discuss policy. In [Section 5](#), we present our conclusions.

## 2. Model setup

In this section, we discuss the following: (i) the key elements of our GARCH in-mean models and the locally risk-neutral valuation relationship (LRNVR), (ii) the relationship between our GARCH models' parameters and the VIX, and (iii) the Monte Carlo simulations approach used to compute other option-based indexes.

### 2.1. GARCH models

In the GARCH models, the log-return of the asset (ie., the log-difference of the asset price,  $P_t$ ), has the following dynamic:

$$\Delta \log P_t = r - \frac{1}{2}h_t + \lambda\sqrt{h_t} + \sqrt{h_t}u_t, \quad (1)$$

where  $r$  is the risk-free short-term interest rate,  $h_t$  is the conditional variance, and  $u_t$  is an independent Gaussian error term (zero mean unit variance) under the physical measure. Depending on  $h_t$ , the log-return for the next period is normally distributed; however, for short or medium runs, the probability distribution function of the cumulated log-returns departs from Gaussian distribution. The equation includes one-half of the conditional variance (the Jensen term) due to the use of logarithmic returns. It also includes the term  $\lambda\sqrt{h_t}$  as the risk-premium component, which is the amount of excess returns that investors require as risk compensation.

The conditional variance is defined recursively through the following equation:

$$h_t = \omega + \alpha h_{t-1} u_{t-1}^2 + \beta h_{t-1} + \gamma h_{t-1} u_{t-1}^2 (u_{t-1} < 0), \quad (2)$$

where  $\omega$  is related to the level of the unconditional variance and is assumed to be positive, and  $\alpha$  controls for the degree of heteroscedasticity and is also assumed to be positive, meaning that when  $\alpha = 0$  the equation collapses to a deterministic expression which implies a constant variance. Moreover,  $\beta$  is used to assess the degree of persistence of the conditional variance and is assumed to be positive, and  $\gamma$  is used to capture the impact of negative shocks in the conditional variance. Thus, we refer to it as a GARCH model when this parameter is zero and as a Threshold GARCH (TGARCH) otherwise.<sup>1</sup> For future reference, we define  $\theta = (\lambda, \omega, \alpha, \beta, \gamma)'$  as the vector of the parameters.

It is important to note that  $h_{t+1}$  is known at time  $t$ . Since  $u_t$  is standard normal we could write the following expression:  $E_t(P_{t+1}) = P_t \exp(r + \lambda\sqrt{h_{t+1}})$ , where the conditional expectation is computed under the physical measure using all of the information available up to time  $t$ . Thus, the parameter  $\lambda$  plays a role in controlling the degree of compensation that investors require to hold this risky asset. It should be noted that in the GARCH and TGARCH models, compensation is expressed in terms of volatility, but it could be defined in other forms. The idea of additional term was introduced by [Engle et al. \(1987\)](#) with the purpose of having a time-variant risk premium. Although the authors suggested using the volatility, they recognized that other functions, such as the logarithm of the

<sup>1</sup> There are alternative functions for the conditional variance in other GARCH models ([StataCorp, 2019](#)). For example, the exponential-GARCH (EGARCH) defines the dynamics of the conditional variance model in logarithms. We leave other GARCH specifications for future research.

**Table 1**  
ML estimates for GARCH models using returns and VIX (Daily data, January 2007 – December 2022).

Column	Returns	Returns & VIX Squared	Returns & VIX Level	Returns & VIX Log			
$\kappa$	0	0.5	1	0.5	1	0.5	1
$\lambda$	0.0931 (0.0147)	0.2656 (0.0214)	1.8709 (0.0228)	0.2090 (0.0258)	1.2711 (0.0402)	0.1767 (0.0256)	0.6137 (0.0398)
$\omega$	2.89.E-06 (2.43.E-07)	2.92.E-06 (1.28.E-07)	2.31.E-06 (4.76.E-08)	2.17.E-06 (7.85.E-08)	1.77.E-06 (3.37.E-08)	1.63.E-06 (7.75.E-08)	1.45.E-06 (3.64.E-08)
$\alpha$	0.1455 (0.0090)	0.0432 (0.0006)	0.0243 (0.0002)	0.0504 (0.0012)	0.0287 (0.0004)	0.0639 (0.0025)	0.0452 (0.0012)
$\beta$	0.8369 (0.0092)	0.9267 (0.0008)	0.8722 (0.0014)	0.9248 (0.0015)	0.9089 (0.0020)	0.9178 (0.0028)	0.9241 (0.0016)
$\rho$	0.9836 (0.0043)	0.9730 (0.0003)	0.9818 (0.0002)	0.9775 (0.0005)	0.9839 (0.0003)	0.9836 (0.0010)	0.9864 (0.0005)
$\sqrt{\bar{h}}$	0.2109 (0.0244)	0.1651 (0.0032)	0.1788 (0.0015)	0.1557 (0.0022)	0.1664 (0.0012)	0.1586 (0.0027)	0.1639 (0.0018)
$VIX^{Mean}$	21.706	20.511	20.837	20.176	20.321	20.301	20.289

Note: This table displays the Maximum Likelihood (ML) estimates obtained from the GARCH model using returns, VIX, or both. Standard errors are in parentheses. Persistence, long-term volatilities, and the mean of the model-implied VIX are also reported.

variance, may fit better.<sup>2</sup> Heston and Nandi (2000) proposed a model based on the conditional variance that particularly specified the dynamic of that variable, which allowed them to have a closed form expression for the European call options using a similar argument as Heston (1993).<sup>3</sup>

