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Isotone Classes of Social Choice Functions with Binary Range

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Abstract: Recently, it has been shown that the characterizations of different classes of non-manipulable social choice functions with binary range can be reduced to a common functional form. In the present paper, we investigate the reasons why this happens. We show that all the classes considered share a common mathematical property. We name this property, which is lattice theoretical in nature, isotonicity.

Keywords: social choice functions; non-manipulability; character; isotonicity

MSC: 91B14

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1. Introduction

The basic mathematical model of the mechanism for selecting an alternative that a society implements in the interest of its members is, simply, a function, called a social choice function. It is a rather abstract concept, which is why several specifications are necessary. Let V be a set that denotes a collectivity of individuals $v \in V$, and let A be the set consisting of all the alternatives that the society V could a priori implement. By \mathcal{P} we denote the domain of a social choice function, which is, therefore, a function $\phi : \mathcal{P} \rightarrow A$. The elements P of the domain \mathcal{P} of ϕ , called preference profiles, describe (*lato sensu*), individual by individual, the “utility” received from implementing one alternative rather than another. Hence, $P = (P_v)_{v \in V}$, where P_v represents the way agent v orders the alternatives, and in other words, their preference relation on A .

In principle, any function ϕ can be used to select a collective choice. However, the fundamental property required of a social choice function is that it avoids the possibility of individuals engaging in strategic behavior. That is, it avoids lying about one’s preferences in order to induce the implementation of an alternative that is more advantageous for oneself. This property is named strategy-proofness.

The consistency of the concept of strategy-proofness is clear: strategy-proof social choice functions exist ¹. However, in the rather natural case that the range of ϕ contains at least three elements, the celebrated Gibbard–Satterthwaite theorem [1,2] states that there are no strategy-proof social choice functions other than those described in Note 1. Thus, this makes the study of the concept uninspiring, unless other elements such as domain restrictions and/or incomplete information are introduced into the scenario (see the survey [3]).

The case of binary collective choices (i.e., the case of functions ϕ whose range has cardinality two) is neither less natural nor less relevant in real life applications. Here, the



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situation drastically changes and many social choice functions verify strategy-proofness (see the papers [4–6], among others).

In the paper [7], various classes of social choice functions with range of cardinality two are considered. Precisely, the classes of all social choice functions are:

- Weakly group strategy-proof;
- Strongly group strategy-proof;
- Weakly group strategy-proof and anonymous;
- Weakly group strategy-proof with domain limited to strict preferences;
- Weakly group strategy-proof and onto²;
- Weakly group strategy-proof, anonymous, and onto.

They are all subclasses of the fundamental class of strategy-proof social choice functions. In the last two decades, several authors have provided functional form characterizations of these classes. Among others, we mention [4–6,8–13].

The main achievement of [7] is that the above classes can all be described by means of a unified functional form. To be precise, what is shown in [7] is as follows. For each of the classes mentioned above, say Φ , it is presented a poset L and a surjective function χ from \mathcal{P} to a L such that a one-to-one correspondence can be established between the elements of Φ and the super order closed subsets of L (see Definition 2, *infra*). Different classes Φ are associated with different pairs L, χ . The structure of the one-to-one correspondence is common to all cases. It is established by writing every element ϕ of Φ , with range $\{a, b\} \subseteq A$, by means of the following formula:

$$(\star) \quad \phi = \phi_{(\chi, C)}$$

where the C is a super order closed subset of L , and the definition of $\phi_{(\chi, C)}$ is

$$\phi_{(\chi, C)}(P) = \begin{cases} a, & \text{if } \chi(P) \in C \\ b, & \text{if } \chi(P) \notin C \end{cases} .$$

Thus, the defining properties of Φ are captured by the function χ (which is the reason why in [7], the function χ is called *character*), while the “list” of the various elements of Φ occurs in one with the list of the super order closed subsets of L .

The results of [7] raise the following questions naturally:

- Since every one of the classes considered in [7] is representable in the aforementioned way, what is it about them that makes them so?
- Is it possible that every class of social choice functions with binary range can be represented this way?
- If not, what conditions must be imposed on a class Φ of social choice functions with binary range for it to be true that a character function exists?

In this paper, we answer these (and related) questions that are not discussed in [7]. What we obtain is the following. Not all Φ can be represented as in [7]. This is possible if and only if the class is *isotone*. Intuitively, this property means that Φ contains the supremum of its subclasses and, for every profile P , the minimum among the social choice functions belonging to Φ for which the collective decision at the profile P is the alternative a^3 . The two conditions (see Proposition 5, *infra*; but also Proposition 6) can be easily checked directly on the class Φ . Indeed, we verify that all classes considered in [7] are isotone. Hence, it is the common property of being isotone that makes [7] succeed in providing, case by case, the character representation of the different classes analyzed.

To isolate isotonicity as the key property, we elaborate on the χ -monotonicity notion presented in [7] (which in turn is inspired by the concept of Maskin monotonicity) and beyond.

Having a character representation (\star) for Φ available, it is possible to argue that Φ is isotone, but here, we have reversed the point of view. Starting with a class Φ , for which we do not know whether it can be represented or not, we find that if Φ is isotone then a

character representation exists. Concerning the specific classes analyzed in [7], since the character we find is not immediately traceable to the specific character presented in [7], we also present a uniqueness result. Moreover, for a possibly non-isotone class Φ , we define the *isotone completion*, as the smallest isotone class including Φ and give a relevant (yet simple) example by showing that the voting by committee social choice functions form the isotone completion of the class of the dictatorial social choice functions.

The paper is organized as follows. Since isotonicity can be considered in relation to families of subsets of an arbitrary set, not necessarily consisting of preference profiles, in Section 3, isotonicity for sets and families of sets is introduced and some characterizations are presented. We also introduce the notion of isotone completion. Preliminarily, in Section 2, the notions of character and monotonicity of sets with respect to a character are defined and expressed in terms of super order closed subsets of a poset. The latter is shown to be essentially unique. In Section 4, the representability of classes of functions with binary range is obtained through the use of characters. Section 5 focuses on classes of social choice functions with binary range. These form the original target of our investigation. We explain why several classes of non-manipulable social choice functions have a unified functional form. The reason is the isotonicity of such classes. Finally, Appendix A contains proofs..

2. Preliminaries

Let T denote a non-empty set.

Inspired by ([7] Definition 3.10), in turn inspired by the notion of Maskin monotonicity of social choice functions, we start by introducing a notion of *monotonicity with respect to a character* for subsets of T .

Let us recall that a partial order on a set L is a reflexive, transitive, and antisymmetric binary relation on L . When a partial order \leq is defined on L , the pair (L, \leq) is called a partially ordered set (or poset, for short).

Definition 1. A function from T onto a poset (L, \leq) is called a **character** on T .

Definition 2. Let (L, \leq) be a partially ordered set. A subset C of L is said to be **super order closed** if:

$$\{x \in L : \exists c \in C, \text{ with } c \leq x\} \subseteq C.$$

We denote by $\text{soc}(L, \leq)$ or, more briefly, by $\text{soc}(L)$ the family of all super order closed subsets of L that are non-empty and proper.

Definition 3. Let χ be a character on T . A proper non-empty subset X of T is said to be **χ -monotone** if:

$$\{t' \in T : \exists t \in X, \text{ with } \chi(t) \leq \chi(t')\} \subseteq X.$$

We denote by $\chi_{\mathcal{M}}$ the family of all non-empty proper χ -monotone subsets of T .

The intuitive relation between the above notion of monotonicity and that of the super order closedness of subsets of a poset is made precise in the next Lemma.

Lemma 1. Let χ be a character on T . A subset X of T is χ -monotone if and only if $\exists! C \in \text{soc}(L)$, such that $X = \chi^{-1}(C)$.

Proof. \Leftarrow Assume $X = \chi^{-1}(C)$ for some $C \in \text{soc}(L)$.

