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Article

# Threshold Protocol Game on Graphs with Magic Square-Generalization Labelings

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**Abstract:** Graphical games describe strategic interactions among a specified network of players. The threshold protocol game is a graphical game that models the adoption of a lesser-used product in a population when individuals benefit by using the same product. The threshold protocol game has historically been considered using infinite, simple graphs. In general, however, players might value some relationships more than others or may have different levels of influence in the graph. These traits are described by weights on graph edges or vertices, respectively. Relative comparisons on arbitrarily weighted graphs have been studied for a variety of graphical games. Alternatively, graph labelings are functions that assign values to the edges and vertices of graphs based on a particular set of rules. This work demonstrates that the outcome of the threshold protocol game can be characterized on a magic square-generalization labeled graph. There are a variety of graph labelings that generalize the concept of magic squares. In each, the labels on similar sets of graph elements sum to a constant. The constant sums of magic square-generalization labelings mean that each player experiences a constant level of influence without needing to specify the value of players relative to one another. The game outcome is compared across different types and features of labelings.

**Keywords:** threshold; coordination; labelings; magic graph; sigma graph; sigma' graph; vertex-magic total graph



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## 1. Introduction

The aim of graphical game theory is to model strategic interactions between players with a specific network of relationships. There is a variety of applications for such systems, from modeling the spread of ideas in a population, to describing social networks, to studying traffic flow in a network of devices. A graph  $G = (V, E)$  with vertex set  $V$  and edge set  $E$  is used to describe the network of players. Each  $v \in V$  is a player and  $uv \in E$  if  $u$  and  $v$  play against each other. The order of  $G$ ,  $n = |V|$ , provides the number of players, and the size of  $G$ ,  $e = |E|$ , represents the number of relationships. The application of both graph and game theoretic tools means that a wide range of questions about graphical games can be addressed. When there are players of greater influence in a graph or players value some relationships more than others, weighted vertices or edges are used to denote the weight placed on an interaction. Discussions of game play on such graphs are necessarily general, since players or relationships may only be compared relative to one another. A strict analysis is usually impossible.

Graph labelings, however, are determined using particular rules on a graph, meaning that much more is known about the vertex or edge values in the graph. Labels are precisely defined and do not need to be defined relative to one another. This means that graph labelings can be reasonably expected to allow for more precise characterizations of graphical game outcomes than general weighted graphs.

A threshold protocol game (TPG) is a graphical game used to model the adoption of products or ideas in a population. It is primarily applied as an economic model, but the

TPG also produces complex dynamics in a graph that can be characterized. In the TPG, players begin in one of two states. The goal of the game is for the minority state to take over the population. The TPG presents a simple game with well-known basic properties, which makes it an ideal framework for studying the effects of labelings on gameplay. Here, attention is paid to outcome of the TPG on magic square-generalization labeled graphs. Such a labeling describes some constant level of influence that all players experience, which may be distributed differently among each player.

Sections 2–4 provide the preliminaries of the work. Section 2 describes the necessary graph labeling background, while Section 3 reviews related graphical game theory work as context for this work. Section 4 introduces the TPG, and the approach used to analyze the TPG on the labeled graphs is outlined in Section 5. Section 6 provides the results of this work on TPGs using labeled graphs, which are discussed in Section 7. Final conclusions and avenues of future work are described in Section 8.

### 2. Magic Square-Generalizing Graph Labelings

Prior to studying the TPG on labeled graphs, the necessary background regarding graph labelings is given. Specifically, the graph labelings that generalize the properties of magic squares are defined. In each labeling, a standard set of graph objects (edges and/or vertices) is specified. The sum of the labels on each such set is a constant.