2.2. The LRNVR assumption

Duan (1995) introduced the LRNVR, which allows us to value options using the GARCH and TGARCH models. If we define  $v_t = u_t + \lambda$  and assume that it follows a standard Gaussian disturbance under the risk-neutral measure, then use Eq. (1), we will generate the following equation:  $\Delta \log P_t = r - 0.5h_t + \sqrt{h_t}v_t$ , where  $r - 0.5h_t$  is the expected value of the asset price under the risk-neutral measure, which implies that asset price is a discounted martingale, ensuring that there is not arbitrage opportunities. This change in the measure also affects the conditional variance:  $h_t = \omega + \alpha h_{t-1}(v_{t-1} - \lambda)^2 + \beta h_{t-1} + \gamma h_{t-1}(v_{t-1} - \lambda)^2(v_{t-1} < \lambda)$ , which adds a non-linear asymmetric component in line with Table 1 from Engle and Ng (1993).<sup>4</sup> After some algebra, under the risk-neutral measure, the equation of the conditional variance can be expressed in the following way:  $h_t = \omega + \rho h_{t-1} + \alpha h_{t-1}e_{t-1}$ , where  $e_{t-1}$  is an error term with a mean of zero and finite variance, and  $\rho$  is the persistence of the conditional variance. Following Duan et al. (2006) and assuming that the disturbances are normally distributed, we utilize the following equation:  $\rho = \alpha(1 + \lambda^2) + \beta + \gamma[\lambda\phi(\lambda) + (1 + \lambda^2)\Phi(\lambda)]$ , where  $\phi()$  and  $\Phi()$  are the density and cumulative functions of the standard normal distribution, respectively. Assuming  $0 < \rho < 1$ , we can also obtain the unconditional variance ( $\bar{h}$ ):  $\bar{h} = \omega/(1 - \rho)$ . Note that the conditional expectation of  $h_{t+2}$  has the following expression:  $E_t^*(h_{t+2}) = \bar{h}(1 - \rho) + \rho h_{t+1}$ , where  $E_t^*(\cdot)$  is the conditional expectation computed under the risk-neutral measure using all of the information available up to time  $t$ . Iterating this expression, we have the following formula:  $E_t^*(h_{t+k}) = \bar{h}(1 - \rho^{k-1}) + \rho^{k-1}h_{t+1}$ .

In summary, under the physical measure, the volatility term in the return equation implies that the model is GARCH and TGARCH in mean, and under the risk-neutral measure, the GARCH and TGARCH model imposes an expected value according to the risk-free rate, but it implies a change in the conditional variance equation.

2.3. The CBOE volatility Index (VIX)

VIX is an index provided by the CBOE that is based on the square root of a weighted average of one-month option prices. The goal of VIX is to approximate the expected value of the variance swap; thus, its interpretation should be in squared terms. Consequently, the relationship between the daily variance based on the VIX ( $X_t$ ) and the conditional variance ( $h_{t+1}$ ) is computed using the following

<sup>2</sup> “The choice of the standard deviation represents the assumption that changes in variance are reflected less than proportionally in the mean. Empirically, the log of h(t) is found to be even better” (Engle et al., 1987).

<sup>3</sup> Given the assumption that the generating function is a closed form, the option price can be expressed in similar terms. However, the procedure requires researchers to solve an integral over the complex space. Several alternatives have been proposed to speed up this part; for example, Mazzoni (2010) suggested an approximation of the generating function and uses it to obtain the skewness and kurtosis in a closed form. Along with these terms, the author proposed using them in a Gram-Charlier density function.

<sup>4</sup> It should be noted that Engel and Ng (1993) did not specify the return equation of the GARCH models. Thus, some of the specifications of the risk premium are not consistent with the conditional variance equation, in the sense that some parameters are not identified.

equation (Kannianen et al., 2014):

$$X_t \equiv \frac{1}{\tau} \left( \frac{VIX_t}{100} \right)^2 \cong \frac{1}{n} \sum_{k=1}^n E_t^*(h_{t+k}), \tag{3}$$

where  $n$  stands for the maturity of the VIX and  $\tau$  stands for the annualizing parameter. We use  $n = 22$  for trading days and  $\tau = 365$  according to VIX calendar count convention. The conditional expectation is computed under the risk-neutral measure and can be obtained in closed form for the GARCH and TGARCH models presented above by plugging in the expression obtained in the previous section:

$$X_t = \left[ 1 - \frac{1 - \rho^n}{n(1 - \rho)} \right] \bar{h} + \left[ \frac{1 - \rho^n}{n(1 - \rho)} \right] h_{t+1}. \tag{4}$$

For the empirical section, we consider the following error term:  $w_t = g(X_t) - g(A + Bh_{t+1})$ , where  $A$  and  $B$  is determined according to (4) and  $g()$  is a given function (identity, square root, or logarithm).

### 2.4. The federal reserve of minneapolis (FRM) indexes

The FRM computes several indexes based on option-prices with 6- and 12-months expirations, including skewness, kurtosis, and probabilities of extreme events. Similar to Shimko (1993) and Malz (1997), the available option prices are summarized into implied-volatilities and strike-price space. A cubic spline is used to characterize this relationship, which is then used to generate call prices. Finally, indexes are computed either in accordance with statistical formulae (eg., skewness and kurtosis) or by conducting a second derivative of call prices (eg., risk-neutral density).

In contrast to the VIX, the option-based indexes obtained from FRM do not have a closed form equation that relate them with the parameters of the GARCH or TGARCH models. Duan et al. (1999) provided formulae for the first four moments of the GARCH model, whereas Hongwiengjan and Thongtha (2021) provided similar expressions for the TGARCH model.<sup>5</sup> For our implementation, we follow Duan (1995) by approximating the expectation with  $I$  in Monte Carlo replications of the asset-price process:

$$E_t^*[f(P_{t+n})] = \frac{1}{I} \sum_{i=1}^I f(P_{(i),t+n}), \tag{5}$$

where  $P_{(i),t+n}$  is the terminal value ( $t + n$ ) of the  $i$ -simulated path, and  $f()$  is an arbitrary function. For the tail-risk, we compute the probability value of falling below a given threshold or the LDP. Following the ones published by the FRM, we choose a threshold value of 80%, which is equivalent to a 20% loss.

## 3. Empirical analysis

In this section, we describe the variables used in the estimation, discuss the procedure and the results obtained under different setups, and compare the results implied by the models with both the CBOE VIX and the FRM LDP.