If $t \in X$, we are saying that $\chi(t) \in C$. By super order closedness of C from $\chi(t) \leq \chi(t')$, it follows that $\chi(t') \in C$; hence, $t' \in X$. C being non-empty, $X \neq \emptyset$ follows. Moreover, since C is proper and the function χ is surjective, it is also true that $X \neq T$.

\Rightarrow Assume X is χ -monotone. Let us show that the desired C is the image set $\chi(X)$.

CLAIM 1: $C = \chi(X) \in \text{soc}(L)$.

The set C is obviously non-empty; moreover, it is a proper subset of L . If not, for any $t \in T$ the element $\chi(t) \in C$ is of the form $\chi(t) = \chi(t')$ for a suitable $t' \in X$. Being the set X monotone, it follows that $t \in X$. The equality $X = T$ contradicts that $X \subset T$ by definition.

The set C is super order closed, and hence, it is an element of $\text{soc}(L)$: Let us suppose $c' \geq c \in C, c' \in L$. Let us rephrase this as $\chi(t') \geq \chi(t), t \in X, t' \in T$. Hence, apply monotonicity, $t' \in X$, i.e., $c' = \chi(t') \in C$.

CLAIM 2: $X = \chi^{-1}(\chi(X))$.

Only the inclusion \supseteq has to be proved, since the reverse is always true. So, take $t \in T$ with $\chi(t) \in C$. Hence, $\chi(t) = \chi(x), x \in X$, and monotonicity gives that also $t \in X$.

CLAIM 3: uniqueness. For two super order closed subsets B, C of L such that $X = \chi^{-1}(B) = \chi^{-1}(C)$, it is obviously $B \subseteq C$. \square

The next proposition is a corollary of Lemma 1.

Proposition 1. *Let $(L, \leq_L), (\Lambda, \leq_\Lambda)$ be posets. Let $\chi : T \rightarrow L$ and $\widehat{\chi} : T \rightarrow \Lambda$ be characters on T . Under the assumption that a subset of T is χ -monotone if and only if it is $\widehat{\chi}$ -monotone, the following is true:*

$$\chi(t) \leq_L \chi(t') \Leftrightarrow \widehat{\chi}(t) \leq_\Lambda \widehat{\chi}(t').$$

Preliminarily to the proof, we introduce a notation: given $\ell \in L$, we denote by $[\ell]_L$ the set $\{x \in L : x \geq \ell\}$.

With this notation, it is obvious that C is super order closed if and only if $C = \bigcup_{\ell \in C} [\ell]_L$, and hence, the family $\text{soc}(L)$ can be described as follows:

$$\text{soc}(L) = \left\{ \bigcup_{\ell \in Y} [\ell]_L : Y \text{ is a non-empty subset of } L \text{ and } \bigcup_{\ell \in Y} [\ell]_L \neq L \right\}.$$

Proof of Proposition 1. Clearly, to prove the stated equivalence, it is enough to prove the “ \Rightarrow ” implication. Let Γ be the element of $\text{soc}(\Lambda)$ equal to $[\widehat{\chi}(t)]_\Lambda$. The set $\widehat{\chi}^{-1}(\Gamma)$ is $\widehat{\chi}$ -monotone, and hence, it is χ -monotone. Take $C \in \text{soc}(L)$ such that $\chi^{-1}(C) = \widehat{\chi}^{-1}(\Gamma)$. Since by definition $t \in \widehat{\chi}^{-1}(\Gamma)$, one has: $\chi(t) \in C \Rightarrow \chi(t') \in C \Rightarrow t' \in \chi^{-1}(C) \Rightarrow t' \in \widehat{\chi}^{-1}(\Gamma)$, which is the assertion. \square

The above proposition allows to define an order isomorphism q between L and Λ as follows: $q(\ell) := \widehat{\chi}(t)$, for $\ell \in L$ equal to $\chi(t)$. Moreover, $\widehat{\chi} = q \circ \chi$. The fact that q is well defined and is an order isomorphism is a straightforward consequence of the proposition. Hence, we have the following.

Corollary 1. *Let $(L, \leq_L), (\Lambda, \leq_\Lambda)$ be posets. Let $\chi : T \rightarrow L$ and $\widehat{\chi} : T \rightarrow \Lambda$ be characters on T . If for the subsets of T the χ -monotonicity is equivalent to the $\widehat{\chi}$ -monotonicity, then the posets L and Λ are order isomorphic under an isomorphism q such that $\widehat{\chi} = q \circ \chi$.*

3. Isotone Sets and Isotone Families of Sets

Throughout the following, let $\mathcal{F}, \mathcal{G}, \dots$ denote non-empty families of non-empty proper subsets of T .

Obviously, any function whose range has cardinality two can be identified with a set: the one in which it takes one of its two values, say a . The other value, say b , is taken on the complementary set. For this reason, our problem about classes of social choice functions can be abstractly stated as follows.

Let T be a set and \mathcal{F} a non-empty family of non-empty proper subsets of T . What conditions must be imposed on \mathcal{F} for it to be true that a character function $\chi : T \rightarrow L$ exists such that sets in \mathcal{F} are in a one-to-one correspondence with the super order closed subsets of L ? Keeping this in mind, let us fix some notations that will be useful next.

For an element t of T set:

- (1) $\mathcal{F}_t := \{S \in \mathcal{F} : t \in S\};$
- (2) $[t]_{\mathcal{F}} := \{t' \in T : \mathcal{F}_t \subseteq \mathcal{F}_{t'}\}.$

Note that for some t , the set \mathcal{F}_t may be empty. However, $[t]_{\mathcal{F}}$ is always non-empty because $t \in [t]_{\mathcal{F}}$. The set $[t]_{\mathcal{F}}$ will be called the \mathcal{F} -upper level of t .

From the definitions above, it follows that for every $t \in T$, one has:

- (3) $[t]_{\mathcal{F}} = \bigcap_{S \in \mathcal{F}_t} S.$

Example 1. Let T be the set $\{1, 2, 3, 4, 5\}$. Consider the family $\mathcal{F} = \{B, C, D, E, F\}$ where:

$$B = \{1, 2, 5\} \quad C = \{1, 2, 4, 5\}$$

$$D = \{3, 4\} \quad E = \{5\} \quad F = \{3, 4, 5\}$$

It turns out that:

$$\mathcal{F}_1 = \mathcal{F}_2 = \{B, C\} \quad \mathcal{F}_3 = \{D, F\}$$

$$\mathcal{F}_4 = \{C, D, F\} \quad \mathcal{F}_5 = \{B, C, E, F\}$$

We list below the \mathcal{F} -upper level sets:

$$[1]_{\mathcal{F}} = [2]_{\mathcal{F}} = \{1, 2, 5\} = B \quad [3]_{\mathcal{F}} = \{3, 4\} = D$$

$$[4]_{\mathcal{F}} = \{4\} \quad [5]_{\mathcal{F}} = \{5\} = E$$

which are all elements of \mathcal{F} , apart from $[4]_{\mathcal{F}}$.

In order to identify a property that a family \mathcal{F} must enjoy to be representable in the way mentioned previously by means of a character function, we introduce the notions of *isotone sets* and *isotone families*. Let us start with the notion of isotone set. In comparing this concept with the previous one of monotone set, observe that the only primitive now is the family \mathcal{F} of subsets of T .

Definition 4. A non-empty proper subset X of T is said to be \mathcal{F} -isotone if:

$$[t \in X, t' \in T, \mathcal{F}_t \subseteq \mathcal{F}_{t'}] \Rightarrow t' \in X.$$

Equivalently, X is \mathcal{F} -isotone if it includes the \mathcal{F} -upper level sets of all its own elements:

$$[t]_{\mathcal{F}} \subseteq X \quad \forall t \in X.$$

It is easy to check that, given a family \mathcal{F} , each set $X \in \mathcal{F}$ is \mathcal{F} -isotone.

Example 1 continuation. Note that the set $\{4\}$ is \mathcal{F} -isotone though it does not belong to \mathcal{F} . The set $\{3, 5\}$ is not \mathcal{F} -isotone.