Graph labelings are a widely studied topic in graph theory. A graph labeling is a mapping  $f : K \rightarrow \mathbb{Z}$ , where  $K \in \{V, E, V \cup E\}$  or, less commonly, another set of graph features. A mapping  $f : V \rightarrow \mathbb{Z}$  is called a vertex labeling, and a mapping  $f : E \rightarrow \mathbb{Z}$  is called an edge labeling. There are typically strict rules governing the labels of a graph. Labelings that generalize the principle of magic squares will be used here.

The most popular of these is a magic labeling. An injective mapping  $f : E \rightarrow \mathbb{Z}^+$  is called a magic labeling of  $G$  for constant  $k$  if  $\sum_{u \in N(v)} f(uv) = k$  for all  $v \in V$ , where  $N(v) = \{u \in V : uv \in E\}$  is the neighborhood of  $v$ . The constant  $k$  is called the magic constant. A magic graph is a graph that has a magic labeling. A magic labeling is called supermagic if the edge labels are consecutive integers, although typically supermagic refers to a magic labeling using integers  $1, 2, \dots, e$ .

Magic graphs get their name as they follow a similar principle to magic squares. Let  $K_{m,p} = (X, Y)$  denote a complete bipartite graph where  $|X| = m, |Y| = p$ . An  $m \times m$  magic square can be used to form a  $K_{m,m}$  magic graph with magic constant  $k = \frac{m(m^2+1)}{2}$ . For example, Figure 1 shows how a  $3 \times 3$  magic square is used to produce a supermagic  $K_{3,3}$  [1].

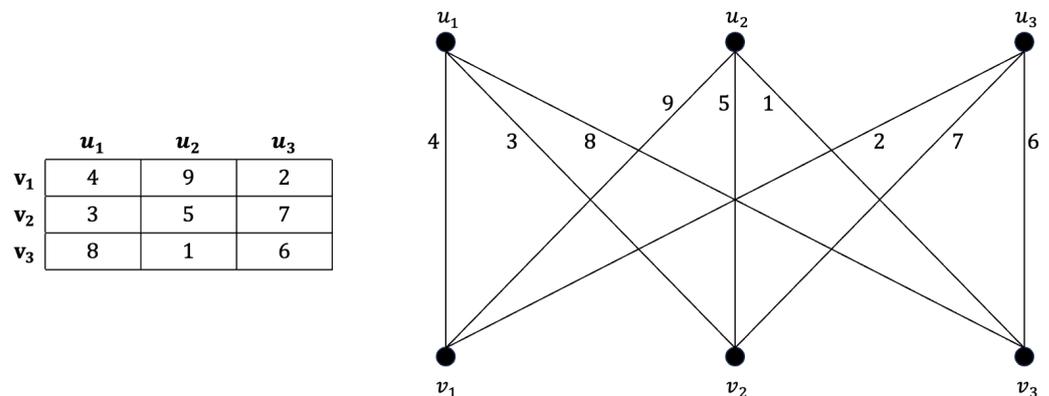
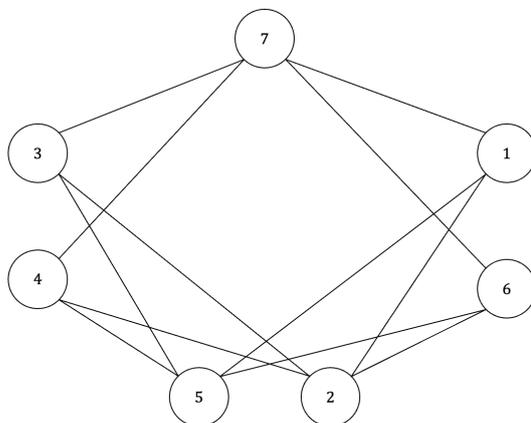


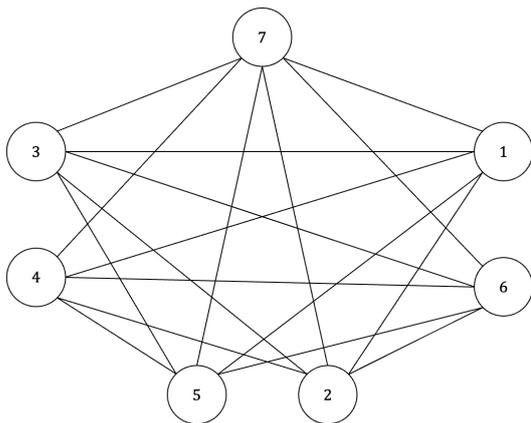
Figure 1. Example of a magic graph of order 6 formed from a  $3 \times 3$  magic square.

