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## Article

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# A Re-Consideration of Money Demand Theory

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**Abstract:** Portfolio models typically ignore precautionary transactions demands for liquid assets, and models of precautionary demands typically ignore asset rate-of-return risk. If asset-holders are risk-averse, however, both transactions risk and rate-of-return risk affect demands for *both* liquid and illiquid assets, *even when* the two risks are independent of each other. We demonstrate this in a four-asset framework, and show that our integrated treatment produces unexpected and instructive results and insights. For example, (a) an increase in the expected return to risky securities *increases* the demand for M1, even when M1 is used entirely for transactions purposes, (b) an increase in the variance of securities returns *reduces* the demand for M1, and (c) an increase in the asset-holders' wealth *reduces* her demand for M1. A broader framework for the study of money demand is thus called for.

**Keywords:** money demand; asset demands; non-separability; expected utility

**JEL Classification:** E4; G1; G2

## 1 Introduction

Consider a risk-averse wealth-holder who can allocate her wealth across four assets – a risky long-term bond or equity, a riskless long-term asset, a lower-yielding riskless short-term asset, and cash. Her investment horizon is shorter than the maturity period of the risky long-term asset, and she wishes to maximize the expected utility of her terminal wealth. However, ‘half-way’ through the period she faces an

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uncertain transactions requirement, which could be  $x$  with a specified probability, and 0 otherwise. Her long-term assets are too costly to liquidate to meet this requirement, which can be met either from cash or from liquidating part or all of her short-term asset, for which she has to incur a transactions cost of, say,  $t$  per dollar liquidated.

As we discuss below, the above setting is, in a stylized way, characteristic of many real-world asset allocation problems. We argue in this article that this fairly elaborate framework, with four assets and two qualitatively different sources of risk, leads to striking new insights, and provides the basis for a re-consideration of money demand theory.

The riskless long-term asset could be fixed deposits or CD's (certificates of deposit), the interest-bearing short-term asset could be savings deposits or other instruments, and 'cash' could represent demand deposits (possibly paying minimal interest) and cash-on-hand. Intuition might suggest that one could partition the asset-holder's decision problem into two components: the choice among the long-term assets on the one hand, and that among the short-term assets on the other, with the obvious restriction being that the sum of her short-term asset-holdings not exceed  $x$  plus the amount of transactions costs incurred in the event of a transactions shock. With this restriction holding, intuition might further suggest that, for example, a mean-preserving change in the variance of the return on the risky long-term asset (we refer to this as a bond henceforth) would affect the choice between bonds and CD's, but would have no bearing on the choice between cash and short-term savings instruments (we refer to these instruments as savings deposits henceforth). *Vice versa* for a change in, say, the interest rate on savings deposits.

In this article, we show that this latter intuition is incorrect. A mean-preserving change in the variance of bond returns *will* affect the choice between cash and savings deposits, notwithstanding that the sum of holdings of cash and savings deposits prior to the transactions shock remains at  $x$  plus required transactions costs, and that the distribution of the transactions shock is independent of that of the bond return. Moreover, the direction of change is quite unexpected. A mean-preserving increase in bond return variance *decreases* the demand for cash, and *increases* the demand for savings deposits, as seen in our simulation analysis below. Perhaps even more surprisingly, an increase in the mean return to bond-holding induces a substitution *towards* cash and *away* from savings deposits.

Risk aversion is a necessary condition for these results. There is a positive probability that the transactions shock  $x$  will not materialize, and her terminal wealth will include her holdings of cash and savings deposits. Randomness of her terminal wealth is thus due to both the randomness of the bond rate of return, and the randomness of her transactions requirement. Being risk-averse, she will thus need

to consider both sources of randomness in her expected-utility-of-wealth calculus, notwithstanding that they are independent of each other.<sup>1</sup> For conciseness, we refer to this as the ‘non-separability property’ of risk-averse expected utility.

Our four-asset framework thus generates interesting patterns of substitutability and complementarity across these assets, which are not induced by patterns of correlation across asset returns as in other models. Our analysis should also provide cautionary advice to asset-holders, including chief financial officers of corporations, against engaging in the partitioning of the decision problem described above, however convenient or logical it might appear to be.

## 2 Literature Review

We seek here to synthesize two hitherto separate strands of literature, albeit in a simplified setting. The first is a discussion originating from James Tobin’s classic 1958 article, ‘Liquidity Preference as Behavior towards Risk,’ which was followed by a critical article by Chang, Hamberg, and Hirata (1983) with the self-explanatory title ‘Liquidity Preference as Behavior toward Risk is a Demand for Short-Term Securities – not Money.’ Chang et al. correctly point out that other ‘safe’ assets, with fixed capital and interest rate values, dominate noninterest-bearing cash for portfolio diversification purposes *alone*. In terms of our four-asset framework, long-term assets such as fixed deposits would dominate short-term assets such as savings deposits for portfolio diversification purposes.

This leaves unanswered two questions, of which the second follows from the first. First, how then do we account for people’s significant holdings of M1 (cash outside banks plus demand deposits) observed all over the world? The obvious answer is that M1 enables transactions demands to be met more cheaply than through the drawing down of other short-term, fixed capital-value assets such as savings deposits. There has developed an extensive literature on this, starting from the classic articles of Baumol (1952) and Tobin (1956), and proceeding through a sequence of complex models allowing for deterministic and stochastic transactions demands of varying forms, as well as various forms of transactions costs. Alvarez, Lippi and Robatto (2019, Section 5) provide a comprehensive overview of this literature (see also Alvarez and Lippi 2017). This literature has indeed assumed that the (single) short-term asset that is alternative to M1 is of fixed capital value and offers a fixed interest rate, while its liquidation to meet transactions demands incurs transactions costs. The asset-holder’s problem is cast as one of expected-cost-minimization over the asset-holder’s infinite lifetime, subject to meeting the transactions demands,

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<sup>1</sup> I owe this interpretation to John Quah.

with exogenous income inflows being represented as negative transactions outlays.<sup>2</sup>

This formulation effectively implies risk-neutrality on the part of asset-holders. However, the assumption of risk-neutrality raises further complications, which do not appear to have been adequately appreciated. In reality, of course, an asset-holder has access to multiple assets, various of which offer stochastic returns. Under risk-neutrality she would, *cet. par.*, channel all her asset holdings in excess of the short-term assets required to meet transactions needs to the asset with the highest expected return: her long-term portfolio holdings would become degenerate, which is clearly counterfactual.

Second, suppose that asset-holders are not risk-neutral, but instead risk-averse. As mentioned in the Introduction, and shown below, it is then no longer the case that the precautionary transactions demands (adapting the terminology of Frenkel and Jovanovic (1980), and others) for *individual* short-term assets is independent of, say, the risk of and return on long-term bonds, which implies the necessity for a more inclusive approach to the determination of optimal asset allocations. This is the synthesis that we seek to effect in this study, in that we allow for both rate-of-return risk and transactions risk, whereas earlier studies have abstracted from one or the other of these (and, when considering transactions risks, have assumed the special case of risk-neutrality).

