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Aggregating credences into beliefs: agenda conditions for impossibility results

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Abstract

Hybrid belief aggregation addresses aggregation of individual probabilistic beliefs into collective binary beliefs. In line with the development of judgment aggregation theory, our research delves into the identification of precise agenda conditions associated with some key impossibility theorems in the context of hybrid belief aggregation. We determine the necessary and sufficient level of logical interconnection between the propositions in an agenda for some key impossibilities to arise. Specifically, we prove three characterization theorems about hybrid belief aggregation: (i) Precisely the path-connected and pair-negatable agendas lead to the ‘oligarchy result’—only oligarchic rules satisfy universal domain, proposition-wise independence, respect for unanimity, and deductive closure of collective beliefs. (ii) Precisely the negation-connected agendas lead to the ‘triviality result’—only unanimity rules satisfy those conditions as well as anonymity. (iii) Precisely the blocked agendas lead to the ‘non-existence result’—no rules satisfy those conditions as well as completeness and consistency of collective beliefs. Furthermore, we compare these novel findings with existing agenda-theoretic characterization theorems in the domains of judgment aggregation and belief binarization.

1 Introduction

The question of how to rationally aggregate individual beliefs into collective beliefs is important and ubiquitous in our society. In this regard, there has been abundant literature on collective decision theory, judgment aggregation, and probabilistic opinion pooling studies. One of the essential features of belief is that there are different types of beliefs. For example, some beliefs may be represented by traditional “logical” languages—she believes that it is raining outside—while other types of beliefs might be modeled by “probability functions”—she believes with 90 percent certainty

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that it is raining outside. Logical languages are similar to our natural languages and are therefore efficient for communicating with human agents, despite the fact that they sometimes suffer from significant information reduction, as in the case of the Lottery paradox. In contrast, probabilistic beliefs hold a fair amount of information to deal with uncertain environments, although people usually do not reach that level of precision. Considering these pros and cons of different types of beliefs, it is not surprising that different types of beliefs may be required at different stages of belief aggregation procedures depending on situations. If objective chances of propositions in question can be given, it is epistemically preferable to report individual opinions in terms of degrees of belief. If the conclusion of an epistemic collective decision guides action (e.g., a jury verdict), it is practically better to report the collective opinion by means of plain logic. Therefore, rational belief aggregation should be able to deal with different types of beliefs. That being said, it is somewhat disappointing that the collective decision problem regarding different types of beliefs has been rarely addressed and studied. Judgment aggregation theory deals with only binary beliefs, probabilistic opinion pooling deals with only probabilistic beliefs, and belief binarization only considers individual beliefs. One important topic in aggregating one type of belief into a different type of belief is aggregating probabilistic beliefs into collective binary beliefs (Ivanovska and Slavkovik 2019; Wang 2024, 2023). We will call this subject matter “hybrid belief aggregation”, or hybrid aggregation for short.¹ We can observe these belief aggregation problems in expert panels, the scientific community, and political parties, whenever individuals’ opinions can be encoded probabilistically, and the group’s beliefs should be more decisive.

Similar to the development of judgment aggregation theory (Dokow and Holzman 2010; Nehring and Puppe 2010), formulating axiomatic requirements, proving impossibility theorems, and identifying exact agenda conditions of impossibility theorems are natural and important research topics in hybrid belief aggregation. Building on our previous research on impossibility theorems, this paper uses an agenda-theoretic approach to determine which level of logical interconnection between the propositions in an agenda is necessary and sufficient for the impossibility theorems to arise. Indeed, our previous paper assumed the agenda to be an algebra, which is the most typical when dealing with probabilistic beliefs. However, in practice, the agenda being an algebra might be quite demanding because we might not be interested in, for example, the conjunction of two propositions when making a collective decision on each of the two propositions. Besides the literature on judgment aggregation, agenda-theoretic approaches can be found in other fields as well. In probabilistic opinion pooling, general agendas were investigated to characterize linear pooling (Dietrich and List 2017a, b). In the belief binarization problem, general agendas were studied to characterize impossibility theorems (Dietrich and List 2018, 2021).

In this study, we demonstrate that (1) precisely the path-connected and pair-negatable agendas lead to the ‘oligarchy result’—only oligarchic rules satisfy universal domain, proposition-wise independence, respect for unanimity, and deductive closure of collective beliefs; (2) precisely the negation-connected agendas lead to the ‘trivial-

¹ In Wang (2024), this was called “binarizing belief aggregation”. We thank the first reviewer for suggesting changing the name in this paper.

ity result’—only unanimity rules satisfy those conditions as well as anonymity; and (3) precisely the blocked agendas lead to the ‘non-existence result’—no rules satisfy those conditions as well as completeness and consistency of collective beliefs. Moreover, we compare these novel findings with existing agenda-theoretic characterization theorems in judgment aggregation and belief binarization.

The remainder of this paper is organized as follows. Section 2 illustrates our setting and reviews the impossibility results proved in the previous paper. Section 3 presents the agenda condition for the oligarchy result. Section 4 presents the agenda condition for the triviality result. Section 5 presents the agenda condition for the non-existence result. Section 6 concludes with comparing our results with certain results in the judgment aggregation and belief binarization literature and suggesting some further research.

2 Hybrid belief aggregation and the impossibility results

We begin by introducing notations and definitions we will use throughout this paper. Let W be a finite non-empty set of possible worlds. An *agenda* \mathcal{A} is a non-empty set of propositions, i.e., subsets of W , that is closed under complement. We denote by $N := \{1, \dots, n\}$ ($n \geq 2$) the set of individuals. A *probabilistic belief* is a function on \mathcal{A} extendable to a probability function on some algebra that includes \mathcal{A} . Each individual $i \in N$ holds a probabilistic belief P_i , and $\vec{P} := (P_1, \dots, P_n) = (P_i)_{i \in N}$ stands for a profile of n individuals’ probabilistic beliefs. Hybrid belief aggregation deals with individuals’ probabilistic beliefs and the group’s binary beliefs. Binary beliefs are represented by a function $Bel : \mathcal{A} \rightarrow \{0, 1\}$. Sometimes, we abuse the notation and denote by Bel the *belief set* $\{A \in \mathcal{A} \mid Bel(A) = 1\}$, and $Bel A$ is a shorthand for $A \in Bel$ or $Bel(A) = 1$. A hybrid belief aggregator, or for short hybrid aggregator (HA) F is a function that takes a profile \vec{P} of n probabilistic beliefs in a given domain and returns a binary belief $F(\vec{P})$.

Now, let us introduce some axiomatic requirements on HA, which will be needed to formulate our impossibility results. First, we need the following rationality requirements on the domain and codomain of a HA.

- Universal Domain (UD): the domain of F is the set of all profiles \vec{P} of n probabilistic beliefs
- Collective Deductive Closure (CDC)/Consistency (CCS)/Completeness (CCP): for all \vec{P} in the domain, the resulting collective beliefs $F(\vec{P})$ is deductively closed/-consistent/complete, respectively.

Note that a binary belief Bel is deductively closed iff for all $A \in \mathcal{A}$ if $Bel \models A$, then $Bel A$, where for $\mathcal{Y} \subseteq \mathcal{A}$, $\mathcal{Y} \models A$ iff $\bigcap \mathcal{Y} \subseteq A$. Moreover, Bel is consistent iff $Bel \neq \emptyset$, and Bel is complete iff $Bel A$ or $Bel \bar{A}$ for all $A \in \mathcal{A}$, where \bar{A} is the complement of A . Second, we enlist different axiomatic requirements on HAs themselves.

- Certainty Preservation (CP)/Zero Preservation (ZP): for all $A \in \mathcal{A}$, if $\vec{P}(A) := (P_1(A), \dots, P_n(A)) = (1, \dots, 1)/(0, \dots, 0)$, then $F(\vec{P})(A) = 1/0$, respectively, for all \vec{P} in the domain of F .

- Anonymity (AN): $F((P_{\pi(i)})_{i \in N}) = F((P_i)_{i \in N})$ for all \vec{P} in the domain of F and all permutation π on N .
- Independence (IND): for all $A \in \mathcal{A}$, there exists a function $G_A : [0, 1]^n \rightarrow \{0, 1\}$ such that $F(\vec{P})(A) = G_A(\vec{P}(A))$ for all \vec{P} in the domain of F .
- Systematicity (SYS): there exists a function $G : [0, 1]^n \rightarrow \{0, 1\}$ such that $F(\vec{P})(A) = G(\vec{P}(A))$ for all $A \in \mathcal{A}$ and for all \vec{P} in the domain of F .

Moreover, we will use the following terminology.

- A HA F is called an oligarchy if there is a non-empty subset M of N such that for all \vec{P} in the domain of F and for all $A \in \mathcal{A}$

$$F(\vec{P})(A) = \begin{cases} 1 & \text{if } P_i(A) = 1 \text{ for all } i \in M \\ 0 & \text{otherwise} \end{cases}$$

If $M = N$, we call it the unanimity rule, which can be seen as a trivial rule.

- An agenda \mathcal{A} is called a non-trivial algebra if \mathcal{A} is an algebra with at least three contingent propositions, where a proposition $A \in \mathcal{A}$ is called contingent if $A \neq \emptyset, W$.

Our previous paper (Wang 2024) proved the following theorems. In this study, we aim to relax the assumption that the agenda \mathcal{A} is a non-trivial algebra.

1. (The Oligarchy Result) If the agenda is a non-trivial algebra, the only HAs satisfying UD, CP, ZP, IND, and CDC are the oligarchies
2. (The Triviality Result) If the agenda is a non-trivial algebra, the only HAs satisfying UD, CP, ZP, IND, CDC, and AN are the unanimity rule.
3. (The Non-existence Result) If the agenda is a non-trivial algebra, there is no HA satisfying UD, CP, IND, CCP, and CCS.

3 The agenda condition for the oligarchy result

How do we capture the type of complexity of the logical interconnection in an agenda? In the judgment aggregation literature, conditional entailment has been one of the preliminary concepts to do it. The notion of entailment only captures the logical interdependence between any antecedents and their consequents. However, two propositions that seem logically unrelated at first glance might become logically related when some other propositions are combined. Conditional entailment can include this kind of indirect entailment. Furthermore, conditional entailment relations can be thought of as a bridge from one proposition to another, and we can start from one proposition and reach other propositions via several conditional entailment relations in a row. The notion of a path from one proposition to another is devised to capture this kind of even more indirect relations.

Definition 1 (Conditional Entailment) Let $A, B \in \mathcal{A}$, $\mathcal{Y} \subseteq \mathcal{A}$, and \mathcal{Y} be consistent with A and \overline{B} . (That is, $\mathcal{Y} \cup \{A\} \neq \emptyset$ and $\mathcal{Y} \cup \{\overline{B}\} \neq \emptyset$.) A entails B conditional on \mathcal{Y} ($A \models_{\mathcal{Y}}^* B$) if $\{A\} \cup \mathcal{Y} \models B$. If \mathcal{Y} is a singleton set, say $\{C\}$, we write \models_C^* . Moreover, A conditionally entails B ($A \models^* B$) if there is a subset $\mathcal{Y} \subseteq \mathcal{A}$ such that A entails B

conditional on \mathcal{Y} . We denote by \models^{**} the transitive and reflexive closure of \models^* . We read $A \models^{**} B$ as there is a path from A to B .

