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Article

Equilibrium Coalition Structures in Three-Player Symmetric Games

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Abstract: In symmetric games with externalities across coalitions, we investigate how three players form coalitions using two solutions: n^* , which is a focal prediction of coalition structure in a class of noncooperative coalitional bargaining games, and equilibrium binding agreements, which represent the cooperative blocking approach. We find that the coarsest equilibrium coalition structure (based on the latter notion) is never finer than n^* , and we provide a sufficient and necessary condition for these two solutions to generate the same coalition structure (i.e., the two solutions coincide if and only if the first coalition to form in n^* is not a two-player coalition or a particular condition about average coalitional worths is satisfied). In symmetric games with more than three players, we demonstrate through a series of examples that any relationship between these two solutions is possible. We also discuss symmetric games with positive externalities or equal division in which these two solutions coincide.

Keywords: coalition formation; symmetry; externalities; efficiency



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1. Introduction

An interesting research question posed in the literature is which coalition structure(s) may form among three players, either in abstract coalitional games (see, e.g., [Binmore, 1985](#); [Moldovanu, 1992](#); [Baron & Herron, 2003](#); [Ray, 2007](#); [Gomes, 2020](#)) or in specific applications (see, e.g., [Riezman, 1985](#); [Krishna, 1998](#); [Kalandrakis, 2004](#); [Aghion et al., 2007](#); [Seidmann, 2009](#)). On one hand, three-player games offer a tractable yet insightful framework for studying coalition formation.¹ On the other hand, many important applications, such as bargaining, oligopoly, and international relations, often involve only a small number of participants.

Following this strand of literature, this paper closely examines how three players form coalitions in symmetric coalitional games with externalities across coalitions and transferable utilities within coalitions. In brief, a game is symmetric if the identities of the players do not matter and the worth of each coalition depends solely on the numerical characteristics of coalition structures (a coalition structure is simply a partition of the players). Many interesting economic applications in the coalition formation literature fall into this category; examples include Cournot oligopoly with homogeneous firms ([Ray & Vohra, 1997, 1999](#)), associations or joint ventures ([Bloch, 1995](#); [Yi, 1998](#); [Greenlee, 2005](#)), customs unions ([Yi, 1996](#)), public good provision ([Ray & Vohra, 2001](#)), and international environmental agreements ([Diamantoudi & Sartzetakis, 2015](#)).² Although the games are symmetric, asymmetric coalition structures often arise.

We assume that utilities are transferable within each coalition for any given coalition structure. While this assumption is frequently used in cooperative game theory, it differs from the commonly used simplifying assumption of equal division among coalition members in the literature on coalition formation (see, e.g., [Yi, 1997](#)). Meanwhile, unlike traditional cooperative game theory, we account for externalities across coalitions rather than assuming them away. (In the terminology of cooperative game theory, the building block here is a partition function, not a characteristic function.) This calls for solution concepts specifically designed to address the strategic effects of these externalities in the process of coalition formation in a consistent way. In this paper, we adopt two widely used solution concepts that have inspired extensive research in the literature on coalition formation.

- \mathbf{n}^* is a coalition structure generated by an algorithm of recursively identifying the coalition with the highest average worth. [Ray and Vohra \(1999\)](#) proved that \mathbf{n}^* is predicted by a broad class of equilibria in a broad class of noncooperative coalitional bargaining games, and it is the only common coalition structure generated in all bargaining protocols.³ Moreover, [Ray and Vohra \(1999\)](#)'s characterization results are particularly sharp in symmetric games—this is another important reason why we only focus on symmetric games.
- Equilibrium binding agreements, proposed by [Ray and Vohra \(1997\)](#). The definition of this concept involves incorporating the basic idea that a coalition structure can only become finer when farsighted players deviate. (Farsightedness means that each perpetrator considers the ultimate consequence of his deviation as further deviations may occur.) Consequently, a coalition structure is called an equilibrium coalition structure (ECS) if there is an equilibrium binding agreement for this structure which is not “blocked” by another equilibrium binding agreement under a finer coalition structure. A recursive method is also employed, starting from the finest structure with all stand-alone players and progressing up to the grand coalition that includes everyone.

With three players, three kinds of coalition structures can be formed: the grand coalition including everyone, the intermediate coalition structure consisting of one stand-alone player and one two-player coalition, and the fragmented coalition structure with all stand-alone players. Proposition 1 finds \mathbf{n}^* and shows that any coalition structure can be \mathbf{n}^* , depending on the magnitude relationships of different average coalitional worths. Though it is a simple characterization, it then helps us to identify the coarsest ECS via its relationship to \mathbf{n}^* . Particularly, Proposition 2 provides a sufficient and necessary condition for \mathbf{n}^* and the coarsest ECS to be identical; i.e., the two solutions coincide if and only if the first coalition to form in \mathbf{n}^* is not a two-player coalition or a particular condition about average coalitional worths is satisfied. As a by-product of our characterization results, we offer two applications: one addressing the widely discussed issue of inefficiency in the literature (Proposition 3), and the other concerning [Bloch \(1995\)](#)'s model of endogenous formation of associations in oligopolistic competition (Proposition 4).⁴

One observation with three players is that the coarsest ECS must be coarser than \mathbf{n}^* if they are not the same (Corollary 1). We explore what the relationship between \mathbf{n}^* and the coarsest ECS is when the number of players is more than three. Though it is difficult to provide a complete answer to this question as we do for three-player symmetric games, we show that any relationship is possible in symmetric games for any number of players by using a series of examples.⁵ We also discuss two special classes of symmetric games: one with positive externalities and the other under the simplifying assumption of equal division of coalitional worths. In both situations with three players, we find that \mathbf{n}^* and the coarsest ECS coincide (Propositions 5 and 6). However, this equivalence may not hold in both classes of symmetric games with more than three players.

As mentioned earlier, there is a body of research on coalition formation in three-player games. Our work is most closely related to [Moldovanu \(1992\)](#) and [Gomes \(2020\)](#). [Moldovanu \(1992\)](#) investigates a noncooperative game of coalition formation and payoff division based on a three-player game in coalitional form. However, there are three major differences between our work and his. First, we assume transferable utilities within coalitions, whereas he considers nontransferable utility games. Second, our model incorporates externalities across coalitions, which are absent in [Moldovanu \(1992\)](#). Third, as the title of [Moldovanu \(1992\)](#) suggests, the solution concepts he uses (i.e., coalition-proof Nash equilibrium and the core) differ substantially from ours. [Gomes \(2020\)](#) studies three-party negotiations in the presence of externalities. However, he focuses on a specific bargaining protocol and explores how the equilibrium expected payoffs relate to classical cooperative solution concepts, such as the Nash bargaining solution, the Shapley value, and the nucleolus. In contrast, our analysis centers on the characterization of coalition structures based on different solution concepts.

It is worth noting that there is a long-standing tension between noncooperative and cooperative approaches to game theory, and their relationship has been investigated extensively in the literature (see, e.g., [Serrano \(2021\)](#) for a review). This tension is also evident in the game-theory literature on coalition formation. Specifically, there are two prominent approaches in the study of how players strategically form coalitions: the bargaining approach, based on noncooperative sequential bargaining, and the blocking approach, more aligned with cooperative game theory (see [Ray \(2007\)](#) and [Ray and Vohra \(2015\)](#) for reviews of these two approaches). As explained earlier, \mathbf{n}^* represents the former approach, while the notion of equilibrium binding agreements represents the latter. In this sense, our analysis also contributes to this paradigm by specifying the conditions under which \mathbf{n}^* provides a strategic foundation of equilibrium binding agreements.