There is a clear vertex labeling analog to magic labelings. A bijective mapping  $f : V \rightarrow \{1, 2, \dots, n\}$  is called a  $\Sigma$ -labeling of  $G$  if, for all  $v \in V$ ,  $\sum_{u \in N(v)} f(u) = k$  for a constant  $k$ . A graph with a  $\Sigma$ -labeling is called a  $\Sigma$ -labeled graph and the constant  $k$  is called the vertex sum.  $\Sigma$ -labeled graphs are motivated by magic squares, like magic

graphs, although vertices are labeled rather than edges. Figure 2 provides examples of two vertex-labeled graphs with  $k = 14$  and  $k = 21$ .  $\Sigma$ -labeled graphs can be constructed using  $\Sigma$ -partitions [2]. A  $\Sigma$ -partition of  $\{1, 2, \dots, n\}$  partitions  $\{1, 2, \dots, n\}$  into  $m$  parts such that the sum of each part is  $k'$ . Labeled vertices are grouped according to the  $\Sigma$ -partition into subsets  $V_1, \dots, V_m$ . Two parts with adjacent vertices form a complete bipartite graph.



(a)  $k = 14, r = 2$



(b)  $k = 21, r = 3$

**Figure 2.** Examples of  $\Sigma$ -labeled graphs.

A hypergraph is a graph generalization such that an edge may connect any number of vertices. Let each part of a  $\Sigma$ -partition form a vertex of a hypergraph  $H$ , so  $V(H) = \{V_1, \dots, V_m\}$  and two vertices in  $V(H)$  are adjacent if any of their elements are adjacent in  $G$ . That is, if  $v_1, v_2 \in V(G)$  are such that  $v_1 \in w_1, v_2 \in w_2, v_1 v_2 \in E(G), w_1, w_2 \in V(H)$ , then  $w_1 w_2 \in E(H)$ . If  $H$  is  $r$ -regular,  $G$  is a  $\Sigma$ -labeled graph with  $k = rk'$ . Figure 2 provides examples with  $k' = 7$ , and  $r = 2$  in Figure 2a and  $r = 3$  in Figure 2b.

A  $\Sigma'$ -labeling is similar to a  $\Sigma$ -labeling, but the labels are summed over every closed neighborhood in the graph.  $N[v] = N(v) \cup \{v\}$  denotes the closed neighborhood of  $v \in V$ . Then, a  $\Sigma'$ -labeling is a bijection  $f : V \rightarrow \{1, 2, \dots, n\}$  such that for all  $v \in V$ ,  $\sum_{u \in N[v]} f(u) = k$ , where  $k$  is a constant called the vertex sum [2,3]. A graph with a  $\Sigma'$ -labeling is called a  $\Sigma'$ -graph.

There are a few magic square-generalization labelings on graph edges and vertices. A vertex-magic total labeling is a bijection  $f : V \cup E \rightarrow \{1, 2, \dots, n + e\}$  such that, for all  $v \in V$ ,  $f(v) + \sum_{u \in N(v)} f(uv) = k$ , where  $k$  is a constant called the magic constant. A graph with a vertex-magic total labeling is called a vertex-magic total graph [4,5].

An a-vertex consecutive magic labeling is a vertex-magic total labeling such that the labels on  $V$  are  $\{a + 1, a + 2, \dots, a + n\}$  for some  $a \in \{0, 1, \dots, e\}$ . A graph that has an a-vertex consecutive magic labeling is called a a-vertex consecutive magic graph [5,6].

