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# Heuristic solutions for nonlinear dynamic pricing in the presence of multiunit demand and two-dimensional customer heterogeneity

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## Abstract

In this paper, we introduce a nonlinear dynamic pricing model in the presence of multiunit demand, enabling firms to quote separate prices for each batch size. This approach diverges from traditional models by accounting for two-dimensional customer heterogeneity in product attraction and batch size preference, each modeled by separate random variables in the calculation of customers' willingness-to-pay. The underlying customer choice model results in a complex formulation of purchase probabilities, necessitating considerable effort for refinements to derive a manageable expression. We present optimality conditions for the state-wise optimization problem and introduce a modified formulation with reduced complexity that serves as an upper bound. We also prove that under specific conditions, the optimal solution to the modified model is optimal for the original problem. In our numerical study, these conditions were consistently met, offering a practical alternative for determining optimal prices. To address the computational challenges of solving the problem to optimality, we develop three efficient heuristics with significantly reduced runtimes. Benchmarking these heuristics against the optimal solution and other mechanisms demonstrates their near-optimal performance. We also evaluate the revenue potential of nonlinear, piecewise linear, and linear pricing schemes, providing firms with tools to weigh revenue maximization against pricing simplicity to inform strategic decisions. Notably, our analysis highlights the strong performance of piecewise linear pricing, offering a practical and easy-to-communicate alternative to full nonlinear pricing while achieving remarkably high revenues.

**Keywords** Revenue management · Dynamic pricing · Nonlinear pricing · Multiunit demand · Customer choice

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## 1 Introduction

Nonlinear pricing—such as volume discounts or special offers like "buy 3, pay 2"—were and still are a commonly applied pricing strategy in the ever-evolving field of retail. This approach is used both for individual products (e.g., cans of soda) and across product lines (e.g., shirts of different sizes, colors, or designs). Enabled by advancements in digital technologies (e.g., e-commerce and digital price tags), businesses can now adjust prices in real-time. This allows for swift responses to varying market demands and inventory levels. However, most traditional dynamic pricing models have a critical limitation: they usually assume that customers purchase only a single unit at a time. This assumption overlooks the complexities and opportunities presented by multiunit demand, which are prevalent in sectors ranging from groceries to clothing.

Addressing this gap, our paper presents an approach that combines nonlinear and dynamic pricing to optimally quote prices for every possible batch size of a product (or variant within a product line). This framework is designed to maximize revenue in scenarios where the selling horizon is finite, the product inventory is scarce, and customers exhibit multiunit demand. In our context, multiunit demand does not necessarily imply that customers are predetermined on the number of units they will purchase. Instead, it refers to settings where customers consider various quantities based on pricing schemes, promotions, or the needs of multiple end-users (e.g., family members, colleagues, or friends).

For instance, consider a supermarket dynamically adjusting batch prices for perishable products, such as bakery items or fresh produce. Depending on realized demand and remaining inventory, retailers dynamically raise batch prices to capitalize on strong demand or lower them to encourage bulk purchases as products approach expiration. Although this strategy naturally reduces waste and mitigates early stockouts, its primary goal remains revenue maximization given limited stock and a finite selling horizon. Similar dynamic nonlinear pricing schemes apply naturally in various other sectors, where the product loses its value at a specific point in time due to factors such as perishability (as in food or hotel stays) or the necessity to clear valuable shelf space for new inventory (as in fashion, seasonal goods, and consumer electronics).

Central to our approach is our customer choice model that recognizes the two-dimensional heterogeneity among customers, both in their attraction to the product and in their inclination to consume. We employ a utility maximization framework, where customer utility is modeled as a function of batch size and the two customer-specific attributes: product attraction and consumption. These attributes are customers' private information and therefore unknown to the firm. However, we assume the firm has knowledge of the distribution of these attributes across the customer population. This allows us to treat these attributes as random variables, forming the basis for our random utility framework.

With these assumptions, we align closely with a recent publication (see Schur 2024), where both studies aim to maximize expected revenues through dynamic nonlinear pricing and employ the same formulation of customers' utility based

on two customer-specific attributes. Accordingly, both publications are rooted in the same problem and use identical mathematical formulations for the choice model and the general dynamic program. However, unlike this publication, we do not make the additional assumption that the firm can identify customers' private information—specifically, their attraction to the product, their consumption indicator, or both. This assumption in Schur (2024) simplifies the optimization problem by utilizing customers' known attributes. In contrast, our model does not rely on observing private information, resulting in a more complex optimization problem. This extends the framework to settings where customer identification is not possible. In this way, our work generalizes the framework of Schur (2024). While techniques similar to those used in Schur (2024), specifically action space reduction and upper bound investigation, can be adapted to reduce complexity in our setting, we must adopt a fundamentally different strategy that focuses on developing effective heuristics.

By not relying on private information, our approach remains also applicable in scenarios where firms must quote prices before any interaction with customers occurs, as is typically the case in stores using digital price tags. This flexibility makes our model particularly valuable for scenarios where advance pricing decisions are necessary, or where the firm lacks access to customers' private information.

## 1.1 Results

Our research achieves three interconnected goals: deriving the optimal solution, developing effective heuristics, and evaluating the trade-offs between nonlinear pricing and simpler pricing schemes.

First, we establish optimality conditions that enable the computation of the optimal solution. While the optimal solution provides an important benchmark for evaluating the effectiveness of our heuristics, its computation remains impractical for real-world applications due to its substantial runtimes, requiring nearly five days in our largest setting.

Second, to overcome these computational challenges, we propose three heuristics that result in well-performing solutions while significantly reducing runtimes. For example, one of these heuristics computes a policy in just 2.5 min with an optimality gap below 0.4% in our largest setting, making it nearly 3000 times faster than computing the optimal solution. Similarly, the other heuristics also perform well, offering near-optimal solutions with runtimes suitable for practical application.

Finally, using the optimal and heuristic solutions as benchmarks, we evaluate the revenue potential of different pricing schemes—nonlinear, piecewise linear, and linear pricing. Our numerical study demonstrates that nonlinear pricing consistently achieves the highest revenue. However, simpler pricing schemes, particularly piecewise linear pricing, can perform very strongly. This observation is supported by an analysis of the strategies employed by our best-performing heuristics, which reveal two key patterns. First, the pricing schemes exhibit an almost piecewise linear structure. This structure explains the strong performance of piecewise linear pricing in our study and suggests that firms could adopt this simpler pricing scheme to

reduce communication complexity. By quoting just two prices—one for the first unit and another for additional units—firms can achieve high revenues while maintaining ease of implementation. Second, in scenarios with high initial inventory ( $\frac{C}{T} \geq 2$ ) prices for smaller batches decrease gradually over time. This gradual adjustment implies that less frequent price changes can still deliver strong performance, making such strategies viable for firms constrained by technical limitations, customer preferences, or strategic considerations. Conversely, in low-stock scenarios ( $\frac{T}{C} \geq 2$ ), the revenue advantage of nonlinear pricing decreases, and standard dynamic pricing approaches become more effective.

## 1.2 Contribution and outline

We contribute to the sparse literature on multiunit dynamic pricing by addressing the challenging optimization problem of dynamically quoting batch prices while considering two-dimensional customer heterogeneity, represented by two random variables in the customer choice model. Our main contributions are as follows:

- C1. *Optimality conditions*: We derive optimality conditions by reducing the complexity of the customer choice model. Notably, we prove that the optimality condition holds whether the optimal solution is located in the interior or on the boundary of the action space, allowing us to focus on this condition without needing to separately investigate boundary cases.
- C2. *Problem modification and upper bound*: We introduce a modified version of the original problem that can be solved more efficiently and serves as an upper bound. We further prove that under specific conditions, the solution of the modified problem is optimal for the original problem.
- C3. *Development of heuristics*: We develop three efficient heuristics that produce well-performing pricing policies across various problem sizes. These heuristics achieve substantially shorter runtimes compared to finding the optimal solution, enabling their use in practical applications.
- C4. *Numerical study*: Our numerical study demonstrates the effectiveness and efficiency of all three heuristics. The study also highlights the revenue potential of linear, piecewise linear, and nonlinear pricing schemes, enabling firms to balance revenue maximization with pricing simplicity.
- C5. *Insights into pricing schemes*: Our study provides insights into the strategy of well-performing pricing schemes. Notably, fewer price changes and simpler pricing schemes, such as piecewise linear pricing, may still perform very well.

In Sect. 2, we review the relevant literature, positioning our research within the context of existing studies and identifying gaps that our work addresses. Section 3 introduces the customer choice model, emphasizing the two-dimensional approach, and presents the optimization model.

In Sect. 4, we reduce the complexity of the probability function, derive optimality conditions, and introduce the modified problem that serves as an upper bound. This section addresses the first two contributions, C1 and C2. It also achieves our first

goal (outlined in Sect. 1.1) by enabling the computation of the optimal solution and builds the foundation for evaluating different pricing schemes, contributing to our third goal.

Section 5 discusses the heuristics we developed, addressing the third contribution, C3. By providing efficient and practical alternatives to computing the optimal solution, this section achieves our second goal from Sect. 1.1.

Finally, Sect. 6 presents the numerical study and its findings. This section addresses the last two contributions, C4 and C5, by demonstrating the practical applicability of the proposed heuristics and gaining insights into their strategies. It also fulfills our third goal by evaluating the revenue potential of nonlinear pricing and simpler pricing schemes.

## 2 Literature review

In this study, we bridge the domains of dynamic pricing and nonlinear pricing, integrating the strengths of both. In Sect. 2.1, we provide a review of works within these two distinct yet interconnected domains. This foundational overview sets the stage for a deeper exploration into the specialized segments of multiunit and multi-product dynamic pricing, as well as bundle pricing, in Sect. 2.2. Multiunit dynamic pricing, which also covers nonlinear pricing, represents a relatively new field of research with sparse literature. Our research contributes to this niche, acknowledging the critical role of nonlinear pricing in catering to multiunit demands. Additionally, we examine multiproduct dynamic pricing and bundle pricing, domains aligned with our research due to their consideration of customer choice, where customers select from multiple options. However, all these studies differ from our approach in a key aspect: they do not account for two-dimensional customer heterogeneity. The assumption of one-dimensional heterogeneity implies that all customers share the same relative utility when comparing two options, which is an unrealistic assumption in multiunit demand scenarios. To underline this gap, we incorporate several studies that, regardless of their specific field of application, address similar choice behaviors. We compare these approaches with our own in Sect. 2.3.

### 2.1 Foundations of nonlinear pricing and dynamic pricing

Nonlinear pricing is a widespread strategy across various sectors, including telecommunications, transportation, energy, supply chains, and retail. Consequently, there is a rich and diverse literature on the subject. Wilson (1993) (Part I) provides a comprehensive overview of the application fields, economic principles, and marketing insights related to nonlinear pricing. While much of the existing literature focuses on static pricing models, a small subset of researchers has turned their attention to dynamic environments, which align more closely with the thematic focus of our study (e.g., Dhebar and Oren 1986, and Braden and Oren 1994).

The conceptual foundations of dynamic pricing trace back to seminal studies on intertemporal price discrimination conducted 30–40 years ago, with notable contributions by Stokey (1979), Landsberger and Meilijson (1985), and Wilson (1988). A significant milestone was achieved by Gallego and van Ryzin (1994), who were the first to explore optimal dynamic pricing for a single product under stochastic demand over a finite selling horizon. This pioneering work led to a vast amount of follow-up research, which was reviewed and summarized by many authors, including Bitran and Caldentey (2003), Chiang et al. (2007), and, with a special focus, Gönsch et al. (2013) and den Boer (2015), as well as in textbooks such as Talluri and van Ryzin (2004) (Chapter 5).

## 2.2 Multiunit dynamic pricing, multiproduct dynamic pricing, and bundle pricing

While dynamic pricing has been extensively explored, the specific area of multiunit dynamic pricing is still emerging. Elmaghraby et al. (2008) introduce markdown pricing mechanisms within a multiunit demand framework, assuming complete information about customers and their willingness-to-pay. Levin et al. (2014) expanded the discussion to a dynamic pricing model characterized by stochastic customer demand for batches. In their framework, customers request a specific batch size and the seller's subsequent pricing response influences whether the requested batch size is purchased. However, unlike their approach, our research introduces a more flexible decision-making process in which customers can review all available prices prior to requesting the purchase quantity. This advancement not only gives customers greater flexibility but also provides firms with a mechanism to strategically steer customers' buying decisions and purchase volumes.

Gallego et al. (2020) investigate three dynamic pricing strategies: nonlinear, linear, and block pricing. In their proposed choice model, customers seek to maximize their utility and are characterized by a single random variable, representing one-dimensional customer heterogeneity. The authors develop optimality conditions and show structural properties. However, our research diverges fundamentally by modeling customer behavior with two independent variables, thereby offering a more granular representation of customer decision-making processes and allowing for two-dimensional heterogeneity. This two-dimensional approach represents a significant departure from the conventional one-dimensional models (refer to Sect. 2.3 for a broader overview of choice models applied in similar demand settings). Schur (2024) explores a scenario similar to ours but diverges in assuming firms' access to some or all private information about arriving customers. This assumption paves the way for personalized pricing strategies and the evaluation of the strategic value of customer information. Contrary to this, our research operates under the premise of private information, enabling the universal applicability of our pricing model without relying on the availability of detailed customer insights.