Note that \mathcal{Y} can be an empty set where $A \models_{\mathcal{Y}}^* B$ becomes $A \models B$ as far as $A \neq \emptyset$ and $\overline{B} \neq \emptyset$ ($B \neq W$). So conditional entailment includes direct entailment as far as propositions are contingent.² Notice that if $A \models_{\mathcal{Y}}^* B$, then $\overline{B} \models_{\mathcal{Y}}^* \overline{A}$, and thus if $A \models^{**} B$, then $\overline{B} \models^{**} \overline{A}$. The following examples illustrate the above notions, where $\pm\{A_1, \dots, A_k\}$ stands for $\{A_1, \dots, A_k, \overline{A_1}, \dots, \overline{A_k}\}$.

Example 1 Let $A, B \subseteq W$ be logically independent, i.e., $A \setminus B, B \setminus A, A \cap B, \overline{A \cup B} \neq \emptyset$. Consider the following agendas:

$$\mathcal{A}_1 = \pm\{A, B\};$$

$$\mathcal{A}_2 = \pm\{A, B, A \cap B\};$$

$$\mathcal{A}_3 = \pm\{A, B, A \leftrightarrow B\}, \text{ where } A \leftrightarrow B := (A \cap B) \cup (\overline{A} \cap \overline{B});$$

$$\mathcal{A}_4 = \pm\{A, B, A \cap B, A \cup B\}.$$

It can be easily seen that \mathcal{A}_1 does not include any path. In the case of \mathcal{A}_2 , for some propositions, we have a conditional entailment relation such as $A \models_{A \cap B}^* \overline{B}$ whereas for some other propositions, say A and \overline{A} , we have neither a conditional entailment relation nor a path. In the case of \mathcal{A}_3 , for some propositions we have a conditional entailment relation such as $A \models_{(A \cap B) \cup (\overline{A} \cup \overline{B})}^* B$ and $A \models_{(A - B) \cup (B - A)}^* \overline{B}$, which eventually leads to the fact that there is a path between every two propositions. In the case of \mathcal{A}_4 , the situation is similar to the case of \mathcal{A}_3 .

The following two agenda conditions, namely path-connectedness and pair-negatability, have been studied a lot as they can characterize the most famous impossibility agendas in judgment aggregation. Let us first define path-connectedness.

Definition 2 (Path-connected Agenda) An agenda \mathcal{A} is path-connected (PC) if for all contingent propositions $A, B \in \mathcal{A}$, $A \models^{**} B$.

Path-connectedness means that every two contingent propositions are connected by a path, i.e., a chain of conditional entailment relations. Next, to define pair-negatability, we need to introduce one more basic notion. A subset $\mathcal{Y} \subseteq \mathcal{A}$ is called minimally inconsistent when it is inconsistent and every proper subset of it is consistent. Let us define a pair-negatable agenda.

Definition 3 (Pair-negatable Agenda) An agenda \mathcal{A} is pair-negatable (PN) if there is a minimally inconsistent set $\mathcal{Y} \subseteq \mathcal{A}$ such that $\mathcal{Y}_{-\mathcal{Z}}$ is consistent for some subset $\mathcal{Z} \subseteq \mathcal{Y}$ with $|\mathcal{Z}| = 2$, where for any $\mathcal{Y}, \mathcal{Z} \subseteq \mathcal{A}$ with $\mathcal{Z} \subseteq \mathcal{Y}$, $\mathcal{Y}_{-\mathcal{Z}}$ is defined by $(\mathcal{Y} \setminus \mathcal{Z}) \cup \{\overline{A} \mid A \in \mathcal{Z}\}$.

So pair-negatability says that a minimally inconsistent subset of the agenda can be made consistent by negating its two elements. Note that $\mathcal{Z} = \mathcal{Y}$ is allowed where $\mathcal{Y}_{-\mathcal{Z}} = \{\overline{A} \mid A \in \mathcal{Z}\}$. The definition of pair-negatable agenda is a little bit involved, but it is well-known that an agenda is pair-negatable unless the propositions in the

² Note that $\emptyset \not\models^* A$ and $A \not\models^* W$ for all $A \in \mathcal{A}$, and hence $\emptyset \not\models^{**} \emptyset$ and $W \not\models^{**} W$. But we have $\emptyset \models^{**} \emptyset$ and $W \models^{**} W$ because \models^{**} is reflexive, which will not be needed for our theorems, though.

agenda are composed only with negation and biconditional from some logically independent propositions. It is worth mentioning that pair-negatability is equivalent to even-negatability in some earlier papers in judgment aggregation, which requires that $|\mathcal{Z}|$ is even instead of $|\mathcal{Z}| = 2$ in the above definition (See Remark 1 in Dietrich and List (2013) or Lemma 7 in the appendix of this paper).³ The following examples illustrate the above notions.

Example 2 (Continued) \mathcal{A}_1 is neither path-connected nor pair-negatable; \mathcal{A}_2 is not path-connected but pair-negatable. Indeed, $\{A, B, \overline{A \cap B}\}$ is the minimally inconsistent set, but $\{\overline{A}, \overline{B}, \overline{A \cap B}\}$ is consistent; \mathcal{A}_3 is path-connected but not pair-negatable; \mathcal{A}_4 is path-connected and pair-negatable.

Note that as the following lemma shows, these two conditions are weaker than the agenda being a non-trivial algebra, which is the assumption on the agenda in Wang (2024).

Lemma 1 *Every non-trivial algebra is path-connected and pair-negatable.*

Proof See the appendix. \square

From now on, we add one more assumption on \mathcal{A} that $\emptyset \notin \mathcal{A}$ (and hence $W \notin \mathcal{A}$).⁴ Thus, our agenda \mathcal{A} is a complement-closed finite non-empty set of some contingent subsets of the underlying set W . The following lemma shows that path-connectedness is sufficient to obtain what is called the contagion lemma.

Lemma 2 (Path-connectedness and Contagion Lemma) *Let \mathcal{A} be path-connected. If a HA F with UD satisfies CDC, CP, and IND, then it satisfies SYS.*

Proof See the appendix. \square

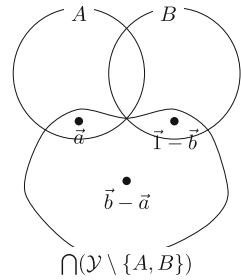
This lemma parallels a characterization theorem in generalized opinion pooling (OP) of Dietrich and List (2017a): path-connectedness characterizes that if generalized OP satisfies CP and IND, then it satisfies SYS. In our lemma as well, its converse—if \mathcal{A} is not path-connected, then there is a HA F on \mathcal{A} satisfying CDC, CP, and IND but not SYS—also holds. The counterexample will be indicated in the proof of Theorem 1.

The following definition and lemma will be needed to prove our succeeding main theorem.

³ Dokow and Holzman (2010) introduced an algebraic characterization of even-negatability, which helps us decide whether a given agenda is even-negatable: For a given agenda \mathcal{A} , the set of all possible binary valuations on the agenda can be represented by a set of 0/1-vectors in $\{0, 1\}^{\frac{|\mathcal{A}|}{2}}$, which can be seen as a vector space over the field $\{0, 1\}$ on which componentwise addition is modulo 2. According to Proposition 2.1 and 4.2 in Dokow and Holzman (2010), an agenda \mathcal{A} is not even-negatable iff the set of all possible binary valuations on the agenda is an affine subspace of $\{0, 1\}^{\frac{|\mathcal{A}|}{2}}$ iff it is closed under addition of odd-tuples. For example, consider the agenda \mathcal{A}_2 in our example. For $(A, B, A \cap B)$, all possible valuations are $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 0)$. Since $(0, 1, 0) + (1, 0, 0) + (0, 0, 0) = (1, 1, 0)$, and $(1, 1, 0)$ is not a possible valuation, \mathcal{A}_2 is even-negatable. In contrast, for the agenda \mathcal{A}_3 with $(A, B, (A \cap B) \cup \overline{(A \cap B)})$, the set of possible valuations on it consists of $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. We can easily see that it is closed under addition of odd-tuples, thus \mathcal{A}_3 is not even-negatable.

⁴ In the following, especially in Theorem 2 and Theorem 3, we will use some results of Nehring and Puppe (2010), where the agenda consists of contingent propositions. To describe our proof more simply, we adopt that assumption.

Fig. 1 A profile of probabilities for a pair-negatable agenda (each point represents an element, to which we can assign a profile of probabilities.)



Definition 4 (Non-simple Agenda) An agenda \mathcal{A} is non-simple (NS) if there is a minimally inconsistent subset $\mathcal{Y} \subseteq \mathcal{A}$ with $|\mathcal{Y}| \geq 3$.

Non-simple agendas can be used as a criterion for determining whether a given agenda has minimal complexity. The following lemma shows that a path-connected agenda already has a fairly complex structure.

Lemma 3 If an agenda \mathcal{A} is path-connected, then it is non-simple.

Proof See Claim 3.2 in Theorem 2.2 in Dokow and Holzman (2010) or the appendix of this paper. \square

Example 3 (Continued) The agendas \mathcal{A}_1 is simple while the agendas \mathcal{A}_2 , \mathcal{A}_3 , and \mathcal{A}_4 are non-simple. In \mathcal{A}_2 , $\{A, B, \overline{A \cap B}\}$ is the minimally inconsistent set including at least three elements.

Now let us prove our first main result that the agenda being path-connected and pair-negatable is the sufficient and necessary condition for the oligarchy result.

Theorem 1 (Agenda Condition for the Oligarchy Result)

- (1) If an agenda \mathcal{A} is path-connected and pair-negatable, then the only HAs on \mathcal{A} satisfying UD, ZP, CP, IND, and CDC are the oligarchies.
- (2) If \mathcal{A} is not path-connected, then there is a non-oligarchic HA on \mathcal{A} satisfying UD, CDC, ZP, CP, IND, and in addition CCS.
- (3) If \mathcal{A} is not pair-negatable, then there is a non-oligarchic HA on \mathcal{A} satisfying UD, CDC, ZP, CP, IND, and in addition CCS.

Proof (1) First of all, by Lemma 2, path-connectedness gives SYS and we can let $F(\vec{P})(A) = G(\vec{P}(A))$. First, we will prove the following two facts about G :

(Fact 1) if $\vec{a} \leq \vec{b}$ and if $G(\vec{a}) = 1$, then $G(\vec{b}) = 1$


(Fact 2) if $\vec{a} + \vec{b} - \vec{1} \geq \vec{0}$ and if $G(\vec{a}) = 1$ and $G(\vec{b}) = 1$, then $G(\vec{a} + \vec{b} - \vec{1}) = 1$

It suffices to prove (Fact 1') and, by Lemma 3, (Fact 2'):

(Fact 1') If \mathcal{A} is pair-negatable, then (Fact 1) holds.