This paper is organized as follows. Section 2 provides all the preliminaries and the definitions of \mathbf{n}^* and equilibrium binding agreements. Section 3 studies coalition formation in three-player symmetric games by using these two solution concepts. Section 4 offers two applications of our main results. Section 5 discusses symmetric games with more than three players and two special classes of symmetric games. Section 6 concludes the paper.

2. The Preliminaries

Though our analysis mainly focuses on three-player games, we present the solution concepts for n -player games ($n \geq 3$) to facilitate our discussion.

2.1. The Set-Up

Let (N, v) denote a (coalitional) game:

- $N = \{1, 2, \dots, n\}$ is the set of players. A coalition structure is a partition of N , and each element in a partition is a coalition. A coalition structure can be written as a set $\pi = \{S_1, S_2, \dots, S_m\}$, where each S_i ($i = 1, 2, \dots, m$) is a coalition. We use the notation s_i to denote the number of players in S_i .
- For every coalition structure π and $S \in \pi$, $v(S, \pi)$ denotes the worth of coalition S in π , which is a nonnegative number. We assume that the worth is transferable among players in each coalition.

We only consider symmetric games: for any coalition structure π and two coalitions $S_i, S_j \in \pi$, $s_i = s_j$ implies $v(S_i, \pi) = v(S_j, \pi)$. Thus, we usually use a numerical coalition structure $\{s_1, s_2, \dots, s_m\}$ for $\{S_1, S_2, \dots, S_m\}$ and write $v(s_i, \pi)$ for $v(S_i, \pi)$.⁶

Let $\mathcal{R}(\pi)$ be the set of refinements of π : $\pi' \in \mathcal{R}(\pi)$ if (1) π' is a coalition structure and (2) $S' \in \pi'$ implies $S' \subseteq S$ for some $S \in \pi$.

2.2. The Bargaining Approach: \mathbf{n}^*

We closely follow Ray and Vohra (1999). For a vector $\mathbf{n} = (n_i)$ of positive integers, we define $K(\mathbf{n}) \equiv \sum n_i$. We use the notation \emptyset to refer to the null vector containing no entries, and let $K(\emptyset) = 0$. Consider a symmetric game (N, v) with $|N| = n$. Let \mathcal{F} be the family of all such vectors \mathbf{n} (including \emptyset) satisfying $K(\mathbf{n}) < n$. We recursively construct a rule $t(\cdot)$ that assigns to each member of \mathcal{F} a positive integer. And \mathbf{n}^* will be generated by repeatedly applying this rule starting from \emptyset .

Step 1. For all \mathbf{n} such that $K(\mathbf{n}) = n - 1$, define $t(\mathbf{n}) \equiv 1$.

Step 2. Recursively, suppose that we have defined $t(\mathbf{n})$ for all \mathbf{n} such that $K(\mathbf{n}) = m + 1, \dots, n - 1$, for some $m \geq 0$. Suppose, moreover, that $K(\mathbf{n}) + t(\mathbf{n}) \leq n$. For any such \mathbf{n} , define

$$c(\mathbf{n}) \equiv (\mathbf{n}, t(\mathbf{n}), t(\mathbf{n}, t(\mathbf{n})), \dots),$$

such that $K(c(\mathbf{n})) = n$, where the notation $\mathbf{n}, t_1, \dots, t_k$ refers to the numerical coalition structure obtained by concatenating \mathbf{n} with the integers t_1, \dots, t_k .

Step 3. For any \mathbf{n} such that $K(\mathbf{n}) = m$, define $t(\mathbf{n})$ to be the largest integer in $\{1, \dots, n - m\}$ that maximizes the expression

$$\frac{v(t, c(\mathbf{n}, t))}{t}.$$

Step 4. Define $\mathbf{n}^* \equiv c(\emptyset)$, and this completes the algorithm.

We provide an example to illustrate how the algorithm operates. (To make our notations consistent, we also write all numerical coalition structures as sets in the algorithm; see footnote 6).

Example 1. Consider (N, v) with $N = \{1, 2, 3\}$, $v(3, \{3\}) = 8.1$, $v(2, \{2, 1\}) = 6$, $v(1, \{2, 1\}) = 0.1$, and $v(1, \{1, 1, 1\}) = 2$. The algorithm works as follows.

Step 1. Define $t(1, 1) = t(2) \equiv 1$.

Step 2. ($m = 1$) Define $t(\mathbf{n})$ for all \mathbf{n} such that $K(\mathbf{n}) = 2$ in Step 1. Then, define $c(1, 1) \equiv (1, 1, t(1, 1)) = \{1, 1, 1\}$ and $c(2) \equiv (2, t(2)) = \{2, 1\}$.

Step 3. ($m = 1$) Define $t(1)$ to be the largest integer in $\{1, 2\}$ that maximizes $v(t, c(1, t))/t$. Because $2 = v(1, c(1, 1))/1 < v(2, \{1, 2\})/2 = 3$, we have $t(1) = 2$.

Step 2'. ($m = 0$) We define $t(\mathbf{n})$ for all \mathbf{n} such that $K(\mathbf{n}) = 1$ and 2 in Steps 3 and 1. Then, we define $c(1) \equiv (1, t(1)) = \{1, 2\}$.

Step 3'. ($m = 0$) Define $t(\emptyset)$ to be the largest integer in $\{1, 2, 3\}$ that maximizes $v(t, c(t))/t$. Because $v(1, c(1))/1 = 0.1$, $v(2, c(2))/2 = 3$ and $v(3, \{3\})/3 = 2.7$, we have $t(\emptyset) = 2$.

Step 4. Define $\mathbf{n}^* \equiv c(\emptyset) = (t(\emptyset), t(t(\emptyset))) = (2, t(2)) = \{2, 1\}$.

It is worth emphasizing that, although equal division of coalitional worths is employed by this algorithm, this assumption is *not* imposed on the bargaining games studied by Ray and Vohra (1999). That is, equal division is an endogenous property emerged in equilibrium.

2.3. The Blocking Approach: Equilibrium Binding Agreements

Consider a game (N, v) . We usually use x for an agreement for some coalition structure π ; i.e., $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ ($x_i \geq 0$ for $i \in N$) and $\sum_{i \in S} x_i = v(S, \pi)$ for every $S \in \pi$. Let x_S denote $(x_j)_{j \in S}$ for $S \subseteq N$. Consider a refinement π' of π . If a coalition in π breaks

into k new coalitions, $k - 1$ of them are labeled as *perpetrators*, and the remaining one is a *residual*. A collection of *perpetrators* and *residuals* in the move from π to π' is any labeling of the relevant elements in π' which satisfies the previous requirement.

