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# Primal and dual optimal stopping with signatures

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## Abstract

We propose two signature-based methods to solve an optimal stopping problem – that is, to price American options – in non-Markovian frameworks. Both methods rely on a global approximation result for  $L^p$ -functionals on rough-path spaces, using linear functionals of robust, rough-path signatures. In the primal formulation, we present a non-Markovian generalisation of the famous Longstaff–Schwartz algorithm, using linear functionals of the signature as regression basis. For the dual formulation, we parametrise the space of square-integrable martingales using linear functionals of the signature and apply a sample average approximation. We prove convergence for both methods and present first numerical examples in non-Markovian and non-semimartingale regimes.

**Keywords** Signature · Optimal stopping · Rough paths · Monte Carlo · Rough volatility

**Mathematics Subject Classification** 60L10 · 60L20 · 91G20 · 91G60

**JEL Classification** C63 · G12

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# 1 Introduction

Stochastic processes with memory play a more and more important role in the modelling of financial markets. In the modelling of equity markets, *rough stochastic volatility models* are now part of the standard toolbox; see e.g. Gatheral et al. [27] and Bayer et al. [4]. In the same area, *path-dependent stochastic volatility models* (e.g. Guyon et al. [30]) are a very powerful alternative for capturing memory effects. Processes with memory are also an essential tool for modelling the microstructure of financial markets, driven by the market practice of splitting large orders into many medium-size ones, as well as by the reaction of algorithmic traders to such orders. Seen from outside, this materialises as self-excitation of the order flow and, consequently, *Hawkes processes* are a fundamental tool for modelling order flows; see e.g. Bouchaud et al. [15, Chap. 9.4]. Beyond finance, processes with memory play an important role in the modelling of many natural phenomena (e.g. earthquakes, see Ogata et al. [35]) or social phenomena.

In this paper, we study optimal stopping problems in non-Markovian frameworks, that is, the underlying price is possibly a stochastic process with memory. For concreteness' sake, let us introduce two processes determining the optimal stopping problem: an underlying *state process*  $X$ , together with its natural filtration  $\mathbb{F}^X$ , and a *reward process*  $Z$ , which is  $\mathbb{F}^X$ -adapted – think about  $X = (S, v)$  for a stock price process  $S$  driven by a stochastic variance process  $v$  and  $Z_t = \phi(t, S_t)$ . The optimal stopping problem then consists of solving the optimisation problem

$$y_0 = \sup_{\tau \in \mathcal{S}_0} E[Z_\tau], \quad (1.1)$$

where  $\mathcal{S}_0$  denotes the set of  $\mathbb{F}^X$ -stopping times valued in  $[0, T]$  for some  $T > 0$ . We merely assume  $\alpha$ -Hölder-continuity for  $X$  in our framework, see Sect. 3.1 below, in particular allowing non-Markovian and non-semimartingale state processes  $X$ .

The lack of a Markov property leads to severe theoretical and computational challenges in the context of optimal control problems, and thus in particular in the optimal stopping problem (1.1). Indeed, the primary analytical and numerical framework for stochastic optimal control problems arguably is the associated Hamilton–Jacobi–Bellman (HJB) PDE, which in the context of optimal stopping leads to so-called *free-boundary problems*; see Peskir and Shiryaev [36, Chap. 4]. When the state process is not a Markov process, such PDEs do not exist a priori. As noted above, infinite-dimensional (BS)PDE formulations can be given; see for instance Bayer et al. [6] for a BSPDE description of the American option price in rough volatility models. When the underlying dynamics is of stochastic Volterra type, *path-dependent* HJB PDEs could probably be derived following the approach of Bonesini and Jacquier [14]. However, most *numerical approximation methods* crucially rely on the Markov property as well.

It should be noted that at least intuitively, all processes with memory can be turned into Markov processes by adding the history to the current state – but see e.g. Carmona and Coutin [16] for a more sophisticated approach in the case of fractional Brownian motion. Hence theoretical and numerical methods from the Markovian world are in principle available, but at the cost of having to work in infinite-dimensional (often very carefully drafted, see e.g. Cuchiero and Teichmann [21])

state spaces. On the other hand, *Markovian approximations*, i.e., finite-dimensional Markov processes closely mimicking the process with memory, can sometimes be a very efficient surrogate model, especially when high accuracy is achievable with low-dimensional Markovian approximations; see e.g. Bayer and Breneis [3].

Inspired by many successful uses in machine learning (for time-series data), Kalsi et al. [31] introduced a model-free method for numerically solving a stochastic optimal execution problem. The method is based on the *path signature*, see e.g. Friz and Victoir [26, Chap. 7], and is applicable in non-Markovian settings. This approach was extended to optimal stopping problems in Bayer et al. [5], where stopping times were parametrised as first hitting times of affine hyperplanes in the signature space. A rigorous mathematical analysis of that method was performed and numerical examples verifying its efficiency were provided.

The signature  $\mathbf{X}^{<\infty}$  of a path  $X : [0, T] \rightarrow \mathbb{R}^d$  is given (at least formally) as the infinite collection of *iterated integrals*, that is, for  $0 \leq t \leq s \leq T$ ,

$$\mathbf{X}_{s,t}^{<\infty} = \left\{ \int_s^t \int_s^{t_k} \cdots \int_s^{t_2} dX_{t_1}^{i_1} \cdots dX_{t_k}^{i_k} : i_1, \dots, i_k \in \{1, \dots, d\}, k \geq 0 \right\}.$$

The signature characterises the history of the corresponding trajectory and hence provides a systematic way of “lifting” a process with memory to a Markov process by adding the past to the state. Relying only on minimal regularity assumptions, the corresponding encoding is efficient and has nice algebraic properties. In many ways, (linear functionals of) the path signature behaves like an analogue of polynomials on path space and can be seen as a canonical choice of basis functions on path space. For example, a *Stone–Weierstrass* type result shows that when restricted to compacts, continuous functionals on path spaces can be approximated by linear functionals of the signature, that is, by linear combinations of iterated integrals; see for instance Kalsi et al. [31, Lemma 3.4].

As a first contribution, we provide in Sect. 2 an abstract approximation result on  $\alpha$ -Hölder rough-path spaces by linear functionals of the *robust* signature, with respect to the  $L^p$ -norm; see Theorem 2.8 below. As a direct consequence and under very mild assumptions, we can show that for any  $\mathbb{R}^X$ -progressive process  $(\xi_t)_{t \in [0, T]}$ , we can find a sequence  $(\ell_n)_{n \in \mathbb{N}}$  of linear functionals on the state space of the signature such that

$$\lim_{n \rightarrow \infty} E \left[ \int_0^T (\xi_t - \langle \mathbf{X}_{0,t}^{<\infty}, \ell_n \rangle)^2 dt \right] = 0; \quad (1.2)$$

see Corollary 2.9 below for details. This result is in marked contrast to the standard universal approximation result for signatures as usually formulated, which only provides uniform convergence on compact subsets of the path space. Let us mention two related works on global approximation with signatures. First, in Cuchiero et al. [20], the authors study global signature approximations based on a version of the Stone–Weierstrass result for so-called *weighted spaces*; see [20, Theorem 3.6]. Compared to our theory, while they do not require a robust version of the signature, such weighted spaces need to be crafted carefully. Secondly, in the recent work by Alaifari and Schell [1], the authors obtain (independently from us) a similar global  $L^p$ -approximation result for robust signatures of bounded variation paths, see [1,

Proposition 4.5], but based on a monotone class rather than a Stone–Weierstrass argument.

Returning to the optimal stopping problem (1.1), we generalise in Sect. 3 two standard techniques from the Markovian to the non-Markovian case by using signatures, namely the *Longstaff–Schwartz algorithm* [33] and Rogers’ *dual martingale method* [37]. Denoting by  $Y$  the *Snell envelope* for the optimal stopping problem, see Sect. 3.2 below for more details, the Longstaff–Schwartz algorithm is based on the *dynamic programming principle*, that is,

$$Y_t = \max(Z_t, E[Y_{t+\Delta t} | \mathcal{F}_t^X]).$$

If  $X$  is a Markov process, then  $E[Y_{t+\Delta t} | \mathcal{F}_t^X] = E[Y_{t+\Delta t} | X_t]$ , which can be efficiently computed by using regression (*least-squares Monte Carlo*). In the non-Markovian case, an application of the global approximation result in Theorem 2.8, i.e., the convergence in (1.2), shows that (under minimal assumptions) a Longstaff–Schwartz algorithm converges when the conditional expectation is approximated by linear functionals of the signature, that is,

$$t \mapsto E[Y_{t+\Delta t} | \mathcal{F}_t^X] \approx \langle \mathbf{X}_{0,t}^{<\infty}, \ell \rangle;$$

see Proposition 3.3.

Regarding the dual method, we rely on Rogers’ characterisation that

$$y_0 = \inf_{M \in \mathcal{M}_0^2} E \left[ \sup_{t \in [0, T]} (Z_t - M_t) \right],$$

where the inf is taken over all square-integrable martingales  $M$  starting at 0. If the underlying filtration is Brownian, such martingales can be written as stochastic integrals with respect to a Brownian motion  $W$ , that is,  $M_t = \int_0^t \xi_s dW_s$  for some  $\mathbb{F}^X$ -progressive process  $\xi$ . The approximation result in Theorem 2.8, i.e., the convergence in (1.2), suggests to approximate the integrand by linear functionals of the signature, that is,

$$t \mapsto \xi_t \approx \langle \mathbf{X}_{0,t}^{<\infty}, \ell \rangle,$$

and we prove convergence after taking the infimum over all linear functionals  $\ell$ , i.e.,

$$y_0 = \inf_{\ell} E \left[ \sup_{t \in [0, T]} \left( Z_t - \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell \rangle dW_s \right) \right]; \quad (1.3)$$

see Proposition 3.8. For numerically solving the dual problem (1.3), we carry out a *sample average approximation* (SAA) with respect to the coefficients of the linear functional of the signature. For a Markovian environment, a related SAA procedure was earlier proposed in Desai et al. [22] and recently refined in Belomestny and Schoenmakers [11] and Belomestny et al. [10] by using a suitable randomisation. An important feature of the SAA method is that it relies on non-nested Monte Carlo simulation and thus is very fast in comparison to the classical nested Monte Carlo method by Andersen and Broadie [2].

For both the Longstaff–Schwartz and dual signature methods, we also prove convergence of the finite-sample approximations when the number of samples grows to infinity; see Propositions 3.4 and 3.10. It is worth noting that after independent re-simulations, the Longstaff–Schwartz algorithm yields lower-biased, whereas the dual method gives upper-biased values to the optimal stopping problem (1.1), and thus applying both methods produces confidence intervals for the true value of  $y_0$ .

Finally, in Sect. 4, we provide first numerical examples based on the primal and dual signature-based approaches in two non-Markovian frameworks. First, in Sect. 4.1, we study the task of optimally stopping *fractional Brownian motion* for a wide range of Hurst parameters  $H \in (0, 1)$ , representing the canonical choice of a state process outside of the Markov regime. The same problem was already studied in Becker et al. [8] and later in Bayer et al. [5], and we compare our lower (resp. upper) bounds with the results therein. Secondly, in Sect. 4.2, we consider the problem of computing American option prices in the *rough Bergomi model* (see Bayer et al. [4]), and we compare our price intervals with Bayer et al. [7] (resp. Goudenege et al. [29]), where lower bounds were computed in the same model.

## 1.1 Notation

For  $d, K \in \mathbb{N}$ , we define the so-called *extended tensor algebra* and the  $K$ -step truncation thereof by

$$T((\mathbb{R}^d)) = \prod_{k \geq 0} (\mathbb{R}^d)^{\otimes k}, \quad T^{\leq K}(\mathbb{R}^d) = \prod_{k=0}^K (\mathbb{R}^d)^{\otimes k},$$

where we use the convention  $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$ . For more details, including natural operations such as sum  $+$  and product  $\star$  on these spaces, see for instance Friz and Victoir [26, Sect. 7.2.1]. For any word  $w = i_1 \dots i_n$  for some  $n \in \mathbb{N}$  with letters  $i_1, \dots, i_n \in \{1, \dots, d\}$ , we define the *degree* of  $w$  as the length of the word, that is,  $\deg(w) = n$ , and denote by  $\emptyset$  the empty word with  $\deg(\emptyset) = 0$ . Moreover, for  $\mathbf{a} \in T((\mathbb{R}^d))$ , we denote by  $\langle \mathbf{a}, w \rangle$  the element of  $\mathbf{a}^{(n)} \in (\mathbb{R}^d)^{\otimes n}$  corresponding to the basis element  $e_{i_1} \otimes \dots \otimes e_{i_n}$ , where  $\{e_1, \dots, e_d\}$  is the standard basis of  $\mathbb{R}^d$ . Denoting by  $\mathcal{W}^d$  the linear span of words, the pairing above can be extended linearly to  $\langle \cdot, \cdot \rangle : T((\mathbb{R}^d)) \times \mathcal{W}^d \rightarrow \mathbb{R}$ . For an element  $\ell \in \mathcal{W}^d$ , i.e.,  $\ell = \lambda_1 w_1 + \dots + \lambda_n w_n$  for some words  $w_1, \dots, w_n$  and scalars  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ , we define the degree of  $\ell$  by  $\deg(\ell) := \max_{1 \leq i \leq n} \deg(w_i)$ , and for  $K \in \mathbb{N}$ , we denote by  $\mathcal{W}_{\leq K}^d \subseteq \mathcal{W}^d$  the subset of elements  $\ell$  with  $\deg(\ell) \leq K$ . For two words  $w$  and  $v$ , we denote by  $\sqcup$  the *shuffle product* given by

$$\begin{aligned} w \sqcup \emptyset &= \emptyset \sqcup w = w, \\ wi \sqcup vj &:= (w \sqcup vj)i + (wi \sqcup v)j, \quad i, j \in \{1, \dots, d\}, \end{aligned} \quad (1.4)$$

which bilinearly extends to the span  $\mathcal{W}^d$  of words. We further define the *free nilpotent Lie group* over  $\mathbb{R}^d$  by

$$G(\mathbb{R}^d) = \{\mathbf{a} \in T((\mathbb{R}^d)) \setminus \{\mathbf{0}\} : \langle \mathbf{a}, w \rangle \langle \mathbf{a}, v \rangle = \langle \mathbf{a}, w \sqcup v \rangle, \forall w, v \in \mathcal{W}^d\};$$

see [26, Chap. 7.5] for details.

