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# Solow-Swan Model of Economic Growth with Allee Effect

Kseniya Akhalaya<sup>1</sup> · Vladimir Shikhman<sup>1</sup>

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## Abstract

In this paper we expand the neoclassical Solow-Swan model of economic growth by introducing population dynamics with Allee effect. Allee effect implies the existence of a threshold for the viability of populations, i.e. a population below this threshold decreases. Above the threshold, the population gradually saturates. We show that the corresponding capital stock per capita may stabilize at two different levels. Both can be expressed in terms of equilibrium points of the standard Solow–Swan model with particular constant population growth rates. Surprisingly enough, the capital stock per capita performs in the long run better if the population becomes extinct, rather than it advances the saturation level. For this conclusion the decrease of population should be relatively moderate compared to the capital depreciation.

**Keywords** Economic growth · Solow-Swan model · Allee effect · Capital stock per capita

**JEL Classification** C62 · O40

## Introduction

In 1956, Solow and Swan proposed a celebrated growth model of an economy (Solow 1956; Swan 1956). The meanwhile classical Solow–Swan model describes how growth of capital and labor force affect the total output of an economy. The output is generated by a neoclassical production function, which exhibits constant returns to scale, positive and diminishing marginal products with respect to capital and labor, and satisfies the so-called Inada conditions. Part of the output will be consumed and the rest will be saved with a constant rate. The gradual decrease in the economic value of the capital stock decreases with the constant rate  $\delta$ . The population is assumed to grow exponentially with some rate  $\lambda$ . By

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using these assumptions, a differential equation for the capital stock per capita can be derived. It essentially determines the performance of the economy in the Solow–Swan model. The main result in Solow (1956) says that there is a unique globally asymptotically stable equilibrium point of the underlying differential equation. In economic terms, the economy's capital stock per capita always stabilizes at a nontrivial level.

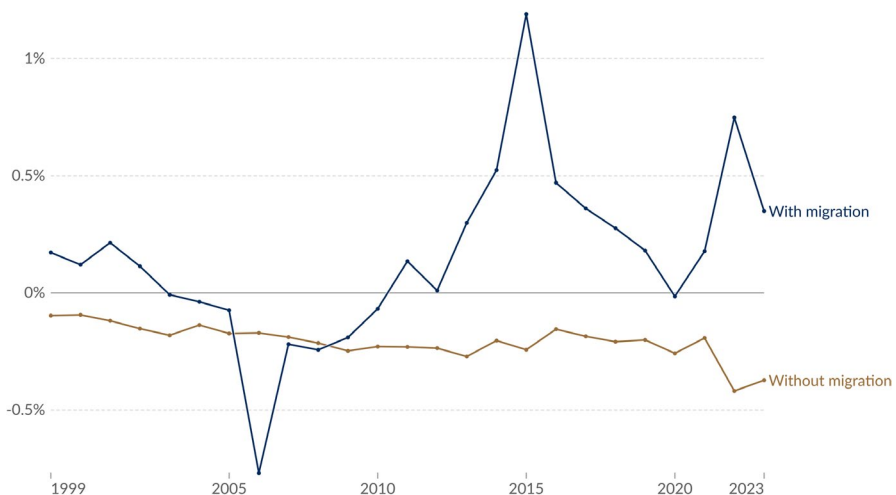
Since its introduction the Solow–Swan model have been generalized in many directions, see e.g. Barro and Sala-i-Martin (1995). One of the main attempts to make the Solow–Swan model more realistic is to modify the underlying population dynamics. In Scarpello and Ritelli (2003), the authors assumed the population dynamics to follow the logistic law. According to the latter, the population grows, but its growth rate decreases to zero. The corresponding capital stock per capita is shown to stabilize at the equilibrium point of the standard Solow–Swan model with zero population growth rate. In Accinelli and Brida (2006), the same general assumptions on the population size and its growth rate are made. Not surprisingly, the achieved stability results persist. Another generalization, which leads mainly to the qualitatively same findings, has been presented in Accinelli and Brida (2005). Here, the so-called Richards law for population dynamics has been incorporated into the Solow–Swan model. In Guerrini (2006), the population dynamics does not follow a prescribed law. Instead, the population grows, but its growth rate has a limit at infinity. The capital stock per capita is shown to stabilize at the equilibrium point of the standard Solow–Swan model with the constant population growth rate equal to this limit.

In this paper, we study the Solow–Swan model with population dynamics underlying the so-called Allee effect. This effect is an important biological phenomenon and was first introduced by Allee in 1930, see e.g. Courchamp et al. (2008). Allee observed during his research on goldfishes that a population with low-density in a certain area leads to the extinction of the species in the long term. Individuals within a population often require the assistance of another individual for more than simple reproductive reasons in order to persist. The most obvious example of this is observed in animals that hunt for prey or defend against predators as a group. In the context of labor, we can think on employees who are relying on the colleagues in their common activities. If some of them depart, the working environment becomes abandoned. This may cause even more people to quit or not to enter the job at all. Allee effect implies the existence of a threshold for the viability of populations, i.e. a population below this threshold decreases and actually becomes extinct over time. However, above the threshold, the population grows in the way similar to the logistic law. Note that the novelty of our approach is two-fold:

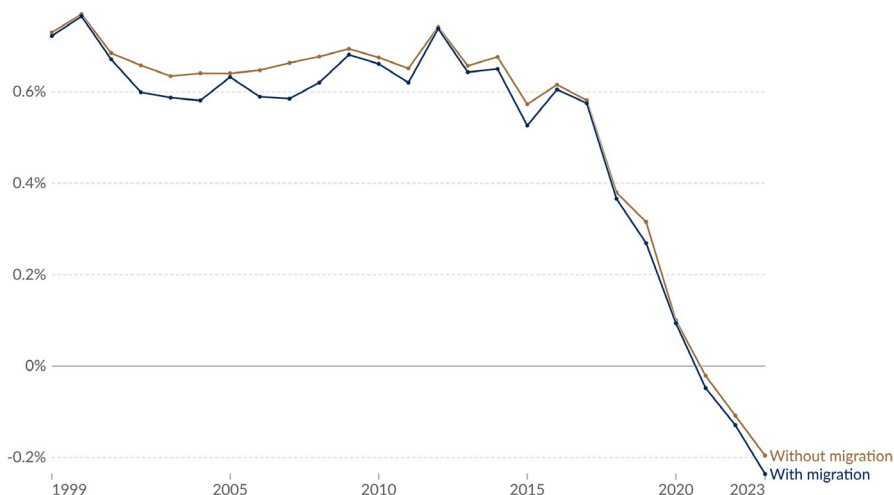
- Into the same Solow–Swan model we incorporate qualitatively different population dynamics. Depending on the initial value, the population may become extinct or saturate, but its growth rate is always decreasing.
- The case of decreasing population is new for the literature on Solow–Swan models. This is if compared to the logistic or Richards law, as well as to implicitly defined population dynamics mentioned above.