Remark 1. Let L be the set:

$$L := \{\mathcal{F}_t : t \in T\}.$$

Ordered by set inclusion, L is a partially ordered set. Define a character χ from T to L by $\chi(t) := \mathcal{F}_t$. It is evident that, with reference to this specific character function, for the subsets of T the concept of \mathcal{F} -isotonicity and the concept of χ -monotonicity coincide.

Definition 5. Let \mathcal{F} be given. \mathcal{F} is said to be **isotone** if all the \mathcal{F} -isotone subsets of T belong to \mathcal{F} . In other words, the family \mathcal{F} is isotone if: $X \in \mathcal{F} \Leftrightarrow X$ is \mathcal{F} -isotone.

The existence of isotone families of sets is guaranteed by Proposition 4, *infra*. On the other hand not all families \mathcal{F} are isotone, as Example 1 shows. Indeed, the family \mathcal{F} considered in the example does not contain the \mathcal{F} -isotone set $\{4\}$.

Collecting some of the previous results, we state the following.

Proposition 2. Let \mathcal{F} be an isotone family of proper non-empty subsets of a set T . Then, there exists a partially ordered set (L, \leq) and a function χ from T onto L (i.e., a character) such that \mathcal{F} is in a one-to-one correspondence with the family $\text{soc}(L)$, by means of the mapping $[X \in \mathcal{F} \mapsto \chi(X) \in \text{soc}(L)]$ with inverse $[C \in \text{soc}(L) \mapsto \chi^{-1}(C) \in \mathcal{F}]$.

Proof. As we have seen in Remark 1, one can use the character $\chi(t) := \mathcal{F}_t$. With respect to this, since the family \mathcal{F} is isotone, it consists exactly of all and only the χ -monotone sets. Hence, by appealing to Lemma 1, the proposition is proved. \square

Given any character χ on T , we recall that by $\chi_{\mathcal{M}}$, we denote the family of all non-empty proper χ -monotone subsets of T . Proposition 2 can be reformulated and strengthened as follows, particularly in light of Corollary 1.

Proposition 3. Let \mathcal{F} be a non-empty family of proper non-empty subsets of a set T . Then, \mathcal{F} is isotone if and only if there exists a character χ , necessarily unique, such that $\mathcal{F} = \chi_{\mathcal{M}}$.

Proof. See Appendix A.1. \square

Our next goal is to present a characterization of isotonicity that can be fruitfully reformulated with reference to functions.

Theorem 1. For a non-empty family \mathcal{F} of proper non-empty subsets of a set T , the following are equivalent:

- The family \mathcal{F} is isotone;
- The family \mathcal{F} verifies the next two conditions:
 - (i) Every union of a non-empty subfamily of elements of \mathcal{F} different from T belongs to \mathcal{F} ;
 - (ii) For every $t \in T$, if the \mathcal{F} -upper level set $\bigcap_{S \in \mathcal{F}_t} S$ is different from T , then it belongs to \mathcal{F} .
- The family $\mathcal{F}' := \mathcal{F} \cup \{\emptyset, T\}$ is closed under unions and intersections;
- There exists a character χ on T , necessarily unique (up to order isomorphisms), such that $\mathcal{F} = \chi_{\mathcal{M}}$;

Proof. See the Appendix A. \square

Observe that both conditions (i), and (ii) are necessary to characterize the isotonicity of a family \mathcal{F} . Indeed, the family \mathcal{F} of Example 1 is not isotone, though condition (i) holds true.

The above theorem answers the question raised at the beginning of this section: Families of sets representable by a character are exactly the ones that verify any of the equivalent conditions in the statement of the previous theorem.

In Section 4, we formulate the answer to the questions raised in the introduction directly with reference to two-valued functions.

We conclude the section by introducing the notion of isotone completion of a family.

Definition 6. The smallest isotone family of non-empty proper subsets of T including a family \mathcal{F} is named the isotone completion of \mathcal{F} . It will be denoted by \mathcal{F}^\uparrow .

The following proposition guarantees the existence of the isotone completion.

Proposition 4. Let \mathcal{F} be a non-empty family of proper non-empty subsets of T .

The family consisting of all non-empty proper subsets of T which are \mathcal{F} -isotone is the isotone completion of \mathcal{F} .

Proof. See the Appendix A. \square

Example 1 continuation. Let us consider the family \mathcal{F} considered in Example 1. As shown in the Appendix A (see Remark A4), the family \mathcal{F}^\uparrow consists of all non-empty proper subsets X of T that verify:

$$X = \bigcup_{t \in X} [t]_{\mathcal{F}}.$$

Of course, since the previous relation is verified by the elements of \mathcal{F} , it is necessary to find the sets X that are not in \mathcal{F} . There are exactly two such sets: $G = \{4\}$ and $H = \{4, 5\}$.

Hence, in this example:

$$\mathcal{F}^\uparrow = \mathcal{F} \cup \{G, H\}.$$

We conclude Example 1 with the illustration of the character considered in Proposition 2. As already observed, the family $\mathcal{G} := \{B, C, D, E, F, G, H\}$ is isotone. Consider the set $L = \{0, 0', 1, 1'\}$, where, by using the notation (1):

$$0 := \mathcal{G}_1 = \mathcal{G}_2 = \{B, C\}, \quad 0' := \mathcal{G}_3 = \{D, F\},$$

$$1 := \mathcal{G}_5 = \{B, C, E, F, H\}, \quad 1' := \mathcal{G}_4 = \{C, D, F, G, H\}.$$

Consider the set theoretical inclusion as the partial order on L , and hence, it is $0 \leq 1$ and $0' \leq 1'$.

The following intuitive picture describes the poset (L, \leq) :



The character $\chi : \{1, 2, 3, 4, 5\} \rightarrow L$ considered in Proposition 2 is defined as follows: $\chi(1) = \chi(2) = 0$, $\chi(3) = 0'$, $\chi(4) = 1'$, and $\chi(5) = 1$.

The elements of $\text{soc}(L)$ are the following:

$$\begin{aligned} &\{0, 1\} \quad \{0, 1, 1'\} \\ &\{0', 1'\} \quad \{0', 1, 1'\} \\ &\{1\} \quad \{1'\} \quad \{1, 1'\} \end{aligned}$$

having as inverse images under $\chi(t)$:

$$\begin{aligned} \chi^{-1}(\{0, 1\}) &= \{1, 2, 5\} &= B \\ \chi^{-1}(\{0, 1, 1'\}) &= \{1, 2, 4, 5\} &= C \\ \chi^{-1}(\{0', 1'\}) &= \{3, 4\} &= D \\ \chi^{-1}(\{1\}) &= \{5\} &= E \\ \chi^{-1}(\{0', 1, 1'\}) &= \{3, 4, 5\} &= F \\ \chi^{-1}(\{1'\}) &= \{4\} &= G \\ \chi^{-1}(\{1, 1'\}) &= \{4, 5\} &= H \end{aligned}$$

These are exactly the elements of \mathcal{G} , as expected.

Example 2. Observe that if for a family \mathcal{F} all the \mathcal{F} -upper level sets are singletons (i.e., $\forall t \in T [t]_{\mathcal{F}} = \{t\}$) then it turns out that:

$$\mathcal{F}^\uparrow = \mathcal{P}(T) \setminus \{\emptyset, T\}.$$

For example, this happens in the following case:

$$\mathcal{F} = \{S \subset \mathbb{N} : S^c \text{ is (empty or) finite}\}$$

where \mathbb{N} denotes the set of the natural numbers, or in the following case:

$$\mathcal{F} = \{S \subseteq T : |S| \geq k\},$$

for a given finite set T and $k \in \{1, 2, \dots, |T| - 1\}$.

4. Functions with Binary Range

In this section, we devote our attention to classes Φ of functions defined on a set T having binary range, say $\{a, b\}$. An element ϕ in such a class Φ is then a (non-constant) function $\phi : T \rightarrow \{a, b\}$.