We now recursively define equilibrium binding agreements for (N, v) . We use the notation $\mathcal{B}(\pi)$ for the set of equilibrium binding agreements for a coalition structure π . For the finest coalition structure π^* , $\mathcal{B}(\pi^*)$ simply contains all agreements for π^* . In a symmetric game, there is a unique agreement in $\mathcal{B}(\pi^*)$: $x^* = (v^*, \dots, v^*) \in \mathbb{R}^n$ where $v^* = v(1, \pi^*)$. Next, consider coalition structures π which have π^* as their only refinement. We say that (π^*, x^*) blocks (π, x) if there exists a perpetrator S such that $x_S^* \gg x_S$.⁷

Recursively, suppose that for some coalition structure π , the set $\mathcal{B}(\pi')$ has been defined for all $\pi' \in \mathcal{R}(\pi)$, and that all (π'', x'') that block (π', x') have been defined for each (π', x') . Then, we say that (π', x') blocks (π, x) if $\pi' \in \mathcal{R}(\pi)$ and there exists a collection of *perpetrators* and *residuals* in the move from π to π' such that

- (B.1) $x' \in \mathcal{B}(\pi')$.
- (B.2) There is a *leading perpetrator* $S \in \pi'$ such that $x_S' \gg x_S$.
- (B.3) Let Γ be the set of all *perpetrators* excluding S in the move from π to π' . Let $\hat{\pi} \in \mathcal{R}(\pi)$ be a coalition structure formed by merging some of the elements in Γ with their respective *residuals*. Then, $\mathcal{B}(\hat{\pi}) = \emptyset$, and there is $(\hat{\pi}, \hat{x})$ and $S' \in \Gamma$, such that (π', x') blocks $(\hat{\pi}, \hat{x})$ with S' as the *leading perpetrator*.

Now, we can complete the recursion. An agreement x is an *equilibrium binding agreement* for π (i.e., $x \in \mathcal{B}(\pi)$) if there is no (π', x') that blocks (π, x) . A coalition structure π is an *equilibrium coalition structure* (ECS) if $\mathcal{B}(\pi) \neq \emptyset$.⁸

3. Three-Player Games

In this section, we assume that $N = \{1, 2, 3\}$. For simplicity, we use the notations in Table 1 for the grand, the intermediate, and the finest coalition structures.

Table 1. Notations for three-player games.

Coalition Structures	Notations	Numerical Structures
$\{\{1, 2, 3\}\}$	π_G	$\{3\}$
$\{\{1\}, \{2, 3\}\}$	π_I^1	$\{1, 2\}$ or $\{2, 1\}$
$\{\{2\}, \{1, 3\}\}$	π_I^2	
$\{\{3\}, \{1, 2\}\}$	π_I^3	
$\{\{1\}, \{2\}, \{3\}\}$	π^*	$\{1, 1, 1\}$

We simply call an intermediate coalition structure π_I when there is no need to emphasize who the stand-alone player is. Also, let the following notations denote the average worths of different coalitions:

$$v_G = \frac{v(3, \pi_G)}{3}, v_I^2 = \frac{v(2, \pi_I)}{2} \text{ and } v_I^1 = v(1, \pi_I).$$

It is not difficult to find \mathbf{n}^* in three-player symmetric games based on the algorithm. For instance, if v_G (v_I^2) is greater than all the other average worths, then \mathbf{n}^* is clearly $\{3\}$ ($\{2, 1\}$). Particularly, \mathbf{n}^* can be any coalition structure depending on the magnitude relationships of different average coalitional worths. For the sake of further analysis, we find \mathbf{n}^* given any possible combination of average worths in three-player symmetric games.

Proposition 1. Let $V \equiv \{v_G, v_I^2, v_I^1, v^*\}$.

$$\mathbf{n}^* = \begin{cases} \{3\} & \text{if } v_G \in \max V \text{ and } v_I^2 \geq v^*; \text{ or } v_G \geq v^* > v_I^2; \\ \{2, 1\} & \text{if } v_I^2 \in \max V \text{ and } v_I^2 > v_G; \\ \{1, 2\} & \text{if } \{v_I^1\} = \max V \text{ and } v_I^2 \geq v^*; \\ \{1, 1, 1\} & \text{otherwise.} \end{cases}$$

Next, we study equilibrium binding agreements. Note that in a three-player symmetric game, (π^*, x^*) blocks (π_G, x) if there is $i \in N$ such that $v^* > x_i$ and $\mathcal{B}(\pi_I^i) = \emptyset$.⁹ The following lemma provides a sufficient and necessary condition for π_I to be an ECS. Particularly, no player in a two-player coalition has incentive to leave this coalition if and only if the average worth of this coalition (i.e., v_I^2) is no lower than what the player can obtain under the finest structure (i.e., v^*).

Lemma 1. *For any $i \in N$, $\mathcal{B}(\pi_I^i) \neq \emptyset$ if and only if $v_I^2 \geq v^*$.*

The next lemma states that the coarsest ECS coincides with \mathbf{n}^* if \mathbf{n}^* is not $\{2, 1\}$.

Lemma 2. *The coarsest ECS is \mathbf{n}^* if $\mathbf{n}^* = \{3\}$, $\{1, 2\}$, or $\{1, 1, 1\}$.*

Lemma 2 is silent on whether $\mathbf{n}^* = \{2, 1\}$ is identical to the coarsest ECS for all possible combinations of average worths. The following example shows that it is not. For notational simplicity, we write $[u_j, u_k]$ for an agreement $(\pi_I^i, (u_i, u_j, u_k))$ where $u_i = v_I^1$ and $u_j + u_k = 2v_I^2$.

Example 2. *We revisit Example 1 and use the notations in this section: $v_G = 2.7$, $v_I^2 = 3$, $v_I^1 = 0.1$ and $v^* = 2$. Recall that $\mathbf{n}^* = \{2, 1\}$. Clearly, $\mathcal{B}(\pi_I^i) = \{[u_j, u_k] : u_j \geq 2, u_k \geq 2\}$ for all $i \in N$. Because $\max\{u_j, u_k\} \leq 4$, no agreement in $\mathcal{B}(\pi_I^i)$ blocks $(x_1, x_2, x_3) = (0.1, 4, 4)$ for π_G . Hence, the coarsest ECS is π_G . Note that an agreement for π_G with equal division 2.7 is not in $\mathcal{B}(\pi_G)$.*

Example 2 demonstrates that the uniquely greatest v_I^2 (which induces $\mathbf{n}^* = \{2, 1\}$) does not exclude the possibility that there may still exist equilibrium binding agreements for π_G . To establish the equivalence of \mathbf{n}^* and the coarsest ECS, those circumstances need to be singled out. This is carried out by the following proposition which characterizes the coarsest ECS via its relationship to \mathbf{n}^* in three-player symmetric games.

Proposition 2. *The coarsest ECS is \mathbf{n}^* if and only if $\mathbf{n}^* \neq \{2, 1\}$ or $3v_G + 2v^* < v_I^1 + 4v_I^2$.*

This proposition states that when $\mathbf{n}^* = \{2, 1\}$, the coarsest ECS coincides with \mathbf{n}^* if and only if one condition about average coalitional worths is satisfied, i.e., $3v_G + 2v^* < v_I^1 + 4v_I^2$. This condition, equivalent to $2v_I^2 - v^* > (3v_G - v_I^1)/2$, is vital since it implies that the agreement $[v^*, 2v_I^2 - v^*]$ for π_I blocks the agreement $(v_I^1, (3v_G - v_I^1)/2, (3v_G - v_I^1)/2)$ for π_G . And this further implies that every agreement for the grand coalition is blocked. Clearly, Example 2 does not satisfy this condition.

Also, it is easy to see (in Proof of Proposition 2) that $\mathbf{n}^* = \{2, 1\}$ implies the existence of equilibrium binding agreements for π_I . Thus, we immediately have the following corollary stating that the coarsest ECS which is not \mathbf{n}^* must be π_G , as we have seen in Example 2. In this sense, the coarsest ECS is never finer than \mathbf{n}^* in three-player symmetric games.