Multiunit dynamic pricing can be compared to the better-explored field of multiproduct dynamic pricing by defining batches of a single product as several different "products". Several articles (see, e.g., Zhang and Cooper 2009, Dong et al. 2009, and Akçay et al. 2010, or, for a review, Chen and Chen 2015) have investigated

dynamic pricing of substitutes. However, studies focusing on horizontally differentiated products are less relevant to our setting, as they lack a common order in product valuations, which violates a core assumption in multiunit demand setting: customers do not pay more for fewer units. In this context, general studies like Maglaras and Meissner (2006) and research on vertically differentiated products, such as Akçay et al. (2010) and Liu and Zhang (2013), are more closely aligned with our setting. While Maglaras and Meissner (2006) do not provide a specific willingness-to-pay function, they demonstrate the asymptotic optimality of a solution of a deterministic fluid approximation. Akçay et al. (2010) and Liu and Zhang (2013) rely on one-dimensional customer heterogeneity regarding quality, which, in a multiunit setting, would imply that marginal willingness-to-pay for additional units decreases at a fixed rate across all customers.

A special case of multiproduct pricing is found in the bundle pricing literature, where multiple items are combined into a bundle. Bundle-size pricing involves the firm quoting a price for each bundle-size, allowing customers to choose which items to include in their bundle. In this context, items can be viewed as single units of the same product, and bundles as batches. Bundle pricing aligns well with multiunit demand, as customer utility typically increases with bundle-size, reflecting one of the core assumptions of multiunit demand. However, related literature, such as Honhon and Pan (2017) and Song and Xue (2021), often assumes one-dimensional customer heterogeneity. Other studies, such as Abdallah et al. (2021), Ettl et al. (2020), and Chen et al. (2024), do not ensure decreasing marginal utilities with increasing bundle size, another key assumption in multiunit demand.

### 2.3 Literature with similar customer choice considerations

Literature generally agrees on the fundamental concept that customers evaluate all available options by assigning a monetary value, often referred to as willingness-to-pay, to each option and comparing these values to quoted prices. The option with the highest utility, i.e. the difference between willingness-to-pay and price, is chosen. Since willingness-to-pay is private information known only to the customer, various models have been introduced, typically accounting for customer heterogeneity using one of two prominent approaches.

The first approach models willingness-to-pay using various customer segment-specific nominal parameters, where heterogeneity within a segment is captured through independently and identically distributed error terms. Hanemann (1984) formulates a willingness-to-pay function that is linear in quantity, leading to constant marginal utilities. Subsequent studies, such as Allenby and Rossi (1991), Chiang (1991), Bell et al. (1999), and Nair et al. (2005), propose non-constant marginal utilities to better reflect decreasing customer valuations for additional quantity. Quadratic formulations for those non-constant marginal utilities can be found in Lambrecht et al. (2007) and Iyengar et al. (2008). Iyengar and Jedidi (2012) introduce a generalization that encompasses both quadratic and power utility functions as special cases. These studies often employ multinomial logit models, assuming Gumbel-distributed error terms. While this model has certain advantages, it presents

a significant limitation: it cannot enforce a common order of valuations, leading to instances where customers may be willing to pay more for fewer units. Similarly, in multiproduct pricing, this model is less suited for vertically differentiated products (refer to Song and Xue 2021).

The second approach often relies on a single random variable to represent customers' preferences, as opposed to the multiple parameters combined with random error terms used in the first approach. This one-dimensional heterogeneity is applied by Mussa and Rosen (1978), where willingness-to-pay increases linearly in quantity. Further applications of this approach, with non-constant marginal utilities, can be seen in Braden and Oren (1994), Sundararajan (2004), Banciu et al. (2010), and Bialogorsky and Koenigsberg (2014). Rochet and Stole (2002) incorporate two-dimensional heterogeneity, where willingness-to-pay is modeled as linear in quantity, weighted by a random variable representing customer type, along with an additive random variable that accounts for brand preferences.

We follow the second approach to model customer choice, addressing two-dimensional heterogeneity through two random variables, but not in the linear form proposed by Rochet and Stole (2002). Instead, our willingness-to-pay function is based on the generalized formulation of Iyengar and Jedidi (2012), incorporating non-linearity in quantity and resulting in non-constant marginal utilities.

## 2.4 Summary

Our literature review highlights a general gap in research on nonlinear dynamic pricing, with only a few exceptions, which differ from our work in their specific assumptions. Broadening the scope, it becomes apparent that even in related fields such as vertically differentiated multiproduct pricing and bundle pricing, two-dimensional customer heterogeneity—allowing for customer-specific, non-fixed rates of marginal utility decrease—is not considered. In this regard, our approach could be of interest to these fields as well.

## 3 Problem definition

We first present the random utility framework that defines our customer choice model in Sect. 2.3. Then, we introduce the dynamic programming formulation that describes our nonlinear dynamic pricing problem.

### 3.1 Customer choice model

In our setting, customers exhibit multiunit demand but are not predetermined in the number of units they consider purchasing. Instead, they evaluate all available options, i.e., possible batch sizes, ranging from a batch size of zero to the entire remaining stock  $c$ . This evaluation results in a personal monetary valuation for each option, which is known as willingness-to-pay and denoted by  $X_j$  for  $j$  units. It is

important to note that willingness-to-pay is private information of the customer and is unknown to the firm.

We assume that willingness-to-pay can be modeled using a formulation inspired by Iyengar and Jedidi (2012) and also applied by Schur (2024). In this formulation, the marginal willingness-to-pay for the  $j$ -th unit is expressed as  $X_j - X_{j-1} = \omega \cdot \lambda^{j-1}$ , where  $\omega \in [0, 1]$  and  $\lambda \in [0, 1]$  are two independent continuous random variables that capture customer heterogeneity in two key aspects: their attraction to the product and their inclination to purchase larger quantities, respectively.

The variable  $\omega$  equals the willingness-to-pay for the first unit, as evident from  $X_1 = \omega \cdot \lambda^0 = \omega$  (with  $X_0 = 0$ ), and linearly affects the marginal willingness-to-pay  $\omega \cdot \lambda^{j-1}$  for  $j \geq 2$ . Thus,  $\omega$  can be interpreted as the product's attractiveness to the customer, or more formally, as the base willingness-to-pay. Conversely,  $\lambda$  does not impact the willingness-to-pay for the first unit but constitutes the rate at which the marginal willingness-to-pay changes in  $j$ . This is illustrated by the relationship  $X_{j+1} - X_j = \omega \cdot \lambda^j = \lambda \cdot (\omega \cdot \lambda^{j-1}) = \lambda \cdot (X_j - X_{j-1})$ , indicating that a customer with  $\lambda$  close to 1 values each additional unit almost as much as the previous one, whereas a customer with  $\lambda$  closer to 0 experiences a sharp drop-off in value for additional units. Consequently,  $\lambda$  serves as a measure of the customer's inclination to consume.

Building on this marginal valuation, the cumulative willingness-to-pay  $X_j$  for a batch of size  $j$  is obtained by summing the valuations for all units up to  $j$ :

$$X_j = \omega \cdot \sum_{k=0}^{j-1} (\lambda)^k \text{ for } j = 1, \dots, c. \tag{1}$$

We further assume that the firm knows the distribution of  $\omega$  and  $\lambda$ , represented by the density functions  $f_\omega$  and  $f_\lambda$ , and cumulative distribution functions  $F_\omega$  and  $F_\lambda$ . Both density functions have their support on  $[0, 1]$ , ensuring that the marginal willingness-to-pay remains non-negative and decreases<sup>1</sup> with respect to the quantity  $j$ . Unless otherwise stated, these distributions may vary over time, allowing for time-inhomogeneous demand. For simplicity, we refrain from including a  $t$ -index to indicate this (possible) time dependence.

This design aligns with a common assumption about customer preferences (see, e.g., Goldman et al. 1984, Baucells and Sarin 2007, and Gallego et al. 2020), effectively capturing the decreasing value of additional units and resulting in willingness-to-pay that is concave in units. Thereby, this formulation clearly differs from linear utility models, where additional units always offer a constant marginal willingness-to-pay. The restriction of  $\omega$  to  $[0, 1]$  serves primarily as a scaling measure, normalizing the marginal willingness-to-pay.

Our approach to modeling customer choice is anchored in the widely accepted random utility model. Within this framework, we define a customer's utility for purchasing  $j$  units at given prices as the difference between their random

<sup>1</sup> We use the terms "increase" and "decrease" in a weak sense, specifying "strictly increase" and "strictly decrease" when needed.

willingness-to-pay and the corresponding price. Specifically, for a batch price  $r_j$ , the utility of purchasing  $j$  units, denoted  $U_j$ , is calculated as:

$$U_j = X_j - r_j \quad \text{for } j = 1, \dots, c. \tag{2}$$

Under the assumption of rational behavior, customers aim to maximize their utility. Therefore, a customer opts to purchase  $j$  units if, and only if,  $U_j$  represents the maximum utility achievable across all possible purchase quantities, including the option not to purchase ( $U_0 = 0$ ). This modeling assumption, in conjunction with the density functions for  $\omega$  and  $\lambda$ , allows us to compute the probability of a customer purchasing  $j$  units. This computation involves determining the probability that a specific realization of the random variables  $\omega$  and  $\lambda$ , which represent a unique customer profile, results in the maximum utility for the purchase of batch size  $j$ .

To visualize this concept, one can imagine mapping out decision regions within the  $[0, 1]^2$  space, identified by combinations of  $w$  and  $l$  (realizations of  $\omega$  and  $\lambda$ , respectively), where the utility of purchasing  $j$  units surpasses that of any other purchase quantity within the set  $\{0, 1, \dots, c\}$ . These regions are determined by where  $U_j$  attains its maximum value across all considered purchasing options ( $U_j = \max_{i=0, \dots, c} \{U_i\}$ ). Since both random variables follow a continuous distribution, customers with a non-unique maximum are almost surely excluded from arriving. This eliminates the need to define tiebreaker rules, which would otherwise be necessary to ensure that customers select only one option. Subsequently, the probability associated with  $\omega$  and  $\lambda$  falling within a particular region reflects the likelihood of an arriving customer, characterized by a specific  $(w, l)$  bundle, opting to purchase  $j$  units. The following figure illustrates these regions, with each one labeled according to the batch size  $j$  that maximizes utility for customers whose preferences are represented by the  $(w, l)$  bundles within that region. The figure is based on a given price vector.

Consequently, the choice probability for a customer to purchase  $j$  units can be calculated by

$$p_j(\mathbf{r}) = \int_0^1 \int_0^1 1_{\left\{U_j = \max_{i=0, \dots, c} \{U_i\}\right\}}(w, l) f_\omega(w) f_\lambda(l) dw dl \quad \text{for } j = 1, \dots, c, \tag{3}$$

where the indicator function  $1_{\left\{U_j = \max_{i=0, \dots, c} \{U_i\}\right\}}(w, l)$  equals 1 if the utility of purchasing  $j$  units,  $U_j$ , is the maximum among all considered batch sizes. Specifically, this condition is satisfied if the utility of purchasing  $j$  units,  $w \cdot \sum_{k=0}^{j-1} l^k - r_j$ , is greater than or equal to the utility of purchasing any other quantity  $i$ ,  $w \cdot \sum_{k=0}^{i-1} l^k - r_i$ , for  $i = 1, \dots, c$ , and also nonnegative. If these conditions are not met, the indicator function is 0.

Alternatively, these selection criteria can be expressed through the relationship between price differences and the cumulative discounting effect of purchasing additional units. A customer with attributes  $w, l \in [0, 1]$  will purchase  $j$  units iff the following condition holds:

$$\max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i} \right\} \leq w \leq \min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i} \right\}.$$

This expression separates both attributes  $w$  and  $l$  and simplifies calculating the choice probabilities. Furthermore, the fractions within this expression define the decision regions as they define the boundaries of these regions (represented as lines in Fig. 1).

To keep notation as short as possible, we further establish the following conventions:  $r_0 = 0$ ,  $\frac{r_1 - r_0}{\sum_{i=0}^0 l^i} = r_1$  for  $l = 0$ ,  $\max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i} \right\} = \infty$  for  $j \neq 1$ , and  $\min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i} \right\} = 1$  for  $j = c$ .

By utilizing the alternative formulation, we can simplify the probability for selling  $j$  units to

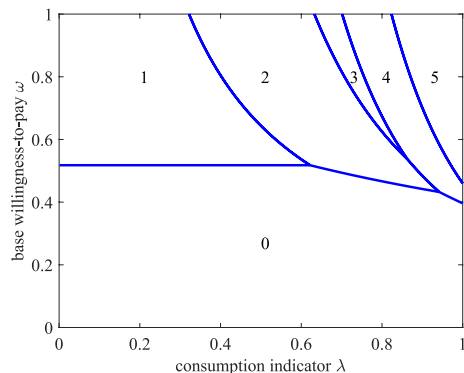
$$p_j(r) = \int_0^1 \left( F_\omega \left( \min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i} \right\} \right) - F_\omega \left( \max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i} \right\} \right) \right)^+ f_\lambda(l) dl \quad \text{for } j = 1, \dots, c, \tag{4}$$

with  $(\cdot)^+ = \max\{\cdot, 0\}$ . While this formulation features only one integral, it remains analytically challenging. Without further knowledge, particularly about which fractions yield the minimum and maximum for any given  $l$ , even calculating the purchase probability for a given price vector  $r$  becomes a tedious task, rendering finding the optimal solution effectively impossible without further improvements. Consequently, we will put considerable effort in refining the formulation and in learning inherent structural properties in Sects. 4.1 and 4.2.