It is worth to mention that the use of Allee effect in the context of Solow-Swan models has been suggested in Khan et al. (2021). There, just a numerical study for the Cobb-Douglas production function with particular attention to fuzzy numbers has been carried out.

Let us empirically justify the introduction of the Allee effect in the framework of economic growth models. For that, it is crucial to focus on the novel case of decreasing population below the Allee threshold, i.e. where the population growth rate is negative. For developed countries this is not unusual if one neglects the migration effects, e.g. see Fig. 1 for Germany in 1999–2023. Mainly due to the relatively low fertility rates in Germany around 1.5 children per woman, the population would decrease without an additional migration. Our interest here lies in the modeling of exactly this population dynamics and its consequences for the economic growth. Note that anti-migration policies are already implemented in some developed countries, such as e.g. the welfare reduction for immigrants in Denmark Agersnap et al. (2020), or are thought to be implemented by far-right political parties all over Europe Abubakara et al. (2024). In developing countries – we think in first place on China – the rise of population has been bounded for decades by governmental birth restrictions. Consequently, the one-child policy lead to the negative population growth rates, see Fig. 2 for China in 1999–2023. Although the Chinese birth policies have been gradually loosened (e.g. in 2013, China allowed couples to have a second child if either parent is an only child, in 2016, it allowed married couples to have two children, and in 2021, it announced support for couples who wish to have a third child), the demographic situation would remain recessionary for decades (Yin 2023). Further, the introduction of Allee effect to human societies can be explained by the depopulation phenomenon from economic geography. There, it is defined as a process in which the population density of an area decreases steadily over time. As briefly summarized in Maya (2024), one distinguishes several reasons for



**Fig. 1** Population growth rate of Germany in 1999–2023 with and without migration processed by UN, World Population Prospects (2024), OurWorldinData.org/population-growth



**Fig. 2** Population growth rate of China in 1999–2023 with and without migration processed by UN, World Population Prospects (2024), OurWorldinData.org/population-growth

depopulation: "There are social reasons, since depopulation is a driver of inequality due to the decrease in services that declining rural populations receive. There are cultural and emotional reasons that link personal and collective roots to a landscape that is disappearing. There are economic reasons, since there are resources in the territory that are no longer exploited. There are environmental reasons since the ability to control certain impacts on the environment, such as forest fires, is reduced. There are geopolitical reasons, since society loses sovereignty over depopulated territories, opening an opportunity window to other power networks, such as crime or other political agents, to come to control them." Overall, the more people abandon an area the more of their fellows decide to do the same. This positive feedback, which accounts for depopulation, motivates our use of Allee effect for modeling the population dynamics here. Additionally, we refer to Merino and Prats (2020) for the impact of economic and governmental factors on depopulation, as well as to Kulcsár (2016) for historical examples at both the national and local levels.

Our findings on the Solow–Swan model with Allee effect are as follows. Depending on the population initial value, the capital stock per capita stabilizes at two different levels. First, if the population dynamics starts below the Allee threshold, the capital stock per capita is shown to stabilize at the equilibrium point of the standard Solow–Swan model with a particular constant population growth rate. The latter constant equals  $-\lambda$ , the negative of the intrinsic population growth rate. More interestingly, the global asymptotic stability is guaranteed iff  $\delta > \lambda$ . In economic terms, the capital stock per capita may stabilize even if the population becomes extinct. However, this happens just if the decrease of population is relatively moderate, i.e. the intrinsic population growth rate  $\lambda$  does not exceed the rate of capital depreciation  $\delta$ . Second, if the population dynamics starts above the Allee threshold, the capital stock per capita is shown to stabilize at the equilibrium point of the standard Solow–Swan model with zero population growth rate. This more or less typical behavior is in accordance with

the results in Accinelli and Brida (2006) and Guerrini (2006). The comparison of both equilibrium points of the Solow-Swan model with Allee effect leads to a surprising conclusion. Namely, the corresponding capital stock per capita performs in the long run better if the population becomes extinct, rather than it advances the saturation level. Finally, it is worth to mention that our model remains neoclassical. Here, the assumption on the population dynamics via Allee effect is exogenous. This is in strong contrast with overlapping-generation models, see e.g. Galor and Weil (2000); Yin (2023), where the growth of population is explained endogenously. In Galor and Weil (2000), the transition from Malthusian stagnation to the demographic transition and beyond is considered. The authors argue that in what they call the Modern Growth Regime by then around 2000 "many rich countries have population growth rates near zero", and further – "over the next several decades much of Western Europe is forecast to have negative population growth." Our paper assumes the Allee effect in order to analyze the economic growth corresponding to the latter demographic trend of diminishing population. By doing so, we enlarge the scope of the analysis in Galor and Weil (2000).

The paper is organized as follows. In Sect. 2, we introduce the Solow-Swan model with Allee effect. Section 3 is devoted to the stability analysis of its equilibrium points. In Sect. 4, we discuss the Solow-Swan model with Allee effect for the Cobb-Douglas production function.