For  $\alpha \in (0, 1)$ , we denote by  $C^\alpha([0, T]; \mathbb{R}^d)$  the space of  $\alpha$ -Hölder-continuous paths  $X$ , that is,  $X : [0, T] \rightarrow \mathbb{R}^d$  such that

$$\|X\|_{\alpha; [0, T]} = \sup_{0 \leq s < t \leq T} \frac{|X_t - X_s|}{|t - s|^\alpha} < \infty,$$

where  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^d$ . Denote by  $\Delta_{[0, T]}^2$  the standard simplex  $\Delta_{[0, T]}^2 := \{(s, t) \in [0, T]^2 : 0 \leq s \leq t \leq T\}$ . For  $L \in \mathbb{N}$  and any two-parameter function on the truncated tensor algebra,

$$\Delta_{[0, T]}^2 \ni (s, t) \mapsto \mathbf{X}_{s,t} = (1, \mathbf{X}_{s,t}^{(1)}, \dots, \mathbf{X}_{s,t}^{(L)}) \in T^{\leq L}(\mathbb{R}^d),$$

we denote by  $\|\cdot\|_{(\alpha, L)}$  the norm given by

$$\|\mathbf{X}\|_{(\alpha, L)} := \max_{1 \leq \ell \leq L} \left( \sup_{0 \leq s < t \leq T} \frac{|\mathbf{X}_{s,t}^{(\ell)}|}{|t - s|^{\ell\alpha}} \right)^{1/\ell}.$$

We denote by  $\mathcal{C}_g^\alpha([0, T]; \mathbb{R}^d)$  the space of geometric  $\alpha$ -Hölder rough paths  $\mathbf{X}$  valued in  $\mathbb{R}^d$ , which is the  $\|\cdot\|_{(\alpha, L)}$ -closure of  $L$ -step signatures of Lipschitz-continuous paths  $X : [0, T] \rightarrow \mathbb{R}^d$  for  $L = \lfloor 1/\alpha \rfloor$ . More precisely, for every  $\mathbf{X} \in \mathcal{C}_g^\alpha$ , there exists a sequence  $(X^n)_{n \in \mathbb{N}} \subseteq \text{Lip}([0, T]; \mathbb{R}^d)$  such that  $\|\mathbf{X}^n - \mathbf{X}\|_{(\alpha, L)} \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\mathbf{X}^n$  is the  $L$ -step signature of  $X^n$ , that is,

$$\mathbf{X}_{s,t}^n := \left( \int_{s < t_1 < \dots < t_\ell < t} \otimes dX_{t_1}^n \cdots \otimes dX_{t_\ell}^n : 0 \leq \ell \leq L \right) \in G^{\leq L}(\mathbb{R}^d),$$

where the integrals are defined in the Riemann–Stieltjes sense. For the rest of the paper, we always fix  $L = \lfloor 1/\alpha \rfloor$  and use the shorter notation  $\|\cdot\|_\alpha := \|\cdot\|_{(\alpha, \lfloor 1/\alpha \rfloor)}$ . For any  $\mathbf{X} \in \mathcal{C}_g^\alpha$ , we denote by  $\mathbf{X}^{<\infty}$  the *rough-path signature*, which is the unique (up to tree-like equivalence  $\sim_t$ , see Boedihardjo et al. [12] for details) path from Lyons' extension theorem (Lyons [34, Theorem 3.7]), that is,

$$\Delta_{[s,t]}^2 : [0, T] \ni (s, t) \mapsto \mathbf{X}_{s,t}^{<\infty} = (1, \mathbf{X}_{s,t}^{(1)}, \dots, \mathbf{X}_{s,t}^{(L)}, \mathbf{X}_{s,t}^{(L+1)}, \dots) \in G(\mathbb{R}^d),$$

such that

$$\|\mathbf{X}^{(k)}\|_{k\alpha} < \infty, \quad \forall k \geq 0, \quad \mathbf{X}_{s,t}^{<\infty} = \mathbf{X}_{s,u}^{<\infty} \star \mathbf{X}_{u,t}^{<\infty}, \quad s \leq u \leq t,$$

where the latter is called *Chen's relation*. Finally, by considering time-augmented paths  $(\hat{X}_t) = (t, X_t)$  and their geometric rough-path lifts  $\hat{\mathbf{X}}$ , the signature map becomes unique due to the strictly monotone time component. We denote by  $\hat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^{d+1})$  the space of geometric  $\alpha$ -Hölder rough-path lifts of  $(\hat{X}_t) = (t, X_t)$ ,  $X \in C^\alpha([0, T]; \mathbb{R}^d)$ . We often use the shorter notation  $\hat{\mathcal{C}}_g^\alpha$  when it is clear from the context that we are working on the fixed time interval  $[0, T]$ .

## 2 Global approximation with rough-path signatures

In this section, we present the theoretical foundation of this paper, which consists of a global approximation result based on *robust* rough-path signatures.

### 2.1 The space of stopped rough paths

For  $\alpha \in (0, 1)$ , we consider an  $\alpha$ -Hölder-continuous path  $X : [0, T] \rightarrow \mathbb{R}^d$  starting at  $X_0 = x_0 \in \mathbb{R}^d$  and denote by  $\mathbf{X}$  a geometric rough-path lift of the time-augmentation  $(t, X_t)$ , that is,  $\mathbf{X} \in \widehat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^{d+1})$ .

**Definition 2.1** For any  $\alpha \in (0, 1)$  and  $T > 0$ , the space of stopped  $\widehat{\mathcal{C}}_g^\alpha$ -paths is defined by the disjoint union

$$\Lambda_T^\alpha := \bigcup_{t \in [0, T]} \widehat{\mathcal{C}}_g^\alpha([0, t]; \mathbb{R}^{d+1}).$$

Moreover, we equip the space  $\Lambda_T^\alpha$  with the final topology induced by the map

$$\phi : [0, T] \times \widehat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^{d+1}) \rightarrow \Lambda_T^\alpha, \quad \phi(t, \mathbf{x}) = \mathbf{x}|_{[0, t]}.$$

The reason to work on this space is the following. If  $X$  is a stochastic process and  $\mathbf{X}$  denotes the random geometric lift of  $(t, X_t)$ , we define  $\mathcal{F}_t^\mathbf{X} = \sigma(\mathbf{X}_{0,s} : s \leq t)$  for  $0 \leq t \leq T$ , i.e., the natural filtration generated by  $\mathbf{X}$ . In Lemma 2.4 below, we show that any  $\mathbb{R}^\mathbf{X}$ -progressive process  $(A_t)_{t \in [0, T]}$  can be written as  $A_t = f(\mathbf{X}|_{[0, t]})$ , where  $f$  is a measurable function on  $\Lambda_T^\alpha$ . Thus progressively measurable processes can be thought of as measurable functionals on  $\Lambda_T^\alpha$ , and we discuss approximation results for the latter below. Similar spaces have already been considered in relation with *functional Itô calculus* in Cont and Fournié [19] and Dupire [23], for  $p$ -rough paths in Kalsi et al. [31], and more recently in relation with optimal stopping in Bayer et al. [5].

**Remark 2.2** One can also introduce a metric  $d_\Lambda$  on the space  $\Lambda_T^\alpha$ . Let  $\mathbf{y} \in \Lambda_T^\alpha$ , that is, there exists  $0 \leq s \leq T$  such that  $\mathbf{y} = \mathbf{y}|_{[0, s]} \in \widehat{\mathcal{C}}_g^\alpha([0, s])$ . Now for any  $t \geq s$ , we can extend  $\mathbf{y}|_{[0, s]}$  to a geometric rough path  $\tilde{\mathbf{y}}|_{[0, t]} \in \widehat{\mathcal{C}}_g^\alpha([0, t])$  as follows. By geometricity, there exists a sequence of smooth paths  $(u, y_u^n)_{u \in [0, s]}$  such that the (canonical) rough-path lifts converge to  $\mathbf{y}|_{[0, s]}$ . Then we define  $\tilde{\mathbf{y}}|_{[0, t]}$  to be the rough-path limit of the canonical lift of  $u \mapsto (u, y_{u \wedge s}^n)$  for  $0 \leq u \leq t$ . This construction can be used to define

$$d_\Lambda(\mathbf{x}|_{[0, t]}, \mathbf{y}|_{[0, s]}) := \|\mathbf{x} - \tilde{\mathbf{y}}\|_{\alpha; [0, t]} + |t - s|, \quad s \leq t.$$

It has been proved in [5, Lemma A.1] that the topology of the metric space  $(\Lambda_T^p, d_\Lambda)$ , where  $\Lambda_T^p$  denotes the space of stopped  $p$ -rough paths (see [5, Sect. 3]), coincides with the final topology, and the space of stopped geometric rough paths is Polish. A similar argument can be done for the  $\alpha$ -Hölder case by replacing the  $p$ -variation norm by the  $\alpha$ -Hölder-norm and using the fact that  $\widehat{\mathcal{C}}_g^\alpha$  is Polish; see Friz and Victoir [26, Proposition 8.27].



**Remark 2.3** Let  $\mathbf{X}$  be a stochastic process valued in  $\widehat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^{d+1})$ . It is discussed in [26, Appendix A.1] that  $\mathbf{X}$  can be regarded as a random variable valued in  $\widehat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^{d+1})$ , and its law  $\mu_{\mathbf{X}}$  is then a Borel measure on the Borel  $\sigma$ -algebra  $\mathcal{B}^\alpha$  with respect to  $\|\cdot\|_\alpha$ . Moreover, define the product measure  $d\mu := dt \otimes d\mu_{\mathbf{X}}$ . For the surjective map  $\phi : [0, T] \times \widehat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^{d+1}) \rightarrow \Lambda_T^\alpha$  defined above, we can define the pushforward measure  $\widehat{\mu}$  on  $\Lambda_T^\alpha$ , in symbols  $\widehat{\mu} := \phi\#\mu = \mu \circ \phi^{-1}$ , which is given by

$$\widehat{\mu}(A) := \mu(\phi^{-1}(A)) \quad \text{for all } A \in \mathcal{B}(\Lambda_T^\alpha).$$

Consider the space  $\mathbb{H}^2$  of  $\mathbb{F}^{\mathbf{X}}$ -progressive processes  $A$  such that

$$\|A\|_{\mathbb{H}^2}^2 := E \left[ \int_0^T A_s^2 ds \right] < \infty. \quad (2.1)$$

The following result justifies the consideration of the space  $\Lambda_T^\alpha$ .

**Lemma 2.4** *For any process  $A \in \mathbb{H}^2$  and  $\alpha \in (0, 1)$ , there exists a measurable function  $f : (\Lambda_T^\alpha, \mathcal{B}(\Lambda_T^\alpha)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that  $A_t = f(\mathbf{X}|_{[0,t]})$  almost everywhere.*

**Proof** Consider the space of elementary  $\mathbb{F}^{\mathbf{X}}$ -progressive processes, that is, processes of the form

$$A_t^n(\omega) := \xi_0^n(\omega) 1_{\{0\}}(t) + \sum_{j=1}^{m_n-1} 1_{(t_j^n, t_{j+1}^n]}(t) \xi_{t_j^n}^n(\omega), \quad (2.2)$$

where  $0 \leq t_0^n < \dots < t_{m_n}^n \leq T$  and  $\xi_{t_j^n}^n$  is an  $\mathcal{F}_{t_j^n}^{\mathbf{X}}$ -measurable, square-integrable random variable. A standard result for the construction of stochastic integrals shows that this space is dense in  $\mathbb{H}^2$ ; see e.g. Karatzas and Shreve [32, Lemma 3.2.4]. Thus we can find  $A^n$  of the form (2.2) such that  $A^n \rightarrow A$  for almost every  $(t, \omega)$ . Since the random variable  $\xi_{t_j^n}^n$  is measurable with respect to the  $\sigma$ -algebra

$$\mathcal{F}_{t_j^n}^{\mathbf{X}} := \sigma(\mathbf{X}_{0,s} : s \leq t_j) = \sigma(\mathbf{X}|_{[0,t_j]}),$$

there exists by the Doob–Dynkin lemma a Borel-measurable function

$$F_j^n : \widehat{\mathcal{C}}_g^\alpha([0, t_j]; \mathbb{R}^{d+1}) \rightarrow \mathbb{R}$$

such that  $\xi_{t_j^n}^n(\omega) = F_j^n(\mathbf{X}|_{[0,t_j]}(\omega))$ . Then the functions

$$[0, T] \times \widehat{\mathcal{C}}_g^\alpha \ni (t, \mathbf{x}) \mapsto 1_{(t_j^n, t_{j+1}^n]}(t) F_j^n(\mathbf{x}|_{[0,t_j]})$$

are  $(\mathcal{B}([0, T]) \otimes \mathcal{F}_T^{\mathbf{X}})$ -measurable, and therefore such is the function

$$F^n(t, \mathbf{x}) := F_0^n(\mathbf{x}_0) 1_{\{0\}}(t) + \sum_{j=1}^{m_n-1} 1_{(t_j^n, t_{j+1}^n]}(t) F_j^n(\mathbf{x}|_{[0,t_j]}).$$

Finally, define the jointly measurable function  $F(t, \mathbf{x}) := \limsup_{n \rightarrow \infty} F^n(t, \mathbf{x})$  and notice that for almost every  $(t, \omega)$ , we have

$$\begin{aligned} F(t, \mathbf{X}(\omega)) &= \limsup_{n \rightarrow \infty} \left( F_0^n(\mathbf{X}_0(\omega)) 1_{\{0\}}(t) + \sum_{j=1}^{m_n-1} 1_{(t_j^n, t_{j+1}^n]}(t) F_j^n(\mathbf{X}|_{[0, t_j]}(\omega)) \right) \\ &= \limsup_{n \rightarrow \infty} \left( \xi_0^n(\omega) 1_{\{0\}}(t) + \sum_{j=1}^{m_n-1} 1_{(t_j^n, t_{j+1}^n]}(t) \xi_{t_j}^n(\omega) \right) = A_t(\omega). \end{aligned}$$

Next, for any  $\mathbf{x}|_{[0, t]} \in \Lambda_T^\alpha$ , we let  $(t, \tilde{\mathbf{x}}) \in [0, T] \times \widehat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^{d+1})$ , where  $\tilde{\mathbf{x}}$  is the geometric rough-path lift of  $[0, T] \ni u \mapsto (u, x_{u \wedge t})$ ; see Remark 2.2. The map  $\Xi : \Lambda_T^\alpha \rightarrow [0, T] \times \widehat{\mathcal{C}}_g^\alpha$  with  $\Xi(\mathbf{x}|_{[0, t]}) := (t, \tilde{\mathbf{x}})$  is continuous and thus measurable. Define the composition  $f := F \circ \Xi$ , which is a measurable map  $f : \Lambda_T^\alpha \rightarrow \mathbb{R}$  with

$$f(\mathbf{X}|_{[0, t]}(\omega)) = A_t(\omega)$$

for almost every  $(t, \omega)$ . This is exactly what was claimed.  $\square$

## 2.2 A Stone–Weierstrass result for robust signatures

The goal of this section is to present a Stone–Weierstrass-type result for continuous functionals  $f : \Lambda_T^\alpha \rightarrow \mathbb{R}$ , which is the key ingredient for the main result in Sect. 2.3. To this end, consider the set of linear functionals of the signature given by

$$L_{\text{sig}}(\Lambda_T^\alpha) = \{\Lambda_T^\alpha \ni \mathbf{X}|_{[0, t]} \mapsto \langle \mathbf{X}_{0, t}^{< \infty}, \ell \rangle : \ell \in \mathcal{W}^{d+1}\} \subseteq C(\Lambda_T^\alpha; \mathbb{R}),$$

where we recall that  $\mathcal{W}^{d+1}$  denotes the linear span of words; see Sect. 1.1. Note that continuity of  $\mathbf{X}|_{[0, t]} \mapsto \langle \mathbf{X}_{0, t}^{< \infty}, \ell \rangle$  follows from continuity of the *truncated* signature, i.e.,  $\mathbf{X}|_{[0, t]} \mapsto \mathbf{X}_{0, t}^{< K}$  – a consequence of Lyons’ extension theorem, see Lyons [34, Theorem 3.7] – for  $K$  large enough, as any  $\ell \in \mathcal{W}^{d+1}$  only has finitely many non-zero coefficients. A similar set was considered in Kalsi et al. [31, Definition 3.3] with respect to  $p$ -rough paths, and the authors prove that when *restricted* to a compact set  $\mathcal{K}$  in the space of time-augmented rough paths, the set  $L_{\text{sig}}$  is dense in  $C(\mathcal{K}; \mathbb{R})$ . In words, when restricted to compacts, continuous functionals on the path space  $\Lambda_T^\alpha$  can be approximated by linear functionals of the signature. However, since such path spaces are not even locally compact, it is desirable to drop the need of a compact set  $\mathcal{K}$ .