### 3.1. Data

To estimate the GARCH models, we use the time-series data from the daily closing prices of the S&P500, VIX, and LDP, ranging from January 2, 2007, to December 30, 2022. For the daily risk-free interest rate, we use the three-month (secondary market) Treasury bill rate retrieved from the Federal Reserve Bank of St. Louis website. These series have 4,175 daily observations available for study.<sup>6</sup>

We obtain the continuously compounded returns of the S&P500 series after log-differencing, which are portrayed in Fig. 1 (left). The daily mean of the series is 0.02%, with a standard deviation of 0.013, exhibiting a negative skewness of -0.521 and kurtosis of 15.344. In Fig. 1 (right), we provide the density distribution of the returns, with a fitted normal (black line). A simple examination suggests that returns do not follow a normal distribution; thus, a GARCH model seems to be appropriate for returns.

Fig. 2(a) presents the VIX series, which is calculated from a portfolio of S&P500 options with a one-month maturity which is associated with perceived risk, presenting easily noticeable spikes at and after the 2008 financial crisis, and at the beginning of 2020 during the surge of the COVID-19 pandemic. The mean of the series is 20.28 (black line), with a standard deviation of 9.3. Fig. 2(b) displays the LDP for 6 months; these estimates were retrieved from the website of the Federal Reserve of Minneapolis, with 383 biweekly observations available for the January 2007 – December 2022 period. The biweekly mean for the sample period is 9.11%, with a standard deviation of 4.67%. A simple examination suggests that VIX is a key component in LDP; thus, a measure of conditional variance can be used to match the LDP measure.

<sup>5</sup> Duan et al. (2006) also provided formulae for the first four moments of the EGARCH model.

<sup>6</sup> For non-trading days related to stock market holidays, we compute the value of the last trading day available.

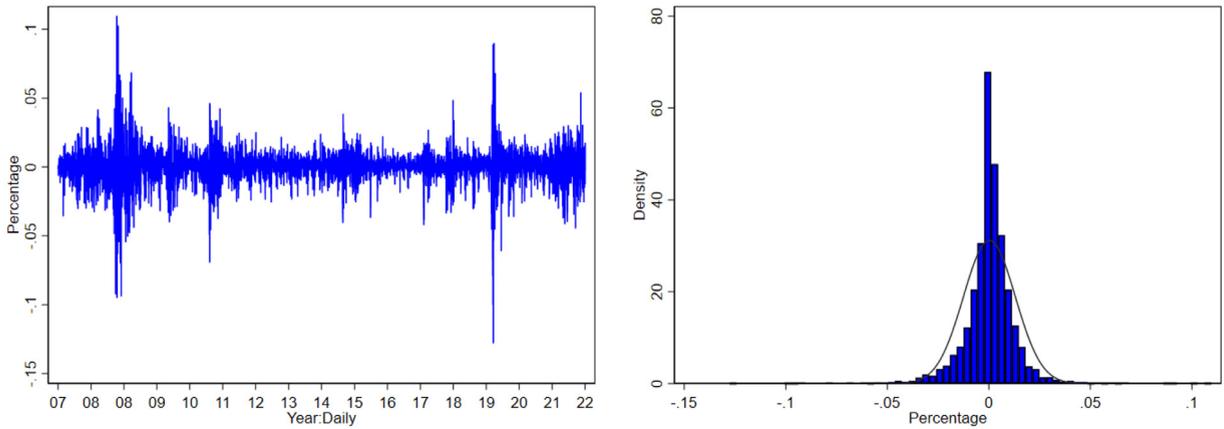
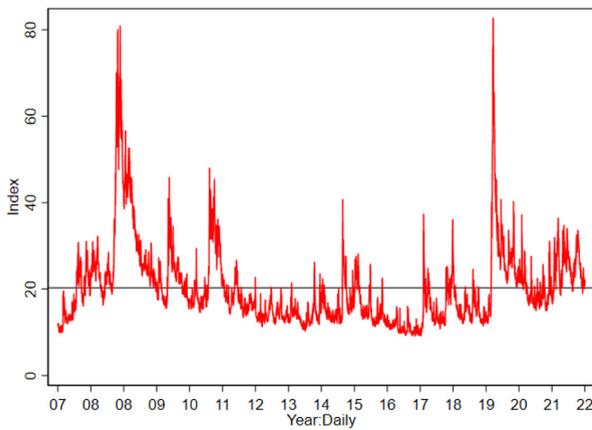


Fig. 1. S&P500 returns.  
Source: Bloomberg

a. CBOE VIX and mean



b. FRM LDP

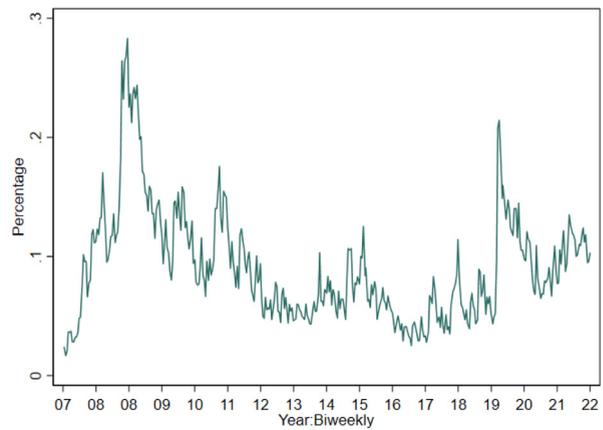


Fig. 2. Option-based Indexes for S&P500.  
Federal Reserve of Minneapolis  
Source: Bloomberg

### 3.2. Joint likelihood

To estimate the GARCH model parameters, we use the Maximum Likelihood (ML) approach for returns and VIX. For the returns data, we use the log-likelihood function  $\log L_{RET}$ :

$$\log L_{RET}(\theta) = -\frac{1}{2} \sum_{t=2}^T \left[ \log h_t + \frac{(\Delta \log P_t - r_t + 0.5h_t - \lambda\sqrt{h_t})^2}{h_t} \right] \tag{6}$$

For the VIX data, we use the following corresponding function:<sup>7</sup>

$$\log L_{VIX}(\theta) = -\frac{1}{2} \left[ (T-1) \log s^2 + \frac{1}{T} \sum_{t=2}^T \frac{w_t^2}{s^2} \right] \tag{7}$$

where  $w_t = g(X_t) - g(A + Bh_{t+1})$  and  $s^2$  is the variance of  $w_t$ . Thus, the joint likelihood of returns and VIX could be defined as a linear combination of both log-likelihood functions:

$$\log L_{TOT}(\theta) = (1 - \kappa) \log L_{RET}(\theta) + \kappa \log L_{VIX}(\theta) \tag{8}$$

<sup>7</sup> Hao and Zhang (2013), Kannianen et al. (2014), and Zhang and Zhang (2020) chose  $g()$  as the squared root function. However, we also consider the identity and the logarithm functions for completeness. Thus, we have results for the squared of VIX, level of VIX, and logarithm of VIX.