Our analysis also differs from three other strands of the literature. The first is that of ‘background risk’ (see in particular Fagereng, Guiso, and Pistaferri 2018) – if, for example, an asset holder is also confronted with labour income risk. By definition, this refers to uninsurable risk – risk that ‘cannot be diversified or avoided’ (*ibid.*, p. 437). In our model, however, the asset-holder can adjust the amount of liquidity risk she effectively bears by changing the ratio of money to savings deposits in her portfolio. We show this more precisely below.

The second is the issue of optimal portfolio allocation across two or more risky assets (see, for example, Hadar and Seo 1990), wherein a major concern is establishing the conditions under which a stochastically dominating shift in the returns to one of the risky assets unambiguously increases the amount invested in that asset. In our model, we also have multiple (two) risks, but one of them is a liquidity risk, which is qualitatively different from rate-of-return risk. This is intuitively evident, and we explicitly discuss the differences below.

Thirdly, and overlapping somewhat with the Baumol-Tobin-type transactions models discussed earlier, there have been studies seeking to distinguish between

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<sup>2</sup> For example, Alvarez, Lippi, and Robatto (2019): ‘The problem for the agent is to minimize the (expected lifetime) cost incurred to finance an exogenous consumption stream’ (p. 211), and ‘The agent withdraws from and deposits to an asset account with real rate of return  $r$ ’ (p. 212).

the transactions roles of currency outside banks, bank deposits, and possibly other short-term monetary instruments such as MMDA's (money market deposit accounts). Freeman and Kydland (2000) assume a single kind of interest-bearing bank deposit (this is characterized as a demand deposit, and included in their definition of M1), but there is a fixed cost of using these for payments purposes (which 'may be thought of as a check-clearing cost or a cost of verifying the identity of the person writing a check or making a withdrawal' (*ibid.*, p. 1126)). By contrast, currency outside banks pays no interest, but incurs no fixed cost when used for payments. They then show that it is optimal to use currency for small purchases, and checks drawn on demand deposits for larger ones,<sup>3</sup> with the purchase threshold between the two determined endogenously. A puzzling, and possibly inconsistent, feature of their study is that they assume infinitely-lived individuals with strictly concave per-period utility functions, as well as technology and money supply shocks (which may be auto-correlated), but do not incorporate any risk premia into their analyses.<sup>4</sup>

Belongia and Ireland (2019) postulate a linear homogeneous monetary services aggregator of currency and a single type of interest-earning bank deposit, and adopt a 'simplified, perfect foresight partial equilibrium framework' (p. 3). Lucas (2000) also adopts a deterministic framework in analyzing money demand, with money either entering the utility function or being explicitly used for transactions purposes. Lucas (1980) allows for random transactions demands, and, working with general utility and transactions risk functions, focuses on characterizing the general equilibrium of the economy without devoting much attention to studying the properties of individual money demand, other than showing that individual Engel curves for real money balances are upward-sloping. He abstracts from asset rate-of-return risk. Finally, Lucas and Nicolini (2015) construct a deterministic model with three monetary assets – currency outside banks, demand deposits, and money-market deposit accounts – with varying transactions costs, reserve requirements, and interest rates, and show that these individual assets are used for transactions of differing sizes. Plotting their NewM1 measure, including money-market deposit account holdings, and ratios of its individual components against short-term Treasury Bill rates, they find that 'while the ability of the model to match the ratios

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3 It is assumed that there are  $n$ , apparently exogenously determined, sub-periods within each period, and household stocks of currency and deposits are replenished after each sub-period, so that currency- and deposit- holdings are constant over the period as a whole.

4 They refer to earlier works by Lacker (1988) and Freeman and Huffman (1991), but these models have overlapping generations of two-period-lived individuals, with risk-neutral second-period utility functions. The latter explicitly note (p. 650) that risk premia become relevant if second-period utility is not linear in consumption. There is a similar puzzle in the infinite-horizon model of Benati et al. (2021), which has productivity, interest rate, and intermediation technology shocks.

between the components of NewM1 is mixed, the behavior of the aggregate is remarkably close to the data' (p. 60).<sup>5</sup>

There is abundant empirical evidence that the four-asset classification we propose corresponds well with reality. In fact, the plausibility of such a correspondence can be established deductively, as well as shown empirically. Since savings deposits offer a higher interest rate than demand deposits, asset-holders would opt entirely for the former unless there is some offsetting disadvantage. Indeed, in the US there is typically a limit of six cheque or debit-card withdrawals per month from savings deposits,<sup>6</sup> any further withdrawals requiring a visit to the bank or to an ATM machine, and minimum-balance requirements are often imposed as well. Banks are obliged to impose such restrictions given that they face lower reserve requirements against savings deposits than against demand deposits: the lower reserve holdings are precisely what enables them to offer higher returns on savings deposits. Similarly, a penalty is often imposed on premature cash withdrawals from fixed deposits and CD's,<sup>7</sup> and again the lower reserve requirements are a major factor, along with investments in longer-term assets, enabling banks to offer higher returns on these than on savings deposits. Lastly, risk-averse individuals will require a risk premium to compensate them for holding bonds or equities, rather than fixed-capital-value fixed deposits and CD's.

There are also restrictions or fees of varying kinds on 'linked savings accounts' (Kagan 2020a), 'high-yield savings accounts' offered by both online and brick-and-mortar institutions (Karl 2020c), and money market deposit accounts and money market mutual funds (Barba 2023). Bankrate (June 21 2020) lists a total of 121 of the 'best available rates across different account types,' and the APY (Annual Percentage Yield), across savings accounts and money market accounts, of the top 15 of these range from 1.15 % to 1.36 %. In contrast, the best CD rates at that time ranged from 1.50 % APY upwards for deposits of 6 months or longer (Karl 2020d).

### 3 The Model

Our analysis is explicitly partial-equilibrium in nature, as in our view various (though not all) general-equilibrium analyses adopt, for tractability reasons,

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<sup>5</sup> New-Monetarist, search-theoretic approaches to the study of money demand include Kim and Marchesiani (2024) and Berentsen, Huber, and Marchesiani (2015, 2018), and abstract entirely from rate-of-return risk.

<sup>6</sup> Fontinelle (2020). We abstract from NOW accounts, which typically have minimum balance requirements and other restrictions. Edmondson (2021) observes, 'Today, there are very few NOW accounts used anymore as many types of checking accounts can bear (low) interest.'

