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Article

A General Model of Bertrand–Edgeworth Duopoly

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Abstract: This paper studies a class of two-player all-pay contests with externalities that encompass a general version of duopoly price competition. This all-pay contest formulation puts little restriction on production technologies, demand, and demand rationing. There are two types of possible equilibria: In the first type of equilibrium, the lower bound to pricing is the same for each firm, and the probability of any pricing tie above this price is zero. Each firm's equilibrium expected profit is their monopoly profit at the lower bound price. In the second type of equilibrium, one firm prices at the lower bound of the other firm's average cost and other firm prices according to a non-degenerate mixed strategy. This type of equilibrium can only occur if production technologies are sufficiently different across firms. We derive necessary and sufficient conditions for the existence of pure strategy equilibrium and use these conditions to demonstrate the fragility of deterministic outcomes in pricing games.

Keywords: price competition; contest; demand rationing; capacity constraints

1. Introduction

The determination of prices in markets with very few sellers has been a central subject of inquiry since the inception of mathematical economics. Edgeworth (1925) moved the understanding of this subject forward by appreciating the impact of consumer rationing and the prominence of price indeterminacy, or pricing cycles.¹ While the conceptual origins of the Bertrand–Edgeworth (hereafter BE) model can be traced back to Edgeworth, his basic insights were first formalized into a game theoretic model by Shubik (1959).² Shubik focused on understanding the range of pricing in mixed strategy equilibrium and the character of pure strategy equilibria when they exist.

These pricing games have been widely studied since Shubik's formalization. The standard BE model in the literature has the following features: firms possess constant marginal costs up to capacity (an absolute limit on production), and consumers are rationed according to either the efficient or proportional rationing rule.^{3,4} This BE model has been used to understand fundamental issues in price determination, including duopoly pricing and capacity investment (Allen & Hellwig, 1993; Davidson & Deneckere, 1986; Deneckere & Kovenock, 1996; Kreps & Scheinkman, 1983; Lepore, 2009; Levitan & Shubik, 1972; Osborne & Pitchik, 1986), sequential pricing (Allen, 1993; Allen et al., 2000; Deneckere & Kovenock, 1992), large markets (Allen & Hellwig, 1986; Dixon, 1987, 1992; Vives, 1986), oligopoly (De Francesco & Salvadori, 2010; Hirata, 2009), and uncertainty (de Frutos & Fabra, 2011; Lepore, 2008, 2012; Reynolds & Wilson, 2000). Only a few papers have investigated BE models with cost structures outside the constant marginal cost case.⁵ Dixon (1987) considers a model of BE oligopoly with strictly convex costs, showing the non-existence of pure strategy equilibrium for the case of efficient and proportional rationing. Yoshida (2006)



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characterizes equilibrium pricing in symmetric duopoly with convex cost and efficient rationing.⁶ While the literature has produced interesting results, the models are restrictive in terms of production technologies and demand rationing of consumers and asymmetries across firms.

The purpose of this paper is to analyze the properties of equilibria in a BE model with a broad range of production technology, minimal restriction demand rationing, and asymmetries across firms. Our approach to the analysis is based on modeling price competition as a particular extension of an all-pay contest (Siegel, 2009, 2010, 2014). In order to contextualize our analysis, we identify some of the abstract properties of the BE model with those of a standard all-pay auction.⁷ In the BE model, firms place bids in the form of a price in an attempt to win the larger share of demand, which goes to the firm with the highest bid (lowest price). There are two fundamental distinctions between the BE model and the all-pay contest or traditional all-pay auction. First, the payoff of the losing player (the firm with the highest price) may depend on the price of the winner through the rationing of residual demand, while, traditionally, the losing player's payoff depends only on her committed bid. Second, the payoff of both the winner and loser can be non-monotonic in her bid, as a reduction in price increases the quantity demanded, possibly raising profits, while an increased bid in traditional contests merely commits the winner or loser to a lower payoff.⁸

In Section 2, we present the general model and introduce key notation. The model is defined based on abstract properties of the front-side profit of the lower-priced winner firm and the residual profit of the higher-priced loser firm. The abstract contest formulation allows us to analyze a model with a broad range of underlying specifications: general production technology including the case of *U*-shaped average cost of production, minimal restriction on demand allowing for a wide swath of demand rationing (including rationing outcomes that are equilibria from consumer search), and asymmetries across firms.⁹ In order to establish the bounds on equilibrium prices and payoffs, we define the following preliminary objects. First, define the *critical judo price* as the highest price either firm can set to guarantee that the other firm would rather maximize its residual profit than undercut. This terminology is based on the sequential pricing model of judo economics by Gelman and Salop (1983).¹⁰ The second important price we define is the *critical safe price*, which is the infimum of all prices at which it earns at least its $\min \hat{\pi} \leq \max \pi$ profit if its rival undercuts.

The general results on the properties of all equilibria are presented in Section 3. After establishing generic equilibrium existence,¹¹ we show that there are only two types of possible equilibria. The first type of equilibrium is such that both firms' pricing distributions have the same lower bound, and ties at prices greater than the lower bound occur with probability zero. This type of equilibrium includes the possibility of pure strategy equilibrium in which firms play the lower bound price with certainty. The second type of equilibrium can only occur if the firms have sufficiently different production technology, and are such that one firm plays a pure strategy price while the other plays a nondegenerate mixed strategy. Since a model specification can have multiple non-payoff equivalent equilibria, we show that, in all equilibria, the expected profits of each firm are bounded between its monopoly profit at the critical safe price and the critical judo price. In the process of establishing the payoff bounds, we provide abstract bounds for the range of equilibrium pricing.

The results particular to pure strategy equilibrium are presented in Section 4. We provide necessary and sufficient conditions for the existence of pure strategy equilibrium. All pure strategy equilibria must be symmetric and only exist under two circumstances. The first case is a symmetric pricing profile at which each firm's residual profit is maximized and equal to the monopoly profit at the same price. The second case is a symmetric pricing

profile at which exactly one firm's monopoly profit exceeds its residual profit (weakly exceeding its maximum residual profit), and the sharing rule is such that this firm receives its monopoly profit with certainty. In this second case, the price must maximize the other firm's residual profit. The existence of a pure strategy equilibrium does not guarantee uniqueness, as there may be additional mixed strategy equilibria that exist concurrently. We show that a pure strategy equilibrium price x^* is unique as the pure strategy equilibrium price x^* is unique if, for each firm, this price is the unique maximizer of residual profit when the other firm prices at x^* , and each firm's residual profit is nonincreasing in the other firm's price.

We present a special case model using a standard BE construction with all assumption made directly on each firm's production technology and demand in Section 5. First, we explore additional properties of pure strategy equilibrium for this special case model, identifying the necessary and sufficient conditions on the underlying primitives of production technology and demand. Second, we examine the impacts of demand and supply shifts on the bounds of equilibrium prices and profits. These shifts can accommodate changes in rationing, production cost, or capacity. We demonstrate that an increase in residual demand will weakly increase the bounds on the lowest equilibrium price along with the bounds on profits; however, by example, we show that the upper bound on pricing may be reduced. An increase in a firm's supply weakly decreases the bounds on the lowest equilibrium price along with the bounds on the other firm's profits. A general prediction cannot be made for the bounds on the profit of the firm with the supply increase, as there are countervailing effects: a direct effect through which lower costs or higher capacities enhance profitability, and an indirect competitive effect through which those changes increase the level of competition, driving down prices and profits.¹²

Finally, all proofs of lemmas and propositions are located in Appendix A.

2. The Model

In this section, we lay out the general model and then provide a subsection of examples to illustrate the scope of the model. All assumptions stated in this section are maintained throughout the remainder of the paper. Consider a homogeneous product industry with two firms $i = 1, 2$. We will use $j = 1, 2$, to refer to the firm other than i . The firms simultaneously and independently announce prices. We denote by p_i the price of firm i and by p the vector of both firms' prices. Since p is the vector of prices (p_1, p_2) , we will use x to unambiguously denote a single price when it is not associated with a particular firm. The profit that each firm receives depends on whether it has a lower price than the other firm. The *front-side profit* of the firm i with a lower price than firm j is $\varphi_i(p_j)$, while the *residual profit* of the firm i with a higher price than firm j is $\psi_i(p)$. The domain of residual profit ψ_i is $\{(p_i, p_j) \in \mathbb{R}_+^2 : p_i \geq p_j\}$, as it need not be defined for prices such that $p_i < p_j$ since the residual profit cannot be obtained at such prices. We make the following assumptions on the profit functions φ_i and ψ_i .

Assumption 1. $\varphi_i(x) \geq \psi_i(x, x') \geq 0$ for all $x \geq x'$.

The assumption that the front-side profit is at least as large as the residual profit is consistent with the notion that customers prefer lower prices, and thus, the firm with the lowest price has weakly greater potential to sell. This assumption also implies that the lower-price firm is not required to sell units that decrease profit.

Assumption 2. For each firm i , there exists the largest $\underline{a}_i$ such that $\varphi_i(x) = \psi_i(x, p_j) = 0$ for all $p_j \leq x \leq \underline{a}_i$. Further, $\psi_i(p_i, x) = \varphi_i(p_i)$ for all $p_i \geq x$ such that $x < \underline{a}_j$.

The price $\underline{a}_i$ is the minimal price for which a firm is willing to produce, typically the infimum of the average cost of production when explicitly modeled. As such, when a firm i prices below $\underline{a}_i$, firm j 's residual profit is equal to its front-side profit because firm i produces nothing.

Assumption 3. φ_i has a unique maximizer $\hat{p}_i > \underline{a}_i$ with $\varphi_i(\hat{p}_i) > 0$. On the interval $(\underline{a}_i, \hat{p}_i)$, φ_i is positive valued, and strictly increasing. Further, $\psi_i(p_i, p_j) = 0$ for all $p_i \geq p_j \geq \hat{p}_j$, and $p_i > \hat{p}_j$.

This assumption on the front-side profit is weaker than assuming the strict quasiconcavity of φ_i as it does not restrict behavior at prices $p_i > \hat{p}_i$. The third part of the assumption is that there is no residual profit for firm i if firm i prices above $\hat{p}_j$. This assumption fits the two cases that demand is continuous at $\hat{p}_j$, or that there is zero quantity demanded above the price $\hat{p}_j$.

The following assumption is important for our characterization.

Assumption 4. For each firm i , there exists a price $\underline{\rho}_i \in [\underline{a}_i, \min\{\hat{p}_i, \hat{p}_j\}]$ such that

$$\begin{aligned}\varphi_i(x) &> \psi_i(x, x) \quad \text{for all } x \in (\underline{\rho}_i, \hat{p}_i], \\ \varphi_i(x) &= \psi_i(x, x') \quad \text{for all } x' \leq x < \underline{\rho}_i.\end{aligned}$$

Further, if $\underline{a}_i > \underline{a}_j$, then $\underline{\rho}_i \leq \underline{\rho}_j$.

The maximum of the two firms' prices $\underline{\rho}_i$ is a primary object used in the analysis that follows; as such, we denote $\underline{\rho} = \max\{\underline{\rho}_1, \underline{\rho}_2\}$.

Remark 1. The existence of the prices $\underline{\rho}_i$ is a natural consequence of traditional constructions of the BE model. It is common that $\underline{\rho}_i$ corresponds to the price at which total industry supply is equal to market demand, as below such a price, the residual demand would exceed the supply of the high-priced firm. Alternatively, in that case that the marginal cost of i is less than the marginal cost of j , $\underline{\rho}_i$ can correspond to the (constant) marginal cost of firm j , as below that price, firm j does not produce, leaving the market demand to the residual, while at or above that price, firm j produces up to its capacity, potentially limiting residual profit below the front side. Assumption 4 accommodates either of these scenarios and generally allows more variety of market structures. For example, it allows for situations in which unsatiated demand is not rationed to other firms, as may be the case with directed search models.

We make the following assumption to rule out the possibility that one firm has a sufficiently competitive advantage to act as a monopoly.

Assumption 5. For each i and j , $\hat{p}_i > \underline{a}_j$.¹³

Each firm i 's profit is specified as follows:

$$u_i(p) = \begin{cases} \varphi_i(p_i) & p_i < p_j \\ \alpha_i(p)\varphi_i(p_i) + (1 - \alpha_i(p))\psi_i(p) & p_i = p_j \\ \psi_i(p) & p_i > p_j \end{cases}, \quad (1)$$

where $\alpha_i(p) \in [0, 1]$ and $\alpha_1(p) + \alpha_2(p) \in (0, 2)$. If we instead assume that $\alpha_1 + \alpha_2 = 1$, then this restricts attention to sharing rules that assign one firm its front-side profit and the other its residual profit, with some randomization over the assignment. By permitting the sum of

the shares to be greater (or less) than one, the model captures any share of demand at ties, which can naturally result in each firm receiving a (non-stochastic) profit strictly between its front-side and residual profits.

We denote the set of maximizers of ψ_i at any p_j by $\tilde{P}_i(p_j)$. Denote the maximized residual profit by $\tilde{\psi}_i(p_j)$, that is,

$$\tilde{\psi}_i(p_j) = \max_{p_i \geq p_j} \psi_i(p_i, p_j).$$

Assumption 6. *There exists the lowest price $\hat{x}$ such that $\varphi_i(\hat{x}) > \varphi_i(x)$ and $\psi_i(\hat{x}, p_j) \geq \psi_i(x, p_j)$ for all prices x and p_j with $x > \hat{x} \geq p_j$.*

Note that $\hat{x} \geq \hat{p}_i$ for each firm i . Given Assumption 6, the price $\hat{x}$ weakly dominates all prices $x > \hat{x}$.

Assumption 7. *Both φ_i and ψ_i are continuous in p_i on $[0, \hat{x})$ and left continuous at $\hat{x}$. ψ_i is right upper semicontinuous in p_j , that is, $\limsup_k \psi_i(p_i, x^k) \leq \psi_i(p_i, x)$ for any sequence $\{x^k\}$ such that $p_i \geq x^k > x$ and $x^k \rightarrow x$.*

These continuity assumptions are satisfied in most BE models previously studied. The right upper semicontinuity captures the notion that a firm does not drastically decrease its production when the other firm's price increases. The potential for discontinuities at prices above $\hat{x}$ allows the model to accommodate settings with box demand.

Define $\bar{r}_i$ to be firm i 's *judo price*, which is the lower bound such that the front-side profit of firm i is greater than the maximal residual profit of firm i when firm j uses any price weakly greater. Formally,

$$\bar{r}_i = \inf\{x | \varphi_i(x) > \sup_{z \geq x} \tilde{\psi}_i(z)\}.$$

Define $\underline{r}_i$ to be firm i 's *safe price*, which is the lower bound of price such that the front-side profit of firm i is greater than the highest profit that firm i can guarantee itself. Formally,

$$\underline{r}_i = \inf\{x | \varphi_i(x) > \underline{u}_i\},$$

where $\underline{u}_i = \inf_{p_j} \sup_{p_i} u_i(p_i, p_j)$.

Define the larger of the two firms' judo prices to be *critical judo price*, denoted by $\bar{r} = \max \bar{r}_i$. Similarly, define the larger of the two firms' safe prices to be the *critical safe price*, denoted by $\underline{r} = \max \underline{r}_i$. Based on the fact that $\underline{u}_i \leq \sup_{z \geq x} \tilde{\psi}_i(z)$ and that φ_i is strictly increasing when positive, the judo price is always weakly greater than the safe price, that is, $\bar{r} \geq \underline{r}$. Note further that $\bar{r} \leq \hat{x}$.

Define firm i 's *judo profit* to be the front-side profit of firm i at the critical judo price, denoted by $\bar{\varphi}_i \equiv \varphi_i(\bar{r})$. Similarly, define firm i 's *safe profit* to be the front-side profit of firm i at the critical safe price, $\varphi_i \equiv \varphi_i(\underline{r})$.

For equilibrium strategies $\mu = (\mu_1, \mu_2)$, we use $\underline{x}_i$ and $\bar{x}_i$ to denote the infimum and supremum of the support of firm i 's strategy, respectively. We will use $\underline{x}$ to denote the minimum of $\underline{x}_1$ and $\underline{x}_2$, and $\bar{x}$ to denote the maximum of $\bar{x}_1$ and $\bar{x}_2$.¹⁴ Further, we define F_i to be the distribution function (CDF) of firm i 's mixed strategies on $[\underline{x}, \bar{x}]$, with $F = (F_1, F_2)$. Additionally, let u_i^* denote firm i 's equilibrium expected profit.

Before proceeding with the analysis of the model, we discuss some underlying specifications that our model contains in the following subsection.

Examples Within Our Framework

To provide context for the abstract model, we present three examples nested within our framework. The framework of our model puts very little restriction on the underlying market demand except continuity on $[0, \hat{x})$, even allowing for increasing demand. As such, the demand structure is easily understood. To help convey the generality of our model, we present the following examples to demonstrate the range of production technologies and demand rationing rules that can be accommodated in our model. With this purpose in mind, we use rectangular unit demand in these numerical examples. Formally, the market demand is

$$D(x) = \begin{cases} 0 & \text{if } x > 1 \\ 1 & \text{if } x \in [0, 1] \end{cases}.$$

The first two examples exhibit production technologies included in our specification. The third example exhibits a residual demand rationing scheme based on directed search.

In each example, we specify each firm's cost of production as a function of quantity produced and use that to derive the supply correspondence of quantities that maximize the direct profit function $\pi_i(x, q) = xq - c_i(q)$.¹⁵ We then use these to derive the corresponding front-side and residual profit functions as well as the prices $\hat{x}$, $\hat{p}_i$, $\underline{a}_i$, and $\underline{p}_i$.

Example 1 (Discontinuous supply). Firm 1's cost of production is

$$c_1(q) = \begin{cases} \frac{1}{2}q - \frac{1}{8} & \text{if } q \geq 1/2 \\ \frac{1}{4}q & \text{if } q \in [0, 1/2] \end{cases},$$

while firm 2's cost of production is $c_2(q) = \frac{1}{3}q$. Neither firm faces a capacity constraint. The supply correspondence of each firm is thus

$$\vartheta_1(p_1) = \begin{cases} \{\infty\} & \text{if } p_1 > 1/2 \\ [\frac{1}{2}, \infty] & \text{if } p_1 = 1/2 \\ \{\frac{1}{2}\} & \text{if } p_1 \in (1/4, 1/2) \\ [0, \frac{1}{2}] & \text{if } p_1 = 1/4 \\ \{0\} & \text{if } p_1 \in [0, 1/4) \end{cases},$$

and

$$\vartheta_2(p_2) = \begin{cases} \{\infty\} & \text{if } p_2 \geq 1/3 \\ [0, \infty] & \text{if } p_2 = 1/3 \\ \{0\} & \text{if } p_2 \in [0, 1/3) \end{cases}.$$

The front-side profits are

$$\varphi_1(p_1) = \begin{cases} 0 & \text{if } p_1 > 1 \\ \left(p_1 - \frac{1}{4}\right)\frac{1}{2} + \left(p_1 - \frac{1}{2}\right)\frac{1}{2} & \text{if } p_1 \in [1/2, 1] \\ \left(p_1 - \frac{1}{4}\right)\frac{1}{2} & \text{if } p_1 \in [1/4, 1/2) \\ 0 & \text{if } p_1 \in [0, 1/4) \end{cases},$$

and

$$\varphi_2(p_2) = \begin{cases} 0 & \text{if } p_2 > 1 \\ p_2 - \frac{1}{3} & \text{if } p_2 \in [1/3, 1] \\ 0 & \text{if } p_2 \in [0, 1/3) \end{cases}.$$

As the supply correspondence is not single-valued, the residual profits of the firms may depend on the particular quantity chosen. Rather than present all possibilities, we use the convention that the firms produce the largest quantity possible in their supply correspondence. With this convention,

the residual profits of the firms are defined with demand rationed according to the efficient (or equivalently proportional) rule

$$\psi_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_2 \geq 1/3 \\ \varphi_1(p_1) & \text{if } p_2 < 1/3 \end{cases},$$

$$\psi_2(p_2, p_1) = \begin{cases} 0 & \text{if } p_2 > 1, \text{ or } p_1 \geq 1/2, \\ & \text{or } p_2 < 1/3, \text{ or } p_1 \in [1/4, 1/2) \\ \left(p_2 - \frac{1}{3}\right)\frac{1}{2} & \text{if } p_2 \geq 1/3, p_1 \in [1/4, 1/2) \\ \varphi_2(p_2) & \text{if } p_1 < 1/4 \end{cases}.$$

The key model parameters for this example are $\hat{x} = \hat{p}_i = 1$, $\underline{a}_1 = 1/4$, $\underline{a}_2 = 1/3$, and $\underline{\rho}_i = \underline{\rho} = 1/3$.

The next example includes firms with a U -shaped average and marginal costs.