Corollary 1. *\mathbf{n}^* is not the coarsest ECS if and only if $\mathbf{n}^* = \{2, 1\}$ and the coarsest ECS is π_G .*

4. Applications

We provide two applications of Proposition 2: one for the so-called Coase theorem [Coase \(1960\)](#) and one for endogenous association formation [Bloch \(1995\)](#).

4.1. Efficiency

The literature on coalition formation demonstrates that even if players can sign binding agreements very freely in the absence of informational imperfections, the outcome may still be inefficient, which is inconsistent with the Coase theorem. We explore this inefficiency issue in three-player symmetric games.

We say that an agreement is *(Pareto) efficient* if there does not exist another agreement under which no player is worse off and some player is better off. The following proposition provides a sufficient and necessary condition under which the Coase theorem holds for the coarsest ECS.

Proposition 3. *There exists an efficient equilibrium binding agreement for the coarsest ECS if and only if one of the following holds:*

- (1) *The coarsest ECS is π_G or π^* ;*
- (2) *$\mathbf{n}^* = \{2, 1\}$ and $v_I^2 = v_I^1 = v^*$;*
- (3) *$\mathbf{n}^* = \{2, 1\}$, $v_I^2 > v^*$, $3v_G + 2v^* < v_I^1 + 4v_I^2$, and $3v_G \leq v_I^1 + 2v_I^2$;*
- (4) *$\mathbf{n}^* = \{1, 2\}$ and $3v_G \leq v_I^1 + 2v_I^2$.*

Condition (1) in Proposition 3 states that once the coarsest ECS is not π_I , efficiency must be established. Note that each of conditions (2)–(4) implies that, by Proposition 2, the coarsest ECS is π_I (i.e., \mathbf{n}^*), which is a more involved situation—efficiency is only achieved for special combinations of average coalitional worths.

4.2. Endogenous Association Formation

Bloch (1995) investigates endogenous formation of associations of firms in oligopolistic competition. This symmetric game is defined as follows.¹⁰ Firms can form associations (i.e., coalitions) in order to decrease their production costs. Let $K_i(\pi)$ denote the size of the association to which firm i belongs under the (numerical) coalition structure π . The marginal cost for firm i is then given by $c_i = \lambda - \mu K_i(\pi)$, where λ and μ are positive parameters. Then, each firm i noncooperatively produces differentiated product, q_i , sold at price p_i . The consumer's linear inverse demand function for firm i is $p_i = \alpha - q_i - \beta \sum_{j \neq i} q_j$ where α measures the market size and $\beta \in (-1/2, 1)$ indicates the degree of product substitutability.¹¹ We assume that profits are transferable across firms within an association after the competition.¹²

Bloch (1995) characterizes \mathbf{n}^* (referred to as the equilibrium association structure) for this game in which firms engage in Cournot or Bertrand competition. The following proposition finds \mathbf{n}^* for both competition styles and all feasible degrees of product substitutability, and asserts that the coarsest ECS is always the same as \mathbf{n}^* for this game with three firms.

Proposition 4. *The coarsest ECS is \mathbf{n}^* for the game described above with three firms. Moreover, $\mathbf{n}^* = \{3\}$ if firms engage in Cournot competition; if firms engage in Bertrand competition, $\mathbf{n}^* = \{3\}$ for $\beta \in (-1/2, \sqrt{2}/2]$ and $\{2, 1\}$ otherwise.*

5. Discussion

5.1. More than Three Players

We have seen that the coarsest ECS is either coarser than or identical to \mathbf{n}^* for a three-player symmetric game. It is difficult to provide a comprehensive analysis as we did for three-player games. However, we can show that, with more than three players, there is a taxonomy of relationships between these two solutions in terms of coarseness. Particularly, for any N with $n \geq 4$, all the following statements hold:

1. There is a symmetric (N, v) for which the coarsest ECS is identical to \mathbf{n}^* ;
2. There is a symmetric (N, v') for which the coarsest ECS and \mathbf{n}^* are incomparable;
3. There is a symmetric (N, v'') for which the coarsest ECS is coarser than \mathbf{n}^* ;
4. There is a symmetric (N, v''') for which the coarsest ECS is finer than \mathbf{n}^* .

The following example simply shows the first statement.

Example 3. Let $v(s, \pi) = 0$ for all π and $s \in \pi$. Obviously, both the coarsest ECS and \mathbf{n}^* are the grand coalition.

For the ease of reading, Examples 4–6 demonstrate the remaining statements (2–4), respectively, in symmetric games with *four* players. Additional examples for symmetric games with n players ($n \geq 5$) are provided in Appendix A (Examples A1–A3).

Example 4. Let

$$v'(s, \pi) = \begin{cases} 6 & \text{if } (s, \pi) = (3, \{3, 1\}), \\ 1 & \text{if } (s, \pi) = (1, \{3, 1\}), \\ 2 & \text{if } (s, \pi) = (2, \{2, 2\}), \\ 10 & \text{if } (s, \pi) = (2, \{2, 1, 1\}), \\ 0.1 & \text{if } (s, \pi) = (1, \{2, 1, 1\}), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\mathbf{n}^* = \{3, 1\}$. The only coarsest ECS is $\{2, 2\}$, because any agreement for $\{3, 1\}$ is blocked by some equilibrium binding agreement for $\{2, 1, 1\}$.

Example 5. Let

$$v''(s, \pi) = \begin{cases} 8 & \text{if } (s, \pi) = (3, \{3, 1\}), \\ 6 & \text{if } (s, \pi) = (2, \{2, 1, 1\}), \\ 1 & \text{if } (s, \pi) = (1, \{3, 1\}) \text{ or } (1, \{2, 1, 1\}), \\ 2.5 & \text{if } (s, \pi) = (1, \pi^*), \\ 0 & \text{otherwise.} \end{cases}$$

It can be straightforwardly checked that $\mathbf{n}^* = \{2, 1, 1\}$. Note that¹³

$$\mathcal{B}(\{2, 1, 1\}) = \{(x, 6 - x, 1, 1) : 2.5 \leq x \leq 3.5\}.$$

Thus, no agreement in $\mathcal{B}(\{2, 1, 1\})$ can block, e.g., $(3.5, 3.5, 1, 1)$ for $\{3, 1\}$. Hence, the coarsest ECS is $\{3, 1\}$.

Example 6. Let

$$v'''(s, \pi) = \begin{cases} 4 & \text{if } (s, \pi) = (4, \{4\}) \text{ or } (1, \{2, 1, 1\}), \\ 6 & \text{if } (s, \pi) = (2, \{2, 2\}), \\ 1 & \text{if } (s, \pi) = (1, \pi^*), \\ 0 & \text{otherwise.} \end{cases}$$

It can be verified that $\mathbf{n}^* = \{4\}$. However, any agreement for $\{4\}$ is blocked by $(3, 3, 3, 3) \in \mathcal{B}(\{2, 2\})$.