Following the terminology used in [7], we consider a canonical way to obtain all functions in Φ on the basis of a given character χ on T . Precisely, for each non-empty proper subset C of the range of χ consider the (non-constant) function $\phi_{(\chi, C)} : T \rightarrow \{a, b\}$ defined by the formula:

$$\phi_{(\chi, C)}(t) = \begin{cases} a & \text{if } \chi(t) \in C \\ b & \text{if } \chi(t) \notin C \end{cases}.$$

Concerning the role of functions structured as above in their capacity of being functional forms of several social choice functions with binary range, we refer to [7].

For elements $\phi, \eta \in \{a, b\}^T$, if we write $\phi \leq_a \eta$ when the inclusion $\phi^{-1}(a) \subseteq \eta^{-1}(a)$ holds, obviously, we define a partial order on $\{a, b\}^T$. Although intuitively clear, we point out that the fact that we privilege alternative a in ordering $\{a, b\}^T$ does not affect the generality of our conclusions.

Let us note that $(\{a, b\}^T, \leq_a)$ is indistinguishable from the power set of T ordered by set inclusion. Similarly, setting $\mathcal{F} := \{\phi^{-1}(a) : \phi \in \Phi\}$, then the pair (Φ, \leq_a) is a poset which is indistinguishable from (\mathcal{F}, \subseteq) .

Notation.

- For a subset Ψ of $\{a, b\}^T$ denote by ψ^\vee (respectively, ψ^\wedge) the function:

$$\psi^\vee(t) = a \Leftrightarrow \exists \phi \in \Psi : \phi(t) = a \text{ (respectively, } \psi^\wedge(t) = a \Leftrightarrow \phi(t) = a \forall \phi \in \Psi).$$

- Given a class Φ and $t \in T$, we define the function $\delta_t : T \rightarrow \{a, b\}$ by setting:

$$\delta_t(t') = \begin{cases} a & \text{if } \{\phi \in \Phi : \phi(t) = a\} \subseteq \{\phi \in \Phi : \phi(t') = a\} \\ b & \text{otherwise.} \end{cases}$$

In case $\Psi \subseteq \Phi$, observe that, if ψ^\vee (respectively, ψ^\wedge) belongs to Φ , it turns out that in (Φ, \leq_a) , it is the supremum (respectively, infimum) of Ψ . The purpose of the next example is to clarify the definition of the function δ_t .

Example 3. The equality $\delta_t(t) = a$ is always true, whatever class Φ is under consideration.

1. It is straightforward to verify that in case Φ contains all non-constant $\{a, b\}$ -valued functions, then for every t the function δ_t takes value a at $t' = t$ and value b at every $t' \neq t$.

2. On the other extreme, if $\Phi = \{\mu\}$ then the function δ_t is the constant function with value a for all t such that $\mu(t) = b$. When $\mu(t) = a$, the function δ_t coincides with μ .

3. Let T be the set $\{-1, 0, 1\} \times \{-1, 0, 1\}$. Consider the class $\Phi = \{\mu, \nu\}$, where $\mu(t) = a \Leftrightarrow t = (1, 1)$, and $\nu(t) = a \Leftrightarrow t \neq (-1, -1)$.

The sets $\{\phi \in \Phi : \phi(t) = a\}$ needed in the definition of δ_t are, respectively: the empty set, $\{\mu, \nu\}$, $\{\nu\}$, as t , respectively, is $(-1, -1)$, $(1, 1)$, any other $t \in T \setminus \{(-1, -1), (1, 1)\}$.

Hence, we have:

- $\delta_{(-1, -1)}$ is the constant function with value a ;
- $\delta_{(1, 1)} = \mu$;
- for $t \in T \setminus \{(-1, -1), (1, 1)\}$, we have $\delta_t = \nu$.

Evidently: setting $\Psi := \{\phi \in \Phi : \phi(t) = a\}$, δ_t is the function ψ^\wedge .

The isotonicity conditions of Theorem 1 can be written in a form which is more appropriate to the present framework of functions. Referring to a class Φ as to an isotone class when the family of sets $\mathcal{F} := \{\phi^{-1}(a) : \phi \in \Phi\}$ is such, we can state the following, for example.

Proposition 5. A class Φ of functions with binary range $\{a, b\}$ on a set T is isotone if and only if:

- (i) For every $\Psi \subseteq \Phi$ such that ψ^\vee is non-constant, then $\psi^\vee \in \Phi$;
- (ii) For every $t \in T$ such that the function δ_t is non-constant, one has $\delta_t \in \Phi$.

So, in the case of an isotone class Φ , we have necessarily that:

- Every time ψ^\vee is non-constant, then $\psi^\vee = \vee \Psi$ (the supremum of Ψ in (Φ, \leq_a));
- The non-constant δ_t values are the minima (in (Φ, \leq_a)) among the elements of Φ that take value a at t .

Also, by considering the two constant functions a and b , the class Φ turns out to be isotone if and only if the class $\Phi \cup \{a, b\}$ is a complete sublattice of $(\{a, b\}^T, \leq_a)$.

Another characterization of isotonicity for a class of functions is presented in the next proposition.

Proposition 6. Let Φ be a class of functions with binary range $\{a, b\}$ defined on a set T . Then, Φ is isotone if and only if it contains the set $\{\eta \in \{a, b\}^T : \eta \text{ is non-constant and } (\pi) \text{ holds}\}$, where:

$$(\pi) \quad [t_1, t_2 \in T, \eta(t_1) = a, \eta(t_2) = b] \Rightarrow [\exists \phi \in \Phi : \phi(t_1) = a, \text{ and } \phi(t_2) = b].$$

Proof. Suppose Φ is isotone and let η be a non-constant function with property (π) . We have to show that $\eta \in \Phi$.

Since η is non-constant, the inverse images $\eta^{-1}(a) =: T^a$ and $\eta^{-1}(b) =: T^b$ are non-empty, and from $t_1 \in T^a, t_2 \in T^b$, since there is some $\phi \in \Phi$ with $\phi(t_1) = a$, and $\phi(t_2) = b$, we get:

$$(\star) \quad t_1 \in T^a, t_2 \in T^b \Rightarrow \delta_{t_1}(t_1) = a, \delta_{t_1}(t_2) = b.$$

In particular, δ_{t_1} is non-constant and, by (ii), belongs to Φ . Hence, the $\Psi = \{\delta_t : t \in T^a\}$ is a non-empty subclass of Φ .

CLAIM: $\psi^\vee = \eta$.

Once the claim is proved, this means that ψ^\vee is non-constant and by (i), we get $\eta \in \Phi$, as desired. The claim to prove is $\psi^\vee(t) = a \Leftrightarrow \eta(t) = a$. Assume $\psi^\vee(t) = a$, then there is $t_1 \in T^a$ such that $\delta_{t_1}(t) = a$. If at the same time $\eta(t) = b$, then by (\star) we must have $\delta_{t_1}(t) = b$, a contradiction. For the converse, assume $\eta(t) = a$. Then, $\delta_t(t) = a$ gives plainly $\psi^\vee(t) = a$.

For the converse, take $\Psi \subseteq \Phi$ and assume ψ^\vee is non-constant. Since ψ^\vee enjoys property (π) , then it belongs to Φ . Similarly, a δ_t verifies (π) , and hence, if it is non-constant, it belongs to Φ . \square

To emphasize the role that the property of isotonicity has relatively to our original representation problem, we state explicitly the following corollary of Theorem 1.

Corollary 2. Suppose that Φ is a class of functions with binary range $\{a, b\}$ defined on a set T . Isotonicity of the class Φ is a necessary and sufficient condition in order to find a character $\chi : T \rightarrow L$ such that:

$$\forall \phi \in \Phi \quad \exists ! C \in \text{soc}(L) : \phi(t) = a \Leftrightarrow \chi(t) \in C,$$

namely such that for every function $\phi \in \Phi$ the following formula holds true:

$$\phi(t) = \begin{cases} a & \text{if } \chi(t) \in C \\ b & \text{if } \chi(t) \notin C \end{cases}$$

In other terms, $\phi = \phi_{(\chi,C)}$. The character with the above property is unique up to order isomorphisms.