Similarly, a  $b$ -edge consecutive magic labeling is a vertex-magic total labeling such that the labels on  $E$  are  $\{b + 1, b + 2, \dots, b + e\}$  for some  $b \in \{0, 1, \dots, n\}$ . A graph that has a  $b$ -edge consecutive magic labeling is called a  $b$ -edge consecutive magic graph [5,6].

Another type of labeling on vertices and edges is an edge-magic total labeling, which is a bijection  $f : V \cup E \rightarrow \{1, 2, \dots, n + e\}$  such that, for all  $uv \in E$ ,  $f(u) + f(v) + f(uv) = k$  where  $k$  is a constant called the magic constant. A graph with an edge-magic total labeling is called an edge-magic total graph [5,7].

The final graph labeling that will be mentioned here is an  $H$ -magic labeling. Let  $G$  and  $H$  be graphs such that each edge in  $G$  is in at least one subgraph isomorphic to  $H$ . An  $H$ -magic labeling is a bijection  $f : V \cup E \rightarrow \{1, 2, \dots, n + e\}$  such that, for all subgraphs  $G'$  of  $G$  where  $G'$  is isomorphic to  $H$ ,  $\sum_{v \in V(G')} f(v) + \sum_{e \in E(G')} f(e) = k$  [5]. While each of these labelings will be referenced, the TPG will be analyzed on magic,  $\Sigma$ -,  $\Sigma'$ -, and vertex-magic total graphs.

### 3. Related Work

The TPG is a coordination and consensus graphical game. Here, specifically labeled graphs are considered. Graphical games were introduced in [8] with the goal of describing a game with a set of players with a specific set of interactions. Defining an  $n$ -player graphical game requires a graph  $G$  of  $n$  vertices, each representing a player, and a set  $M$  of  $n$  payoff matrices, each associated with a player. A graphical game is then defined as an ordered pair  $(G, M)$ .

The equilibria of graphical games are of primary interest. A population-wide consensus is a strong equilibrium in a coordination game like the TPG. In a population-wide consensus, all players take the same strategy [9]. When studying consensus, it is often of interest at what rate consensus occurs [10–18] or whether a population can be motivated to reach consensus on a particular state [19]. There are several well-known graphical game features that influence the rate of consensus, including graph connectivity and the efficiency of a strategy. The focus will be on when consensus occurs for the TPG, since the goal of the game is to obtain a population-wide consensus on the minority state. In order for consensus to be achieved, graphical games are usually repeated, with players' strategies being updated upon each repetition. Iterations of the game may occur continuously [20] or at discrete time steps [21]. With discrete repetitions, players may select updated strategies either synchronously or asynchronously [22–24]. The TPG uses synchronous, discrete updates, and is repeated an infinite number of times.

Weighted graphs are useful when players do not value all relationships equally, and are applied to a graphical game in [25–27]. It can also be said that players in a graphical game have weights, which denote the level of influence a particular player exerts on each of its neighbors. This type of weighting is studied in [28–33]. These weightings typically weight graph features relative to one another, while a more uniform weighting can be achieved using graph labelings.

Graph labelings occasionally occur in the graphical game theory literature. Where they do, the games are used as a way to generate graph labelings, known as “labeling games.” In these games, two players take turns labeling edges. The winner may be the last player who is able to place a label, players may have competing goals, such as creating or preventing a labeling, etc. There is limited research surrounding such games, but they are studied in [34–40]. There has not been work on the play of graphical games on labeled graphs. Rather than using gameplay to develop labelings, this work examines the impact of labelings on gameplay.