## Model Description

We consider a closed economy consisting of a single good  $Y(t)$ , the community's real income. The output  $Y(t)$  is the measure term for economy growth and will be generated by capital  $K(t)$  and labor force  $L(t)$ . We assume the neoclassical production function  $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  with the following properties:

(P1) The function  $F$  exhibits constant returns to scale, i.e.

$$F(aK, aL) = aF(K, L) \quad \text{for all } a > 0.$$

(P2) The function  $F$  exhibits positive and diminishing marginal products with respect to each input  $K > 0$  and  $L > 0$ :

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial F}{\partial L} > 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial K^2} < 0, \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

(P3) The function  $F$  fulfils the Inada conditions (Inada 1963):

$$\lim_{K \rightarrow 0} \frac{\partial F}{\partial K} = \lim_{L \rightarrow 0} \frac{\partial F}{\partial L} = \infty, \quad \lim_{K \rightarrow \infty} \frac{\partial F}{\partial K} = \lim_{L \rightarrow \infty} \frac{\partial F}{\partial L} = 0.$$

Note that the neoclassical properties of the production function imply that the two inputs,  $K$  and  $L$ , are each essential for production, see e.g. Färe and Primont (2002):

$$F(K, 0) = F(0, L) = 0.$$

Let the level of technological possibilities  $T > 0$  be constant. We have

$$Y(t) = T \cdot F(K(t), L(t)). \quad (1)$$

Part of each instant's output will be consumed and the rest with constant rate  $s \in [0, 1]$  saved or invested. We also assume the gradual decrease in the economic value of the capital stock with the rate  $\delta > 0$ . The following increase of capital stock is thus given by

$$K'(t) = s \cdot Y(t) - \delta \cdot K(t). \quad (2)$$

We intend to investigate the per capita income which can be simplified by using (P1):

$$y(t) = \frac{Y(t)}{L(t)} = T \cdot F\left(\frac{K(t)}{L(t)}, 1\right) = T \cdot F(k(t), 1), \quad (3)$$

where the capital stock per unit of effective labor is set as

$$k(t) = \frac{K(t)}{L(t)}.$$

Per capita considerations help to measure a sectors' average incomes and improve the comparison of different economies. The main interest of the model is the dynamics of  $k(t)$ . Taking derivatives with respect to time  $t$ , we obtain

$$k'(t) = \frac{K'(t) \cdot L(t) - K(t) \cdot L'(t)}{L^2(t)} = s \cdot y(t) - (\delta + n(t)) \cdot k(t), \quad (4)$$

where

$$n(t) = \frac{L'(t)}{L(t)}$$

is the growth rate of population. The differential equation in (4) is the economic key of the Solow-Swan model.

Now, we turn our attention to the assumption on the population dynamics. In the canonical Solow-Swan model (Solow 1956; Swan 1956), the growth rate  $n(t)$  is assumed to be constant, i.e.

$$L'(t) = \lambda \cdot L(t),$$

where  $\lambda > 0$  is an intrinsic growth rate. Hence, the population size yields exponential behavior over time and for any initial level  $L_0 > 0$ , at time  $t$  the level of labor force is

$$L(t) = L_0 e^{\lambda t}.$$

This behavior represents a useful model for simple populations over relatively short periods of time without accommodation of growth reductions due to limited resources. In Scarpello and Ritelli (2003), a modification to the canonical Solow-Swan model was proposed by assuming a logistic population growth. This is an

alternative model, which states that a stable population would have a characteristic saturation level  $M > 0$ . The corresponding equation is the well-known logistic differential equation

$$L'(t) = \lambda \cdot L(t) \cdot \left(1 - \frac{L(t)}{M}\right), \quad (5)$$

which can be solved by separation of variables. The analytical solution is given by

$$L(t) = \frac{M}{1 + e^{-\lambda t} \left(\frac{M}{L_0} - 1\right)}. \quad (6)$$

Under this assumption on population growth the economy has a unique non-trivial equilibrium point in terms of the capital stock per capita.

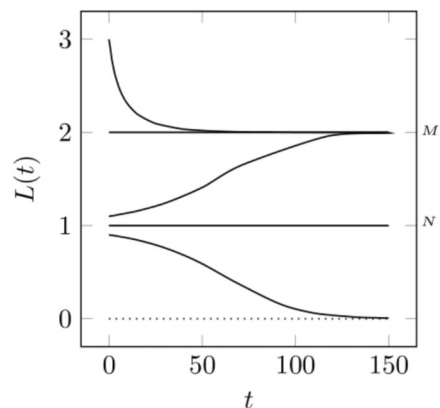
In this paper, we go a step further and extend the logistic population growth with the Allee effect, see e.g. Courchamp et al. (2008). A mathematical representation of the Allee effect for population size is given by

$$L'(t) = \lambda \cdot L(t) \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right). \quad (7)$$

The parameter  $N$  is the critical value or the so-called Allee threshold. We couple (7) with the initial condition  $L(t_0) = L_0 > 0$  to form the initial value problem. Unfortunately, the unique solution  $L(t)$  of (7) cannot be given explicitly in the general case. Therefore, we look at its qualitative behavior depending on the initial value  $L_0$ , see Fig. 3 with  $N = 1$  and  $M = 2$ . We observe asymptotically stable equilibrium points  $M$  and  $0$  as well as an unstable equilibrium point  $N$ .

Let us now assume the population law with Allee effect from (7) for the Solow-Swan model (4). We see that the economy of this modified Solow-Swan model is described by the differential equation

**Fig. 3** Population growth (7) with Allee effect





$$k'(t) = s \cdot T \cdot F(k(t), 1) - (\delta + n(t)) \cdot k(t), \quad (8)$$

where

$$n(t) = \frac{L'(t)}{L(t)} = \lambda \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right).$$

In what follows, we shall study the corresponding initial value problem of the Solow-Swan model with Allee effect for given capital stock per capita  $k(t_0) = k_0 > 0$ .