An elegant way to circumvent the requirement of a compact set is to consider the so-called *robust signatures* introduced in Chevyrev and Oberhauser [17]. Loosely speaking, the authors construct on the state space  $T((\mathbb{R}^{d+1}))$  the so-called *tensor normalisation*  $\lambda$ , see [17, Proposition 14 and Example 4], which is a continuous and injective map

$$\lambda : T((\mathbb{R}^{d+1})) \rightarrow \{\mathbf{a} \in T((\mathbb{R}^{d+1})) : \|\mathbf{a}\| \leq R\} \quad \text{for some fixed } R > 0,$$

and they call  $\lambda(\mathbf{X}^{<\infty}) := \lambda^R(\mathbf{X}^{<\infty})$  the robust signature. The continuity of the latter is with respect to the Banach-space norm  $\|\mathbf{a}\| = (\sum_{k \geq 0} \|\mathbf{a}^{(k)}\|_{(\mathbb{R}^{d+1})^{\otimes k}}^2)^{1/2}$  for  $\mathbf{a} \in T((\mathbb{R}^{d+1}))$ ; see [17, Definition 10]. This motivates us to define the set

$$L_{\text{sig}}^\lambda(\Lambda_T^\alpha) = \{\Lambda_T^\alpha \ni \mathbf{X}|_{[0,t]} \mapsto \langle \lambda(\mathbf{X}_{0,t}^{<\infty}), \ell \rangle : \ell \in \mathcal{W}^{d+1}\} \subseteq C_b(\Lambda_T^\alpha; \mathbb{R}).$$

A general version of the Stone–Weierstrass result given in Giles [28] leads to the following result, which was stated already in Chevyrev and Oberhauser [17, Theorem 26], and we present the proof here for completeness.

**Lemma 2.5** *Let  $\alpha \in (0, 1)$ . Then the set  $L_{\text{sig}}^\lambda(\Lambda_T^\alpha)$  is dense in  $C_b(\Lambda_T^\alpha; \mathbb{R})$  with respect to the strict topology. More precisely, for any  $f \in C_b(\Lambda_T^\alpha; \mathbb{R})$ , we can find a sequence  $(f_n)_{n \in \mathbb{N}} \subseteq L_{\text{sig}}^\lambda(\Lambda_T^\alpha)$  such that*

$$\|f - f_n\|_{\infty, \psi} := \sup_{\mathbf{x} \in \Lambda_T^\alpha} |\psi(\mathbf{x})(f(\mathbf{x}) - f_n(\mathbf{x}))| \longrightarrow 0 \quad \text{as } n \rightarrow \infty, \forall \psi \in B_0(\Lambda_T^\alpha),$$

where  $B_0(\Lambda_T^\alpha)$  denotes the set of functions  $\psi : \Lambda_T^\alpha \rightarrow \mathbb{R}$  such that for all  $\epsilon > 0$ , there exists a compact set  $K \subseteq \Lambda_T^\alpha$  with  $\sup_{\mathbf{x} \in \Lambda_T^\alpha \setminus K} |\psi(\mathbf{x})| < \epsilon$ .

**Proof** This result is a consequence of the general Stone–Weierstrass result proved in [28]; see also [17, Theorem 9]. From the latter, we only need to check that the inclusion  $L_{\text{sig}}^\lambda \subseteq C_b(\Lambda_T^\alpha; \mathbb{R})$  is a subalgebra such that

- 1)  $L_{\text{sig}}^\lambda$  separates points, i.e., for all  $x \neq y$ , there exists  $f \in L_{\text{sig}}^\lambda$  with  $f(x) \neq f(y)$ ;
- 2)  $L_{\text{sig}}^\lambda$  contains non-vanishing functions, i.e., for all  $x$ , there exists  $f \in L_{\text{sig}}^\lambda$  with  $f(x) \neq 0$ .

As mentioned at the beginning of Sect. 2.2,  $\mathbf{X}|_{[0,t]} \mapsto \langle \mathbf{X}_{0,t}^{<\infty}, \ell \rangle$  is continuous for any  $\ell \in \mathcal{W}^{d+1}$ , and as the tensor normalisation is continuous and bounded, it follows that  $\mathbf{X}|_{[0,t]} \mapsto \langle \lambda(\mathbf{X}_{0,t}^{<\infty}), \ell \rangle$  is continuous and bounded. To see that  $L_{\text{sig}}^\lambda \subseteq C_b(\Lambda_T^\alpha; \mathbb{R})$  is a subalgebra, we fix  $\phi_1, \phi_2 \in L_{\text{sig}}^\lambda$ . By definition, there exists a linear combination of words  $\ell_1, \ell_2 \in \mathcal{W}^{d+1}$  such that  $\phi_i(\mathbf{X}|_{[0,s]}) = \langle \lambda(\mathbf{X}_{0,s}^{<\infty}), \ell_i \rangle$ . We clearly have

$$\phi(\mathbf{X}|_{[0,s]}) := \phi_1(\mathbf{X}|_{[0,s]}) + \phi_2(\mathbf{X}|_{[0,s]}) = \langle \lambda(\mathbf{X}_{0,s}^{<\infty}), \ell_1 + \ell_2 \rangle \in L_{\text{sig}}^\lambda.$$

Now assume that  $\ell_1, \ell_2$  are words  $\ell_1 = w$  and  $\ell_2 = v$ . By the definition of the tensor normalisation [17, Definition 12], for some positive function  $\Psi : T((\mathbb{R}^{d+1})) \rightarrow (0, \infty)$  we have that

$$\begin{aligned} \phi_1(\mathbf{X}|_{[0,s]})\phi_2(\mathbf{X}|_{[0,s]}) &= \langle \lambda(\mathbf{X}_{0,s}^{<\infty}), w \rangle \langle \lambda(\mathbf{X}_{0,s}^{<\infty}), v \rangle \\ &= \Psi(\mathbf{X}_{0,s}^{<\infty})^{|w|+|v|} \langle \mathbf{X}_{0,s}^{<\infty}, w \rangle \langle \mathbf{X}_{0,s}^{<\infty}, v \rangle \\ &= \Psi(\mathbf{X}_{0,s}^{<\infty})^{|w|+|v|} \langle \mathbf{X}_{0,s}^{<\infty}, w \sqcup v \rangle, \end{aligned}$$

where we used for the last equality that  $\mathbf{X}^{<\infty} \in G(\mathbb{R}^d)$ . But by the definition (1.4) of the shuffle product, it follows that  $w \sqcup v = \sum_j u_j$ , where  $u_j$  are words with

$|u_j| = |w| + |v|$ , and hence

$$\phi_1(\mathbf{X}|_{[0,s]})\phi_2(\mathbf{X}|_{[0,s]}) = \sum_j \Psi(\mathbf{X}_{0,s}^{<\infty})^{|w|+|v|} \langle \mathbf{X}_{0,s}^{<\infty}, u_j \rangle = \langle \lambda(\mathbf{X}_{0,s}^{<\infty}), w \sqcup v \rangle,$$

so that the product lies in  $L_{\text{sig}}^\lambda$ . The same reasoning can be extended to linear combination of words  $\ell_1, \ell_2$ , and thus the set  $L_{\text{sig}}^\lambda$  is indeed a subalgebra in  $C_b(\Lambda_T^\alpha; \mathbb{R})$ .

Now take  $\mathbf{X}, \mathbf{Y} \in \Lambda_T^\alpha$  with  $\mathbf{X} \neq \mathbf{Y}$ . As remarked in Sect. 1.1, since we are working with rough-path lifts of time-augmented paths  $(t, X_t)$ , the signature map is injective. Moreover, by definition (see [17, Definition 12]), the map  $\lambda$  is also injective, and therefore  $\lambda(\mathbf{X}^{<\infty}) \neq \lambda(\mathbf{Y}^{<\infty})$ . This in particular implies that there exists  $\ell \in \mathcal{W}^{d+1}$  with  $\langle \lambda(\mathbf{X}^{<\infty}), \ell \rangle \neq \langle \lambda(\mathbf{Y}^{<\infty}), \ell \rangle$ . Defining  $f \in L_{\text{sig}}^\lambda$  by  $f(\mathbf{x}) = \langle \lambda(\mathbf{x}^{<\infty}), \ell \rangle$ , it follows that  $f(\mathbf{X}) \neq f(\mathbf{Y})$  and thus  $L_{\text{sig}}^\lambda$  separates points. Finally, since  $1 = \langle \lambda(\mathbf{X}^{<\infty}), \emptyset \rangle \in L_{\text{sig}}^\lambda$ , the claim follows.  $\square$

### 2.3 Approximation with robust signatures

We are now ready to state and prove the main result of this section. We impose the following

**Assumption 2.6** (i)  $\alpha \in (0, 1)$  is such that  $q := \frac{1}{\alpha} \notin \mathbb{N}$ .

(ii)  $\mu$  is a measure on the Borel space  $(\widehat{\mathcal{C}}_g^\alpha, \mathcal{B}(\widehat{\mathcal{C}}_g^\alpha))$  such that

$$\mu(\widehat{\mathcal{C}}_g^\alpha) < \infty \quad \text{and} \quad \mu(\widehat{\mathcal{C}}_g^\alpha \setminus \widehat{\mathcal{C}}_g^\beta) = 0, \quad \forall \alpha < \beta < \frac{1}{[q]}.$$

Before stating the main result of this section, we give an example to clarify the role of Assumption 2.6.

**Example 2.7** Let  $X$  be a  $d$ -dimensional Brownian motion and fix  $\alpha \in (1/3, 1/2)$ , that is, (i) in Assumption 2.6 holds with  $2 < q < 3$ . Denote by  $\mathbf{X}$  the geometric  $\alpha$ -Hölder rough-path lift of the time-augmentation  $(t, X_t)$ , that is, define the second level using Stratonovich integration; see Friz and Hairer [25, Proposition 3.5]. As discussed in Remark 2.3, we can see  $\mathbf{X}$  as a  $\widehat{\mathcal{C}}_g^\alpha$ -valued random variable with law  $\mu_{\mathbf{X}}$ . Applying a Kolmogorov-type result for rough paths, see for instance [25, Theorem 3.1], it follows that  $\mu_{\mathbf{X}}$  assigns full measure to  $\widehat{\mathcal{C}}_g^\beta$  for all  $\beta \in (\alpha, 1/2)$ , and thus Assumption 2.6 (ii) holds. In a similar way, we can treat geometric lifts of more general processes such as semimartingales, but also fractional Brownian motion by replacing  $1/2$  by the Hurst parameter  $H$ .

The following result shows that under Assumption 2.6, we can approximate any functional in  $L^p(\Lambda_T^\alpha, \widehat{\mu})$ , where we recall from Remark 2.3 that  $\widehat{\mu}$  is the pushforward measure of  $dt \otimes d\mu$  on  $\Lambda_T^\alpha$ , by linear functionals of the robust signature with respect to the  $L^p$ -norm.

**Theorem 2.8** Let  $\alpha \in (0, 1)$  and let  $\mu$  be a measure on  $(\widehat{\mathcal{C}}_g^\alpha, \mathcal{B}(\widehat{\mathcal{C}}_g^\alpha))$  such that Assumption 2.6 holds true. Then for all  $f \in L^p(\Lambda_T^\alpha, \widehat{\mu})$ ,  $1 \leq p < \infty$ , there exists a sequence  $(f_n)_{n \in \mathbb{N}} \subseteq L_{\text{sig}}^\lambda(\Lambda_T^\alpha)$  such that  $\|f - f_n\|_{L^p} \rightarrow 0$  as  $n \rightarrow \infty$ .

Before proving this result, let us show the following immediate consequence for random geometric rough paths, which will be of particular importance in Sect. 3.

**Corollary 2.9** Let  $\mathbf{X}$  be a stochastic process valued in  $\widehat{\mathcal{C}}_g^\alpha$  such that Assumption 2.6 holds for its law  $\mu_{\mathbf{X}}$ . Then for all  $A \in \mathbb{H}^2$ , see (2.1), there exists a sequence of linear functionals  $(f_n)_{n \in \mathbb{N}} \subseteq L_{\text{sig}}^\lambda(\Lambda_T^\alpha)$  such that  $\|A - f_n(\mathbf{X}|_{[0, \cdot]})\|_{\mathbb{H}^2} \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proof** By Lemma 2.4, there exists a measurable function  $f : \Lambda_T^\alpha \rightarrow \mathbb{R}$  such that  $A_t = f(\mathbf{X}|_{[0, t]})$ . Applying a standard change of measure result for the pushforward measure, see for example Bogachev and Ruas [13, Theorem 3.6.1], and denoting by  $\phi$  the quotient map given in Definition 2.1, we have

$$\begin{aligned} \|A_t - f_n(\mathbf{X}|_{[0, \cdot]})\|_{\mathbb{H}^2}^2 &= E \left[ \int_0^T (f(\mathbf{X}|_{[0, t]}) - f_n(\mathbf{X}|_{[0, t]}))^2 dt \right] \\ &= \int_{\widehat{\mathcal{C}}_g^\alpha} \int_0^T (f \circ \phi - f_n \circ \phi)(t, \mathbf{X})^2 dt d\mu_{\mathbf{X}} \\ &= \int_{\Lambda_T^\alpha} (f - f_n)^2 d\widehat{\mu} \\ &= \|f - f_n\|_{L^2}^2 \longrightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where the convergence follows from Theorem 2.8.  $\square$

The proof of Theorem 2.8 makes use of two lemmas. The first is very elementary, and in the language of probability theory, it states that for every random variable  $\xi$  valued in  $\mathbb{R}_+$ , we can find a strictly increasing and integrable function  $\eta$ , that is,  $E[\eta(\xi)] < \infty$ .

**Lemma 2.10** Let  $(E, \mathcal{E}, \mu)$  be a finite measure space and consider a measurable function  $\xi : (E, \mathcal{E}) \rightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ . Then there exists a strictly increasing function  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\eta(x) \rightarrow \infty$  as  $x \rightarrow \infty$  and  $\int_E (\eta \circ \xi) d\mu < \infty$ .