The TPG is a threshold game, which is a category of games described in [41]. In a threshold game, each player  $v \in V$  has some weight that must be acted on it to motivate  $v$  to change state or to allow  $v$  to receive a payoff [42]. There are many generalizations of threshold games, which are discussed in [32,43]. These include alternative methods of aggregating the payoff  $v$  receives from playing against all of its neighbors  $u \in N(v)$  [19], or unique or time-dependent threshold values for each player [27,42]. The TPG is a standard

threshold game, but these generalizations can be applied based on the application. A simple threshold game (the TPG) is used so that the influence of graph labelings on game play remains clear.

#### 4. Threshold Protocol Game

The TPG was introduced in [44], and was discussed and expanded on in [21,43]. As noted, Ref. [43] particularly focused on generalizing the game, while Ref. [21] primarily used the game as an illustrative example of graphical games.

In the TPG, players are initially in state  $A$  or state  $B$  at time  $t_0$ . The game is repeated over infinite discrete time steps, and at each step all players simultaneously choose state  $A$  or state  $B$ . Table 1 is the payoff matrix of the TPG stage game, and a player’s expected payoff is based on the aggregate expected payoff of playing the stage game against all  $u \in N(v)$ . Let the degree of  $v \in V$  be denoted  $d(v) = |N(v)|$ . Then,  $d_A^i(v), d_B^i(v)$  denote the number of neighbors of  $v$  of state  $A$  and state  $B$  at time  $t_i$ , respectively.  $V_A^i, V_B^i$  are the sets of vertices of state  $A$  and state  $B$  at  $t_i$ , respectively, and  $|V_A^i| = n_A^i$  and  $|V_B^i| = n_B^i$ . At time  $t_{i+1}$ ,  $v \in V$  selects state  $A$  if  $qd_A^i(v) > (1 - q)d_B^i(v)$  and state  $B$  if  $qd_A^i(v) \leq (1 - q)d_B^i(v)$ . This update rule is applied as follows:  $v \in V$  selects state  $B$  if  $q \leq \frac{d_B^i(v)}{d(v)}$  and  $A$  otherwise.

**Table 1.** TPG stage game payoff matrix.

	$A$	$B$
$A$	$q$	$0$
$B$	$0$	$1 - q$

The payoff matrix is determined by parameter  $q \in (0, 1)$  to generalize the game, but players are incentivized to agree with their neighbors. In general, the game does not include  $q = 0$  or  $q = 1$ , since it may be assumed that if  $q = 0$ , all players select  $B$ , and if  $q = 1$ , all players select  $A$ .

State  $B$  begins in the minority, but  $V_B^0$  is nonempty. The TPG may be played on an infinite or finite graph, so in an infinite graph,  $n_B^0 < \infty$ , and in a finite graph,  $n_B^0 < \frac{n}{2}$ . The preferred outcome of the TPG is for state  $B$  to spread throughout the whole graph. This means there exists a time step  $t_i$  such that  $V_B^i = V$ . If this occurs,  $B$  is said to be completely contagious in  $G$  with respect to  $q$ , or just completely contagious when  $G$  and  $q$  are obvious. An initial set of state  $B$  vertices,  $V_B^0$ , is called contagious in  $G$  with respect to  $q$ , or just contagious, if  $B$  is completely contagious from  $V_B^0$ .

Historically, the TPG has been studied exclusively on infinite graphs. Here, the use of a TPG will be considered exclusively on finite graphs, since this work seeks to characterize the TPG on connected and labeled (and therefore finite) graphs. Since the TPG is typically used for economic models where each vertex is a member of the population, a finite graph also allows for more accurate modeling.

The TPG is deterministic, and the game outcome is determined using the graph  $G$  on which the game is played, with the set of vertices initially in state  $B$ ,  $V_B^0$ , and the parameter  $q$ . A TPG is then defined as an ordered triple  $(G, V_B^0, q)$ .