Example 2 (U-shaped average and marginal costs). Each firm i 's cost of production is

$$c_i(q) = \begin{cases} \frac{2}{3}q^3 - \frac{1}{4}q^2 + \frac{1}{16}q + \frac{1}{20} & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases}.$$

Neither firm faces a capacity constraint. The supply correspondence of each firm i can be expressed as the function

$$\vartheta(p_i) = \begin{cases} \frac{1 + \sqrt{32p_i - 1}}{8} & \text{if } p_i \geq 0.194 \\ 0 & \text{if } p_i \in [0, 0.194] \end{cases}.$$

Thus, the front-side profit of firm i is

$$\varphi_i(p_i) = \begin{cases} 0 & \text{if } p_i > 1 \\ p_i \vartheta(p_i) + c_i(\vartheta(p_i)) & \text{if } p_i \in [0.194, 1] \\ 0 & \text{if } p_i \in [0, 0.194] \end{cases},$$

and the residual profit of firm i ($p_i \geq p_j$) is (again using efficient rationing)¹⁶

$$\psi_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_j \geq 0.397 \\ p_i \min\{\vartheta(p_i), 1 - \vartheta(p_j)\} + c_i(\min\{\vartheta(p_i), 1 - \vartheta(p_j)\}) & \text{if } p_j \in (0.194, 0.397) \\ \varphi_i(p_i) & \text{if } p_j < 0.194 \end{cases}.$$

The key model parameters for this example are $\hat{x} = \hat{p}_i = 1$, $\underline{a}_i = 0.194$, and $\underline{\rho}_i = \underline{\rho} = 0.3125$.

The final example has demand rationing determined by a directed consumer search game.

Example 3 (Search). Each firm has zero cost of production. Both firms have the same capacity $k \in (1/2, 1)$, with each firm limited to producing a quantity of at most k . Demand rationing is determined by the equilibrium of a directed consumer search game. There is a unit mass of consumers, each of which demands a single unit of the good, which they value at 1. The consumers observe the prices of the firms, and then simultaneously choose a firm to visit. If a mass $W_i \leq k$ of consumers visit firm i , each of those consumers receives a good at price p_i . If a mass $W_i > k$ of consumers visits firm i , then each of those consumers receives a good at price p_i with probability k/W_i . Consumers that do not receive goods obtain a payoff of zero. In any pure strategy equilibrium of the consumer

game, if $p_i < p_j$, then either all consumers shop at firm i , or the mass W_i of consumers that shop at firm i satisfies

$$\frac{k}{W_i}(1 - p_i) = 1 - p_j.$$

Therefore, we can write

$$W_i = \begin{cases} 0 & \text{if } p_i > 1 \\ \min\left\{1, k \frac{1-p_i}{1-p_j}\right\} & \text{if } p_i \in [0, 1] \end{cases},$$

and the mass of consumers who go to firm j are $W_j = 1 - W_i$. Notice that $W_i \geq k$ for all prices $p_i < p_j$. Thus, we can write the front-side and residual profit of firm i as follows:

$$\varphi_i(p_i) = \begin{cases} 0 & \text{if } p_i > 1 \\ p_i k & \text{if } p_i \in [0, 1] \end{cases},$$

and

$$\psi_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_i \geq 1 \\ p_i \max\left\{0, 1 - \frac{k(1-p_j)}{1-p_i}\right\} & \text{if } p_i \in [p_j, 1] \end{cases}.$$

The key model parameters for this example are $\hat{x} = \hat{p}_i = 1$, $\underline{a}_i = \underline{\rho}_i = \underline{\rho} = 0$.

3. Mixed Strategy Equilibria

In this section, we establish some abstract properties of all equilibria. We begin the analysis by dealing with the problem of existence of equilibrium. As long as each firm's residual profit is lower semicontinuous in the other firm's price, we are able to show that an equilibrium exists if $\underline{\rho}_1 = \underline{\rho}_2$. When there is firm i such that $\underline{\rho}_i < \underline{\rho}$, our proof requires an additional condition that this firm i receives its front-side profit with certainty at ties below $\underline{\rho}$.

Proposition 1. Assume that (1) $\alpha_i(x, x) = 1$ for any x such that $\varphi_i(x) > \psi_i(x, x)$ and $\varphi_j(x) = \psi_j(x, x)$ and (2) that ψ_i is lower semicontinuous in p_j . Then a mixed strategy equilibrium of the BE game exists.

If the residual profits ψ_i are continuous, then the existence of equilibrium for the BE duopoly follows directly from Proposition 2 in Allen and Lepore (2014). A generalization of this proposition is presented in Appendix A.1, which applies to the case in which the residual profits are not continuous. The requirement that ψ_i is lower semicontinuous is added not out of necessity for the existence of equilibrium, but out of necessity for the abstract verification of the existence of equilibrium without explicit calculation.¹⁷

We now turn to the analysis of the set of equilibria of the BE game. The following proposition partitions the set of equilibria into two possible types and provides a partial characterization of each type.

Proposition 2. There are two types of equilibria: symmetric lower bound and asymmetric lower bound.

Symmetric lower bound equilibria are such that

- $\underline{x}_1 = \underline{x}_2 = \underline{x} \geq \underline{\rho}$;
- The probability of an atom at any price $x > \underline{\rho}$ is zero;
- The probability of a tie at $\underline{x}$ is positive only if $\varphi_i(\underline{x}) = \psi_i(\underline{x}, \underline{x})$ for each firm i ; and
- At most one firm can price higher than $\hat{x}$.

Asymmetric lower bound equilibria are only possible for the case that $\underline{a}_i > \underline{a}_j$ and are such that

- $\underline{x}_i < \underline{x}_j = \underline{a}_i = \underline{\rho}$;
- $\mu_j(\{\underline{\rho}\}) = 1$;
- $\mu_i([\underline{\rho}, \underline{\rho} + \varepsilon)) > 0$ for any $\varepsilon > 0$; and
- $u_i^* = 0, u_j^* > 0$.

Notice that every pure strategy equilibrium must be a symmetric lower bound equilibrium. Either type of equilibrium can be mixed, although the asymmetric lower bound allows only one player to use a nondegenerate mixed strategy.

The proof of Proposition 2 is extensive and based on the series of Lemmas 1–4. The first of these lemmas provides an upper bound for at least one firm's prices.

Lemma 1. *In any equilibrium, at most one firm can select prices above $\hat{x}$. That is, in any equilibrium $\mu_i([0, \hat{x}]) = 1$ for at least one firm i .*

The following lemma is instrumental in many of the proofs of this paper. This lemma establishes that relevant ties ($p_1 = p_2 > \underline{\rho}$) occur with probability zero in all equilibria of the BE game.¹⁸ It is worth noting that μ_i may have an atom $\mu_i(\{x\}) > 0$ at a price $x > \underline{\rho}$, although $\mu_1(\{x\})\mu_2(\{x\}) = 0$. That is, ties above the price such that the front-side profit equals residual profit occur with probability zero in equilibrium.

Lemma 2. *In any equilibrium, if firm i 's strategy has mass at a price x such that $\varphi_i(x) > \psi_i(x, x)$, then either $\alpha_i(x, x) = 1$ or firm j 's strategy does not have mass at x . Consequently, in any equilibrium, both firms cannot simultaneously have an atom at a price $p_1 = p_2 > \underline{\rho}$, and the equilibrium strategies and payoffs are unaffected by the choice of sharing rule α at any price $x > \underline{\rho}$.*

The next lemma constrains the possibilities for equilibria in which the infimum of each firm i 's strategy is not the same.

Lemma 3. *In any equilibrium, if $\underline{x}_i < \underline{x}_j$, then firm j 's strategy is degenerate, with $\underline{x}_j = \underline{\rho}$ and $\mu_j(\{\underline{\rho}\}) = 1$. In any such equilibrium, $\underline{a}_j < \underline{a}_i = \underline{\rho}$, $u_i^* = 0$, $u_j^* > 0$, and $\mu_i([\underline{\rho}, \underline{\rho} + \varepsilon)) > 0$ for any $\varepsilon > 0$.*

We use the following example to help illustrate the type of equilibria characterized by Lemma 3. The types of equilibria exhibited in this example are not novel observations and were comprehensively treated in Deneckere and Kovenock (1996). The purpose of this example is simply to provide a concrete illustration of the statement of Lemma 3.

Example 4. *Consider a market in which firm 1 has zero cost of production and firm 2 has a constant marginal cost of $c > 0$. Demand is given by $D(x)$, a continuous and nonincreasing function with $D(0) < \infty$ and $D(c) > 0$. Given such a market, the front-side profits are $\varphi_1(x) = D(x)x$ and $\varphi_2(x) = D(x)(x - c)$. Residual profits are*

$$\psi_1(x, p_2) = \begin{cases} D(x)x & \text{if } p_2 < c \\ 0 & \text{if } p_2 \geq c \end{cases}$$

and $\psi_2(x, p_1) = 0$. Assume that $D(x)x$ is maximized at some price $\hat{p}_1 > c$.

In this game, $\underline{a}_1 = 0$, $\underline{a}_2 = c$, and $\underline{\rho}_1 = \underline{\rho}_2 = \underline{\rho} = c$. There are infinitely many equilibria in which firm 1 sets a price $p_1^* = c$. An example of such equilibrium strategies for firm 2 is to set $\mu_2([c, c + \varepsilon]) \geq 1 - \varphi_1(c) / \varphi_1(\hat{p}_1)$ for all $\varepsilon > 0$, with the remaining mass placed at prices below c and possibly above $\hat{p}_1$. To see that such strategy profiles are equilibria, note that $u_2(c, p_2) = 0$ for

all p_2 , so firm 2 has no incentive to deviate. Firm 1 has no incentive to deviate to any price other than $\hat{p}_1$. Such a deviation is not profitable since $[1 - \varphi_1(c) / \varphi_1(\hat{p}_1)]\varphi_1(\hat{p}_1) \leq \varphi_1(c)$.

The next lemma characterizes equilibria in which the infimum of the support of each firm's strategy is identical.

Lemma 4. *The following conditions hold at any equilibrium in which both firms use nondegenerate mixed strategies: (i) $\underline{x}_1 = \underline{x}_2 = \underline{x} \geq \underline{p}$, and (ii) if there is a firm i such that $\varphi_i(\underline{x}) > \psi_i(\underline{x}, \underline{x})$, then neither firm's equilibrium strategy may have an atom at $\underline{x}$.*

An important implication of Lemma 4 is that, in such an equilibrium, each firm selects prices arbitrarily close to $\underline{x}$ with positive probability, receiving its front-side profit with probability arbitrarily close to one. Consequently, each firm i 's expected profit in equilibrium is $u_i^* = \varphi_i(\underline{x})$.

Because of the general structure of each firm's residual profit, there can be multiple non-payoff equivalent equilibria. The following example is a simple BE game with two non-payoff equivalent mixed strategy equilibria.

Example 5. *Consider a market with a mass of 1 of consumers all with a maximum willingness to pay of 1 for the homogeneous product. The perceived value of the good to the consumers is based on the observed prices. When prices are low (high), consumers have a low (high) willingness to pay. The willingness to pay of any consumer is*

$$v(p) = \begin{cases} 1 & \text{if } \min\{p_1, p_2\} > 1/2 \\ 1/2 & \text{if } \min\{p_1, p_2\} \leq 1/2 \end{cases}.$$

Thus, if both firms set prices weakly less than 1/2, then willingness to pay is 1/2. If both firms price above 1/2, then willingness to pay is 1.

Each firm i has zero production cost but is limited to producing a quantity no greater than its capacity $k_i < 1$, where $k_1 + k_2 \geq 1$. The demand system follows standard "efficient rationing", although this is determined by Nash equilibrium play of the consumer search game.

The front-side profit of firm i is

$$\varphi_i(x) = \begin{cases} 0 & \text{if } x > 1 \\ xk_i & \text{if } x \in [0, 1] \end{cases},$$

and the residual profit of firm i is

$$\psi_i(p_i, p_j) = \begin{cases} 0 & \text{if } p_i > 1, \text{ or } p_i > 1/2 \geq p_j \\ p_i(1 - k_j) & \text{if } 1 \geq p_i \geq p_j > 1/2 \text{ or } 1/2 \geq p_i \geq p_j \end{cases}.$$

Let $k_1 = k_2 = 5/8$. There are two mixed strategy equilibria. Each equilibrium is symmetric, with both firms employing the CDF $F(x)$ or both firms employing the CDF $G(x)$ defined as follows:

$$F(x) = \begin{cases} 0 & \text{if } x < 3/10 \\ \frac{10x-3}{4x} & \text{if } x \in [3/10, 1/2] \\ 1 & \text{if } x > 1/2 \end{cases},$$

$$G(x) = \begin{cases} 0 & \text{if } x < 3/5 \\ \frac{5x-3}{2x} & \text{if } x \in [3/5, 1] \\ 1 & \text{if } x > 1 \end{cases}.$$

In the first equilibrium, each firm i has the expected payoff $u_i^* = 3/16$, while in the second equilibrium, each firm i has the expected payoff $u_i^* = 3/8$.

Note that, for this example, $\bar{r} = 3/5$ and $\underline{r} = 3/10$. Consequently, $\bar{\varphi}_i(x) = 3/8$ and $\underline{\varphi}_i(x) = 3/16$, which coincide with the two equilibrium expected payoffs.

In the following proposition, we show that the expected profits of all equilibria lie between the safe profit and the judo profit. The bounds on the equilibrium expected payoffs follow immediately from showing that the lower bound on pricing of any equilibrium must lie between the critical safe price and the critical judo price.

Proposition 3. *All equilibria are such that $u_i^* \in [\underline{\varphi}_i, \bar{\varphi}_i]$.*

These payoff bounds provide a solid foundation for understanding the properties of the equilibria of BE games in a general setting. While the literature on BE games has, in some cases, been able to provide precise payoff predictions, the bounds presented here apply to a much larger class of games than previously studied. The proposition gives precise predictions of the equilibrium profits when the judo price and safe price coincide ($\underline{r} = \bar{r}$). While it may seem restrictive to characterize only the bounds on the equilibrium profits along with the lowest equilibrium price, this result is inherently valuable as a precise computation of the equilibrium bounds and payoffs can only be obtained by first calculating the equilibrium strategies. Without very precise and often simplistic classes of functions, such a computation can be prohibitively difficult, and we are unaware of any method that would allow one to solve for the equilibrium strategies in general.

In order to bound the upper bound of equilibrium pricing, we need to formally define maximizers of the residual profit function conditional on the other firm's mixed strategy. Given a distribution of prices F_j , define the set of *conditional residual maximizers* $\tilde{P}_i(F_j) \equiv \arg \max_x E_{F_j}[\psi_i(x, p_j) | p_j \leq x]$. The following lemma demonstrates that the upper bound on pricing must lie between the smallest and largest of all firms' conditional residual maximizers.

Lemma 5. *For any equilibrium $F = (F_1, F_2)$, $\bar{x} \in \tilde{P}_1(F_2) \cup \tilde{P}_2(F_1)$.*

Lemma 5 is useful for the technical analysis but has a limited predictive value for two reasons. First, this result requires knowledge of equilibrium pricing distributions, and second, it is not possible to infer payoffs from the upper bound on pricing because of the dependence of the residual profit on the lower price.

Remark 2. *In contrast with the literature on BE duopoly, the generality of our specification introduces an additional level of pricing indeterminacy. The first level of indeterminacy, in the previous literature, is based on the equilibrium being in non-degenerate mixed strategies. The second level of indeterminacy present in our framework is driven by the fact that there can be multiple non-payoff equivalent equilibria.*

4. Pure Strategy Equilibria

The objective of this section is to understand when price indeterminacy is resolved by equilibrium play. To that end, we present necessary and sufficient conditions for the existence of pure strategy equilibrium. Under additional restrictions, our conditions become necessary and sufficient for this to be the unique equilibrium. When these conditions fail to hold, all equilibria of the pricing game must be in mixed strategies.

We show that the only possible pure strategy equilibrium is both firms pricing at $\underline{p}$. This allows us to derive necessary and sufficient conditions for pure strategy equilibrium.

There are two different sets of conditions for the two cases: $\varphi_i(\underline{\rho}) = \psi_i(\underline{\rho}, \underline{\rho})$ for all i , and for some i , $\varphi_i(\underline{\rho}) > \psi_i(\underline{\rho}, \underline{\rho})$. The only condition for the case that $\varphi_i(\underline{\rho}) = \psi_i(\underline{\rho}, \underline{\rho})$ for all i is that $\underline{\rho}$ is a residual maximizer for both firms. For the case that there is a firm i with $\varphi_i(\underline{\rho}) > \psi_i(\underline{\rho}, \underline{\rho})$, firm i must obtain the front-side profit for sure at the tie and cannot have a residual maximizer that is better than this front-side profit, while for firm j , it must be that $\varphi_j(\underline{\rho}) = \psi_j(\underline{\rho}, \underline{\rho})$ and $\underline{\rho}$ is a residual maximizer for firm j .

Proposition 4. Any pure strategy equilibrium (x^*, x^*) must be symmetric and $x^* = \underline{\rho}$. The following are two types of possible pure strategy equilibrium:

1. If $\varphi_i(\underline{\rho}) = \psi_i(\underline{\rho}, \underline{\rho})$ for all i , then $x^* = \underline{\rho}$ is a pure strategy equilibrium if and only if $\underline{\rho} \in \tilde{P}_i(\underline{\rho})$ for all i ;
2. If $\varphi_i(\underline{\rho}) > \psi_i(\underline{\rho}, \underline{\rho})$ for some i , then $x^* = \underline{\rho}$ is a pure strategy equilibrium if and only if $\varphi_j(\underline{\rho}) = \psi_j(\underline{\rho}, \underline{\rho})$, $\underline{\rho} \in \tilde{P}_j(\underline{\rho})$, $\alpha_i(\underline{\rho}, \underline{\rho}) = 1$, and $\varphi_i(\underline{\rho}) \geq \tilde{\psi}_i(\underline{\rho})$.

Intuitively, it is easy to understand why both firms setting a price of $\underline{\rho}$ is the only possible pure strategy equilibrium. Each firm i must price at least at $\underline{a}_i$, as otherwise, firm j would act as a monopoly and then firm i would have a profitable deviation to undercut the monopoly price of firm j . Next, each firm must price at least at $\underline{\rho}_j$, as otherwise, the firm with the lowest price could increase its price and still receive a payoff equal to their front-side profit. This is a profitable deviation since the front-side profit is increasing. Similarly, both firms must set the same price, as otherwise, the firm with the low price could improve by increasing its price. Lastly, if both firms set a higher price than $\underline{\rho}$, then each firm's front-side profit is higher than their respective residual profit. Since both firms cannot receive their front-side profit with certainty, at least one firm has incentive to undercut.

We now proceed to the proof of Proposition 4.

Proposition 4 is important in that it specifies the exact circumstances that Edgeworth's concerns about price indeterminacy can be alleviated. However, in this general setting, the existence of a pure strategy equilibrium does not guarantee the uniqueness of this equilibrium. Since $\underline{\rho}$ is uniquely defined, it is the only pure strategy equilibrium candidate; however, it may be that a mixed strategy equilibrium concurrently exists. The following example is a BE game with both a pure strategy equilibrium and a mixed strategy equilibrium.

Example 6. Consider an industry with a market with demand

$$D(x) = \begin{cases} 0 & \text{if } x > 10 \\ 10 & \text{if } x \in [0, 10] \end{cases}$$

Both firms have zero cost of production up to the capacity constraint $k_1 = k_2 = 10$. The front-side profit of firm i is

$$\varphi_i(x) = \begin{cases} 0 & \text{if } x > 10 \\ 10x & \text{if } x \in [0, 10] \end{cases}.$$

There is an additional mass of 8 consumers with no income for whom the government will fully subsidize the purchase of this good up to a price of 10 dollars, but only if the lowest posted price is at least 3. Based on this program, the residual profit of each firm i is

$$\psi_i(p_i, p_j) = \begin{cases} 8p_i & \text{if } 10 \geq p_i \geq p_j \geq 3 \\ 0 & \text{if } p_j < 3 \text{ or } p_i > 10 \end{cases}.$$

This game has the following two equilibria:

EQ 1 (pure strategy): Both firms set a price of zero, with corresponding equilibrium profits of zero.

EQ 2 (mixed strategy): Each firm i employs the CDF $F(x)$ defined by

$$F(x) = \begin{cases} 0 & \text{if } x < 8 \\ \frac{5x-40}{x} & \text{if } x \in [8, 10] \\ 1 & \text{if } x > 10 \end{cases}$$

and earns an equilibrium profit of $u_i^* = 80$.

The following proposition demonstrates that no other equilibrium in pure or mixed strategies may exist as long as residual profit is nonincreasing in the other firm's price and $\underline{\rho}$ is the only residual maximizer for each firm when the other firm sets a price of $\underline{\rho}$.

Proposition 5. Suppose that there is a pure strategy equilibrium. If $\underline{\rho}$ is the unique maximizer of $\psi_i(p_i, \underline{\rho})$ and ψ_i is nonincreasing in p_j for each firm i , then both firms pricing at $\underline{\rho}$ is the unique equilibrium of the BE game.

Remark 3. Based on Proposition 5, in the case that residual profits are nonincreasing in the rival's price and have the unique residual maximizer of $\underline{\rho}$ to the price $\underline{\rho}$, a non-degenerate mixed strategy and pure strategy equilibrium cannot coexist for the same parameters.¹⁹ Thus, in this environment, the necessary and sufficient condition for the existence of pure strategy equilibrium also guarantees its uniqueness. In getting back to the theme of price indeterminacy, Proposition 5 provides precise conditions for determinate pricing.