<sup>7</sup> Karl (2020b).

simplifying assumptions that do not do full justice to the richness of the determinants of individual asset demands. A general-equilibrium extension of our elaborately-specified partial-equilibrium analysis is postponed to future research. We consider a risk-averse asset-holder who has an initial wealth endowment of  $W_0$  at time 0, and seeks to maximize the expected utility of her terminal wealth,  $W_2$ , at the end of period 2.<sup>8</sup> She receives no income in period 1. At the beginning of period 1 she faces an uncertain liquidity demand, which is  $x > 0$  with probability  $p_t$  and 0 with probability  $1 - p_t$ . This is an essential ‘maintenance’ expense (e.g. a medical expenditure), but does not otherwise enhance her utility: it could simply be viewed as forestalling a sharp decline in her utility that would otherwise occur.

She can invest her initial wealth across four assets: (1) cash or demand deposits  $M_0$  (assumed noninterest bearing for convenience<sup>9</sup>), which can be liquidated (utilized to pay for the expense) without any transactions cost at time 1, (2) a short-term savings deposit  $S_0$ , on which, following the discussion earlier, an inconvenience or transactions cost of  $t$  per dollar withdrawn at time 1 is incurred, (3) a long-term fixed deposit or CD  $F_0$ , which matures at time 2, and (4) a risky security  $B_0$ , which yields an interest payment at time 2, as well as any capital gain or loss then.<sup>10</sup> For simplicity, we assume that the penalty for a premature withdrawal from the fixed deposit at time 1, inclusive of the foregone interest, is sufficiently high that the asset holder does not entertain this possibility. Similarly, we assume that foregoing any interest payment if the risky security is liquidated at time 1, together with the brokerage or transactions cost incurred, is sufficiently costly that this too is not a worthwhile option for her.<sup>11</sup> We believe these last two specifications accord well with reality.

In principle, should the liquidity demand not transpire, the asset-holder may wish to convert her  $M$  and  $S$  holdings into holdings of  $F$  and  $B$  at time 1. However, the returns from converting into  $F$  would be sharply reduced on account of her reduced holding period, as well as the brokerage cost incurred. This is particularly

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<sup>8</sup> This is specified as the terminal period for notational convenience. Below, we point out that our analysis is equally applicable in a suitably-specified infinite-horizon optimization framework.

<sup>9</sup> As will be evident from the subsequent analysis, introducing a further distinction between cash holdings outside banks – which pay no interest and incur no transactions costs – and demand deposits – which pay lower interest than savings deposits, and incur positive but smaller transactions costs than the latter – will not affect our qualitative insights.

<sup>10</sup> As mentioned we assume that the bond’s maturity date is at some date beyond time 2, so that it is not redeemable at par at time 2.

<sup>11</sup> We also assume a prohibitively high cost of short sales of any asset (borrowing), so as not to unduly complicate the discussion. Our assumption that the liquidity demand can only occur once, at time 1, is simply designed as an approximation to the fact that in reality there may be myriad small liquidity demands between times 0 and 2, and the asset holder may well have to incur fixed ‘attention costs’, as well as brokerage costs, each time she has to liquidate small amounts of bonds. As such, she does not entertain this option.

so if, following the argument in *fn.* 11 above, she is uncertain as to precisely when all liquidity demands might be incurred. Regarding  $B$ , the closer she is to time 2 at the time she converts from  $M$  and  $S$  into this asset, the lower will be her gain as the price of the security would, with adjustment for risk, tend to converge towards the expected gross return, inclusive of interest payment, at time 2 so as to rule out ‘abnormal’ expected capital gains or losses.<sup>12</sup> In addition she would still have to incur the brokerage cost of such conversion. Thus, we rule out the possibility of re-optimization of asset holdings at time 1. Allowing for re-optimization would significantly complicate the analysis without generating new insights, since  $M$  would still command a liquidity advantage over  $S$  given that she would first have to incur a transactions cost to convert from  $S$  to  $M$  before investing in either  $B$  or  $F$ . Our specification is equivalent to assuming, as in infinite-horizon models such as the one we outline briefly below, that asset-holdings are only optimized at the beginning of each period (where the ‘period’ corresponds to the dual periods in our current model).

Her wealth constraint at time 0 is

$$W_0 = M_0 + S_0 + F_0 + B_0 \quad (1)$$

Since  $M$  pays no interest, its gross return (1 plus interest) is simply 1, and the gross returns on  $S$ ,  $F$ , and  $B$ , net of any brokerage costs incurred in investing in them at time 0 (assumed proportional to the amounts invested), are, respectively,  $r_s$ ,  $r_f$ , and, for  $B$ ,  $r_1 > 1$  with probability  $p_b$  and  $r_2 < 1$  with probability  $1 - p_b$ . For obvious reasons we assume:

$$1 < r_s < r_f < p_b r_1 + (1 - p_b) r_2. \quad (2)$$

Given these inequalities, it would clearly not be optimal for her to hold any  $M$  or  $S$  above the sums required to meet the liquidity demand at time 1. This does not imply the restriction  $M_0 + S_0 = x$ , however, as we also have to account for the transactions costs incurred if any  $S$  is utilized for meeting the liquidity demand at that time. Interestingly, it can be shown that, provided that  $t < 1$ , it is a matter of indifference whether these transactions costs are incurred out of  $M$  or out of further drawdown of  $S$ ,<sup>13</sup> and for concreteness we assume the former. Let  $M_1$  denote the

<sup>12</sup> This ‘no-arbitrage’ argument would also ensure that bond prices could not be overly high at time 1 (as there would then be high expected capital losses going forward), again serving to dissuade the use of bonds to meet liquidity demands at that time.

<sup>13</sup> It can be shown that for any given  $M_0$  the amount of  $S_0$  held is the same in either case. Formally, if  $S$  is used to defray the transactions costs, then we have  $(1 - t)S_0 = x - M_0$ , and if  $M$  is used to defray the transactions costs then we have  $S_0 = x - M_1 = x - (M_0 - tS_0)$ , which generates the same equation for  $S_0$ . The restriction  $t < 1$  ensures that holdings of  $S$  do not ‘explode’ if  $S$

amount of  $M$  directly allocated to meeting the liquidity demand, so that:

$$S_0 = x - M_1, \quad (3)$$

and

$$M_0 = M_1 + tS_0. \quad (4)$$

Then, the 4 possible outcomes for wealth at time 2 are, making use of (1), (3), and (4) to substitute out  $S_0$ ,  $M_1$ , and  $F_0$  are

$$W_{11} = M_0 + r_s \frac{x - M_0}{1 - t} + r_1 B_0 + r_f \left( W_0 - B_0 - x - t \frac{x - M_0}{1 - t} \right), \quad (5)$$

$$W_{21} = r_1 B_0 + r_f \left( W_0 - B_0 - x - t \frac{x - M_0}{1 - t} \right), \quad (6)$$

$$W_{12} = M_0 + r_s \frac{x - M_0}{1 - t} + r_2 B_0 + r_f \left( W_0 - B_0 - x - t \frac{x - M_0}{1 - t} \right), \quad (7)$$

$$W_{22} = r_2 B_0 + r_f \left( W_0 - B_0 - x - t \frac{x - M_0}{1 - t} \right). \quad (8)$$

$W_{11}$  and  $W_{12}$  are the outcomes when there is no precautionary demand at time 1, and the bond returns are  $r_1$  and  $r_2$  respectively, while  $W_{21}$  and  $W_{22}$  are the outcomes when there is a precautionary demand at time 1, and the possible bond return outcomes are the same as in the preceding two cases. Thus, the first two expressions on the right sides of (5) and (7) disappear from (6) and (8) since all of  $M$  and  $S$  are used in meeting the precautionary demand and the associated transactions costs. However, it is important to note that  $M_0$  and  $S_0$  (substituted using (3) and (4)) *do not* disappear from the identical bracketed final expression on the right sides of all these equations, since this expression is equal to  $F_0$  (multiplied by  $r_f$ ), and the decision as to how much wealth to allocate to  $F$  is made at time 0, along with the decisions as to how much wealth to allocate to the other three assets.