Since the TPG is deterministic for a particular  $G, V_B^0$ , and  $q$ , optimization questions are of greater interest than the outcome of a fixed  $(G, V_B^0, q)$ . There are two optimization questions of interest in the TPG:

1. What is the smallest contagious  $V_B^0$ ?
2. What is the largest value of  $q$  such that  $B$  is completely contagious? This value of  $q$  is called the contagion threshold and is denoted  $q^*$ .

Note that it is beneficial to the spread of state  $B$  for  $n_B^0$  to be large and  $q$  to be small, but these are often expensive circumstances to implement. Hence, it is desirable for  $B$  to be completely contagious from a small  $V_B^0$  with large  $q$ .

An example of TPG play is given in Figure 3. Player 1 has one neighbor, which is initially of state B. This means player 1 expects a payoff of zero from playing A and a payoff of  $1 - q = \frac{2}{3}$  from playing B. Player 2 has four neighbors, two of which are initially of state A and two of which are initially of state B. Thus, player 2 expects a payoff of  $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$  from playing state A and a payoff of  $\frac{2}{3} + \frac{2}{3} = \frac{4}{3}$  from playing state B. The graph and initial states are symmetric, so all players of state A are in the same situation as player 1, and all players of state B are in the same situation as player 2. All players prefer state B to state A, so B is completely contagious after one time step.

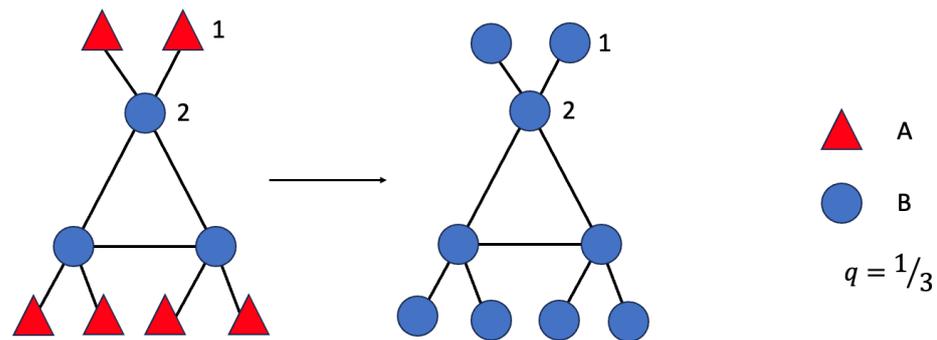


Figure 3. Example of TPG play on an unlabeled, finite graph with  $q = \frac{1}{3}$ .

The existence of a contagious  $V_B^0$  for a particular value of  $q$  has been the focus of previous work, with play occurring on infinite, connected, undirected, simple graphs. The graphs considered here differ only in that they are labeled and finite. Long-term game dynamics in a population are the focus of this work. Exceptions can arise when the number of players is small. However, the dynamics of the game using a small population are not generally of interest, either in theory or application. For these reasons, assume  $n \geq 5$ . All interactions are two-way and a player does not play against themselves, so  $G$  is assumed to be undirected and does not have loops. It does not make sense to study the spread of states on a disconnected graph, so assume  $G$  is connected.

### 5. Method for Analyzing End Behavior of the Threshold Protocol Game on Labeled Graphs

In order to compare the influence of different types of labelings on the outcome of the TPG, the game is played on graphs with labelings that generalize magic squares. Such labelings also have the clearest interpretation within the game, and are thus a clear starting place. Several of the most direct generalizations are applied. The edge labeling that is considered is a magic labeling.  $\Sigma$ - and  $\Sigma'$ -labelings are vertex labelings that are both used, since these labelings are supported by different graph topologies. Finally, vertex-magic total labeling is used as a labeling on edges and vertices. Complete, complete bipartite, and similar graphs are examined for each labeling as appropriate. The fixed graph topologies are exploited to characterize the possible outcomes of the TPG played on the labeled graphs. Existing results regarding the topological features of graphs with certain labelings are used to determine “appropriately similar”.