## Stability Analysis

Let us show that the differential equation (8) is uniquely solvable. For that, we first study the behavior of the population growth rate  $n(t)$  depending on the initial value  $L_0 > 0$ . From

$$n(t) = \lambda \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right)$$

we obtain:

$$n'(t) = \lambda \cdot L'(t) \cdot \left(\frac{1}{M} + \frac{1}{N} - 2\frac{L(t)}{MN}\right).$$

By additionally using the properties of  $L(t)$  and  $L'(t)$ , we may distinguish the following cases for the population growth rate:

(1) If  $L_0 \in (0, N)$ , then it holds:

$$n(t) < 0, \quad n'(t) < 0, \quad \lim_{t \rightarrow \infty} n(t) = -\lambda.$$

(2a) If  $L_0 \in \left(N, \frac{N+M}{2}\right]$ , then it holds:

$$n(t) > 0, \quad n'(t) \begin{cases} > 0 & \text{if } t < \bar{t} \\ = 0 & \text{if } t = \bar{t} \\ < 0 & \text{if } t > \bar{t} \end{cases}, \quad \lim_{t \rightarrow \infty} n(t) = 0,$$

where  $\bar{t}$  is the unique solution of  $L(t) = \frac{N+M}{2}$ .

(2b) If  $L_0 \in \left(\frac{N+M}{2}, M\right)$ , then it holds:

$$n(t) > 0, \quad n'(t) < 0, \quad \lim_{t \rightarrow \infty} n(t) = 0.$$

(3) If  $L_0 \in (M, \infty)$ , then it holds:

$$n(t) < 0, \quad n'(t) > 0, \quad \lim_{t \rightarrow \infty} n(t) = 0.$$

(4) If  $L_0 \in \{N, M\}$ , then it holds:

$$n(t) \equiv 0.$$

**Theorem 1** (Existence and uniqueness) For any  $k_0 > 0$ , there is a unique solution  $k(t, t_0, k_0)$  of the Solow-Swan model with Allee effect (8) on the time interval  $[t_0, \infty)$ .

**Proof** Let us denote the right-hand side of (8) as

$$f(k, t) = s \cdot T \cdot F(k, 1) - (\delta + n(t)) \cdot k.$$

For its partial derivative with respect to  $k$  we have:

$$\frac{\partial f(k, t)}{\partial k} = s \cdot T \cdot \frac{\partial F(k, 1)}{\partial k} - (\delta + n(t)).$$

Since the function  $F$  exhibits diminishing marginal products, it holds with some fixed  $0 < \varepsilon < k_0$ :

$$\frac{\partial F(k, 1)}{\partial k} \leq \frac{\partial F(k_0 - \varepsilon, 1)}{\partial k} \quad \text{for all } k \geq k_0 - \varepsilon.$$

Depending on  $L_0$  from the cases (1)–(4), it is straightforward to deduce:

- (1) If  $L_0 \in (0, N)$ , then  $-\lambda < n(t) < 0$ .
- (2a) If  $L_0 \in \left(N, \frac{N+M}{2}\right]$ , then  $0 < n(t) \leq n(\bar{t})$ .
- (2b) If  $L_0 \in \left(\frac{N+M}{2}, M\right)$ , then  $0 < n(t) \leq n(t_0)$ .
- (3) If  $L_0 \in (M, \infty)$ , then  $n(t_0) < n(t) < 0$ .
- (4) If  $L_0 \in \{N, M\}$ , then  $n(t) \equiv 0$ .

In any case, the absolute value of the population growth rate  $n(t)$  is uniformly bounded with respect to  $t$ , i.e. there exists  $\eta > 0$  with

$$|n(t)| \leq \eta \quad \text{for all } t \geq t_0.$$

Overall, we obtain for all  $(k, t) \in [k_0 - \varepsilon, \infty) \times [t_0, \infty)$ :

$$\left| \frac{\partial f(k, t)}{\partial k} \right| \leq s \cdot T \cdot \frac{\partial F(k_0 - \varepsilon, 1)}{\partial k} + \delta + \eta.$$

Hence,  $f(k, t)$  is not only continuous in both variables, but also satisfies the Lipschitz condition with respect to  $k \in [k_0 - \varepsilon, \infty)$  uniformly for all  $t \in [t_0, \infty)$ . The Picard-Lindelöf theorem provides then a unique solution  $k(t, t_0, k_0)$  of the initial value problem (8) with the domain  $[t_0, \infty)$ .  $\square$

Let us determine the equilibrium points of the Solow-Swan model with Allee effect. For that, we equivalently write (8) as a two-dimensional system of differential equations with respect to both variables  $k$  and  $L$ :

$$\begin{cases} k' = s \cdot T \cdot F(k, 1) - \left( \delta + \lambda \cdot \left( 1 - \frac{L}{M} \right) \cdot \left( \frac{L}{N} - 1 \right) \right) \cdot k, \\ L' = \lambda \cdot L \cdot \left( 1 - \frac{L}{M} \right) \cdot \left( \frac{L}{N} - 1 \right). \end{cases} \quad (9)$$

In order to describe its equilibrium points, we consider a family of standard Solow-Swan models with constant population growth rates  $n \in \mathbb{R}$ :

$$k'(t) = s \cdot T \cdot F(k(t), 1) - (\delta + n) \cdot k(t). \quad (10)$$

For the initial value  $k(t_0) = k_0 > 0$ , its unique solution  $k_n(t, t_0, k_0)$  exists on the time interval  $[t_0, \infty)$ . In case of  $n > -\delta$  the right-hand side of (10) has a unique positive root  $k_n^* > 0$ , i.e.

$$s \cdot T \cdot F(k_n^*, 1) - (\delta + n) \cdot k_n^* = 0.$$

We mention that  $k_n^*$  is strictly decreasing and continuous with respect to the population growth rate  $n \in (-\delta, \infty)$ . Additionally, it holds:

$$\lim_{n \rightarrow -\delta} k_n^* = \infty.$$

Note that the equilibrium point  $k_n^*$  is globally asymptotically stable for (10). In particular, it holds for any  $k_0 > 0$ :

$$\lim_{t \rightarrow \infty} k_n(t, t_0, k_0) = k_n^*.$$

In case of  $n \leq -\delta$ , the only equilibrium point of (10) is the origin. The origin is always unstable for (10). All of this follows mainly in view of the assumptions (P1)-(P3), see e.g. Simonovits (2000).

By using the introduced notation, the equilibrium points of (9) with positive capital stock per capita are  $(k_0^*, M)$ ,  $(k_0^*, N)$  and, if  $\delta > \lambda$ , also  $(k_{-\lambda}^*, 0)$ . The equilibrium points with zero capital stock per capita are  $(0, M)$ ,  $(0, N)$ , and  $(0, 0)$ . Our first attempt to study stability of the equilibrium points of (9) is based on the linearization technique.