We conclude the section by introducing the notion of isotone completion of a class Φ of functions with binary range $\{a, b\}$. If we take the isotone completion \mathcal{F}^\uparrow of the family of sets $\{\phi^{-1}(a) : \phi \in \Phi\} =: \mathcal{F}$, we can correspondingly introduce the class Φ^\uparrow of functions with range $\{a, b\}$ defined by the relation $\phi \in \Phi^\uparrow \Leftrightarrow \phi^{-1}(a) \in \mathcal{F}^\uparrow$.

Of course, Φ^\uparrow is the smallest isotone class including the class Φ . So, we fix the notion in the following definition.

Definition 7. The smallest isotone class including a class Φ is named the isotone completion of Φ .

5. Collective Binary Choice: Isotone Classes of Social Choice Functions

This section specializes the considerations of the preceding one to the case when two-valued functions model the collective binary choice. We shall see that what makes possible that the same functional form $\phi = \phi_{(\chi,C)}$ characterizes all different classes considered in [7], is that they are all isotone. This property can be declined in several ways.

5.1. The Model of Collective Binary Choice

Let V be a set that represents a collectivity of decision makers. The set of alternatives is A , whereas $\mathcal{W}(A)$ denotes the set of all complete and transitive binary relations over A . Unless specified differently, there are no restrictions on the sets V and A . We refer to the elements of $\mathcal{W}(A)$ as *preferences*. For a given preference $W \in \mathcal{W}(A)$:

- The notation $x \underset{\sim_W}{\succ} y$ stands for $(x, y) \in W$;
- The notation $x \succ_W y$ stands for $[(x, y) \in W \text{ and } (y, x) \notin W]$;
- The notation $x \sim_W y$ stands for $[(x, y) \in W \text{ and } (y, x) \in W]$.

The elements $P \in \mathcal{W}(A)^V$, i.e., the functions P from V to $\mathcal{W}(A)$, are the preference profiles, and a collective choice is a mechanism that selects an element of A starting from a preference profile.

Let \mathcal{P} be a subset of $\mathcal{W}(A)^V$. A function $\phi : \mathcal{P} \rightarrow A$ is called *social choice function* (SCF, for short). The peculiarity of the framework of *collective binary choice* we want to deal with is that one limits to consider SCFs whose range has cardinality two. Namely, we shall consider subclasses of the class $\{\phi \in A^{\mathcal{P}} : |\text{range}(\phi)| = 2\}$.

As remarked in ([7] Example 1.1), in relevant cases, the fact that A can have in principle cardinality greater than two, plays such a role that, in general, the investigation cannot be systematically traced back to the case $|A| = 2$.

On the other hand, our interest is in the **functional form** of SCFs with range of cardinality two and this will be common to the functional form of the elements of the following set:

$$\{\phi \in A^{\mathcal{P}} : \text{range}(\phi) = \{a, b\}\}$$

for fixed distinct alternatives $a, b \in A$. Hence, we shall simply be in the setting of the previous Section 4.

We shall consider first illustrative examples.

Example 4. Take $A = \{a, b\}$. Assume that the domain of preferences is $\mathcal{P} \subseteq \mathcal{W}(A)^V$.

Let Φ_{pwa} be the class of SCFs that satisfy the property PWA (positive welfare association) introduced in [8].

Let $P \in \mathcal{P}$, v_0, Q_{v_0} denote, respectively, a preference profile, an agent, and a preference such that the profile $(Q_{v_0}, P_{-v_0}) \in \mathcal{P}$.

We recall that a SCF ϕ satisfies PWA if the following implication holds true:

If $a \sim_{P_{v_0}} b$ and $\phi(Q_{v_0}, P_{-v_0}) \succ_{Q_{v_0}} \phi(P)$, then $\phi(Q_{v_0}, P_{-v_0}) \succ_{\sim_{P_v}} \phi(P)$, for all $v \in V \setminus \{v_0\}$.

Proposition 7. The class Φ_{pwa} is a complete sublattice of $\{a, b\}^{\mathcal{P}}$.

Proof. For every subclass Ψ of Φ_{pwa} , one has to verify that both functions ψ^\vee and ψ^\wedge belong to Φ_{pwa} .

For example, let us suppose that $a \sim_{P_{v_0}} b$ and $\psi^\vee(Q_{v_0}, P_{-v_0}) \succ_{Q_{v_0}} \psi^\vee(P)$. We show from this that $\psi^\vee(Q_{v_0}, P_{-v_0}) \succ_{\sim_{P_v}} \psi^\vee(P)$, for all $v \in V \setminus \{v_0\}$.

Case $\psi^\vee(Q_{v_0}, P_{-v_0}) = a$.

In this case, there is $\phi \in \Psi$ such that $\phi(Q_{v_0}, P_{-v_0}) = a$. Since necessarily $\psi^\vee(P) = b$, we also have $\phi(P) = b$. Then, by the PWA property of ϕ , we have $\phi(Q_{v_0}, P_{-v_0}) \succ_{\sim_{P_v}} \phi(P)$, for all $v \in V \setminus \{v_0\}$ i.e., $a \succ_{\sim_{P_v}} b$, for all $v \in V \setminus \{v_0\}$, which is what was to be proved.

Case $\psi^\vee(Q_{v_0}, P_{-v_0}) = b$.

In this case, each function in Ψ attains value b at (Q_{v_0}, P_{-v_0}) . The relation $\psi^\vee(Q_{v_0}, P_{-v_0}) \succ_{Q_{v_0}} \psi^\vee(P)$ implies that $\psi^\vee(P) = a$, hence for at least one function ϕ in Ψ it is $\phi(P) = a$. The fact that the function ϕ satisfies WPA, implies that $b \succ_{\sim_{P_v}} a$, for all $v \in V \setminus \{v_0\}$, as desired.

The argument to get that $\psi^\wedge \in \Phi_{pwa}$ is similar. \square

Example 5. In this example, we present the smallest isotone class containing the dictatorial SCFs. The social choice model is the basic one: Take $A = \{a, b\}$, $V = \{1, 2, \dots, n\}$, $\mathcal{P} = \{a, b\}^V$, where \mathbf{a} (respectively, \mathbf{b}) denotes the strict (i.e., antisymmetric) preference $a \succ b$ (respectively, $b \succ a$).

For every agent $v \in V$, the dictatorial choice between a and b is the SCF ϕ_v defined by $\phi_v(P) = a \Leftrightarrow P_v = \mathbf{a}$.

For every $\mathcal{C} \in \text{soc}(2^V, \subseteq)$, we can define as follows a SCF $\phi_{\mathcal{C}}$:

$$\phi_{\mathcal{C}}(P) = a \Leftrightarrow D(a, P) \in \mathcal{C}, \text{ where } D(a, P) \text{ is the set } \{v \in V : P_v = \mathbf{a}\}.$$

This is the voting by committee SCF defined in ([6] Section 4). Of course, a special case of $\phi_{\mathcal{C}}$ is when for a non-empty subset S of V we consider $\mathcal{C} = \{D \subseteq V : D \supseteq S\}$. In this case, we use the notation ϕ_S and $\phi_S(P) = a \Leftrightarrow S \subseteq D(a, P)$.

Proposition 8. The voting by committee SCFs form the smallest isotone class containing all dictatorial SCFs ϕ_v , i.e., they form the isotone completion of the class $\{\phi_v : v \in V\}$.