The circumstances under which B is completely contagious are of primary interest, but of secondary interest is the long-term behavior of the game. Say that  $v \in V$  has static end behavior if there exists  $t_j$  such that, for all  $i \geq j$ ,  $v$  does not change state at  $t_i$ . Then,  $v$  has static A end behavior if  $v \in V_A^i$  and static B end behavior if  $v \in V_B^i$ . Alternatively,  $v$  has 2-periodic end behavior if there exists  $t_j$  such that, for  $i = 0, 1, \dots$ ,  $v \in V_A^{j+2i}$  and  $v \in V_B^{j+2i+1}$ , meaning  $v$  alternates states. The end behavior is a vertex property, but the end behavior of all vertices being static A is equivalent to A being completely contagious in  $G$ , and all vertices having a static B end behavior is equivalent to B being completely contagious in  $G$ .

Along with the graph features, the threshold value  $q$  is key to determining the end behavior of the TPG in any graph and  $q^*$  may change based on the graph labeling. Thus, the majority of determinations regarding the TPG end behavior are dependent on these components. Often, a bound on the value of  $q$  is selected to motivate  $B$  to be completely contagious. Here, a low, obvious bound on  $q$  is provided to first find a minimum-sized contagious  $V_B^0$ . When  $V_B^0$  or  $G$  are chosen carefully, a higher upper bound on  $q$  can be provided such that  $B$  is completely contagious. Where possible, this value is contrasted with a lower bound on  $q$  such that  $A$  is completely contagious, or with bounds that will allow for an alternative end behavior to occur. Clearly, each bound is applicable only if it is in  $(0, 1)$ .  $G$  is fixed with a simple topology, such as a complete or complete bipartite graph. Then, bounds on  $q$  that cause a particular end behavior to occur are provided for an arbitrary  $V_B^0$  in terms of  $V_B^0$  and the graph labeling. Details regarding the approaches used for each graph labeling are discussed in the corresponding section.

### 6. Results

The outcome of the TPG is considered using magic graphs,  $\Sigma$ -graphs,  $\Sigma'$ -graphs, and vertex-magic total graphs. For each labeling, a completely contagious  $V_B^0$  can be identified for a sufficiently small  $q$ . It is also shown that, for each labeling, a characterization of the TPG outcomes is possible on complete and complete bipartite graphs in terms of the graph labeling. Some labelings have additional features that can be exploited to draw further conclusions about TPG behavior when using the labeled graphs.

#### 6.1. Threshold Protocol Game on Magic Graphs

Let  $G$  be a weighted graph with  $\lambda_{uv}$  denoting the weight of  $uv \in E$ . When the TPG is played on a weighted graph,  $v \in V$  will take state  $B$  at  $t_{i+1}$  if  $q \sum_{u \in N_A^i(v)} \lambda_{uv} \leq (1 - q) \sum_{u \in N_B^i(v)} \lambda_{uv}$ , or  $q \leq \frac{\sum_{u \in N_B^i(v)} \lambda_{uv}}{\sum_{u \in N(v)} \lambda_{uv}}$ , and  $A$  otherwise.

The definitions of graph labelings mean that there are very specific rules governing a graph labeling. Taking edge labels as edge weights, the TPG can be played on the resulting weighted graph. In the case of magic graphs,  $v \in V$  will take state  $B$  at  $t_{i+1}$  if  $q \leq \frac{\sum_{u \in N_B^i(v)} \lambda_{uv}}{\sum_{u \in N(v)} \lambda_{uv}} = \frac{\sum_{u \in N_B^i(v)} \lambda_{uv}}{k}$ , and  $A$  otherwise. An example of TPG play on a magic graph is provided in Figure 4. The magic graph has  $k = 15$  and six players, two of whom begin in state  $B$ . Then, with  $q = \frac{1}{15}$ ,  $B$  is completely contagious after one time step.