### 3.2 Dynamic programming formulation

We adapt the standard setting of dynamic pricing to cope with multiunit purchases. To do so, we consider the following framework: A firm sells a single product over a finite selling horizon. The selling horizon is divided into  $T$  periods and indexed

Fig. 1 An example of the location of decision regions for  $c = 5$



backward in time, i.e., periods  $T$  and  $0$  mark the beginning and the end of the horizon, respectively. The initial stock of  $C$  units of the product is nonreplenishable and any capacity remaining after the selling horizon is worthless. Nonetheless, with a straightforward modification, this research can be easily extended to include salvage values. We assume that exactly one customer arrives in each period  $t \in \{T, \dots, 1\}$ . However, our framework can be straightforwardly extended to a non-stationary arrival process. At any period, the firm knows the remaining capacity  $c \in \{1, \dots, C\}$  and sets a price vector  $\mathbf{r} = (r_1, r_2, \dots, r_c)^T$  with  $r_j$  marking the price the customer has to pay for  $j$  units of the product, i.e. for a batch with batch size  $j \leq c$ . Depending on the prices the firm quotes, this customer may purchase zero, one or more (up to  $c$ ) units of the product.

This optimization problem can be formulated as a dynamic program using the Bellman equation (see, e.g., Gallego et al. 2020 for a similar setting, and Powell 2011, Sect. 3.10.1, for a general proof). Since the remaining capacity  $c$  (and, thus, the maximum number of purchasable units) varies over time, we define a state-dependent action space  $\mathcal{R}_c = \{\mathbf{r} \in \mathbb{R}^c : r_j \geq 0, j = 1, \dots, c\}$  with  $\mathcal{R}_0 = \emptyset$ . The overall target is the maximization of the total expected revenue and can be achieved through the following dynamic program:

$$V_t(c) = \max_{\mathbf{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^c p_j(\mathbf{r}) \cdot (r_j + V_{t-1}(c-j)) + \left( 1 - \sum_{j=1}^c p_j(\mathbf{r}) \right) \cdot V_{t-1}(c) \right\}, \tag{5}$$

with boundary conditions  $V_0(c) = 0$  for  $c \geq 0$  and  $V_t(0) = 0$  for  $t \geq 0$ . In this dynamic program,  $V_t(c)$  denotes the optimal expected revenue-to-go from period  $t$  onwards (before the arrival of a customer in  $t$ ).

With this formulation, we consider that customers might be willing to buy more than just one unit. Moreover, we allow the firm to set batch prices to take full advantage of a nonlinear pricing scheme. The expectation captures  $c + 1$  possible events: With probability  $p_j(\mathbf{r})$ , the firm sells  $j$  units and immediately earns  $r_j$ . Additionally, it can expect future revenues amounting to  $V_{t-1}(c-j)$  with a reduced stock of  $c-j$  units from the next period onwards. With probability  $1 - \sum_{j=1}^c p_j(\mathbf{r})$ , the firm sells nothing and only faces expected revenue  $V_{t-1}(c)$  from stock  $c$  and period  $t-1$  onwards. We denote the optimal batch prices selected in a state  $(t, c)$  by  $\mathbf{r}_t(c) \in \mathcal{R}_c$ .

An alternative formulation of (5) relies on opportunity costs  $\Delta_j V_t(c)$  for selling  $j$  units, i.e.

$$\Delta_j V_t(c) = V_t(c) - V_t(c-j) \quad \text{for } j = 1, \dots, c, \tag{6}$$

and can be written as

$$V_t(c) = \max_{\mathbf{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^c p_j(\mathbf{r}) \cdot (r_j - \Delta_j V_{t-1}(c)) \right\} + V_{t-1}(c). \tag{7}$$

This formulation presents several benefits compared to Eq. (5). The most immediate benefit is the clear indication that optimal prices must exceed opportunity costs. Not meeting this criterion could mean that sales do not contribute to an increase

in expected revenue, or worse, could result in a decrease in overall expected revenue. Furthermore, this formulation highlights the critical role of opportunity costs. As the only component that varies with state, opportunity costs are fundamentally responsible for the dynamic adjustments in optimal pricing strategies over time.

Summarizing, the optimization breaks down to maximizing the expected additional gain realized by selling something between 1 and  $c$  units in period  $t$  instead of retaining the capacity for later sales.

## 4 Structural properties and optimality conditions

In this section, we outline how we facilitate the process of finding the optimal solution. While we successfully derive optimality conditions, the computational effort required to solve the corresponding equations is too high for practical applications, particularly given the need for dynamic pricing decisions to be made online and in real time. Nevertheless, knowing the optimal solution is crucial for evaluating the heuristics that we will introduce in Sect. 5.

To complement this evaluation, we modify the optimization problem to create a formulation that serves as an upper bound to the original problem and is computationally more efficient to solve. A notable strength of this specific modification is our ability to establish easily verifiable conditions under which the solution to the modified problem is also optimal for the original problem.

In Sects. 4.1 and 4.2, our objective is to simplify the formulation of the probability function in two steps. Initially, we introduce appropriate notation and show inherent structural properties, thereby formulating a more concise function. Subsequently, we eliminate prices from the action space that do not contribute to solving the optimization problem. This allows us to gain additional insights into the probability function, leading to further refinement.

This refinement enables us, in Sect. 4.3, to establish optimality conditions for solving the state-wise optimization problem. Additionally, in Sect. 4.4, we propose a modified formulation that is simpler to solve than our original problem and constitutes an upper bound. Moreover, we prove that both the modified and the original formulation result in the same optimal solution under specific conditions. Thereby, we offer an alternative approach to finding the optimal solution.

### 4.1 Refining the probability function

Despite simplifying the model by eliminating one of the integrals at the end of Sect. 3.1, the resulting probability function remains challenging to manage. Specifically, incorporating the minimum and maximum functions throughout the analysis introduces significant complexity. To address this, we propose dividing the integration region into distinct segments. This approach allows us to streamline the calculation process by substituting  $\min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} \beta^i} \right\}$  and  $\max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} \beta^i} \right\}$  with more manageable expressions for each segment.

We start with introducing the following sets, designed to sort  $l$  in such a way that on each set the minimum or maximum operator can be substituted:

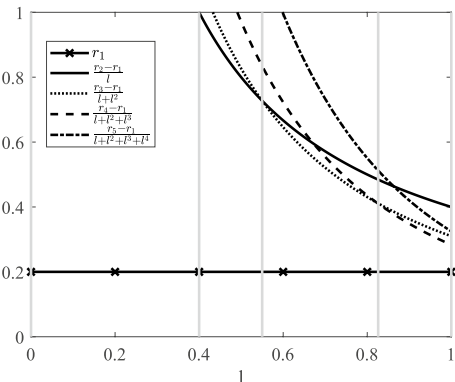
$$\Lambda_j(\mathbf{r}) = \left\{ l \in [0, 1] \mid \max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \right\} \leq 1 \leq \min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \right\} \right\},$$

$$\begin{aligned} \Lambda_{ij}^{min}(\mathbf{r}) &= \left\{ l \in [0, 1] \mid \max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \right\} \leq \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \right. \\ &= \left. \min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \right\} \leq 1 \right\}, i > j, \text{ and} \end{aligned}$$

$$\begin{aligned} \Lambda_{ji}^{max}(\mathbf{r}) &= \left\{ l \in [0, 1] \mid \max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \right\} \right. \\ &= \left. \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} \leq \min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \right\}, 1 \right\}, i < j. \end{aligned}$$

These sets are either empty or an interval. This follows immediately from Lemma 1 and is illustrated in Fig. 2 for a given price vector. In this figure, we can identify specific intervals for the sets under consideration for  $j = 1$ :  $\Lambda_{10}^{max}(\mathbf{r}) = [0, 1]$ ,  $\Lambda_1(\mathbf{r}) = [0, 0.4]$ ,  $\Lambda_{21}^{min}(\mathbf{r}) = [0.4, 0.55]$ ,  $\Lambda_{31}^{min}(\mathbf{r}) = [0.55, 0.826]$ ,  $\Lambda_{41}^{min}(\mathbf{r}) = [0.826, 1]$ , and  $\Lambda_{51}^{min}(\mathbf{r}) = \emptyset$ , with their boundaries marked by the vertical lines in the figure. Many of these intervals (particularly,  $\Lambda_{i1}^{min}(\mathbf{r})$  with  $i > 1$ ) are defined by the intersection points of the curves  $\frac{r_k - r_1}{\sum_{m=1}^{k-1} l^m}$  ( $k > 1$ ), where the curve that constitutes the minimum shifts. To calculate the probability of selling a single unit batch ( $j = 1$ ) in this scenario, it is necessary to determine the probability of realizations  $(w, l)$  falling within the region above the horizontal line at  $w = 0.2$  and below all curves. By

**Fig. 2** An example of intervals  $\Lambda_j(\mathbf{r})$  and  $\Lambda_{ij}^{min}(\mathbf{r})$  with  $j = 1$  and  $c = 5$



identifying these specific intervals, we know precisely which curve defines the upper bound for realizations  $w$  at any given realization  $l$ .

**Lemma 1** For every  $i \neq k$  with  $i, k > j$  and  $r_i - r_j \neq 0 \neq r_k - r_j$ , there is at most one  $l \in (0,1)$  where  $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$ . For every  $i \neq k$  with  $i, k < j$  and  $r_i - r_j \neq 0 \neq r_k - r_j$ , there is at most one  $l \in [0,1]$  where  $\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} = \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m}$ .

**Proof:** See Supplement S.1.

Lemma 1, in conjunction with the observation that the equation  $\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} = 1$  is satisfied by at most one  $l \in [0,1]$ , implies that each curve can only serve as the minimum or maximum exclusively within a specific interval. At some point, it will intersect with another curve and will consistently remain either above or below the intersecting curve. Consequently, for the various sets of interest – namely  $\Lambda_j(\mathbf{r})$  for instances where  $l \geq \min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \right\}$ ,  $\Lambda_{ij}^{min}(\mathbf{r})$  for cases where  $i > j$ , and  $\Lambda_{ji}^{max}(\mathbf{r})$  for cases where  $j > i$ —we define their corresponding intervals as  $[l_j(\mathbf{r}), \bar{l}_j(\mathbf{r})]$ ,  $[l_{-ij}^{min}(\mathbf{r}), \bar{l}_{-ij}^{min}(\mathbf{r})]$ , and  $[l_{-ji}^{max}(\mathbf{r}), \bar{l}_{-ji}^{max}(\mathbf{r})]$ , respectively. It is important to note that some of these intervals may be empty, and thus, boundary values have to be selected accordingly.

**Remark 1** By definition,  $\max_{i>j} \{ \bar{l}_{ij}^{min}(\mathbf{r}) \} = \max_{i<j} \{ \bar{l}_{ji}^{max}(\mathbf{r}) \}$ . We denote this largest boundary as  $\bar{l}_j^{end}(\mathbf{r}) = \max_{i>j} \{ \bar{l}_{ij}^{min}(\mathbf{r}) \}$ . Moreover, it holds that:

$$\begin{aligned} \bigcup_{i>j} [l_{-ij}^{min}(\mathbf{r}), \bar{l}_{-ij}^{min}(\mathbf{r})] &= [\bar{l}_j(\mathbf{r}), \bar{l}_j^{end}(\mathbf{r})] \\ \bigcup_{i<j} [l_{-ji}^{max}(\mathbf{r}), \bar{l}_{-ji}^{max}(\mathbf{r})] &= [l_j(\mathbf{r}), \bar{l}_j^{end}(\mathbf{r})] \end{aligned}$$

Currently, our model requires extensive notation to accurately represent the probability function. Nevertheless, the introduction of the subsequent lemma will streamline the notation required, thereby enhancing the brevity and clarity of our presentation.

**Lemma 2** For every  $i > j$  it holds that  $[l_{-ij}^{min}(\mathbf{r}), \bar{l}_{-ij}^{min}(\mathbf{r})] = [l_{-ij}^{max}(\mathbf{r}), \bar{l}_{-ij}^{max}(\mathbf{r})]$ .