Proof. Let $\widehat{\Phi}$ be the class of the voting by committee SCFs. With reference to the properties (i) and (ii) of Proposition 5, the fact that $\widehat{\Phi}$ verifies (i) is straightforward. To show (ii), consider $\delta_{P^*} =: \delta^*$, assume it is non-constant. By definition $\delta^*(P) = a \Leftrightarrow \{\mathcal{C} : D(a, P^*) \in \mathcal{C}\} \subseteq \{\mathcal{C} : D(a, P) \in \mathcal{C}\}$, hence $D(a, P^*)$ is non-empty. It is immediate to show that:

$$\delta^* = \phi_{\mathcal{C}^*} \text{ where } \mathcal{C}^* = \{D \subseteq V : D \supseteq D(a, P^*)\}.$$

Thus, we have isotonicity of the class $\widehat{\Phi}$. Finally we prove that if a class Φ is isotone and includes $\{\phi_1, \dots, \phi_n\}$, then $\widehat{\Phi} \subseteq \Phi$. Indeed, take $\mathcal{C} \in \text{soc}(2^V, \subseteq)$, consider the set of its minimal elements $\{S_1, \dots, S_p\}$, and the corresponding SCFs $\{\phi_{S_\ell} : \ell = 1, \dots, p\} =: \Psi$.

We prove that $\Psi \subseteq \Phi$. More generally, every SCF ϕ_S belongs to Φ . This is a consequence of Proposition 6: since ϕ_S verifies property (π) and Φ is isotone, then ϕ_S turns out to be in Φ . The fact that ϕ_S verifies property (π) is due to the assumption $\{\phi_1, \dots, \phi_n\} \subseteq \Phi$.

Now, we apply (i) from which we get that the SCF ψ^V belongs to the isotone class Φ . Finally, observe that $\psi^V = \phi_C$. \square

5.2. Non-Manipulable Collective Binary Choices

Let us assume that \mathcal{P} is a Cartesian restricted domain, namely that $\mathcal{P} = \times_{v \in V} \mathcal{W}_v$ with $\emptyset \neq \mathcal{W}_v \subseteq \mathcal{W}(A)$. We emphasize the role of groups in manipulating SCFs. Naturally, a special case is the individual manipulation when groups are singletons. If two profiles P, Q of preferences coincide outside a set $D \subset V$ (i.e., every agent $v \in D^c$ has the same preference in both P and Q : $P_v = Q_v$), we shortly write $Q = (Q_D, P_{D^c})$.

Definition 8. Let ϕ be a SCF. Let $D - (P, Q)$ denote, respectively, a coalition of agents and a pair of feasible preference profiles.

– We say that the coalition D **strongly manipulates**, under ϕ , a profile $P \in \mathcal{P}$ if $\exists Q = (Q_D, P_{D^c}) \in \mathcal{P}$ such that:

- Every agent v in D prefers $\phi(Q)$ to $\phi(P)$ according to P_v , i.e., $\phi(Q) \succ_{P_v} \phi(P)$.

– We say that the coalition D **weakly manipulates**, under ϕ , a profile $P \in \mathcal{P}$ if $\exists Q = (Q_D, P_{D^c}) \in \mathcal{P}$ such that:

- For every agent v in D , $\phi(Q) \succ_{\sim_{P_v}} \phi(P)$, and at least for one agent v_0 of D , $\phi(Q) \succ_{P_{v_0}} \phi(P)$.

In other words, in case of weak manipulation, all members of the manipulating coalition D are not worse off and at least one is better off, whereas in case of strong manipulation, every member of D must be better off.

If at least a coalition D can strongly (respectively, weakly) manipulate at least a profile $P \in \mathcal{P}$ under ϕ , then we say that ϕ can be strongly (respectively, weakly) manipulated. The impossibility of the above forms of manipulation leads to the notions of strategy-proofness.

Definition 9. An SCF ϕ is:

- **Weakly group strategy-proof** if it cannot be strongly manipulated in any profile by any coalition;
- **Strongly group strategy-proof** if it cannot be weakly manipulated in any profile by any coalition;

Moreover, we shorten the expression “weakly (respectively, strongly) group strategy-proof” as wGSP (respectively, as sGSP).

Replacing, in the previous Definition 9, coalitions with singletons, the corresponding notions coincide and one speaks of *individual strategy-proofness* (ISP, for short). Whereas the implications $sGSP \Rightarrow wGSP \Rightarrow ISP$ are trivial, they are not equivalences in general, even when the set V of agents is finite. Concerning this aspects, one can refer to [4,14,15].

5.3. Representability of the Classes of SCFs Considered in [7]

The purpose of the present section is to show that the classes of SCFs with range of cardinality two that were considered in [7] are isotone, that is they enjoy the properties (i) and (ii) of Proposition 5. Considering Corollary 2, we can say that this is the reason why they all can be represented in the canonical functional form based on the character function approach.

The investigation conducted in [7] concerns the classes of all social choice functions with range of cardinality two and which are:

- (i) Weakly group strategy-proof;
- (ii) Strongly group strategy-proof;
- (iii) Weakly group strategy-proof and anonymous;
- (iv) Weakly group strategy-proof with domain limited to strict preferences;
- (v) Weakly group strategy-proof and onto;

(vi) Weakly group strategy-proof, anonymous, and onto.

We recall that the ontteness property required in the last two classes just means that in the model $A = \{a, b\}$ is assumed. Thus, these last two classes are particular cases of (i) and (iii).

Clearly, every class can be partitioned into its subclasses consisting of the $\{a, b\}$ -valued SCFs if $\{a, b\}$ runs in the family of subsets of A of cardinality two. To be precise, what we show is that these subclasses are isotone. This is enough to determine the functional form of every SCF of the considered class.

Let us set for two different alternatives $a, b \in A$, and for $\ell \in \{i, ii, iii, iv\}$:

$$\Phi_\ell := \left\{ \phi \in A^{\mathcal{P}} : \text{range}(\phi) = \{a, b\}, \text{ and } \phi \text{ is } (\ell) \right\}.$$

Proposition 9. *The class Φ_ℓ is isotone.*

Proof. We show property (i) of Proposition 5.

Consider an arbitrary non-empty subclass $\{\phi_i \in \Phi_\ell : i \in I\}$ of SCFs, and set:

$$\phi := \bigvee_{i \in I} \phi_i.$$

We shall show that if we assume that ϕ is not (ℓ) , then we shall find $i_0 \in I$ with ϕ_{i_0} not in Φ_ℓ , a contradiction.

If ϕ can be manipulated (weakly or strongly), there exists a coalition D , two profiles P and $Q = (Q_D, P_{D^c})$ and at least an agent $v_0 \in D$ with:

$$(*) \quad \phi(Q) \succ_{P_{v_0}} \phi(P) \quad \text{for some } v_0 \in D.$$

Case 1: $\phi(P) = a$

In this case $\phi(Q) = b$, obviously. Moreover, from (*), it is $b \succ_{P_{v_0}} a$, whereas for all the remaining agents v in the coalition D , either $b \succ_{\sim P_v} a$, or $b \succ_{P_v} a$ holds, according, respectively, to weak or strong manipulation performed by D .

Using the definition of ϕ , the relation $\phi(P) = a$ says that there exists at least an index i_0 with $\phi_{i_0}(P) = a$. On the other side the relation $\phi(Q) = b$ says that $\phi_i(Q) = b$ for each $i \in I$. This means that ϕ_{i_0} is manipulated.

Case 2: $\phi(P) = b$

The argument in this case is identical to the previous one, *mutatis mutandis*.

Observe that it is obvious that ϕ is anonymous if all ϕ_i are, so property (i) of Proposition 5 is achieved.

Now, we show property (ii) of Proposition 5.

Consider the function δ_{P^*} for a fixed P^* . For brevity, we write $\delta^* = \delta_{P^*}$. Let us verify that the SCF δ^* is (ℓ) .

Assume by contradiction that for a certain coalition D and profiles P and $Q = (Q_D, P_{D^c})$, the following results:

$$\delta^*(Q) \succ_{P_{v_0}} \delta^*(P) \quad \text{for at least some } v_0 \in D.$$

Without loss of generality consider the case $\delta^*(P) = b$. Then $\delta^*(Q) = a$, and hence, $a \succ_{P_{v_0}} b$, whereas for all the remaining elements v of D it is $a \succ_{\sim P_v} b$ or, in case of strong manipulation: $a \succ_{P_v} b, \forall v \in D$.