**Theorem 2** (Asymptotic stability) The equilibrium points  $(k_0^*, M)$  and, if  $\delta > \lambda$ , also  $(k_{-\lambda}^*, 0)$  of the Solow-Swan model (9) with Allee effect are asymptotically stable.

**Proof** Let us denote the right-hand side of (9) as

$$G(k, L) = \begin{pmatrix} s \cdot T \cdot F(k, 1) - \left( \delta + \lambda \cdot \left( 1 - \frac{L}{M} \right) \cdot \left( \frac{L}{N} - 1 \right) \right) \cdot k \\ \lambda \cdot L \cdot \left( 1 - \frac{L}{M} \right) \cdot \left( \frac{L}{N} - 1 \right) \end{pmatrix}.$$

We compute the Jacobian of  $G$  at the first equilibrium point under consideration:

$$DG(k_0^*, M) = \begin{pmatrix} s \cdot T \cdot \frac{\partial F(k_0^*, 1)}{\partial k} - \delta & \frac{\lambda}{M} \cdot \left(\frac{M}{N} - 1\right) \cdot k_0^* \\ 0 & -\lambda \cdot \left(\frac{M}{N} - 1\right) \end{pmatrix}.$$

Since  $k_0^*$  is asymptotically stable for (10) with  $n = 0$ , we have:

$$s \cdot T \cdot \frac{\partial F(k_0^*, 1)}{\partial k} - \delta \leq 0.$$

Let us assume for a moment that

$$s \cdot T \cdot \frac{\partial F(k_0^*, 1)}{\partial k} - \delta = 0.$$

Then, the concave function  $h(k) = s \cdot T \cdot F(k, 1) - \delta \cdot k$  would have its global maximum at  $k_0^* > 0$  with the vanishing optimal value. This contradicts with the fact that  $h(0) = 0$ , and the set of maxima of a concave function is convex. We conclude that

$$s \cdot T \cdot \frac{\partial F(k_0^*, 1)}{\partial k} - \delta < 0.$$

It follows that both eigenvalues of the matrix  $DG(k_0^*, M)$  are negative, hence,  $(k_0^*, M)$  is asymptotically stable. Now, we assume that  $\delta > \lambda$  and compute the Jacobian of  $G$  at the other equilibrium point:

$$DG(k_{-\lambda}^*, 0) = \begin{pmatrix} s \cdot T \cdot \frac{\partial F(k_{-\lambda}^*, 1)}{\partial k} - (\delta - \lambda) & -\left(\frac{\lambda}{M} + \frac{\lambda}{N}\right) \cdot k_{-\lambda}^* \\ 0 & -\lambda \end{pmatrix}.$$

As above, it is straightforward to show that

$$s \cdot T \cdot \frac{\partial F(k_{-\lambda}^*, 1)}{\partial k} - (\delta - \lambda) < 0.$$

Both eigenvalues of the matrix  $DG(k_{-\lambda}^*, 0)$  are negative, hence,  $(k_{-\lambda}^*, 0)$  is asymptotically stable.  $\square$

As an auxiliary tool to relate differential equations (8) and (10), a comparison result from Waltma (2004) is needed. It turns out to be crucial for our global stability analysis.

**Lemma 1** (Comparison result, Waltma (2004)) Let us given continuous functions  $f, g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  with  $f(t, x) \leq g(t, x)$ . Suppose that  $g(t, x)$  (or  $f(x, t)$ ) satisfies a Lipschitz condition with respect to  $x$  uniformly for all  $t \in [t_0, t_1]$  with  $t_1 > t_0 > 0$ . If  $x(t)$  and  $y(t)$  satisfy

$$x' = f(t, x) \quad \text{and} \quad y' = g(t, y)$$

for all  $t \in [t_0, t_1]$ , respectively, and  $x(t_0) \leq y(t_0)$ , then it holds:

$$x(t) \leq y(t) \quad \text{for all} \quad t \in [t_0, t_1].$$

The comparison technique for studying stability of Solow-Swan models with variable population growth rate has been already successively applied in Accinelli and Bida (2006); Guerrini (2006). Let us recall, in particular, the stability result from Guerrini (2006), which we shall use for our purposes. For that, we consider a differential equation for the capital stock per capita

$$k'(t) = s \cdot T \cdot F(k, 1) - (\bar{\delta} + \bar{n}(t)) \cdot k, \quad (11)$$

where the depreciation rate is positive, i.e.  $\bar{\delta} > 0$ , and the variable population growth rate is bounded and exhibits limiting behavior, i.e.

$$0 \leq \bar{n}(t) \leq \bar{n}, \quad \lim_{t \rightarrow \infty} \bar{n}(t) = n_{\infty}. \quad (12)$$

The solution of the initial value problem corresponding to (11) will be denoted by  $\bar{k}(t, t_0, k_0)$ .

**Theorem 3** (Stability result, Guerrini (2006)) The capital stock per capita from the Solow-Swan model (11) with the variable population growth rate stabilizes, namely for any  $k_0 > 0$  it holds:

$$\lim_{t \rightarrow \infty} \bar{k}(t, t_0, k_0) = \bar{k}_{n_{\infty}}^*,$$

where  $\bar{k}_{n_{\infty}}^*$  is the unique equilibrium of the standard Solow-Swan model with the constant population growth rate:

$$k'(t) = s \cdot T \cdot F(k, 1) - (\bar{\delta} + \bar{n}_{\infty}) \cdot k.$$

Now, we are ready to handle the case of extinguishing population for the Solow-Swan model with Allee effect.