Assume that $\mathcal{Y}(\subseteq \mathcal{A})$ is minimally inconsistent and $\mathcal{Y}_{-\{A, B\}}$ is consistent. Let $\vec{a} \leq \vec{b}$. Since $\cap \mathcal{Y}_{-\{B\}}$, $\cap \mathcal{Y}_{-\{A\}}$, and $\cap \mathcal{Y}_{-\{A, B\}}$ are not empty, we can assign to them profiles of probabilities \vec{b} , $\vec{1} - \vec{a}$, and $\vec{b} - \vec{a}$, respectively. Thus, there exists a profile \vec{P} of probabilistic beliefs such that $\vec{P}(A) = \vec{a}$, $\vec{P}(B) = \vec{1} - \vec{b}$, and $\vec{P}(Y) = \vec{1}$ for all

The diagram shows a large circle representing the set Z . Inside, three overlapping circles represent the sets A , B , and C . The regions of Z are labeled as follows: $Z \cap A$ (top left), $Z \cap B$ (top right), and $Z \cap C$ (bottom). The intersection of Z and A is labeled $a + \bar{b}$. The intersection of Z and B is labeled $\bar{a} + b$. The intersection of Z and C is labeled $\bar{a} + \bar{b}$. The intersection of all three sets A , B , and C is labeled $\bar{a} + b + \bar{b}$. The intersection of Z and A and B is labeled $a + b$. The intersection of Z and A and C is labeled $a + \bar{b}$. The intersection of Z and B and C is labeled $\bar{a} + b$. The intersection of Z and A , B , and C is labeled $a + b + \bar{b}$.

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as follows. By (Fact 1), we can substitute all other components a_l (i.e., $l \neq i$) that is not 1 with 1 and so we have $G((1, \dots, 1, a_i, 1, \dots, 1)) = 1$. This process enables us to focus on i -th component of vectors when we apply Statement (1) because $1+1-1 = 1$. Now apply (Fact 1) and Statement (1). (i) If $a_i \leq 0.5$, we have $G(1, \dots, 1, 0.5, 1, \dots, 1) = 1$ by (Fact 1) and $G(1, \dots, 1, 0, 1, \dots, 1) = 1$ by Statement (1). (ii) Now let $a_i > 0.5$. After applying Statement (1) to $G((1, \dots, 1, a_i, 1, \dots, 1)) = 1$ k times, we have $G((1, \dots, 1, a_i^{(k)}, 1, \dots, 1)) = 1$ where $a_i^{(k)} = 1 - 2^k(1 - a_i)$ and there must be k such that $a_i^{(k)} \leq 0.5$. Then, we can apply (i) and obtain $G(1, \dots, 1, 0, 1, \dots, 1) = 1$. This holds for arbitrary i . So we have $G((0, 1, \dots, 1)) = G((1, 0, 1, \dots, 1)) = \dots = G((1, \dots, 1, 0))$ by [Step 1].

[STEP 2] We show, by induction, $G((0, \dots, 0)) = 1$, which contradicts ZP.

Let \vec{a}_k be a vector $(0, \dots, 0, 1, \dots, 1)$ where the first k components are zero and one elsewhere. For $k = 1$, we have $G(\vec{a}_1) = 1$. Assume $G(\vec{a}_k) = 1$. Since $G((1, \dots, 1, 0, 1, \dots, 1)) = 1$ where all components of the input are one except for the $(k+1)$ -th one, which is zero, we have, by (Fact 2), $G(\vec{a}_k + (1, \dots, 1, 0, 1, \dots, 1) - \vec{1}) = G(\vec{a}_{k+1}) = 1$.

(ii) $a_i = 1$ for all $i \in M$ iff $G(\vec{a}) = 1$.

(\leftarrow) It is obvious by the construction of M . (\rightarrow) By (Fact 1), it is enough to show that $G((\mathbb{1}_M(i))_{i \in N}) = 1$ where $\mathbb{1}_M(i) = 1$ if $i \in M$, otherwise $\mathbb{1}_M(i) = 0$. For any $j \notin M$, there is \vec{a} such that $G(\vec{a}) = 1$ and $a_j \neq 1$, by definition of M . Similar to [STEP 2], we can prove the following:

$$G(\vec{a}[a_j \mapsto 0, a_l \mapsto 1 \text{ for all } l \neq j]) = 1 \quad (2)$$

Now enumerate individuals who are not in M , like $j_1, j_2, \dots, j_{|N|-|M|}$ and let \vec{a}_k be a profile where $a_{j_1} = 0, \dots, a_{j_k} = 0$ and other components are all 1. For $k = 1$, we have $G(\vec{a}_1) = 1$. Assume $G(\vec{a}_k) = 1$. Since by Equation (2) we have $G(1, \dots, 1, 0, 1, \dots, 1) = 1$ where 0 is j_{k+1} -th component, by (Fact 2), we have $G(\vec{a}_{k+1}) = 1$. Therefore, we have $G(\vec{a}_{|N|-|M|}) = G((\mathbb{1}_M(i))_{i \in N}) = 1$.

(2) Suppose that \mathcal{A} is not path-connected. Then there are propositions, say P, Q , such that there is no path from P to Q . Now we can partition \mathcal{A} into two subsets $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{A}$ such that $\mathcal{X}_1 = \{A \in \mathcal{A} \mid P \models^{**} A\}$ and $\mathcal{X}_2 = \{B \in \mathcal{A} \mid P \not\models^{**} B\}$. Note that there is no path from a proposition in \mathcal{X}_1 to a proposition in \mathcal{X}_2 . Let us define F as follows:

(i) For every proposition $A \in \mathcal{X}_1$, $F(\vec{P})(A) = 1$ iff $P_1(A) = 1$.

(ii) For every proposition $B \in \mathcal{X}_2$, $F(\vec{P})(B) = 1$ iff $\vec{P}(B) = 1$. (See Fig. 3.)

It is easy to check that the above F satisfies UD, ZP, CP, IND, and non-oligarchy. Let $P_1^{-1}(1)$ denote the set $\{C \in \mathcal{A} \mid P_1(C) = 1\}$. Since $F(\vec{P}) \subseteq P_1^{-1}(1)$ and $P_1^{-1}(1)$ is consistent, it follows that F satisfies CCS. Now let us prove that F satisfies CDC. Suppose towards contradiction that there is a proposition, say C , such that $F(\vec{P})(C) = 0$ but $F(\vec{P}) \models C$. Then there is a minimal subset $\mathcal{T} \subseteq F(\vec{P})$ such that $\bigcap \mathcal{T} \subseteq C$. The remaining proof will be divided into three steps:

(i) there is a proposition, say D , in $\mathcal{T} \cap \mathcal{X}_1$,

(ii) $C \in \mathcal{X}_2$,

(iii) there is a path from D to C ,

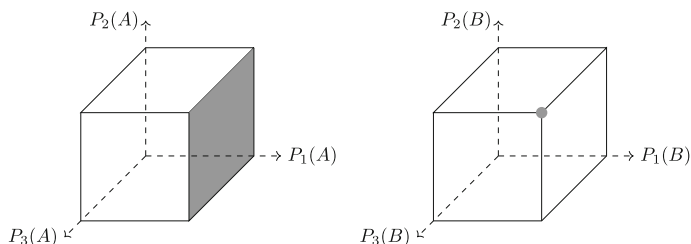
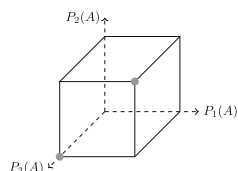


Fig. 3 This represents the HA in (2), provided $N = \{1, 2, 3\}$. Grey points represent where 1 is assigned. The left figure is for any $A \in \mathcal{X}_1$ and the right one for any $B \in \mathcal{X}_2$

Fig. 4 This illustrates the HA in (3) provided $N = \{1, 2, 3\}$. Grey points represent where 1 is assigned



which contradicts our assumption that there is no path from \mathcal{X}_1 to \mathcal{X}_2 .

We first prove (i). The set $\mathcal{T} \cap \mathcal{X}_1$ is not empty since otherwise \mathcal{T} would be included in $F(\vec{P}) \cap \mathcal{X}_2 (= \{B \in \mathcal{X}_2 \mid \vec{P}(B) = \vec{1}\})$, which is actually deductively closed, and consequently $F(\vec{P})(C) = 1$, a contradiction.

We now turn to (ii). It suffices to show that $P_1(C) = 1$ for it would force $C \notin \mathcal{X}_1$, since otherwise $F(\vec{P})(C) = 1$. As $P_1^{-1}(1)$ includes $F(\vec{P})$, we have $P_1^{-1}(1) \models C$. As $P_1^{-1}(1)$ is deductively closed we obtain $C \in P_1^{-1}(1)$.

It remains to show (iii). Since $\mathcal{T} \cup \{\bar{C}\}$ is minimally inconsistent, we see that $\mathcal{T} \setminus \{D\}$ is consistent with D and \bar{C} . As $\mathcal{T} \models C$ we get $D \models_{\mathcal{T} \setminus \{D\}}^* C$, and hence $D \models^{**} C$.

(3) Suppose that \mathcal{A} is not pair-negatable, i.e., not even-negatable. Let us define F as follows:

For every $A \in \mathcal{A}$, $F(\vec{P})(A) = 1$ iff $\vec{P}(A) = \vec{1}$ or $\vec{P}(A) = (0, 0, 1, \dots, 1)$. (See Fig. 4.)

It is easily seen that F satisfies UD, ZP, CP, IND, and non-oligarchy. Note that the above F will fail to satisfy ZP if we drop the assumption that $|N| \geq 3$. We will denote by \mathcal{X} the set $\{A \in \mathcal{A} \mid \forall i \in N \setminus \{1, 2\} P_i(A) = 1\}$. Since $F(\vec{P}) \subseteq \mathcal{X}$ and \mathcal{X} is consistent, it follows that F satisfies CCS.

Now let us prove that F satisfies CDC. Suppose that, contrary to our claim, F does not satisfy CDC. We will prove that \mathcal{A} is even-negatable. The remaining proof will be divided into two steps:

(i) we will find a minimally inconsistent subset $\mathcal{Y} \subseteq \mathcal{A}$ that has a subset $\mathcal{Z} \subseteq \mathcal{Y}$ and an element $X \in \mathcal{Y}$ such that $\mathcal{Z} \neq \emptyset$, and $X \notin \mathcal{Z}$,

(ii) we will prove that both $\mathcal{Y}_{-\mathcal{Z}}$ and $\mathcal{Y}_{-(\mathcal{Z} \cup \{X\})}$ are consistent.

Then we would get the desired result since either \mathcal{Z} or $\mathcal{Z} \cup \{X\}$ must have even number elements.

We first construct \mathcal{Y} . Since F does not satisfy CDC, there is a proposition, say B , such that $F(\vec{P})(B) = 0$ but $F(\vec{P}) \models B$. From this we see that there is a minimal subset $\mathcal{T} \subseteq F(\vec{P})$ such that $\bigcap \mathcal{T} \subseteq B$. Set $\mathcal{Y} = \mathcal{T} \cup \{\bar{B}\}$, which is minimally inconsistent.