By the definition of δ^* , there exists $\hat{\phi} \in \Phi_\ell$ with $\hat{\phi}(P^*) = a$ and $\hat{\phi}(P) = b$. On the other side, since $\delta^*(Q) = a$, from $\hat{\phi}(P^*) = a$, it follows that $\hat{\phi}(Q) = a$. This allows to rewrite the previous relations either in the form:

$$\hat{\phi}(Q) \succ_{\sim P_v} \hat{\phi}(P) \quad \forall v \in D \quad \text{and} \quad \hat{\phi}(Q) \succ_{P_{v_0}} \hat{\phi}(P)$$

or in the form:

$\widehat{\phi}(Q) \succ_{P_v} \widehat{\phi}(P) \forall v \in D$
 meaning that $\widehat{\phi} \notin \Phi_\ell$, which is a contradiction.

Of course, in the anonymous case, one has to observe that δ^* is anonymous. \square

Remark 2. One may think that Corollary 2 generalizes the results in [7]. However, this is not the case. A finer approach to the comparison of Corollary 2 with the results in [7] helps understanding this point. Let us consider, for example, the case in which $A = \{a, b\}$ and:

$$\Phi = \left\{ \phi \in A^{\mathcal{P}} : \phi \text{ is weakly group strategy-proof and onto} \right\}.$$

In this framework, which is also the one of [9], the comparison can take place in simpler terms.

For the isotone class Φ , Corollary 2 ensures the existence of a character χ that allows to represent the SCFs in the class by means of the canonical functional form $\phi_{(\chi, C)}$. However, the poset L and χ are quite abstract and one can hardly trace them back to the primitives of the model. This is not the case in ([7] Section 5.2) where the role of L is played by the following set ⁵:

$$\{(S, W) \in 2^V \times 2^V : S \subseteq W\}$$

whose elements have the following veto interpretation: a pair of coalitions (S, W) of voters determines a non-manipulable rule for a collective choice, and expresses veto against one of the two alternatives. For example, the veto against alternative b corresponds to the fact that to block b , it is enough that all members of S strictly prefer a and the other agents (if any) in $W \setminus S$ either prefer a or are indifferent. Accordingly, the character proposed therein associates a profile P of preferences with the pair whose first component is the set of voters that, under P , strictly prefer a to b and the second component is the set of voters that, under P , either strictly prefer a to b or are indifferent.

Even if we take into account the uniqueness part of Corollary 2, there is no immediate conversion of the outcome of the application of Corollary 2 into the intuitive results of ([7] Section 5.2).

This scheme of comparison between results in [7] and here applies for all the other classes of SCFs. At the same time, the relevance of what we have obtained in this paper should be clear. Once a certain class of SCFs is considered, determining that it is isotone tells us that it is representable in the canonical way. This leads us to seek for a character that best expresses the defining property of the SCFs of the class.

Remark 3. The above isotone class Φ_i of the $(A \supseteq \{a, b\})$ -valued weakly group strategy-proof SCFs, which has been characterized in ([7] Theorem 3.15) by resorting to the character χ defined in ([7] Definitions 3.13 and 3.12), has also been characterized in ([11] Theorem 2, and Corollary 1) by means of the notion (see [11] page 662) of the upper level set with respect to the relation $\preceq_{(b,a)}$ in

\mathcal{P} . The validity (for $P, Q \in \mathcal{P}$) of the equivalence $Q \preceq_{(b,a)} P \Leftrightarrow \chi(Q) \leq_{\mathcal{D}} \chi(P)$ may lead one to think that identity map can be used as an alternative character to that of ([7] Definitions 3.13 and 3.12). However, this is not the case since the relation $\preceq_{(b,a)}$ is not antisymmetric. i.e., $(\mathcal{P}, \preceq_{(b,a)})$ is not a poset.

6. Conclusions

In the literature, functional form characterizations exist for classes of two-valued social choice functions, such as weakly group strategy-proof functions, strongly group strategy-proof functions, anonymous and weakly group strategy-proof functions, and others.

We have investigated the reason why it is possible that these characterizations can all be reduced to a unique functional form that works for all classes, a result that was shown in [7].

Dealing with functions with range of cardinality two, the study has been preliminarily conducted abstractly (i.e., without any reference to social choice) on families of sets. Obviously, any function whose range has a cardinality two can be identified with a set.

The discovery is that a family of sets can be described in a one-to-one way by the super order closed parts of a partially ordered set if and only if it enjoys a property we have named as isotonicity. The functional counterpart of this result is that a class Φ of two-valued functions admits the canonical representation identified in [7] if and only if it enjoys the following two properties

(i) For every $\Psi \subseteq \Phi$, the class Φ contains the function ψ^\vee , defined as:

$$\psi^\vee(t) = a \Leftrightarrow \exists \phi \in \Psi : \phi(t) = a$$

if ψ^\vee is non-constant;

(ii) For every $t^* \in T$, the class Φ contains the function δ_{t^*} defined as:

$$\delta_{t^*}(t) = \begin{cases} a & \text{if } [\phi \in \Phi, \phi(t^*) = a] \Rightarrow \phi(t) = a \\ b & \text{otherwise.} \end{cases}$$

if such a function is non-constant.

In this case, the functions ψ^\vee and δ_{t^*} can be interpreted, respectively, as the supremum of Ψ and the minimum among the elements of Φ that take value a at t^* .

We can also say that the mentioned representability of Φ is equivalent to the fact that by adding the two constant functions a and b to Φ , the larger class $\Phi \cup \{a, b\}$ is a complete sublattice of $(\{a, b\}^T, \leq_a)$.

Stepping to social choice functions, it is then possible to prove that all classes considered in [7] enjoy the above properties. This sheds light on the unifying results achieved in [7].

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Appendix A. Proofs

We recall that $\mathcal{F}, \mathcal{G}, \dots$ are non-empty families of non-empty proper subsets of T . Some helpful observations are in order.

Remark A1. The set \mathcal{F}_t is empty if and only if $[t]_{\mathcal{F}} = T$ ⁶.

Also observe that:

$$(4) \quad \mathcal{F} \subseteq \mathcal{G} \Rightarrow [t]_{\mathcal{F}} \supseteq [t]_{\mathcal{G}}.$$

Since every \mathcal{F} -upper level set $[t]_{\mathcal{F}}$ contains the element t , for a proper non-empty subset X of T , it turns out that:

$$(5) \quad X \text{ is } \mathcal{F}\text{-isotone if and only if } X = \bigcup_{t \in X} [t]_{\mathcal{F}}.$$

Appendix A.1. Proof of Proposition 3

Obviously, equivalence (5) implies what follows.

Proposition A1. A family \mathcal{F} is isotone if and only if the following equivalence holds true:

$$\emptyset \subset X \subset T, X = \bigcup_{t \in X} [t]_{\mathcal{F}} \Leftrightarrow X \in \mathcal{F},$$

i.e., the following implication holds true:

$$\emptyset \subset X \subset T, X \supseteq \bigcup_{t \in X} [t]_{\mathcal{F}} \Rightarrow X \in \mathcal{F}.$$

Proof. Now, to prove Proposition 3, it needs to be verified that if for some character χ it is $\mathcal{F} = \chi_{\mathcal{M}}$, then the family \mathcal{F} is isotone. Because of Proposition A1, this means we must prove that:

$$\bigcup_{t \in X} [t]_{\mathcal{F}} = X \Rightarrow X \in \mathcal{F}.$$

On the other hand, Lemma 1 tells us that the elements of \mathcal{F} are all and only the inverse images of the elements of $\text{soc}(L)$. Hence, by (3):

$$\begin{aligned} X &= \bigcup_{t \in X} [t]_{\mathcal{F}} = \bigcup_{t \in X} \left[\bigcap_{S \in \mathcal{F}, t \in S} S \right] = \bigcup_{t \in X} \left[\bigcap_{C \in \text{soc}(L), \chi(t) \in C} \chi^{-1}(C) \right] = \\ &= \bigcup_{t \in X} \chi^{-1} \left[\bigcap_{C \in \text{soc}(L), \chi(t) \in C} C \right] = \chi^{-1} \left\{ \bigcup_{t \in X} \left[\bigcap_{C \in \text{soc}(L), \chi(t) \in C} C \right] \right\}. \end{aligned}$$

The fact that the set $\bigcup_{t \in X} \left[\bigcap_{C \in \text{soc}(L), \chi(t) \in C} C \right]$ belongs to $\text{soc}(L)$ completes the proof. \square

Appendix A.2. Proof of Theorem 1

To prove the theorem, we first point out some propositions and observations.