**Table 2**  
ML estimates for TGARCH models using returns and VIX (Daily data, January 2007 – December 2022).

Column	Returns	Returns & VIX Squared	Returns & VIX Level	Returns & VIX Log			
$\kappa$	0	0.5	1	0.5	1	0.5	1
$\lambda$	0.0490 (0.0160)	0.1638 (0.0245)	2.4342 (0.0143)	0.1125 (0.0289)	1.0152 (0.0589)	0.0832 (0.0288)	0.4642 (0.0545)
$\omega$	2.84.E-06 1.98.E-07	1.85.E-06 (8.98.E-08)	2.34.E-06 (4.83.E-08)	1.73.E-06 (6.74.E-08)	1.52.E-06 (3.14.E-08)	1.49.E-06 (6.94.E-08)	1.38.E-06 (3.50.E-08)
$\alpha$	0.0218 (0.0046)	-0.0242 (0.0011)	0.4336 (0.0203)	-0.0034 (0.0025)	-0.0221 (0.0031)	0.0198 (0.0051)	0.0252 (0.0040)
$\beta$	0.8575 (0.0077)	0.9554 0.0012	0.8298 0.0013	0.9471 0.0015	0.9273 0.0025	0.9334 0.0026	0.9326 0.0018
$\gamma$	0.1903 (0.0126)	0.0790 (0.0013)	-0.4118 (0.0203)	0.0631 (0.0036)	0.0530 (0.0034)	0.0542 (0.0068)	0.0235 (0.0053)
$\rho$	0.9821 (0.0038)	0.9815 (0.0003)	0.9814 (0.0001)	0.9812 (0.0004)	0.9860 (0.0003)	0.9842 (0.0008)	0.9866 (0.0005)
$\sqrt{h}$	0.2003 (0.0241)	0.1586 (0.0033)	0.1782 (0.0015)	0.1522 (0.0022)	0.1656 (0.0014)	0.1544 (0.0027)	0.1608 (0.0020)
$VIX^{Mean}$	21.71	20.51	20.84	20.18	20.33	20.30	20.29

Note: This table displays the Maximum Likelihood (ML) estimates obtained from the TGARCH model using returns, VIX, or both. Standard errors are in parentheses. Persistence, long-term volatilities, and the mean of the model-implied VIX are also reported.

where  $\kappa$  is the parameter controlling the weight assigned to the VIX data. Hao and Zang (2013), Kannianen et al. (2014), and Zhang and Zhang (2020) consider three cases ( $\kappa = 0$ ,  $\kappa = 0.5$ , and  $\kappa = 1$ ) that are associated with only returns, returns and VIX (equally weighted), and only VIX. We extend these choices by running the optimization problem with several values of  $\kappa$ .

### 3.3. Main results

Table 1 provides a selected set of parameters for the GARCH models that weight the returns and VIX series and impose  $\gamma = 0$ , with corresponding standard errors in parentheses. First, we provide estimates using only returns. Second, we increase the relative importance of the VIX (by raising  $\kappa$ ) and show the different combinations of weights and VIX types (squares, levels, and logarithms).

As we put more relative weight onto the VIX data compared to the returns, the risk-premium parameter ( $\lambda$ ) increases from 9% (column 1) to 187% (column 3), which implies that the adjusted GARCH model (under the risk-neutral measure) has a high degree of asymmetry. However, this is compensated by the heteroscedasticity parameter ( $\alpha$ ), which decreases from 15% to 2%. The increase in the risk-premium parameter is in line with the results of previous papers, and comprises an increase in the persistence parameter ( $\rho$ ) under the risk-neutral measure. These findings are somewhat robust for different functions applied to the VIX; however, the use of logarithms implies a smooth transition in the parameter estimates, which we will discuss next.

The parameters for the TGARCH models ( $\gamma \neq 0$ ) are provided in Table 2. Like the previous table, we start by using only returns, then we increase the relative importance of the VIX and present different combinations of weights and VIX types. All columns show that parameter  $\gamma$  is significant. However, before choosing between the GARCH or TGARCH models, we will discuss which function of VIX seems to be the most appropriate.