It may be seen that each unit of  $M$  ( $S$ ) will 'yield' a gross payoff of 0 ( $-t$ ) at the end of period 2 if there is a precautionary demand for it, and a gross payoff of 1 ( $r_s$ ) if there is no precautionary demand. There is thus a wider dispersion of gross payoffs from  $S$  than from  $M$ , and so, substantiating a point made in Section 2, the wealth holder can reduce the amount of precautionary risk she faces by holding more  $M$  and less  $S$ .<sup>14</sup> Also, the expected gross payoff from holding  $M$  is  $(1 - p_\ell)$ , while that from holding  $S$  is  $p_\ell(-t) + (1 - p_\ell)r_s$ , and a necessary condition for any

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is used to defray transactions costs (as each accumulation of  $S$  that has to be liquidated to meet the transactions demand incurs further transactions costs upon liquidation).

<sup>14</sup> Formally, the variance of the gross return to  $M$  is  $p_\ell(1 - p_\ell)$ , and that of the gross return to  $S$  is higher, at  $p_\ell(1 - p_\ell)(r_s + t)^2$ .

$S$  to be held is that the latter exceed the former, as compensation for the increased risk that utilizing  $S$  entails. (Digressing slightly, a *risk-neutral* individual would only hold positive amounts of both  $M$  and  $S$  if, say,  $r_s$  were to adjust until the expected gross payoffs from both assets were equal, in which case her actual holding of each of these would, within limits, be indeterminate – not a particularly appealing outcome.)

Explicating another point made in Section 2, there are clear qualitative differences between asset holdings in response to precautionary risk and in response to the usual portfolio optimization considerations. First, the holdings of  $M$  and  $S$  are not independent of each other, since the more  $M$  is held the less  $S$  needs to be held to meet precautionary demands (this is additional to the non-independence induced by the overall wealth constraint). Second, any holdings of  $M$  and  $S$  in excess of the amounts required to meet precautionary demands will yield different, and certain, payoffs of 1 and  $r_s$  respectively, which are dominated by the return from holding  $F$ . Thus, precautionary risk is not isomorphic to rate-of-return risk as formulated by Hadar and Seo (*op. cit.*) and others.

Proceeding, we assume an iso-elastic (Constant Relative Risk Aversion, *CRRA*) utility function for terminal wealth:

$$EU(W_2) = E \frac{W_2^{1-\sigma} - 1}{1-\sigma}, \quad (9)$$

where  $E$  is the expectation operator and  $\sigma > 0$  is the coefficient of relative risk aversion (with  $EU(W_2) = \ln W_2$  when  $\sigma = 1$ ). Thus,

$$\begin{aligned} E \frac{W_2^{1-\sigma} - 1}{1-\sigma} &= (1-p_\ell) p_b \frac{W_{11}^{1-\sigma} - 1}{1-\sigma} + p_\ell p_b \frac{W_{21}^{1-\sigma} - 1}{1-\sigma} + (1-p_\ell)(1-p_b) \frac{W_{12}^{1-\sigma} - 1}{1-\sigma} \\ &\quad + p_\ell(1-p_b) \frac{W_{22}^{1-\sigma} - 1}{1-\sigma}. \end{aligned} \quad (10)$$

Given our substitutions, this has to be maximized with respect to  $M_0$  and  $B_0$ . Also, we note that since borrowing is not permitted, holdings of all assets are required to be non-negative. There is then a large number of possible combinations of positive and zero holdings of the various assets, depending on parameter values. We wish to avoid an unduly taxonomic discussion here, as it does not yield substantive additional insights. We thus confine our attention primarily to parameter configurations that generate strictly interior (positive) optimal solutions for all asset holdings. Since it is not possible to obtain closed-form solutions for the various asset holdings, even in this interior case, we solve the model through numerical simulations under various parameter configurations, and we verify in the various cases except a couple (discussed below) that all optimal asset holdings are strictly positive.

Accordingly, the first-order conditions for expected utility maximization with respect to  $M_0$  and  $B_0$  are, respectively:

$$p_\ell r_f t [p_b W_{21}^{-\sigma} + (1 - p_b) W_{22}^{-\sigma}] + (1 - p_\ell)(1 - t - r_s + r_f t) \\ \times [p_b W_{11}^{-\sigma} + (1 - p_b) W_{12}^{-\sigma}] = 0 \quad (11)$$

$$p_b(r_1 - r_f) [(1 - p_\ell) W_{11}^{-\sigma} + p_\ell W_{21}^{-\sigma}] + (1 - p_b)(r_2 - r_f) \\ \times [(1 - p_\ell) W_{12}^{-\sigma} + p_\ell W_{22}^{-\sigma}] = 0 \quad (12)$$

For these equations to hold, without the necessity of adding any Kuhn–Tucker multiplier terms, we require as necessary conditions that in each equation one expression on the left side be positive, and one negative. This is automatically ensured for (12), given our earlier rates of return restrictions, while for (11) we require  $(1 - t - r_s + r_f t) < 0$ , which holds for small  $t$  (note that the gross return  $r_s$  exceeds unity). Equations (11) and (12) are highly nonlinear, and analytical solutions are not possible. We proceed in Section 4 to numerically simulate the model.

**Proposition 1.** *The first-order conditions for an optimum, in the case of strictly interior solutions for  $M_0 \in (tx, x)$  and  $B_0 \in (0, W_0 - x)$ , are given by (11) and (12) above.*

**A Small Digression:** A referee has insightfully suggested that we compare the results under *CRRA* expected utility with those under *CARA* (Constant Absolute Risk Aversion) expected utility. Accordingly, as an alternative to (9) above, we posit

$$EU(W_2) = E[1 - e^{-kW_2}], \quad k > 0. \quad (13)$$

In (5)–(8) above, every wealth term can be partitioned into terms involving  $B_0$ , and all other terms, e.g. (5) above can be expressed as

$$W_{11} = (r_1 - r_f)B_0 + W_{11}^C, \quad (14)$$

where the complementary term  $W_{11}^C = M_0 + r_s \frac{x - M_0}{1 - t} + r_f (W_0 - x - t \frac{x - M_0}{1 - t})$ . Likewise for the other wealth terms (we also note that  $W_{11}^C = W_{12}^C$ , and  $W_{21}^C = W_{22}^C$ ).  $-e^{-kW_{11}}$  can then be expressed as  $-e^{-k[(r_1 - r_f)B_0]} e^{-kW_{11}^C}$ , which is separable *except* for the presence of  $r_f$  in both of these constituent terms. This separability is the major difference between the *CARA* and *CRRA* specifications, and also permits a closed-form solution under *CARA*, as follows.