Proposition A2. Let \mathcal{F} be a non-empty family of proper non-empty subsets of a set T . Assume that: (i) The union of the elements of every non-empty subfamily of \mathcal{F} , if it is different from T , belongs to \mathcal{F} .

Under this assumption the following two conditions are equivalent: (ii) For every $t \in T$, if the \mathcal{F} -upper level set $[t]_{\mathcal{F}} = \bigcap_{S \in \mathcal{F}_t} S$ is different from T , then it belongs to \mathcal{F} ; (ii \bullet) The intersection of the elements of every non-empty subfamily of \mathcal{F} , if it is non-empty, belongs to \mathcal{F} .

Proof. The implication (ii \bullet) \Rightarrow (ii) is plain, each being an upper level non-empty set. For the converse suppose $\emptyset \neq F = \bigcap_{i \in I} F_i$ with $F_i \in \mathcal{F}$. For every $t \in F$, obviously $\mathcal{F}_t \neq \emptyset$, Remark A1 gives $[t]_{\mathcal{F}} \neq T$, and (ii) gives $[t]_{\mathcal{F}} \in \mathcal{F}$. Notice that by the definition of F we have $[t]_{\mathcal{F}} \subseteq F$. Finally $F = \bigcup_{t \in F} [t]_{\mathcal{F}}$ and applying (i) we get $F \in \mathcal{F}$. \square

Remark A2. Because of the equivalence above, we may observe that for a family \mathcal{F} , the validity of both conditions (i) and (ii) is equivalent to the fact that the family $\mathcal{F}' := \mathcal{F} \cup \{\emptyset, T\}$ is closed under (arbitrary) unions and intersections. In other words, by adding the empty set and T to \mathcal{F} , we produce a complete sublattice⁷ of the complete lattice consisting of the power set of T ordered by inclusion.

Remark A3. In a poset L , it is immediate to verify that $\text{soc}(L)$ contains both:

- All the unions of its elements when such unions are proper subsets of L ;
- All the intersections of its elements when such intersections are non-empty.

The above remark, Proposition 3, the equivalence between (ii) and (ii \bullet) in Proposition A2, and the union-intersection preserving properties of the inverse image give straightforward the following proposition.

Proposition A3. Let \mathcal{F} be a non-empty family of proper non-empty subsets of a set T . \mathcal{F} is isotone if and only if it verifies the following two conditions:

- (i) The union of the elements of every non-empty subfamily of \mathcal{F} , if it is different from T , belongs to \mathcal{F} ;
- (ii) For every $t \in T$, if the \mathcal{F} -upper level set $\bigcap_{S \in \mathcal{F}_t} S$ is different from T , then it belongs to \mathcal{F} .

At this point, it is clear that Theorem 1 collects the above Proposition A3, Remark A2, and Proposition 3.

Appendix A.3. Proof of 4

Let us introduce the family \mathcal{F}^\uparrow consisting of all non-empty proper subsets of T which are \mathcal{F} -isotone. Note that by definition, the family \mathcal{F} is isotone if and only if $\mathcal{F}^\uparrow = \mathcal{F}$. Set $\mathcal{G} := \mathcal{F}^\uparrow$. Since $\mathcal{G} \subseteq \mathcal{G}^\uparrow = \mathcal{F}^{\uparrow\uparrow}$, it is sufficient to verify the relation $\mathcal{F}^{\uparrow\uparrow} \subseteq \mathcal{F}^\uparrow$, to obtain $\mathcal{G} = \mathcal{G}^\uparrow$.

Let us then check that from $X \in \mathcal{F}^{\uparrow\uparrow}$, it follows that $X \in \mathcal{F}^\uparrow$.

Assuming that the set X is \mathcal{F}^\uparrow -isotone, which means \mathcal{G} -isotone, it is necessary to verify that X is also \mathcal{F} -isotone. In other words, we need to prove the implication:

$$t \in X, \quad \mathcal{F}_t \subseteq \mathcal{F}_{t'} \Rightarrow t' \in X.$$

We verify that from $t \in X$ and $\mathcal{F}_t \subseteq \mathcal{F}_{t'}$, it follows that $\mathcal{G}_t \subseteq \mathcal{G}_{t'}$. At this point, X being \mathcal{G} -isotone, the relation $t' \in X$ holds.

Let us then assume that $S \in \mathcal{G}_t$, which means that t belongs to S and that S is \mathcal{F} -isotone. From:

$$t \in S \quad \text{and} \quad \mathcal{F}_t \subseteq \mathcal{F}_{t'}$$

it follows that $t' \in S$, which means $S \in \mathcal{G}_{t'}$, as desired.

To prove the second part of the statement, it is sufficient to prove that any isotone family $\widehat{\mathcal{F}}$ that includes \mathcal{F} verifies $\widehat{\mathcal{F}} \supseteq \mathcal{F}^\uparrow$.

Once we prove that from $X \in \mathcal{G} = \mathcal{F}^\uparrow$, it follows that $X \supseteq [t]_{\widehat{\mathcal{F}}} \forall t \in X$. Proposition A1 implies that $X \in \widehat{\mathcal{F}}$.

From (4), it follows that $[t]_{\widehat{\mathcal{F}}} \subseteq [t]_{\mathcal{F}}$. Relation $X \in \mathcal{G} = \mathcal{F}^\uparrow$ implies $X \supseteq [t]_{\mathcal{F}} \forall t \in X$, as desired.

Remark A4. Proposition A1 allows to obtain the description of the elements of \mathcal{F}^\uparrow .

The elements of \mathcal{F}^\uparrow are all and only the proper non-empty solutions X of the equation:

$$X = \bigcup_{t \in X} [t]_{\mathcal{F}}.$$

In other words:

$$\mathcal{F}^\uparrow = \{ \bigcup_{t \in X} [t]_{\mathcal{F}} : X \subset T \} \setminus \{ \emptyset, T \}.$$

Notes

- ¹ Let $v \in V$ be an individual, and consider the function ϕ which, heedless of other individuals, associates the profile P with an alternative which is most preferred by v under the profile P . In case only strict preferences are admitted, such a ϕ is obviously strategy-proof.
- ² The ontoness property that is added corresponds to the assumption that the set A of alternatives is itself of cardinality two.
- ³ The reference to the alternative a does not affect the generality of the result.
- ⁴ From $\phi_S(P) = a, \phi_S(Q) = b$, one gets by definition $S \subseteq D(a, P)$ and $S \not\subseteq D(a, Q)$. Hence, for $v \in S \setminus D(a, Q)$, we have $\phi_v \in \Phi, \phi_v(P) = a, \phi_v(Q) = b$.
- ⁵ Endowed with the component-wise set inclusion as partial order.

- ⁶ Indeed, if $\mathcal{F}_t = \emptyset$, the equality $[t]_{\mathcal{F}} = T$ immediately follows from the definition of $[t]_{\mathcal{F}}$. Conversely, if for some S in \mathcal{F} it is $t \in S$, then using (3), we obtain $[t]_{\mathcal{F}} \subseteq S \subset T$.
- ⁷ We are adopting the terminology of [16] (Definition 4.3).

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