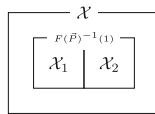


Fig. 5 $\mathcal{X} = \{A \in \mathcal{A} \mid \forall i \in N \setminus \{1, 2\} P_i(A) = 1\}$, $\mathcal{X}_1 = \{A \in \mathcal{A} \mid \vec{P}(A) = \vec{1}\}$, and $\mathcal{X}_2 := \{A \in \mathcal{A} \mid \vec{P}(A) = (0, 0, 1, \dots, 1)\}$

Second, let $\mathcal{Z} = \mathcal{T} \cap \mathcal{X}_2$ where $\mathcal{X}_2 := \{A \in \mathcal{A} \mid \vec{P}(A) = (0, 0, 1, \dots, 1)\}$. (See Fig. 5.) It is not empty set since otherwise \mathcal{T} would be included in the set $\{A \in \mathcal{A} \mid \vec{P}(A) = \vec{1}\} (=:\mathcal{X}_1)$ that is deductively closed, and hence includes B , a contradiction. Third, we set $X = \overline{B}$. As $\mathcal{Z} \cup \{B\}$ is consistent, we obtain $\overline{B} \notin \mathcal{Z}$.

We now turn to (ii). To prove it, we claim that $\forall i \in N \setminus \{1, 2\} P_i(B) = 1$. Since \mathcal{X} includes $F(\vec{P})$ we have $\mathcal{X} \models B$. As \mathcal{X} is deductively closed, we obtain the claim. From this and the fact that $F(\vec{P})(B) = 0$, it follows that

$$P_1(B) \neq 0 \text{ or } P_2(B) \neq 0 \quad (3)$$

and

$$P_1(\overline{B}) \neq 0 \text{ or } P_2(\overline{B}) \neq 0 \quad (4)$$

Now consider the set

$$\mathcal{Y}_{\neg \mathcal{Z}} = \{\overline{B}\} \cup \{\overline{A} \mid A \in \mathcal{T} \cap \mathcal{X}_2\} \cup (\mathcal{T} \cap \mathcal{X}_1)$$

$$\mathcal{Y}_{\neg(\mathcal{Z} \cup \{X\})} = \{B\} \cup \{\overline{A} \mid A \in \mathcal{T} \cap \mathcal{X}_2\} \cup (\mathcal{T} \cap \mathcal{X}_1)$$

Observe that $P_1(\overline{A}) = 1 = P_2(\overline{A})$ for every $A \in \mathcal{T} \cap \mathcal{X}_2$ and $P_1(C) = 1 = P_2(C)$ for every $C \in \mathcal{T} \cap \mathcal{X}_1$. From (4) we can assert that $P'_1(\bigcap \mathcal{Y}_{\neg \mathcal{Z}}) > 0$ or $P'_2(\bigcap \mathcal{Y}_{\neg \mathcal{Z}}) > 0$ where P'_1 and P'_2 are probability measures that are extensions of P_1 and P_2 , respectively. Furthermore, from (3) we see that $P'_1(\bigcap \mathcal{Y}_{\neg(\mathcal{Z} \cup \{\overline{B}\})}) > 0$ or $P'_2(\bigcap \mathcal{Y}_{\neg(\mathcal{Z} \cup \{\overline{B}\})}) > 0$, and consequently $\bigcap \mathcal{Y}_{\neg \mathcal{Z}} \neq \emptyset$, and $\bigcap \mathcal{Y}_{\neg(\mathcal{Z} \cup \{X\})} \neq \emptyset$, which is the desired conclusion. \square

Part (1) of the theorem generalizes the oligarchy result in Wang (2024) and shows that even if the agenda satisfies a weaker condition—path-connectedness and pair-negatability—than a non-trivial algebra, the oligarchy result holds. In Wang (2024), the agenda condition being a non-trivial algebra was used solely to establish the following two facts, where G is a function satisfying $F(\vec{P})(A) = G(\vec{P}(A))$:

(Fact 1) if $\vec{a} \leq \vec{b}$ and if $G(\vec{a}) = 1$, then $G(\vec{b}) = 1$.

(Fact 2) if $\vec{a} + \vec{b} - \vec{1} \geq \vec{0}$ and if $G(\vec{a}) = 1$ and $G(\vec{b}) = 1$, then $G(\vec{a} + \vec{b} - \vec{1}) = 1$.

Therefore, to prove Part (1), it is enough to derive (Fact 1) from pair-negatability and (Fact 2) from path-connectedness. The agenda conditions are only relevant to (Fact 1) and (Fact 2), and once we see that they hold, we can prove the rest similarly to the way we proved the oligarchy result in Wang (2024).

Our proof also reveals that if we assume the stronger property of SYS instead of IND, then Lemma 2 is not needed, and non-simplicity (NS) is sufficient to obtain the oligarchy result. This observation indicates that stronger properties of a HA lead

Table 1 Agenda Conditions for the Oligarchy result with and without SYS/MON

There is no HA satisfying..	Agenda Conditions
UD, ZP, CP, and IND + CDC + Non-oligarchy	path-connected, pair-negatable
UD, ZP, CP, and SYS + CDC + Non-oligarchy	non-simple, path-connected
UD, ZP, CP, IND, and MON + CDC + Non-oligarchy	path-connected
UD, ZP, CP, SYS, and MON + CDC + Non-oligarchy	non-simple

to weaker agenda conditions for achieving the oligarchy result. To provide additional agenda conditions for the oligarchy result, let us introduce the concept of monotonicity (MON) for a HA as follows:

$$(\text{MON}) \text{ If } \vec{P}(A) \leq \vec{P}'(A) \text{ and } F(\vec{P})(A) = 1, \text{ then } F(\vec{P}')(A) = 1$$

where \leq is applied to each component of two vectors. If we assume MON, we can bypass the need to prove (Fact 1), thereby eliminating the requirement for the agenda to be pair-negatable (PN). This is because (Fact 1) is already implied by SYS and MON.

The following corollary illustrates the agenda conditions that are not only sufficient but also necessary to achieve the oligarchy result based on different properties of HA. (See Table 1)

Corollary 1 (1) Let $|N| \geq 3$. An agenda \mathcal{A} is non-simple and pair-negatable iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, and CDC are the oligarchies.

(2) An agenda \mathcal{A} is path-connected iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, IND, MON, and CDC are the oligarchies.

(3) An agenda \mathcal{A} is non-simple iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, MON, and CDC are the oligarchies.

Proof See the appendix. \square

It is noteworthy that the agenda condition required for our oligarchy result is the same as the one for the dictatorship and oligarchy results in judgement aggregation (Dokow and Holzman 2010; Dietrich and List 2008). One might wonder whether we can obtain our result by restricting a HA to the domain of 0/1-valued probabilistic beliefs, i.e., complete and consistent binary beliefs, and by obtaining the oligarchies in the restricted domain due to the oligarchy result in judgement aggregation. However, this is not possible, because the oligarchies in the restricted domain do not result in oligarchies in the domain of HA. (For more details, see Sect. 5 in Wang (2024).)

Part (2) suggests a counterexample to the oligarchy result when the agenda is not path-connected. Notice that the counterexample does not satisfy SYS and thus, it also serves as a counterexample to Lemma 2 as mentioned earlier. In the proof, some methods in the proofs of Theorem 3 (c) in Dietrich and List (2008) and Claim 3.6 in Dokow and Holzman (2010) are carried over to our domain. We extend the judgment aggregator (JA) in the counterexample in Dietrich and List (2008) to our domain so that UD, ZP, CP, IND, CDC, and CCS are satisfied. Notice that the non-oligarchy of an extension follows from the non-oligarchy of the JA. Our extension is a minimal

extension satisfying monotonicity (MON). It is not that any extension works. For example, if we extended the JA by assigning 0 to every input outside the domain of the JA, which violates MON, the extension would not satisfy CDC. (Concerning Step (ii) in the proof of Part (2), from $P_1(C) = 1$ it would not follow that $C \notin \mathcal{X}_1$.) Indeed, since we do not exclude the case where \mathcal{A} is pair-negatable, the extension should satisfy MON.

Part (3) gives a counterexample to the oligarchy result when the agenda is not pair-negatable. The counterexample is an extension of the counterexample discussed in Theorem 3 (b) in Dietrich and List (2008). Contrary to Part (2), it is not an extension satisfying MON, which is not required because the agenda is not pair-negatable. In contrast, we do not exclude the agenda being path-connected, which forces the counterexample to satisfy SYS. The proof of Part (3) is similar to the one in Dietrich and List (2008). But the last steps in our proof include novel ideas that are needed because of the difference between binary and probabilistic beliefs. In particular, Step (ii) in the proof of Part (3) utilized our own methodology: to show the consistency of a set of propositions we use the fact that there is an agent who would assign a positive probability to the intersection of the propositions.

To summarize, we proved that our oligarchy result can be obtained in more general settings than non-trivial algebras. Furthermore, we showed that the necessary and sufficient agenda condition for the oligarchy result with CDC and the dictatorship result with CCS and CCP in judgment aggregation—path-connectedness and pair-negatability—is also the one for the oligarchy result with CDC in hybrid belief aggregation.

4 The agenda condition for the triviality result

This section presents and proves our second main result: the agenda condition for the triviality result. Stronger properties of a HA yield weak agenda conditions. Thus, one might ask whether the agenda condition for the oligarchy result can be weakened, if we add anonymity (AN). We will demonstrate that the agendas that yield the triviality result can be characterized by negation-connectedness, which is also the agenda condition for an impossibility result of belief binarization methods as shown in Dietrich and List (2021).

Definition 5 (Negation-connected Agenda) An agenda \mathcal{A} is negation-connected (NC) if for every contingent proposition $A \in \mathcal{A}$ it holds that $A \models^{**} \bar{A}$.

So negation-connectedness means that every contingent proposition has a path to its complement. According to Proposition 1 in Dietrich and List (2021), the agenda being negation-connected is equivalent to the agenda being partitioned into subagendas each of which is path-connected, where a subagenda is a non-empty subset of the agenda that is closed under complementation.⁶

⁶ To see the reason why a negation-connected agenda is partitioned into path-connected subagendas, it is instructive to see that a path-relation in negation-connected agenda is actually an equivalence relation. Indeed, for any propositions A, B in a negation-connected agenda \mathcal{A} if $A \models^{**} B$, then $B \models^{**} \bar{B} \models^{**} \bar{A} \models^{**} A$, which implies that a path-relation in \mathcal{A} satisfies symmetry; reflexivity and transitivity are trivial.

Example 4 (Continued) Consider the following agenda as well as \mathcal{A}_1 - \mathcal{A}_4 :

$$\mathcal{A}_5 = \pm\{A, B, A \cap B, A \cup B, C, D, C \cap D, C \cup D\}$$

where every two propositions among A, B, C , and D are logically independent. The agenda \mathcal{A}_1 and \mathcal{A}_2 are not negation-connected while $\mathcal{A}_3, \mathcal{A}_4$, and \mathcal{A}_5 are negation-connected. Note that \mathcal{A}_5 is partitioned into two path-connected subagendas, namely $\pm\{A, B, A \cap B, A \cup B\}$ and $\pm\{C, D, C \cap D, C \cup D\}$.