After some manipulations, the first-order conditions analogous to (11) and (12) can, assuming again strictly interior solutions for  $M_0$  and  $B_0$ , be expressed as:

$$\left[ p_\ell r_f t e^{-kW_{21}^C} + (1 - p_\ell)(1 - t - r_s + r_f t) e^{-kW_{11}^C} \right] = 0, \quad (15)$$

(recall that  $(1 - t - r_s + r_f t) < 0$ ), and

$$\left[ (r_1 - r_f) e^{-k(r_1 - r_f)B_0} p_b + (r_2 - r_f) e^{-k(r_2 - r_f)B_0} (1 - p_b) \right] = 0. \quad (16)$$

Using the equations for  $W_{11}^C$  and  $W_{21}^C$ , we may obtain the following solution for  $M_0$ :

$$M_0^* = \frac{r_s X}{r_s + t - 1} + \frac{(1 - t)}{k(r_s + t - 1)} \ln \frac{p_l r_f t}{(1 - p_l)(r_s + t - 1 - r_f t)}, \quad (17)$$

while that for  $B_0$  is:

$$B_0^* = \frac{1}{k(r_1 - r_2)} \ln \frac{p_b(r_1 - r_f)}{(1 - p_b)(r_f - r_2)}. \quad (18)$$

(The logarithmic term above is strictly positive if, as we require for a positive  $B_0^*$ , the expected return to  $B$  exceeds that to  $F$ .) We examine the implications of these solutions below. For ease of reference, at various places below we present the results of the *CARA* analysis under the sub-heading ‘A Small Digression, Continued’.

## 4 Simulations

We adopt the MultiStart function in Matlab (MathWorks 2024a), based on Matlab’s Optimization Toolbox, which determines the Global Maximum of expected utility within the permitted ranges of  $M_0$  and  $B_0$ .<sup>15</sup> Our primary interest is to examine the implications, for patterns of substitutability and complementarity among assets, of non-separability of risk-averse expected utility as described in the Introduction, and as exemplified by *CRR*A utility.<sup>16</sup> Accordingly, in our initial calibration we choose parameter values that are empirically reasonable, and generate

<sup>15</sup> Earlier, we had adopted the Adaptive Grid Search algorithm of Parkhomenko (2017), which has a faster convergence speed than MultiStart, but only identifies local optima. It turned out that both algorithms provided the same solutions in all our simulations below. I am indebted to Wang Yu Han for bringing the MultiStart function to my attention.

<sup>16</sup> Thus, our assumption of independence of precautionary risk and rate-of-return risk is motivated, not by its empirical realism, but by the desire to have as sharp a theoretical analysis as possible, not confounded by any correlation between these risks. Similarly with our simulation analysis: we do not aim at empirical comprehensiveness, but at highlighting theoretical results or possibilities on money demand which scholars have not hitherto been aware of.

asset allocations that are close to reality, even though they may not exactly match all aspects of real-world allocations.

Our initial calibration is:  $W_0 = 1$ ;  $r_s = 1.01499$ ;  $r_f = 1.02465$ ;  $r_1 = 1.11$ ;  $r_2 = 0.95$ ;  $p_\ell = 0.5$ ;  $p_b = 0.5$ ;  $x = 0.199$ ;  $t = 0.0115$ ;  $\sigma = 1$ . The choice of ‘unrounded’ values of  $r_s$  and  $x$  is due to our having to confine the solution values of  $M_0$  and  $S_0$  to the interior of small intervals, and these solution values are highly sensitive to tiny perturbations of parameter values. The solution values we obtain, denoted by asterisks, are:  $M_0^* = 0.0771$ ,  $S_0^* = 0.1233$ ,  $F_0^* = 0.0451$ , and  $B_0^* = 0.7544$ . These solutions are fairly reflective of actual US values around 2015.<sup>17</sup> The square root of the sum of squared deviations of the left sides of (11) and (12) from 0, denoted  $D^*$ , is  $3.0301e-15$ , indicating the high accuracy of the solution.

We come now to the crux of our analysis. This is best demonstrated by two comparative-static analyses, followed by additional investigations designed to explore further significant implications of the model:

(A) Let there be a mean-preserving increase in the variance of bond returns. Specifically, we let  $r_1 = 1.115$  and  $r_2 = 0.945$  with the same probabilities as before.  $B_0^*$  falls to 0.6679, and  $F_0^*$  rises to 0.1316. These changes are in line with what one would expect from Tobin’s classic work (*op. cit.*), modified by the argument of Chang et al. (*op. cit.*), although the magnitudes of the changes, after a very small increase in the variance of returns, is striking.<sup>18</sup> What is surprising, however, is the next pair of results.  $M_0^*$  decreases to 0.0734, and  $S_0^*$  increases to 0.1271.  $D^*$  is  $1.5332e-14$ , indicating again a very high accuracy of the solution.

These results are surprising on two counts. The fact that  $M_0^*$  and  $S_0^*$  change at all is quite unexpected, since these assets are essentially designed to meet the asset-holder’s precautionary requirements, rather than for portfolio diversification purposes, for which one would expect  $B_0$  and  $F_0$  to suffice.

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17 We consider only the shares of ‘marketable wealth’ components reported in Srinivas (2019), abstracting from pension entitlements, unincorporated business equity, and life insurance reserves (small), add the value of currency outside banks (not included in Srinivas’ chart), and disaggregate the deposits component into (a) savings accounts and CDs using as an approximation the weights for 2013 for each of these derived from Table 45 of Panis and Brien (2016), and (b) total checkable deposits, data on which is given in the St Louis Fed’s database. The resulting estimates, using the same notation as above and normalizing total initial portfolio size to 1, are:  $M_0 = 0.083$ ,  $S_0 = 0.13$ ,  $F_0 = 0.047$ , and  $B_0$  (corporate equities and bonds) = 0.74. Also, we adopt somewhat higher values of  $r_s$  and  $r_f$  than those reported in Section 2 above to better match the data for 2015, while our selected values for  $r_1$  and  $r_2$  are admittedly low, as we make no attempt here to resolve the equity premium puzzle. (Below, even a doubling of the coefficient of relative risk aversion does not noticeably affect the values of  $r_1$  and  $r_2$  that generate a good fit with the data.)

18 Unfortunately, empirical studies of interest rate volatility generally focus on the implications of this for bond risk premia, rather than for asset demands.