5.2. Positive Externalities

Section 6 of Ray and Vohra (1997) explored a class of symmetric games (with transferable utilities) with *positive externalities*. In such games, every coalition does not lose from the merger of other coalitions. Precisely speaking, whenever $S \in \pi$, $\pi' \in \mathcal{R}(\pi)$ and

$S \in \pi'$, we have $x \in \mathcal{B}(\pi')$, implying that there exists $y \in \mathcal{B}(\pi)$ such that $\sum_{i \in S} y_i \geq \sum_{i \in S} x_i$, whenever $\mathcal{B}(\pi)$ is nonempty.¹⁴ Proposition 6.3 of Ray and Vohra (1997) showed that in this class of symmetric games, the set of equilibrium coalition structures is identical to that obtained in a simplified game where the worth of every coalition is equally divided among its members.

When $n = 3$, this property of positive externalities establishes the equivalence of the coarsest ECS and \mathbf{n}^* , as stated by the following proposition.

Proposition 5. *Suppose that $N = \{1, 2, 3\}$ and there are positive externalities. Then, the coarsest ECS and \mathbf{n}^* are identical.*

However, Example 6 above illustrates that in symmetric games with positive externalities, the coarsest ECS may not be \mathbf{n}^* when there are more than three players.

5.3. Equal Division

In this subsection, we depart from our original assumption of transferable utilities. Instead, we assume that there is a unique agreement under each coalition structure: the worth of every coalition is equally divided among its members. There are two major reasons why this assumption is often adopted in the literature. First, in many economic applications, the symmetry of the game seems to justify equal allocation within the coalitions; see Yi (1997). Second, equal division can sometimes be endogenously vindicated in symmetric games; see Ray and Vohra (1997) and Ray and Vohra (1999).

Compared with Proposition 2, the following proposition shows that the equivalence of the coarsest ECS and \mathbf{n}^* can always be established when there are three players.

Proposition 6. *Suppose that $N = \{1, 2, 3\}$ and the worth of each coalition is equally divided. Then, the coarsest ECS and \mathbf{n}^* are identical.*

Assuming equal division, our observations in Examples 3, 4, and 6 still hold in terms of the relationship between the coarsest ECS and \mathbf{n}^* (i.e., the former is identical to, incomparable to, or finer than the latter). The following example shows that the coarsest ECS can also be coarser than \mathbf{n}^* .

Example 7. *Suppose $n = 5$. Let*

$$v(s, \pi) = \begin{cases} 9 & \text{if } (s, \pi) = (3, \{3, 2\}), \\ 4 & \text{if } (s, \pi) = (1, \{3, 1, 1\}), \\ 3 & \text{if } (s, \pi) = (2, \{2, 2, 1\}), \\ 2 & \text{if } (s, \pi) = (1, \{2, 2, 1\}), (2, \{3, 2\}) \text{ or } (2, \{2, 1, 1, 1\}), \\ 1 & \text{if } (s, \pi) = (1, \{2, 1, 1, 1\}), \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to check that $\mathbf{n}^ = \{1, 2, 2\}$ and the coarsest ECS is $\{3, 2\}$ with equal division.*

6. Conclusions

There is no consensus on the most suitable solution concept for coalition formation. Undoubtedly, \mathbf{n}^* and equilibrium binding agreements are among the most well known. Our study examines the coalition formation among (three) players using these two solution concepts. A notable observation is that the coalition structures they generate are usually nonidentical, particularly when the number of players is more than three. This divergence is not unexpected, as the underlying logics of these solution concepts differ significantly,

each representing a distinct approach to coalition formation. In this regard, our equivalence results contribute to identifying the conditions under which these two approaches may align (i.e., with three players, $\mathbf{n}^* \neq \{2, 1\}$ or $3v_G + 2v^* < v_I^1 + 4v_I^2$). Additionally, from the perspective of the Nash program, our results help to clarify the precise circumstances under which \mathbf{n}^* can be considered a strategic foundation for equilibrium binding agreements, though our analysis offers only a partial answer.

There are several other directions for future research. First, economic applications typically rely on a single solution concept within one approach. Our analysis could help derive new results by applying an alternative approach. (Proposition 4 is a simple attempt.) Second, we exclusively focus on symmetric games, which, while often useful, can be somewhat restrictive in certain situations. It would be interesting to extend the analysis to general games that are not necessarily symmetric. Third, there are various other solution concepts in the literature on coalition formation [e.g., [Diamantoudi and Xue \(2007\)](#)’s extended equilibrium binding agreements; see, also, [Ray and Vohra \(2015\)](#) for a review and the works cited therein], and similar analyses could be applied to these as well.

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Appendix A

Proof of Proposition 1. Suppose that $v_I^2 \geq v^*$. Then, $t(1) = 2$. Thus, $t(\emptyset) = 3$ if $v_G \geq v_I^2$ and $v_G \geq v_I^1$; $t(\emptyset) = 2$ if $v_I^2 > v_G$ and $v_I^2 \geq v_I^1$; $t(\emptyset) = 1$ if $v_I^1 > v_G$ and $v_I^1 > v_I^2$. Suppose, otherwise, that $v^* > v_I^2$. Then, $t(1) = 1$. Thus, $t(\emptyset) = 3$ if $v_G \geq v^*$; $t(\emptyset) = 1$ if $v^* > v_G$. \square

Proof of Lemma 1. (If) Clearly, $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^i)$ for any $i \in N$. (Only if) Say, $[u_j, u_k] \in \mathcal{B}(\pi_I^i)$. Then, $u_j \geq v^*$ and $u_k \geq v^*$. Thus, $2v_I^2 = u_j + u_k \geq 2v^*$. \square

Proof of Lemma 2. There are four cases to consider by Proposition 1.

(1) Consider $\mathbf{n}^* = \{3\}$ with $v_G \in \max V$ and $v_I^2 \geq v^*$. By Lemma 1, $\mathcal{B}(\pi_I^i) \neq \emptyset$ for all $i \in N$. Because $v_G \geq v_I^1$ and $v_G \geq v_I^2$, there are no $i \in N$ and $[u_j, u_k] \in \mathcal{B}(\pi_I^i)$ such that $v_I^1 > v_G$ or $(u_j, u_k) \gg (v_G, v_G)$. Hence, $(v_G, v_G, v_G) \in \mathcal{B}(\pi_G)$.

(2) Consider $\mathbf{n}^* = \{3\}$ with $v_G \geq v^* > v_I^2$. By Lemma 1, $\mathcal{B}(\pi_I^i) = \emptyset$ for all $i \in N$. Because $v_G \geq v^*$, $(v_G, v_G, v_G) \in \mathcal{B}(\pi_G)$.

(3) Consider $\mathbf{n}^* = \{1, 2\}$, i.e., $\max V = \{v_I^1\}$ and $v_I^2 \geq v^*$. By Lemma 1, $\mathcal{B}(\pi_I^i) \neq \emptyset$ for all $i \in N$. Suppose, by contradiction, that there exists an agreement $(x_1, x_2, x_3) \in \mathcal{B}(\pi_G)$. Note that there exists $j \in N$ such that $x_j \leq v_G$. Consider $[v_I^2, v_I^2]$ which is in $\mathcal{B}(\pi_I^j)$ due to $v_I^2 \geq v^*$. Then, $v_I^1 > v_G \geq x_j$ implies that $(x_1, x_2, x_3) \notin \mathcal{B}(\pi_G)$. This is a contradiction.