The following lemma will be needed for the proof of the first part of the succeeding theorem. Part (1) allows us to consider the stronger condition, namely path-connectedness, than negation-connectedness to prove the triviality result. Part (2) will be used when the agenda is path-connected and not pair-negatable.

Lemma 4 (1) *If the triviality result holds—i.e., the only HA on \mathcal{A} satisfying UD, CDC, ZP, CP, IND, and AN is the unanimity rule—for any path-connected agenda \mathcal{A} , then the same holds for any negation-connected agenda.*

(2) *If an agenda \mathcal{A} is not pair-negatable, then for any minimally inconsistent subset $\mathcal{Y} \subseteq \mathcal{A}$ and any two-member subset $\mathcal{Z} \subseteq \mathcal{Y}$ it holds that $\mathcal{Y} \setminus \mathcal{Z}$ is also minimally inconsistent.*

Proof (1) See the appendix for the proof.

(2) See Lemma 14 in Dietrich and List (2021), which is a corollary of Proposition 4.2 in Dokow and Holzman (2010). \square

The following lemma will be needed for the proof of the second part of the succeeding theorem.

Lemma 5 *Let \mathcal{H}_0 be the set $\{A \in \mathcal{A} \mid A \models^{**} \bar{A} \text{ and } \bar{A} \models^{**} A\}$. If \mathcal{A} is not negation-connected, then there is a non empty subset $\mathcal{M} \subseteq \mathcal{A} \setminus \mathcal{H}_0$ such that for any minimally inconsistent set $\mathcal{Y} \subseteq \mathcal{A}$ it holds that $|\mathcal{Y} \cap \mathcal{M}| \leq 1$. Furthermore, for any minimally inconsistent set $\mathcal{Y} \subseteq \mathcal{A}$ intersecting \mathcal{H}_0 it holds that $|\mathcal{Y} \cap \mathcal{M}| = 0$. In addition, for $B \in \mathcal{A} \setminus \mathcal{H}_0$, it holds that $B \in \mathcal{M}$ iff $\bar{B} \notin \mathcal{M}$.*

Proof See Proposition 3.1 in Nehring and Puppe (2010). \square

This lemma looks technical but it is closely related to the notion of median point in the next section. Indeed, if \mathcal{H}_0 is the empty set, then $\bigcap \mathcal{M}$ is the set of all median points where \mathcal{H}_0 and \mathcal{M} are defined in the following lemma. Now let us prove the theorem that negation-connectedness is the sufficient and necessary condition for the triviality result.

Theorem 2 (Agenda Condition for the Triviality Result)

- (1) *If an agenda \mathcal{A} is negation-connected, then the only HA on \mathcal{A} satisfying UD, ZP, CP, CDC, IND, and AN is the unanimity rule.*
- (2) *If \mathcal{A} is not negation-connected, then there is a non-trivial⁷ HA on \mathcal{A} satisfying UD, CDC, ZP, CP, IND, AN, and in addition CCS.*

⁷ A HA is called nontrivial if it is not the unanimity rule.

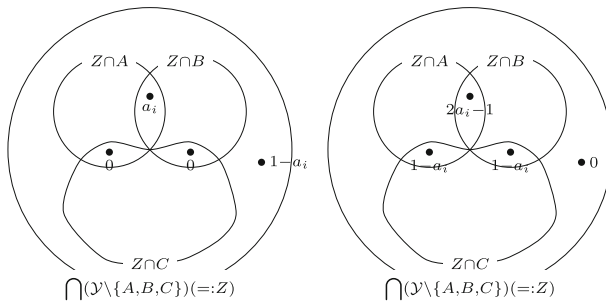


Fig. 6 In the left/right figure, $P'_i(\bigcap \mathcal{Y}_{\neg\{A\}}) = P'_i(\bigcap \mathcal{Y}_{\neg\{B\}}) = 0/1 - a_i$, respectively, and $P_i(\overline{C}) = 1/2a_i - 1$

Proof (1) It suffices to show the claim under the assumption of \mathcal{A} being path-connected and not pair-negatable. This is because we have Lemma 4 (1) and we can apply Theorem 1, if the agenda is pair-negatable. Since \mathcal{A} is path-connected, by Lemma 2 we can set $F(\vec{P})(A) = G(P(A))$. Moreover, by Lemma 3, \mathcal{A} is non-simple and thus we have (Fact 2) by (Fact 2') in Theorem 1.

Now we will prove the following using \mathcal{A} being non-simple and not pair-negatable: (Fact 1'') If $G(\vec{a}) = 1$, then $G(\vec{c}) = 1$ for all $\vec{c} \geq |2\vec{a} - \vec{1}|$

where $|\vec{x}|$ is defined to be $(|x_i|)_{i \in N}$. By non-simplicity and Lemma 4 (2), it follows that there is a minimally inconsistent \mathcal{Y} that has more than three elements, say $A, B, C \in \mathcal{Y}$ such that $\bigcap \mathcal{Y}_{\neg\{A\}}, \bigcap \mathcal{Y}_{\neg\{B\}}, \bigcap \mathcal{Y}_{\neg\{C\}}$, and $\bigcap \mathcal{Y}_{\neg\{A, B, C\}}$ are not empty. Consider individual probability functions P'_i on the algebra generated by \mathcal{A} satisfying $P'_i(\bigcap (\mathcal{Y} \setminus \{A, B, C\})) = 1$ and $P'_i(A) = P'_i(B) = a_i$.

(Case 1) $2a_i \geq 1$

$P'_i(\bigcap \mathcal{Y}_{\neg\{A\}})$ and $P'_i(\bigcap \mathcal{Y}_{\neg\{B\}})$ might have the value between 0 and $1 - a_i$, from which it follows that $P_i(\overline{C}) = P'_i(\bigcap \mathcal{Y}_{\neg\{C\}}) + P'_i(\bigcap \mathcal{Y}_{\neg\{A, B, C\}}) = 1 - P'_i(\bigcap \mathcal{Y}_{\neg\{A\}}) - P'_i(\bigcap \mathcal{Y}_{\neg\{B\}}) \in [2a_i - 1, 1]$ where P_i is the probabilistic belief that can be extended to P'_i . The left/right figure in Fig. 6 illustrates P'_i when $P'_i(\bigcap \mathcal{Y}_{\neg\{A\}})$ and $P'_i(\bigcap \mathcal{Y}_{\neg\{B\}})$ have the minimum/maximum value, respectively.

(Case 2) $2a_i < 1$

$P'_i(\bigcap \mathcal{Y}_{\neg\{A\}})$ and $P'_i(\bigcap \mathcal{Y}_{\neg\{B\}})$ might have the value between 0 and a_i , from which it follows that $P_i(\overline{C}) \in [1 - 2a_i, 1]$ (See Fig. 7.)

Thus, we can assert that there can be probabilistic beliefs P_i satisfying

- (i) $P_i(A) = P_i(B) = a_i$
- (ii) $P_i(\overline{C}) \in [2a_i - 1, 1]$
- (iii) $P_i(Y) = 1$ for all $Y \in \mathcal{Y} \setminus \{A, B, C\}$

By CP, we have $F(\vec{P})(Y) = 1$ for all $Y \in \mathcal{Y} \setminus \{A, B, C\}$. Note that we have $\mathcal{Y} \setminus \{C\} \models \overline{C}$, for \mathcal{Y} is inconsistent. From this we conclude that if $G(\vec{a}) = F(\vec{P})(A) = F(\vec{P})(B) = 1$, then $F(\vec{P})(\overline{C}) = G(\vec{c}) = 1$ for all $\vec{c} \in [|2\vec{a} - \vec{1}|, \vec{1}]$ by CDC, which completes the proof of (Fact 1'').

We now apply [Step 1] and [Step 2] in the proof of Theorem 1 again, with (Fact 1) replaced by (Fact 1''). Since the oligarchies satisfying AN are the unanimity rule, we obtain the triviality result.

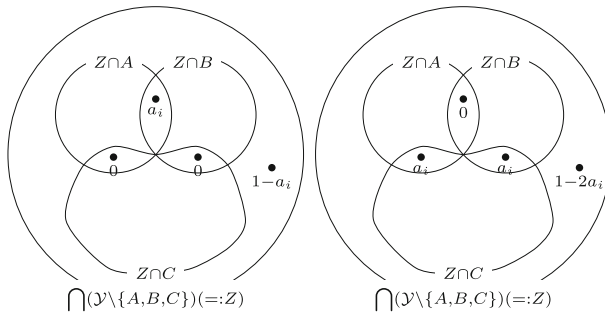


Fig. 7 In the left/right figure, $P'_i(\bigcap \mathcal{Y}_{\neg\{A\}}) = P'_i(\bigcap \mathcal{Y}_{\neg\{B\}}) = 0/a_i$, respectively, and $P_i(\bar{C}) = 1/1 - 2a_i$

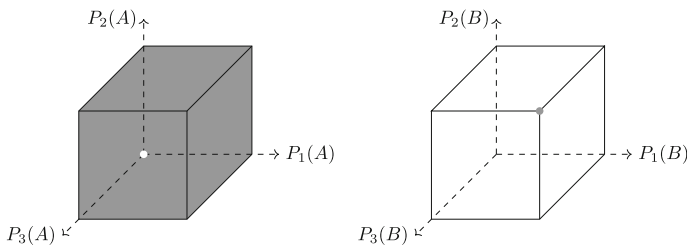


Fig. 8 This represents the HA in (2), provided $N = \{1, 2, 3\}$. Grey points represents where 1 is assigned. The left figure is for any $A \in \mathcal{M}$ and the right one for any $B \notin \mathcal{M}$

(2) Suppose that \mathcal{A} is not negation-connected. Then there is a subset $\mathcal{M} \subseteq \mathcal{A}$ satisfying properties in Lemma 5. Let us define a HA F as follows:

- (i) For every proposition $A \in \mathcal{M}$, $F(\vec{P})(A) = 1$ iff $\vec{P}(A) \neq \vec{0}$.
- (ii) For every proposition $B \notin \mathcal{M}$, $F(\vec{P})(B) = 1$ iff $\vec{P}(A) = \vec{1}$. (See Fig. 8.)