Propositions 4 and 5 provide the basic character of all pure strategy equilibria of the BE game. We turn now to a discussion of the nature of pure strategy equilibrium in our model and classifying all pure strategy equilibria of this game as one of two distinct types. The first type of pricing requires that price equals marginal cost, a la Bertrand pricing. The second type of equilibrium includes pricing above marginal cost with supply to equal demand, a la Cournot pricing. These types are defined formally as follows.

Type B: $\underline{\rho} = \max \underline{a}_i$

Type C: $\underline{\rho} > \max \underline{a}_i$

It is important to point out that our structure permits pure strategy equilibrium pricing not of classical Bertrand (price equals marginal cost) or classical Cournot (market clearing) character. For example, a model with underlying cost functions that have constant marginal cost c from 0 to q and then a jump in constant marginal cost to $c' > c$ for all quantities greater than q . Although a pure strategy equilibrium with each firm pricing at marginal cost c' has more of the flavor of Bertrand pricing, it actually falls in the category of Type C. This is based on the fact that the model $\underline{a}_i = c$ to satisfy our abstract assumptions. It is worth noting that the conditions of Proposition 5 can only apply to Type C pure strategy equilibrium based on the unique maximizer restriction. The reason for this is that, in any Type B equilibrium, at least one firm i has an equilibrium profit equal to zero, which means that firm i 's residual profit is maximized at any $x \geq \underline{\rho}$.

Additional characterization of the set of pure strategy equilibrium is provided in Section 5 to follow for a special case model where we use a traditional construction of the market based on primitive assumptions on demand and production technology.

5. Special Case Model

In this section, we construct a BE duopoly model from assumptions on cost, demand, and residual demand. The purpose of the section is two-fold: First, show a model with underlying production technology and demand assumptions sufficient for the abstract front-side and residual profit functions to follow Assumptions 1–8. Second, provide results for

this model that cannot be shown in the abstract formulation. These results pertain to the conditions for pure strategy equilibrium and how changes in residual demand rationing and individual supply change the bounds on equilibrium prices and payoffs. We now turn to establishing the basic structure of the special case model.

Each firm i has a cost of production $c_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying the following properties.

Condition 1. Each c_i is continuous and nondecreasing with $c_i(0) = 0$.

Define the profit function for firm i to be $\pi_i(x, z) = xz - c_i(z)$. Firm i has a capacity denoted by $k_i > 0$ that serves as the upper bound on the quantity that can be produced.²⁰ The supply function of each firm i , denoted by $s_i(x)$, is a quantity that would maximize $\pi_i(x, z)$. The supply function $s_i(x)$ is any selection from the supply correspondence $\vartheta_i(x) = \arg \max_{z \in [0, k_i]} xz - c_i(z)$, which must exist due to the continuity of c_i from Condition 1 and the compactness of $[0, k_i]$.

Condition 2. $\pi_i(x, z)$ is quasiconcave in z .

The key parameter $\underline{a}_i$ is the infimum of firm i 's average cost, defined formally as $\underline{a}_i = \inf_{z > 0} c_i(z)/z$. The following condition is that each firm's lowest average cost is as the production quantity approaches zero.

Condition 3. $\inf_{z > 0} \frac{c_i(z)}{z} = \liminf_{z \rightarrow 0} \frac{c_i(z)}{z}$.

The market demand is denoted by $D : \mathbb{R}_+ \mapsto \mathbb{R}_+$. We make the following assumptions about the properties of market demand.

Condition 4. D is continuous and nonincreasing. There is a choke price $\bar{p}^c = \inf\{x : D(x) = 0\} < \infty$, with $\max\{\underline{a}_1, \underline{a}_2\} < \bar{p}^c$.

The residual demand of firm i is the demand available to the firm i if $p_i \geq p_j$ and is denoted by $d_i : \{(p_i, p_j) \in \mathbb{R}_+^2 : p_i \geq p_j\} \mapsto \mathbb{R}_+$. We make the following assumptions about the properties of each firm i 's residual demand.

Condition 5. $d_i(p_i, p_j)$ is continuous and nonincreasing p_i , and right continuous and nonincreasing in p_j .²¹ Furthermore, $d_i(p_i, p_j) \leq D(p_i)$ for all $p_i \geq p_j$.

Given Conditions 1–5, at any price p_i , the optimal production quantity for firm i with the lower price given $D(p_i)$ is $Q_i(p_i) = \min\{s_i(p_i), D(p_i)\}$. Similarly, the optimal production quantity for the firm with the higher price given residual demand $d_i(p_i, p_j)$ is $q_i(p_i, p_j) = \min\{s_i(p_i), d_i(p_i, p_j)\}$. The following condition relates the quantity produced by the firm with the lower price to the residual demand.

Condition 6. If $Q_j(x) = D(x)$, then $d_i(x, x) = 0$. If $Q_j(x') = 0$, then $d_i(x, x') = D(x)$ for all $x \geq x'$. For any price x , $d_i(x, x) = D(x) - Q_j(x)$. If $Q_j(x) > 0$, then $d_i(x, x) < D(x)$ for all $x < \bar{p}^c$.

We define the front-side and residual profit functions of each firm i , respectively, by

$$\begin{aligned}\varphi_i(p_i) &= p_i Q_i(p_i) - c_i(Q_i(p_i)) \\ \psi_i(p_i, p_j) &= p_i q_i(p_i, p_j) - c_i(q_i(p_i, p_j))\end{aligned}$$

It will be useful to denote the lower bound of all prices such that firm i 's supply is at least as big as the demand by τ_i . Formally, $\tau_i = \inf\{x : s_i(x) \geq D(x)\}$.

Condition 7. $p_i D(p_i) - c_i(D(p_i))$ is strictly quasiconcave on $[\tau_i, \bar{p}^c]$.

The restriction of the strict quasiconcavity to only the interval $[\tau_i, \bar{p}^c]$ allows some additional freedom for the demand function at prices below τ_i . The properties of $p_i D(p_i) - c_i(D(p_i))$ at prices $p_i < \tau_i$ are irrelevant, as the the profit of the firm does not correspond to this expression at such prices.

An immediate consequence of Condition 7 is that there is a unique maximizer $\hat{p}_i$ of the front-side profit for each firm i . The next condition is a restatement of Assumption 5, which we still need to guarantee that no single firm will monopolize the market.

Condition 8. $\hat{p}_i > a_j$ for each firm i .

The following lemma shows that Conditions 1–8 imply Assumptions 1–7 of the general model.

Lemma 6. If Conditions 1–8 are satisfied, then Assumptions 1–7 are satisfied.

In this special case model, we can define the judo and safe price of each firm i in a slightly simplified way. The judo price of firm i is

$$\bar{r}_i = \inf\{x | \varphi_i(x) > \tilde{\psi}_i(x)\}.$$

Note that, based on the assumptions of this section, either $\varphi_i(\bar{r}_i) = \tilde{\psi}_i(\bar{r}_i)$, or $\bar{r}_i = a_j$. The safe price of firm i is

$$r_i = \min\{x | \varphi_i(x) \geq \underline{u}_i\}.$$

5.1. Special Results for Pure Strategy Equilibrium

As we described in the general model section on pure strategy equilibrium, there is an intuitive way to classify the pure strategy equilibrium into two types. Particularly, Type B with $x^* = \max\{a_1, a_2\}$ and Type C with $x^* > \max\{a_1, a_2\}$. The structure added in this section allows us to say more about these types of equilibrium. Particularly, in the following proposition, we present the necessary and sufficient conditions on supply, demand, and residual demand for the existence of pure strategy equilibrium.

Proposition 6. The price $x^* = \underline{\rho}$ is a pure strategy Nash equilibrium if and only if one of the following three conditions holds:

- B.1 $\underline{\rho} = a_1 = a_2$ and $d_i(x', \underline{\rho}) = 0$ for each firm i , and all $x' > \underline{\rho}$.
- B.2 $\underline{\rho} = a_i > a_j$, $d_i(x, \underline{\rho}) = 0$ for all $x \geq \underline{\rho}$, $u_j(\underline{\rho}, \underline{\rho}) = \varphi_j(\underline{\rho})$, and $\psi_j(x, \underline{\rho}) \leq \varphi_j(\underline{\rho})$ for all $x \geq \underline{\rho}$.
- C $\underline{\rho} \in (\max\{a_1, a_2\}, \min\{\hat{p}_1, \hat{p}_2\}]$, $\underline{\rho} \in \tilde{P}_i(\underline{\rho})$ for each firm i , and $\underline{s}_i(\underline{\rho}) + s_j(\underline{\rho}) \leq D(\underline{\rho})$ for any firm i with $\alpha_i(\underline{\rho}, \underline{\rho}) < 1$, where $\underline{s}_i(x) = \min \vartheta_i(x)$.

The specificity of the model in this section allows us to break Type B equilibrium into two categories. Type B.1 requires that each firm's infimum of average cost is the same, $a_1 = a_2$. In all Type B.1 pure strategy equilibrium, both firms make zero profit. There are two different possibilities for B.1 pure strategy pricing. The first case, akin to Classical Bertrand marginal cost pricing, is the case in which $\min\{s_1(\underline{\rho}), s_2(\underline{\rho})\} \geq D(\underline{\rho})$, and thus by Condition 6, $d_i(\underline{\rho}, \underline{\rho}) = 0$ for both i . The second case allows for a firm's supply to not cover all of demand $s_i(\underline{\rho}) < D(\underline{\rho})$ as long as there is no residual demand for the other firm

$d_j(x', \underline{\rho}) = 0$ for all $x' > \underline{\rho}$. This case relies on no consumers going to the higher-priced firm j when firm i prices at $\underline{\rho}$.

Type B.2 is pure strategy pricing that can only occur in the case that the two firms have different infima of their average costs. In such an equilibrium, both firms price at the higher of the two infimum average costs, and it must be that the firm with the lower infimum average cost obtains its full front-side profit. This means that the lower-cost firm must obtain a large-enough share of demand at this price to achieve its front-side profit, while the higher-cost firm makes zero profit. This type of pure strategy equilibrium was shown in Theorem 2 of [Deneckere and Kovenock \(1996\)](#) for a model of firms with asymmetric constant marginal costs.

In all Type C equilibria, both firms make positive profits. Like B.1, Type C pricing has two different possibilities. The first case is similar to Classical Cournot market clearing prices ($s_1(\underline{\rho}) + s_2(\underline{\rho}) = D(\underline{\rho})$) above the infimum of average cost. The second case is such that total supply is less than demand $s_1(\underline{\rho}) + s_2(\underline{\rho}) < D(\underline{\rho})$ and residual demand rationing does not permit a profit increase by pricing higher than $\underline{\rho}$. This case of Type C pricing requires that each firm's residual demand is sufficiently low at prices greater than $\underline{\rho}$.

Next, we establish a condition such that the only possible pure strategy equilibrium is Type C.

Proposition 7. *Index the firms such that $a_1 \geq a_2$. If $s_1(a_1) = 0$, then only Type C pure strategy pricing equilibria are possible.*

From this proposition, in order to have a Type B equilibrium, we see that the minimum average cost of the higher-cost firm must be achieved by at least one positive production quantity (it could be equal to the infimum of the average cost at zero as well). This necessitates either a flat section of this firm's marginal cost or a U-shaped marginal cost curve.

Now we make a remark regarding the character of Type C pricing in a differentiable model.

Remark 4. *By adding standard differentiability assumptions, it becomes clear that the existence of a pure strategy equilibrium is remarkably fragile. If all key components of the model are differentiable (demand, residual demand, cost, and supply), then any Type C equilibrium must be such that $s_1(\underline{\rho}) + s_2(\underline{\rho}) = D(\underline{\rho})$. This basic insight on the non-existence of pure strategy equilibrium is first found by Shubik (1959).*

5.2. Some Comparative Statics on Equilibrium Bounds

We examine the effects of changes in residual demand rationing and individual supply on the bounds of equilibrium prices and payoffs. It should be clear that, for small changes to these components, the actual equilibrium payoffs need not follow the bounds (though perhaps they are likely to). However, for sufficiently large changes, when the new range of prices or profits does not intersect the old, we are able to precisely conclude how the actual equilibrium profits are affected.

We begin by examining the role of changes in the demand side of the market. Specifically, we consider an increase in residual demand rationing of consumers, whereby at least one firm has an increase in their residual demand. We use residual demands d_i and d'_i ; let $\varphi = (\varphi_1, \varphi_2)$ and φ' denote the corresponding front-side profits and $\psi = (\psi_1, \psi_2)$ and ψ' denote the corresponding residual profits. Our first result establishes that an increase in the residual demand of either firm weakly increases the bounds on equilibrium payoffs.

Proposition 8. *If $d'_i \geq d_i$, then $\bar{r}' \geq \bar{r}$, $\underline{r}' \geq \underline{r}$, $\bar{\varphi}' \geq \bar{\varphi}$ and $\underline{\varphi}' \geq \underline{\varphi}$.*

Simply put, this proposition establishes that a shift to a more generous rationing scheme increases the bounds on the profits of each firm.

Now we turn our attention to understanding the impact of changes in production technology. The following proposition shows conditions such that an increase in a firm's supply, which could result from an increase in capacity or reduction in costs, will weakly reduce its price and weakly reduce the profit of the other firm. Given supply functions $s = (s_1, s_2)$ and s' , let $\varphi = (\varphi_1, \varphi_2)$ and φ' denote the corresponding front-side profits and $\psi = (\psi_1, \psi_2)$ and ψ' denote the corresponding residual profits.

Proposition 9. *Consider a weak cost reduction for firm i that results in a weak supply increase so that $s'_i \geq s_i$. Suppose further that this results in a weak reduction in residual demand for firm j , $d_j(x, y) \geq d'_j(x, y)$ for all $x \geq y$. Lastly, suppose that $c_i(q) - c'_i(q)$ is nondecreasing in q . Then $\bar{r}' \leq \bar{r}$, $\underline{r}' \leq \underline{r}$, $\bar{\varphi}'_j \geq \bar{\varphi}_j$, and $\underline{\varphi}'_j \geq \underline{\varphi}_j$.*

Note that this proposition does not make any statements regarding the profits of the firm whose supply shifts. The reason is that the effect is ambiguous. That is, a technological increase for a firm does not necessarily imply an increase in equilibrium profits for that firm. The direction of the change in profits is instead determined by the nature of the shift and market conditions. Two extreme examples illustrate this point.

Consider a duopoly in which identical firms have constant marginal costs and capacities equal to half the monopoly quantity. In such a setting, pure strategy pricing can be sustained with each firm earning half the monopoly profit. Now consider a technology shock that increases the capacity of both firms so that their capacity is nonbinding at any price. This technological increase actually lowers each firm's profit from something strictly positive to zero.

The previous example involved an industry-wide capacity shock; however, the same result may occur as a result of a cost reduction for a single firm. Consider a duopoly in which firm 1 has constant marginal cost $c = 0$ while firm 2 has a strictly convex cost of production with supply $s_2(x) > 0$ for all $x > 0$ and $s_2(0) = 0$. Suppose that the firms are not capacity constrained. It follows that $\underline{p} = 0$. Note that by choosing a price x arbitrarily close to zero, the right continuity of s_2 guarantees that $\psi_1(x, 0) > 0$. Thus, $p_1 = p_2 = 0$ cannot be an equilibrium, and any equilibrium must be in mixed strategies. Therefore, it must be that firm 2 receives its front-side profit in equilibrium with positive probability, in which case, it earns positive profits. Consider a technology increase of firm 2 that reduces its cost to zero. Then the game becomes the classic Bertrand duopoly with zero profits. Thus, a reduction in one firm's cost may actually reduce its profits.

These examples highlight that there are countervailing effects associated with a change in technology. There is a primary cost effect or capacity effect that allows a firm to earn a higher profit margin or produce more at any given price, both of which increase the profits of that firm. Alternatively, there is a secondary competition effect, whereby the change in cost or capacity alters the strategic environment and incentivizes the other firm to price more competitively, driving the prices of both firms down and thereby reducing profits. Whether the net change in profits is positive or negative depends on the relative strength of these two effects.

6. Concluding Remarks

Reformulating price competition as an all-pay contest with externalities has allowed us to derive results on the nature of equilibrium for a class of BE pricing games far more general than using conventional methods. The broad range of underlying specifications includes many new specifications (as U-shaped average cost of production, minimal restriction on demand, demand rationing based on consumer search, and technology asymmetries

across firms) that expand the possible policy applications of the BE model. Further, we have presented a methodology that can be extended to analyze BE oligopoly including the possibility of incomplete information.

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Appendix A

Appendix A.1. Existence of Equilibrium (Proof of Proposition 1)

We prove the existence of equilibrium using the following result based on the works of [Reny \(1999\)](#) and [Bagh and Jofre \(2006\)](#).

Fact A1. *If the mixed extension of a compact game is payoff secure and satisfies weak reciprocal upper semicontinuity (WRUSC), then the game has a mixed strategy Nash equilibrium.*

Verifying payoff security and WRUSC in the mixed extension of a game is burdensome, and so we rely on the recent results of [Allen and Lepore \(2014\)](#) and [Allison et al. \(2018\)](#), which provide easily verifiable conditions for games that imply that these properties are satisfied in the mixed extension.

The following definition is from [Allen and Lepore \(2014\)](#). Let X_i and u_i denote player i 's strategy set and utility function, respectively. Define the discontinuity mapping $\mathbb{D}_i : X_i \rightarrow X_{-i}$ such that

$$\mathbb{D}_i(x_i) = \{x_{-i} \in X_{-i} : u_i(x_i, x_{-i}) \text{ is discontinuous in } x_{-i} \text{ at } (x_i, x_{-i})\}.$$

Definition A1. *A game satisfies disjoint payoff matching (DPM) if, for each player i and all $x_i \in X_i$, there exists a sequence $\{x_i^k\} \subset X_i$ such that*

- (1) $\liminf_k u_i(x_i^k, x_{-i}) \geq u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$; and
- (2) $\limsup_k \mathbb{D}_i(x_i^k) = \emptyset$.²²

Fact A2 ([Allen and Lepore \(2014\)](#)). *If a compact game satisfies DPM, then the mixed extension of the game is payoff secure.*

The problem in using this definition of DPM to verify the existence of equilibrium in our model is that the payoff function u_i can be discontinuous in p_j through ψ_j at some prices p regardless of the choice of p_i .²³ As such, it may be impossible to satisfy part 2 of the definition. A trivial modification is sufficient to generalize the existence result. Define the discontinuity map $\mathbb{D}'_i : X_i \rightarrow X_{-i}$ such that

$$\mathbb{D}'_i(x_i) = \{x_{-i} \in X_{-i} : u_i(x_i, x_{-i}) \text{ is not lower semicontinuous in } x_{-i} \text{ at } (x_i, x_{-i})\}.$$

By replacing $\mathbb{D}_i$ with $\mathbb{D}'_i$ in the definition of DPM, the proof of the main result of [Allen and Lepore \(2014\)](#) is unaffected. Since ψ_i is assumed to be lower semicontinuous in the statement of Proposition 1, it follows that the discontinuity sets $\mathbb{D}_i(x_i)$ and $\mathbb{D}'_i(x_i)$ coincide, and so we will be able to use this modified definition of DPM in our model.

Verifying that the BE game satisfies DPM is quite simple: for any price $p_i > 0$, the sequence of deviations $p_i^k = p_i - 1/k$ satisfies the definition, as these deviations result in either the same profit in the limit or a higher profit by guaranteeing the front-side profit if there would be a tie at p_i . Further, the only points of discontinuity at which the payoffs are not lower semicontinuous are ties, and $\limsup_k \mathbb{D}_i(x_i^k) = \emptyset$ since $\mathbb{D}_i(p_i^k) = \{p_i^k\}$, and thus, $\mathbb{D}_i(p_i^k) \cap \mathbb{D}_i(p_i^{k'}) = \emptyset$ for all $k \neq k'$. If $p_i = 0$, then $u_i(p_i, p_j) = 0$ for all p_j and $\mathbb{D}_i(p_i) = \emptyset$, so $p_i^k = 0$ for all k trivially satisfies the definition. Thus, the mixed extension of the BE game is payoff secure.

We now verify that the mixed extension of the BE game satisfies WRUSC. It will be useful to define the object $\bar{u}_i(x) = \limsup_{x' \rightarrow x} u_i(x')$ and $\bar{u}$ to be the vector valued functions whose individual components are each $\bar{u}_i$.