**Theorem 4** (Global stability for extinction) Let  $L_0 \in (0, N)$ . Then, the capital stock per capita from the Solow-Swan model (8) with Allee effect stabilizes iff  $\delta > \lambda$ , namely for any  $k_0 > 0$  it holds:

$$\lim_{t \rightarrow \infty} k(t, t_0, k_0) = k_{-\lambda}^*.$$

**Proof** We know that for  $L_0 \in (0, N)$  the population growth rate has the following properties, see case (1) from above:

$$n(t) < 0, \quad n'(t) < 0, \quad \lim_{t \rightarrow \infty} n(t) = -\lambda. \quad (13)$$

(i) We assume first that  $\delta > \lambda$ . Let us shift the depreciation rate and the growth population rate by setting

$$\bar{\delta} = \delta - \lambda, \quad \bar{n}(t) = n(t) + \lambda.$$

Clearly, (8) can be equivalently written as (11), where (12) holds with

$$\bar{n} = n(t_0) + \lambda, \quad n_\infty = 0.$$

Since the shifted depreciation rate  $\bar{\delta}$  remains positive, the stability analysis from Theorem 3 applies. It says that the capital stock per capita for (11) stabilizes at  $\bar{k}_0^*$ . Recalling  $\bar{\delta} = \delta - \lambda$ , this equilibrium point corresponds to  $k_{-\lambda}^*$ , and the assertion follows.

(ii) Now, we assume that  $\delta < \lambda$ . Due to (13), there exists  $t_2 \geq t_0$  such that  $-\delta > n(t_2) > -\lambda$ . By setting  $n_2 = n(t_2)$ , we have for all  $k > 0$  and  $t \geq t_2$ :

$$s \cdot T \cdot F(k, 1) - (\delta + n_2) \cdot k \leq s \cdot T \cdot F(k, 1) - (\delta + n(t)) \cdot k.$$

By applying Lemma 1, we get for  $t \geq t_2$ :

$$k_{n_2}(t, t_0, k_0) \leq k(t, t_0, k_0). \quad (14)$$

Since  $\delta + n_2 < 0$ , the capital stock per kapita of the standard Solow-Swan model with the constant population growth rate  $n_2$  blows up, i.e.

$$\lim_{t \rightarrow \infty} k_{n_2}(t, t_0, k_0) = \infty,$$

and the reverse implication follows from (14).

(iii) Finally, we assume that  $\delta = \lambda$ . Let  $M \geq 1$  be arbitrary, but fixed. Recall that  $k_n^*$  is strictly decreasing with respect to  $n \in (-\delta, \infty)$  and it holds:

$$\lim_{n \rightarrow -\delta} k_n^* = \infty.$$

Hence, we can find  $n_3 \in (-\delta, n(t_0)]$  with

$$k_{n_3}^* \geq 2M. \quad (15)$$

Due to (13), there exists  $t_3 \geq t_0$  such that  $n(t_3) > -\lambda$ . By setting  $n_3 = n(t_3)$ , we have for all  $k > 0$  and  $t \geq t_3$ :

$$s \cdot T \cdot F(k, 1) - (\delta + n_3) \cdot k \leq s \cdot T \cdot F(k, 1) - (\delta + n(t)) \cdot k.$$

By applying Lemma 1 once again, we get for  $t \geq t_3$ :

$$k_{n_3}(t, t_0, k_0) \leq k(t, t_0, k_0). \quad (16)$$

Since  $\delta + n_3 > 0$ , we know from the theory of the standard Solow-Swan model with constant population growth rate that

$$\lim_{t \rightarrow \infty} k_{n_3}(t, t_0, k_0) = k_{n_3}^*.$$

In particular, there exists  $\bar{t} \geq t_3$  such that for all  $t \geq \bar{t}$  it holds:

$$\left| k_{n_3}(t, t_0, k_0) - k_{n_3}^* \right| \leq \frac{1}{M}. \quad (17)$$

Overall, we obtain for all  $t \geq \bar{t}$ :

$$k(t, t_0, k_0) \stackrel{(16)}{\geq} k_{n_3}(t, t_0, k_0) \stackrel{(17)}{\geq} k_{n_3}^* - \frac{1}{M} \stackrel{(15)}{\geq} 2M - \frac{1}{M} \geq M.$$

Hence, we have:

$$\lim_{t \rightarrow \infty} k(t, t_0, k_0) = \infty,$$

and the reverse implication follows also in this case.  $\square$

Let us briefly comment on the condition  $\delta > \lambda$  from Theorem 4.

**Remark 1** (Condition  $\delta > \lambda$ ) If the population dynamics starts below the Allee threshold  $N$ , the global asymptotic stability of the equilibrium  $k_{-\lambda}^*$  is guaranteed iff  $\delta > \lambda$ . In economic terms, the capital stock per capita may stabilize despite the fact that the population becomes extinct. However, this may happen just if the intrinsic population growth rate  $\lambda$  does not exceed the rate of capital depreciation  $\delta$ . In this case, the decrease of population is rather moderate in comparison to the capital depreciation.

Next Theorem 5 also uses the comparison technique. Its proof can be easily deduced from Guerrini (2006).

**Theorem 5** (Global stability for saturation) Let  $L_0 \in [N, \infty)$ . Then, the capital stock per capita from the Solow-Swan model with Allee effect (8) stabilizes, namely for any  $k_0 > 0$  it holds:

$$\lim_{t \rightarrow \infty} k(t, t_0, k_0) = k_0^*.$$

**Proof** (i) We know that for  $L_0 \in (N, M)$  the population growth rate has the following properties – at least for sufficiently large  $t \geq \bar{t}$ , – see cases (2a) and (2b) from above:

$$n(t) > 0, \quad n'(t) < 0, \quad \lim_{t \rightarrow \infty} n(t) = 0.$$

This allows to apply the stability analysis from Theorem 3. In fact, the population growth rate is bounded and exhibits limiting behavior, i.e. by setting

$$\bar{\delta} = \delta, \quad \bar{n}(t) = n(t),$$

we have (12) with

$$\bar{n} = n(\bar{t}), \quad n_\infty = 0.$$

The capital stock per capita then stabilizes at the equilibrium point of the standard Solow–Swan model (10) with zero population growth rate.