It is easy to check that F satisfies UD, ZP, CP, IND, AN, and non-triviality. Let us prove that F satisfies CCS. Suppose, contrary to our claim, that $F(\vec{P})$ is inconsistent. Then there is a minimally inconsistent subset, say $\mathcal{Y} \subseteq F(\vec{P})$. By lemma 5 we have $|\mathcal{Y} \cap \mathcal{M}| \leq 1$ and hence we can find a proposition, say C , such that $\mathcal{Y} \setminus \{C\}$ has no intersection with \mathcal{M} . This leads to $\vec{P}(Y) = \vec{1}$ for all propositions $Y \in \mathcal{Y} \setminus \{C\}$ by the definition of F . Note that \mathcal{Y} is inconsistent and so $\mathcal{Y} \setminus \{C\} \models \bar{C}$. From this we conclude that $\vec{P}(\bar{C}) = \vec{1}$ and thus $\vec{P}(C) = \vec{0}$, which contradicts $C \in \mathcal{Y} \subseteq F(\vec{P})$. It remains to show that F satisfies CDC. Suppose that $F(\vec{P}) \models D$. Then there is a subset $\mathcal{Y} \subseteq F(\vec{P})$ such that $\mathcal{Y} \cup \{\bar{D}\} (=:\mathcal{Z})$ is minimally inconsistent. By Lemma 5 it holds that $|\mathcal{Z} \cap \mathcal{M}| \leq 1$. First, consider the case where $|\mathcal{Z} \cap \mathcal{M}| = 0$. Then for any $Y \in \mathcal{Y}$ it holds that $\vec{P}(Y) = \vec{1}$, and hence $\vec{P}(D) = \vec{1}$, which implies that $F(\vec{P})(D) = 1$. Now consider the case where $|\mathcal{Z} \cap \mathcal{M}| = 1$. If $\mathcal{Z} \cap \mathcal{M} = \{\bar{D}\}$, then similar arguments to the former case can be applied. In the case where $\bar{D} \notin \mathcal{M}$, we see that $D \in \mathcal{M}$ from the last part of Lemma 5, because $|\mathcal{Z} \cap \mathcal{M}| = 1$ implies $\bar{D} \notin \mathcal{H}_0$ by the second part of Lemma 5. Thus, we need to show that $\vec{P}(D) \neq \vec{0}$. Since $\vec{P}(\bigcap \mathcal{Y}) \leq \vec{P}(D)$, it is enough to show that $\vec{0} < \vec{P}(\bigcap \mathcal{Y})$. Denote by E the unique element in $\mathcal{Y} \cap \mathcal{M}$. By

Table 2 Agenda Conditions for the Triviality result with and without SYS/MON

There is no HA satisfying..	Agenda Conditions
UD, ZP, CP, and IND + CDC + AN + Non-triviality	negation-connected
UD, ZP, CP, and SYS + CDC + AN + Non-triviality	non-simple
UD, ZP, CP, IND, and MON + CDC + AN + Non-triviality	negation-connected
UD, ZP, CP, SYS, and MON + CDC + AN + Non-triviality	non-simple

the definition of F , we have $\vec{P}(E) > \vec{0}$ and $\vec{P}(Y) = \vec{1}$ for all $Y \in \mathcal{Y} \setminus \{E\}$. Thus we have $\vec{P}(\bigcap \mathcal{Y}) > \vec{0}$. \square

Part (1) of the theorem shows that the triviality result holds if the agenda is negation-connected, which is a generalization of the triviality result in Wang (2024). The proof suggests further that, if we assume SYS, then non-simplicity (NS) becomes the sufficient condition to obtain the triviality result. In this case, neither PN nor PC is needed, unlike in the case of Corollary 1. The following corollary illustrates the agenda conditions that are not only sufficient but also necessary to achieve the triviality result based on different properties of HA. (See Table 2)

Corollary 2 (1) An agenda \mathcal{A} is non-simple iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, CDC, and AN are the unanimity rule.

(2) An agenda \mathcal{A} is negation-connected iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, IND, MON, CDC, and AN are the unanimity rule.

(3) An agenda \mathcal{A} is non-simple iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, MON, CDC, and AN are the unanimity rule.

Proof See the appendix. \square

Compared to the case of the oligarchy result, when we add AN, we obtain the triviality result even under a weaker agenda condition: (i) instead of requiring path-connectedness (PC), negation-connectedness (NC) is sufficient, and (ii) the triviality result holds even when the agenda is not pair-negatable (PN). The difference mentioned in (i) does not play a role in finding the sufficient condition according to Lemma 4. However, the necessary condition is not path-connectedness but negation-connectedness. In cases where the agenda is PC and PN, we can apply Theorem 1 since the oligarchy satisfying AN is the unanimity rule (i.e., the oligarchy with $M = N$). Thus, we only need to focus on the cases where the agenda is PC and not PN. When the agenda is assumed to be not PN, we encounter the following difficulty: to show the triviality result, we used (Fact 1), which could be proved if the agenda was assumed to be PN. Our strategy here is to prove a weaker claim than (Fact 1):

(Fact 1'') If $G(\vec{a}) = 1$, then $G(\vec{c}) = 1$ for all $\vec{c} \geq |2\vec{a} - \vec{1}|$.

The new claim (Fact 1'') is weaker than (Fact 1), as it only guarantees that vectors greater than $|2\vec{a} - \vec{1}|$ are mapped to 1, rather than all vectors greater than \vec{a} .

One may ask why (Fact 1), which follows from pair-negatability, is required for the oligarchy result, whereas (Fact 1'') is enough for the triviality result to hold. (Fact

1) and (Fact 1'') differ in the following way: for example, if $(0, 1, \dots, 1)$ is mapped to 1 then $(r, 1, \dots, 1)$ with any $r > 0$ is mapped to 1 by (Fact 1), but by (Fact 1'') only $(1, 1, \dots, 1)$ is mapped to 1, which is enough to prove the triviality result. For the oligarchy result, we need (Fact 1), because without it we cannot deduce the fact that if $a_i = 1$ for all $i \in M$, M being the set of oligarchs, then $G(\vec{a}) = 1$ from $G((\mathbb{1}_M(i))_{i \in N}) = 1$, which can be proved not by (Fact 1'') but by (Fact 1).

It is worth noting that negation-connectedness is the same agenda condition for the triviality result on belief binarization, which is shown in Theorem 2* in Dietrich and List (2021). Their result can be restated using our terminology as follows: there is no belief binarization rule satisfying UD, CCS, CP, IND, CDC, and non-triviality iff the agenda is negation-connected. One might ask whether we can follow their proof for our theorem or the other way around. On the one hand, we cannot use their proof because whereas they deal with probabilistic beliefs, we are dealing with profiles of probabilistic beliefs and so their reasoning cannot be applied. In particular, for PC and not PN agendas we proved (Fact 1''), which is a new idea and what has done in our own way.

On the other hand, since we have not used the fact that $|N| \geq 2$ in the proof of part (1), our proof can be applied for the case where $|N| = 1$ as well, which is the same problem as belief binarization. Notice that Dietrich and List (2021) used CCS (stronger assumption) for belief binarization while our result assumes only ZP (weaker assumption). Therefore, if we apply our proof to the triviality result in belief binarization, then we can prove it using only ZP without requiring CCS. Thus we have the stronger claim (the triviality result with ZP) than Dietrich and List (2021)'s claim (the triviality result with CCS). On top of that, we can use this argument for judgment aggregation, since anonymous independent judgment aggregation can be thought of as belief binarization. Thus, we also have the following result: there is no judgment aggregation satisfying UD, ZP, CP, IND, CDC, AN, and non-triviality iff the agenda is negation-connected. This is a stronger claim than the one in Dietrich and List (2008). They dealt with the agenda conditions—PC and PN—for the oligarchy result of judgment aggregation and derive the triviality result as a corollary under those agenda conditions. Our argument shows that the agenda condition can be weakened to negation-connectedness.

Part (2) gives a counterexample to the triviality result when an agenda is not negation-connected, which implies the agenda being not path-connected. The counterexample presented in Theorem 1 is not applicable in this case because it does not satisfy AN. Moreover, there would be no counterexample if we only assumed an agenda to be not path-connected. This is the reason why we need to weaken path-connectedness to negation-connectedness, even though they fulfill the same role concerning the sufficient agenda condition for the triviality result. Our counterexample is an extension of the belief binarization rule in Theorem 2* in Dietrich and List (2021), which can be viewed as an anonymous judgement aggregation. Extending a counterexample in judgment aggregation to satisfy UD, CDC, ZP, CP, IND, AN is the key for hybrid belief aggregation, since non-triviality is directly satisfied. Notice that we have not excluded pair-negatable agenda, thus MON must be satisfied and one can see that our example satisfies MON. However, it is not minimal among such extensions, which differs from the way of the extension—minimal extension with

MON—in Theorem 1 (2). On the other hand, SYS is not forced since we do not have path-connectedness and our example does not satisfy SYS.

The upshot is that negation-connectedness is the necessary and sufficient condition not only for the triviality result in belief binarization and in judgment aggregation but also for our triviality result in hybrid belief aggregation.

5 The agenda condition for the non-existence result

Now we will show that the agendas for the non-existence result can be characterized by blocked agendas.

Definition 6 (Blocked Agenda) An agenda \mathcal{A} is blocked if there is a proposition $A \in \mathcal{A}$ such that $A \models^{**} \bar{A}$ and $\bar{A} \models^{**} A$.

So a blocked agenda contains a proposition that has a path to its complement. Recall that \mathcal{H}_0 is defined by the set $\{A \in \mathcal{A} \mid A \models^{**} \bar{A} \text{ and } \bar{A} \models^{**} A\}$. Then \mathcal{A} is negation-connected iff $\mathcal{H}_0 = \mathcal{A}$, and \mathcal{A} is blocked iff $\mathcal{H}_0 \neq \emptyset$. If \mathcal{A} is negation-connected, then it is blocked.

Example 5 (Continued) Consider the following agenda as well as \mathcal{A}_1 – \mathcal{A}_5 :

$$\mathcal{A}_6 = \pm\{A, B, A \cap B, A \cup B, C\}$$

where every two propositions among A , B , and C are logically independent. The agenda \mathcal{A}_1 and \mathcal{A}_2 are not blocked while the agendas \mathcal{A}_3 , \mathcal{A}_4 , \mathcal{A}_5 , and \mathcal{A}_6 are blocked. Notice that \mathcal{A}_6 is not negation-connected.

The following definition and lemma will be needed for the succeeding theorem.

Definition 7 (Median Point) Let \mathcal{A} be an agenda on the set W of possible worlds. A possible world $m \in W$ is a median point if for any minimally inconsistent subset $\mathcal{Y} \subseteq \mathcal{A}$, it holds that $|\{A \in \mathcal{Y} \mid m \in A\}| \leq 1$.

So a median point is a possible world that is contained in at most one proposition in every minimally inconsistent set.⁸

Example 6 (Continued) For the agenda \mathcal{A}_1 , every world is a median point; for \mathcal{A}_2 , any world in $\bar{A} \cap \bar{B}$ is a median point.

It is well-known in judgment aggregation that if a median point is guaranteed to exist, then we can easily construct an anonymous, complete, and consistent judgment aggregator where a median point is thought of as a default collective judgment unless everybody believes the proposition being true/false at the median point to be false/true,

⁸ Although this definition is a little bit involved, it has a geometrical meaning (Nehring and Puppe 2007): first of all, let us define a betweenness relation on worlds and introduce a useful notation. A world c is between a and b iff for all propositions A such that $a, b \in A$, $c \in A$; $[a, b]$ denotes the set of all worlds between a and b . Then the geometrical definition of a median point says that a world m is a median point iff for any worlds v, w the three worlds m, v, w admit a median where a world m' is a median of m, v, w iff $m' \in [m, v] \cap [v, w] \cap [w, m]$. According to Lemma 5 in Nehring and Puppe (2005), the geometrical definition of a median point is equivalent to Definition 7.

respectively (Nehring and Puppe 2007). The following lemma states that the agenda not being blocked is the necessary and sufficient condition for the existence of a median point.