Secondly, given that bonds have become more risky, one would expect, if anything, a shift towards increased holdings of  $M_0$ , the less risky asset, and reduced holdings of  $S_0$ , whereas the opposite occurs.

The economic rationale for these results can be understood from an additional finding. Although the variance of individual bond returns has increased,  $B_0^*$  decreases to such an extent that the realized variance of total bond returns,  $(B_0^*)^2 [p_b(r_1 - \bar{r})^2 + (1 - p_b)(r_2 - \bar{r})^2]$ , where  $\bar{r}$  is the mean of  $r_1$  and  $r_2$ , decreases, from 0.0036 to 0.0032. Although asset allocations are determined jointly, heuristically it appears that, having reduced her risk exposure to bonds significantly, the asset-holder is prepared to assume a slightly greater exposure to liquidity risk given that as shown earlier  $S$  yields a higher expected return than  $M$  in any interior solution. The overall ‘risk-return trade-off’ shifts in the direction of increased holding of  $S$ . Our analysis thus demonstrates, even in the case where bond rate-of-return risk and precautionary risk are independent of each other, the non-separability property of risk-averse expected utility, in the sense that precautionary demands for assets are not independent of bond rate-of-return risk.

There is some controversy as to an empirically accurate estimate of the coefficient of relative risk aversion, with some studies viewing it as close to unity, the value we have employed above, and others as significantly different.<sup>19</sup> To examine the robustness of our result above, we follow Christiano, Eichenbaum, and Rebelo (2011, cited in Rouleau-Pasdeloup 2018) and double  $\sigma$ , from 1 to 2. We also adjusted  $r_s$ ,  $r_f$ , and  $r_1$  slightly, to 1.0185, 1.0195, and 1.1112 respectively, to obtain solutions for the various asset-holdings that are close to those in our baseline calibration. In particular,  $B_0^*$  is now 0.7664. We then as previously increase  $r_1$  by 0.005 and decrease  $r_2$  by the same amount, and the results are qualitatively unchanged, with  $B_0^*$  again decreasing, to 0.6783,  $F_0^*$  increasing from 0.0336 to 0.1216,  $M_0^*$  decreasing from 0.1094 to 0.1061, and  $S_0^*$  increasing from 0.0907 to 0.0940. Again, the realized variance of total bond returns goes down, from 0.0038 to 0.0034. This and a further result reported in (B) below attest to the robustness of our non-separability results.

**A Small Digression, Continued.**<sup>20</sup> In (17) both  $r_1$  and  $r_2$  are absent, and so changing them has no effect on  $M_0^*$  under *CARA*, unlike the *CRRRA* case.

<sup>19</sup> Gandelman and Hernandez-Murillo (2014) provide a useful survey.

<sup>20</sup> In the *CARA* case, we set  $k = 2$  (MathWorks 2024b), and small adjustments had to be made to  $r_1$  (to 1.114),  $r_2$  (to 0.954), and  $r_s$  (to 1.01785) to ensure interior optima for all variables that are reasonably close to those under *CRRRA*. Our solutions are  $M_0^* = 0.0725$ ,  $S_0^* = 0.1280$ ,  $F_0^* = 0.0657$ , and  $B_0^* = 0.7338$ .

Differentiating (18) with respect to  $r_1$  and setting  $dr_2/dr_1 = -1$  it is readily shown that  $dB_0^*/dr_1 < 0$ , as expected, and hence  $dF_0^*/dr_1 > 0$ .

Although there is separability in that  $r_1$  and  $r_2$  do not affect  $M_0^*$  and  $S_0^*$  under *CARA*, the same is not true of  $r_f$ . From (18) we have that  $dB_0^*/dr_f < 0$ , and from (17) it is readily shown that  $dM_0^*/dr_f > 0$ . The rationale for this latter result is novel. Manipulating (3) and (4) we easily obtain that  $S_0^* + M_0^* = \frac{x-tM_0^*}{1-t}$ . Thus, when  $M_0^*$  goes up  $S_0^* + M_0^*$  goes down, and so some further funds are released for investment in  $F_0$ . This is the main source of non-separability in the *CARA* case.<sup>21</sup>

It should be noted that in the present experiment of a mean-preserving increase in bond returns, there are no wealth effects. Thus, the key difference between *CARA* and *CRRA* is not the *DARA* (Decreasing Absolute Risk Aversion) entailed by *CRRA* utility (which plays some part below), but the much more limited degree of non-separability under *CARA* utility.

- (B) Returning to the *CRRA* case, the next comparative-static experiment is perhaps even more stark. Let there instead be a marginal increase in the expected bond return. We raise  $r_1$  to 1.1102, and  $r_2$  to 0.9502. As expected,  $B_0^*$  rises, to 0.7829, and  $F_0^*$  falls, to 0.0167. What is entirely unexpected is that  $M_0^*$  increases, to 0.0795, and  $S_0^*$  decreases, to 0.1208.  $D^*$  is again extremely low, at 1.3204e-14. In this case, the realized variance of total bond returns has gone up, to 0.0039, and heuristically the asset-holder mitigates the increase in her overall risk exposure by substituting away from  $S_0$  and towards  $M_0$  in meeting precautionary demands.<sup>22</sup> Remarkably, a positive relationship between US money (M1) demand and the 10-year Government Bond Rate was indeed observed in the 1990–2019 period for interest rates above 4 % *p.a.*, as Kim and Marchesiani (2024) document. Not only does our result further exemplify the non-separability property of risk-averse expected utility, it also shows that the conventional wisdom that expected bond returns should enter negatively in the demand function for M1 need not always hold, and the effect is instead model-dependent. Moreover, we obtain the same qualitative outcomes when  $\sigma = 2$ . Henceforth, we maintain  $\sigma$  at unity.<sup>23</sup>

<sup>21</sup> Another source, which results rather mechanically from the wealth constraint, has to do with the effect of  $r_s$  on  $F_0^*$ , the analysis of which is very similar to the effect in the *CRRA* case (*fn.* 25 below).

<sup>22</sup> There is an offsetting wealth effect: evaluated at her initial level of bond-holding, an increase in the mean bond return increases the asset-holder's wealth, and with *DARA* she would wish to increase her holding of  $S$  and reduce her holding of  $M$ . It turns out that this only partially offsets the effect described above.