**Proof:** See Supplement S.3.

This lemma not only simplifies our notation but also carries another implication: for  $i > j$ , the interval  $[l_{-ij}^{min}(\mathbf{r}), \bar{l}_{-ij}^{min}(\mathbf{r})]$  defines the region for realization  $l$  where the curve  $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}$  marks the upper bound for all realizations  $w$  that specify, in combina-

tion with  $l$ , all customers who choose to purchase  $j$  units. From Lemma 2, we know that  $\left[ \underline{l}_{ij}^{min}(\mathbf{r}), \bar{l}_{ij}^{min}(\mathbf{r}) \right] = \left[ \underline{l}_{ij}^{max}(\mathbf{r}), \bar{l}_{ij}^{max}(\mathbf{r}) \right]$ , which implies that  $\Lambda_{ij}^{min}(\mathbf{r}) = \Lambda_{ij}^{max}(\mathbf{r})$ . By definition of  $\Lambda_{ij}^{max}(\mathbf{r})$ , the same curve  $\left( \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \right)$  also serves as the lower bound for all realizations  $w$  that, in combination with  $l$ , represent all customers who prefer purchasing  $i$  units. Consequently, this curve marks the boundary between two decision regions (refer to Fig. 1): one associated with customers opting to purchase  $j$  units and the other with those preferring  $i$  units.

We can now shorten our notation to  $\underline{l}_{ij}(\mathbf{r}) = \underline{l}_{ij}^{min}(\mathbf{r}) = \underline{l}_{ij}^{max}(\mathbf{r})$  and  $\bar{l}_{ij}(\mathbf{r}) = \bar{l}_{ij}^{min}(\mathbf{r}) = \bar{l}_{ij}^{max}(\mathbf{r})$ . The probability function can be written as:

$$\begin{aligned}
 p_j(\mathbf{r}) = & \int_{\underline{l}_j(\mathbf{r})}^{\bar{l}_j(\mathbf{r})} f_\lambda(l) dl + \sum_{i=j+1}^c \int_{\underline{l}_j(\mathbf{r})}^{\bar{l}_j(\mathbf{r})} F_\omega \left( \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \right) f_\lambda(l) dl \\
 & - \sum_{i=0}^{j-1} \int_{\underline{l}_j(\mathbf{r})}^{\bar{l}_{ji}(\mathbf{r})} F_\omega \left( \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} \right) f_\lambda(l) dl \quad \text{for } j = 1, \dots, c.
 \end{aligned}
 \tag{8}$$

With this refinement of the probability function, we can better explore how prices and their variations impact selling probability. This is crucial for determining optimal prices in the state-wise optimization (7). However, some complexity remains due to the implicit definition of the bounds of the integrals. To address this, it is reasonable to gain deeper insights into these bounds. One way to achieve this is by reducing the action space, which will result in a closed-form expression of some of the bounds.

### 4.2 Action space reduction

Upon closer examination of our choice model, it becomes evident that we only need to consider a specific subset of prices to maximize expected revenue. Consequently, we aim to refine the definition of the action space  $\mathcal{R}_c$  by excluding prices that do not lead to a unique outcome in terms of selling probabilities and expected revenues. By doing so, we simplify our optimization problem without the risk of excluding a potentially unique optimal solution.

The argumentation for deeming certain prices as irrelevant is as follows: Maintaining multiple different values for price  $r_j$  that effectively nullify demand for  $j$  units (i.e.,  $p_j(\mathbf{r}) = 0$ ) is unnecessary. It suffices to have at least one value for  $r_j$  (depending on  $r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_c$ ) to preserve the option of pricing out  $j$  units. Irrelevant prices can be identified by any of the following four criteria.

1. *Exclusion of higher prices for smaller batches:* We exclude any price  $r_j$  with  $r_j > r_{j+1}$  because customers almost surely have a higher willingness-to-pay for

- $j + 1$  units than for  $j$  units. Therefore, setting  $r_j = r_{j+1}$  is sufficient to eliminate demand for  $j$  units.
2. *Exclusion of prices exceeding batch size threshold:* Any price  $r_j$  exceeding  $j$  is irrelevant, as the maximal willingness-to-pay for  $j$  units is  $j$ .
  3. *Exclusion of prices with excessive margins:* We exclude any price  $r_j$  for which  $r_j - r_{j-1} > 1$ , since the maximal marginal utility for the  $j$ -th unit is one.
  4. *Exclusion of prices with excessive comparative margins:* We omit prices  $r_j$  that satisfy the condition  $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}}$ . This inequality implies that a customer with a positive marginal utility for purchasing the  $j$ -th unit has almost surely also a positive marginal utility for purchasing the  $(j + 1)$ -th unit.

The following lemma confirms that reducing the action space based on these four criteria only eliminates prices that do not result in unique outcomes in terms of selling probabilities and revenues.

**Lemma 3** *The search for the optimal price vector can be restricted to the set .*

$$\mathcal{R}_c = \left\{ \mathbf{r} \in \mathbb{R}^c : 0 \leq r_1 \leq \dots \leq r_c \leq c, r_j \leq j \forall j, r_j - r_{j-1} \leq 1 \right. \\ \left. \text{for } j \geq 2, \text{ and } (r_j - r_{j-1})^{\frac{1}{j-1}} \leq (r_{j+1} - r_j)^{\frac{1}{j}} \text{ for } 2 \leq j \leq c - 1 \right\}.$$

**Proof:** See Supplement S.4.

In Sect. 3.1, we have seen that  $l_k$  and  $\bar{l}_k$  play a crucial role in calculating selling probabilities. With the action space reduction, we are now able to shed more light on the definition of these parameters.

**Lemma 4** *It holds that  $l_1(\mathbf{r}) = 0, l_k(\mathbf{r}) = (r_k - r_{k-1})^{\frac{1}{k-1}} = \bar{l}_{k-1}(\mathbf{r}), 2 \leq k \leq c, \bar{l}_c(\mathbf{r}) = 1$*

for all  $\mathbf{r} \in \mathcal{R}_c$ .

**Proof:** See Supplement S.5.

With Lemma 4, we obtain a closed-form expression for  $l_k(\mathbf{r})$  and  $\bar{l}_k(\mathbf{r})$  ( $1 \leq k \leq c$ ), which simplifies our probability function. Additionally, the proof of this lemma shows that  $l_{k,k-1}(\mathbf{r}) = l_k(\mathbf{r})$ , extending the closed-form expression to these specific bounds. While the implicit definitions of other bounds, such as  $l_{kj}(\mathbf{r})$  and  $\bar{l}_{kj}(\mathbf{r})$ , generally cannot be replaced by explicit definitions, we can still explore how these bounds change with small variations in prices, laying the groundwork for calculating partial derivatives and finding optimal solutions.

These bounds correspond to the intervals  $\Lambda_{kj}^{min}(\mathbf{r})$  and  $\Lambda_{kj}^{max}(\mathbf{r})$  for  $k > j$  and are defined as the intersection points of two curves, provided these intervals are non-empty (for an illustrative reference, see Fig. 2). By identifying these bounds as

intersection points, we establish that for every upper bound  $\bar{l}_{kj}(\mathbf{r})$  (with a single exception as noted in Remark 1), there exists a corresponding lower bound  $l_{ij}(\mathbf{r})$  such that  $\bar{l}_{kj}(\mathbf{r}) = l_{ij}(\mathbf{r})$  and  $\frac{r_k - r_j}{\sum_{m=j}^{k-1} (\bar{l}_{kj}(\mathbf{r}))^m} = \frac{r_i - r_j}{\sum_{m=j}^{i-1} (l_{ij}(\mathbf{r}))^m}$ .

We now want to discuss the impact of a small price variation, specifically changing  $r_m$  while keeping other batch prices constant. For  $m \in \{j, k, i\}$ , this would obviously shift the location where both curves intersect. However, as long as the price variation is sufficiently small, it would not affect which intervals are adjacent and connected at the intersection point, i.e. the equation  $\bar{l}_{kj}(\mathbf{r}) = l_{ij}(\mathbf{r})$  would still hold, albeit with a different value. For  $m \notin \{j, k, i\}$ , a sufficiently small price variation would neither change the matching of  $\bar{l}_{kj}(\mathbf{r})$  and  $l_{ij}(\mathbf{r})$  nor affect the value of these two bounds.

However, there are rare instances of ambiguity that we have not explicitly addressed. Specifically, when the price vector results in three curves intersecting at a single point, any two of these three curves can be used to determine the intersection point. In this scenario, one of the corresponding intervals consists of exactly one element, which is the intersection point. This ambiguity resolves with even arbitrarily small changes in one of the prices  $r_m$  associated with these curves, as the three curves no longer intersect at the same spot, causing the one-element interval to either become empty or expand to cover an infinite number of elements. In the first case, we would use the other two curves to determine the intersection point. In the second case, we would determine two intersection points using the combinations of the curve belonging to the one-element interval with both other curves. While these cases of ambiguity are not explicitly covered in the remainder of this section, please note that they can be easily resolved.

The following remark summarizes the observations above and will come in handy in the development of optimality conditions.

**Remark 2** *It holds that:*

- For every  $\bar{l}_{kj}(\mathbf{r}) \neq \bar{l}_j^{\text{end}}(\mathbf{r})$  there is  $l_{ij}(\mathbf{r})$  such that  $\bar{l}_{kj}(\mathbf{r}) = l_{ij}(\mathbf{r})$  and  $\frac{r_k - r_j}{\sum_{m=j}^{k-1} (\bar{l}_{kj}(\mathbf{r}))^m} = \frac{r_i - r_j}{\sum_{m=j}^{i-1} (l_{ij}(\mathbf{r}))^m}$ . Moreover,  $\frac{\partial}{\partial r_m} \bar{l}_{kj}(\mathbf{r}) = \frac{\partial}{\partial r_m} l_{ij}(\mathbf{r})$  for all  $m$  and  $\frac{\partial}{\partial r_m} \bar{l}_{kj}(\mathbf{r}) = \frac{\partial}{\partial r_m} l_{ij}(\mathbf{r}) = 0$  for  $m \notin \{j, k, i\}$ .
- $l_{j+1}(\mathbf{r}) = l_{j+1,j}(\mathbf{r})$  with  $\frac{r_{j+1} - r_j}{(l_{j+1}(\mathbf{r}))^j} = 1$ . Moreover,  $\frac{\partial}{\partial r_m} l_{j+1}(\mathbf{r}) = \frac{\partial}{\partial r_m} l_{j+1,j}(\mathbf{r})$  for all  $m$  and  $\frac{\partial}{\partial r_m} l_{j+1}(\mathbf{r}) = \frac{\partial}{\partial r_m} l_{j+1,j}(\mathbf{r}) = 0$  for  $m \notin \{j, j + 1\}$ .
- $l_{j-1}(\mathbf{r}) = l_{j,j-1}(\mathbf{r})$  with  $\frac{r_j - r_{j-1}}{(l_{j-1}(\mathbf{r}))^{j-1}} = 1$ . Moreover,  $\frac{\partial}{\partial r_m} l_{j-1}(\mathbf{r}) = \frac{\partial}{\partial r_m} l_{j,j-1}(\mathbf{r})$  for all  $m$  and  $\frac{\partial}{\partial r_m} l_{j-1}(\mathbf{r}) = \frac{\partial}{\partial r_m} l_{j,j-1}(\mathbf{r}) = 0$  for  $m \notin \{j - 1, j\}$ .

### 4.3 Optimality conditions

With the previous section, we gathered enough information regarding the probability function to advance to one of our main goal: finding the optimal solution to (7).

Before we engage in the partial differentiation of the objective function, we first want to elaborate more on the partial differentiation of probability function (8). The calculation of  $\frac{\partial}{\partial r_i} p_j(\mathbf{r})$  varies a little depending on the following three cases:  $i > j$ ,  $i < j$ , and  $i = j$ .