Lemma 6 *An agenda \mathcal{A} is not blocked iff there is a median point.*

Proof See Proposition 3.1 in Nehring and Puppe (2010). \square

Now let us formulate and prove our last theorem.

Theorem 3 (Agenda Condition for the Non-existence Result)

- (1) *If an agenda \mathcal{A} is blocked, then there is no HA on \mathcal{A} satisfying UD, CP, IND, CCP, and CCS.*
- (2) *If \mathcal{A} is not blocked, then there is a HA on \mathcal{A} that satisfies UD, CCP, CCS, CP, IND, and additionally AN, and therefore is non-dictatorial.*

Proof (1) Assume that $A \models^{**} \bar{A}$ and $\bar{A} \models^{**} A$. To obtain a contradiction, suppose that there is a HA F satisfying UD, CCP, CCS, CP, and IND. By IND, we can let $F(\vec{P})(A) = G_A(\vec{P}(A))$ for all \vec{P} and $A \in \mathcal{A}$. In Lemma 2, we made use of UD and CCD, which can be followed from CCP and CCS, and proved that for all $A, B \in \mathcal{A}$ if $A \models^* B$, then for all \vec{a} it holds that if $G_A(\vec{a}) = 1$, then $G_B(\vec{a}) = 1$. Therefore, we have $G_A = G_{\bar{A}}$. Since every proposition is contingent, $A, \bar{A} \neq \emptyset$. Hence there is a profile \vec{P} of probabilistic belief satisfying $\vec{P}(A) = \vec{P}(\bar{A}) = (0.5, \dots, 0.5)$ in our domain by UD, which yields $F(\vec{P})(A) = F(\vec{P})(\bar{A})$. This contradicts our assumption that F satisfies CCP and CCS.

(2) Suppose that \mathcal{A} is not blocked. By the above lemma, there is a median point m . Let us define a HA F as follows:

- (i) For every proposition A with $m \in A$, $F(\vec{P})(A) = 1$ iff $\vec{P}(A) \neq \vec{0}$.
- (ii) For every proposition B with $m \notin B$, $F(\vec{P})(B) = 1$ iff $\vec{P}(B) = \vec{1}$.

It is easily seen that F satisfies UD, CP, IND, AN, and non-dictatorship. Let us prove that F satisfies CCP. For any proposition C we have $m \in C$ or $m \in \bar{C}$. Without loss of generality, we can assume that $m \in C$. If $\vec{P}(C) = (0, \dots, 0)$, then $\vec{P}(\bar{C}) = (1, \dots, 1)$ and hence $F(\vec{P})(\bar{C}) = 1$, and if otherwise, then $F(\vec{P})(C) = 1$. It remains to show that F satisfies CCS. Suppose, contrary to our claim, that $F(\vec{P})$ is inconsistent. Then there is a minimally inconsistent subset $\mathcal{Y} \subseteq F(\vec{P})$. Since m is a median point, we see that $|\{A \in \mathcal{Y} \mid m \in A\}| \leq 1$. Hence we can find a proposition, say D , such that none of the sets Y in $\mathcal{Y} \setminus \{D\}$ contains m . From the construction of F , it follows that $\vec{P}(Y) = 1$ for any proposition $Y \in \mathcal{Y} \setminus \{D\}$, which implies that $\vec{P}(\bigcap(\mathcal{Y} \setminus \{D\})) = 1$. By the inconsistency of \mathcal{Y} , we obtain $D \cap [\bigcap(\mathcal{Y} \setminus \{D\})] = \emptyset$. From this we conclude that $\vec{P}(D) = \vec{0}$, which implies that $F(\vec{P})(D) = 0$, by the construction of F . This contradicts $F(\vec{P})(D) = 1$. \square

Part (1) asserts that the non-existence result holds even when the agenda is blocked. Indeed, CCS and CCP together are stronger assumptions than CDC. As a result, we obtain the non-existence result more easily, without assuming AN and non-dictatorship, and with a more relaxed agenda condition. The proof demonstrates that by adding SYS, the non-existence result still holds even without CP and even when no agenda condition is assumed—e.g., even when $\mathcal{A} = \{A, \bar{A}\}$.

Table 3 Classification of Agenda Conditions for Impossibility Results in Hybrid Aggregation, Judgement aggregation and Belief Binarization

There is no HA satisfying..	Agenda Conditions
(1) UD, ZP, CP, and IND + CDC + Non-oligarchy	path-connected, pair-negatable
(2) UD, ZP, CP, and IND + CDC + AN + Non-triviality	negation-connected
(3) UD, CP, and IND + CCS and CCP	blocked
There is no judgment aggregator satisfying..	Agenda Conditions
(1') UD, ZP, CP, and IND + CDC + Non-oligarchy	path-connected, pair-negatable (Dietrich and List 2008)
(2') UD, ZP, CP, and IND + CDC + AN + Non-triviality	negation-connected
(3') UD, CP, and IND + CCS and CCP + non-dictatorship	path-connected, pair-negatable (Dokow and Holzman 2010)
(4') UD, CP, and IND + CCS and CCP + AN	blocked (Nehring and Puppe 2010)
There is no belief binarization rule satisfying..	Agenda Conditions
(2'') UD, CCS, CP, and IND + CDC + Non-triviality	negation-connected (Dietrich and List 2021)
(4'') UD, CCS, CP, and IND + CCP	blocked (Dietrich and List 2018)

The blocked agenda is also the agenda condition for the impossibility results on judgment aggregation with AN in Nehring and Puppe (2010) and belief binarization in Dietrich and List (2018). Our counterexample for non-blocked agenda is an extension of the counterexample in Dietrich and List (2018). It is an extension that satisfies MON, but not minimally so. This is the same as the extension in Theorem 2, but different from the one in Theorem 1. Note that the median point m in the proof of this theorem plays the same role as \mathcal{M} in the proof of Theorem 2. The only difference is that m is a possible world and \mathcal{M} is a set of propositions. This difference arises from assuming CDC versus assuming CCS and CCP.

6 Discussions

All the results in this paper are stated in Table 3: (1) path-connectedness and pair-negatability constitute the exact agenda condition for the oligarchy result; (2) negation-connectedness is for the triviality result; and (3) blockedness is for the non-existence result. These new findings can be compared to the existing characterization theorems in judgment aggregation and belief binarization. Regarding (1), it has the same agenda condition as (1') (Dietrich and List 2008) and (3') (Dokow and Holzman 2010) in judgment aggregation. For (2), it is similar to (2'') (Dietrich and List 2021) in belief binarization, with the difference being the use of ZP instead of CCS for (2''). Since applying our proofs can weaken CCS to ZP, the agenda condition for (2'), which has

not been discussed in the literature, is also negation-connected because an anonymous and independent judgment aggregator can be viewed as a belief binarization function. As for (3), it is similar to (4') (Nehring and Puppe 2010) in judgment aggregation and (4'') (Dietrich and List 2018) in belief binarization.

Let us mention some further research topics. One might think that the rationality norms for collective binary beliefs could be weakened since adhering to deductive closure might be too demanding for group agents. Instead, we could focus on requiring group beliefs to respect consistency or pairwise consistency. By exploring these weaker norms, we can investigate stronger impossibility results. Furthermore, let us discuss how to obtain possibility results. For this purpose, it is advantageous that hybrid belief aggregation provides a framework that generalizes the problem of judgment aggregation or belief binarization. As in judgment aggregation, we can employ and study premise-based hybrid belief aggregation methods. Alternatively, we can combine an individual belief binarization procedure with judgment aggregation. If we assume that linear or geometric pooling methods are very natural given individual credences, we can apply belief binarization methods to the pooled group credence. Of course, we can also come up with new procedures that cannot be reduced to existing methods. Ultimately, we should keep in mind that hybrid belief aggregation is an *epistemic* collective decision problem. Therefore, we should be concerned about which methods accurately track the truth. One natural approach would be to investigate belief binarization methods that minimize the expected distance from the truth in light of the group's pooled credence.

Beyond belief aggregation, hybrid belief aggregation can be applied to general group decision problems that involve not only descriptive propositions but also normative propositions.⁹ In existing theories, such as Savage's frameworks, the collective utility function typically needs to be derived from some aggregation of individual preferences/utilities, and we can see a fundamental dichotomy between beliefs and preferences/utilities: beliefs are formed over the set of possible worlds, while values are formed over outcomes. However, hybrid belief aggregation can address this group decision problem in a more unified manner. It is because the output of the aggregation is not collective probabilistic beliefs but collective binary beliefs. In the former case, collective preferences, utilities or values should be somehow formed to make a collective decision. In the latter case, binary beliefs can be formed over an agenda extended to include normative propositions, which are action-guiding. Thus, hybrid belief aggregation might be seen as encompassing an entire collective decision process on an extended agenda, eliminating the need to separately form collective probabilistic beliefs and collective values.

In conclusion, hybrid belief aggregation opens a new research area in which various procedures of belief aggregation, different studies on the relation between credences and beliefs, and collective decision theory can be combined and explored.

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⁹ We thank the first reviewer for indicating this research direction.

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Declarations

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Appendix

Lemma 1 *Every non-trivial algebra is path-connected and pair-negatable.*

Proof (1) Let A, B be two contingent propositions in the algebra.

(Case 1) $\emptyset \neq A \subseteq B \neq W$.

Since (i) $A \neq \emptyset$, (ii) $B \neq W$ and (iii) $A \models B$, we have $A \models^* B$. Moreover, we have $B \models_{\{A \cup \bar{B}\}}^* A$, for (i) $(A \cup \bar{B}) \cap B = A \neq \emptyset$, (ii) $(A \cup \bar{B}) \cap \bar{A} = \bar{B} \neq \emptyset$ and (iii) $B \cap (A \cup \bar{B}) = A \subseteq A$. Thus, $A \models^* B$ and $B \models^* A$.

(Case 2) $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$.

Let $v \in A \setminus B$ and $w \in B \setminus A$. We can use $C \in \mathcal{A}$ such that $\{v, w\} \subseteq C \neq W$ since \mathcal{A} has at least three contingent propositions.

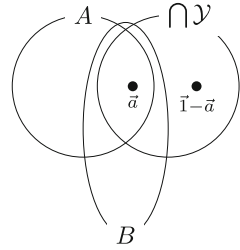
By the result of (Case 1), $A \models^* A \cap C \models^* C \models^* C \cap B \models^* B$. Thus, $A \models^{**} B$.