<sup>23</sup> A seemingly puzzling feature of our result is the very high interest-elasticity of money demand that is generated, of the order, in fact, of 155. Berentsen, Huber, and Marchesiani (2015) obtain an

A third experiment yields somewhat less clear-cut but nonetheless highly useful additional insights. We return to our original calibration, except that we increase  $t$  marginally to 0.01152: at the same time, in order to keep  $M_0^* + S_0^*$  unchanged, at 0.2004, we increase  $x$ , and to 4 decimal places the required value of  $x$  is also 0.2004. Now  $M_0^*$  rises drastically, to 0.198, and  $S_0^*$  falls drastically, to practically 0 (actual value is 0.0024). Following our earlier argument, one might expect  $B_0^*$  to increase and  $F_0^*$  to decrease, since the asset-holder can now ‘afford’ to take on additional risk. Instead, however,  $B_0^*$  decreases marginally, to 0.7540, and  $F_0^*$  increases marginally, to 0.0456. These very small effects are explained by the wealth effect: evaluated at her initial levels of  $M_0$  and  $S_0$ , increases in  $t$  and  $x$  reduce the asset-holder’s expected terminal wealth net of expected precautionary costs, and with *DARA* she chooses to marginally decrease her holdings of  $B$  and increase her holdings of  $F$ . Any further increase in  $t$  would result in a corner solution for  $S_0$ , which would make comparative-static experiments difficult to interpret. Evidently, the optimal level of  $S_0$  is highly sensitive to  $t$ .<sup>24</sup>

One may examine the effects of varying other parameters, such as the rates of return on  $S$  and  $F$ , but the results are similar.<sup>25</sup> Our final thought-provoking experiment is to examine the effects of changes in  $W_0$  relative to changes in  $x$ . There are two noteworthy cases here:

- (a) Suppose we increase only  $W_0$  in our baseline calibration, to 1.005. The results are, quite remarkably, that  $M_0^*$  decreases quite significantly to 0.0345, and  $S_0^*$  increases to 0.1664.  $B_0^*$  and  $F_0^*$  increase slightly, to 0.7587 and 0.0454 respectively. We note that  $M_0^*$  has decreased, by over 50 %, notwithstanding that

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estimate of this elasticity in the upward-sloping region of money demand of 0.16. However, in our model the increase in money demand is *more* than offset by a concomitant decrease in demand for  $S$  (from (3) and (4) above we have that  $(M_0 + S_0) = (x - tM_0)/(1 - t)$ ), while the resultant decrease in interest earnings from  $S$  is ‘compensated for’ by the increased bond expected return. As further discussed below, this result underscores the necessity of a ‘systems’ or structural approach to the study of money demand.

**24** In order to generate more variation in  $t$  we changed the initial calibration to  $x = 0.38874$  and  $t = 0.00878$ , resulting in an initial value of  $S_0^*$  of 0.3526 (we also increased  $r_f$  to 1.0265 to generate an interior solution for  $F_0$ ). However, even here we could only increase  $t$  and  $x$  very slightly before  $S_0^*$  approached the corner at 0, and the results are qualitatively unchanged.

**25** We increased  $r_s$  marginally to 1.0151, keeping all other parameters at their baseline values, and there occurs a large increase in  $S_0^*$ , a virtually zero (0.0023) solution for  $M_0^*$ , and very small decreases in  $B_0^*$  and  $F_0^*$ . These small declines are mainly due to the larger value of  $M_0^* + S_0^*$  owing to the higher transactions costs incurred when  $S_0^*$  goes up. Similarly, a small increase in  $r_f$ , to 1.025, generates as expected an increase in  $F_0^*$  and decrease in  $B_0^*$ , both of which are fairly large (to 0.0948 and 0.7050 respectively), as well as a moderate decrease in  $S_0^*$  (to 0.1085) and increase in  $M_0^*$  (to 0.0917). (With  $F_0^* + B_0^*$  increasing,  $M_0^* + S_0^*$  has to decrease, which is effected by reducing  $S_0^*$  and increasing  $M_0^*$  ((3) and (4) above).)

the initial overall portfolio size  $W_0$  has increased. There are a number of noteworthy implications of these results:

- (i) With *DARA* the demand for the risky assets has gone up. However, strikingly, the demand for  $S$  rises much more than the demand for  $B$ , in both absolute and relative terms. This is due to a feature of the model that has been alluded to earlier: investing in  $S$  reduces the demand for  $M$  for precautionary purposes almost *pari passu*, and the funds thus released can be utilized for other investments. No such offsetting release occurs upon increasing holdings of  $B$ .
- (ii) The fact that the demand for  $M$  has gone down marks a notable departure from the well-known result expressed in Theorem 1 of Cass and Stiglitz (1972) and in earlier literature, in which, in a pure ‘portfolio’ analysis of the demand for one risky asset and money, the wealth elasticity of the demand for money is unity under constant relative risk aversion. With our more elaborate asset structure, the use of  $M$  for portfolio diversification purposes is dominated by  $F$ , and the demand for  $M$ , *vis-à-vis*  $S$ , is governed by precautionary requirements and expected-utility maximization. We thus support the opinion of Cass and Stiglitz (*ibid.* p. 331): ‘without stringent conditions it does not appear to be possible to derive a simple theory of the demand for money from portfolio analysis’ (p. 331).

**A Small Digression, Continued:**  $W_0$  does not appear in (17) and (18), and hence  $M_0^*$  (and  $S_0^*$ ) as well as  $B_0^*$  are invariant to any changes in  $W_0$ , which simply change  $F_0^*$  *pari passu*. This result clearly exhibits the role played by *DARA* in the analysis of wealth effects above. In view of the well-known analytical limitations of *CARA* utility (exemplified, in fact, by the result here), we will continue to work with *CRRA* utility.<sup>26</sup>

- (b) Suppose that the increase in  $W_0$  is accompanied by an increase in  $x$  such that the initial portfolio size, *net of* the expected size of the precautionary shock, is unchanged. In our example, with  $p_\ell = 0.5$ , this requires that  $x$  increases by twice the increase in  $W_0$ , to 0.209. The asset-holder’s expected initial net wealth has not changed, but she confronts higher risk. In this case, we find that, notwithstanding the small absolute size of these changes,  $S_0^*$  *decreases*, to such an extent that the zero lower bound on it becomes binding. The final solution then is that  $M_0^*$  increases substantially, also to 0.209 (since with  $S_0^* = 0$

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<sup>26</sup> Kim and Marchesiani (p. 1223) observe that assuming a *CRRA* specification for the utility function ‘is a standard practice in the literature.’ *Obiter dictum* our analysis should provide a cautionary note against uncritical adoption of specifications (such as *CARA* utility) which give rise to ‘neat’ closed-form solutions, in preference to *CRRA* utility, the analysis of which is more ‘messy’ but also more realistic.

there are no transactions costs to be incurred), and with increased provision for the possible precautionary shock owing to the increase in  $x$  both  $B_0^*$  and  $F_0^*$  decrease marginally. We discuss the implications of these results below.

The analysis thus far has been conducted within a single- (or dual-) period optimization framework. Under simplifying assumptions, the results also hold in an infinite-horizon framework, noting that as mentioned we confine ourselves in this study to a partial-equilibrium analysis of asset demands. We assume that the asset-holder receives an exogenous endowment each period, which, consistent with the data, may be growing over time, and is sufficiently impatient that she wishes to consume her entire end-of-period wealth each period. (From the definition of the stochastic discount factor, such impatience is even more pronounced if consumption is generally rising over time.) We further assume that she faces a borrowing constraint that does not permit any borrowing in any period. Finally, we assume that the rate-of-return risk and the precautionary risk are independent over time, and independent of each other. Under these specifications, it is easy to see that our one-period analysis is replicated every period, with possibly different values of beginning-of-period wealth, and of  $t$  and  $x$  each period. If desired, the various asset rates of return can also be treated as exogenously varying over time.