(ii) We know that for  $L_0 \in (M, \infty)$  the population growth rate has the following properties, see case (3) from above:

$$n(t) < 0, \quad n'(t) > 0, \quad \lim_{t \rightarrow \infty} n(t) = 0.$$

The proof of the assertion here goes along the same lines as in Theorem 4. We just add  $n(\bar{t})$  to the depreciation rate and subtract it from the population growth rate:

$$\bar{\delta} = \delta + n(\bar{t}), \quad \bar{n}(t) = n(t) - n(\bar{t}).$$

Here,  $\bar{t}$  is taken sufficiently large to guarantee that  $\delta + n(\bar{t}) > 0$ . Clearly, (8) can be equivalently written as (11), where (12) holds with

$$\bar{n} = -n(\bar{t}), \quad n_\infty = -n(\bar{t}).$$

Since the shifted depreciation rate  $\bar{\delta}$  remains positive, the stability analysis from Theorem 3 applies. It says that the capital stock per capita for (11) stabilizes at  $\bar{k}_{-n(\bar{t})}^*$ . Recalling  $\bar{\delta} = \delta + n(\bar{t})$ , this equilibrium point corresponds to  $k_0^*$ , and the assertion follows.

(iii) We know that for  $L_0 \in \{N, M\}$  the population growth rate vanishes, see case (4) from above:

$$n(t) \equiv 0.$$

Then, (8) becomes the standard Solow-Swan model with zero population growth rate. Thus, the assertion trivially follows.  $\square$

From Theorems 4 and 5 the instability of the remaining equilibrium points of (9) follows.

**Corollary 1** (Instability) The equilibrium points  $(k_0^*, N)$ ,  $(0, M)$ ,  $(0, N)$ , and  $(0, 0)$  of the Solow-Swan model (9) with Allee effect are unstable.

Let us compare the performance of the capital stock per capita in the long run depending on the population growth.

**Remark 2** (Comparison of equilibria) We assume that  $\delta > \lambda$ . Then, according to Theorems 4 and 5, the Solow-Swan model (8) with Allee effect has two globally asymptotically stable equilibrium points  $k_{-\lambda}^*$  and  $k_0^*$ . Since  $k_n^*$  is strictly decreasing with respect to  $n \in (-\delta, \infty)$ , we have:

$$k_{-\lambda}^* > k_0^*.$$

From the economic point of view, the capital stock per capita performs in the long run better if the population becomes extinct, rather than it advances the saturation level.



Let us establish bounds on the capital stock per capita in terms of the standard Solow-Swan model with constant population growth rates.

**Remark 3** (Bounds on capital stock per capita) We assume that  $\delta > \lambda$ . From the proof of Theorem 4 we see that in case of  $L_0 \in (0, N)$  it holds for all  $t \geq t_0$ :

$$k_{n(t_0)}(t, t_0, k_0) \leq k(t, t_0, k_0) \leq k_{-\lambda}(t, t_0, k_0).$$

Analogously, from the proof of Theorem 5 we see that in case of  $L_0 \in (N, M)$  it holds for all sufficiently large  $t \geq t_0$ :

$$k_{n(t_0)}(t, t_0, k_0) \leq k(t, t_0, k_0) \leq k_0(t, t_0, k_0).$$

In case of  $L_0 \in (M, \infty)$  it holds for all  $t \geq t_0$ :

$$k_0(t, t_0, k_0) \leq k(t, t_0, k_0) \leq k_{n(t_0)}(t, t_0, k_0).$$

In economic terms, the capital stock per capita is bounded by that of the standard Solow-Swan model with constant population growth rates  $-\lambda$ ,  $0$ , and  $n(t_0)$ , respectively.

## Discussion for Cobb-Douglas Production Function

In this section, we verify the results from Sect. 3 on an illustrative example. For that, the two-dimensional system of differential equation (9) will be solved numerically for a given production function satisfying (P1)-(P3). We choose the well-known Cobb-Douglas production function:

$$F(K, L) = K^\alpha \cdot L^{1-\alpha}$$

with  $\alpha \in (0, 1)$ . The Solow-Swan model (9) with Allee effect then becomes

$$\begin{cases} k' = s \cdot T \cdot k^\alpha - \left( \delta + \lambda \cdot \left( 1 - \frac{L}{M} \right) \cdot \left( \frac{L}{N} - 1 \right) \right) \cdot k, \\ L' = \lambda \cdot L \cdot \left( 1 - \frac{L}{M} \right) \cdot \left( \frac{L}{N} - 1 \right). \end{cases} \quad (18)$$

Additionally, we make a comparison of (18) with the Solow-Swan model which assumes the population dynamics according to the logistic law, cf. Guerrini (2006); Scarpello and Ritelli (2003):

$$\begin{cases} k' = s \cdot T \cdot k^\alpha - \left( \delta + \lambda \cdot \left( 1 - \frac{L}{M} \right) \right) \cdot k, \\ L' = \lambda \cdot L \cdot \left( 1 - \frac{L}{M} \right). \end{cases} \quad (19)$$

Assuming  $\delta > \lambda$ , we get the equilibrium points of the capital stock per capita for (18) in explicit form, cf. Theorems 4 and 5:

$$k_{-\lambda}^* = \left( \frac{s \cdot T}{\delta - \lambda} \right)^{\frac{1}{1-\alpha}}, \quad k_0^* = \left( \frac{s \cdot T}{\delta} \right)^{\frac{1}{1-\alpha}}.$$

The equilibrium point of the capital stock per capita for (19) is just  $k_0^*$ , see (Guerini 2006; Scarpello and Ritelli 2003). We aim to investigate the dependence of the solutions of (18) and (19) on the population initial value. The initial value  $L_0$  can be below the Allee threshold  $N$ , between the Allee threshold  $N$  and the saturation level  $M$ , and above the saturation level  $M$ . We refer to Figs. 4–6 for these three cases. Here, the solid line corresponds to the Solow-Swan model (18) with Allee effect, and the dotted line to the Solow-Swan model (19) with logistic law. The parameters related to capital are chosen as follows:

$$\alpha = 0.3, \quad s = 0.025, \quad T = 1, \quad \delta = 0.075.$$

The parameters determining the population dynamics are set as

$$\lambda = 0.025, \quad N = 1, \quad M = 2.$$

The main difference between assuming Allee effect or logistic law in the Solow-Swan model is if starting below the Allee threshold  $N$ , see Fig. 4. It turns out that then the capital stock per capita stabilizes at a higher level  $k_{-\lambda}^*$  for the Allee effect in comparison to the lower level  $k_0^*$  for the logistic law. If starting above the Allee threshold  $N$ , see Figs. 5 and 6, the behavior of the capital stock per capita for the Allee effect is as expected similar to that for the logistic law. In order to illustrate the blow-up effect, which occurs if the stability condition  $\delta > \lambda$  is violated, let us take