(2) Since \mathcal{A} is a non-trivial algebra, we can pick three non-empty propositions $A, B, C \in \mathcal{A}$ that have no intersections with each other. Set $\mathcal{Y} = \{A, B\}$. Then \mathcal{Y} is minimally inconsistent set. Set $\mathcal{Z} = \mathcal{Y}$. Then $\mathcal{Y}_{\neg \mathcal{Z}} = \{\bar{A}, \bar{B}\}$. Since $\bar{A} \cap \bar{B} \supseteq C \neq \emptyset$, $\mathcal{Y}_{\neg \mathcal{Z}}$ is consistent. \square

Lemma 2 (Path-connectedness and Contagion Lemma) *Let \mathcal{A} be path-connected. If a HA F with UD satisfies CDC, CP, and IND, then it satisfies SYS.*

Proof By IND, we can let $F(\vec{P})(A) = G_A(\vec{P}(A))$ for all \vec{P} and $A \in \mathcal{A}$. We need to show that $G_A = G_B$ for all $A, B \in \mathcal{A}$. Since \mathcal{A} is path-connected, it is enough to show that if $A \models^* B$, then for all \vec{a} , it holds that if $G_A(\vec{a}) = 1$, then $G_B(\vec{a}) = 1$. Assume that $\{A\} \cup \mathcal{Y} \models B$ where \mathcal{Y} is consistent with A and \bar{B} , i.e., $A \cap \bigcap \mathcal{Y} \subseteq B$, $\bigcap \mathcal{Y} \cap A \neq \emptyset$ and $\bigcap \mathcal{Y} \cap \bar{B} \neq \emptyset$. By UD, we can take as an input a profile of probabilistic beliefs \vec{P} that can be extended to a profile of probability functions \vec{P}' such that $\vec{P}'(\bigcap \mathcal{Y} \cap A) = \vec{a}$ and

Fig. 9 A profile of probabilities for a path-connected agenda



$\vec{P}'(\bigcap \mathcal{Y} \cap \bar{B}) = \bar{1} - \bar{a}$. (See Fig. 9.) Notice that $\vec{P}(Y) = \bar{1}$ for all $Y \in \mathcal{Y}$, $\vec{P}(A) = \bar{a}$ and $\vec{P}(B) = \bar{a}$. By CP, $F(\vec{P})(Y) = 1$ for all $Y \in \mathcal{Y}$. Thus, if $G_A(\bar{a}) = F(\vec{P})(A) = 1$, then $F(\vec{P})(B) = G_B(\bar{a}) = 1$ by CDC, for $\{A\} \cup \mathcal{Y} \models B$. \square

Lemma 3 *If an agenda \mathcal{A} is path-connected, then it is non-simple.*

Proof Let $A \in \mathcal{A}$ be contingent, and $A(= A_1) \vdash^* A_2 \vdash^* A_3 \vdash^* \dots A_{n-1} \vdash^* \neg A (= A_n)$. Since $A \not\vdash^* \neg A$, $n \geq 3$ and there is $1 \leq j \leq n-1$ with $A_j \not\vdash A_{j+1}$ and $A_j \vdash_{\mathcal{Y}} A_{j+1}$ such that $\{A_j\} \cup \mathcal{Y}$ and $\{\neg A_{j+1}\} \cup \mathcal{Y}$ are consistent, and $\mathcal{Y} \neq \emptyset$. Note that $\{A_j, \neg A_{j+1}\} \cup \mathcal{Y}$ is inconsistent. Then there is a minimally inconsistent set $\{A_j, \neg A_{j+1}\} \cup \mathcal{Z}$ where $\emptyset \neq \mathcal{Z} \subseteq \mathcal{Y}$, because $\{A_j, \neg A_{j+1}\}$ and any subsets of $\{A_j\} \cup \mathcal{Y}$ or $\{\neg A_{j+1}\} \cup \mathcal{Y}$ are consistent. \square

Corollary 1 (1) *Let $|N| \geq 3$. An agenda \mathcal{A} is non-simple and pair-negatable iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, and CDC are the oligarchies.*

(2) *An agenda \mathcal{A} is path-connected iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, IND, MON and CDC are the oligarchies.*

(3) *An agenda \mathcal{A} is non-simple iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, MON and CDC are the oligarchies.*

Proof For all of (1), (2) and (3), we can check that (\rightarrow) holds from the proofs of Lemma 2 and Theorem 1. For (\leftarrow) , it is sufficient to construct a counterexample for non-simple agendas, because the counterexamples in the proof of Theorem 1 work for path-connected and pair-negatable agendas.

Suppose that \mathcal{A} is not non-simple. Then for all $\mathcal{Y} \subseteq \mathcal{A}$ if \mathcal{Y} is minimally inconsistent, then $|\mathcal{Y}| \leq 2$. We define $F(\vec{P})(A) = 1$ iff for all $i \in N$ $P_i(A) \geq r(\neq 1) > 0.5$ for all $A \in \mathcal{A}$. It is easy to see that F satisfies UD, ZP, CP, SYS, and Non-oligarchy. It remains to show that F satisfies CCS and CDC. Suppose that $F(\vec{P})$ is inconsistent for some \vec{P} . Then, there is a minimally inconsistent set $\mathcal{Y} \subseteq F(\vec{P})$. By the assumption, $|\mathcal{Y}| = 2$ ($|\mathcal{Y}| \neq 1$ due to $\emptyset \notin F(\vec{P})$), say $\mathcal{Y} = \{A, B\}$ with $A \models \bar{B}$. Since $A \in F(\vec{P})$, we have $r \leq P_i(A) \leq P_i(\bar{B})$ for all $i \in N$, which contradicts $B \in F(\vec{P})$, i.e., $P_i(B) \geq r$ for all $i \in N$. Now consider CDC. Suppose that F does not satisfy CDC. Then there is a proposition A such that $F(\vec{P}) \models A$ and $F(\vec{P})(A) = 0$. Then there is $i \in N$ with $P_i(A) < r$. Since $F(\vec{P}) \cup \bar{A}$ is inconsistent, there is a subset $\mathcal{Z} \subseteq F(\vec{P})$ such that $\mathcal{Z} \cup \{\bar{A}\}$ is minimally inconsistent. By the assumption of not being non-simple, it must be that $|\mathcal{Z} \cup \{\bar{A}\}| = 2$, i.e., say $\mathcal{Z} = \{Z\}$. Since $\{Z\} \models A$ and $Z \in F(\vec{P})$, we know that $r \leq P_i(Z) \leq P_i(A)$, a contradiction. \square

Lemma 4 (1) *If the triviality result holds—i.e., the only HA on \mathcal{A} satisfying UD, CDC, ZP, CP, IND, and AN is the unanimity rule—for any path-connected agenda \mathcal{A} , then the same holds for any negation-connected agenda.*

(2) *If an agenda \mathcal{A} is not pair-negatable, then for any minimally inconsistent subset $\mathcal{V} \subseteq \mathcal{A}$ and any two-member subset $\mathcal{Z} \subseteq \mathcal{V}$ it holds that $\mathcal{V}_{\neg\mathcal{Z}}$ is also minimally inconsistent.*

Proof (1) Assume that the triviality result holds for any path-connected agenda. Let \mathcal{A} be a negation-connected agenda. Then \mathcal{A} can be partitioned into path-connected subagendas, say, $\mathcal{A}_1, \dots, \mathcal{A}_m$. Further, assume that a non-trivial BA F on \mathcal{A} satisfies UD, CDC, ZP, CP, IND, and AN. Then, it can be easily seen that $F \upharpoonright \mathcal{A}_k$, treated as a HA on \mathcal{A}_k , satisfies UD, ZP, CP, IND, and AN for any $k \leq m$. Thus, if we prove that $F \upharpoonright \mathcal{A}_k$ satisfies CDC for any $k \leq m$, then we see that $F \upharpoonright \mathcal{A}_k$ is the unanimity rule for all $k \leq m$, since \mathcal{A}_k is path-connected. This implies that F is the unanimity rule.

Thus, it is enough to show $F \upharpoonright \mathcal{A}_k$ satisfy CDC for any $k \leq m$. Indeed, if $F \upharpoonright \mathcal{A}_k(\vec{P}) \models B$ for any proposition $B \in \mathcal{A}$, then by CDC of F , $F(\vec{P}')(B) = 1$, \vec{P}' being any extension of \vec{P} to \mathcal{A} . Thus, we only need to show that $B \in \mathcal{A}_k$. If we choose a proposition $B_1 \in F \upharpoonright \mathcal{A}_k(\vec{P})(\subseteq \mathcal{A}_k)$, we have $B_1 \models_{F \upharpoonright \mathcal{A}_k(\vec{P}) \setminus \{B_1\}}^* B$, which implies $B \in \mathcal{A}_k$ since \mathcal{A}_k is an equivalence class with respect to \models^{**} and $B_1 \in \mathcal{A}_k$. \square

Corollary 2 (1) *An agenda \mathcal{A} is non-simple iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, CDC and AN are the unanimity rule.*

(2) *An agenda \mathcal{A} is negation-connected iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, IND, MON, CDC and AN are the unanimity rule.*

(3) *An agenda \mathcal{A} is non-simple iff the only HAs on \mathcal{A} satisfying UD, ZP, CP, SYS, MON, CDC and AN are the unanimity rule.*

Proof For all of (1), (2) and (3), we can check that (\rightarrow) holds from the proof of Theorem 2. For (\leftarrow) concerning negation-connected agendas, the counterexample in the proof of Theorem 2 works. For non-simple agendas, the counterexample in the proof of Corollary 1 works. \square

Lemma 7 *Let \mathcal{A} be an agenda. \mathcal{A} is pair-negatable (PN) iff \mathcal{A} is evenly-negatable (EN).*

Proof Assume PN is not true ($=: \neg\text{PN}$). We show that for all minimally inconsistent \mathcal{V} , for all $A, B \in \mathcal{V}$, $\mathcal{V}_{\neg\{A,B\}}$ is minimally inconsistent, from which $\neg\text{EN}$ follows. Assume \mathcal{V} is minimally inconsistent. By $\neg\text{PN}$, $\mathcal{V}_{\neg\{A,B\}}$ is inconsistent. Then, there is a minimally inconsistent set $\mathcal{V} \subseteq \mathcal{V}_{\neg\{A,B\}}$. Note that $\neg A, \neg B \in \mathcal{V}$ (due to minimal inconsistency of \mathcal{V} , $\mathcal{V}_{\neg\{B\}}$ and $\mathcal{V}_{\neg\{A\}}$ are consistent, and so are $\mathcal{V}_{\neg\{A,B\}} \setminus \{\neg A\}$ and $\mathcal{V}_{\neg\{A,B\}} \setminus \{\neg B\}$). By $\neg\text{PN}$, $\mathcal{V}_{\neg\{\neg A, \neg B\}}$ is inconsistent. Since $\mathcal{V}_{\neg\{\neg A, \neg B\}} \subseteq \mathcal{V}$ and \mathcal{V} is minimally inconsistent, $\mathcal{V}_{\neg\{\neg A, \neg B\}} = \mathcal{V}$, i.e., $\mathcal{V} = \mathcal{V}_{\neg\{A,B\}}$.

The converse direction is trivial. \square

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