Replication, *mutatis mutandis*, of our one-period analysis may help to explain certain noteworthy broad trends over time.<sup>27</sup> Federal Reserve data (Federal Reserve 2020) show a pronounced decline in the M1/GDP ratio in the US from about 0.275 in 1959 to about 0.09 in 2007, prior to the onset of the Global Financial Crisis. (There were upward movements in the late-1980s and mid-1990s.) Absolute levels of M1 rose throughout this period (*ibid.*). Basing on our various results above, our model is in principle capable of accounting for these, and concurrent developments in other asset-holdings, by suitable combinations of rises in GDP (a proxy for yearly endowments), smaller rises in  $x$ , some declines in  $t$  in accordance with the data, and, although this is not essential, some adjustments in asset returns. The first two of these changes could produce a falling M1/GDP ratio and a rising M1,<sup>28</sup> and the

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27 We are abstracting here from savings over time, implicitly assuming, as do Lucas (2000) and Lucas and Nicolini (2015), that all income is consumed. The US national savings rate has been below 10 % from the mid-1960's till 2007, except for one year in the early 1970's (Peterson 2020). The personal savings rate has been below 11.4 % (8.8 %) since 1980 (1990) up to 2007 (Statistica Research Department 2020), and part of these savings, e.g. those placed in Individual Retirement Accounts, are for longer-term retirement purposes.

28 There is a possible additional reason for these first two developments. In 2012 the Federal Reserve introduced the annual *Diary of Consumer Payment Choice*, one of the findings of which has been, 'Historically, cash has been the most used payment instrument for small-value payments (payments under \$25)' (Cubides and O'Brien (2023), p. 11). One might in fact extend the argument to M1 as a whole, since small-value payments can also be defrayed through debit cards and ATM

third of these could help to better match the generally rising behaviour of  $S$  over time. By contrast, the framework of Belongia and Ireland (*op. cit.*), building on the earlier work by Lucas (2000, 1980), abstracts completely from rate of return and precautionary risks, and imposes an unchanging equilibrium value of  $M1/GDP$  over time *if* user costs of holding money are constant. This does not appear to do justice to the complexity of the determinants and behaviour of money demand.<sup>29</sup>

Beyond the foregoing observations, formal econometric modelling of money demand over time is outside the scope of this essentially conceptual study. Such modelling would have to take into consideration changes in financial regulations, ‘the introduction of more innovative financial products’ (Kim and Marchesiani, *op. cit.*), measurement issues, mechanism design issues (*ibid.*), and other factors. It is unclear as to what extent various of these considerations can simply be subsumed under the rubric of changes in our  $t$  parameter. Changes in both  $t$  and  $x$  over time would have to be identified and estimated, as also would possibly time-varying patterns of correlation, if any, between differing sources of risk. There is also a further consideration, which none of the studies cited above have taken into account: money demand is affected, not only by the average level of bond returns, but also, we have shown, by a mean-preserving increase in the variance of bond returns. Relatedly, one must be cognizant of the fact that in our setup it is not possible to vary  $M_0$  independently of  $S_0$ , and, again, a structural approach to financial asset demands (viewing an asset-holder’s portfolio in its entirety) is called for. Lastly, to the extent that parsimony is an advantage, the fact that our model can parsimoniously match the observed positive relationship, over a certain range, between money demand and the mean bond return gives it an edge over the extraordinarily complicated (and highly stylized) New-Monetarist-Mechanism-Design article by Kim and Marchesiani (*op. cit.*), the only other study among those cited that can generate such a relationship.<sup>30</sup> Our model thus, we believe, provide a valuable ‘springboard’ for further studies, both theoretical and empirical.

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cards. One could for convenience then model the use of  $M1$  for this purpose as a concave deterministic function of  $GDP$ , which could easily be introduced into our model as an additional source of demand for  $M1$ , thus helping to account for the first two developments mentioned above. Unfortunately, data limitations preclude an empirical examination of this hypothesis over the 1959–2007 period.

<sup>29</sup> Lucas (2000) himself notes (looking at the 1900–94 period), ‘The money-income ratio is essentially trendless over the entire century, although there has been a strong downward trend since World War II’ (p. 249).

<sup>30</sup> The following comment by Kim and Marchesiani (p. 1232) should, however, be acknowledged: ‘Of course, we do not claim that the mechanism is the only determinant of the observed money demand behavior after the 1990s. We only argue that the mechanism, in the form of  $MI$  (Market Intelligence), could have, together with other factors studied in the literature ... explained part of the behavior of the money demand. Our paper complements this literature.’

## 5 Conclusions

Why is a four-asset framework more suitable for the study of money demand, and demand for other assets, than a two- or three- asset framework? There are two independent sources of risk in the model – a longer-term portfolio or rate-of-return risk, and a shorter-term transactions risk. With four assets, two of them can be ‘assigned’ to deal with each source of risk, in the specific senses that  $B$  and  $F$  are not deployed to handle transactions risk, while there would be no demand for  $M$  and  $S$  if  $x$  were 0. By contrast, if there were for example only three assets, one of them, namely  $S$ , would have to help satisfy both portfolio diversification and transactions motives. As such, one would expect the demand for  $S$  to go down by less (and hence the demand for  $M$  to go up by less) when  $t$  goes up, since  $S$  is also held for portfolio diversification purposes. Asset-holders are in reality aware of the benefits of optimizing across four assets, as evidenced by positive real-world holdings of all these assets, and if one instead works with a three- or two- asset framework, one is likely to obtain an inaccurate characterization of asset demands.

Quite remarkably, even though the demands for  $M$  and  $S$  arise only when  $x$  is positive, the optimal allocation *between*  $M$  and  $S$  does depend on characteristics of long-term assets, such as the rate of return on bonds and the variance of bond returns, as we have demonstrated. Similarly characteristics of short-term assets such as the unit transactions cost  $t$  do affect the demands for long-term assets. Changes in the asset-holder’s wealth also affects the demands for various assets in unexpected ways. These results are due to the non-separability property of risk-averse expected utility of terminal wealth under *DARA*, explained earlier. The juxtaposition of this property with a four-asset framework thus yields novel and important results and insights.

In an intertemporal context, our model is potentially capable of matching the pronounced decline in the M1/GDP ratio over the 1959–2007 period, which various other works cited above do not. There is, finally, a need to extend the analysis to a dynamic, stochastic general-equilibrium framework, without imposing simplifying assumptions that yield, often counterfactually, an unchanging equilibrium M1/GDP ratio over time.

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