**Proof** Let  $\nu$  be the pushforward of  $\mu$  under  $\xi$ , that is,  $\nu(A) := \mu(\xi^{-1}(A))$  for all  $A \in \mathcal{B}(\mathbb{R}_+)$ . Then for any  $\epsilon > 0$ , we can find  $R > 0$  large enough such that  $\nu((R, \infty)) \leq \epsilon$ . In particular, for any strictly decreasing sequence  $(a_n)_{n \in \mathbb{N}_0}$  with  $a_n \searrow 0$ , we can find a strictly increasing sequence  $(R_n)_{n \in \mathbb{N}}$  with  $R_1 > 0$  such that  $\nu((R_n, \infty)) \leq \frac{a_n}{n^2}$ . Now we can define a strictly increasing function  $\eta$  as follows. Let  $\eta(0) = 0$  and for all  $n \in \mathbb{N}$ , define  $\eta(R_n) = \frac{1}{a_{n-1}}$  and linearly interpolate on the intervals  $[R_n, R_{n+1})$ . Then setting  $R_0 = 0$  and using a change of measure as in [13, Theorem 3.6.1], we have

$$\int_E (\eta \circ \xi) d\mu = \int_0^\infty \eta d\nu \leq \sum_{n \geq 0} \frac{1}{a_n} \nu((R_n, \infty)) \leq \frac{\nu([0, \infty))}{a_0} + \sum_{n \geq 1} \frac{1}{n^2} < \infty.$$

□

The next result is the key ingredient to apply the Stone–Weierstrass result from Lemma 2.5 in the main result.

**Lemma 2.11** *Let  $\alpha < \beta$  and define  $\bar{\psi} : \Lambda_T^\alpha \rightarrow \mathbb{R}$  by  $\bar{\psi}(\mathbf{x}) := 1_{\Lambda_T^\beta}(\mathbf{x}) \left( \frac{1}{1 + \eta(\|\tilde{\mathbf{x}}\|_\beta)} \right)^{1/p}$ , where  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is strictly increasing such that  $\eta(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Then we have  $\bar{\psi} \in B_0(\Lambda_T^\alpha)$ , that is, for all  $\epsilon > 0$ , there exists  $K \subseteq \Lambda_T^\alpha$  compact such that  $\sup_{\mathbf{x} \in K^c} \bar{\psi}(\mathbf{x}) \leq \epsilon$ .*

**Proof** Recall that an element  $\mathbf{x} \in \Lambda_T^\alpha$  can be written as  $\mathbf{x} = \mathbf{x}|_{[0,t]} \in \widehat{\mathcal{C}}_g^\alpha([0, t])$ , where  $\mathbf{x}|_{[0,t]}$  is the geometric rough-path lift of some time-augmented,  $\alpha$ -Hölder-continuous  $[0, t] \ni u \mapsto (u, \omega_u)$ . Moreover, recall that we define  $\tilde{\mathbf{x}} \in \widehat{\mathcal{C}}_g^\alpha([0, T])$  to be the geometric rough-path lift of  $u \mapsto (u, x_{t \wedge u})$ . If we can show that for any  $R > 0$ , the sets

$$B_R = \{\mathbf{x} \in \Lambda_T^\beta : \|\tilde{\mathbf{x}}\|_\beta \leq R\} \subseteq \Lambda_T^\alpha$$

are compact, then we are done. Indeed, in this case, we have that for any  $\epsilon \in (0, 1)$ , we can choose  $\widehat{R} \geq \eta^{-1}(\frac{1-\epsilon^p}{\epsilon^p})$  and then

$$\bar{\psi}(\mathbf{x}) = \left( \frac{1}{1 + \eta(\|\tilde{\mathbf{x}}\|_\beta)} \right)^{1/p} \leq \frac{1}{\sqrt[1 + \widehat{R}]} \leq \epsilon, \quad \forall \mathbf{x} \in \Lambda_T^\alpha \setminus B_{\widehat{R}},$$

and therefore  $\bar{\psi} \in B_0(\Lambda_T^\alpha)$ . Now to prove compactness, we can notice that by the definition of the quotient map  $\phi$  in Definition 2.1, we have

$$B_R \subseteq \phi([0, T] \times \{\mathbf{x} \in \widehat{\mathcal{C}}_g^\beta([0, T]; \mathbb{R}^{d+1}) : \|\mathbf{x}\|_\beta \leq R\}),$$

since for all  $\mathbf{x} = \mathbf{x}|_{[0,t]} \in B_R$ , we have  $\mathbf{x} = \phi(t, \tilde{\mathbf{x}})$  by construction. As  $\phi$  is continuous, it is enough to show that  $[0, T] \times \{\mathbf{x} \in \widehat{\mathcal{C}}_g^\beta : \|\mathbf{x}\|_\beta \leq R\}$  is compact in  $[0, T] \times \widehat{\mathcal{C}}_g^\alpha$ , which by Tychonoff's theorem is true if the sets  $\{\mathbf{x} \in \widehat{\mathcal{C}}_g^\beta : \|\mathbf{x}\|_\beta \leq R\}$  are compact in  $\widehat{\mathcal{C}}_g^\alpha$ . But the latter follows from the general fact that  $\beta$ -Hölder spaces are compactly embedded in  $\alpha$ -Hölder spaces for  $\alpha < \beta$ . This can be proved by applying the Arzelà–Ascoli theorem together with an interpolation argument for the equicontinuous and  $\|\cdot\|_\beta$ -bounded subsets of  $\widehat{\mathcal{C}}_g^\alpha$ , which was carried out for example in Cuchiero et al. [20, Theorem A.3]. Thus we conclude that  $B_R \subseteq \Lambda_T^\alpha$  is compact, which finishes the proof. □

Finally, we are ready to prove the main result.

**Proof of Theorem 2.8** First, recall from Remark 2.3 that the measure  $\widehat{\mu}$  is defined as the pushforward of the product measure  $dt \otimes d\mu$  via the surjective map  $\phi$  from Definition 2.1. We can easily see that for  $\alpha, \beta$  as in Assumption 2.6, we have

$$\phi^{-1}(\Lambda_T^\alpha \setminus \Lambda_T^\beta) = \phi^{-1}(\Lambda_T^\alpha) \setminus \phi^{-1}(\Lambda_T^\beta) \subseteq [0, T] \times (\widehat{\mathcal{C}}_g^\alpha \setminus \widehat{\mathcal{C}}_g^\beta)$$

and thus  $\widehat{\mu}(\Lambda_T^\alpha \setminus \Lambda_T^\beta) \leq (dt \otimes d\mu)([0, T] \times (\widehat{\mathcal{C}}_g^\alpha \setminus \widehat{\mathcal{C}}_g^\beta)) = 0$ . Fix  $\epsilon > 0$ . For any  $K > 0$ , we define the function  $f_K(x) := 1_{\{|f(x)| \leq K\}}(x)f(x)$  and notice that we have  $\|f - f_K\|_{L^p} \rightarrow 0$  as  $K \rightarrow \infty$  by dominated convergence. Hence we can find  $K_\epsilon > 0$  such that  $\|f - f_{K_\epsilon}\|_{L^p} \leq \epsilon/3$ . Since  $\widehat{\mu}$  is a finite measure on  $\Lambda_T^\alpha$ , we can by Lusin's theorem find a closed set  $C_\epsilon \subseteq \Lambda_T^\alpha$  such that  $f_{K_\epsilon}$  restricted to  $C_\epsilon$  is continuous and  $\widehat{\mu}(\Lambda_T^\alpha \setminus C_\epsilon) \leq \epsilon^p/(6K_\epsilon)^p$ . By Tietze's extension theorem, we can find a continuous extension  $\widehat{f}_\epsilon \in C_b(\Lambda_T^\alpha; [-K_\epsilon, K_\epsilon])$  of  $f_{K_\epsilon}$  such that

$$\|f_{K_\epsilon} - \widehat{f}_\epsilon\|_{L^p}^p = \int_{\Lambda_T^\alpha \setminus C_\epsilon} |f_\epsilon - f_{K_\epsilon}|^p d\widehat{\mu} \leq (2K_\epsilon)^p \widehat{\mu}(\Lambda_T^\alpha \setminus C_\epsilon) = (\epsilon/3)^p.$$

We are left with approximating  $\widehat{f}_\epsilon \in C_b(\Lambda_T^\alpha; \mathbb{R})$  by linear functionals of the robust signature, that is, with applying Lemma 2.5. By Assumption 2.6, we can choose  $\beta \in (\alpha, \frac{1}{[q]})$  and from Lemma 2.10, we know that there exists an increasing function  $\eta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\eta(x) \rightarrow \infty$  as  $x \rightarrow \infty$  and  $\int_{\Lambda_T^\alpha} \eta(\|\tilde{\mathbf{x}}\|_\beta) d\widehat{\mu}(x) < \infty$ , where  $\tilde{\mathbf{x}}$  is the extension of the stopped rough path from the interval  $[0, t]$  to  $[0, T]$ ; see also Remark 2.2. Define the function  $\psi: \Lambda_T^\beta \rightarrow \mathbb{R}_+$  by  $\psi(\mathbf{x}|_{[0,t]}) := (\frac{1}{1+\eta(\|\tilde{\mathbf{x}}\|_\beta)})^{1/p}$ . In Lemma 2.11, we saw that  $\bar{\psi}(\mathbf{x}) := 1_{\Lambda_T^\beta}(\mathbf{x})\psi(\mathbf{x})$  belongs to  $B_0(\Lambda_T^\alpha)$ , that is, for all  $\delta > 0$ , there exists a compact set  $K \subseteq \Lambda_T^\alpha$  such that  $\sup_{x \in K^c} \bar{\psi}(x) \leq \delta$ . Notice that Lemma 2.10 yields an increasing function  $\eta$ , which is used to build  $\bar{\psi}$  in Lemma 2.11, in a way that  $\Xi := \int_{\Lambda_T^\beta} \frac{1}{\psi^p} d\widehat{\mu}$  is finite, which is needed below. If  $\eta$  is an arbitrary increasing function, the integrability of  $\frac{1}{\psi^p}$  needs to be assumed (e.g. for  $\eta(x) = x$ , we should need a finite first moment of the rough-path norm). By Lemma 2.5, we can find  $f_\epsilon \in L_{\text{sig}}^\lambda(\Lambda_T^\alpha)$  such that

$$\|\widehat{f}_\epsilon - f_\epsilon\|_{\infty, \bar{\psi}}^p \leq \epsilon^p/(3^p \Xi).$$

Using that  $\mu$  assigns full measure to the subspace  $\Lambda_T^\beta \subseteq \Lambda_T^\alpha$ , we have

$$\begin{aligned} \|\widehat{f}_\epsilon - f_\epsilon\|_{L^p}^p &= \int_{\Lambda_T^\beta} |\widehat{f}_\epsilon - f_\epsilon|^p d\widehat{\mu} \leq \sup_{x \in \Lambda_T^\beta} \left( \psi(x)(\widehat{f}_\epsilon(x) - f_\epsilon(x)) \right)^p \int_{\Lambda_T^\beta} \frac{1}{\psi^p} d\widehat{\mu} \\ &\leq \|\widehat{f}_\epsilon - f_\epsilon\|_{\infty, \bar{\psi}}^p \Xi \\ &\leq (\epsilon/3)^p. \end{aligned}$$

Finally, we conclude by the triangle inequality that

$$\|f - f_\epsilon\|_{L^p} \leq \|f - f_{K_\epsilon}\|_{L^p} + \|f_{K_\epsilon} - \widehat{f}_\epsilon\|_{L^p} + \|\widehat{f}_\epsilon - f_\epsilon\|_{L^p} \leq \epsilon. \quad \square$$

### 3 Optimal stopping with signatures

In this section, we exploit the signature approximation theory presented in Sect. 2.3 in order solve the optimal stopping problem in a general setting.

### 3.1 Framework and problem formulation

Suppose we have a complete filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, P)$  for some  $T > 0$ , fulfilling the usual conditions, and fix  $\alpha \in (0, 1)$ . For any  $\mathbb{F}$ -adapted and  $\alpha$ -Hölder-continuous stochastic process  $(X_t)_{t \in [0, T]}$  taking values in  $\mathbb{R}^d$  with  $X_0 = x_0$ , we consider

- $\mathbf{X} \in \widehat{\mathcal{C}}_g^\alpha$ , a geometric  $\alpha$ -Hölder rough-path lift of  $(t, X_t)$  such that its law  $\mu_{\mathbf{X}}$  fulfils Assumption 2.6;
- $\mathbf{X}^{<\infty}$ , the robust rough-path signature introduced in Sect. 2.3;
- $(Z_t)_{t \in [0, T]}$ , a real-valued  $\mathbb{F}^{\mathbf{X}}$ -adapted process such that  $\sup_{t \in [0, T]} |Z_t| \in L^2$ .

The optimal stopping problem then reads

$$y_0 = \sup_{\tau \in \mathcal{S}_0} E[Z_\tau], \quad (3.1)$$

where  $\mathcal{S}_0$  denotes the set of  $\mathbb{F}^{\mathbf{X}}$ -stopping-times valued in  $[0, T]$ .

**Remark 3.1** Notice that the framework described above is very general in two ways. First, we only assume  $\alpha$ -Hölder-continuity for the state process  $X$ , including in particular non-Markovian and non-semimartingale regimes which one encounters for instance in rough volatility models; see Sect. 4.2. Secondly, by considering the projection  $\mathbf{X} \mapsto (t, X_t)$  onto the first coordinate, our framework includes for any payoff function  $\phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  the more common form of the optimal stopping problem

$$y_0 = \sup_{\tau \in \mathcal{S}_0} E[\phi(\tau, X_\tau)].$$

**Remark 3.2** In general, there is no canonical way of lifting a process  $(t, X_t)$  to a random rough path  $\mathbf{X}$ , and often careful justification is required, for instance based on a rough-path version of the Kolmogorov criterion; see Friz and Hairer [25, Theorem 3.1]. However, for a big class of processes (e.g. semimartingales, Gaussian processes, one-dimensional processes), there are canonical choices for random geometric rough-path lifts, and we explain below in Sect. 4 how to do it. Moreover, we always look at lifts such that  $\mathbb{F}^{\mathbf{X}} = \mathbb{F}^X$ , that is, the optimal stopping problem has the same underlying information when observing  $\mathbf{X}$  or  $X$ .

### 3.2 Primal optimal stopping with signatures

First, we present a method to compute a lower-biased approximation  $y_0^L \leq y_0$  to the optimal stopping problem (3.1). More precisely, we construct a regression-based approach, generalising the famous algorithm from Longstaff and Schwartz [33], returning a suboptimal exercise strategy. Let us first quickly describe the main idea of most regression-based approaches.