(4) Consider $\mathbf{n}^* = \{1, 1, 1\}$, i.e., $v^* > v_I^2$ and $v^* > v_G$. By Lemma 1, $\mathcal{B}(\pi_I^i) = \emptyset$ for all $i \in N$. For any agreement $(\pi_G, (x_1, x_2, x_3))$, there exists $j \in N$ such that $x_j \leq v_G$. Then, $v^* > v_G \geq x_j$ implies that $(x_1, x_2, x_3) \notin \mathcal{B}(\pi_G)$. Hence, $\mathcal{B}(\pi_G) = \emptyset$. \square

Proof of Proposition 2. (If) If $\mathbf{n}^* \neq \{2, 1\}$, by Lemma 2, the coarsest ECS is \mathbf{n}^* . Now, suppose $\mathbf{n}^* = \{2, 1\}$ and $3v_G + 2v^* < v_I^1 + 4v_I^2$ (i.e., $2v_I^2 - v^* > (3v_G - v_I^1)/2$). By Proposition 1 and Lemma 1, $\mathbf{n}^* = \{2, 1\}$ implies that $\mathcal{B}(\pi_I^i) \neq \emptyset$ for all $i \in N$. Consider an agreement $(\pi_G, (x_1, x_2, x_3))$. Without loss of generality, assume that $x_1 = \min\{x_1, x_2, x_3\}$ and $x_3 = \max\{x_1, x_2, x_3\}$. Clearly, $x_1 \leq v_G \leq x_3$. If $x_1 < v_I^1$, then $(x_1, x_2, x_3) \notin \mathcal{B}(\pi_G)$.

because it is blocked by $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^1)$. Otherwise, there are three scenarios to consider. (1) If $v^* > x_2 \geq x_1$, then (x_1, x_2, x_3) is blocked by $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^3)$ because $v_I^2 \geq v^*$. (2) If $x_2 \geq v^* > x_1$, we can show that (x_1, x_2, x_3) is blocked by $[v^*, 2v_I^2 - v^*] \in \mathcal{B}(\pi_I^3)$. (Clearly, $2v_I^2 - v^* \geq v^*$.) Since $x_1 \geq v_I^1$, we have $x_2 + x_3 = 3v_G - x_1 \leq 3v_G - v_I^1$ and, hence, $x_2 \leq (3v_G - v_I^1)/2$. Thus, $x_2 < 2v_I^2 - v^*$. Hence, $(v^*, 2v_I^2 - v^*) \gg (x_1, x_2)$. (3) If $x_2 \geq x_1 \geq v^*$, we can show that $(\pi_G, (x_1, x_2, x_3))$ is blocked by an agreement $(\pi_I^3, [x_1 + \epsilon/2, x_2 + \epsilon/2])$ where $\epsilon = 2v_I^2 - (x_1 + x_2)$. Note that, by Proposition 1, $v_I^2 > v_G$. Then, $x_1 + x_2 = 3v_G - x_3 \leq 2v_G < 2v_I^2$, implying that $\epsilon > 0$. Hence, $[x_1 + \epsilon/2, x_2 + \epsilon/2] \in \mathcal{B}(\pi_I^3)$ and $(x_1 + \epsilon/2, x_2 + \epsilon/2) \gg (x_1, x_2)$.

(Only if) Suppose, by contradiction, that $\mathbf{n}^* = \{2, 1\}$ and $3v_G + 2v^* \geq v_I^1 + 4v_I^2$. By Proposition 1, $v_I^2 \in \max V$. Then, by Lemma 1, $\mathcal{B}(\pi_I^i) \neq \emptyset$ for all $i \in N$. Note that $(3v_G - v_I^1)/2 \geq 2v_I^2 - v^* \geq v_I^2 \geq v_I^1 \geq 0$. Then, $(v_I^1, (3v_G - v_I^1)/2, (3v_G - v_I^1)/2)$ is an agreement for π_G . We show that this agreement is in $\mathcal{B}(\pi_G)$. Consider three possible blockings. (1) There is $[u_2, u_3] \in \mathcal{B}(\pi_I^1)$ such that $(u_2, u_3) \gg ((3v_G - v_I^1)/2, (3v_G - v_I^1)/2)$. This is impossible since it implies $2v_I^2 = u_2 + u_3 > 3v_G - v_I^1$. (2) There is $[u_1, u_3] \in \mathcal{B}(\pi_I^2)$ such that $(u_1, u_3) \gg (v_I^1, (3v_G - v_I^1)/2)$. This is impossible since it implies $2v_I^2 = u_1 + u_3 > v^* + (3v_G - v_I^1)/2$. (3) There is $[u_1, u_2] \in \mathcal{B}(\pi_I^3)$ such that $(u_1, u_2) \gg (v_I^1, (3v_G - v_I^1)/2)$. The preceding analysis applies. \square

Proof of Corollary 2. If $\mathbf{n}^* \neq \{2, 1\}$, then the coarsest ECS is \mathbf{n}^* by Proposition 2. If $\mathbf{n}^* = \{2, 1\}$, then $v_I^2 \geq v^*$ and $v_I^2 > v_G$ by Proposition 1. This implies that $3v_G + v^* < 4v_I^2$. Thus, by Proposition 2, the coarsest ECS is \mathbf{n}^* . \square

Proof of Proposition 3. (If) (1) Suppose that the coarsest ECS is π_G . Then, there are three cases. (i) Consider $\mathbf{n}^* = \{3\}$ with $v_G \in \max V$ and $v_I^2 \geq v^*$. Proof of Lemma 2 shows that $(v_G, v_G, v_G) \in \mathcal{B}(\pi_G)$. Also, this agreement is efficient. (ii) Consider $\mathbf{n}^* = \{3\}$ with $v_G \geq v^* > v_I^2$. Then, $(v_G, v_G, v_G) \in \mathcal{B}(\pi_G)$ is efficient. (iii) Consider $\mathbf{n}^* = \{2, 1\}$, and $3v_G + 2v^* \geq v_I^1 + 4v_I^2$. Proof of Proposition 2 shows that $(x_1, x_2, x_3) = (v_I^1, (3v_G - v_I^1)/2, (3v_G - v_I^1)/2) \in \mathcal{B}(\pi_G)$. Also, this agreement is efficient because $(3v_G - v_I^1)/2 \geq v_I^2 > v_G > v_I^1$ and $(3v_G - v_I^1)/2 \geq v_I^2 \geq v^*$.

Suppose that the coarsest ECS is π^* , i.e., $\mathbf{n}^* = \{1, 1, 1\}$. Then, x^* is efficient because $v^* > v_I^2$ and $v^* > v_G$.

(2) If $v_I^2 = v_I^1 = v^* > v_G$, the coarsest ECS is π_I by Proposition 2. Clearly, $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^i)$ ($i \in N$) is efficient.

(3) By Proposition 2, the coarsest ECS is π_I . Because $v_I^2 > v^*$ and $3v_G \leq v_I^1 + 2v_I^2$, $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^i)$ ($i \in N$) is efficient.

(4) By Lemma 2, the coarsest ECS is π_I . Because $\max V = \{v_I^1\}$ and $3v_G \leq v_I^1 + 2v_I^2$, $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^i)$ ($i \in N$) is efficient.