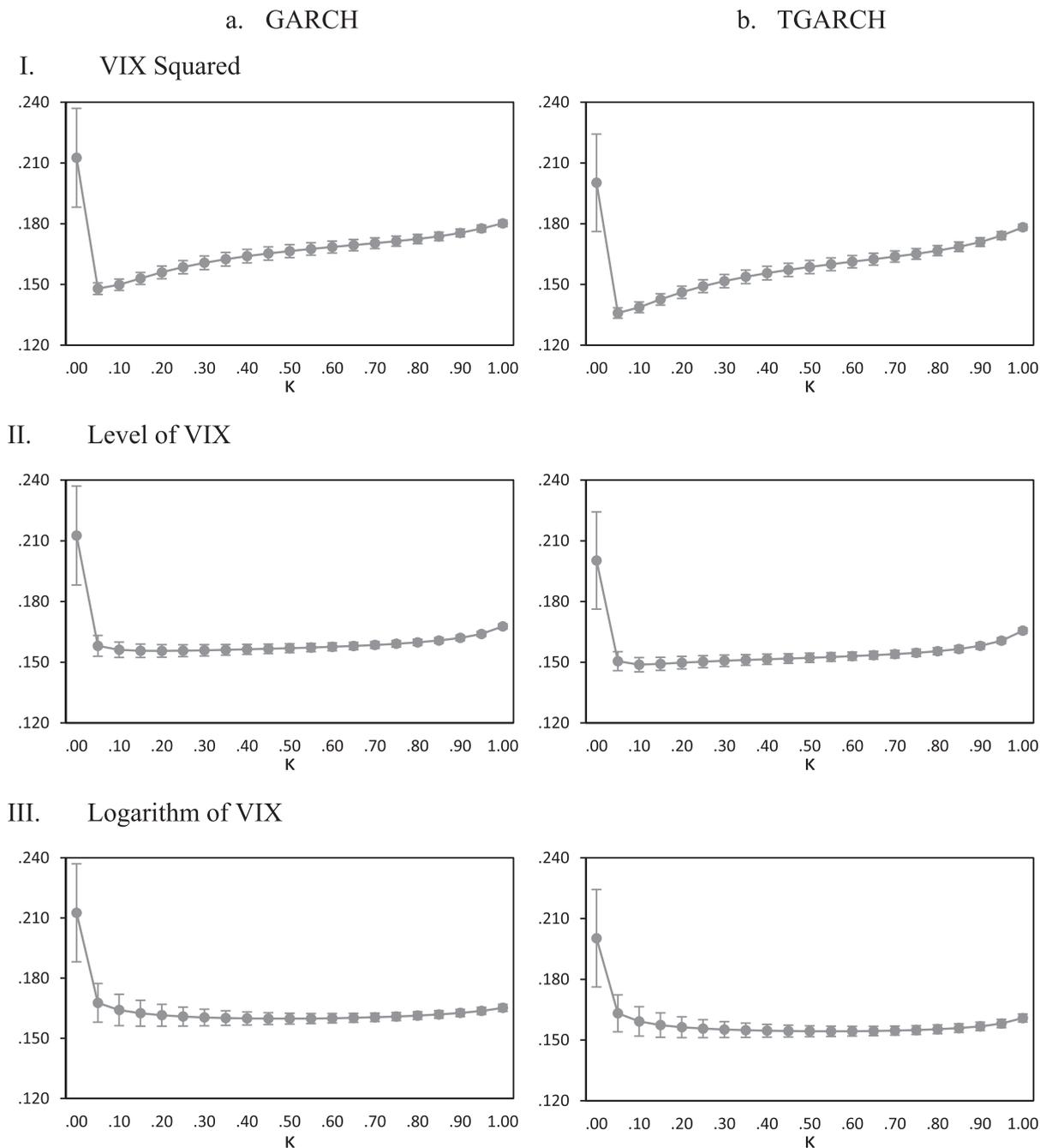
Similar to the GARCH model, the TGARCH model yields an increase in the risk-premium parameter ( $\lambda$ ), showing that equal-weights ( $\kappa = 0.5$ ) provide reasonable estimates for this parameter. For example, when we consider returns only, it stands at 4.9%, whereas for  $VIX^2$  only, it reaches 243% (for this case,  $\kappa = 0.5$  implies 16%). Similar to GARCH model results, the logarithm of the VIX seems to be the most appropriate; therefore, we will expand on this argument a bit further.

Fig. 3 shows the estimates of long-term volatility (LTV) at different values of  $\kappa$  for both the GARCH and TGARCH models, with their respective confidence intervals.<sup>8</sup> We observe that weighting the information of the VIX a bit reduces the LTV. When more weight is considered, these estimates increase, but the LTV remains significantly low compared to the cases when only returns are used for the estimation.<sup>9</sup>

Using the parameters in Tables 1 and 2, we estimate the model-implied VIX using Eq. (4) and compare it to the original VIX series. This is portrayed in Fig. 4, with the returns only case in panel a and the VIX functions in panels b to d; the latter panels all include equally weighted returns and VIX ( $\kappa = 0.5$ ) for ease of comparison. The correlation among the VIX series and the implied VIX are displayed under each figure. Tables 3 and 4 present the corresponding descriptive statistics. Fig. 4 shows that for both models,

<sup>8</sup> Notably, by increasing the importance of the VIX information, we yield more precise estimates of both measures. This is in line with previous papers and is derived by adding the information of the VIX, which has a similar effect that “increases” the sample size.

<sup>9</sup> Indeed, when  $\kappa = 0.05$ , we have a LTV of 14.8% using VIX squared and the GARCH model. The same occurs for other VIX functions and for the TGARCH model.

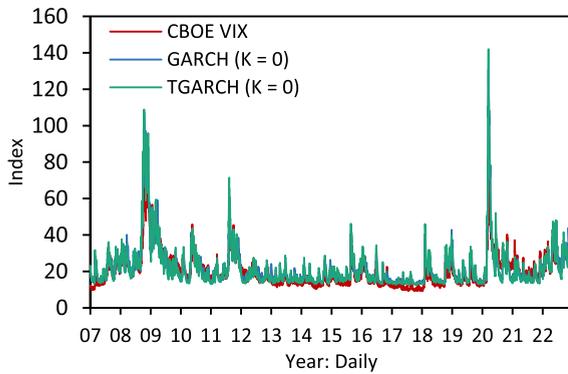


**Fig. 3.** Estimated LTV using returns and VIX  
 Note: This figure displays the estimated long-term volatility (LTV) obtained from the GARCH and TGARCH models at different VIX weights and VIX functions. It also provides 95% confidence intervals for the estimates.

the parameters obtained from equally weighting returns and the logarithm of VIX yield the highest correlation among the series. Additionally, both models yield similar results, which is in line with previous papers.