**Lemma 5** *It holds:*

$$\begin{aligned}
 (1) \quad & \text{For } i > j, \frac{\partial}{\partial r_i} p_j(\mathbf{r}) = \int_{L_{ij}(\mathbf{r})}^{\bar{l}_{ij}(\mathbf{r})} \frac{1}{\sum_{m=j}^{i-1} l^m} f_{\omega} \left( \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \right) f_{\lambda}(l) dl. \\
 (2) \quad & \text{For } i < j, \frac{\partial}{\partial r_i} p_j(\mathbf{r}) = \int_{L_{ji}(\mathbf{r})}^{\bar{l}_{ji}(\mathbf{r})} \frac{1}{\sum_{m=i}^{j-1} l^m} f_{\omega} \left( \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} \right) f_{\lambda}(l) dl. \\
 (3) \quad & \text{For } i = j, \frac{\partial}{\partial r_i} p_j(\mathbf{r}) = - \sum_{k=i+1}^c \int_{L_{ki}(\mathbf{r})}^{\bar{l}_{ki}(\mathbf{r})} \frac{1}{\sum_{m=i}^{k-1} l^m} f_{\omega} \left( \frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} \right) f_{\lambda}(l) dl - \sum_{k=0}^{i-1} \int_{L_{ik}(\mathbf{r})}^{\bar{l}_{ik}(\mathbf{r})} \frac{1}{\sum_{m=k}^{i-1} l^m} f_{\omega} \left( \frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m} \right) f_{\lambda}(l) dl.
 \end{aligned}$$

**Proof:** See Supplement S.6.

With additional knowledge about the probability function, we now can turn our focus on the first-order condition. Therefore, we calculate the partial derivatives of the objective function of (7):

$$\begin{aligned}
 & \frac{\partial}{\partial r_i} \left( \sum_{j=1}^c p_j(\mathbf{r}) \cdot (r_j - \Delta_j V_{t-1}(c)) \right) \\
 &= p_i(\mathbf{r}) + \sum_{j=1}^c \left( \frac{\partial}{\partial r_i} p_j(\mathbf{r}) \right) \cdot (r_j - \Delta_j V_{t-1}(c)) \\
 &= p_i(\mathbf{r}) + \sum_{j=1}^{i-1} \int_{L_{ij}(\mathbf{r})}^{\bar{l}_{ij}(\mathbf{r})} \frac{1}{\sum_{m=j}^{i-1} l^m} f_{\omega} \left( \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \right) f_{\lambda}(l) dl \cdot (r_j - \Delta_j V_{t-1}(c)) \\
 &\quad - \sum_{k=i+1}^c \int_{L_{ki}(\mathbf{r})}^{\bar{l}_{ki}(\mathbf{r})} \frac{1}{\sum_{m=i}^{k-1} l^m} f_{\omega} \left( \frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} \right) f_{\lambda}(l) dl \cdot (r_i - \Delta_i V_{t-1}(c)) \\
 &\quad - \sum_{k=0}^{i-1} \int_{L_{ik}(\mathbf{r})}^{\bar{l}_{ik}(\mathbf{r})} \frac{1}{\sum_{m=k}^{i-1} l^m} f_{\omega} \left( \frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m} \right) f_{\lambda}(l) dl \cdot (r_i - \Delta_i V_{t-1}(c)) \\
 &\quad + \sum_{j=i+1}^c \int_{L_{ji}(\mathbf{r})}^{\bar{l}_{ji}(\mathbf{r})} \frac{1}{\sum_{m=i}^{j-1} l^m} f_{\omega} \left( \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} \right) f_{\lambda}(l) dl \cdot (r_j - \Delta_j V_{t-1}(c)).
 \end{aligned}$$

While the first-order condition is a necessary condition for local maxima, the global maximum over a closed and bounded space does not generally need to satisfy this condition. However, the following propositions states that, regardless of whether the optimal solution lies on the boundary of the action space  $\mathcal{R}_c$  or within its interior, it always satisfies the first-order condition.

**Proposition 1** *The optimal solution of (7) meets for every batch size  $i$  the first-order condition:*

$$\frac{\partial}{\partial r_i} \left( \sum_{j=1}^c p_j(\mathbf{r}) \cdot (r_j - \Delta_j V_{t-1}(c)) \right) = 0.$$

**Proof:** See Supplement S.7.

**Remark 3** *For  $\omega \sim U[0, 1]$  the first-order condition simplifies to*  

$$\frac{\partial}{\partial r_i} \left( \sum_{j=1}^c p_j(\mathbf{r}) \cdot (r_j - \Delta_j V_{t-1}(c)) \right) = 2p_i(\mathbf{r}) - (F_\lambda(l_{i+1}(\mathbf{r})) - F_\lambda(l_i(\mathbf{r})))$$

$$- \sum_{k=i+1}^c \int_{l_{-k}(\mathbf{r})}^{\bar{l}_{ik}(\mathbf{r})} \frac{\Delta_k V_{t-1}(c) - \Delta_i V_{t-1}(c)}{\sum_{m=i}^{k-1} l^m} f_\lambda(l) dl + \sum_{k=0}^{i-1} \int_{l_{-k}(\mathbf{r})}^{\bar{l}_{ik}(\mathbf{r})} \frac{\Delta_i V_{t-1}(c) - \Delta_k V_{t-1}(c)}{\sum_{m=k}^{i-1} l^m} f_\lambda(l) dl .$$

*Moreover, it holds that  $\sum_{i=1}^c p_i(\mathbf{r}^*) = \frac{1}{2} - \sum_{k=1}^c \int_{l_{k0}(\mathbf{r}^*)}^{\bar{l}_{k0}(\mathbf{r}^*)} \frac{\Delta_k V_{t-1}(c)}{\sum_{m=0}^{k-1} l^m} f_\lambda(l) dl$ . Thus, the overall selling probability is less than or equal to 0.5 and decreasing with opportunity costs.*

In the final period ( $t = 1$ ), there are no opportunity costs, so Proposition 1 applies with  $\Delta_j V_{t-1}(c) = 0$  for all  $j$  in these states. To conclude this section, we focus on a scenario without opportunity costs, where both random variables  $\lambda$  and  $\omega$  are uniformly distributed. Under these conditions, finding the optimal solution in the final states further simplifies. Moreover, the optimal solution does not lie on the boundary of our action space, implying that none of the corresponding selling probabilities are reduced to zero.

**Proposition 2** *Let  $\omega, \lambda \sim U[0, 1]$ . If  $\Delta_k V_{t-1}(c) = 0$  for every  $k \leq c$ , the optimal solution for (7) is an interior point of  $\mathcal{R}_c$  and fulfills  $p_i(\mathbf{r}) = \frac{1}{2} (l_{i+1}(\mathbf{r}) - l_{-i}(\mathbf{r}))$  for every  $i \leq c$ .*

**Proof:** See Supplement S.8.

Although we have formulated the optimality conditions, finding the solution that fulfills these equations within each state is a difficult task. We are facing a system with  $c$  nonlinear equations that are additionally plagued by integrals and implicitly defined variables. Many of these challenges arise from the analytical intractability of the choice model. To address this, we explore a modified optimization problem in the following section. This alternative formulation is simpler to solve and provides

an upper bound for the original problem. Notably, under specific conditions, the optimal solution of the modified formulation also constitutes the optimal solution to our original problem.

### 4.4 Modified formulation

In this section, we propose a modification to our choice model, which leads to a simplified problem that acts as an upper bound to our original problem. The main aspect of this formulation, however, is the potential for its optimal solution to also be the optimal solution to the original problem. Specifically, we develop easy-to-verify conditions for this solution to ensure its optimality with respect to the original problem.

The modification of the customer choice model involves reducing the competition between different options. In particular, instead of calculating the probability that  $U_k = \max_{j=0, \dots, c} \{U_j\}$ , we now calculate the probability that  $U_k = \max_{j=0, k-1, k, k+1} \{U_j\}$ . Thereby, we reduce competition between available options and, thus, only compare four instead of  $c + 1$  options. We denote demand by this modified choice model as  $p_j^m(\mathbf{r})$  and write  $p_j^m(\mathbf{r}) = \mathbb{P}\left(U_j = \max_{k=0, j-1, j, j+1} \{U_k\}\right) \geq p_j(\mathbf{r})$ . A state-wise optimization with  $p_j^m(\mathbf{r})$  therefore serves as an upper bound to (7).

Technically, through the modification, the choice model is no longer a choice model. By summing up all probabilities, we get a value greater than or equal to 1, i.e.  $\sum_{j=0}^C p_j^m(\mathbf{r}) \geq \sum_{j=0}^C p_j(\mathbf{r}) = 1$ , which does not satisfy one of the main properties of choice models.

Similar considerations as in Sect. 4.3 lead to the following remark.

**Remark 4** When  $p_j^m(\mathbf{r})$  is used instead of  $p_j(\mathbf{r})$ , Propositions 1 and 2 still hold.

This modification significantly reduces complexity in our choice model (see Supplement S.9 for an exemplary calculation of  $p_j^m(\mathbf{r})$  for a scenario where  $\omega, \lambda \sim U[0, 1]$ ), thus improving the determination of the optimal solution  $\mathbf{r}^m$  for the modified formulation compared to finding the optimal solution of the original problem. Utilizing the fact that the modified model constitutes an upper bound to the original model, it suffices to verify that its optimal solution  $\mathbf{r}^m$  is feasible for the original model and results in the same objective value. To facilitate this verification, we propose an approach that avoids the exhaustive checking whether  $p_j^m(\mathbf{r}^m) = p_j(\mathbf{r}^m)$  for every  $j$ . Instead, we introduce a set of straightforward and verifiable conditions, designed to simplify the process of proving  $\mathbf{r}^m$ 's optimality for the original model.

**Proposition 3** If  $\mathbf{r}$  fulfills  $\bar{l}_j^{end}(\mathbf{r}) \leq \bar{l}_{j+1}^{end}(\mathbf{r})$  for every  $j$ , with  $1 \leq j < c$ , then  $p_j^m(\mathbf{r}) = p_j(\mathbf{r})$  for every  $j$ .

**Proof:** See Supplement S.10.

**Remark 5** *The conditions of Proposition 3 are satisfied by linear pricing schemes. Additionally, piecewise linear pricing schemes, where a distinct price is set for the first unit and a constant marginal price applies to any additional unit, also satisfy the conditions of Proposition 3.*

In our numerical study, we calculated the optimal solution for the modified model for every combination of  $T \leq 40$  and  $C \leq 120$ , with  $\omega, \lambda \sim U[0, 1]$ . The conditions of Proposition 3 were always satisfied, showing that, in these instances, the optimal solution obtained from the modified model is also optimal for the original problem. However, our numerical study reveals that solving each state of the modified model leads to long computation times for medium-sized problems (see Sect. 6.2). Therefore, we aim to address these drawbacks by developing efficient and effective heuristics.

## 5 Heuristic design

Dynamic pricing decisions often need to be made online and in real-time. Given the long computation times required to find the optimal solution, we developed three heuristics to tackle the dynamic optimization problem (5). These heuristics aim to compute well-performing policies efficiently, with short runtimes, ensuring practical applicability in dynamic nonlinear pricing scenarios.

In this section, we introduce three heuristics to address the dynamic optimization problem (5). Two of these approaches, presented in Sect. 5.1, build on the results of Schur (2024) and use the optimal solution in a setting where the firm has access to customers' private information, i.e., their base willingness-to-pay and their consumption indicator, respectively. The third approach, presented in Sect. 5.2, can be described as a decomposition in units. Thereby, we allow customers to buy the  $j$ -th unit of the product without buying the units  $1, 2, \dots, j-1$ . Even though this does not reflect reality, it constitutes an easy to solve optimization problem, enabling us to devise batch pricing strategies for the original problem based on the solutions obtained.

### 5.1 Approaches 1 and 2: expected optimal batch prices

We can efficiently compute realization-dependent optimal batch prices  $r_{jt}(c|w)$  and  $r_{jt}(c|l)$  for realization  $w$  and  $l$ , respectively. Technically, these realization-dependent batch prices are themselves random variables, raising the following idea: By calculating the expected value of these realization-dependent optimal batch prices, we construct a policy for optimization problem (5).

Both approaches follow the same idea, differing only in the determination of the realization-dependent optimal batch prices:  $r_{jt}(c|w)$  for Approach 1 and  $r_{jt}(c|l)$  for Approach 2. Beyond this distinction, both approaches follow the same subsequent steps. Consequently, we will explain the remaining steps without distinguishing

between both approaches and write  $r_{jt}(c|x)$  instead of  $r_{jt}(c|w)$  and  $r_{jt}(c|l)$  to denote the realization-dependent optimal batch prices.

This framework is applicable across a wide spectrum of distribution functions, including, but not limited to, uniform, triangular, normal, exponential, Weibull, Gumbel, and gamma distributions, along with their truncated versions, albeit with some constraints on parameter selections. Notably, for scenarios where  $\omega \sim U[0, 1]$ , Approach 2 provides a closed-form expression for the realization-dependent optimal batch prices:  $r_{jt}(c|l) = \frac{1}{2} \left( \sum_{k=0}^{j-1} l^k + \Delta_j V_{t-1}^E(c) \right)$ , with  $j \leq N_t(c|l) = \max_{j=1, \dots, c} \{j : \Delta_1 V_{t-1}^E(c-j+1) < l^{j-1}\}$ .

Building on these realizations-dependent optimal batch prices, we compute expected optimal batch prices:

$$r_{jt}^E(c) = \frac{\int_0^1 r_{jt}(c|x) \cdot 1_{\{j < N_t(c|x)\}} f(x) dx}{\int_0^1 1_{\{j < N_t(c|x)\}} f(x) dx} \quad \text{for } j = 1, \dots, c. \tag{9}$$

More precisely, this formulation results in conditional expected optimal batch prices, where we only take realizations of  $\lambda$  and  $\omega$  into account that lead to possible economic sales, i.e.  $r_{j+1,t}(c|\cdot) - r_{jt}(c|\cdot) \geq \Delta_1 V_{t-1}^E(c+1-j)$ . Other realizations are economically irrelevant and can distort results, given the lack of a clear pricing strategy in these cases. Thus, these events where we refrain from selling are not used to compute our policy.

Except for the scenario where we have a closed-form expression of  $r_{jt}(c|l)$ , we resort to numerically calculating  $r_{jt}(c|x)$  for several realizations  $x$  to accurately derive  $r_{jt}^E(c)$  (in Sect. 6, we use a sample size of 100; for an analysis of the tradeoff between accuracy and runtimes, refer to Supplement S.11). Finally, we calculate the expected revenue-to-go derived by expected optimal batch prices  $r_{jt}^E(c)$ :

$$V_t^E(c) = \sum_{j=1}^c p_j(r_t^E(c)) \cdot \left( r_{jt}^E(c) + V_{t-1}^E(c-j) \right) + \left( 1 - \sum_{j=1}^c p_j(r_t^E(c)) \right) \cdot V_{t-1}^E(c), \tag{10}$$

with the same boundary conditions as the original problem (5).