Replacing the interval  $[0, T]$  by a finite grid  $\{0 = t_0 < t_1 < \dots < t_N = T\}$ , the discrete optimal stopping problem reads

$$y_0^N = \sup_{\tau \in \mathcal{S}_0^N} E[Z_\tau],$$



where  $\mathcal{S}_n^N$  is the set of stopping times taking values in  $\{t_n, \dots, t_N\}$  for  $n = 0, \dots, N$ , with respect to the discrete filtration  $\mathbb{F}^{\mathbf{X}, N} = (\mathcal{F}_{t_n}^{\mathbf{X}})_{n=0, \dots, N}$ . We define the discrete Snell envelope by

$$Y_{t_n}^N = \operatorname{ess\,sup}_{\tau \in \mathcal{S}_n^N} E[Z_\tau | \mathcal{F}_{t_n}^{\mathbf{X}}], \quad 0 \leq n \leq N, \quad (3.2)$$

and one can show that  $Y^N$  satisfies the discrete dynamic programming principle (DPP)

$$Y_{t_n}^N = \max(Z_{t_n}, E[Y_{t_{n+1}}^N | \mathcal{F}_{t_n}^{\mathbf{X}}]), \quad n = 0, \dots, N-1; \quad (3.3)$$

see for instance Peskir and Shiryaev [36, Theorem 1.2]. Now the key idea of most regression-based approaches, such as for instance Longstaff and Schwartz [33], is that by assuming that  $X$  is a Markov process, one can choose a suitable family of basis functions  $(b^k)$  and apply least-squares regression to approximate

$$E[Y_{t_{n+1}}^N | \mathcal{F}_{t_n}^{\mathbf{X}}] \approx \sum_{k=0}^D \alpha_k b_n^k(X_{t_n}), \quad 0 \leq n \leq N-1, \alpha_k \in \mathbb{R}, \forall k \leq D, \quad (3.4)$$

and then make use of the DPP to recursively approximate  $Y_0^N = y_0^N$ . Of course, the approximation of the conditional expectations in (3.4) heavily relies on the Markov property, and thus one cannot expect such an approximation to converge in non-Markovian settings.

Returning to our framework, we need to replace (3.4) by a suitable approximation for the conditional expectations

$$E[Y_{t_{n+1}}^N | \mathcal{F}_{t_n}^{\mathbf{X}}] = f_n(\mathbf{X}|_{[0, t_n]}), \quad 0 \leq n \leq N-1.$$

The universality result in Theorem 2.8 now suggests approximating  $f_n$  by a sequence of linear functionals of the robust signature, that is,

$$f_n(\mathbf{X}|_{[0, t_n]}) \approx \langle \mathbf{X}_{0, t_n}^{<\infty}, \ell \rangle, \quad \ell \in \mathcal{W}^{d+1},$$

where  $\mathcal{W}^{d+1}$  is the linear span of words introduced in Sect. 1.1.

### 3.2.1 Longstaff–Schwartz with signatures

In this section, we present a version of the Longstaff–Schwartz (LS) algorithm [33], using signature-based least-squares regression. A convergence analysis for the LS algorithm was presented in Clément et al. [18], and combining their techniques with the universality of the signature allows us to recover a convergent algorithm.

The main idea of the LS algorithm is to reformulate the DPP (3.3) for stopping times by taking advantage of the fact that optimal stopping times can be expressed in terms of the Snell envelope. More precisely, it is proved in Peskir and Shiryaev [36, Theorem 1.2] that the stopping times  $\tau_n := \min\{t_m \geq t_n : Y_{t_m}^N = Z_{t_m}\}$  for  $0 \leq n \leq N$  are optimal in (3.2), and hence one recursively defines

$$\begin{aligned} \tau_N &= t_N, \\ \tau_n &= t_n 1_{\{Z_{t_n} \geq E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]\}} + \tau_{n+1} 1_{\{Z_{t_n} < E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]\}}, \quad n = 0, \dots, N-1. \end{aligned}$$

Now for any truncation level  $K \in \mathbb{N}$  for the signature and some fixed  $n = 0, \dots, N-1$ , assume we are given an approximation  $\tau_{n+1}^K$  of  $\tau_{n+1}$ . Then we approximate the conditional expectation  $E[Z_{\tau_{n+1}^K} | \mathcal{F}_{t_n}^{\mathbf{X}}]$  by solving the minimisation problem

$$\ell^* := \ell^{*,n,K} = \arg \min_{\ell \in \mathcal{W}_{\leq K}^{d+1}} \|Z_{\tau_{n+1}^K} - \langle \mathbf{X}_{0,t_n}^{\leq K}, \ell \rangle\|_{L^2}, \quad n = 0, \dots, N-1. \quad (3.5)$$

Setting  $\psi^{n,K}(\mathbf{x}) = \langle \mathbf{x}^{\leq K}, \ell^{*,n,K} \rangle \in L_{\text{sig}}^\lambda$ , we define the approximating sequence of stopping times

$$\begin{aligned} \tau_N^K &= t_N, \\ \tau_n^K &= t_n 1_{\{Z_{t_n} \geq \psi^{n,K}(\mathbf{X}|_{[0,t_n]})\}} + \tau_{n+1}^K 1_{\{Z_{t_n} < \psi^{n,K}(\mathbf{X}|_{[0,t_n]})\}}, \quad n = 0, \dots, N-1. \end{aligned}$$

The following result shows convergence as the depth of the signature goes to infinity, and the proof is given in Appendix A.1.

**Proposition 3.3** *For all  $n = 0, \dots, N$ , we have*

$$\lim_{K \rightarrow \infty} E[Z_{\tau_n^K} | \mathcal{F}_{t_n}^{\mathbf{X}}] = E[Z_{\tau_n} | \mathcal{F}_{t_n}^{\mathbf{X}}] \quad \text{in } L^2.$$

*In particular, we have  $y_0^{K,N} = \max(Z_{t_0}, E[Z_{\tau_1^K}]) \rightarrow y_0^N$  as  $K \rightarrow \infty$ .*

Let us now describe how to numerically solve (3.5) by using Monte Carlo simulations. Besides the truncation of the signature at some level  $K$ , we introduce two further approximations steps. First, we replace the signature  $\mathbf{X}^{<\infty}$  by some discretised version  $\mathbf{X}^{<\infty}(J)$ , for example piecewise linear approximation of the iterated integrals, on some fine grid  $s_0 = 0 < s_1 < \dots < s_J = T$ , such that  $\langle \mathbf{X}_{0,t}^{<\infty}(J), v \rangle \rightarrow \langle \mathbf{X}_{0,t}^{<\infty}, v \rangle$  in  $L^2$  as  $J \rightarrow \infty$  for all words  $v$  and  $t \in [0, T]$ . Second, for  $i = 1, \dots, M$  i.i.d. sample paths of  $Z$  and the discretised and truncated signature  $\mathbf{X}^{\leq K} = \mathbf{X}^{\leq K}(J)$ , assuming that  $\tau_{n+1}^{K,J}$  is known, we estimate  $\ell^*$  by solving (3.5) via linear least-squares regression. This yields an estimator  $\ell^* = \ell^{*,n,J,K,M}$ . Defining  $\psi^{n,J,K,M}(\mathbf{x}) = \langle \mathbf{x}^{\leq K}, \ell^{*,n,J,K,M} \rangle$  leads to a recursive algorithm for stopping times, for  $i = 1, \dots, M$ ,

$$\begin{aligned} \tau_N^{K,J,(i)} &= t_N, \\ \tau_n^{K,J,(i)} &= t_n 1_{\{Z_{t_n}^{(i)} \geq \psi^{n,J,K,M}(\mathbf{X}^{(i)}|_{[0,t_n]})\}} + \tau_{n+1}^{K,J,(i)} 1_{\{Z_{t_n}^{(i)} < \psi^{n,J,K,M}(\mathbf{X}^{(i)}|_{[0,t_n]})\}}. \end{aligned} \quad (3.6)$$

Then the following law-of-large-numbers-type result holds true, which almost directly follows from Clément et al. [18, Theorem 3.2]; see Appendix A.1.

**Proposition 3.4** *For fixed  $K, J$ , we have*

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=1}^M Z_{\tau_n^{K,J,(i)}}^{(i)} = E[Z_{\tau_n^{K,J}}] \quad a.s.$$

Moreover, setting  $y_0^{N,K,J,M} := \max(Z_{t_0}, \frac{1}{M} \sum_{i=1}^M Z_{\tau_1^{K,J,(i)}}^{(i)})$ , we have

$$\lim_{K \rightarrow \infty} \lim_{J \rightarrow \infty} \lim_{M \rightarrow \infty} y_0^{K,N,J,M} = y_0^N,$$

where the convergence with respect to  $M$  is almost sure convergence.

**Remark 3.5** The recursion (3.6) of stopping times, resp. the resulting linear functionals of the signature  $\psi^{n,K,M}$ , provide a stopping policy for each sample path of  $Z$ . By re-simulating  $\tilde{M}$  i.i.d. samples of  $Z$  and the signature  $\mathbf{X}^{<\infty}$ , we can notice that the resulting estimator  $y_0^{K,N,J,\tilde{M}}$  is lower-biased, that is,  $y_0^{K,N,J,\tilde{M}} \leq y_0^N$ , since  $y_0^N$  is defined by taking the supremum over all possible stopping policies.

### 3.3 Dual optimal stopping with signatures

In this section, we approximate solutions to the optimal stopping problem in its dual formulation, leading to upper bounds  $y_0^U \geq y_0$  for (3.1). The dual representation goes back to Rogers [37], where the author shows that under the assumption  $\sup_{0 \leq t \leq T} |Z_t| \in L^2$ , the optimal stopping problem (3.1) is equivalent to

$$y_0 = \inf_{M \in \mathcal{M}_0^2} E \left[ \sup_{t \leq T} (Z_t - M_t) \right], \quad (3.7)$$

where  $\mathcal{M}_0^2$  denotes the space of square-integrable  $\mathbb{F}^{\mathbf{X}}$ -martingales starting from 0. Assuming that  $\mathbb{F}^{\mathbf{X}}$  is generated by a Brownian motion  $W$ , we can prove the following equivalent formulation of (3.7).

**Remark 3.6** From a financial modelling perspective, assuming that the filtration is generated by an  $m$ -dimensional Brownian motion  $W$  is natural in the context of diffusive or rough volatility modelling. In this setting,  $W$  represents the driver of  $m/2$  assets (semimartingales) and their corresponding  $m/2$  variance processes.

**Theorem 3.7** Assume  $\mathbb{F}^{\mathbf{X}}$  is generated by an  $m$ -dimensional Brownian motion  $W$ . Then for all  $M \in \mathcal{M}_0^2$ , there exist sequences  $\ell^i = (\ell_n^i)_{n \in \mathbb{N}} \subseteq \mathcal{W}^{d+1}$  for  $i = 1, \dots, m$  such that

$$\int_0^\cdot \langle \mathbf{X}_{0,s}^{<\infty}, \ell_n \rangle^\top dW_s := \sum_{i=1}^m \int_0^\cdot \langle \mathbf{X}_{0,s}^{<\infty}, \ell_n^i \rangle dW_s^i \longrightarrow M. \quad \text{ucp as } n \rightarrow \infty.$$

In particular, the minimisation problem (3.7) can be equivalently formulated as

$$\begin{aligned} y_0 &= \inf_{\ell \in (\mathcal{W}^{d+1})^m} E \left[ \sup_{t \leq T} \left( Z_t - \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell \rangle^\top dW_s \right) \right] \\ &= \inf_{\ell^1, \dots, \ell^m \in \mathcal{W}^{d+1}} E \left[ \sup_{t \leq T} \left( Z_t - \sum_{i=1}^m \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell^i \rangle dW_s^i \right) \right]. \end{aligned} \quad (3.8)$$

**Proof** By the martingale representation theorem, any square-integrable  $\mathbb{F}^{\mathbf{X}}$ -martingale can be written as

$$M_t = \int_0^t \alpha_s^\top dW_s = \sum_{i=1}^m \int_0^t \alpha_s^i dW_s^i,$$

where  $(\alpha_s)_{s \in [0, T]}$  is  $\mathbb{F}^{\mathbf{X}}$ -progressively measurable and square-integrable. Moreover, since  $M \in \mathcal{M}_0^2$ , it follows that  $E[M_T^2] = E[\int_0^T |\alpha_t|^2 dt] < \infty$ . From Corollary 2.9, we know that there exist sequences  $\ell^i = (\ell_n^i)_{n \in \mathbb{N}} \subseteq \mathcal{W}^{d+1}$  for  $i = 1, \dots, m$  such that for  $\alpha_t^{i,n} := \langle \mathbf{X}_{0,t}^{<\infty}, \ell_n^i \rangle$ , we have  $\|\alpha^{i,n} - \alpha^i\|_{\mathbb{H}^2} \rightarrow 0$  as  $n \rightarrow \infty$ . Using Doob's inequality, we in particular have

$$\begin{aligned} E \left[ \left( \sup_{t \leq T} \int_0^t (\alpha_s^n - \alpha_s)^\top dW_s \right)^2 \right] &\leq C \sum_{i=1}^m E \left[ \int_0^T |\alpha_s^{i,n} - \alpha_s^i|^2 dt \right] \\ &= \sum_{i=1}^m \|\alpha^{i,n} - \alpha^i\|_{\mathbb{H}^2}^2 \rightarrow 0. \end{aligned}$$

But this readily implies the first claim, that is,

$$\int_0^\cdot \langle \mathbf{X}_{0,s}^{<\infty}, \ell_n \rangle^\top dW_s \rightarrow \int_0^\cdot \alpha_s dW_s = M. \quad \text{ucp.}$$

To show (3.8), since  $\int_0^\cdot \langle \mathbf{X}_{0,s}^{<\infty}, \ell_n \rangle^\top dW_s$  are clearly square-integrable  $\mathbb{F}^{\mathbf{X}}$ -martingales, we can notice that

$$\inf_{\ell \in (\mathcal{W}^{d+1})^m} E \left[ \sup_{t \leq T} \left( Z_t - \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell \rangle^\top dW_s \right) \right] \geq \inf_{M \in \mathcal{M}_0^2} E \left[ \sup_{t \leq T} (Z_t - M_t) \right] = y_0.$$

On the other hand, for any fixed square-integrable martingale  $M$ , we know there exist sequences of words  $\ell^i = (\ell_n^i)_{n \in \mathbb{N}} \subseteq \mathcal{W}^{d+1}$  such that

$$\lim_{n \rightarrow \infty} \sup_{t \leq T} \left( Z_t - \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell_n \rangle^\top dW_s \right) = \sup_{t \leq T} (Z_t - M_t) \quad \text{in } L^2.$$

Therefore

$$\begin{aligned} E \left[ \sup_{t \leq T} (Z_t - M_t) \right] &= \lim_{n \rightarrow \infty} E \left[ \sup_{t \leq T} \left( Z_t - \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell_n \rangle^\top dW_s \right) \right] \\ &\geq \inf_{\ell \in (\mathcal{W}^{d+1})^m} E \left[ \sup_{t \leq T} \left( Z_t - \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell \rangle^\top dW_s \right) \right]. \end{aligned}$$

Taking the infimum over all  $M \in \mathcal{M}_0^2$  yields the claim.  $\square$

Next, similarly to the primal case, we translate the minimisation problem (3.8) into a finite-dimensional optimisation problem by discretising the interval  $[0, T]$  and

truncating the signature to some level  $K$ . More precisely, for  $0 = t_0 < \dots < t_N = T$  and some  $K \in \mathbb{N}$ , we reduce the minimisation problem (3.8) to

$$y_0^{K,N} = \inf_{\ell \in (\mathcal{W}_{\leq K}^{d+1})^m} E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell) \right], \quad (3.9)$$

where for any  $\ell = (\ell^1, \dots, \ell^m) \in (\mathcal{W}_{\leq K}^{d+1})^m$ , we define

$$M_t^\ell = \int_0^t \langle \mathbf{X}_{0,s}^{\leq K}, \ell \rangle^\top dW_s = \sum_{i=1}^m \int_0^t \langle \mathbf{X}_{0,s}^{\leq K}, \ell^i \rangle dW_s^i.$$

The discrete version of the dual formulation (3.7) is given by

$$y_0^N = \inf_{M \in \mathcal{M}_0^{2,N}} E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}) \right],$$

where  $\mathcal{M}_0^{2,N}$  denotes the space of discrete square-integrable  $\mathbb{F}^{\mathbf{X},N}$ -martingales. The following result shows that the minimisation problem (3.9) has a solution and the value converges to  $y_0^N$  as the level  $K$  of the signature goes to infinity. The proof can be found in Appendix A.2.