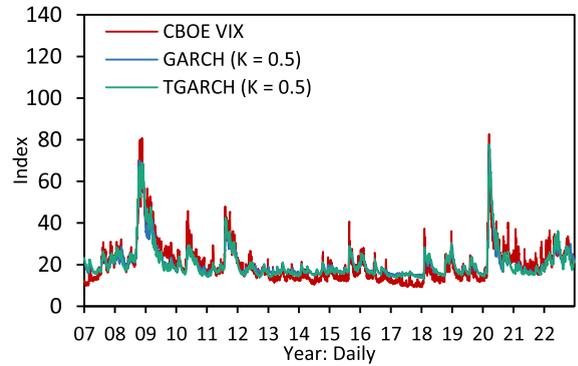
Depending on the chosen model and the VIX function, the implied VIX matches certain aspects of the CBOE VIX distribution and its statistical properties. As shown in Table 3, the logarithm of VIX produces an implied VIX with the least difference between the original series in terms of its mean, skewness, and kurtosis, along with the lowest part of the distribution. However, as shown in Table 4, the TGARCH model using the logarithm of VIX function closely matches the mean, the lowest part of the distribution, and the 90<sup>th</sup> percentile.

a. Returns



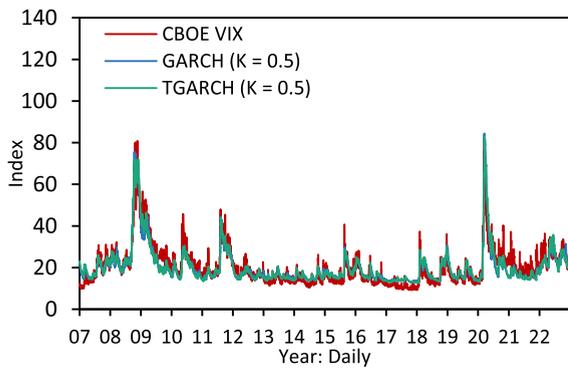
corr = 0.8977 / 0.8972

b. VIX Squared



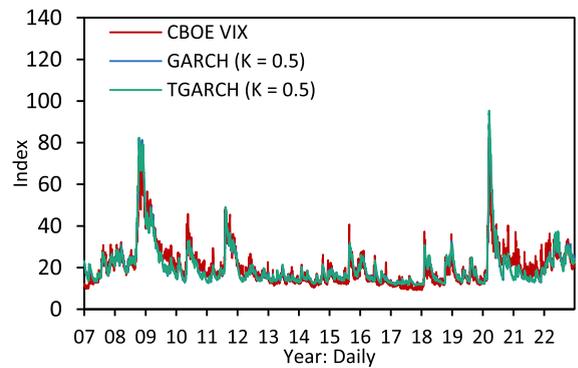
corr = 0.9099 / 0.9170

c. Level of VIX



corr = 0.9131 / 0.9218

d. Logarithm of VIX



corr = 0.9250 / 0.9231

Fig. 4. Implied VIX compared to CBOE VIX

Note: This figure displays implied VIX in the GARCH and TGARCH models by equally weighing-in returns and VIX, compared with the CBOE VIX. Under each figure, we report the correlation among the series, first between GARCH implied VIX and CBOE VIX, and second between TGARCH implied VIX and CBOE VIX.

Table 3

GARCH implied VIX distribution compared to CBOE VIX.

	CBOE VIX	a. Returns	b. VIX Squared	c. Level VIX	d. Log VIX
Mean	20.281	21.888	20.645	20.130	<b>20.281</b>
SD	9.297	11.345	7.772	<b>8.864</b>	10.221
Skewness	2.291	3.825	3.700	3.549	<b>3.291</b>
Kurtosis	7.759	20.336	17.157	16.033	<b>13.851</b>
AR1	0.977	0.980	0.994	0.994	0.993
AR10	0.865	0.807	0.899	0.898	0.890
AR30	0.666	0.480	0.608	0.609	0.597
Min	9.140	12.711	14.707	12.950	<b>11.525</b>
p10	12.160	14.235	15.680	14.216	<b>13.149</b>
p25	13.950	15.388	16.474	15.239	<b>14.450</b>
Median	17.840	18.519	<b>18.242</b>	17.408	17.087
p75	23.700	<b>24.219</b>	22.012	21.956	22.705
p90	30.608	<b>31.599</b>	27.075	27.783	29.342
Max	82.690	131.238	77.681	<b>84.379</b>	89.223

Note: This table displays the GARCH model-implied VIX and compares it with the CBOE VIX in regards to distribution and statistical properties. The bold values indicate the least percentage difference compared to the original series.

**Table 4**  
TGARCH implied VIX distribution compared to CBOE VIX.

	CBOE VIX	a. Returns	b. VIX Squared	c. Level VIX	d. Log VIX
Mean	20.281	21.706	20.511	20.176	<b>20.301</b>
SD	9.297	11.822	8.204	<b>8.922</b>	10.281
Skewness	2.291	3.801	<b>3.223</b>	3.299	3.321
Kurtosis	7.759	20.691	<b>12.867</b>	13.686	14.283
AR1	0.977	0.974	0.991	0.993	0.993
AR10	0.865	0.770	0.900	0.904	0.895
AR30	0.666	0.445	0.659	0.649	0.620
Min	9.140	12.734	13.227	12.766	<b>11.623</b>
p10	12.160	13.795	14.946	14.107	<b>13.110</b>
p25	13.950	14.883	15.889	15.152	<b>14.386</b>
Median	17.840	17.982	<b>17.762</b>	17.301	17.121
p75	23.700	<b>24.315</b>	22.111	22.064	22.706
p90	30.608	32.387	27.671	27.737	<b>29.284</b>
Max	82.690	142.007	77.764	<b>83.187</b>	95.461

Note: This table displays the GARCH model-implied VIX and compares it with CBOE VIX in regards to distribution and statistical properties. The bold values indicate the least percentage difference compared to the original series.