**Remark 6** *By using suboptimal batch prices, we get a lower bound to optimization problem (5). Thus, it holds that  $V_t(c) \geq V_t^E(c)$  for every  $(t, c)$ .*

We sum up the first two heuristics that are based upon the idea of expected optimal batch prices by the following pseudo code:

---

**Pseudo code Approaches 1 and 2: Expected optimal batch prices**


---

**Input:** time horizon  $T$ , starting stock  $C$ 
**Output:** batch prices for every batch size  $j$  and state  $(t, c)$ :  $r_j^E(c)$ 


---

1. Initialize  $V_0^E(c) = 0$  for every  $c = 0, 1, \dots, C$  and  $V_t^E(0) = 0$  for every  $t = 0, 1, \dots, T$   $\triangleright$  initialize boundary conditions
  2. **For**  $t = 1, 2, \dots, T$  **do**  $\triangleright$  loop over time horizon
    - 2.1. **For**  $c = 1, 2, \dots, C$  **do**  $\triangleright$  loop over capacity
      - 2.1.1. Calculate  $r_j^E(c)$  for every  $j = 1, 2, \dots, c$   $\triangleright$  calculate expected optimal batch prices
      - 2.1.2. Calculate  $V_t^E(c)$  with (10)  $\triangleright$  calculate expected revenue-to-go
- 

## 5.2 Approach 3: decomposition in units

Our next algorithm employs a decomposition strategy. The basic idea is that customers have the flexibility to buy the  $j$ -th unit of the product even though they might not buy units 1 to  $j - 1$ . As every unit of the product is the same, there is no distinction between the 1st, 2nd or  $j$ -th unit other than the number customers have already in their basket. Thus, this decomposition is merely theoretical without having immediate practical applicability. However, it results in a greatly simplified optimization problem. A hypothetical customer now faces  $c$  distinct binary decisions instead of one decision with  $c + 1$  options. This, in turn, enables us to solve  $c$  distinct and rather simple independent optimization problems instead of one complex problem. With this simplification, we can derive batch prices that can form a policy to address optimization problem (5).

To consider this decomposition, we must change the customer choice model. Customers still strive to maximize their utility. But, instead of purchasing  $j$  units if and only if  $U_j = \max_{j=0, \dots, c} \{U_j\}$  with  $U_0 = 0$  denoting the no-purchase option, they decide for every single unit whether they want to purchase it or not. This decision is based upon whether the additional willingness-to-pay for the  $j$ -th unit is at least as high as the additional price the customer has to pay, i.e.  $X_j - X_{j-1} = \omega \cdot (\lambda)^{j-1} \geq r_j - r_{j-1}$ . If the customers decide to purchase the  $j$ -th unit, they must pay  $r_j - r_{j-1}$ . For example, for given batch prices, a customer might only be willing to purchase the second and fourth unit due to the willingness-to-pay curve. In this case, the customer pays  $(r_2 - r_1) + (r_4 - r_3)$  to get 2 units of the product in total.

The decomposition approach reduces the complexity of the choice model. The model itself becomes easier as the decision between several options is broken down to several binary independent decisions. This method avoids the need to determine a single price vector that encompasses all batch prices and to predict the customer's

response to this vector comprehensively. Instead, the firm’s pricing strategy is divided into  $c$  separate decisions, each focused on setting the price for the  $j$ -th unit, while considering the probability of its purchase (which happens solely based on its price). Therefore, the decision variable becomes the price of the  $j$ -th unit,  $r^{Dj} = r_j - r_{j-1}$ . The optimization problem we are now focusing on is given by:

$$\sum_{j=1}^c \left( \max_{r^{Dj} \geq 0} \{ p^{Dj}(r^{Dj}) \cdot (r^{Dj} - \Delta_1 V_{t-1}^D(c - (j - 1))) \} \right) + V_{t-1}^D(c), \tag{11}$$

with  $p^{Dj}(r^{Dj})$  denoting the probability that a customer is purchasing the  $j$ -th unit of the product and  $V_{t-1}^D(c)$  denoting the expected revenue-to-go.

Probability  $p^{Dj}(r^{Dj})$  only depends on  $r^{Dj}$  and can be calculated by  $p^{Dj}(r^{Dj}) = \int_0^1 \int_0^1 \mathbb{1}_{\{w \cdot (l)^{j-1} \geq r^{Dj}\}}(w, l) f_\omega(w) f_\lambda(l) dw dl = \int_0^1 1 - F_\omega\left(\frac{r^{Dj}}{l^{j-1}}\right) f_\lambda(l) dl$ . Optimization problem (11), which is the sum of  $c$  single unit dynamic pricing problems, greatly simplifies the search for the optimal solution. With  $r_t^{Dj}(c)$  denoting the optimal solution to (11), we can use  $r_{jt}^D(c) = \sum_{i=1}^j r_t^{D,i}(c)$  to design a policy for the original optimization problem (5). Lastly, applying our original probability function  $p_j(\mathbf{r})$ , we calculate the expected revenue-to-go derived by  $r_{jt}^D(c)$ :

$$V_t^D(c) = \sum_{j=1}^c p_j(r_{jt}^D(c)) \cdot (r_{jt}^D(c) + V_{t-1}^D(c - j)) + \left( 1 - \sum_{j=1}^c p_j(r_{jt}^D(c)) \right) \cdot V_{t-1}^D(c). \tag{12}$$

The boundary conditions are again given by  $V_0^D(c) = 0$  for  $c \geq 0$  and  $V_t^D(0) = 0$  for  $t \geq 0$ .

**Remark 7** *By using suboptimal batch prices, we get a lower bound to optimization problem (5). Thus, it holds that  $V_t(c) \geq V_t^D(c)$  for every  $(t, c)$ .*

The heuristic can be summed up by the following pseudo code:

---

**Pseudo code Approach 3: Decomposition of units**

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**Input:** time horizon  $T$ , starting stock  $C$

**Output:** batch prices for every batch size  $j$  and state  $(t, c)$ :  $r_{jt}^D(c)$

---

1. Initialize  $V_0^D(c) = 0$  for every  $c = 0, 1, \dots, C$  and  $V_t^D(0) = 0$  for every  $t = 0, 1, \dots, T$  ▷ initialize boundary conditions
2. **For**  $t = 1, 2, \dots, T$  **do** ▷ loop over time horizon
  - 2.1. **For**  $c = 1, 2, \dots, C$  **do** ▷ loop over capacity
    - 2.1.1. Solve optimization problem (11) to get  $r_{jt}^{D,j}(c)$  for every  $j = 1, 2, \dots, c$  ▷ calculate optimal price for the  $j$ th unit for every  $j$
    - 2.1.2. Calculate  $r_{jt}^D(c) = \sum_{i=1}^j r_{jt}^{D,i}(c)$  for every  $j = 1, 2, \dots, c$  ▷ calculate proxy for optimal batch prices
    - 2.1.3. Calculate  $V_{t-1}^D(c)$  with (12) ▷ calculate expected revenue-to-go

---

## 6 Numerical studies

In this section, we evaluate the performance of the heuristics introduced in the previous section by comparing them with the results of the optimal solution. This analysis offers insights into the effectiveness of the proposed heuristics and their ability to deliver well-performing policies across various scenarios. We assess the revenue potential of different pricing schemes by comparing the following mechanisms:

- *Opt*: This mechanism employs the optimal solution to (5) (facilitated by Sect. 4)
- $E(\lambda)$  and  $E(\omega)$ : These mechanisms are based on the concept of expected optimal batch prices (Sect. 5.1).
- *D*: This mechanism is based on a decomposition in units (Sect. 5.2).
- *FA*: This mechanism employs a static policy based on the optimal solution of a fluid approximation (refer to Supplement S.12)
- *S*: This mechanism solves a standard singleunit dynamic pricing problem and, thereby, ignores the fact that customers might be willing to purchase more than just one unit. As a singleunit dynamic pricing procedure results in a price for only one unit of the product, we then extend it in a linear manner to get batch prices.
- *L*: This mechanism solves (5) but with the additional constraint that batch prices follow a linear pricing scheme, i.e.  $r_j = j \cdot r_1$  for every  $j$ . This simplifies the optimization problem by reducing it to a single decision variable.
- *PL*: This mechanism solves (5) but with the additional constraint that batch prices follow a piecewise linear pricing scheme, with  $r_1$  and  $r_j = (j - 1) \cdot (r_2 - r_1) + r_1$  for  $j \geq 2$ .

Besides our heuristics, we selected *FA* as a benchmark because the static policy derived by the solution of a fluid approximation is asymptotically optimal. Comparing *FA* and *Opt* highlights the advantage of dynamically adjusting batch prices over time. Notably, the analytical results from Sect. 4 facilitate finding the optimal solution to the fluid approximation. Additionally, we included *S* and *L* in our numerical studies, as they highlight the importance of considering multiunit demand (by comparing the results of *S* and *L*) and nonlinear pricing (by comparing the results of *L* and *Opt*). Both mechanisms result in a linear pricing scheme, which mirrors a common pricing structure often observed in retail when no special volume discounts are applied. These schemes also have the benefit of being easy-to-communicate. Mechanism *S* is conducting standard dynamic pricing and, thus, ignoring multiunit demand. In contrast, firms that recognize customers may purchase more than a single unit are likely to adopt mechanism *L*, as this is the obvious choice without specialized optimization problems that merely exist in literature. The last mechanism, denoted as *PL*, leads to a pricing scheme with a simple structure, making it easy to implement in practice. In this scheme, the firm only needs to quote a price for the first and a price for subsequent units. Thereby, *PL* offers a balanced approach,

combining the applicability of pricing schemes like those from  $S$  and  $L$  with the potential performance advantage of nonlinear pricing schemes.

Every mechanism provides a policy that contains batch prices. To evaluate the mechanisms, we perform simulation studies. We choose the setting for our study similar to Gallego et al. (2020), resulting in the choice of  $T = 1, \dots, 40$ ,  $C = 1, \dots, 120$ , and the uniform distribution of  $\omega$  and  $\lambda$ , i.e.  $\omega, \lambda \sim U[0, 1]$ . This setting forms the basis for all subsequent analyses in this section. We generated 10,000 customer streams in advance and applied every policy we derived from the mechanisms to these streams separately. One simulation run consists of a complete sales process containing  $T$  specific customers and a batch price quoted in every period (depending on the capacity left) according to the mechanism investigated. After observing the decision of the current customer, a new batch price is set for the next customer. Repeating this procedure until the end of the selling horizon leads to a total revenue for this simulation run. By averaging the total revenues from 10,000 simulation runs, we obtain a mean revenue for each mechanism. Since all mechanisms are evaluated using the same 10,000 customer streams, their simulated revenues are directly comparable. To assess the reliability of the mean revenues as an approximation for the expected revenues, we analyze the 95% confidence intervals in Supplement S.13 and find them to be very tight. The experiments were implemented using MATLAB version R2022a and conducted on a PC equipped with an Intel Core i9-12,900 processor and 128 GB of RAM, running Microsoft Windows 11.

In Sect. 6.1, we present the simulated revenues of all mechanisms and compare them to the revenues generated by the optimal solution. Thereby, we observe that mechanisms  $D$  and  $E(\omega)$  yield the highest revenues, with  $E(\lambda)$  performing slightly less effectively. Notably, a piecewise linear pricing scheme also performs remarkably well. In Sect. 6.2, we analyze the runtimes of all mechanisms, highlighting the varying levels of efficiency across them. Finally, Sect. 6.3 discusses selling strategies and evolutions of batch prices of the two best performing mechanisms,  $D$  and  $E(\omega)$ .

## 6.1 Comparison in terms of revenue earned

In this section, we show the simulated revenues of the mechanisms for the investigated scenarios. To shorten tables and give a more lucid overview of the study, we provide only a subset of the studied settings in the following table.

Table 1 shows simulated revenues of all mechanisms with  $C \in \{1, 30, 60, 90, 120\}$ . For  $C = 1$ , every mechanism (except of the static policy derived from  $FA$ ) performs identical as there is not enough stock to sell more than one unit, and thus, the consideration of multiunit purchases is not possible. As we can see, the developed heuristics (mechanisms  $E(\cdot)$  and  $D$ ) are also able to calculate the optimal price for the singleunit case. The more capacity, the higher the importance of attending customers' demand for more than one unit. This can be seen by comparing mechanisms  $S$  and  $L$  where the only difference between those mechanisms is that  $L$  is aware of customers' multiunit demand while  $S$  is ignoring it. Except  $C = 1$ ,  $L$  is outperforming  $S$  in every scenario. This holds all the same for the cases we did not incorporate in this

paper. On the other hand,  $L$  is dominated by our heuristics. Even though  $E(\cdot)$  and  $D$  are only heuristics while  $L$  finds optimal linear batch prices, the advantage of setting nonlinear batch prices (like  $E(\cdot)$  and  $D$  do) is overcompensating the incapability to solve optimization problem (5) to optimality. Mechanism  $PL$  falls between our heuristics and mechanism  $L$  in terms of revenue earned. While the gap between  $D$  and  $PL$  is notable,  $PL$  performs very well overall, suggesting that this easy-to-communicate pricing scheme could offer a reasonable trade-off between simplicity and performance. Finally, when comparing our heuristics to each other, we can observe that  $E(\lambda)$  results in the lowest expected revenues while  $E(\omega)$  and  $D$  are performing nearly identical.