(Only if) Suppose that none of the conditions (1)–(4) hold. Because (1) does not hold, by Proposition 2, either $\mathbf{n}^* = \{1, 2\}$ or $\mathbf{n}^* = \{2, 1\}$ with $3v_G + 2v^* < v_I^1 + 4v_I^2$. Suppose $\mathbf{n}^* = \{1, 2\}$. Because (4) does not hold, we have $3v_G > v_I^1 + 2v_I^2$. Thus, for any agreement for π_I , you can always find an agreement for π_G which is a Pareto improvement, a contradiction. Now, suppose $\mathbf{n}^* = \{2, 1\}$ and $3v_G + 2v^* < v_I^1 + 4v_I^2$. There are two cases. (i) Consider $v_I^2 > v^*$. Because (3) does not hold, we have $3v_G > v_I^1 + 2v_I^2$. A similar argument as above leads to a contradiction. (ii) Consider $v_I^2 = v^*$. Note that $[v^*, v^*]$ is the only agreement in $\mathcal{B}(\pi_I^i)$ for $i \in N$. Because (2) does not hold, we have $v^* > v_I^1$. Thus, x^* is a Pareto improvement. \square

Proof of Proposition 4. (Cournot) By (Bloch (1995) Proposition 3), $\mathbf{n}^* = \{3\}$. Thus, by Lemma 2, the coarsest ECS is \mathbf{n}^* .

(Bertrand) By (Bloch (1995), Proposition 4), $\mathbf{n}^* = \{3\}$ for $\beta \in (-1/2, \sqrt{2}/2]$ and $\{2, 1\}$ otherwise. Thus, by Lemma 2, the coarsest ECS is \mathbf{n}^* for $\beta \in (-1/2, \sqrt{2}/2]$. Bloch (1995) shows that the worth of size- s association for $s \in \pi$ is

$$v(s, \pi) = \frac{s(1+\beta)}{(1-\beta)(1+2\beta)} [p^*(s, \pi) - \lambda + \mu s]^2 \quad (*)$$

where

$$p^*(s, \pi) = \frac{\alpha(1-\beta) + \lambda(1+\beta)}{2} - \frac{\mu(1+\beta)s}{2+3\beta} - \frac{\mu\beta(1+\beta) \sum_j K_j(\pi)}{2(2+3\beta)}.$$

Thus, by algebra, we have

$$3v_G + 2v^* - (v_I^1 + 4v_I^2) = \frac{2\mu}{(2+3\beta)} [s(\alpha - \lambda) + t\mu]$$

where $s = 2 + \beta - 7\beta^2 - 2\beta^3 + 6\beta^4$ and $t = 6 + 3\beta - 22\beta^2 - 8\beta^3 + 17\beta^4$. For $\beta \in (\sqrt{2}/2, 1)$, s and t are both negative, implying that $3v_G + 2v^* < v_I^1 + 4v_I^2$. Hence, by Proposition 2, the coarsest ECS is \mathbf{n}^* . \square

Proof of Proposition 5. If $\mathbf{n}^* \neq \{2, 1\}$, the conclusion holds by Proposition 2. If $\mathbf{n}^* = \{2, 1\}$, then $v_I^2 > v_G$ and $v_I^2 \geq v^*$ by Proposition 1. Note that $\mathcal{B}(\pi_I)$ is nonempty by Lemma 1. Consider the stand-alone player in both π_I and π^* . Then, by positive externalities, we have $v_I^1 \geq v^*$. Thus, $3v_G + 2v^* < v_I^1 + 4v_I^2$. By Proposition 2, the coarsest ECS and \mathbf{n}^* are identical. \square

Proof of Proposition 6. There are five cases to consider by Proposition 1.

(1) If $\mathbf{n}^* = \{3\}$ with $v_G \in \max V$ and $v_I^2 \geq v^*$, then $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^i)$ for all $i \in N$ and $(v_G, v_G, v_G) \in \mathcal{B}(\pi_G)$.

(2) If $\mathbf{n}^* = \{3\}$ with $v_G \geq v^* > v_I^2$, then $[v_I^2, v_I^2] \notin \mathcal{B}(\pi_I^i)$ for all $i \in N$ and $(v_G, v_G, v_G) \in \mathcal{B}(\pi_G)$.

(3) If $\mathbf{n}^* = \{2, 1\}$ (i.e., $v_I^2 \in \max V$ and $v_I^2 > v_G$), then $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^i)$ for all $i \in N$ and $(v_G, v_G, v_G) \notin \mathcal{B}(\pi_G)$. Thus, $\mathcal{B}(\pi_G)$ is empty.

(4) If $\mathbf{n}^* = \{1, 2\}$ (i.e., $\max V = \{v_I^1\}$ and $v_I^2 \geq v^*$), then $[v_I^2, v_I^2] \in \mathcal{B}(\pi_I^i)$ for all $i \in N$ and $(v_G, v_G, v_G) \notin \mathcal{B}(\pi_G)$.

(5) If $\mathbf{n}^* = \{1, 1, 1\}$ (i.e., $v^* > v_I^2$ and $v^* > v_G$), then $[v_I^2, v_I^2] \notin \mathcal{B}(\pi_I^i)$ for all $i \in N$ and $(v_G, v_G, v_G) \notin \mathcal{B}(\pi_G)$. Thus, only $\mathcal{B}(\pi^*)$ is nonempty. \square

Example A1. Suppose $n \geq 5$. Let

$$v'(s, \pi) = \begin{cases} 6 & \text{if } (s, \pi) = (3, \{3, 1, \dots, 1\}), \\ 1 & \text{if } (s, \pi) = (1, \{3, 1, \dots, 1\}), \\ 2 & \text{if } (s, \pi) = (2, \{2, 2, 1, \dots, 1\}), \\ 10 & \text{if } (s, \pi) = (2, \{2, 1, \dots, 1\}), \\ 0.1 & \text{if } (s, \pi) = (1, \{2, 2, 1, \dots, 1\}) \text{ or } (1, \{2, 1, \dots, 1\}), \\ 0 & \text{otherwise.} \end{cases}$$

We can show by induction that

$$\begin{aligned} t\left(2, \underbrace{1, \dots, 1}_{n-2-k \text{ terms}}\right) &= 1 \text{ for } k \in \{1, \dots, n-2\}, \\ t\left(3, \underbrace{1, \dots, 1}_{n-3-k \text{ terms}}\right) &= 1 \text{ for } k \in \{1, \dots, n-3\}, \end{aligned}$$

and

$$t\left(\underbrace{1, \dots, 1}_{n-k \text{ terms}}\right) = 2 \text{ for } k \in \{2, \dots, n-1\}.$$

Thus, $t(\emptyset) = 2$ and $\mathbf{n}^* = \{2, 1, \dots, 1\}$.

Next, we show that the coarsest ECS is $\{2, 2, 1, \dots, 1\}$. Note that $\mathcal{B}(\{2, 1, \dots, 1\})$ and $\mathcal{B}(\{2, 2, 1, \dots, 1\})$ are nonempty. Consider an arbitrary agreement $(x_i, x_j, x_k, 1, \dots, 1)$ for $\{3, 1, \dots, 1\}$ (i.e., players i, j and k form a coalition). Without loss of generality, we assume that player k 's utility is $\max\{x_i, x_j, x_k\} \geq 2$. Hence, because $x_i + x_j = 6 - \max\{x_i, x_j, x_k\} \leq 4 < 10$, there exists an agreement in $\mathcal{B}(\{2, 1, \dots, 1\})$ that blocks $(\{3, 1, \dots, 1\}, (x_i, x_j, x_k, 1, \dots, 1))$ with $\{i, j\}$ being the leading perpetrator. Thus, $\mathcal{B}(\{3, 1, \dots, 1\}) = \emptyset$. Also, any agreement for any other structure except π^* is blocked by an agreement in $\mathcal{B}(\{2, 2, 1, \dots, 1\})$.