Fact A3 (Allison et al. (2018)). *Let $G = (N, X, u)$ be a compact game. Suppose that, (1) for each player i , there exists a sequence of Borel measurable functions $T_i^k : X_i \rightarrow X_i$ such that, for all $x \in X$, $\liminf_k u_i(T_i^k(x_i), x_{-i}) \geq \bar{u}_i(x)$, and (2) for any strategy profile $x \in X$, if there is some sequence $\{x^k\}$ with $\lim_k u(x^k) = \bar{u}(x)$, then $u(x) = \bar{u}(x)$. Then the mixed extension of the game satisfies WRUSC.*

These two conditions intuitively state that (1) each player can deviate from any strategy so that, given any strategy profile of the other players, the deviating player obtains the highest feasible payoff near that strategy profile, and (2) if it is feasible for all players to simultaneously obtain their highest feasible payoff near a strategy profile, then the payoffs specify that they all receive such a payoff at that strategy. In the context of our BE model, (1) is satisfied by the same deviations as with DPM: $T_i^k(p_i) = \max\{p_i - 1/k, 0\}$. This sequence of deviations maximizes the firm's chances of obtaining the front-side payoff, which corresponds to $\bar{u}_i$. For (2), observe that since ψ_i is lower semicontinuous in p_j , then $\bar{\psi}_i = \psi_i$, where ψ_i is derived from ψ_i as $\bar{u}_i$ is derived from u_i . Thus, $u_i(p_i, p_j) = \psi_i(p_i, p_j) = \bar{u}_i(p_i, p_j)$ for any $p_i > p_j$. If $p_i < p_j$, then $u_i(p_i, p_j) = \varphi_i(p_i) = \bar{u}_i(p_i, p_j)$. Any violation of condition (2) can thus only be at ties. Note that if $\varphi_i(x) > \psi_i(x, x)$ for both firms i , it is not feasible that both firms i simultaneously obtain $\bar{u}_i$. If $\varphi_i(x) = \psi_i(x, x)$ for both firms i , then $u_i(x, x) = \bar{u}_i(x, x)$. Lastly, if $\varphi_i(x) > \psi_i(x, x)$ and $\varphi_j(x) = \psi_j(x, x)$, then by the assumption in the statement of the proposition, $\alpha_i(x, x) = 1$, and so $u_i(x, x) = \varphi_i(x) = \bar{u}_i(x, x)$ and $u_j(x, x) = \varphi_j(x) = \bar{u}_j(x, x)$. Thus, condition (2) is satisfied. We conclude that the mixed extension of the BE game satisfies WRUSC.

Appendix A.2. Proof of Lemmas and Propositions

Lemma A1. $\sup_{p_j \in [x_0, x]} \psi_i(x, p_j)$ is right upper semicontinuous in x at $x = x_0$.

Proof of Lemma A1. Let $x_0 \geq 0$ and observe that $\sup_{p_j \in [x_0, x]} \psi_i(x, p_j) = \psi_i(x_0, x_0)$ at $x = x_0$. Let $\varepsilon > 0$ and $x^n \rightarrow x_0$ be such that $x^n > x$ for each n . For each n , let y_n^k be a sequence in k such that $\lim_k \psi_i(x^n, y_n^k) = \sup_{p_j \in [x_0, x^n]} \psi_i(x^n, p_j)$. For each n , let $K(n)$ be such that $|\psi_i(x^n, y_n^k) - \sup_{p_j \in [x_0, x^n]} \psi_i(x^n, p_j)| < \varepsilon/3$ for all $k > K(n)$ and choose $y^n = y_n^k$ for some $k > K(n)$. Then note that

$$\begin{aligned} \sup_{p_j \in [x_0, x^n]} \psi_i(x^n, p_j) - \psi_i(x_0, x_0) &\leq \sup_{p_j \in [x_0, x^n]} \psi_i(x^n, p_j) - \psi_i(x^n, y^n) + \psi_i(x^n, y^n) - \psi_i(x_0, x_0) \\ &< \frac{\varepsilon}{3} + \psi_i(x^n, y^n) - \psi_i(x^n, x_0) + \psi_i(x^n, x_0) - \psi_i(x_0, x_0). \end{aligned}$$

Since $y^n \in [x_0, x^n]$, it follows that $y^n \rightarrow x_0$. Thus, by the right upper semicontinuity of ψ_i in p_j , there exists an N_1 such that $\psi_i(x^n, y^n) - \psi_i(x^n, x_0) < \varepsilon/3$ for all $n > N_1$. Similarly, by

the continuity of ψ_i in p_i , there exists an N_2 such that $|\psi_i(x^n, x_0) - \psi_i(x_0, x_0)| < \varepsilon/3$ for all $n > N_2$. Thus, for all $n > \max\{N_1, N_2\}$, it must be that

$$\sup_{p_j \in [x_0, x^n]} \psi_i(x^n, p_j) - \psi_i(x_0, x_0) < \frac{\varepsilon}{3} + \psi_i(x^n, y^n) - \psi_i(x^n, x_0) + \psi_i(x^n, x_0) - \psi_i(x_0, x_0) < \varepsilon.$$

Therefore,

$$\sup_{p_j \in [x_0, x^n]} \psi_i(x^n, p_j) < \psi_i(x_0, x_0) + \varepsilon,$$

so by definition, $\sup_{p_j \in [x_0, x]} \psi_i(x, p_j)$ is right upper semicontinuous in x at $x = x_0$. $\square$

Proof of Lemma 1. Suppose to the contrary that $\mu_i([0, \hat{x}]) < 1$ for each firm i . Then for any firm i and any $x > \hat{x}$ in support of μ_i , Assumption 6 guarantees that $\varphi_i(\hat{x}) > \varphi_i(x)$ and $\psi_i(\hat{x}, p_j) \geq \psi_i(x, p_j)$ for all $p_j \leq \hat{x}$. Observe that

$$\begin{aligned} \int u_i(\hat{x}, p_j) d\mu_j &\geq (1 - F_j(\hat{x}))\varphi_i(\hat{x}) + \int_{[0, \hat{x}]} \psi_i(\hat{x}, p_j) dF_j \\ &\geq (1 - F_j(\hat{x}))\varphi_i(\hat{x}) + \int_{[0, \hat{x}]} \psi_i(x, p_j) dF_j \end{aligned}$$

and $(1 - F_j(\hat{x})) > 0$ since $\mu_j([0, \hat{x}]) < 1$. Thus, we have

$$\begin{aligned} \int u_i(\hat{x}, p_j) d\mu_j &\geq (1 - F_j(\hat{x}))\varphi_i(\hat{x}) + \int_{[0, \hat{x}]} \psi_i(x, p_j) dF_j \\ &> (1 - F_j(\hat{x}))\varphi_i(x) + \int_{[0, \hat{x}]} \psi_i(x, p_j) dF_j \\ &\geq (1 - F_j(x) + \mu_j(\{x\}))\varphi_i(x) + \int_{[0, x)} \psi_i(x, p_j) dF_j \\ &= \int u_i(x, p_j) d\mu_j. \end{aligned}$$

The last inequality follows from the fact that F_j is a CDF and thus nondecreasing and from Assumption 1 guaranteeing that $\varphi_i(x) \geq \varphi_i(x, x')$ for $x' \leq x$. This contradicts all $x > \hat{x}$ as equilibrium strategies. $\square$

Proof of Lemma 2. Let μ be an equilibrium and x be such that $\varphi_i(x) > \psi_i(x, x)$ and $\mu_i(\{x\}) > 0$. Suppose that $\mu_j(\{x\}) > 0$ and $\alpha_i(x, x) < 1$. Consider a sequence of deviations by firm i to $\tilde{\mu}_i^n$ defined by

$$\tilde{\mu}_i^n(E) = \begin{cases} \mu_i(E \cup \{x\}) & \text{if } x - \delta_n \in E \\ \mu_i(E \setminus \{x\}) & \text{otherwise} \end{cases},$$

where each δ_n is chosen so that $0 < \delta_n < 1/n$ and $\mu_j(\{x - \delta_n\}) = 0$. That is, $\tilde{\mu}_i^n$ is the measure created from μ_i by shifting all mass from the price x to the price $x - \delta_n$. Then note that

$$\int u_i(p) d\tilde{\mu}_i^n \times \mu_j = \int u_i(p) d\mu + \mu_i(\{x\}) \int (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j.$$

We will show that $\lim_n \int (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j > 0$ for sufficiently large n , which will guarantee a profitable deviation for firm i , violating μ_i as an equilibrium strategy. Note that

$$u_i(x - \delta_n, p_j) - u_i(x, p_j) = \begin{cases} \psi_i(x - \delta_n, p_j) - \psi_i(x, p_j) & \text{if } p_j < x - \delta_n \\ \varphi_i(x - \delta_n) - \psi_i(x, p_j) & \text{if } x - \delta_n \leq p_j < x \\ \varphi_i(x - \delta_n) - \alpha_i(x, x)\varphi_i(x) & \text{if } p_j = x \\ -(1 - \alpha_i(x, x))\psi_i(x, x) & \text{if } p_j > x \end{cases}.$$

It follows that the pointwise limit as $n \rightarrow \infty$ is

$$\lim_n (u_i(x - \delta_n, p_j) - u_i(x, p_j)) = \begin{cases} 0 & \text{if } p_j < x \\ (1 - \alpha_i(x, x))(\varphi_i(x) - \psi_i(x, x)) & \text{if } p_j = x \\ 0 & \text{if } p_j > x \end{cases}.$$

Thus, since $|u_i| \leq \varphi_i(\hat{p}_i)$, then by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_n \int (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j &= \int \lim_n (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j \\ &= \mu_j(\{x\})(1 - \alpha_i(x, x))(\varphi_i(x) - \psi_i(x, x)). \end{aligned}$$

Since $\varphi_i(x) > \psi_i(x, x)$ and $\alpha_i(x, x) < 1$, μ_i^n is a profitable deviation for firm i for sufficiently large n , violating μ as an equilibrium. We conclude that either $\alpha_i(x, x) = 1$ or $\mu_j(\{x\}) = 0$.

From Lemma 1, we know that $\mu_i([0, \hat{x}]) = 1$. Observe that if $x \in (\rho, \hat{x}]$, it must be that $\mu(\{(x, x)\}) = 0$ since $\varphi_i(x) > \psi_i(x, x)$ for each firm i at any price $x \in (\rho, \hat{x}]$ and $\alpha_i(x, x) < 1$ for some firm i at any price x .

Next, we show that the equilibrium is invariant to the choice of α at prices $x \in (\rho, \hat{x}]$. Let μ be an equilibrium given the sharing rule α with expected profits $v = (v_1, v_2)$ and consider another sharing rule α' such that $\alpha(x, x) = \alpha'(x, x)$ for all $x \leq \rho$. Let $u_i(x, \mu_j)$ denote firm i 's expected payoff when choosing a price x given α and $u'_i(x, \mu_j)$ the corresponding payoff given α' . To show that μ is an equilibrium for the game with sharing rule α' , it will suffice to show that, for each player i , (i) $u'_i(x, \mu_j) = v_i$ μ_i -almost everywhere and (ii) $u'_i(x, \mu_j) \leq v_i$ for all prices x .

(i) Note that the sharing rule does not influence the payoffs at any price x such that $\mu_j(\{x\}) = 0$, and so $u_i(x, \mu_j) = u'_i(x, \mu_j)$ at all such prices. Further, at all prices $x \leq \rho$, $u_i(x, \mu_j) = u'_i(x, \mu_j)$ since $\alpha(x, x) = \alpha'(x, x)$. The first part of this lemma demonstrates that $\mu_i(\{x\}) = 0$ for all $x \in (\rho, \hat{x}]$ such that $\mu_j(\{x\}) > 0$. Since μ_j has at most countably many atoms, then $\mu_i(\{x : \mu_j(\{x\}) > 0\}) = 0$. It follows that $u'_i(x, \mu_j) = v_i$ μ_i -almost everywhere.

(ii) As we have shown in part (i), $u_i(x, \mu_j) = u'_i(x, \mu_j)$ except possibly at prices $x \in (\rho, \hat{x}]$ such that $\mu_j(\{x\}) > 0$. Since price above $\hat{x}$ is weakly dominated by $\hat{x}$, it is sufficient to examine prices in $[0, \hat{x}]$. Consider any such price x and let $\{x^k\}$ be a sequence such that $x^k \rightarrow x$, $x^k < x$ for all k , and $\mu_j(\{x^k\}) = 0$ for all k . Then note that the continuity of φ_i and ψ_i in p_i on $[0, \hat{x}]$ from Assumption 7 implies that $\lim_k u'_i(x^k, \mu_j) \geq u'_i(x, \mu_j)$. Since $\mu_j(\{x^k\}) = 0$ for all k , then $u_i(x^k, \mu_j) = u'_i(x^k, \mu_j)$ for all k . If $u'_i(x, \mu_j) > v_i$, then $u_i(x^k, \mu_j) > v_i$ for sufficiently large k , violating μ_i as an equilibrium strategy with the sharing rule α . Therefore, $u'_i(x, \mu_j) \leq v_i$ for all x .

We conclude that μ is an equilibrium given the sharing rule α' . $\square$

Proof of Lemma 3. We first argue that, in any equilibrium, $\underline{x}_i \geq \underline{a}_i$ for at least one firm i . Suppose to the contrary that $\underline{x}_i < \underline{a}_i$ for each firm i . Then Assumption 2 implies that $\int u_i(x, p_j) d\mu_j = \varphi_i(x) = 0$ and $\psi_j(p_j, x) = \varphi_j(p_j)$ for all $p_j \geq x$ and all $x \in [\underline{x}_i, \underline{a}_i]$. It follows that either player i could choose a price of $\hat{p}_i$ and receive a payoff of $\varphi_i(\hat{p}_i) > 0$ with positive probability since $\mu_j([\underline{x}_i, \underline{a}_i]) > 0$. This contradicts prices in $[\underline{x}_i, \underline{a}_i]$ as equilibrium strategies. We conclude that $\underline{x}_i \geq \underline{a}_i$ for at least one firm i .

Let μ be an equilibrium with $\underline{x}_i < \underline{x}_j$. Note that $\int u_i(x, p_j) d\mu_j = \varphi_i(x)$ for all $x \in [\underline{x}_i, \underline{x}_j]$. If $\underline{x}_j > \underline{a}_i$, then Assumptions 2 and 3 imply that φ_i , and thus $\int u_i(x, p_j) d\mu_j$ is strictly increasing on $[\underline{a}_i, \underline{x}_j]$, violating prices in $[\underline{x}_i, \underline{x}_j]$ as equilibrium strategies for firm i . Thus, $\underline{x}_j \leq \underline{a}_i$ and so $\underline{x}_i < \underline{a}_i$. Assumption 2 thus implies that $\varphi_i(x) = 0$ for all $x \in [\underline{x}_i, \underline{x}_j]$, so firm i 's equilibrium profit must be zero. From the result proved immediately above, it must be that $\underline{x}_j \geq \underline{a}_j$. Suppose that $\underline{x}_j < \underline{a}_i$. Then Assumption 2 implies that $\int u_j(x, p_i) d\mu_i = \varphi_j(x)$ for all $x \in [\underline{x}_j, \underline{a}_i]$. Therefore, since $\underline{x}_j \geq \underline{a}_j$, Assumption 3 guarantees that φ_j , and thus, u_j is strictly increasing on $[\underline{x}_j, \underline{a}_i]$, violating these prices as equilibrium strategies for firm j . Thus, it must be that $\underline{x}_j = \underline{a}_j$. If μ_j is nondegenerate, then there is some $x > \underline{a}_i$ such that firm j prices strictly higher than x with positive probability. If firm i sets a price of this x , then with positive probability, firm i will receive $\varphi_i(x)$, which is strictly positive by Assumption 3, contradicting zero as its equilibrium profit. Therefore, μ_j is degenerate with $\mu_j(\{\underline{a}_i\}) = 1$. Since there is a positive probability that firm i chooses a price $x < \underline{a}_i$, Assumption 2 implies that there is a positive probability that firm j will obtain a profit $\varphi_j(\hat{p}_j)$ if it sets its price at $\hat{p}_j$. Consequently, firm j 's equilibrium profit must be positive, and thus $\underline{a}_j < \underline{a}_i$.

Suppose that $\underline{a}_i < \underline{\rho} = \underline{\rho}_j$, since $\underline{a}_j < \underline{a}_i$ by Assumption 4. Assumptions 3 and 4 guarantee that $\int u_j(x, p_j) d\mu_j = \varphi_j(x)$ for all $x \in (\underline{a}_i, \underline{\rho})$ and that φ_j is strictly increasing on this interval. Thus, it must be that $\underline{a}_i = \underline{\rho}$.

Finally, suppose that $\mu_i([\underline{\rho}, \underline{\rho} + \varepsilon)) = 0$ for some $\varepsilon > 0$. Then firm j could set any price $x \in [\underline{\rho}, \underline{\rho} + \varepsilon)$ and still receive $\varphi_j(x)$ with certainty. Since φ_j is strictly increasing, this would violate $\underline{\rho}$ as an equilibrium strategy for firm j . We conclude that $\mu_i([\underline{\rho}, \underline{\rho} + \varepsilon)) > 0$ for any $\varepsilon > 0$. $\square$

Proof of Lemma 4. Lemma 3 implies that any nondegenerate mixed strategy equilibrium requires that $\underline{x}_1 = \underline{x}_2 = \underline{x}$. Let μ be an equilibrium with $\underline{x}_1 = \underline{x}_2 = \underline{x}$. Recall from the proof of Lemma 3 that at least one firm i must have $\underline{x}_i \geq \underline{a}_i$.

First, we show that $\underline{x} \geq \max \underline{a}_i$. Suppose to the contrary that $\underline{x} < \underline{a}_i$ for some firm i . Then it must be that $\underline{x} \geq \underline{a}_j$. By Assumption 2, firm j can set any price $x \in [\underline{x}, \underline{a}_i]$ and obtain a profit of $\varphi_j(x)$ with certainty. From Assumption 3, φ_j is strictly increasing on this interval, contradicting these as equilibrium strategies. Therefore, it must be that $\underline{x} \geq \max \underline{a}_i$.

Second, we show that $\underline{x} \geq \underline{\rho}$. Suppose to the contrary that $\underline{x} < \underline{\rho}$. Then by Assumption 4, $\psi_i(\underline{\rho}, p_j) = \varphi_i(\underline{\rho})$ for all $p_j < \underline{\rho}$. Consequently, either firm i could choose any price $x \in (\underline{x}, \underline{\rho})$ and earn $\int u_i(x, p_j) d\mu_j = \varphi_i(x)$. Since $\underline{x} \geq \underline{a}_i$, Assumption 3 guarantees that $\varphi_i(x)$ is strictly increasing on this interval, violating these prices as equilibrium strategies. Therefore, it must be that $\underline{x} \geq \underline{\rho}$.

Third, we show that neither firm i can have an atom at an $\underline{x}$ if $\varphi_i(\underline{x}) > \psi_i(\underline{x}, \underline{x})$. Suppose to the contrary that firm j has an atom at $\underline{x}$, $\mu_j(\{\underline{x}\}) > 0$, noting that $\underline{x} \leq \hat{x}$ since it is in support of each firm's strategy. From Lemma 2, we know that $\mu_i(\{\underline{x}\}) = 0$; however, as we have just shown, $\underline{x}_i = \underline{x}$. Thus, $\mu_i((\underline{x}, \underline{x} + \delta)) > 0$ for all $\delta > 0$. We will show that there is some $x' < \underline{x}$ and neighborhood $(\underline{x}, \underline{x} + \delta)$ such that $\int u_i(x', p_j) d\mu_j > \int u_i(x, p_j) d\mu_j$ for all $x \in (\underline{x}, \underline{x} + \delta)$.

Define $\beta = \mu_j(\{\underline{x}\}) > 0$ and let $\varepsilon > 0$ be such that $\varepsilon < \beta(\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x}))$. If firm i sets a price $x < \underline{x}$, then its profit will be $\varphi_i(x)$ with certainty. Since $\underline{x} \leq \hat{x}$, Assumption 7 guarantees that φ_i is left continuous, so there exists a $\delta_1 > 0$ such that $|\varphi_i(x) - \varphi_i(\underline{x})| < \varepsilon/2$ for all $x \in (\underline{x} - \delta_1, \underline{x})$. Note that $\int u_i(x, p_j) d\mu_j \leq (1 - \beta)\varphi_i(x) + \beta \sup_{p_j \in [\underline{x}, x]} \psi_i(x, p_j)$ at any price $x > \underline{x}$ and observe that $\sup_{p_j \in [\underline{x}, x]} \psi_i(x, p_j) = \psi_i(\underline{x}, \underline{x})$ at $x = \underline{x}$. Note that $\sup_{p_j \in [\underline{x}, x]} \psi_i(x, p_j)$ is right upper semicontinuous by Lemma A1. If $\underline{x} = \hat{x}$, then Assumption 6 guarantees that $\varphi_i(x) < \varphi_i(\underline{x})$ for all $x > \underline{x}$, and thus, φ_i is right upper semicontinuous at $\underline{x}$. Alternatively, if $\underline{x} < \hat{x}$, then Assumption 7 guarantees that

φ_i is continuous and thus right upper semicontinuous. In either case, the function $(1 - \beta)\varphi_i(x) + \beta \sup_{p_j \in [\underline{x}, x]} \psi_i(x, p_j)$ is right upper semicontinuous in x at $\underline{x}$, so there exists $\delta_2 > 0$ such that

$$(1 - \beta)\varphi_i(x) + \beta \sup_{p_j \in [\underline{x}, x]} \psi_i(x, p_j) < (1 - \beta)\varphi_i(\underline{x}) + \beta\psi_i(\underline{x}, \underline{x}) + \frac{\varepsilon}{2}$$

for all $x \in (\underline{x}, \underline{x} + \delta_2)$. Let $x' \in (\underline{x} - \delta_1, \underline{x})$ and note that $\varphi_i(x') > \varphi_i(\underline{x}) - \varepsilon/2$. Thus,

$$\begin{aligned} \varphi_i(x') &> (1 - \beta)\varphi_i(\underline{x}) + \beta\varphi_i(\underline{x}) - \frac{\varepsilon}{2} \\ &= (1 - \beta)\varphi_i(\underline{x}) + \beta\psi_i(\underline{x}, \underline{x}) + \beta[\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x})] - \frac{\varepsilon}{2}. \end{aligned}$$

For all $x \in (\underline{x}, \underline{x} + \delta_2)$, we have

$$\begin{aligned} &(1 - \beta)\varphi_i(\underline{x}) + \beta\psi_i(\underline{x}, \underline{x}) + \beta[\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x})] - \frac{\varepsilon}{2} \\ &> (1 - \beta)\varphi_i(x) + \beta \sup_{p_j \in [\underline{x}, x]} \psi_i(x, p_j) + \beta[\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x})] - \varepsilon \end{aligned}$$

and therefore,

$$\varphi_i(x') > (1 - \beta)\varphi_i(x) + \beta \sup_{p_j \in [\underline{x}, x]} \psi_i(x, p_j) + \beta[\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x})] - \varepsilon.$$

Since $\varepsilon < \beta(\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x}))$, this implies that $\int u_i(x', p_j) d\mu_j > \int u_i(x, p_j) d\mu_j$ for all $x \in (\underline{x}, \underline{x} + \delta_2)$. This violates such prices as equilibrium strategies for firm i . We conclude that $\mu_j(\{\underline{x}\}) = 0$, so neither firm i can have an atom at $\underline{x}$ if $\varphi_i(\underline{x}) > \psi_i(\underline{x}, \underline{x})$. $\square$

Proof of Proposition 3. We will prove the proposition for the two types of equilibria from Proposition 2 separately.