Applying  $E(\cdot)$  and  $D$  is clearly beneficial in comparison to  $L$  and even more so to  $S$ . To get a feeling for the performance of all mechanisms, we present the simulated revenues as a percentage of the optimal objective value of optimization problem (5). Therefore, we divided the simulated revenues received from each mechanism by the revenues from  $Opt$ , thereby showing the percentage every mechanism obtains. These percentages are shown in the following figure. In Table 1, we have seen that  $E(\omega)$  and  $D$  are performing nearly identical. Thus, we opted to only show the line of  $D$  representing both mechanisms. We let the ordinate start at 0.8 to demonstrate differences between the curves more clearly.

Figure 3 shows the same order in the mechanisms for every capacity like Table 1, i.e.  $D \geq E \geq PL \geq L \geq S$ . The only exception is  $FA$ , whose revenues surpass the simulated revenues of two other mechanisms ( $S$  and  $L$ ) as capacity increases. This behavior indicates that the disadvantage of employing a static policy decreases with higher capacity. The gap between  $Opt$  and any other mechanism is increasing in capacity. However, all heuristics (especially  $D$  and  $E(\omega)$ ) are very close to the optimal objective value of optimization problem (5) and, thus, are very good approximations. In each scenario, mechanism  $D$  achieves more than 99.6% of the optimal mean revenues, highlighting this mechanism's excellent performance. Finally, we note that our heuristics are performing remarkably better (with a small plus for  $D$ ) than the mechanisms that result in a linear pricing scheme,  $L$  and  $S$ , although a piecewise linear pricing scheme is not far behind. Overall,  $D$  and  $E(\omega)$  are outperforming  $E(\lambda)$  particularly for higher capacities.

Moreover, Fig. 3 reveals an unusual behavior for mechanism  $S$  compared to the others (particularly mechanism  $L$ , which is its most similar mechanism). Since mechanism  $S$  ignores multiunit demand, it suffers from two opposing effects: it

**Table 1** Revenues for  $C \leq 120$ ,  $T = 40$ , and  $\omega, \lambda \sim U[0, 1]$

$T = 40$	$Opt$	$E(\lambda)$	$E(\omega)$	$D$	$FA$	$S$	$L$	$PL$
$C = 1$	0.91	0.91	0.91	0.91	0.80	0.91	0.91	0.91
$C = 30$	16.29	16.18	16.23	16.27	14.89	15.12	15.74	15.94
$C = 60$	24.17	23.81	24.06	24.12	22.82	22.16	22.86	23.53
$C = 90$	29.17	28.56	29.02	29.08	27.80	25.24	27.32	28.39
$C = 120$	32.66	31.83	32.50	32.54	30.98	27.02	30.43	31.85

underestimates future demand (leading to prices that are too low at low capacities) and simultaneously misses opportunities to actively influence purchase quantities (resulting in prices that are too high at high capacities). At an initial capacity of  $C = 40$ , the opposing effects in mechanism  $S$  partially cancel each other out, yielding prices similar to those of mechanism  $L$  (which anticipates multiunit demand) as the influence of both effects diminishes at moderate capacities.

### 6.2 Comparison in terms of runtimes

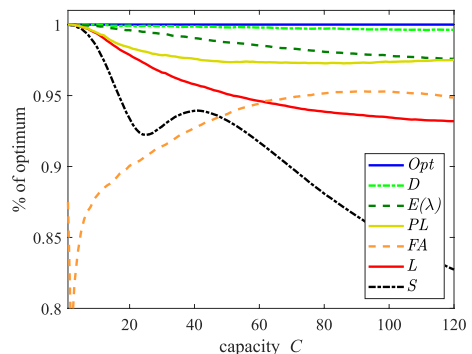
In the previous section, we demonstrated the effectiveness of mechanisms  $D$  and  $E(\omega)$  in finding near-optimal policies, alongside strong performance by our third heuristic and the piecewise linear pricing scheme. To assess their efficiency, we compare the runtimes (in seconds) required to compute these policies.

Table 2 presents the runtimes of all mechanisms described in Sect. 6 for the settings  $C \in \{30, 60, 90, 120\}$  and  $T = 40$ . While the optimal linear solution ( $L$ ) can be computed efficiently in seconds ( $\leq 107s$ ), finding the optimal piecewise linear ( $PL$ ) and nonlinear solutions ( $Opt$ ) takes significantly longer, ranging from hours ( $\leq 97,916s \approx 27.2h$ ) to several days ( $\leq 416,419s \approx 4.82d$ ), respectively. Given that dynamic pricing often needs to be optimized in real-time, these mechanisms are impractical for real-world applications due to their slow runtimes.

In contrast, our heuristics are substantially faster, with runtimes ranging from under 36 s ( $E(\lambda)$ ) to less than 26 min ( $E(\omega)$ ). Mechanism  $D$ , despite its near-optimal revenues, runs in about 2.5 min–10 times faster than  $E(\omega)$  and 636 times faster than  $PL$ . This makes  $D$  the best trade-off between speed and performance. For scenarios with stricter runtime constraints,  $E(\lambda)$  is the most viable choice due to its exceptional speed.

Figure 4 illustrates runtime scaling with problem size, where  $(C, T) = (3 \cdot i, i)$  for  $i = 1, \dots, 40$ . Mechanism  $E(\omega)$  shows the steepest runtime increase, reflecting its high sensitivity to problem size. Mechanisms  $D$  and  $E(\lambda)$  scale more moderately, demonstrating greater robustness. Unlike these mechanisms,  $FA$  computes a static policy through a single optimization, yet it is more sensitive to problem size than  $D$  and  $E(\lambda)$ , suggesting that the growing number of prices disproportionately impacts  $FA$ 's performance.

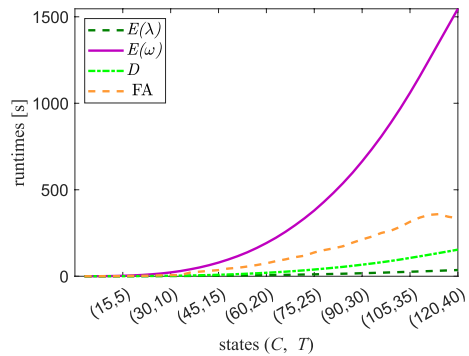
**Fig. 3** Performance of all mechanisms relative to the optimum for  $C \leq 120, T = 40$ , and  $\omega, \lambda \sim U[0, 1]$



**Table 2** Runtimes [s] for  $C \leq 120, T = 40$ , and  $\omega, \lambda \sim U[0, 1]$

$T = 40$	$Opt$	$E(\lambda)$	$E(\omega)$	$D$	$FA$	$S$	$L$	$PL$
$C = 30$	5,360	4	71	11	15	<1	10	1829
$C = 60$	49,440	11	345	41	76	<1	28	13,277
$C = 90$	176,392	21	848	90	211	<1	60	42,833
$C = 120$	416,419	36	1546	154	336	<1	107	97,916

**Fig. 4** Runtimes of heuristics and fluid approximation for  $(C, T) = (3i, i)$ , where  $i = 1, \dots, 40$ , and  $\omega, \lambda \sim U[0, 1]$



### 6.3 Selling strategies and pricing paths

Since both heuristics,  $D$  and  $E(\omega)$ , result in near optimal revenues, we analyze their strategies to gain insights into the behavior of well-performing policies. Therefore, we have a closer look at the simulated mean revenue and mean purchases at every point in time. We perform this (and any following) analysis on a smaller setting, to enhance the visibility of effects, and start the observation in state  $(C, T) = (20, 10)$ . Figure 5 illustrates the revenues earned and units purchased in each period, averaged across all customer streams. For each stream, we apply the policies derived by our mechanisms, record the outcomes for each period as the system transitions between states, and compute the overall averages.

Although  $D$  and  $E(\omega)$  perform similarly overall, their strategies clearly differ. Figure 5 shows that  $D$  is superior to  $E(\omega)$  in terms of mean revenue at the end of the selling horizon, while the opposite is true at the beginning. Mean revenues of each heuristic are decreasing over time. While  $D$  remains relatively stable throughout the whole selling process, the curve representing  $E(\omega)$  is declining at a more pronounced rate. Comparing both strategies to the optimal policy reveals that  $D$  closely mimics the behavior of the optimal solution, with mean revenues in each period closely aligned.

Mean purchases are around 1.5 for every mechanism. The curve for  $E(\omega)$  declines over time, while the curves of the other two mechanisms remain relatively stable. Throughout the entire selling horizon,  $E(\omega)$  sells more units than the other two mechanisms.  $D$  and  $Opt$  are selling similar quantities, with  $D$  consistently slightly below  $Opt$  in each period. This suggests that  $E(\omega)$  prices units lower than the optimal policy, while  $D$  appears to price them slightly too high.

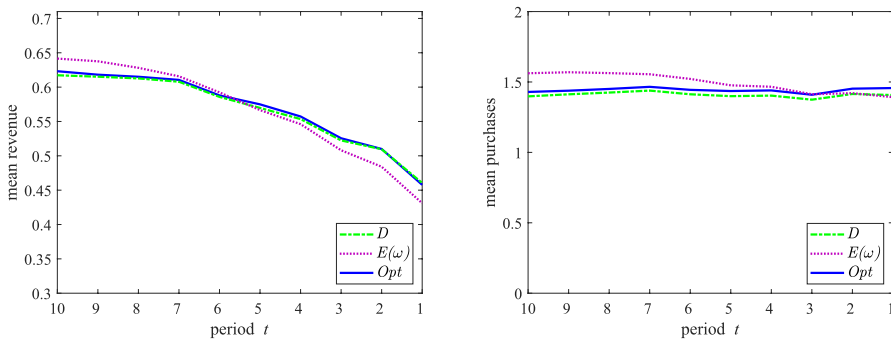
To summarize, mechanism  $D$  closely aligns with the optimal mechanism, mirroring mean revenues and mean purchases over time. In contrast, mechanism  $E(\omega)$  adopts a more aggressive pricing strategy, selling more units than  $D$  and the optimal policy over the entire selling horizon, seemingly by offering lower prices.

This pattern is also evident in Fig. 6, where we have depicted the evolution of batch prices for the scenario where no customer makes a purchase. Specifically, the figure shows batch prices in the states  $(c, t)$  for  $c = 20$  and  $t = 10, 9, \dots, 1$ . In the figure, the lowest curve corresponds to the price for a batch of size one, the second-lowest line represents the price for a batch of size two, and so on. Over the entire selling horizon, mechanism  $D$  quotes higher prices than mechanism  $E(\omega)$ . Additionally, prices from both mechanisms are decreasing over time. Despite the distinct approaches used to calculate  $r_t^D(c)$  and  $r_t^{E(\omega)}(c)$ , the resulting curves share a similar structure, indicating an underlying structure well-performing policies have in common.

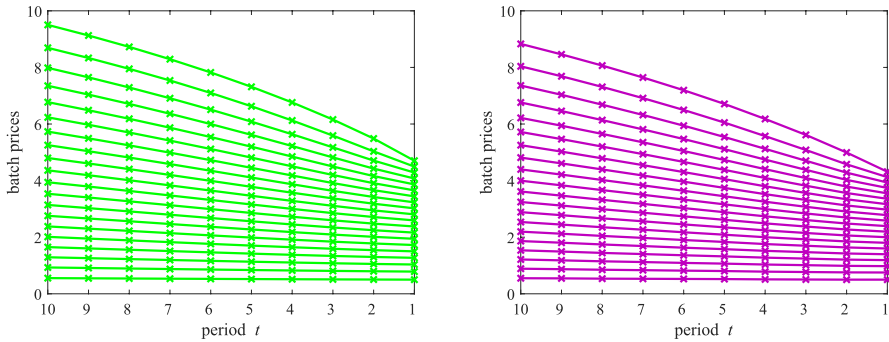
Batch prices for small numbers of units are virtually linear in batch size (apart from the first unit, the second, third, etc. units cost nearly the same). For large numbers of units, batch prices are convexly increasing in batch size. With a concavely increasing willingness-to-pay curve, these pricing schemes automatically prevent selling a large batch or even the whole stock ( $C = 20$ ) to only one customer. Finally, it is notable that prices for small batches merely change over time whereas prices for big batches noticeably decrease.

In Fig. 5, we have seen that both heuristics result in selling processes where a customer purchases approximately 1.5 units on average. Therefore, we also want to examine the evolution of batch prices in a scenario where alternately two units and one unit are sold, starting with a purchase of 2 units in  $t = 10$ . As a result, we analyze the states  $(c, t) = (20, 10), (18, 9), (17, 8), \dots, (6, 1)$ . As the firm cannot offer batches that are not covered by capacity any longer, most of the curves are terminated during the selling horizon.