Example A2. Suppose $n \geq 5$. Let

$$v''(s, \pi) = \begin{cases} 8 & \text{if } (s, \pi) = (3, \{3, 1, \dots, 1\}), \\ 6 & \text{if } (s, \pi) = (2, \{2, 1, \dots, 1\}), \\ 1 & \text{if } (s, \pi) = (1, \{3, 1, \dots, 1\}) \text{ or } (1, \{2, 1, \dots, 1\}), \\ 2.5 & \text{if } (s, \pi) = (1, \pi^*), \\ 0 & \text{otherwise.} \end{cases}$$

We can show by induction that

$$\begin{aligned} t\left(2, \underbrace{1, \dots, 1}_{n-2-k \text{ terms}}\right) &= 1 \text{ for } k \in \{1, \dots, n-2\}, \\ t\left(3, \underbrace{1, \dots, 1}_{n-3-k \text{ terms}}\right) &= 1 \text{ for } k \in \{1, \dots, n-3\}, \end{aligned}$$

and

$$t\left(\underbrace{1, \dots, 1}_{n-k \text{ terms}}\right) = 2 \text{ for } k \in \{2, \dots, n-1\}.$$

Thus, $t(\emptyset) = 2$ and $\mathbf{n}^* = \{2, 1, \dots, 1\}$.

Next, we show that the coarsest ECS is $\{3, 1, \dots, 1\}$. It is easy to see that $\mathcal{B}(\{2, 1, \dots, 1\}) \neq \emptyset$, and $\mathcal{B}(\{3, 1, \dots, 1\}) \neq \emptyset$ (e.g., $(1, 3.5, 3.5, 1, \dots, 1)$ for $\{3, 1, \dots, 1\}$ is an equilibrium binding agreement). Also, any agreement for any other coalition structure except π^* is blocked by an agreement in either $\mathcal{B}(\{2, 1, \dots, 1\})$ or $\mathcal{B}(\{3, 1, \dots, 1\})$. Hence, the coarsest ECS is $\{3, 1, \dots, 1\}$.

Example A3. Suppose $n \geq 5$. Let

$$v'''(s, \pi) = \begin{cases} 3 & \text{if } (s, \pi) = (3, \{3, 2, 1, \dots, 1\}), \\ 3 & \text{if } (s, \pi) = (3, \{3, 1, \dots, 1\}), \\ 10 & \text{if } (s, \pi) = (2, \{2, 1, \dots, 1\}), \\ 1 & \text{if } (s, \pi) = (1, \{3, 1, \dots, 1\}), (1, \{2, 1, \dots, 1\}), \text{ or } (1, \{3, 2, 1, \dots, 1\}), \\ 0 & \text{otherwise.} \end{cases}$$

Particularly, $v(3, \{3, 2\}) = 3$ when $n = 5$. When $n = 5$, it is routine to check that $\mathbf{n}^* = \{3, 1, 1\}$. When $n \geq 6$, we can show by induction that

$$\begin{aligned} t\left(3, \underbrace{1, \dots, 1}_{n-3-k \text{ terms}}\right) &= 1 \text{ for } k \in \{1, \dots, n-3\}, \\ t\left(3, 2, \underbrace{1, \dots, 1}_{n-5-k \text{ terms}}\right) &= 1 \text{ for } k \in \{1, \dots, n-5\}, \\ t\left(2, \underbrace{1, \dots, 1}_{n-2-k \text{ terms}}\right) &= \begin{cases} 1 & \text{for } k \in \{1, 2\}, \\ 3 & \text{for } k \in \{3, \dots, n-2\}, \end{cases} \end{aligned}$$

and

$$t\left(\underbrace{1, \dots, 1}_{n-k \text{ terms}}\right) = \begin{cases} 2 & \text{for } k \in \{2, 3, 4\}, \\ 3 & \text{for } k \in \{5, \dots, n-1\}. \end{cases}$$

Thus, $t(\emptyset) = 3$ and $\mathbf{n}^* = \{3, 1, \dots, 1\}$.

Next, we show that the coarsest ECS is $\{2, 1, \dots, 1\}$. This can be easily verified when $n = 5$. When $n \geq 6$, note that $\mathcal{B}(\{2, 1, \dots, 1\}) \neq \emptyset$ and $\mathcal{B}(\{3, 2, \dots, 1\}) = \mathcal{B}(\{3, 1, \dots, 1\}) = \emptyset$. Also, any agreement for any other structure except π^* is blocked by an agreement in $\mathcal{B}(\{2, 1, \dots, 1\})$.

Notes

- ¹ Even in a simple Cournot oligopoly with a linear demand function, the endogenous coalition structure is highly sensitive to the number of firms, and a general characterization remains an open question. See Section 7 of [Ray and Vohra \(1997\)](#).
- ² See, also, [Yi \(1997\)](#) for many examples.
- ³ [Bloch \(1996\)](#) assumes a fixed division of coalitional worths in the game set-up (and, thus, equal division in symmetric games). In this sense, he studies games with nontransferable utilities. But this does not change the fact that \mathbf{n}^* is still a focal prediction in symmetric games; see [Bloch \(1996\)](#). Note that [Bloch \(1996\)](#) identifies \mathbf{n}^* with the subgame perfect equilibrium in a sequential game of choosing coalition sizes.
- ⁴ See, e.g., [Ray and Vohra \(1997\)](#) and [Diamantoudi and Xue \(2007\)](#) for the inefficiency issue.
- ⁵ Symmetric games with two players are simple to analyze: the coarsest ECS and \mathbf{n}^* are always identical.
- ⁶ It is a slight abuse of notation to write $\{s_1, s_2, \dots, s_m\}$ as a set, as some elements in it can be identical. For example, in a three-player symmetric game, we write $\{1, 1, 1\}$ for the fragmented coalition structure. Also, as the set language suggests, $\{1, 2\}$ and $\{2, 1\}$ are the same numerical coalition structure. But we use both to refer to different orders of coalition formation for \mathbf{n}^* .
- ⁷ $x_S \gg y_S$ if $x_i > y_i$ for all $i \in S$.
- ⁸ In the original definition of equilibrium binding agreements, an agreement satisfies the best response property (p. 37 of [Ray and Vohra \(1997\)](#)). This property is absent in our definition, because we do not use a normal-form game as our primitive.
- ⁹ $\mathcal{B}(\pi_i^i) = \emptyset$ iff (π^*, x^*) blocks all agreements for π_i^i . See (B.3).
- ¹⁰ See (*) in Proof of Proposition 4 for the reason why this game is symmetric.
- ¹¹ Parameters are chosen so that all firms produce positive quantities in equilibrium under any association structure. See [Bloch \(1995\)](#) for details.

- ¹² Bloch (1995) adopts the assumption of equal division, instead. However, this does not change the equivalence result in Proposition 4 (see Proposition 6).
- ¹³ We always adopt this notational convention: When we write an agreement for some numerical coalition structure $\{m, n, \dots\}$, the first m components in this agreement are the utilities for m players in the first size- m coalition and the next n components are the utilities for n players in the second size- n coalition, and so on.
- ¹⁴ A strong version of positive externalities is that whenever $S \in \pi$, $\pi' \in \mathcal{R}(\pi)$ and $S \in \pi'$, we have $v(S, \pi) \geq v(S, \pi')$.

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