Case 1: Let μ be a symmetric lower bound equilibrium.

For this case, we first show that $\underline{x} \in [\underline{r}, \bar{r}]$ in two parts. First, we argue that $\underline{x} \leq \bar{r}$. Suppose to the contrary that $\underline{x} > \bar{r}$. From Lemma 2, at most one firm can have an atom at $\bar{x}$. Without loss of generality, let firm i be such that that $\bar{x}$ is in support of μ_i and $\mu_j(\{\bar{x}\}) = 0$. Choose $\{x_i^k\}$ in support of μ_i such that $x_i^k \rightarrow \bar{x}$; then note that

$$\lim_{k \rightarrow \infty} \int_{\underline{x}}^{\bar{x}} u_i(x_i^k, p_j) d\mu_j = \int_{\underline{x}}^{\bar{x}} \psi_i(\bar{x}, p_j) d\mu_j$$

since ψ_i is continuous in p_i from Assumption 7. Next, since each x_i^k is in support of μ_i , it must be that each $\int_{\underline{x}}^{\bar{x}} u_i(x_i^k, p_j) d\mu_j = u_i^*$. Note that

$$\int_{\underline{x}}^{\bar{x}} \psi_i(\bar{x}, p_j) d\mu_j \leq \int_{\underline{x}}^{\bar{x}} \tilde{\psi}_i(p_j) d\mu_j$$

by definition of $\tilde{\psi}_i$. By definition of $\bar{r}_i$ and the fact that $\underline{x} > \bar{r}_i$, $\varphi_i(\bar{r}_i) > \tilde{\psi}_i(x)$ for all $x \geq \bar{r}_i$ for each firm i . Further, Assumptions 2 and 3 imply that φ_i is nondecreasing, so $\varphi_i(\underline{x}) > \tilde{\psi}_i(x)$ for all $x \geq \underline{x}$. This implies that $\int_{\underline{x}}^{\bar{x}} \tilde{\psi}_i(p_j) d\mu_j < \varphi_i(\underline{x})$, and thus $u_i^* < \varphi_i(\underline{x})$. This violates μ_i as an equilibrium strategy since firm i has a profitable deviation to $\underline{x} - \varepsilon$ for sufficiently small ε that would guarantee a payoff of $\varphi_i(\underline{x})$. We conclude that $\underline{x} \leq \bar{r}$.

Second, we argue that $\underline{x} \geq \underline{r}$. Suppose to the contrary that $\underline{x} < \underline{r}$. Let firm i be such that $\underline{r}_i = \underline{r}$. By definition of $\underline{r}_i$ and the continuity of φ_i from Assumption 7, it must be that $\underline{u}_i = \varphi_i(\underline{r})$. From Lemma 4, $\underline{x} \geq \underline{\rho}$, and since $\underline{\rho} \geq \underline{a}_i$, Assumption 2 implies that $\varphi_i(x) > \varphi_i(\underline{x})$ for all $x > \underline{x}$. Thus, $\varphi_i(\underline{x}) < \varphi_i(\underline{r}) = \underline{u}_i$. The continuity of φ_i and ψ_i in p_i

thus guarantee that $\varphi_i(x) < \underline{u}_i$ and $\psi_i(x, x') < \underline{u}_i$ for all prices $x \in [\underline{x}, r]$ with $x' \leq x$, so $\int u_i(x, p_j) d\mu_j < \underline{u}_i$ for all such x . This violates all $x \in [\underline{x}, r]$ as equilibrium strategies. We conclude that $\underline{x} \geq r$.

It follows from Lemma 4 that each firm i 's equilibrium expected profit is $u_i^* = \varphi_i(\underline{x})$. The statement of the proposition thus follows from $\underline{x} \in [\underline{r}, \bar{r}]$ and the facts that φ_i is strictly increasing on $[\rho, \hat{p}_i]$ and that $\rho \leq \underline{r}$.

Case 2: Let μ be an asymmetric lower bound equilibrium.

First, consider player i . The lower bound payoff $\underline{u}_i = 0$, which implies that $\varphi_i(\underline{r}) = 0$. Further, we know from Lemma 3 that $u_i^* = 0$. By construction, $\varphi_i(\bar{r}) \geq 0$; therefore, $u_i^* \in [\varphi_i, \bar{\varphi}_i]$. Next, consider player j . From Lemma 3, $u_j^* = \varphi_j(\rho) = \varphi_j(\underline{a}_i)$. This payoff is possible for player j for any pricing by player i . Thus, $u_j^* = \underline{u}_j = \varphi_j(r)$. By construction, $\varphi_j(\bar{r}) \geq 0$; therefore, $u_j^* \in [\varphi_j, \bar{\varphi}_j]$. $\square$

Proof of Lemma 5. We use $\underline{p}_i(F_j)$ and $\bar{p}_i(F_j)$ to denote the smallest and largest conditional residual maximizer, respectively. That is, $\underline{p}_i(F_j) = \min \tilde{P}_i(F_j)$ and $\bar{p}_i(F_j) = \sup \tilde{P}_i(F_j)$, where the right continuity of F_j ensures that $\tilde{P}_i(F_j)$ contains a minimal element, while it need not contain a maximal element.

Let μ be an equilibrium with the corresponding CDF's F . If $\bar{x} = \rho$, then from Lemmas 3 and 4, either the equilibrium is degenerate or $\underline{x}_i < \underline{x}_j$ for some firm i . If the equilibrium is degenerate, then the statement of Proposition 4 (in Section 4) guarantees that $\rho \in \tilde{P}_i(\rho)$ for some firm i . It trivially follows that $\min\{\underline{p}_1(F_2), \underline{p}_2(F_1)\} \leq \bar{x} \leq \max\{\bar{p}_1(F_2), \bar{p}_2(F_1)\}$. Alternatively, $\underline{x}_i < \underline{x}_j$ for some firm i ; then Lemma 3 guarantees that $\mu_j(\{\rho\}) = 1$ and $u_i^* = 0$. In order for μ to be an equilibrium, firm i cannot have any profitable deviations, so it must be that $\psi_i(x, \rho) = 0$ for all $x \geq \rho$. Thus, by definition, $\tilde{P}_i(\rho) = [\rho, \infty)$, so $\min\{\underline{p}_1(F_2), \underline{p}_2(F_1)\} \leq \bar{x} \leq \max\{\bar{p}_1(F_2), \bar{p}_2(F_1)\}$.

Finally, let $\bar{x} > \rho$ and suppose that either $\bar{x} < \min\{\underline{p}_1(F_2), \underline{p}_2(F_1)\}$ or $\bar{x} > \max\{\bar{p}_1(F_2), \bar{p}_2(F_1)\}$. From Lemma 2, at most one firm may have an atom at $\bar{x}$. Let firm i be such that $\bar{x}_i = \bar{x}$ and $\mu_j(\{\bar{x}\}) = 0$. Since $\bar{x}$ is in support of firm i 's strategy, we may choose $\{x_i^k\}$ in support of μ_i such that $x_i^k \rightarrow \bar{x}$; then note that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\underline{x}}^{\bar{x}} u_i(x_i^k, p_j) d\mu_j &= \int_{\underline{x}}^{\bar{x}} \psi_i(\bar{x}, p_j) d\mu_j \\ &= E_{F_j}[\psi_i(\bar{x}, p_j) | p_j \leq \bar{x}]. \end{aligned}$$

Since each x_i^k is a best response for firm i , this implies that $\bar{x}$ is also a best response for firm i . It follows from our supposition that $\bar{x} \notin \tilde{P}_i(F_j)$. Note that, for any price x ,

$$\begin{aligned} u_i(x, F_j) &\geq (1 - F_j(x))\varphi_i(x) + \int_{[\underline{x}, x]} \psi_i(x, p_j) dF_j \\ &= (1 - F_j(x))\varphi_i(x) + F_j(x)E_{F_j}[\psi_i(x, p_j) | p_j \leq x]. \end{aligned}$$

By definition of $\tilde{P}_i(F_j)$, $E_{F_j}[\psi_i(x, p_j) | p_j \leq x] > E_{F_j}[\psi_i(\bar{x}, p_j) | p_j \leq \bar{x}]$ for all $x \in \tilde{P}_i(F_j)$. Thus, $u_i(x, F_j) > u_i(\bar{x}, F_j)$ for all $x \in \tilde{P}_i(F_j)$ since $\varphi_i(x) \geq \varphi_i(\bar{x})$ for all p_j . This contradicts $\bar{x}$ as a best response. We conclude that $\min\{\underline{p}_1(F_2), \underline{p}_2(F_1)\} \leq \bar{x} \leq \max\{\bar{p}_1(F_2), \bar{p}_2(F_1)\}$. $\square$

Proof of Proposition 4. First, observe that, in any equilibrium (p_1^*, p_2^*) with corresponding profits (v_1^*, v_2^*) , it must be that $v_i^* \geq \varphi_i(\min\{p_j^*, \hat{x}\})$. To see why, note that Assumption 7 guarantees that φ_i is left continuous at $\min\{p_j^*, \hat{x}\}$. Thus, firm i can guarantee itself a payoff of $\varphi_i(\min\{p_j^*, \hat{x}\} - \varepsilon)$ by deviating to $p_i = \min\{p_j^*, \hat{x}\} - \varepsilon$, with the guarantee that $\varphi_i(\min\{p_j^*, \hat{x}\} - \varepsilon) \rightarrow \varphi_i(\min\{p_j^*, \hat{x}\})$.

We now argue that any pure strategy equilibrium must be symmetric. Suppose to the contrary that there is an asymmetric equilibrium with $p_i^* < p_j^*$. This means that firm i obtains $\varphi_i(p_i^*)$ with certainty. There are two cases to consider: (i) $p_i^* < \hat{p}_i$ and (ii) $p_i^* = \hat{p}_i$. We may ignore the case in which $p_i^* > \hat{p}_i$ since firm i would trivially be better off with a price of $\hat{p}_i$.

In case (i), if $p_i^* \geq \underline{a}_i$, then Assumption 3 guarantees that some price $p'_i \in (p_i^*, p_j^*)$ is strictly better for firm i since its front-side profit is strictly increasing on $(p_i^*, \hat{p}_i)$. Alternatively, if $p_i^* < \underline{a}_i$, then Assumption 2 guarantees that $\varphi_i(p_i^*) = 0$ and $\psi_j(p_j, p_i^*) = \varphi_j(p_j)$ for any $p_j > p_i^*$. It follows that $p_j^* = \hat{p}_j$; else firm j has a profitable deviation to $p_j = \hat{p}_j$. Thus, since $\hat{p}_j \leq \hat{x}$, firm i 's equilibrium profits must be such that $v_i^* \geq \varphi_i(\hat{p}_j)$. It follows that $\varphi_i(\hat{p}_j) = 0$, contradicting Assumption 5.

In case (ii), Assumption 3 guarantees that firm j has zero profit. Thus, since Assumption 5 guarantees that $\varphi_j(\hat{p}_j) > 0$, firm j would be better off charging some price $p'_j < \hat{p}_j$ in order to obtain a positive profit. We conclude that any equilibrium must be symmetric.

Let (x^*, x^*) be an equilibrium. Now we show that the only possibility is $x^* = \underline{\rho}$. It follows immediately from Lemma 2 that $x^* \leq \underline{\rho}$. Therefore, it remains to show that $x^* \geq \underline{\rho}$. Suppose to the contrary that $x^* < \underline{\rho}_i$ for some firm i . Assumption 4 guarantees that $\varphi_i(p_i) = \psi_i(p_i, x^*)$ for all $p_i \in [x^*, \underline{\rho}_i]$. If $x^* \geq \underline{a}_i$, then Assumption 3 guarantees that φ_i is strictly increasing, thus violating (x^*, x^*) as an equilibrium. Therefore, it must be that $x^* < \underline{a}_i$, and thus, $\varphi_i(x^*) = 0$. Assumption 2 then guarantees that $\psi_j(\hat{p}_j, x^*) = \varphi_j(\hat{p}_j)$, so it must be that $x^* = \hat{p}_j$; else firm j has a profitable deviation. However, Assumption 3 guarantees that $\varphi_i(\hat{p}_j) > 0$, a contradiction. We conclude that $x^* \geq \underline{\rho}_i$ for each firm i , so $x^* = \underline{\rho}$.

Now that we have established that the only possible pure strategy equilibrium is such that $p_1 = p_2 = \underline{\rho}$, we show that the conditions of the proposition are necessary and sufficient for this to be a pure strategy equilibrium price. We begin with sufficiency by showing that the conditions rule out any profitable defections. For any defection $x > \underline{\rho}$, the condition $\underline{\rho} \in \tilde{P}_i(\underline{\rho})$ immediately rules out a profit increase for firm i . The other conditions $\alpha_i(\underline{\rho}, \underline{\rho}) = 1$ and $\varphi_i(\underline{\rho}) \geq \tilde{\psi}_i(\underline{\rho})$ imply that $u_i(\underline{\rho}, \underline{\rho}) = \varphi_i(\underline{\rho}) \geq \tilde{\psi}_i(\underline{\rho}) \geq u_i(x, \underline{\rho})$ for all $x > \underline{\rho}$. Notice that the conditions make it such that $u_i(\underline{\rho}, \underline{\rho}) = \varphi_i(\underline{\rho})$ for all firms i . Any defection to $x < \underline{\rho}$ results in $u_i(x, \underline{\rho}) = \varphi_i(x) \leq \varphi_i(\underline{\rho})$ based on the facts that, from Assumption 3, φ_i is strictly increasing on the interval $[\underline{a}_i, \underline{\rho}]$, and from Assumption 2, $\varphi_i(x') = 0$ for all $x' \leq \underline{a}_i$.

Next, we prove that the conditions are necessary. Suppose to the contrary that $p_1 = p_2 = \underline{\rho}$ is an equilibrium and neither condition holds. First, we consider the case $\varphi_i(\underline{\rho}) = \psi_i(\underline{\rho}, \underline{\rho})$ for all i . If there is a firm i such that $\underline{\rho} \notin \tilde{P}_i(\underline{\rho})$, then for any $x \in \tilde{P}_i(\underline{\rho})$, $\psi_i(x, \underline{\rho}) > u_i(\underline{\rho}, \underline{\rho})$, a contradiction. Second, consider the case that there is a firm i such that $\varphi_i(\underline{\rho}) > \psi_i(\underline{\rho}, \underline{\rho})$. We will go through a violation of each of four conditions in turn. If $\varphi_i(\underline{\rho}) < \tilde{\psi}_i(\underline{\rho})$, then for any $x \in \tilde{P}_i(\underline{\rho})$, $\psi_i(x, \underline{\rho}) > u_i(\underline{\rho}, \underline{\rho})$, a contradiction. If $\alpha_i(\underline{\rho}, \underline{\rho}) < 1$, then $u_i(\underline{\rho}, \underline{\rho}) < \varphi_i(\underline{\rho})$, and based on the continuity of φ_i for $x < \underline{\rho}$ sufficiently close to $\underline{\rho}$, $u_i(x, \underline{\rho}) = \varphi_i(x) > u_i(\underline{\rho}, \underline{\rho})$. If $\varphi_j(\underline{\rho}) > \psi_j(\underline{\rho}, \underline{\rho})$, then $\alpha_j(\underline{\rho}, \underline{\rho}) < 1$ because $\alpha_i(\underline{\rho}, \underline{\rho}) = 1$ and $\alpha_1(\underline{\rho}, \underline{\rho}) + \alpha_2(\underline{\rho}, \underline{\rho}) < 2$. Then $u_j(\underline{\rho}, \underline{\rho}) < \varphi_j(\underline{\rho})$, and based on the continuity of φ_j for $x < \underline{\rho}$ sufficiently close to $\underline{\rho}$, $u_j(x, \underline{\rho}) = \varphi_j(x) > u_j(\underline{\rho}, \underline{\rho})$. Finally, if $\underline{\rho} \notin \tilde{P}_j(\underline{\rho})$ and from the previous step, it must be that $\varphi_j(\underline{\rho}) = \psi_j(\underline{\rho}, \underline{\rho})$, then for any $x \in \tilde{P}_j(\underline{\rho})$, $\psi_j(x, \underline{\rho}) > u_j(\underline{\rho}, \underline{\rho})$, a contradiction. $\square$

Proof of Proposition 5. Given Proposition 4, we need only show that there is no equilibrium other than $(\underline{\rho}, \underline{\rho})$. Let μ be a nondegenerate mixed strategy equilibrium. We consider two cases corresponding to whether $\underline{x}_1 < \underline{x}_j$ or $\underline{x}_1 = \underline{x}_j$.

Suppose first that $\underline{x}_1 = \underline{x}_2 = \underline{x}$. Lemmas 3 and 4 imply that $\underline{x} \geq \underline{\rho}$. Using Lemma 2, we may, without loss of generality, assume that $\bar{x}_i \geq \bar{x}_j$ and that firm j 's

strategy does not have an atom at $\bar{x}_i$. Then note that when choosing a price at or near $\bar{x}_i$, firm i earns a profit of approximately $\int \psi_i(\bar{x}_i, p_j) d\mu_j \leq \int \psi_i(\bar{x}_i, \underline{\rho}) d\mu_j = \psi_i(\bar{x}_i, \underline{\rho})$, where the inequality follows from the fact that ψ_i is nonincreasing in p_j . Since $\underline{\rho}$ is the unique maximizer of $\psi_i(p_i, \underline{\rho})$, it must be that $\psi_i(\bar{x}_i, \underline{\rho}) < \psi_i(\underline{\rho}, \underline{\rho})$. Further, since $\varphi_i(\underline{\rho}) \geq \psi_i(\underline{\rho}, \underline{\rho})$, it must be that $\int u_i(\underline{\rho}, p_j) d\mu_j \geq \psi_i(\underline{\rho}, \underline{\rho}) > \int \psi_i(\bar{x}_i, p_j) d\mu_j$. This contradicts prices at or near $\bar{x}_i$ as equilibrium strategies.

Next suppose $\underline{x}_i < \underline{x}_j$. From Lemma 3, firm j 's strategy must be degenerate with $p_j = \underline{\rho}$. If $\underline{a}_i < \underline{\rho}$, then Assumption 2 guarantees that φ_i is strictly increasing on $(\underline{x}_i, \underline{\rho})$, and so this interval contains no best responses to $p_j = \underline{\rho}$. Thus, it must be that $\underline{a}_i = \underline{\rho}$ and therefore that $\varphi_i(\underline{\rho}) = 0$. This further implies that $\psi_i(\underline{\rho}, \underline{\rho}) = 0$, so $\psi_i(x, \underline{\rho}) \geq \psi_i(\underline{\rho}, \underline{\rho})$ for all $x > \underline{\rho}$. This contradicts $\underline{\rho}$ as the unique maximizer of $\psi_i(p_i, \underline{\rho})$. We conclude that $(\underline{\rho}, \underline{\rho})$ is the unique equilibrium. $\square$

Proof of Lemma 6. We go through the verification of each assumption in turn.

Verification of Assumption 1: $\varphi_i(x) \geq \psi_i(x, x') \geq 0$ for all $x \geq x'$.

First, note that since each $\pi_i(x, z)$ is quasiconcave in z , it follows that $\pi_i(x, z) \geq \pi_i(x, z')$ for all $z' \leq z \leq s_i(x)$. Observe that $d_i(x, x) \leq D(x)$, so $q_i(x, x) \leq Q(x) \leq s_i(x)$. Thus, $\varphi(x) = \pi_i(x, Q_i(x)) \geq \pi_i(x, q_i(x, x)) = \psi_i(x, x)$.

Verification of Assumption 2: For each firm i , there exists an $\underline{a}_i$ such that $\varphi_i(x) = \psi_i(x, p_j) = 0$ for all $p_j \leq x \leq \underline{a}_i$. Further, $\psi_i(p_i, x) = \varphi_i(p_i)$ for all $p_i \geq x$ such that $x < \underline{a}_i$.