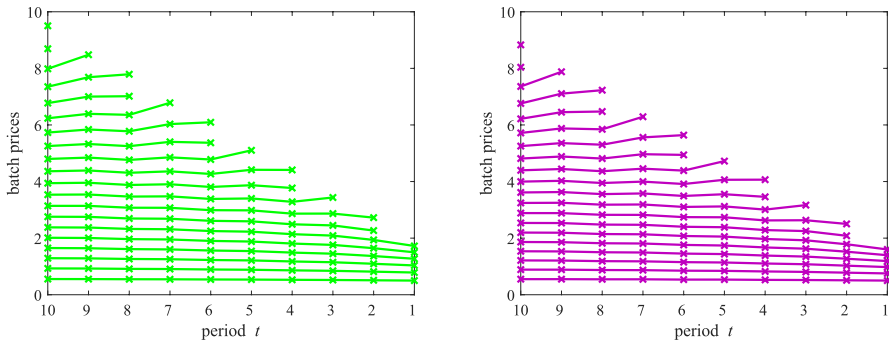
The pattern the curves draw looks nearly the same for both heuristics. Batch prices obtained by  $E(\omega)$  are lower than those obtained by  $D$ . The gaps between batch prices are nearly same-sized for smaller batches (again, starting with the two-unit



**Fig. 5** Mean revenues (left) and mean purchases (right) at every point in time,  $t = 10, 9, \dots, 1$  with  $C = 20$ , and  $\omega, \lambda \sim U[0, 1]$



**Fig. 6** Evolution of batch prices without a purchase for  $D$  (left) and  $E(\omega)$  (right) over  $t = 10, 9, \dots, 1$  with  $C = 20$ , and  $\omega, \lambda \sim U[0, 1]$



**Fig. 7** Evolution of batch prices with purchases at every period for  $D$  (left) and  $E(\omega)$  (right) over  $t = 10, 9, \dots, 1$  with  $C = 20$ , and  $\omega, \lambda \sim U[0, 1]$

batch) and are increasing for bigger batches. After selling, batch prices for bigger batches increase. This effect is more pronounced after selling two units in comparison to selling one unit. It is a well-observed pricing behavior in (standard) dynamic pricing that prices increase after a sale took place. However, this only holds partially in our multiunit setting as prices for small batches usually decrease slowly and steady over the selling horizon.

In conclusion, we have observed two notable effects in Figs. 6 and 7. First, prices for small batches (relative to the remaining stock) decrease over time, regardless of whether a sale takes place or not. This holds for the discussed settings with a reasonably large stock, i.e.,  $\frac{C}{T} = 2$ . In scenarios with a small stock, i.e.,  $\frac{T}{C} = 2$ , prices for small batches do not always decrease from one period to the next; to maintain brevity, we have excluded the corresponding figures. Second, prices for small and medium batches increase approximately linearly with batch size, starting from the two-unit batch. This linearity explains the strong performance of a piecewise linear pricing scheme observed in our numerical study. Such a pricing scheme is easy to communicate, as the firm can quote two prices—one for the first unit and one for each additional unit—instead of a long list of prices for every possible batch size.

## 7 Conclusion

In this study, we introduced a nonlinear dynamic pricing model, based on a customer choice model that captures two-dimensional customer heterogeneity in terms of product attraction and consumption inclination. Despite the complexity of the resulting probability function, we were able to simplify it by leveraging structural properties and removing irrelevant batch prices from the action space. We then presented optimality conditions for the state-wise optimization model. Additionally, we introduced a modified model that, under certain conditions, yields an optimal solution that is also optimal for the original problem. However, finding the optimal solution for the state-wise optimization model remains computationally challenging. To address these difficulties, we developed three novel heuristics: one that uses a decomposition approach and two that calculate expected optimal prices.

In our numerical study, two heuristics performed exceptionally well, with an optimality gap of less than 0.4% and 0.6%, while the third heuristic was only slightly behind. Moreover, they significantly outperformed other mechanisms. The results also highlight the revenue potential of nonlinear pricing, allowing firms to compare straightforward pricing schemes, such as linear and piecewise linear pricing, with nonlinear pricing. We further analyzed both best performing heuristics and found several interesting characteristics of well-performing pricing policies: In cases with reasonably large stocks (in our setting with, e.g.,  $\frac{C}{T} \geq 2$ ), batch prices for small batches are slowly decreasing over time. This is particularly interesting as it is an indicator that changing prices at a lower rate (not after every customer) might still perform well in the presence of multiunit demand. This makes the obtained policies also applicable in settings where the firm cannot sustain frequent changes in batch prices due to, e.g., technical reasons, customers' reluctance, or strategic considerations. On the other hand, in scenarios with a limited stock (in our setting with,  $\frac{T}{C} \geq 2$ ), the importance of nonlinear pricing is declining, whereas a typical (standard) dynamic pricing structure becomes more and more relevant.

Another finding is that batch prices are nearly linear for low- and medium-sized batches starting with the two-unit batch. This explains the strong performance of a piecewise linear pricing scheme in our numerical study, which has the benefit of being easy-to-communicate. Instead of displaying a long list containing prices for every possible batch size, the firm could quote two prices—one for the first unit and one for additional units. Computing the optimal piecewise linear pricing scheme efficiently remains a challenge at this stage.

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## References

- Abdallah T, Asadpour A, Reed J (2021) Large-scale bundle-size pricing: a theoretical analysis. *Oper Res* 69(4):1158–1185. <https://doi.org/10.1287/opre.2020.2097>
- Akçay Y, Natarajan HP, Xu SH (2010) Joint dynamic pricing of multiple perishable products under consumer choice. *Manag Sci* 56(8):1345–1361. <https://doi.org/10.1287/mnsc.1100.1178>
- Allenby GM, Rossi PE (1991) Quality perceptions and asymmetric switching between brands. *Market Sci* 10(3):185–204. <https://doi.org/10.1287/mksc.10.3.185>
- Banciu M, Gal-Or E, Mirchandani P (2010) Bundling strategies when products are vertically differentiated and capacities are limited. *Manag Sci* 56(12):2207–2223. <https://doi.org/10.1287/mnsc.1100.1242>
- Baucells M, Sarin RK (2007) Satiation in discounted utility. *Oper Res* 55(1):170–181. <https://doi.org/10.1287/opre.1060.0322>
- Bell DR, Chiang J, Padmanabhan V (1999) The decomposition of promotional response: an empirical generalization. *Market Sci* 18(4):504–526. <https://doi.org/10.1287/mksc.18.4.504>
- Bitran G, Caldentey R (2003) An overview of pricing models for revenue management. *M&SOM* 5(3):203–229. <https://doi.org/10.1287/msom.5.3.203.16031>
- Biyalogorsky E, Koenigsberg O (2014) The design and introduction of product lines when consumer valuations are uncertain. *Prod Oper Manag* 23(9):1539–1548. <https://doi.org/10.1111/poms.12167>
- Braden DJ, Oren SS (1994) Nonlinear pricing to produce information. *Market Sci* 13(3):310–326. <https://doi.org/10.1287/mksc.13.3.310>
- Chen M, Chen Z-L (2015) Recent developments in dynamic pricing research: multiple products, competition, and limited demand information. *Prod Oper Manag* 24(5):704–731. <https://doi.org/10.1111/poms.12295>
- Chen N, Li X, Li Z, Wang C (2024) Component pricing with bundle size discount. *SSRN J*. <https://doi.org/10.2139/ssrn.4032247>
- Chiang J (1991) A simultaneous approach to the whether, what and how much to buy questions. *Market Sci* 10(4):297–315. <https://doi.org/10.1287/mksc.10.4.297>
- Chiang WC, Chen JC, Xu X (2007) An overview of research on revenue management: current issues and future research. *IJRM* 1(1):97. <https://doi.org/10.1504/IJRM.2007.011196>
- den Boer AV (2015) Dynamic pricing and learning: Historical origins, current research, and new directions. *Surv Oper Res Manag Sci* 20(1):1–18. <https://doi.org/10.1016/j.sorms.2015.03.001>
- Dhebar A, Oren SS (1986) Dynamic nonlinear pricing in networks with interdependent demand. *Oper Res* 34(3):384–394. <https://doi.org/10.1287/opre.34.3.384>
- Dong L, Kouvelis P, Tian Z (2009) Dynamic pricing and inventory control of substitute products. *M&SOM* 11(2):317–339. <https://doi.org/10.1287/msom.1080.0221>
- Elmaghraby W, Güllü A, Keskinocak P (2008) Designing optimal preannounced markdowns in the presence of rational customers with multiunit demands. *M&SOM* 10(1):126–148. <https://doi.org/10.1287/msom.1070.0157>
- Ettl M, Harsha P, Papush A, Perakis G (2020) A data-driven approach to personalized bundle pricing and recommendation. *Manuf Serv Oper Manag* 22(3):461–480. <https://doi.org/10.1287/msom.2018.0756>
- Gallego G, van Ryzin G (1994) Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Manag Sci* 40(8):999–1020. <https://doi.org/10.1287/mnsc.40.8.999>
- Gallego G, Li MZF, Liu Y (2020) Dynamic nonlinear pricing of inventories over finite sales horizons. *Oper Res* 68(3):655–670. <https://doi.org/10.1287/opre.2019.1891>

- Goldman MB, Leland HE, Sibley DS (1984) Optimal nonuniform prices. *Rev Econ Stud* 51(2):305. <https://doi.org/10.2307/2297694>
- Gönsch J, Klein R, Neugebauer M, Steinhardt C (2013) Dynamic pricing with strategic customers. *J Bus Econ* 83(5):505–549. <https://doi.org/10.1007/s11573-013-0663-7>
- Hanemann WM (1984) Discrete/continuous models of consumer demand. *Econometrica* 52(3):541. <https://doi.org/10.2307/1913464>
- Honhon D, Pan XA (2017) Improving profits by bundling vertically differentiated products. *Prod Oper Manag* 26(8):1481–1497. <https://doi.org/10.1111/poms.12686>
- Iyengar R, Jedidi K (2012) A conjoint model of quantity discounts. *Market Sci* 31(2):334–350. <https://doi.org/10.1287/mksc.1110.0702>
- Iyengar R, Jedidi K, Kohli R (2008) A conjoint approach to multipart pricing. *J Market Res* 45(2):195–210. <https://doi.org/10.1509/jmkr.45.2.195>
- Lambrecht A, Seim K, Skiera B (2007) Does uncertainty matter? Consumer behavior under three-part tariffs. *Market Sci* 26(5):698–710. <https://doi.org/10.1287/mksc.1070.0283>
- Landsberger M, Meilijson I (1985) Intertemporal price discrimination and sales strategy under incomplete information. *RAND J Econ* 16(3):424. <https://doi.org/10.2307/255568>
- Levin Y, Nediak M, Bazhanov A (2014) Quantity premiums and discounts in dynamic pricing. *Oper Res* 62(4):846–863. <https://doi.org/10.1287/opre.2014.1285>
- Liu Q, Zhang D (2013) Dynamic pricing competition with strategic customers under vertical product differentiation. *Manag Sci* 59(1):84–101. <https://doi.org/10.1287/mnsc.1120.1564>
- Maglaras C, Meissner J (2006) Dynamic pricing strategies for multiproduct revenue management problems. *M&SOM* 8(2):136–148. <https://doi.org/10.1287/msom.1060.0105>
- Mussa M, Rosen S (1978) Monopoly and product quality. In *Journal of Economic Theory* 18(2):301–317. [https://doi.org/10.1016/0022-0531\(78\)90085-6](https://doi.org/10.1016/0022-0531(78)90085-6)
- Nair H, Dubé J-P, Chintagunta P (2005) Accounting for primary and secondary demand effects with aggregate data. *Market Sci* 24(3):444–460. <https://doi.org/10.1287/mksc.1040.0101>
- Powell WB (2011) *Approximate dynamic programming*. Wiley, Hoboken
- Rochet J-C, Stole LA (2002) Nonlinear pricing with random participation. *Rev Econ Stud* 69(1):277–311. <https://doi.org/10.1111/1467-937X.00206>
- Schur R (2024) Multiunit dynamic pricing with different types of observable customer information. *Or Spectrum* 46(2):589–636. <https://doi.org/10.1007/s00291-024-00759-x>
- Song J-S, Xue Z (2021) Demand shaping through bundling and product configuration: a dynamic multi-product inventory-pricing model. *Oper Res* 69(2):525–544. <https://doi.org/10.1287/opre.2020.2062>
- Stokey NL (1979) Intertemporal price discrimination. *Quarterly J Econ* 93(3):355. <https://doi.org/10.2307/1883163>
- Sundararajan A (2004) Nonlinear pricing of information goods. *Manag Sci* 50(12):1660–1673. <https://doi.org/10.1287/mnsc.1040.0291>
- Talluri, KT, van Ryzin GJ (2004) *The theory and practice of revenue management* (International series in operations research & management science)
- Wilson CA (1988) On the optimal pricing policy of a monopolist. *J Polit Econ* 96(1):164–176. <https://doi.org/10.1086/261529>
- Wilson R (1993): *Nonlinear pricing*. 1. publ. New York, NY: Oxford Univ. Press. Available online at <http://www.loc.gov/catdir/enhancements/fy0639/91032603-d.html>
- Zhang D, Cooper WL (2009) Pricing substitutable flights in airline revenue management. *Eur J Oper Res* 197(3):848–861. <https://doi.org/10.1016/j.ejor.2006.10.067>

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