$$\lambda = 0.085.$$

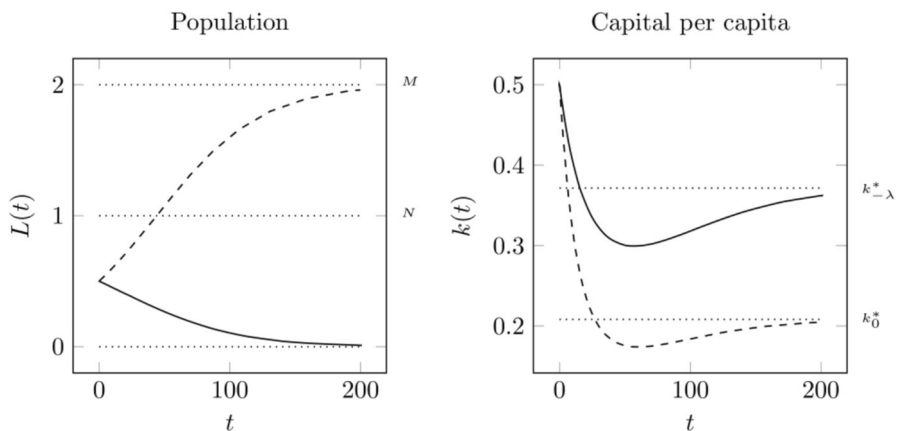
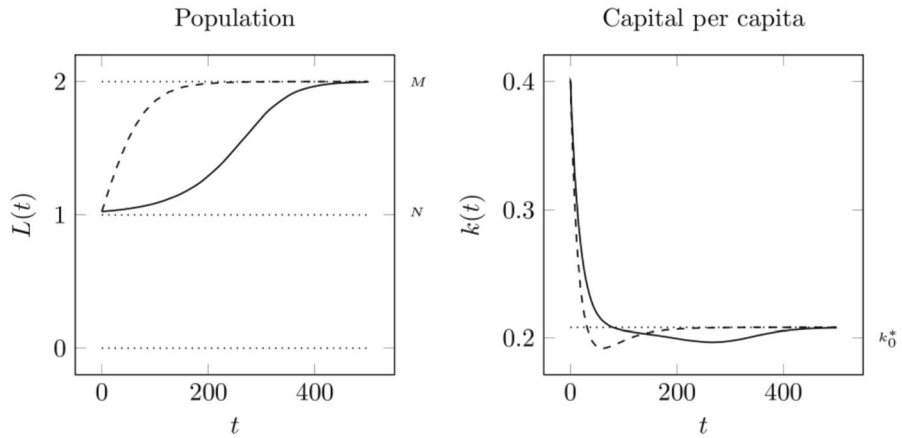
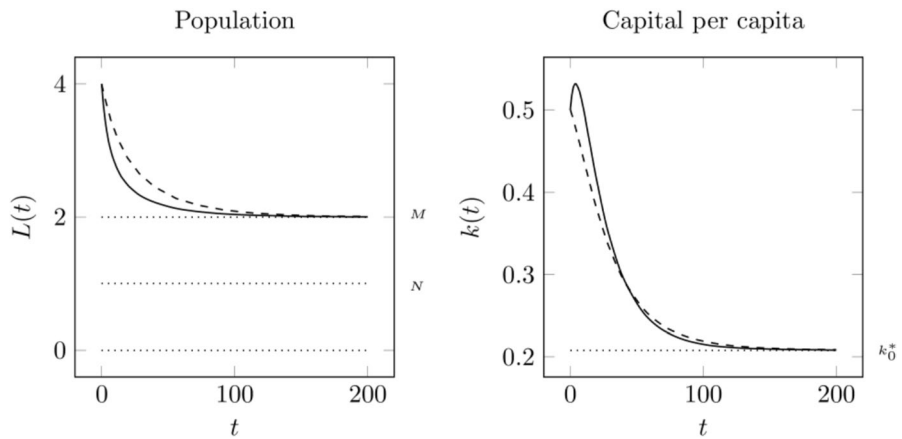


Fig. 4 Case  $L_0 \in (0, N)$

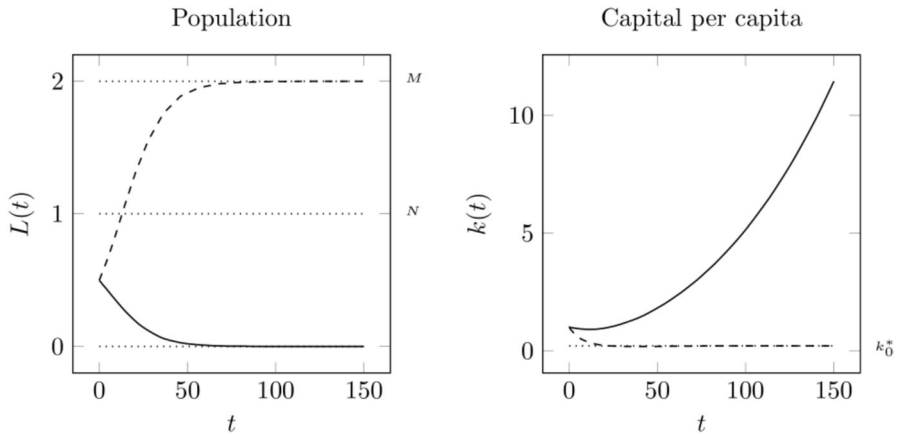


**Fig. 5** Case  $L_0 \in (N, M)$



**Fig. 6** Case  $L_0 \in (M, \infty)$

If starting below the Allee threshold  $N$ , the capital stock per capita explodes in this case, see Fig. 7. This is in contrast with the logistic law, where the corresponding capital stock per capita stabilizes at  $k_0^*$  once again.



**Fig. 7** Blow-up for  $\delta < \lambda$

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