As defined above, in the special case of the model each $\underline{a}_i = \inf_{x>0} \frac{c_i(z)}{z}$. Thus, for any price $x \leq \underline{a}_i$ and any quantity $z > 0$, it follows that $xz - c_i(z) \leq 0$, and thus, $xs_i(x) - c_i(s_i(x)) = 0$. It follows immediately that $\varphi_i(x) = 0$ and $\psi_i(x, p_j) = 0$ for all $p_j \leq x$. This further implies that $s_i(x) = 0$ for all $x < \underline{a}_i$, so $Q_i(x) = 0$. As such, Condition 6 implies that $d_j(p_j, x) = D(p_j)$ for all $x < \underline{a}_i$. Consequently, $q_j(p_j, x) = \min\{s_j(p_j), d_j(p_j, x)\} = \min\{s_j(p_j), D(p_j)\} = Q_j(p_j)$, and so it must be that $\psi_j(p_j, x) = \varphi_j(x)$.

Verification of Assumption 3: φ_i has a unique maximizer $\hat{p}_i > \underline{a}_i$ with $\varphi_i(\hat{p}_i) > 0$. φ_i is strictly increasing at any price $p_i \in (\underline{a}_i, \hat{p}_i)$. Further, $\psi_i(p_i, p_j) = 0$ for all $p_i \geq p_j \geq \hat{p}_i$.

From Condition 7, $p_i D(p_i) - c_i(D(p_i))$ is strictly quasiconcave and thus has a unique maximizer $\hat{p}_i$ on $[\tau_i, \bar{p}^c]$. We will argue that $\hat{p}_i$ satisfies the statement of Assumption 3. To do so, we will argue that $\varphi_i(p_i) = p_i D(p_i) - c_i(D(p_i))$ on $[\tau_i, \bar{p}^c]$. We begin by showing that the correspondence $\theta_i(x) = \arg \max_{z \in [0, k_i]} xz - c_i(z)$ is nondecreasing.²⁴ Let $x < x'$ and $z = \max \theta_i(x)$, noting that $z \in \theta_i(x)$ since c_i is continuous by Condition 1. Define $z' = \max \theta_i(x')$ and suppose to the contrary that $z' < z$. Then it must be that $x'z - c_i(z) < x'z' - c_i(z')$, and thus $x'(z - z') < c_i(z) - c_i(z')$. By definition, $xz' - c_i(z') \leq xz - c_i(z)$, and thus $c_i(z) - c_i(z') \leq x(z - z')$. Together with the previous inequality, this implies that $x'(z - z') < x(z - z')$, requiring that $x' < x$, a contradiction.

Next, observe that $\theta_i(x)$ is convex valued due to the quasiconcavity of $xz - c_i(z)$ in z guaranteed by Condition 2. Let $p_i \in [\tau_i, \bar{p}^c]$. If $s_i(p_i) \geq D(p_i)$, then by definition, $Q_i(p_i) = D(p_i)$, so $\varphi_i(p_i) = p_i D(p_i) - c_i(D(p_i))$. Suppose that $s_i(p_i) < D(p_i)$. Since $\theta_i(x)$ is nondecreasing, $D(x)$ is nonincreasing from Condition 4, and $s_i(\tau_i) \geq D(\tau_i)$, it follows that, for any price $x \in [\tau_i, \bar{p}^c]$, there is some quantity $z \in \theta_i(x)$ such that $z \geq D(x)$. Further, since $\theta_i(x)$ is convex valued, it follows that $D(p_i) \in \theta_i(p_i)$, and so $\varphi_i(p_i) = p_i s_i(p_i) - c_i(s_i(p_i)) = p_i D(p_i) - c_i(p_i)$. Thus, $\varphi_i(p_i) = p_i D(p_i) - c_i(D(p_i))$ and is thus strictly quasiconcave on $[\tau_i, \bar{p}^c]$, and so $\hat{p}_i$ is the unique maximizer of φ_i on $[\tau_i, \bar{p}^c]$.

Next, we argue that φ_i is strictly increasing on $(\underline{a}_i, \tau_i)$, which may be empty. Suppose that $(\underline{a}_i, \tau_i)$ is nonempty. Let $x \in (\underline{a}_i, \tau_i)$. Since $x > \underline{a}_i$, the definition of $\underline{a}_i$ implies that there is some quantity $z > 0$ such that $\frac{c_i(z)}{z} < x$, and so $xz - c_i(z) > 0$. This implies that $xs_i(x) - c_i(s_i(x)) > 0$, and so it must be that $s_i(x) > 0$. By definition of τ_i , $s_i(x) < D(x)$. Since D is continuous by Condition 4, it follows that $s_i(x) < D(x')$ for some $x' \in (x, \tau_i)$.

Thus, $\varphi_i(x') \geq x's_i(x) - c_i(s_i(x)) > xs_i(x) - c_i(s_i(x)) = \varphi_i(x)$. We conclude that φ_i is strictly increasing on $(\underline{a}_i, \tau_i)$. Since φ_i is strictly quasiconcave on $[\tau_i, \bar{p}^c]$, it follows that φ_i is strictly increasing on $[\tau_i, \hat{p}_i)$, and therefore that φ_i is strictly increasing on $(\underline{a}_i, \hat{p}_i)$. We conclude that $\hat{p}_i$ is the unique maximizer of φ_i .

It remains to demonstrate that $\hat{p}_i > \underline{a}_i$. From Condition 4, there is some $x > \underline{a}_i$ such that $D(x) > 0$. For such a price x , Condition 3 implies that there is some quantity $z < D(x)$ such that $x > \frac{c_i(z)}{z}$, implying that $xz - c_i(z) > 0$. Thus, $\varphi_i(x) > 0 = \varphi_i(\underline{a}_i)$. It follows that $\hat{p}_i > \underline{a}_i$.

Lastly, we must argue that $\psi_i(p_i, p_j) = 0$ for all $p_i \geq p_j \geq \hat{p}_j$. To do so, we will show that $d_i(p_i, p_j) = 0$ for all $p_i \geq p_j \geq \hat{p}_j$, as this will imply that $q_i(p_i, p_j) = 0$ and thus $\psi_i(p_i, p_j) = 0$. From Condition 6, it is sufficient to show that $Q_j(p_j) = D(p_j)$ for all $p_j \geq \hat{p}_j$. Suppose to the contrary that $Q_j(p_j) = s_j(p_j) < D(p_j)$ for some $p_j \geq \hat{p}_j$. Since D is nondecreasing by Condition 4, it follows that $s_j(p_j) < D(\hat{p}_j)$. Observe that

$$\begin{aligned} \varphi_j(p_j) &= p_j s_j(p_j) - c_j(s_j(p_j)) \\ &\geq p_j s_j(\hat{p}_j) - c_j(s_j(\hat{p}_j)) \\ &> p_j s_j(\hat{p}_j) - c_j(s_j(\hat{p}_j)) \\ &= \varphi_j(\hat{p}_j). \end{aligned}$$

This contradicts $\hat{p}_j$ as the maximizer of φ_j . We conclude that $\psi_i(p_i, p_j) = 0$ for all $p_i \geq p_j \geq \hat{p}_j$.

Verification of Assumption 4: For each firm i , there exists a price $\underline{\rho}_i \in [\underline{a}_i, \min\{\hat{p}_i, \bar{p}_i\}]$ such that

$$\begin{aligned} \varphi_i(x) &> \psi_i(x, x) \quad \text{for all } x \in (\underline{\rho}_i, \hat{p}_i], \\ \varphi_i(x) &= \psi_i(x, x') \quad \text{for all } x' \leq x < \underline{\rho}_i. \end{aligned}$$

Define the set $A_i = \{x : \varphi_i(x) > \psi_i(x, x)\}$ and $\underline{\rho}_i = \inf A_i$. We will first show that $\hat{p}_i, \bar{p}_i \in A_i$, implying that $\underline{\rho}_i \leq \min\{\hat{p}_i, \bar{p}_i\}$. We will then show that if $x \in A_i$ and $x' \in (x, \hat{p}_i)$, then $x' \in A_i$. Lastly, we will show that if $\varphi_i(x) = \psi_i(x, x)$, then $\varphi_i(x) = \psi_i(x, x')$ for all $x' < x$.

First, we have already verified Assumption 3, which guarantees that $\psi_i(x, x) = 0$ for all $x \geq \hat{p}_j$. By Condition 8, $\hat{p}_j > \underline{a}_i$, so $\varphi_i(\hat{p}_j) > 0$. Thus, $[\hat{p}_j, \bar{p}^c] \subset A_i$.

We argue that for any quantity $Q < D(\hat{p}_i)$, $\hat{p}_i Q - c_i(Q) < \hat{p}_i D(\hat{p}_i) - c_i(D(\hat{p}_i))$. Suppose to the contrary that there exists a quantity $z < D(\hat{p}_i)$ such that $\varphi_i(\hat{p}_i) = \hat{p}_i z - c_i(z)$. Since D is continuous by Condition 4, it follows that there exists a price $x > \hat{p}_i$ such that $z < D(x)$, and thus $\varphi_i(x) \geq xz - c_i(z) > \hat{p}_i z - c_i(z) = \varphi_i(\hat{p}_i)$, contradicting $\hat{p}_i$ as the unique maximizer. Thus, $D(\hat{p}_i) \leq \min \theta_i(\hat{p}_i)$. It follows that, for any quantity $Q < D(\hat{p}_i)$, $\hat{p}_i Q - c_i(Q) < \hat{p}_i D(\hat{p}_i) - c_i(D(\hat{p}_i))$.

As we have argued that $\varphi_i(\hat{p}_j) > 0$, it follows that $\varphi_j(\hat{p}_i) > 0$, which requires that $Q_j(\hat{p}_i) > 0$. Thus, from Condition 6, $d_i(\hat{p}_i, \hat{p}_i) < D(\hat{p}_i)$. It follows that $q_i(\hat{p}_i, \hat{p}_i) < D(\hat{p}_i)$, and thus $\psi_i(\hat{p}_i, \hat{p}_i) = \hat{p}_i q_i(\hat{p}_i, \hat{p}_i) - c_i(q_i(\hat{p}_i, \hat{p}_i)) < \hat{p}_i D(\hat{p}_i) - c_i(D(\hat{p}_i)) = \varphi_i(\hat{p}_i)$. Thus, $\hat{p}_i \in A_i$.

Let $x \in A_i$ and $x' \in (x, \hat{p}_i)$. Suppose that $x' \notin A_i$. Then by definition, $\psi_i(x', x') \geq \varphi_i(x')$, and thus $\psi_i(x', x') = \varphi_i(x')$ by Assumption 1. We first argue that $q_i(x, x) \leq q_i(x', x')$. Suppose to the contrary that $q_i(x, x) > q_i(x', x')$. Then observe that

$$\begin{aligned} \psi_i(x', x') &= x' q_i(x', x') - c_i(q_i(x', x')) \\ &\geq x' q_i(x, x) - c_i(q_i(x, x)). \end{aligned}$$

Thus,

$$\begin{aligned} c_i(q_i(x, x)) - c_i(q_i(x', x')) &\geq x'(q_i(x, x) - q_i(x', x')) \\ &> x(q_i(x, x) - q_i(x', x')). \end{aligned}$$

Further, note that

$$\begin{aligned} \psi_i(x, x) &= xq_i(x, x) - c_i(q_i(x, x)) \\ &\geq xq_i(x', x') - c_i(q_i(x', x')). \end{aligned}$$

Thus,

$$x(q_i(x, x) - q_i(x', x')) \geq c_i(q_i(x, x)) - c_i(q_i(x', x')).$$

This is a contradiction, and so we conclude that $q_i(x, x) \leq q_i(x', x')$.

Second, we show that $q_i(x', x') < D(x')$. Suppose to the contrary that $q_i(x', x') = D(x')$. Then from Condition 6, it must be that $Q_j(x') = 0$, so $\varphi_j(x') = \pi_j(x', 0) = 0$. Since Condition 3 implies that $\pi_j(x'', z) > 0$ for all $x'' > \underline{a}_j$ for sufficiently small z , then it must be that $x' \leq \underline{a}_j$. This further implies that $x < \underline{a}_j$, and since we already verified Assumption 2, this implies that $\varphi_i(x) = \psi_i(x, x) = 0$, so $x \notin A_i$. This is a contradiction, so we conclude that $q_i(x', x') < D(x')$.

Observe that since $\psi_i(x', x') = \varphi_i(x')$, then $\psi_i(x', x') = \pi_i(x', q_i(x', x')) = \pi_i(x', Q_i(x')) = \varphi_i(x')$. This further implies that $\pi_i(x', q_i(x', x')) \geq \pi_i(x', z)$ for any quantity $z \leq D(x')$. The following are equivalent:

$$\begin{aligned} \pi_i(x', q_i(x', x')) &\geq \pi_i(x', z) \\ x'q_i(x', x') - c_i(q_i(x', x')) &\geq x'z - c_i(z) \\ 0 &\geq x'(z - q_i(x', x')) - (c_i(z) - c_i(q_i(x', x'))). \end{aligned}$$

Let $z \in (q_i(x', x'), D(x'))$ and observe that

$$\begin{aligned} 0 &\geq x'(z - q_i(x', x')) - (c_i(z) - c_i(q_i(x', x'))) \\ &> x(z - q_i(x', x')) - (c_i(z) - c_i(q_i(x', x'))). \end{aligned}$$

This can be solved to find $xz - c_i(z) < xq_i(x', x') - c_i(q_i(x', x'))$. Thus, $\pi_i(x, q_i(x', x')) > \pi_i(x, z)$. Since π_i is quasiconcave by Condition 2, it follows that $\pi_i(x, q_i(x', x')) > \pi_i(x, z)$ for all $z > q_i(x', x')$. Thus, it must be that $s_i(x) \in [0, q_i(x', x')]$. Since $q_i(x', x') \leq d_i(x', x')$ and d_i is nonincreasing by Condition 5, it follows that $q_i(x', x') \leq d_i(x', x')$. Therefore, $s_i(x) \leq d_i(x, x)$, and so $\psi_i(x, x) = \pi_i(x, s_i(x)) \geq \pi_i(x, Q_i(x)) = \varphi_i(x)$. Thus, $x \notin A_i$, a contradiction. We conclude that $x' \in A_i$.

Finally, we argue that if $\varphi_i(x) = \psi_i(x, x)$, then $\varphi_i(x) = \psi_i(x, x')$ for all $x' < x$. Let $x < \underline{\rho}_i$. Since $x \notin A_i$, it follows by definition of A_i that $\varphi_i(x) \leq \psi_i(x, x)$. Thus, since we already verified Assumption 1, we conclude that $\varphi_i(x) = \psi_i(x, x)$. Let $q = \min\{Q_i(x), q_i(x, x)\}$ and observe that $\varphi_i(x) = \psi_i(x, x) = xq - c_i(q)$, where $q \leq d_i(x, x)$. By Condition 5, d_i is nonincreasing in p_j , so $d_i(x, x) \leq d_i(x, x')$ for all $x' < x$. Thus, since $q \in \arg \max_{z \in [0, \min\{k_i, d_i(x, x)\}]} xz - c_i(x)$, it follows that $\psi_i(x, x') = \max_{z \in [0, \min\{k_i, d_i(x, x')\}]} xz - c_i(x) \geq \psi_i(x, x) = \varphi_i(x)$. Further, since $d_i(x, x') \leq D(x)$ by Condition 5, it follows that $\varphi_i(x) = \max_{z \in [0, \min\{k_i, D(x)\}]} xz - c_i(x) \geq \max_{z \in [0, \min\{k_i, d_i(x, x')\}]} xz - c_i(x) = \psi_i(x, x')$. Therefore, $\varphi_i(x) = \psi_i(x, x')$ for all $x' \leq x < \underline{\rho}_i$.

Verification of Assumption 5: $\hat{p}_i > \underline{a}_j$ for each firm i .

This is identical to the statement of Condition 8.

Before proceeding with the verification of Assumptions 6 and 7, we will show that $\varphi_i(p_i)$ and $\psi_i(p_i, p_j)$ are continuous in p_i on $[0, \bar{p}^c]$.

We have already shown that $\varphi_i(p_i) = \psi_i(p_i, p_j) = 0$ for all $p_i \leq \underline{a}_i$. Thus, we need only check continuity on $[\underline{a}_i, \bar{p}^c]$. We will first show that φ_i is continuous in p_i on $[\underline{a}_i, \tau_i)$ and $(\tau_i, \bar{p}^c]$ separately, and finally, that φ_i is continuous at $p_i = \tau_i$.

By the definition of τ_i , $s_i(x) < D(x)$ for all $x < \tau_i$. It follows that $\varphi_i(x) = \max_{z \in [0, k_i]} xz - c_i(z)$. Since c_i is continuous by Condition 1, the theorem of the maximum implies that $\max_{z \in [0, k_i]} xz - c_i(z)$ is continuous in x for all x , and thus, φ_i is continuous on $[\underline{a}_i, \tau_i)$.

Define $\bar{s}_i(x) = \max \vartheta_i(x)$, noting that $\bar{s}_i$ exists since c_i is continuous by Condition 1. Note further that $\bar{s}_i$ is nondecreasing, as we showed in the verification of Assumption 3 that $\vartheta_i(x)$ is a nondecreasing correspondence. Define $\bar{Q}_i(x) = \min\{\bar{s}_i(x), D(x)\}$ and $\bar{q}_i(x, x') = \min\{\bar{s}_i(x), d_i(x, x')\}$. From Condition 2, $xz - c_i(z)$ is quasiconcave in z , and thus $xz - c_i(z) \leq xz' - c_i(z')$ for any $z \leq z' \leq \bar{s}_i(x)$. In particular, by choosing $z = Q_i(x)$ and $z' = \bar{Q}_i(x)$, this implies that $\varphi_i(x) = x\bar{Q}_i(x) - c_i(\bar{Q}_i(x))$. Similarly, by choosing $z = q_i(x, x')$ and $z' = \bar{q}_i(x, x')$, this implies that $\psi_i(x, x') = x\bar{q}_i(x, x') - c_i(\bar{q}_i(x, x'))$. Note that, for all $x \geq \tau_i$, $\bar{s}_i(x) \geq D(x)$. Thus, since D is continuous by Condition 4, it follows immediately that $x\bar{Q}_i(x) - c_i(\bar{Q}_i(x))$ is continuous at all $x > \tau_i$, and thus, φ_i is continuous on $(\tau_i, \bar{p}^c]$.

Next, let $\varphi_i^-(\tau_i) = \lim_{x \rightarrow \tau_i^-} \varphi_i(x)$. Since $\max_{z \in [0, k_i]} xz - c_i(z)$ is continuous in x as noted above, it follows that $\varphi_i^-(\tau_i) = \max_{z \in [0, k_i]} \tau_i z - c_i(z) \geq \tau_i \bar{Q}_i(\tau_i) - c_i(\bar{Q}_i(\tau_i)) = \varphi_i(\tau_i)$. We have already demonstrated above that φ_i is strictly quasiconcave and that $\hat{p}_i \geq \tau_i$. Thus, φ_i is strictly increasing up to the unique maximizer $\hat{p}_i$. Therefore, it must be that $\varphi_i(\tau_i) = \varphi_i^-(\tau_i)$. We conclude that φ_i is continuous at τ_i and thus continuous on $[0, \bar{p}^c]$.

The proof that $\psi_i(p_i, p_j)$ is continuous in p_i is similar to that of $\varphi_i(p_i)$. Define $\tau'_i = \inf\{p_i : d_i(p_i, p_j) \leq \bar{s}_i(p_i)\}$.²⁵ For $x > \tau'_i$, $\bar{s}_i(x) \geq d_i(p_i, p_j)$. Thus, $\psi_i(x, p_j) = xd_i(x, p_j) - c_i(d_i(x, p_j))$. Condition 5 implies that d_i is continuous in p_i , so ψ_i is continuous in p_i at any $p_i \in (\tau'_i, \bar{p}^c]$ given $p_j \leq p_i$ since c_i is continuous by Condition 1. For any $x < \tau'_i$, $\bar{s}_i(x) < d_i(x, p_j)$. As such, $\psi_i(x, p_j) = \max_{z \in [0, k_i]} xz - c_i(z)$. Thus, the theorem of the maximum implies that $\max_{z \in [0, k_i]} xz - c_i(z)$ is continuous in x , and so $\psi_i(p_i, p_j)$ is continuous in p_i at any price $p_i \in [0, \tau'_i]$ such that $p_j \leq p_i$.