**Table 5**  
GARCH estimated probability descriptives compared to FRM LDP.

	FRM LDP	$\kappa = 0.25$	$\kappa = 0.50$	$\kappa = 0.75$	$\kappa = 1.00$
Mean	0.091	0.063	0.065	0.068	<b>0.074</b>
SD	0.047	0.037	0.039	<b>0.039</b>	0.039
Skewness	1.340	3.831	3.660	3.501	<b>3.248</b>
Kurtosis	2.350	18.003	16.315	14.764	<b>12.607</b>
Min	0.017	<b>0.031</b>	0.031	0.035	0.042
p10	0.043	0.040	0.039	<b>0.043</b>	0.050
p25	0.057	0.046	0.047	0.048	<b>0.054</b>
Median	0.080	0.053	0.055	0.056	<b>0.062</b>
p75	0.116	0.065	0.070	0.072	<b>0.079</b>
p90	0.147	0.097	0.100	0.106	<b>0.112</b>
Max	0.283	0.314	0.324	0.319	<b>0.308</b>

Note: This table displays the GARCH model estimated tail probability and compares it with the LDP informed by the FRM in regards to its distribution and statistical properties. We highlight the values with the least percentage difference compared with the original series.

As result of this comparison, using the logarithm of the VIX seems to be more appropriate in both the GARCH and TGARCH models, although the former has a slightly higher correlation between implied VIX and CBOE VIX. To complete the diagnostic, we identified the Schwarz criteria of these models as -19683 and -19801, respectively. Thus, for the overall fit (returns and VIX), the TGARCH model is more appropriate, and it is still good enough for approximating the VIX.

### 3.4. External validity exercise

In this section, we compare the probability of a 20% decrease using the parameters obtained from different weights of the logarithm of VIX (as it fits the VIX best) against the LDP series. Fig. 5 presents our models implied probabilities, and Tables 5 and 6 describe the distributions and properties.<sup>10</sup> A visual inspection suggests that there is room for improvement regarding the model used to approximate the FRM LDP series. The highest correlation between the FRM LDP and the model-implied 20% LDP is obtained using the TGARCH model and VIX only information ( $\kappa = 1$ ), demonstrating that the GARCH model offers slightly lower correlations in all cases (see Fig. 5). Moreover, when  $\kappa = 1$ , the TGARCH model more closely fits the mean, skewness, and kurtosis of the original series, as well as the 25<sup>th</sup>, 50<sup>th</sup>, 75<sup>th</sup>, and 90<sup>th</sup> percentile of its distribution (see Table 6). Thus, the TGARCH model seems to be more appropriate.

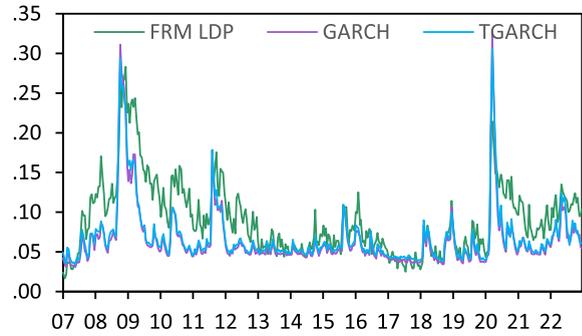
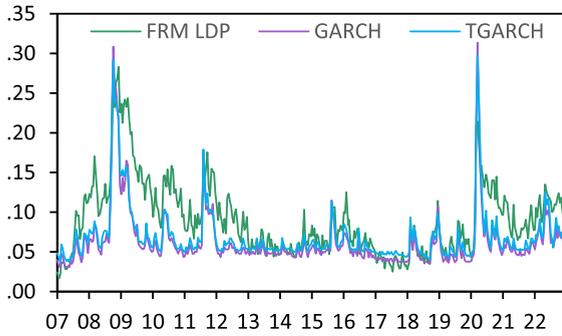
## 4. Policy discussion

In the previous sections, we show that, based on the work of Duan (1995), the GARCH and TGARCH models can be used to combine two sources of information: returns and option-prices (VIX). This has practical applications because, in general, there are

<sup>10</sup> For each implied LDP, we use the actual returns and risk-free interest rate described in Section 3.1, along with 2000 Monte Carlo simulations.

a.  $\kappa = 0.25$

b.  $\kappa = 0.50$

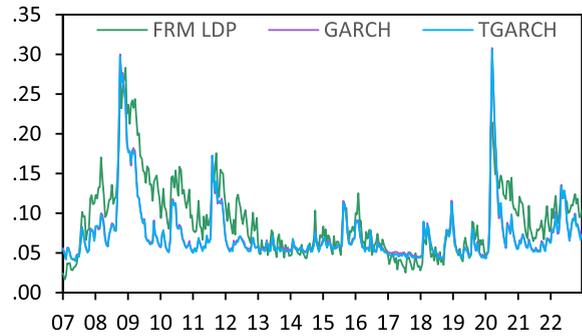
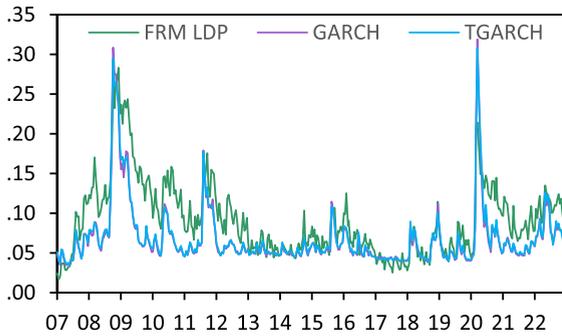


corr = 0.791 / 0.800

corr = 0.807 / 0.819

c.  $\kappa = 0.75$

d.  $\kappa = 1.00$



corr = 0.816 / 0.826

corr = 0.832 / 0.835

**Fig. 5.** Estimated tail probabilities compared to FRM LDP

Note: This figure displays the GARCH and TGARCH models implied 20% decrease probability by weighing-in different combinations of returns and VIX against the FRM LDP. Under each figure, we report the correlation between the series, first between GARCH implied probabilities and FRM LDP, and second between TGARCH implied probabilities and FRM LDP.

**Table 6**

TGARCH estimated probability descriptives compared to FRM LDP.