**Proposition 3.8** *There exists a minimiser  $\ell^*$  to (3.9) and*

$$|y_0^N - y_0^{K,N}| \longrightarrow 0 \quad \text{as } K \rightarrow \infty.$$

**Remark 3.9** In a financial context, Propositions 3.3 and 3.8 tell us that  $y_0^{K,N}$  converges to the Bermudan option price as  $K \rightarrow \infty$ . Moreover, the triangle inequality gives

$$|y_0 - y_0^{K,N}| \leq |y_0 - y_0^N| + |y_0^N - y_0^{K,N}|,$$

and hence the finite-dimensional approximations converge to  $y_0$  as  $K, N \rightarrow \infty$  whenever the Bermudan price converges to the American price. For our numerical examples, we always approximate  $y_0^N$  for some fixed  $N$ , and therefore we do not further investigate the latter convergence here.

### 3.3.1 Sample average approximation (SAA)

We now present a method to approximate the value  $y_0^{K,N}$  in (3.9) by using Monte Carlo simulations. This procedure is called *sample average approximation* (SAA) and we refer to Shapiro [38] for a general and extensive study of this method. Similarly to the primal case, we introduce two further approximation steps. First, let  $0 = s_0 < \dots < s_J = T$  be a finer discretisation of  $[0, T]$  and denote by  $M^{\ell,J}$ , resp.  $\mathbf{X}^{<\infty}(J)$ , an approximation of the stochastic integral  $\int \langle \mathbf{X}_{0,s}^{<\infty}, \ell \rangle dW_s$ , resp. the signature  $\mathbf{X}^{<\infty}$ , by using an Euler scheme. Second, we consider  $i = 1, \dots, M$  i.i.d.

sample paths  $Z^{(i)}$ ,  $M^{(i),\ell,J}$  and replace the expectation in (3.9) by a sample average, leading to the empirical minimisation problem

$$y_0^{K,N,J,M} = \inf_{\ell \in (\mathcal{W}_{\leq K}^{d+1})^m} \frac{1}{M} \sum_{i=1}^M \max_{0 \leq n \leq N} (Z_{t_n}^{(i)} - M_{t_n}^{(i),\ell,J}). \quad (3.10)$$

The following result can be deduced from Shapiro [38, Theorem 4] combined with Proposition 3.8. We refer to Appendix A.2 for the details.

**Proposition 3.10** *For  $M$  large enough, there exists a minimiser  $\beta^*$  to (3.10), and*

$$\lim_{K \rightarrow \infty} \lim_{J \rightarrow \infty} \lim_{M \rightarrow \infty} y_0^{K,N,J,M} = y_0^N,$$

where the convergence with respect to  $M$  is almost sure convergence.

**Remark 3.11** Let us quickly describe how we solve (3.10) numerically. Consider the number  $D := \sum_{k=0}^K (d+1)^k$ , which corresponds to the number of entries of the  $K$ -step signature. Notice that for any element  $\ell \in \mathcal{W}_{\leq K}^{d+1}$ , we have the representation  $\ell = \lambda_1 w_1 + \dots + \lambda_D w_D$ , where  $w_1, \dots, w_D$  are all possible words of length at most  $K$ . Since  $\langle \mathbf{X}_{0,t}^{\leq K}, \ell \rangle = \sum_{r=1}^D \lambda_r \langle \mathbf{X}_{0,t}^{\leq K}, w_r \rangle$ , the minimisation (3.10) has the equivalent formulation

$$y_0^{K,N,J,M} = \inf_{\lambda \in (\mathbb{R}^D)^m} \frac{1}{M} \sum_{i=1}^M \max_{0 \leq n \leq N} \left( Z_{t_n}^{(i)} - \sum_{r=1}^D \lambda_r M_{t_n}^{(i),w_r,J} \right).$$

As described in Desai et al. [22], the above minimisation problem is equivalent to the linear program

$$\min_{x \in \mathbb{R}^{M+D}} \frac{1}{M} \sum_{j=1}^M x_j \quad \text{subject to } Ax \geq b,$$

where  $A \in \mathbb{R}^{M(N+1) \times (M+D)}$  with  $A = [A^1, \dots, A^M]^\top$ , and  $Ax \geq b$  represents the constraints

$$x_i \geq Z_{t_n}^{(i)} - \sum_{r=1}^D M_{t_n}^{(i),w_r,J}, \quad i = 1, \dots, M, n = 0, \dots, N.$$

**Remark 3.12** A solution  $\ell^*$  to (3.10) yields the square-integrable  $\mathbb{F}^{\mathbf{X},N}$ -martingale  $M^{\ell^*}$ , and by re-simulating  $\tilde{M}$  i.i.d. samples of  $Z$ , the Brownian motion  $W$  and the signature  $\mathbf{X}^{<\infty}$ , we can notice that the resulting estimator  $y_0^{K,N,J,\tilde{M}}$  is upper-biased, that is,  $y_0^{K,N,J,\tilde{M}} \geq y_0^N$ , since the latter is defined by taking the infimum over all square-integrable  $\mathbb{F}^{\mathbf{X},N}$ -martingales.

## 4 Numerical examples

In this section, we study two non-Markovian optimal stopping problems and test our methods to approximate lower and upper bounds for the optimal stopping value. The details for the implementations and examples can be found at [https://github.com/lucafelizzari/Optimal\\_Stopping\\_with\\_signatures](https://github.com/lucafelizzari/Optimal_Stopping_with_signatures).

**Remark 4.1** In all the numerical experiments performed below, we did not observe any significant difference when using the normalised signature (choosing the same normalisation the authors introduce in Chevyrev and Oberhauser [17, Sect. 3]) and when using the standard, non-normalised signature. Since the tensor normalisation increases the complexity of the algorithms, all the results presented here were obtained using the standard signature.

### 4.1 Optimal stopping of fractional Brownian motion

We start with the task of optimally stopping a fractional Brownian motion (fBm), which represents the canonical choice of a framework leaving the Markov and semimartingale regimes. Recall that an fBm with Hurst parameter  $H \in (0, 1)$  is the unique continuous Gaussian process  $(X_t^H)_{t \in [0, T]}$  with

$$\begin{aligned} E[X_t^H] &= 0, \quad \forall t \geq 0, \\ E[X_s^H X_t^H] &= \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \quad \forall s, t \geq 0; \end{aligned}$$

see e.g. Friz and Hairer [25, Chap. 9] for more details. We wish to approximate the value

$$y_0^H = \sup_{\tau \in \mathcal{S}_0} E[X_\tau^H], \quad H \in (0, 1), \quad (4.1)$$

from below and above. This example has already been studied in Becker et al. [8, Sect. 4.3] as well as in Bayer et al. [5, Sect. 8.1], and we compare the results below.

Since  $X^H$  is one-dimensional and  $\alpha$ -Hölder-continuous for any  $\alpha < H$ , its (scalar) rough-path lift is given by

$$\left(1, X_{s,t}^H, \frac{1}{2}(X_{s,t}^H)^2, \dots, \frac{1}{L!}(X_{s,t}^H)^L\right) \in \mathcal{C}_g^\alpha([0, T]; \mathbb{R}),$$

where  $L = \lfloor \frac{1}{\alpha} \rfloor$ . We can extend it to a geometric rough-path lift  $\mathbf{X}^H \in \widehat{\mathcal{C}}_g^\alpha([0, T]; \mathbb{R}^2)$  of the time-augmentation  $(t, X_t^H)$ , as for instance described in [5, Example 2.4]. To numerically solve (4.1), we replace the continuous-time interval  $[0, T]$  by some finite grid points  $0 = t_0 < t_1 < \dots < t_N = T$ . Below we compare our results with [8, Sect. 4.3], where the authors chose  $N = 100$ . Before doing so, an important remark about the difference of our problem formulation is in order.

**Remark 4.2** In [8], the authors lift  $X^H$  to a 100-dimensional Markov process of the form  $\widehat{X}_{t_k} = (X_{t_k}^H, \dots, X_{t_1}^H, 0, \dots, 0) \in \mathbb{R}^{100}$ , and they consider the corresponding discrete (!) filtration  $\widehat{\mathcal{F}}_k = \sigma(X_{t_k}^H, \dots, X_{t_1}^H)$ ,  $k = 0, \dots, 100$ . Notice that this differs from our setting as we consider the bigger filtration  $\mathcal{F}_k = \sigma(X_s^H : s \leq t_k)$  (see Sects. 3.2 and 3.3) that contains the whole past of  $X^H$ , not only the information at the past exercise dates. Thus in general  $y_0^H$  dominates the lower bounds from [8], simply because our filtration contains more stopping times. Similarly, the (very sharp) upper bounds in [8] were obtained by using a nested Monte Carlo approach, which constructs  $(\widehat{\mathcal{F}}_k)$ -martingales that are not martingales in our filtration, and thus their upper bounds are not necessarily upper bounds for (4.1).

In Table 1, we present intervals for the optimal stopping values  $y_0^H$  for Hurst parameters  $H \in \{0.01, 0.05, \dots, 0.95\}$ , where the lower (resp. upper) bounds were approximated by using Longstaff–Schwartz with signatures, resp. the SAA approach described in Sect. 3. We truncate the signature at the level  $K = 6$  and apply the primal approach using  $M = 10^6$  samples for both the regression and the re-simulation, and for the dual approach, we choose  $M = 15'000$  to solve the linear program from Remark 3.11 and re-simulate with  $M = 10^5$  samples. In the first column, we choose the time discretisation for the signature equal to the number of exercise dates as  $J = N = 100$ . While the lower bounds are very close, our upper bounds exceed those from [8]. This observation matches with the comments made in Remark 4.2 as

**Table 1** Intervals for optimal stopping of fBm: we show  $H \mapsto y_0^H$  with  $N = 100$  exercise dates and discretisation  $J = 100$  (left column),  $J = 500$  (middle column), and intervals from [8] (right column). The overall Monte Carlo error is below 0.003

$H$	$J = 100$	$J = 500$	Becker et al. [8]
0.01	[1.518, 1.645]	[1.545, 1.631]	[1.517, 1.52]
0.05	[1.293, 1.396]	[1.318, 1.382]	[1.292, 1.294]
0.1	[1.045, 1.129]	[1.065, 1.117]	[1.048, 1.05]
0.15	[0.83, 0.901]	[0.847, 0.895]	[0.838, 0.84]
0.2	[0.654, 0.706]	[0.663, 0.698]	[0.657, 0.659]
0.25	[0.507, 0.538]	[0.510, 0.533]	[0.501, 0.505]
0.3	[0.363, 0.396]	[0.371, 0.392]	[0.368, 0.371]
0.35	[0.248, 0.272]	[0.255, 0.270]	[0.254, 0.257]
0.4	[0.153, 0.168]	[0.155, 0.165]	[0.154, 0.158]
0.45	[0.069, 0.077]	[0.068, 0.076]	[0.066, 0.075]
0.5	[−0.001, 0]	[−0.002, 0]	[0, 0.005]
0.55	[0.061, 0.071]	[0.060, 0.066]	[0.057, 0.065]
0.6	[0.112, 0.133]	[0.112, 0.124]	[0.115, 0.119]
0.65	[0.163, 0.187]	[0.163, 0.175]	[0.163, 0.166]
0.7	[0.203, 0.234]	[0.205, 0.220]	[0.206, 0.208]
0.75	[0.242, 0.273]	[0.240, 0.260]	[0, 242, 0.245]
0.8	[0.275, 0.306]	[0.281, 0.298]	[0.276, 0.279]
0.85	[0.306, 0.335]	[0.301, 0.324]	[0.307, 0.31]
0.9	[0.331, 0.357]	[0.337, 0.356]	[0.335, 0.339]
0.95	[0.367, 0.381]	[0.366, 0.381]	[0.365, 0.367]



we consider the filtration  $\widehat{\mathbb{F}}$  in this case for the lower bounds; but our upper bounds are by construction upper bounds for the continuous problem with filtration  $\mathbb{F}$ , and the continuous martingale is approximated only at the exercise dates. By increasing the discretisation to  $J = 500$  and thereby adding information to the filtration between exercise dates, one can see for small  $H$  ( $\leq 0.2$ ) that the lower bounds exceed the intervals from [8], showing that even for  $N = 100$  points in  $[0, 1]$ , the information between exercise dates is relevant for optimally stopping the fBm.

## 4.2 American options in rough volatility models

The second example we present is the problem of pricing American options in rough volatility models. More precisely, we consider the one-dimensional asset-price model

$$X_0 = x_0, \quad dX_t = rX_t dt + X_t v_t (\rho dW_t + \sqrt{1 - \rho^2} dB_t), \quad 0 < t \leq T,$$

where  $W$  and  $B$  are two independent Brownian motions, the volatility  $(v_t)_{t \in [0, T]}$  is an  $\mathbb{F}^W$ -adapted continuous process,  $\rho \in [-1, 1]$  and  $r > 0$  is the interest rate. For any payoff function  $\phi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , we want to approximate the optimal stopping problem

$$y_0 = \sup_{\tau \in \mathcal{S}_0} E[e^{-r\tau} \phi(\tau, X_\tau)], \quad (4.2)$$

where  $\mathcal{S}_0$  is the set of  $\mathbb{F} := (\mathbb{F}^W \vee \mathbb{F}^B)$ -stopping times valued in  $[0, T]$ . It is worth noting that our method does not depend on the specification of  $v$ , and as soon as we can sample from  $(X, v)$ , we can apply the method to approximate values of American options.

In the following, we focus on the rough Bergomi model (see Bayer et al. [4]), that is, we specify the volatility as

$$v_t = \xi_0 \mathcal{E} \left( \eta \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \right),$$

where  $\mathcal{E}$  denotes the stochastic exponential, and we consider the parameters  $x_0 = 100$ ,  $r = 0.05$ ,  $\eta = 1.9$ ,  $\rho = -0.9$ ,  $\xi_0 = 0.09$ . Moreover, we consider put options with  $\phi(t, x) = (K - x)^+$  for different strikes  $K \in \{70, 80, \dots, 120\}$ , with maturity  $T = 1$  and  $N = 12$  exercise dates. Thus we write (4.2) as the discrete optimal stopping problem

$$y_0^N = \sup_{\tau \in \mathcal{S}_0^N} E[e^{-r\tau} (K - X_\tau)^+],$$

where  $\mathcal{S}_0^N$  is described at the beginning of Sect. 3.2. Moreover, for some finer grid  $s_0 = 0 < s_1 < \dots < s_J = 1$ ,  $J \in \mathbb{N}$ , and fixed signature level  $K$  and sample size  $M$ , we denote by  $y_0^{LS}$  the value  $y_0^{K, N, J, M}$  defined in Proposition 3.4 and by  $y_0^{\text{SAA}}$  the value  $y_0^{K, N, J, M}$  defined in (3.10). We compare our results with the lower bounds obtained in Bayer et al. [7] for  $H = 0.07$ , resp. in Goudenege et al. [29] with  $H = 0.07$  and  $H = 0.8$ .