Suppose that $p_j < \tau'_i$. As was the case for φ_i , $\lim_{x \rightarrow \tau'_i^-} \psi_i(x, p_j) = \max_{z \in [0, k_i]} \tau'_i z - c_i(z) \geq \tau'_i q_i(\tau'_i, p_j) - c_i(q_i(\tau'_i, p_j)) = \psi_i(\tau'_i, p_j)$. Suppose that $\lim_{x \rightarrow \tau'_i^-} \psi_i(x, p_j) > \psi_i(\tau'_i, p_j)$ and let $\{x^k\}$ be a sequence of prices such that $x^k < \tau'_i$ for each k and $x^k \rightarrow \tau'_i$. Then from the continuity of ψ_i in p_i at prices $p_i < \tau'_i$, there exists a $K \in \mathbb{N}$ such that $\psi_i(x^k, p_j) > \psi_i(\tau'_i, p_j)$ for all $k > K_1$. Let $k > K_1$. If $\bar{q}_i(x^k, p_j) \leq d_i(\tau'_i, p_j)$, then

$$\begin{aligned} \psi_i(x^k, p_j) &= x^k \bar{q}_i(x^k, p_j) - c_i(\bar{q}_i(x^k, p_j)) \\ &< \tau'_i \bar{q}_i(x^k, p_j) - c_i(\bar{q}_i(x^k, p_j)) \\ &\leq \tau'_i q_i(\tau'_i, p_j) - c_i(q_i(\tau'_i, p_j)) \\ &= \psi_i(\tau'_i, p_j), \end{aligned}$$

contradicting $\psi_i(x^k, p_j) > \psi_i(\tau'_i, p_j)$. Thus, it must be that $\bar{q}_i(x^k, p_j) > d_i(\tau'_i, p_j)$. It follows from the continuity of d_i in p_i by Condition 5 that $\bar{q}_i(x^k, p_j) \rightarrow d_i(\tau'_i, p_j)$. Observe that $xz - c_i(z)$ is continuous in (x, z) since c_i is continuous by Condition 1, and thus, $\psi_i(x^k, p_j) = x^k \bar{q}_i(x^k, p_j) - c_i(\bar{q}_i(x^k, p_j)) \rightarrow \tau'_i q_i(\tau'_i, p_j) - c_i(q_i(\tau'_i, p_j)) = \psi_i(\tau'_i, p_j)$. This contradicts the supposition that $\lim_{x \rightarrow \tau'_i^-} \psi_i(x, p_j) > \psi_i(\tau'_i, p_j)$. We conclude that $\psi_i(p_i, p_j)$ is continuous at $p_i = \tau'_i$ given $p_j < \tau'_i$.

Lastly, suppose that $p_j = \tau'_i$. Observe that this means that $\bar{s}_i(p_j) \geq d_i(p_j, p_j)$ and thus, $\bar{s}_i(x) \geq d_i(x, p_j)$ for all $x > p_j$ since d_i is nonincreasing in p_i by Condition 5 and $\bar{s}_i$ is nondecreasing. It follows that $\bar{q}_i(x, p_j) \geq d_i(x, p_j)$ for all $x \geq p_j$. The continuity of ψ_j is thus guaranteed by the fact that d_i is continuous in p_i by Condition 5, while $xz - c_i(z)$ is continuous in (x, z) since c_i is continuous by Condition 1.

Verification of Assumption 6: There exists a lowest price $\hat{x}$ such that for some price $x^* \leq \hat{x}$, such that $\varphi_i(x^*) > \varphi_i(x)$ and $\psi_i(x^*, p_j) \geq \psi_i(x, p_j)$ for all $x > \hat{x}$ and $p_j \leq x^*$.

Define the set $E_i = \{p_i : \exists x^* \leq p_i \text{ such that } \varphi_i(x^*) > \varphi_i(x) \text{ and } \psi_i(x^*, p_j) \geq \psi_i(x, p_j) \forall x > p_i \text{ and } \forall p_j \leq x^*\}$; then define $\hat{x}_i = \inf E_i$ and $\hat{x} = \max\{\hat{x}_1, \hat{x}_2\}$. It is sufficient to show that $\hat{x} \in E_i$ for each firm i . We first argue that each E_i is nonempty, and thus, $\hat{x}$ is finite. We will then show that $\hat{x}_i \in E_i$, and lastly that $\hat{x} \in E_i$.

To demonstrate that each E_i is nonempty, we will show that $\bar{p}^c \in E_i$. Define $x^* = \hat{p}_i$ and observe that $\varphi_i(x^*) > 0$. By the definition of $\bar{p}^c$, $D(x) = 0$ for all $x > \bar{p}^c$. From Condition 5, $d_i(x, p_j) \leq D(x) = 0$ for all $p_j \leq x$. As such, since $Q_i(x) = \min\{s_i(x), D(x)\}$ and $q_i(x, p_j) = \min\{s_i(x), d_i(x, p_j)\}$, it follows that $Q_i(x) = q_i(x, p_j) = 0$, and thus, $\varphi_i(x) = \psi_i(x, p_j) = 0 < \varphi_i(x^*)$. Thus, each E_i is nonempty and $\hat{x}_i$ is finite.

For each $p_i \in E_i$, let $x^*(p_i)$ be as in the definition of E_i and let $x^* = \liminf_{p_i \in E_i, p_i \rightarrow \hat{x}_i^+} x^*(p_i)$ and let $\{p_i^k\}$ be a sequence of prices such that $p_i^k \in E_i$ for all k , $p_i^k \rightarrow \hat{x}_i$, and $x^*(p_i^k) \rightarrow x^*$. Since $x^*(p_i) \leq p_i$, it follows that $x^* \leq \hat{x}_i$. Observe that $\hat{p}_i \leq \hat{x}_i$, and since φ_i is strictly quasiconcave on $[\tau_i, \bar{p}^c]$, it follows that φ_i is strictly decreasing on $(\hat{p}_i, \bar{p}^c)$. Thus, it must be that $\varphi_i(x^*) > \varphi_i(x)$ for all $x > \hat{x}_i$. By definition, $\psi_i(x^*(p_i), p_j) \geq \psi_i(x, p_j)$ for all $x > p_i$ and all $p_j \leq x^*(p_i)$. It follows that $\liminf_k \psi_i(x^*(p_i^k), p_j) \geq \psi_i(x, p_j)$ for all $x > \hat{x}_i$ and all $p_j \leq x^*$. As shown above, $\psi_i(p_i, p_j)$ is continuous in p_i , and so $\liminf_k \psi_i(x^*(p_i^k), p_j) = \psi_i(x^*, p_j)$. This implies that $\psi_i(x^*, p_j) \geq \psi_i(x, p_j)$ for all $p_j \leq x^*$. Therefore, $\hat{x} \in E_i$.

Lastly, consider the price $\hat{x}_i$ and corresponding x^* as in the definition of E_i . By definition, $\varphi_i(x^*) > \varphi_i(x)$ and $\psi_i(x^*, p_j) \geq \psi_i(x, p_j)$ for all $x > \hat{x}_i$ and $p_j \leq x^*$. In particular, this holds for all $x > \max\{\hat{x}_1, \hat{x}_2\}$, and thus, $\hat{x} \in E_i$.

Verification of Assumption 7: Both φ_i and ψ_i are continuous in p_i on $[0, \hat{x}]$. ψ_i is right upper semicontinuous in p_j , that is, $\limsup_k \psi_i(p_i, x^k) \leq \psi_i(p_i, x)$ for any sequence $\{x^k\}$ such that $p_i \geq x^k > x$ and $x^k \rightarrow x$.

We have already shown that φ_i and ψ_i are continuous in p_i on $[0, \bar{p}^c]$ and thus are continuous on $[0, \hat{x}]$. It remains to show that ψ_i is right upper semicontinuous in p_j . By Condition 5, d_i is right continuous in p_j . Let $\{x^k\}$ be a sequence such that $x^k > p_j$ for all k and $x^k \rightarrow p_j$, noting that $d_i(p_i, x^k) \rightarrow d_i(p_i, p_j)$. Observe that $\psi_i(p_i, x^k) = \max_{z \in [0, \min\{k_i, d_i(p_i, x^k)\}]} p_i z - c_i(z)$. Since d_i is nonincreasing in p_j by Condition 5, it follows that $\psi_i(p_i, x^k) \neq \psi_i(p_i, p_j)$ only if $s_i(p_i) > d_i(p_i, p_j)$. In this case, $q_i(p_i, x^k) \in (d_i(p_i, p_j), d_i(p_i, x^k)]$. Note that

$$\begin{aligned} \left| \psi_i(p_i, x^k) - \psi_i(p_i, p_j) \right| &\leq \left| p_i d_i(p_i, p_j) - c_i(d_i(p_i, p_j)) - p_i q_i(p_i, x^k) + c_i(q_i(p_i, x^k)) \right| \\ &\leq p_i \left| d_i(p_i, p_j) - q_i(p_i, x^k) \right| + \left| c_i(d_i(p_i, p_j)) - c_i(q_i(p_i, x^k)) \right|. \end{aligned}$$

Since $q_i(p_i, x^k) \leq d_i(p_i, x^k)$ and $d_i(p_i, x^k) \rightarrow d_i(p_i, p_j)$, it follows that $q_i(p_i, x^k) \rightarrow d_i(p_i, p_j)$. Thus, since c_i is continuous by Condition 1, we may conclude that $\left| \psi_i(p_i, x^k) - \psi_i(p_i, p_j) \right| \rightarrow 0$, so $\psi_i(p_i, p_j)$ is right continuous (and thus right upper semicontinuous) in p_j . $\square$

Proof of Proposition 6. We begin by showing that under each set of condition $x^* = \underline{\rho}$ is a pure strategy equilibrium.

B.1 Consider any firm i . From assumption 2, $\varphi_i(x) = \psi_i(x, x') = 0$ for all $x' \leq x \leq \underline{\rho}$. From the assumption of B.1, $d_i(x', \underline{\rho}) = 0$ for all $x' > \underline{\rho}$, and so $q_i(x', \underline{\rho}) = 0$ for all $x' > \underline{\rho}$. It

follows that $\psi_i(x', \underline{\rho}) = \pi_i(x', 0) = 0$ for all $x' > \underline{\rho}$. Thus, neither firm possesses a profitable deviation, so each firm pricing at $x^* = \underline{\rho}$ is an equilibrium.

B.2 Consider firm i with $\underline{a}_i > \underline{a}_j$. As noted in the (B.1) case, firm i has no profitable deviations. From the assumptions of B.2, observe that $u_j(\underline{\rho}, \underline{\rho}) = \varphi_j(\underline{\rho})$. From Assumptions 2 and 3, φ_j is nondecreasing on $[0, \hat{p}_j]$, and so $\varphi_j(\underline{\rho}) \geq \varphi_j(x)$ for all $x < \underline{\rho}$. Lastly, the assumption in B.2 that $\psi_j(x, \underline{\rho}) \leq \varphi_j(\underline{\rho})$ for all $x \geq \underline{\rho}$ guarantees that there are no profitable deviations for firm j to prices higher than $\underline{\rho}$. Thus, neither firm possesses a profitable deviation, so each firm pricing at $x^* = \underline{\rho}$ is an equilibrium.

C. From Condition 6, $d_i(\underline{\rho}, \underline{\rho}) = D(\underline{\rho}) - Q_j(\underline{\rho})$. Since $Q_j(\underline{\rho}) \leq s_j(\underline{\rho})$, it follows that $d_i(\underline{\rho}, \underline{\rho}) \geq D(\underline{\rho}) - s_j(\underline{\rho})$. Observe that if $\alpha_i(\underline{\rho}, \underline{\rho}) = 1$, then by definition $u_i(\underline{\rho}, \underline{\rho}) = \varphi_i(\underline{\rho}, \underline{\rho})$. Suppose that $\alpha_i(\underline{\rho}, \underline{\rho}) < 1$. Then from the assumption of C, $\underline{s}_i(\underline{\rho}) + s_j(\underline{\rho}) \leq D(\underline{\rho})$, so $D(\underline{\rho}) - s_j(\underline{\rho}) \geq \underline{s}_i(\underline{\rho})$. It follows that $\pi_i(\underline{\rho}, Q_j(\underline{\rho})) = \pi_i(\underline{\rho}, \min\{\underline{s}_i(\underline{\rho}), d_i(\underline{\rho}, \underline{\rho})\})$ for each firm i , and so $u_i(\underline{\rho}, \underline{\rho}) = \varphi_i(\underline{\rho})$. As noted above, φ_i is nondecreasing on $[0, \hat{p}_i]$, so there are no profitable deviations to prices $x < \underline{\rho}$. Further, since $\underline{\rho} \in \tilde{P}_i(\underline{\rho})$, there are no profitable deviations to prices $x > \underline{\rho}$. Thus, neither firm possesses a profitable deviation, so each firm pricing at $x^* = \underline{\rho}$ is an equilibrium.

Next, we prove that any pure strategy equilibrium must satisfy either B.1, B.2, or C.

Proposition 4 implies that any pure strategy equilibrium must be symmetric with $x^* = \underline{\rho}$. Further, by definition, it must be that $\underline{\rho} \geq \underline{a}_i$ and $\underline{\rho} < \hat{p}_i$ for each firm i .

B.1 Suppose that $x^* = \underline{\rho} = \underline{a}_1 = \underline{a}_2$ and that $d_i(x, \underline{\rho}) > 0$ for some $x > \underline{\rho}$ for some firm i . Then since $x > \underline{a}_i$, from Condition 3, there is some quantity $z \in (0, \underline{\rho})$ such that $x > \frac{c_i(z)}{z}$, so $\pi_i(x, z) > 0$. Since $\psi_i(x, \underline{\rho}) \geq \pi_i(x, z)$, it follows that $\psi_i(x', \underline{\rho}) > 0$. This contradicts x^* as an equilibrium since $\varphi_i(x^*) = 0$.

B.2 Suppose that $x^* = \underline{\rho} = \underline{a}_i > \underline{a}_j$. If $d_i(x, \underline{\rho}) > 0$ for some $x > \underline{\rho}$, then the preceding argument for the (B.1) case applies and rules out x^* as an equilibrium. If $u_j(\underline{\rho}, \underline{\rho}) < \varphi_j(\underline{\rho})$, then by continuity of φ_j by Assumption 7, there exists a price $x < \underline{\rho}$ such that $u_j(x, \underline{\rho}) = \varphi_j(x) > u_j(\underline{\rho}, \underline{\rho})$. This contradicts x^* as an equilibrium. Lastly, suppose that $\psi_j(x, \underline{\rho}) > \varphi_j(\underline{\rho})$ for some $x \geq \underline{\rho}$. Then since $u_i(x, \underline{\rho}) \geq \psi_i(x, \underline{\rho})$, then x is a profitable deviation from x^* , violating x^* as an equilibrium.

C. Suppose that $\underline{\rho} \in (\max\{\underline{a}_1, \underline{a}_2\}, \min\{\hat{p}_1, \hat{p}_2\}]$. If $\underline{\rho} \notin \tilde{P}_i(\underline{\rho})$ for some firm i , then by definition, any price $x \in \tilde{P}_i(\underline{\rho})$ is a profitable deviation for firm i , violating x^* as an equilibrium. Lastly, suppose that $\underline{s}_i(\underline{\rho}) + s_j(\underline{\rho}) > D(\underline{\rho})$ for some firm i with $\alpha_i(\underline{\rho}, \underline{\rho}) < 1$. Then since $d_i(\underline{\rho}, \underline{\rho}) = D(\underline{\rho}) - s_j(\underline{\rho})$ by Condition 6, it must be that $d_i(\underline{\rho}, \underline{\rho}) < \underline{s}_i(\underline{\rho})$, so $q_i(\underline{\rho}, \underline{\rho}) \notin \arg \max_z \pi_i(\underline{\rho}, z)$. Therefore, since $\psi_i(\underline{\rho}, \underline{\rho}) = \pi_i(\underline{\rho}, q_i(\underline{\rho}, \underline{\rho}))$ and $\alpha_i(\underline{\rho}, \underline{\rho}) < 1$, it must be that $u_i(\underline{\rho}, \underline{\rho}) = \alpha_i(\underline{\rho}, \underline{\rho})\varphi_i(\underline{\rho}) + (1 - \alpha_i(\underline{\rho}, \underline{\rho}))\psi_i(\underline{\rho}, \underline{\rho}) < \varphi_i(\underline{\rho})$. Then, since φ_i is continuous by Assumption 7, there exists a price $x < \underline{\rho}$ such that $u_i(x, \underline{\rho}) = \varphi_i(x) > u_i(\underline{\rho}, \underline{\rho})$. This violates x^* as an equilibrium. $\square$

Proof of Proposition 7. Without loss of generality, we assume $\underline{a}_1 \geq \underline{a}_2$. Suppose to the contrary that there is an equilibrium price $x^* = \underline{a}_1$ with $s_1(x^*) = 0$. Then $Q_1(x^*) = 0$, so Condition 6 guarantees that $d_2(x, \underline{a}_1) = D(x)$ for all $x > \underline{a}_1$. It follows immediately that $\psi_2(x, \underline{a}_1) = \varphi_2(x)$ for all $x \geq \underline{a}_1$. From Condition 8, $\hat{p}_2 > \underline{a}_1$. Therefore, by definition of $\hat{p}_2$, we have $\psi_2(\hat{p}_2, \underline{a}_1) = \varphi_2(\hat{p}_2) > \varphi_2(\underline{a}_1)$, contradicting $x^* = \underline{a}_1$ as an equilibrium price. $\square$

Proof of Proposition 8. Since D and c_i are unchanged, then $\varphi_i(x) = \varphi'_i(x)$ for all x . Note that

$$\begin{aligned}\psi_i(p_i, p_j) &= \max_{z \in [0, \min\{k_i, d_i(p_i, p_j)\}]} xz - c_i(z) \text{ and} \\ \psi'_i(p_i, p_j) &= \max_{z \in [0, \min\{k_i, d'_i(p_i, p_j)\}]} xz - c_i(z).\end{aligned}$$

It follows immediately that $\psi'_i \geq \psi_i$. The fact that $\bar{r}'_i \leq \bar{r}_i$ and $r'_i \leq r_i$ follows immediately from their definitions. $\square$

Proof of Proposition 9. The fact that $\bar{r}'_j \leq \bar{r}_j$ and $r'_j \leq r_j$ follows directly from Proposition 8 since any change in firm i 's supply has no impact on the front-side profit of firm j , so $\varphi_j(x) = \varphi'_j(x)$ for all x . It remains to show that $\bar{r}'_i \leq \bar{r}_i$ and $r'_i \leq r_i$. Observe that $d_i(x, y) = d'_i(x, y)$ for all prices $x \geq y$.

Define $\bar{s}_i(x) = \sup \vartheta_i(x)$, with $\bar{s}'_i$ defined analogously for π'_i . Let $\bar{Q}_i(x) = \min\{\bar{s}_i(x), D(x)\}$ and $\bar{q}_i(x, y) = \min\{\bar{s}_i(x), d_i(x, y)\}$, with $\bar{Q}'_i$ and $\bar{q}'_i$ defined analogously. Then note that $\varphi_i(x) = \pi_i(x, \bar{Q}_i(x))$. Note that $\bar{s}_i$ is nondecreasing.

Part 1: $\bar{r}'_i \leq \bar{r}_i$

The proof that $\bar{r}'_i \leq \bar{r}_i$ is conducted in four steps. In Step 1, we argue that $\pi_i(\hat{p}_i, z) < \pi_i(\hat{p}_i, D(\hat{p}_i))$ for all $z < D(\hat{p}_i)$ and then use that fact to argue that we show that $\bar{r}_i < \hat{p}_i$. In Step 2, we show that $\bar{q}_i(x, y) \leq Q_i(y)$ for all $x \geq y > \bar{r}_i$. In Step 3, we argue that $\bar{q}_i(x, y) \geq q'_i(x, y)$ for all $x \geq y > \bar{r}_i$, implying that $q'_i(x, y) \leq Q_i(\bar{r}_i)$ for all $x \geq y > \bar{r}_i$. Finally, in Step 4, we show that if $\bar{r}'_i > \bar{r}_i$, we can find prices $x \geq y > \bar{r}_i$ such that $c_i(Q_i(x)) - c'_i(Q_i(x)) < c_i(q'_i(\tilde{x}, x)) - c'_i(q'_i(\tilde{x}, x))$, contradicting the assumption that $c_i(q) - c'_i(q)$ is nondecreasing in q .

Step 1: We first show that $\pi_i(\hat{p}_i, z) < \pi_i(\hat{p}_i, D(\hat{p}_i))$ for all $z < D(\hat{p}_i)$. Suppose to the contrary that $\pi_i(\hat{p}_i, z) \geq \pi_i(\hat{p}_i, D(\hat{p}_i))$ for some $z < D(\hat{p}_i)$. As noted in the proof of Proposition 6, $Q_i(\hat{p}_i) = D(\hat{p}_i)$. Thus, $\pi_i(\hat{p}_i, z) = \varphi_i(\hat{p}_i)$. Since φ_i is continuous by Assumption 7, there exists a price $x > \hat{p}_i$ such that $z < D(x)$. Observe that $\varphi_i(x) \geq \pi_i(x, z) = xz - c_i(z) > \hat{p}_i z - c_i(z) = \varphi_i(\hat{p}_i)$. This contradicts $\hat{p}_i$ as the maximizer of φ_i . We conclude that $\pi_i(\hat{p}_i, z) < \pi_i(\hat{p}_i, D(\hat{p}_i))$.