	FRM LDP	$\kappa = 0.25$	$\kappa = 0.50$	$\kappa = 0.75$	$\kappa = 1.00$
Mean	0.091	0.069	0.069	0.069	<b>0.073</b>
SD	0.047	0.035	0.038	0.039	<b>0.039</b>
Skewness	1.340	3.553	3.369	3.298	<b>3.193</b>
Kurtosis	2.350	15.314	13.627	12.925	<b>12.130</b>
Min	0.017	0.037	0.035	<b>0.035</b>	0.040
p10	0.043	0.047	0.043	<b>0.043</b>	0.048
p25	0.057	0.052	0.050	0.049	<b>0.052</b>
Median	0.080	0.060	0.058	0.059	<b>0.061</b>
p75	0.116	0.072	0.073	0.074	<b>0.079</b>
p90	0.147	0.099	0.103	0.107	<b>0.113</b>
Max	0.283	<b>0.298</b>	0.306	0.308	0.306

Note: This table displays the TGARCH model estimated tail probability and compares it with the LDP informed by the FRM in regards to its distribution and statistical properties. We highlight the values with the least percentage difference compared with the original series.

several indicators for any given financial market. Therefore, we propose using the GARCH and TGARCH models as a framework for combining these inputs. In our empirical section, we expanded on the results from other studies available in the literature by (i) updating the sample (2007-2022), (ii) considering different ways to incorporate the VIX information (level, squares, logarithms), and (iii) using different weights in the joint likelihood. Thus, our results can be used as robustness tests for the GARCH and TGARCH models that we examined. Finally, we use an additional option-based indicator (ie., LDP) reported by the FRM to provide an external validation of the models. Based on all of these exercises, we conclude that the proper way to combine the information is to consider the logarithm of the VIX and to use the TGARCH model as a framework for filtering the S&P500 data. A similar approach can be applied to any other financial market.

## 5. Conclusions

GARCH models combined with Monte Carlo simulations are useful tools for modelling asset prices, which can, in turn, be used to compute tail-risk measures. We use log-return information and VIX to estimate GARCH models and find empirical results that are in line with previous papers in the area (Hao and Zhang, 2013; Kannianen et al., 2014; Zhang and Zhang, 2020) including VIX information, and contribute to the literature by broadening this approach with different combinations of VIX and returns. We propose a calibrate TGARCH model that can be used to compute tail-risk measures that match current estimates of large decreases. Our framework can be applied to other financial markets, although there is room for improvement. We are considering this for future developments, as well as other GARCH models such as EGARCH.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References

- Duan, J.C., 1995. The GARCH option pricing model. *Math. Finance* 5 (1), 13–32. doi:[10.1111/j.1467-9965.1995.tb00099.x](https://doi.org/10.1111/j.1467-9965.1995.tb00099.x).
- Duan, J.C., Gauthier, G., Simonato, J.G., 1999. An analytical approximation for the GARCH option pricing model. *J. Comput. Finance* 4 (2), 75–116. doi:[10.21314/JCF.1999.033](https://doi.org/10.21314/JCF.1999.033).
- Duan, J.C., Gauthier, G., Simonato, J.G., Sasseville, C., 2006. Approximating the GJR-GARCH and EGARCH option pricing models analytically. *J. Comput. Finance* 9 (3), 1755–2850. doi:[10.21314/JCF.2006.156](https://doi.org/10.21314/JCF.2006.156).
- Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50 (4), 987–1008. doi:[10.2307/1912773](https://doi.org/10.2307/1912773).
- Engle, R.F., Lilien, D.M., Robins, R.P., 1987. Estimating time varying risk Premia in the term structure: the ARCH-M Model. *Econometrica* 55 (2), 391–407. doi:[10.2307/1913242](https://doi.org/10.2307/1913242).
- Engle, R.F., Ng, V., 1993. Measuring and testing the impact of news on volatility. *J. Finance* 48 (5), 1749–1778. doi:[10.1111/j.1540-6261.1993.tb05127.x](https://doi.org/10.1111/j.1540-6261.1993.tb05127.x).
- Heston, S.L., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.* 6, 327–343. doi:[10.1093/RFS/6.2.327](https://doi.org/10.1093/RFS/6.2.327).
- Heston, S.L., Nandi, S., 2000. A closed-form GARCH option valuation model. *Rev. Financial Studies* 13, 585–625. doi:[10.2139/ssrn.210009](https://doi.org/10.2139/ssrn.210009).
- Hao, J., Zhang, J., 2013. GARCH option pricing models, the CBOE VIX, and variance risk premium. *J. Financ. Econ.* 11 (3), 556–580. doi:[10.1093/jfinec/nbs026](https://doi.org/10.1093/jfinec/nbs026).
- Hongwiengjan, W., Thongtha, D., 2021. An analytical approximation of option prices via TGARCH model. *Econ. Res.* 34 (1), 948–969. doi:[10.1080/1331677X.2020.1805636](https://doi.org/10.1080/1331677X.2020.1805636).
- Kannianen, J., Lin, B., Yang, X., 2014. Estimating and using GARCH models with VIX data for option valuation. *J. Bank. Finance* 43, 200–211. doi:[10.1016/j.jbankfin.2014.03.035](https://doi.org/10.1016/j.jbankfin.2014.03.035).
- Malz, A., 1997. Option-implied probability distributions and currency excess return. *Staff Rep.* 32. doi:[10.2139/ssrn.943500](https://doi.org/10.2139/ssrn.943500), Federal Reserve New York.
- Mazzoni, T., 2010. Fast analytic option valuation with GARCH. *J. Deriv.* 18 (1), 18–38. doi:[10.3905/jod.2010.18.1.018](https://doi.org/10.3905/jod.2010.18.1.018).
- Shimko, D., 1993. Bounds of probability. *Risk* 6, 33–37.
- Singleton, K. (2002) “Empirical Dynamic Asset Pricing: Model Specification and Econometric Assessment”.
- StataCorp, L.L.C., 2019. *Stata Time-Series Reference Manual, Release 16*. Stata Press.
- Zhang, W., Zhang, J.E., 2020. GARCH option pricing models and the variance risk premium. *J. Risk Financ. Manag.* 13 (51), 1–21. doi:[10.3390/jrfm13030051](https://doi.org/10.3390/jrfm13030051).