**Table 2** Put option price intervals with  $J = 48$  and  $H = 0.07$ , for the two choices of basis functions described in  $(\mathbf{B}_1)$  (resp.  $(\mathbf{B}_2) = \{(\mathbf{P}_2), (\mathbf{D}_2)\}$ ) and lower-bound reference values

$K$	Basis $(B_1)$	Basis $(B_2)$	Bayer et al. [7]	Goudenege et al. [29]
70	(1.83, 2.59)	(1.85, 2.04)	1.88	1.88
80	(3.18, 4.44)	(3.18, 3.44)	3.22	3.25
90	(5.19, 7.66)	(5.25, 5.68)	5.30	5.34
100	(8.33, 13.16)	(8.44, 9.13)	8.50	8.53
110	(13.02, 21.38)	(13.18, 14.18)	13.23	13.28
120	(20.20, 30.91)	(20.22, 21.40)	20	20.20

**Table 3** Put option price intervals with  $J = 600$  and  $H = 0.07$ , for the two choices of basis functions described in  $(\mathbf{B}_1)$  (resp.  $(\mathbf{B}_2) = \{(\mathbf{P}_2), (\mathbf{D}_2)\}$ ) and lower-bound reference values

$K$	Basis $(B_1)$	Basis $(B_2)$	Bayer et al. [7]	Goudenege et al. [29]
70	(1.90, 2.38)	(1.92, 1.99)	1.88	1.88
80	(3.25, 4.13)	(3.27, 3.37)	3.22	3.25
90	(5.34, 7.17)	(5.37, 5.49)	5.30	5.34
100	(8.51, 12.55)	(8.57, 8.77)	8.50	8.53
110	(13.24, 20.79)	(13.29, 13.59)	13.23	13.28
120	(20.22, 29.90)	(20.24, 20.66)	20	20.20

In the first numerical experiments, we again simply consider the signature of the time-augmented path  $(t, X_t)$ , that is, we choose the basis functions for the least-squares regression (3.5), resp. for the SAA minimisation problem in (3.9), to be

$$(\mathbf{B}_1) \quad \{(\mathbf{X}_{0,t}^{<\infty}, \ell) : \ell \in \mathcal{W}_{\leq K}^2\}, \quad K \in \mathbb{N}.$$

In the first column of Tables 2 and 3, we report the price intervals  $[y_0^{\text{LS}}, y_0^{\text{SAA}}]$  for the Hurst parameter  $H = 0.07$  and different discretisations  $J = 48$  and  $J = 600$ . The degree of the signature is fixed at  $K = 4$ , and we apply the primal algorithm described in Sect. 3.2 for  $M = 10^6$  samples. For the obtained stopping policies, we re-simulate with again  $M = 10^6$  samples to obtain true lower bounds  $y_0^{\text{LS}}$ . For the upper bounds, we solve the linear program described in Sect. 3.3 for  $M = 10^4$  samples, and then re-simulate with  $M = 10^5$  samples to obtain true upper bounds  $y_0^{\text{SAA}}$ . In the first column of Table 4, we consider the same problem for  $H = 0.8$  and  $J = 600$ . Similarly as in Sect. 4.1, we observe that the price intervals shrink when we increase the number of discretisation points between exercise dates. However, even for  $J = 600$ , we still observe a significant gap between lower and upper bounds. It was already observed in Markovian frameworks that the approximation of the Doob martingale usually requires a careful and specific choice of basis functions (e.g. in Belomestny et al. [9], the authors use European deltas). This motivates us to extend the basis  $(\mathbf{B}_1)$  in two ways. First, we add Laguerre polynomials of the states  $(X_t, v_t)$ , which would be a natural choice in a Markovian framework, for instance used in the original work by Longstaff and Schwartz in [33]. Additionally, for the dual problem, we add the

**Table 4** Put option price intervals with  $J = 600$  and  $H = 0.8$ , for the two choices of basis functions described in  $(\mathbf{B}_1)$  (resp.  $(\mathbf{B}_2) = \{(\mathbf{P}_2), (\mathbf{D}_2)\}$ ) and lower-bound reference values

$K$	Basis $(B_1)$	Basis $(B_2)$	Goudenege et al. [29]
70	(1.83, 2.39)	(1.83, 1.90)	1.84
80	(3.08, 4.13)	(3.08, 3.19)	3.10
90	(5.04, 7.38)	(5.07, 5.17)	5.08
100	(8.11, 12.84)	(8.15, 8.27)	8.19
110	(12.89, 20.77)	(12.97, 13.09)	13.00
120	(20.16, 30.21)	(20.21, 20.51)	20.28

payoff process  $Z$  to the path, that is, we lift  $(t, X_t, Z_t)$  to the signature  $\mathbf{Z}^{<\infty}$ . Since  $(t, X_t, Z_t)$  is a semimartingale, the signature  $\mathbf{Z}^{<\infty}$  is given as the sequence of iterated Stratonovich integrals as explained in Friz et al. [24]. Of course, adding basis functions in the primal and dual approach does not change the convergence. To summarise, for the least-squares regression (3.5), resp. for the SAA minimisation problem in (3.9), we use the extended basis  $(\mathbf{B}_2) = \{(\mathbf{P}_2), (\mathbf{D}_2)\}$  of the form

$$\begin{aligned}
 (\mathbf{P}_2) \quad & \{L_i(X_t, v_t), \langle \mathbf{X}_{0,t}^{<\infty}, \ell \rangle : i = 1, \dots, m_p, \ell \in \mathcal{W}_{\leq K_p}^{d+1}\}, \quad m_p, K_p \in \mathbb{N}, \\
 (\mathbf{D}_2) \quad & \{L_i(X_t, v_t), \langle \mathbf{Z}_{0,t}^{<\infty}, \ell \rangle : i = 1, \dots, m_d, \ell \in \mathcal{W}_{\leq K_d}^{d+1}\}, \quad m_d, K_d \in \mathbb{N},
 \end{aligned}$$

where  $(L_k)_{k \geq 0}$  are the Laguerre polynomials. In the second columns of Tables 2–4 we report the price intervals for the extended basis  $(\mathbf{B}_2)$ . We consider polynomials of degree 3 for the primal and 5 for the dual approach, and the signature levels and number of samples are the same as before. Especially for  $J = 600$ , we observe a significant reduction of the upper bounds, which are now only 2%–3% higher than the lower bounds. We expect these margins to shrink more when either further increasing all parameters or when choosing nonlinear basis functions, such as for instance deep neural networks on the log-signature or signature-kernel-based methods. These two ideas are currently under development, and more details will appear in future works.

## Appendix: Technical details for Sect. 3

### A.1 Proofs in Sect. 3.2

**Proof of Proposition 3.3** The proof is based on the same ideas as the proof in Clément et al. [18, Theorem 3.1]. We proceed by induction over  $n$ . For  $n = N$ , the claim trivially holds true. Assume it holds for  $0 \leq n+1 \leq N-1$ . Let us recall from (3.5) that for fixed  $K$  and  $n$ , we define  $\psi^{n,K}(\mathbf{x}) = \langle \mathbf{x}^{\leq K}, \ell^{*,n,K} \rangle$ , where

$$\ell^* := \ell^{*,n,K} = \arg \min_{\ell \in \mathcal{W}_{\leq K}^{d+1}} \|Z_{\tau_{n+1}^K} - \langle \mathbf{X}_{0,t_n}^{\leq K}, \ell \rangle\|_{L^2}.$$

In particular, by the Hilbert projection theorem,  $\psi^{n,K}(\mathbf{X}|_{[0,t]})$  is the orthogonal projection of  $Z_{\tau_{n+1}^K}$  onto the subspace  $\{\langle \mathbf{X}_{0,t_n}^{\leq K}, \ell \rangle : \ell \in \mathcal{W}^{d+1}\}$  of  $L^2$ . Define the events

$$A(n) := \{Z_{t_n} \geq E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]\}, \quad A(n, K) := \{Z_{t_n} \geq \psi^{n,K}(\mathbf{X}|_{[0,t_n]})\}.$$

By definition, we can write

$$\tau_n^K = t_n 1_{A(n,K)} + \tau_{n+1}^K 1_{A(n,K)^c}, \quad \tau_n = t_n 1_{A(n)} + \tau_{n+1} 1_{A(n)^c}.$$

Using this, it is possible to check that

$$\begin{aligned} E[Z_{\tau_n^K} - Z_{\tau_n} | \mathcal{F}_{t_n}^{\mathbf{X}}] &= (Z_{t_n} - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}])(1_{A(n,K)} - 1_{A(n)}) \\ &\quad + E[Z_{\tau_{n+1}^K} - Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}] 1_{A(n,K)^c}. \end{aligned}$$

The second term converges to 0 by the induction hypothesis, and we only need to show

$$L_n^K := (Z_{t_n} - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}])(1_{A(n,K)} - 1_{A(n)}) \longrightarrow 0 \quad \text{in } L^2 \text{ as } K \rightarrow \infty.$$

Now on  $A(n, K) \cap A(n)$  and  $A(n, K)^c \cap A(n)^c$ , we clearly have  $L_n^K = 0$ . Moreover,

$$1_{A(n,K)^c \cap A(n)} |L_n^K| \leq 1_{A(n,K)^c \cap A(n)} |\psi^{n,K}(\mathbf{X}|_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]|$$

since  $\psi^{n,K}(\mathbf{X}|_{[0,t_n]}) > Z_{t_n} \geq E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]$  on  $A(n, K)^c \cap A(n)$ . Similarly, one can show that

$$1_{A(n,K) \cap A(n)^c} |L_n^K| \leq 1_{A(n,K) \cap A(n)^c} |\psi^{n,K}(\mathbf{X}|_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]|,$$

and thus

$$|L_n^K| \leq |\psi^{n,K}(\mathbf{X}|_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]|. \quad (\text{A.1})$$

As mentioned above,  $\psi^{n,K}$  is the orthogonal projection in  $L^2$  of the random variable  $Z_{\tau_{n+1}^K}$  onto the subspace  $\{\langle \mathbf{X}_{0,t_n}^{\leq K}, \ell \rangle : \ell \in \mathcal{W}^{d+1}\}$ , and we similarly denote by  $\hat{\psi}^{n,K}$  the orthogonal projection of  $Z_{\tau_{n+1}}$  onto the same space. Then we have

$$\begin{aligned} \|L_n^K\|_{L^2} &\leq \|\psi^{n,K}(\mathbf{X}|_{[0,t_n]}) - \hat{\psi}^{n,K}(\mathbf{X}|_{[0,t_n]})\|_{L^2} \\ &\quad + \|\hat{\psi}^{n,K}(\mathbf{X}|_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]\|_{L^2} \\ &\leq \|E[Z_{\tau_{n+1}^K} | \mathcal{F}_{t_n}^{\mathbf{X}}] - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]\|_{L^2} \\ &\quad + \|\hat{\psi}^{n,K}(\mathbf{X}|_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]\|_{L^2}. \end{aligned}$$

Now the first term converges to 0 by the induction hypothesis. For the second term, the conditional expectation of the random variable  $Z_{\tau_{n+1}} \in L^2$  is nothing else than the orthogonal projection onto the space  $L^2(\mathcal{F}_{t_n}^{\mathbf{X}})$ . But by Theorem 2.8, for any  $\epsilon > 0$ , we can find  $\phi \in L_{\text{sig}}^{\lambda}$  such that  $\|\phi(\mathbf{X}|_{[0,t_n]}) - Z_{\tau_{n+1}}\|_{L^2} \leq \epsilon$ . For  $K$  large enough, we have  $\phi(\mathbf{X}|_{[0,t_n]}) \in \{\langle \mathbf{X}_{0,t_n}^{\leq K}, \ell \rangle : \ell \in \mathcal{W}^{d+1}\}$ , and thus

$$\|\hat{\psi}^{n,K}(\mathbf{X}|_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]\|_{L^2} \leq \|\hat{\psi}^{n,K}(\mathbf{X}|_{[0,t_n]}) - Z_{\tau_{n+1}}\|_{L^2} \leq \epsilon$$

since  $\hat{\psi}^{n,K}$  is such that the distance is minimal.  $\square$

**Proof of Proposition 3.4** First, consider the sequence  $(\tau_n^{K,J})$  of stopping times defined in (3.6). One can then rewrite the proof of Proposition 3.3 for  $Z_{\tau_n^{K,J}}$ , writing  $\psi^{n,K,J}$  for the orthogonal projection of  $Z_{\tau_{n+1}^{K,J}}$  onto  $\{\langle \mathbf{X}_{0,t_n}^{\leq K}(J), \ell \rangle : \ell \in \mathcal{W}^{d+1}\}$ . Instead of (A.1), we have

$$\begin{aligned} L_n^{K,J} &\leq |\psi^{n,K,J}(\mathbf{X}_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]| \\ &\leq |\psi^{n,K}(\mathbf{X}_{[0,t_n]}) - E[Z_{\tau_{n+1}} | \mathcal{F}_{t_n}^{\mathbf{X}}]| + |\psi^{n,K,J}(\mathbf{X}_{[0,t_n]}) - \psi^{n,K}(\mathbf{X}_{[0,t_n]})|, \end{aligned}$$

where the first term converges to 0 in  $L^2$  by the same argument as in the proof of Proposition 3.3. Then since we assume that  $\langle \mathbf{X}_{0,t}^{<\infty}(J), v \rangle \rightarrow \langle \mathbf{X}_{0,t}^{<\infty}, v \rangle$  in  $L^2$  as  $J \rightarrow \infty$  for all words  $v$ , the second term converges to 0 in  $L^2$  for any fixed  $K$ . Thus it follows for all  $n = 0, \dots, N$  that

$$\lim_{K \rightarrow \infty} \lim_{J \rightarrow \infty} E[Z_{\tau_n^{K,J}} | \mathcal{F}_{t_n}^{\mathbf{X}}] = E[Z_{\tau_n} | \mathcal{F}_{t_n}^{\mathbf{X}}] \quad \text{in } L^2. \quad (\text{A.2})$$

Next, we want to show that

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=1}^M Z_{\tau_n^{K,J,(i)}}^{(i)} = E[Z_{\tau_n^{K,J}}] \quad \text{a.s.} \quad (\text{A.3})$$