We next argue that $\bar{r}_i < \hat{p}_i$. Suppose to the contrary that $\bar{r}_i \geq \hat{p}_i$. Since $\underline{a}_j < \hat{p}_i$ by Assumption 5, we may choose a strictly increasing sequence $\{y^k\}$ such that $y^0 > \underline{a}_j$ and $y^k \rightarrow \hat{p}_i$. By definition of $\bar{r}_i$, $\varphi_i(x) \leq \tilde{\psi}_i(x)$ for all $x < \bar{r}_i$. Thus, since $\tilde{\psi}_i(x)$ is nonincreasing as noted earlier, this implies that $\tilde{\psi}_i(y^0) \geq \varphi_i(y^k)$ for all k . By continuity of φ_i by Assumption 7, $\lim_k \varphi_i(y^k) = \varphi_i(\hat{p}_i)$, and so $\tilde{\psi}_i(y^0) \geq \varphi_i(\hat{p}_i)$. Let $\tilde{x} \in \tilde{P}_i(y^0)$ and note that $\psi_i(\tilde{x}, y^0) \leq \varphi_i(\tilde{x})$ by Assumption 1. Observe that $\varphi_i(\hat{p}_i) \geq \varphi_i(\tilde{x}) \geq \tilde{\psi}_i(y^0) \geq \varphi_i(\hat{p}_i)$. Since $\hat{p}_i$ is the unique maximizer of φ_i , it follows that $\tilde{x} = \hat{p}_i$. As demonstrated in the proof of Proposition 6, $Q_i(\hat{p}_i) = D(\hat{p}_i)$ and $s_j(\hat{p}_i) > 0$ since $\hat{p}_i > \underline{a}_j$. By Condition 6, this implies that $d_i(\hat{p}_i, y_0) < D(\hat{p}_i)$, so $q_i(\hat{p}_i, y^0) < D(\hat{p}_i)$. From the first paragraph of this step, this implies that $\pi_i(\hat{p}_i, q_i(\hat{p}_i, y^0)) < \pi_i(\hat{p}_i, D(\hat{p}_i))$ and thus that $\psi_i(\hat{p}_i, y^0) < \varphi_i(\hat{p}_i)$. A contradiction to $\tilde{\psi}_i(y^0) \geq \varphi_i(\hat{p}_i)$. We conclude that $\bar{r}_i < \hat{p}_i$.

Step 2: We show that $\bar{q}_i(x, y) \leq Q_i(y)$ for all $x \geq y > \bar{r}_i$. Suppose to the contrary that $\bar{q}_i(x, y) > Q_i(y)$ for some $x \geq y > \bar{r}_i$. Note that

$$\psi_i(x, y) = \max_{z \in [0, \min\{k_i, d_i(x, y)\}]} \pi_i(x, z).$$

Since $Q_i(\bar{r}_i) < \bar{q}_i(x, y) \leq d_i(x, y)$, it follows that

$$\begin{aligned} \psi_i(x, y) &\geq xQ_i(y) - c_i(Q_i(y)) \\ &\geq yQ_i(y) - c_i(Q_i(y)) \\ &= \varphi_i(y). \end{aligned}$$

This contradicts the definition of $\bar{r}_i$ as $\bar{r}_i = \sup\{x | \varphi_i(x) \leq \tilde{\psi}_i(x)\}$. We conclude that $\bar{q}_i(x, y) \leq Q_i(\bar{r}_i)$ for all $x \geq y > \bar{r}_i$.

Step 3: We argue that $\bar{q}_i(x, y) \geq q'_i(x, y)$ for all $x \geq y > \bar{r}_i$. Let $x \geq y > \bar{r}_i$ and suppose to the contrary that $\bar{q}_i(x, y) < q'_i(x, y)$. Then since $q'_i(x, y) \leq d_i(x, y)$, it must be that $\bar{q}_i(x, y) < d_i(x, y)$. It follows that $\bar{q}_i(x, y) = \bar{s}_i(x)$, and so $\psi_i(x, y) = \pi_i(x, \bar{s}_i(x)) = \varphi_i(x)$.

Since $\bar{s}_i$ is nondecreasing and d_i is nonincreasing by Condition 5, it follows that $\bar{s}_i(p_i) < d_i(p_i, p_j)$ for all $p_i < x$ such that $p_i \geq p_j > \bar{r}_i$. Thus, $\psi_i(p_i, p_j) = \pi_i(p_i, \bar{s}_i(p_i)) = \varphi_i(p_i)$ for all p_i and p_j such that $x > p_i \geq p_j > \bar{r}_i$. Thus, $\tilde{\psi}_i(p_j) \geq \psi_i(p_j, p_j) = \varphi_i(p_j)$ for all $p_j \in (\bar{r}_i, x)$, contradicting the definition of $\bar{r}_i$. We conclude that $\bar{q}_i(x, y) \geq q'_i(x, y)$ for all $x \geq y > \bar{r}_i$.

In summary we have now established that, since $\bar{s}_i$ is nondecreasing, then for any $x \geq y > \bar{r}_i$, it follows that $q'_i(x, y) \leq \bar{q}_i(x, y) \leq Q_i(y)$.

Step 4: We argue that $\bar{r}'_i \leq \bar{r}_i$. Suppose to the contrary that $\bar{r}'_i > \bar{r}_i$. Then for any price $x \in (\bar{r}_i, \bar{r}'_i)$, it must be that $\varphi'_i(x) \leq \tilde{\psi}'_i(x)$. Let $x \in (\bar{r}_i, \bar{r}'_i)$ and $\tilde{x} \in \tilde{P}'_i(x)$ and note that from above, $q_i(\tilde{x}, x) \leq Q_i(x)$. Note that $\tilde{\psi}'_i(x)$ is nonincreasing in x since

$$\tilde{\psi}'_i(x) = \max_{p_i} \max_{z \in [0, \min\{k_i, d_i(p_i, x)\}]} \pi_i(p_i, z)$$

and $d_i(p_i, x)$ is nonincreasing in x by Condition 5. From above, we may choose the price $x \in (\bar{r}_i, \bar{r}'_i)$ such that $x < \hat{p}_i$. Since φ_i is strictly increasing on $(\underline{a}_i, \hat{p}_i)$ by Assumption 3 and from continuity of D from Condition 4, we may choose x so that $\varphi'_i(x) \leq \tilde{\psi}'_i(x)$ and $\varphi_i(x) > \tilde{\psi}_i(x)$. Thus, we have

$$\varphi'_i(x) - \varphi_i(x) < \tilde{\psi}'_i(x) - \tilde{\psi}_i(x). \quad (\text{A1})$$

Note that $\varphi_i(x) = xQ_i(x) - c_i(Q_i(x))$ and $\varphi'_i(x) \geq xQ_i(x) - c'_i(Q_i(x))$. Putting these together, we have

$$\varphi'_i(x) - \varphi_i(x) \geq c_i(Q_i(x)) - c'_i(Q_i(x)). \quad (\text{A2})$$

Next, note that $\tilde{\psi}'_i(x) = \tilde{x}q'_i(\tilde{x}, x) - c'_i(q'_i(\tilde{x}, x))$ and $\tilde{\psi}_i(x) \geq \tilde{x}q'_i(\tilde{x}, x) - c_i(q'_i(\tilde{x}, x))$. Putting these together yields

$$\tilde{\psi}'_i(x) - \tilde{\psi}_i(x) \leq c_i(q'_i(\tilde{x}, x)) - c'_i(q'_i(\tilde{x}, x)). \quad (\text{A3})$$

The inequalities (A1), (A2), and (A3) together imply that

$$c_i(Q_i(x)) - c'_i(Q_i(x)) < c_i(q'_i(\tilde{x}, x)) - c'_i(q'_i(\tilde{x}, x)),$$

which contradicts the assumption that $c_i(z) - c'_i(z)$ is nondecreasing in $z \geq 0$. We conclude that $\bar{r}'_i \leq \bar{r}_i$.

Part 2: $\underline{r}'_i \leq \underline{r}_i$

The proof that $\underline{r}'_i \leq \underline{r}_i$ is also done by contradiction. Suppose to the contrary that $\underline{r}'_i > \underline{r}_i$. Recall that $\underline{u}_i = \sup_{p_i} \inf_{p_j} u_i(p_i, p_j)$. As noted above, ψ_i is nonincreasing in p_j . Further, by Assumption 1, we can conclude that $\inf_{p_j} u_i(p_i, p_j) = \psi_i(p_i, p_i)$, and thus $\underline{u}_i = \sup_{p_i} \psi_i(p_i, p_i)$. Additionally, since φ_i is continuous by Assumption 7, it follows that $\varphi_i(\underline{r}_i) = \underline{u}_i$.

Let $\{x^k\}$ be a sequence such that $\psi'_i(x^k, x^k) \rightarrow \underline{u}'_i$. We may without loss of generality choose this sequence such that $x^k \rightarrow x^*$ for some price x^* . It follows that $\underline{u}'_i = \pi'_i(x^*, q^*)$, where $q^* = \lim_k q'(x^k, x^k)$. By definition of $\underline{u}_i$, it must be that $\underline{u}_i \geq \psi_i(x^k, x^k)$ for all k . Further $\psi_i(x^k, x^k) = \pi_i(x^k, q_i(x^k, x^k)) \geq \pi_i(x^k, q'_i(x^k, x^k))$, and so $\underline{u}_i \geq \pi_i(x^*, q^*)$. Therefore,

$$\underline{u}'_i - \underline{u}_i \leq \pi'_i(x^*, q^*) - \pi_i(x^*, q^*) = c_i(q^*) - c'_i(q^*). \quad (\text{A4})$$

We briefly argue that $x^* \geq \underline{r}'_i$. To see this, suppose to the contrary that $x^* < \underline{r}'_i$. Then note that $\varphi'_i(\underline{r}'_i) = \underline{u}'_i = \pi'_i(x^*, q^*) \leq \varphi'_i(x^*)$. By definition of $\underline{a}_i$, it must be that $\underline{a}'_i \leq \underline{a}_i$, and since $\underline{a}_i \leq \underline{r}_i < \underline{r}'_i$, it follows that $\underline{r}'_i > \underline{a}'_i$. Thus, Assumption 3 implies that φ'_i is strictly

increasing on $(\max\{a'_i, x^*\}, r'_i)$. Since $\varphi'_i(x) = 0$ for $x < a'_i$ by Assumption 2, it follows that $\varphi'_i(x^*) < \varphi'_i(r'_i)$, a contradiction. We conclude that $x^* \geq r'_i$.

We will now argue that $q^* \leq \bar{Q}_i(x)$ for all x in some neighborhood $(r_i, r_i + \delta)$.

We begin by arguing that $\bar{Q}_i(y) \geq d_i(x, x)$ for all prices $x > y > r_i$. Suppose to the contrary that $\bar{Q}_i(y) < d_i(x, x)$ for some $x > y > r_i$. Recall that $\varphi_i(r_i) = \pi_i(r_i, \bar{Q}_i(r_i))$. Next, since $\pi_i(x, z)$ is quasiconcave in z by Condition 2, it follows that $\psi_i(x, x) = x\bar{q}_i(x, x) - c_i(\bar{q}_i(x, x))$, where $\bar{q}_i(x, x) = \min\{\bar{s}_i(x), d_i(x, x)\}$. Note that by definition of $\underline{u}_i$ and $\bar{s}_i$,

$$\begin{aligned} \underline{u}_i &\geq \psi_i(x, x) \\ &= x\bar{q}_i(x, x) - c_i(\bar{q}_i(x, x)) \\ &\geq x\bar{Q}_i(y) - c_i(\bar{Q}_i(y)). \end{aligned}$$

Since $y > r_i$ and $r_i \geq a_i$, it follows that $\varphi_i(y) > 0$, and so $\bar{Q}_i(y) > 0$. Thus, $x\bar{Q}_i(y) > y\bar{Q}_i(y)$. Therefore,

$$\begin{aligned} \underline{u}_i &\geq x\bar{Q}_i(y) - c_i(\bar{Q}_i(y)) \\ &> y\bar{Q}_i(y) - c_i(\bar{Q}_i(y)) \\ &= \varphi_i(y) \\ &\geq \varphi_i(r_i) \\ &= \underline{u}_i. \end{aligned}$$

This is a contradiction. We conclude $\bar{Q}_i(y) \geq d_i(x, x)$ for all prices $x > y > r_i$.

Now, suppose to the contrary that there exists a sequence $\{y^n\}$ with $y^n \rightarrow r_i$ and $y^n > r_i$ such that $q^* > \bar{Q}_i(y^n)$ for all n . Since $x^* \geq r'_i > r_i$, we may without loss of generality assume that $y^n < \min\{x^k, x^*\}$ for all k and n . Thus, from above, $\bar{Q}_i(y^n) \geq d_i(x^k, x^k)$ for all k and n . Since $q'_i(x^k, x^k) \leq d_i(x^k, x^k)$, this implies that $\bar{Q}_i(y^n) \geq q^*$, a contradiction. We conclude that $q^* \leq \bar{Q}_i(x)$ for all x in some neighborhood $(r_i, r_i + \delta)$. Choose such a δ .

Now observe that by definition of r'_i , $\varphi'_i(r_i) \leq \underline{u}'_i$ for any price $x \in (r_i, r'_i)$. Thus, for any price $x \in (r_i, \hat{p}_i)$, it must be that $\varphi_i(x) > \underline{u}_i$ since φ_i is strictly increasing by Assumption 3. Recall that $r_i \leq \bar{r}_i$, and as shown above, $\bar{r}_i < \hat{p}_i$, so $(r_i, \hat{p}_i)$ is nonempty. Let $x \in (r_i, \min\{r'_i, \hat{p}_i, r_i + \delta\})$ with $\delta > 0$ picked such that $q^* \leq \bar{Q}_i(x)$ for all x in some neighborhood $(r_i, r_i + \delta)$. Note that

$$\varphi'_i(x) - \varphi_i(x) < \underline{u}'_i - \underline{u}_i.$$

Observe that

$$\varphi_i(x) = xQ_i(x) - c_i(Q_i(x)),$$

and

$$\begin{aligned} \varphi'_i(x) &= xQ'_i(x) - c'_i(Q'_i(x)) \\ &\geq xQ_i(x) - c'_i(Q_i(x)). \end{aligned}$$

Putting these together, we have

$$\varphi'_i(x) - \varphi_i(x) \geq c_i(Q_i(x)) - c'_i(Q_i(x)),$$

and thus

$$c_i(Q_i(x)) - c'_i(Q_i(x)) < \underline{u}'_i - \underline{u}_i.$$

Recall that from (A4)

$$u'_i - u_i \leq c_i(q^*) - c'_i(q^*).$$

It follows that

$$c_i(\bar{Q}_i(x)) - c'_i(\bar{Q}_i(x)) < c_i(q^*) - c'_i(q^*),$$

which contradicts the assumption that $c_i(z) - c'_i(z)$ is nondecreasing in $z \geq 0$ since $\bar{Q}_i(x) \geq q^*$. We conclude that $\bar{r}'_i \leq \bar{r}_i$. $\square$

Notes

- ¹ Vives (1986, 1993) both provide excellent context for Edgeworth's contribution to oligopoly.
- ² Before Shubik (1959), Shapley (1957) published an abstract with a description of results derived from a game theoretic model of pricing. Other early contributions to BE competition were made by Beckmann and Hochstadter (1965), Shapley and Shubik (1969), and Levitan and Shubik (1972).
- ³ To contextualize the different rationing schemes, imagine that demand is composed of a continuum of consumers with different levels of willingness to pay for a single unit of the good. The efficient rationing rule specifies that the low price firm serves the consumers with the highest willingness to pay. That is, all rationed consumers have a weakly lower willingness to pay than all consumers that purchase from the low price firm. The proportional rule specifies that all consumers willing to pay the low price are equally likely to be served by the low price firm, resulting in a proportion of high willingness to pay being rationed and thus a larger residual demand than the efficient rule.
- ⁴ Almost all of the BE literature also assumes that the firms have a symmetric, constant marginal cost up to capacity. Deneckere and Kovenock (1996) and Allen et al. (2000) are the notable exceptions. These papers focus on the interesting case in which firms have constant marginal costs that are asymmetric. Additionally, the bulk of this literature further restricts demand to be such that a firm's monopoly profit is concave in its price. Our analysis is based on the considerably weaker assumption that a firm's monopoly profit is strictly increasing in its own price up to its unique profit-maximizing monopoly price.
- ⁵ Hoernig (2007) provides a treatment of classical Bertrand price competition with general cost structure and sharing rules. In the classical Bertrand specification, any firm that does not have the lowest price receives no residual demand.
- ⁶ Yoshida (2002) provides a similar treatment to Yoshida (2006) for a model with linear demand and quadratic cost.
- ⁷ The relationship between BE games and all-pay auctions is discussed in Baye et al. (1996), the first comprehensive treatment of all-pay auctions. More recently, Chowdhury (2017) rely on techniques from the analysis of BE duopoly by Osborne and Pitchik (1986) and Deneckere and Kovenock (1996) to examine all-pay auctions with non-monotonic payoffs.
- ⁸ The two distinctions between a traditional all-pay auction and our BE game have been each treated individually in the all-pay auction literature. In Baye et al. (2012), the issue of externalities of bids (contingent on being a winner or a loser) has been addressed in the context of all-pay auctions. Chowdhury (2017) provides a treatment of all-pay auctions in which the winning payoff is nonmonotonic in a player's own bid.
- ⁹ In terms of market demand restrictions, we require only that the monopoly profit be strictly increasing at prices above minimum average total cost up to its unique maximizer.
- ¹⁰ Gelman and Salop (1983) show that, in a two-period sequential game, a single potential entrant can use judo capacity restriction and pricing to induce an unconstrained monopolist to allow entry. The mathematical object that we have denoted as the critical judo price has played a critical role in the analysis of BE price competition since it was first used to characterize the Edgeworth range of price fluctuation in (Shubik, 1959, p. 96).
- ¹¹ Progress with the analysis of models with more general costs and demand rationing has been hindered by theoretical problems with the existence of equilibrium (pure or mixed) in this setting. However, we utilize advances in the literature on the existence of equilibrium in discontinuous games by Bagh (2010) and Allen and Lepore (2014) that allow for the straightforward verification of the existence of equilibrium in vast generalizations of BE oligopoly.
- ¹² This countervailing effect of a supply increase is immediate in the existing BE literature with regard to an increase in a firm's capacity. This is also related to the impact of an import trade quota in a duopoly with an international and domestic firm, for example Krishna (1989).
- ¹³ When this assumption fails to hold, the equilibrium is trivial: one firm charges its monopoly price, and the other firm charges any price and does not produce. While it would be easy to conduct the analysis in this paper without this assumption, it would take away from the clarity of the results and would not meaningfully contribute to the study of duopoly.
- ¹⁴ Here, the bounds $\underline{x}_i$ and $\bar{x}_i$ are inherently dependent on the equilibrium strategies, though we suppress notation indicating this for clarity as there is no ambiguity as to which strategies they correspond to.
- ¹⁵ The supply correspondence is taken to be a subset of the extended real line.

- 16 Observe that the residual profit may be zero despite the presence of residual demand as the cost of engaging in low levels of
production may exceed the associated revenues, thus inducing the firms to not produce.
- 17 For readers familiar with [Simon and Zame \(1990\)](#), their result can be used to guarantee that an equilibrium exists for some sharing
rule of the game. The complication in this setting is that the sharing rule in the Simon and Zame framework does not correspond
only to the division α_i between q_i and ψ_i at pricing ties. The sharing rule in this setting also reflects specifications of payoffs at
points of discontinuity of ψ_i in p_j . The results of [Simon and Zame \(1990\)](#) do not give any way of identifying the sharing rule for
which an equilibrium exists. If the sharing rule with an equilibrium fits the specifications of Proposition 1 and the corresponding
equilibrium happens to place mass at price ties as in Proposition 1 or at a point of discontinuity of ψ_i , then this strategy profile
would not be an equilibrium of a sharing rule that violated the conditions of Proposition 1 at those prices. However, if the sharing
rule with an equilibrium violates the conditions of Proposition 1, then the corresponding equilibrium will also be an equilibrium
for a sharing rule that satisfies the conditions of Proposition 1. Other applicable results that guarantee the existence of equilibrium
typically necessitate the same conditions. We have found more general results to be infeasible for application to our model.
- 18 Ties at a price $x < \underline{\rho}$ are irrelevant since each firm's front-side profit is identical to its residual profit. Ties at prices $x = \underline{\rho}$ may be
relevant by this notion and are covered by the lemma.
- 19 The efficient and proportional rationing rules that have typically been used in studying price competition both lead to residual
profit functions being nonincreasing in the rival firm's price.
- 20 In this formulation of the model, demand is finite at all prices, so a firm that is not capacity-constrained can be accommodated via
an arbitrarily large capacity.
- 21 With any continuous rationing rule, such as efficient or proportional rationing, firm i 's residual demand will be lower semicontin-
uous in p_j so long as s_j is upper semicontinuous.
- 22 Here, the limit superior of the sequence of sets A_k refers to the set $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.
- 23 This occurs at prices p_j such that firm j 's cost of production is constant and equal to p_j for some levels of production. As such, it
is possible that firm j 's quantity jumps up at such a price, causing a discrete drop in the residual profit for firm i .
- 24 A correspondence ϑ is nondecreasing if, for any $x \leq x'$ and any $y \in \vartheta(x)$, there exists a $y' \in \vartheta(x')$ such that $y' \geq y$.
- 25 Note that τ'_i is inherently a function of p_j . We choose not to introduce notation to express this as p_j is fixed for the duration of the
proof that utilizes τ'_i , and thus, there is no possibility for ambiguity.

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