Now for any  $\ell \in \mathcal{W}^{d+1}$ , we can write  $\ell = \lambda_1 w_1 + \dots + \lambda_D w_D$ , where the number of words is given by  $D = \sum_{k=0}^K (d+1)^k$ , that is, we sum over all possible words of length at most  $K$ . One can therefore notice that minimising  $\langle \mathbf{X}^{\leq K}, \ell \rangle$  over  $\ell \in \mathcal{W}_{\leq K}^{d+1}$  is equivalent to minimising  $\sum_{i=1}^D \lambda_i \langle \mathbf{X}^{\leq K}, w_i \rangle$  over all vectors  $\lambda \in \mathbb{R}^D$ . Define the basis  $e_k(\mathbf{x}) := \langle \mathbf{x}^{\leq K}, w_k \rangle$  for  $k = 1, \dots, D$ , and set  $X_n := \mathbf{X}_{[0,t_n]}$ . Then we are exactly in the framework of Clément et al. [18, Sect. 3], and the result follows from [18, Theorem 3.2] under the following remark. The authors make for the set of basis functions the assumption, denoted by (A.2) there, that

$$\text{for all } t : \quad \text{if } \sum_j \alpha_j e_j(X_t) = 0 \text{ almost surely, then } \alpha = 0, \quad (\text{A.4})$$

which allows an explicit representation of the coefficient  $\ell^*$  in (3.5). Of course, in our framework, such an assumption cannot hold true, as this would correspond to

$$\sum_{\ell=1}^D \alpha_\ell \langle \mathbf{X}_{0,t}^{<\infty}, w_\ell \rangle = 0 \text{ a.s.} \quad \implies \quad \alpha_\ell = 0, \forall \ell = 1, \dots, D.$$

Since we consider the signature of the time-augmented path  $(t, X_t)$ , the purely deterministic components of the signature contradict this assumption. However, for a fixed signature level  $K$ , we can always discard linearly dependent (in the sense of (A.4)) components of the signature, that is, minimise over the basis functions

$$\{\tilde{e}_1, \dots, \tilde{e}_{\tilde{D}}\} \subseteq \{e_1, \dots, e_D\} \text{ such that (A2) holds,}$$

for the largest possible  $\tilde{D} \leq D$ . The resulting least-squares problem (3.5) over  $\mathbb{R}^{\tilde{D}}$ , with respect to  $\{\tilde{e}_1, \dots, \tilde{e}_{\tilde{D}}\}$ , has an explicit representation for its solution, and since the two sets of basis functions generate the same subspace of  $L^2$ , the explicit solution is also optimal for the original problem. Thus for a fixed level  $K$ , we can proceed with the reduced set of basis functions for which the assumption (A2) holds by definition, and we can apply [18, Theorem 3.2]. Finally, the asserted convergence directly follows by combining (A.2) and (A.3).  $\square$

## A.2 Proofs in Sect. 3.3

**Proof of Proposition 3.8** The existence of a minimiser is proved below in Lemma A.1. We can find the discrete Doob martingale  $M^{\star, N}$  and write

$$y_0^N = E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^{\star, N}) \right].$$

Define the continuous-time  $\mathbb{R}^X$ -martingale  $M_t := E[M_T^{\star, N} | \mathcal{F}_t^X]$  and notice that we have  $M_{t_n} = M_{t_n}^{\star, N}$ . Recall the following notation from Sect. 3.3: for any  $\ell \in (\mathcal{W}^{d+1})^m$ , we define the martingale  $M^\ell$  to be

$$M_t^\ell = \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell \rangle^\top dW_s = \sum_{i=1}^m \int_0^t \langle \mathbf{X}_{0,s}^{<\infty}, \ell^i \rangle dW_s^i.$$

An application of the martingale approximation in Theorem 3.7 shows, for all  $\epsilon > 0$ , that there exists an  $\ell^\epsilon = (\ell^{i,\epsilon})_{i=1,\dots,m}$  in  $(\mathcal{W}^{d+1})^m$  such that

$$E \left[ \max_{0 \leq n \leq N} (M_{t_n}^{\star, N} - M_{t_n}^{\ell^\epsilon}) \right] \leq \epsilon.$$

Thus we have

$$y_0^N = E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^{\star, N}) \right] \geq E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^{\ell^\epsilon}) \right] - \epsilon.$$

Now since  $y_0^{K,N} \geq y_0^N$ , we can find  $K$  large enough such that

$$\begin{aligned} 0 \leq y_0^{K,N} - y_0^N &\leq \inf_{\ell \in (\mathcal{W}_{\leq K}^{d+1})^m} E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell) \right] \\ &\quad - E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^{\ell^\epsilon}) \right] + \epsilon \\ &\leq \epsilon, \end{aligned}$$

where the last inequality follows from that fact that  $\ell^\epsilon \in (\mathcal{W}_{\leq K}^{d+1})^m$  for  $K$  large.  $\square$

**Lemma A.1** *The minimisation problem*

$$y_0^{K,N} = \inf_{\ell \in (\mathcal{W}_{\leq K}^{d+1})^m} E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell) \right]$$

has a solution.

**Proof** First notice that  $\ell \mapsto E[\max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell)]$  is convex. Then we have

$$\begin{aligned} E\left[\max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell)\right] &\geq E[\max(Z_T - M_T^\ell, 0)] \\ &= \frac{1}{2}E[Z_T - M_T^\ell + |M_T^\ell - Z_T|] \\ &\geq \frac{1}{2}E[|M_T^\ell|] + E[\max(-Z_T, 0)], \end{aligned}$$

where the equality uses  $\max(A - B, 0) = \frac{1}{2}(A - B + |B - A|)$ , and the second inequality follows via  $E[M_T^\ell] = 0$  by the martingale property and, for  $A = Z_T$  and  $B = M_T^\ell$ , the identity  $A + |B - A| \geq A - |A| + |B| = |B| + 2\max(-A, 0)$ . Now for any word  $\ell = \lambda_1 w_1 + \dots + \lambda_n w_n$ , we set  $|\ell| = \sum_{i=1}^n |\lambda_i|$  and notice that

$$\frac{1}{2}E[|M_T^\ell|] = \frac{1}{2}|\ell|E[|M_T^{\ell/|\ell|}|] \geq \frac{|\ell|}{2} \inf_{\widehat{\ell} \in (\mathcal{W}_{\leq K}^{d+1})^m, |\widehat{\ell}|=1} E[|M_T^{\widehat{\ell}}|]. \quad (\text{A.5})$$

Since  $\widehat{\ell} \mapsto E[|M_T^{\widehat{\ell}}|]$  is continuous and the set  $\{\widehat{\ell} \in (\mathcal{W}_{\leq K}^{d+1})^m : |\widehat{\ell}| = 1\}$  is compact, the minimum on the right-hand side of (A.5) is attained. Assume now that we have  $\inf_{\widehat{\ell} \in (\mathcal{W}_{\leq K}^{d+1})^m, |\widehat{\ell}|=1} E[|M_T^{\widehat{\ell}}|] = 0$ . Then there exists an  $\widehat{\ell}^*$  with  $|\widehat{\ell}^*| = 1$  and  $|M_T^{\widehat{\ell}^*}| = 0$  almost surely. But notice that  $M^{\widehat{\ell}^*}$  is a true martingale with quadratic variation given by  $[M^{\widehat{\ell}^*}]_T = \sum_{i=1}^m \int_0^T \langle \mathbf{X}_{0,s}^{\leq K}, \widehat{\ell}^{\star,i} \rangle^2 ds$ . Since in particular  $M_t^{\widehat{\ell}^*} = 0$  almost surely for all  $t$ , the same is true for the quadratic variation, and hence we have for each  $i = 1, \dots, m$  that  $\int_0^T \langle \mathbf{X}_{0,s}^{\leq K}, \widehat{\ell}^{\star,i} \rangle^2 ds = 0$  almost surely. This implies that for all  $i = 1, \dots, m$ ,

$$\langle \mathbf{X}_{0,s}^{\leq K}, \widehat{\ell}^{\star,i} \rangle = 0 \quad \text{for almost every } s \in [0, T], \text{ almost surely.}$$

But this is only possible if  $\widehat{\ell}^* = 0$ , contradicting the fact that  $|\widehat{\ell}^*| = 1$ . Hence the infimum in (A.5) is positive and we conclude that the function

$$\ell \mapsto E\left[\max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell)\right] \longrightarrow \infty \quad \text{as } |\ell| \rightarrow \infty,$$

which implies the existence of the minimiser.  $\square$

Finally, in order to prove Proposition 3.10, we quickly introduce the general idea of sample average approximation (SAA), for which we refer to Shapiro [38, Chap. 6] for details. Assume  $\mathcal{X}$  is a closed and convex subset of  $\mathbb{R}^N$  and  $\xi$  a random vector valued in  $\mathbb{R}^d$  for some  $d, N \in \mathbb{N}$ , and  $F$  is some function  $F : \mathbb{R}^N \times \mathbb{R}^d \rightarrow \mathbb{R}$ . We are interested in approximating the stochastic programming problem

$$y_0 = \min_{x \in \mathcal{X}} E[F(x, \xi)]. \quad (\text{A.6})$$

To that end, we define the sample average function  $F^M(x) = \frac{1}{M} \sum_{j=1}^M F(x, \xi^j)$ , where  $\xi^j, j = 1, \dots, M$ , are i.i.d. samples of the random vector  $\xi$ . The sample aver-

age approximation of  $y_0$  is then given by

$$y_0^M = \min_{x \in \mathcal{X}} F^M(x). \quad (\text{A.7})$$

The following result provides sufficient conditions for the convergence  $y_0^M \rightarrow y_0$  as  $M \rightarrow \infty$ , and a more general version can be found in [38, Theorem 4].

**Theorem A.2** *Suppose that*

- (1)  $F$  is measurable and  $x \mapsto F(x, \eta)$  is lower semicontinuous for all  $\eta \in \mathbb{R}^d$ ;
- (2)  $x \mapsto F(x, \eta)$  is convex for almost every  $\eta \in \mathbb{R}^d$ ;
- (3)  $\mathcal{X}$  is closed and convex;
- (4)  $f(x) := E[F(x, \xi)]$  is lower semicontinuous and  $f(x) < \infty$  for all  $x \in \mathcal{X}$ ;
- (5) the set  $S$  of solutions to (A.6) is non-empty and bounded.

Then  $y_0^M \rightarrow y_0$  as  $M \rightarrow \infty$ .

Similarly as in the proof of Proposition 3.4, for a fixed truncation level  $K \in \mathbb{N}$ , we denote by  $D$  the number of components of the truncated signature, which is given by  $D = \sum_{k=0}^K (d+1)^k$ . Now every  $\ell \in \mathcal{W}_{\leq K}^{d+1}$  is of the form  $\ell = \sum_{i=1}^D \beta_i w_i$  for some  $\beta \in \mathbb{R}^D$ , where  $w_1, \dots, w_D$  are all words of length at most  $K$ . In particular, for every  $\ell \in (\mathcal{W}_{\leq K}^{d+1})^m$ , there is a  $\beta \in \mathbb{R}^{m \times D}$  and our usual notation reads

$$M_t^\ell = \int_0^t \langle \mathbf{X}_{0,s}^{\leq K}, \ell \rangle^\top dW_s = \sum_{j=1}^D (\beta^j)^\top \int_0^t \langle \mathbf{X}_{0,s}^{\leq K}, w_j \rangle dW_s,$$

so that instead of minimising over  $\ell$ , we can minimise over  $\beta$ . Let us now formulate the minimisation problem in Proposition 3.10 in the language of Theorem A.2 as

$$\begin{aligned} E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell) \right] &= E \left[ \max_{0 \leq n \leq N} \left( Z_{t_n} - \sum_{i=1}^m \sum_{j=1}^D \beta^{ij} \int_0^{t_n} \langle \mathbf{X}_{0,s}^{\leq K}, w_j \rangle dW_s^i \right) \right] \\ &= E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - \boldsymbol{\beta}^\top \mathbf{M}_{t_n}) \right], \end{aligned}$$

where we identify  $\beta \in \mathbb{R}^{m \times D} \cong \mathbb{R}^{mD}$  and  $\mathbf{M}_t$  is the  $(mD)$ -dimensional vector

$$\left( \int_0^t \langle \mathbf{X}_{0,s}^{\leq K}, w_j \rangle dW_s^i : 1 \leq i \leq m, 1 \leq j \leq D \right).$$

Finally, defining the random vector

$$\xi := (Z_{t_0}, \mathbf{M}_{t_0}^\top, \dots, Z_{t_N}, \mathbf{M}_{t_N}^\top) \in \mathbb{R}^{(D+1)(N+1)m},$$

we notice that

$$\inf_{\ell \in (\mathcal{W}_{\leq K}^{d+1})^m} E \left[ \max_{0 \leq n \leq N} (Z_{t_n} - M_{t_n}^\ell) \right] = \min_{x \in \mathbb{R}^{mD}} E[F(x, \xi)], \quad (\text{A.8})$$



where

$$F(x, \eta) := \max_{0 \leq n \leq N} \left( \eta_{n(mD+1)} - x^\top \begin{pmatrix} \eta_{n(mD+1)+1} \\ \vdots \\ \eta_{n(mD+1)+1+mD} \end{pmatrix} \right). \quad (\text{A.9})$$

Therefore, setting  $\mathcal{X} = \mathbb{R}^{Dm}$  and  $d = (D+1)(N+1)m$ , the minimisation problem for  $y_0^{K,N,J,M}$  in Proposition 3.10 can simply be written in the SAA formulation (A.7), where additionally the stochastic integrals in  $\mathbf{M}$  are replaced by the discretised versions  $\mathbf{M}^J$ .

**Proof of Proposition 3.10** First, it is possible to rewrite the proof of Lemma A.1 for  $F^M$  instead of the expectation, to show that there exists a minimiser  $\ell^*$  to (3.10). Moreover, denote by  $y_0^{K,N,J}$  the minimisation problem (A.8), where we replace  $M^\ell$  by the discretised version  $M^{\ell,J}$  described in Sect. 3.3.1. By the same reasoning as in Proposition 3.8, we can show the limit  $\lim_{K \rightarrow \infty} \lim_{J \rightarrow \infty} y_0^{K,N,J} = y_0^N$ . Now for fixed  $K, J, N$ , we are left with showing almost sure convergence of  $y_0^{K,N,J,M}$  to  $y_0^{K,N,J}$ . But this can be deduced from Theorem A.2 if we can show that (1)–(5) hold true for our  $F$  in (A.9). Clearly,  $F$  is measurable and it is easy to see that  $x \mapsto F(x, \eta)$  is continuous and convex; thus (1) and (2) readily follow. Moreover, (3) holds true, and in order to show (4), set  $f(x) = E[F(x, \xi)]$  and notice that for  $x_1, x_2 \in \mathcal{X}$ , we have

$$|f(x_1) - f(x_2)| \leq E \left[ \max_{0 \leq n \leq N} ((x_1 - x_2)^\top \mathbf{M}_{t_n}^J) \right] \leq |x_1 - x_2| E \left[ \max_{0 \leq n \leq N} |\mathbf{M}_{t_n}^J| \right],$$

where we simply applied the triangle and Cauchy–Schwarz inequalities. Since we have  $M^\ell \in L^2$  for all  $\ell$ , an application of Doob’s inequality shows for the right-hand side that  $E[\max_{0 \leq n \leq N} |\mathbf{M}_{t_n}^J|] < \infty$ , and therefore (4) follows. Finally, non-emptiness of  $S$  follows from Lemma A.1, and the proof of the latter reveals that  $E[F(x, \xi)] \rightarrow \infty$  as  $|x| \rightarrow \infty$ . Thus  $S$  must be bounded, which finishes the proof.  $\square$

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## Declarations

**Competing interests** The authors declare no competing interests.

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