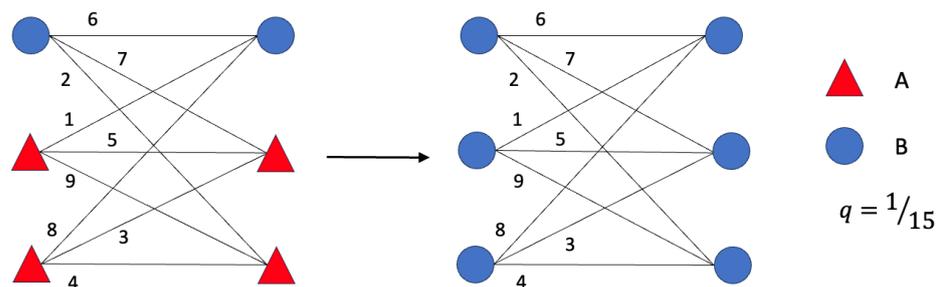


Figure 4. Example of TPG play on a magic graph with  $q = \frac{1}{15}$  and  $k = 15$ .

It is first shown that, for small enough  $q$ , there exists a contagious  $V_B^0$  for all  $G$ .

**Theorem 1.** Let  $G$  be a magic graph with magic constant  $k$ . Let  $m = \min\{\lambda_{uv} : uv \in E\}$  and  $q \leq \frac{m}{k}$ .

- (a) If  $G$  is non-bipartite,  $B$  is completely contagious for  $V_B^0 = \{v\}$  for all  $v \in V$ .
- (b) If  $G = (X, Y)$  is bipartite, with partite sets  $X$  and  $Y$ ,  $B$  is completely contagious for  $V_B^0 = \{v_1, v_2\}$  for all  $v_1 \in X, v_2 \in Y$ .

**Proof.** Note that for all  $v \in V$ ,  $q \leq \frac{m}{k} \leq \frac{\sum_{uv \in F} \lambda_{uv}}{k}$  for all  $F \subseteq E$ . Thus, if  $v \in N(V_B^i)$ , then  $v \in V_B^{i+1}$ .

- (a) Suppose  $G$  is non-bipartite and  $n_B^0 = 1$ . Note the common result that states that a graph is non-bipartite if and only if it contains an odd cycle. Let  $C$  be an odd cycle in  $G$ . Since  $G$  is connected and  $v \in V_B^{i+1}$  if  $v \in N(V_B^i)$ , there exists  $i_1$  such that there exists  $v \in C$ , such that  $v \in V_B^{i_1}$ . Clearly,  $|C \cap V_B^{i_1}| > 1$  is more contagious in  $C$  than  $|C \cap V_B^{i_1}| = 1$ , so consider the worst-case scenario  $|C \cap V_B^{i_1}| = 1$ . Since  $C$  is an odd cycle, there exists  $i_2 > i_1$  such that  $v_1, v_2 \in C$ ,  $v_1, v_2 \in V_B^{i_2}$ , and  $v_1 \sim v_2$ . For all  $i \geq i_2$ ,  $v_1, v_2 \in V_B^i$ . Then, for all  $k \in \mathbb{N}$ ,  $N^{(k)}(\{v_1, v_2\}) \subseteq V_B^{i_2+k}$ , where  $N^{(k)}(\{v_1, v_2\}) = N(N(\dots N(\{v_1, v_2\}) \dots))$ . Since  $G$  is connected, there exists  $k'$  such that  $N^{(k')}(\{v_1, v_2\}) = V$ . Hence,  $B$  is completely contagious in  $G$ .
- (b) Suppose  $G$  is bipartite and  $V_B^0 = \{v_1, v_2\}$  with  $v_1 \in X, v_2 \in Y$ . Note that for all  $k \in \mathbb{N}$ ,  $N^{(k)}(v_1) \subseteq V_B^k$  and  $N^{(k)}(v_2) \subseteq V_B^k$ . Since  $G$  is connected, there exists  $k'$  such that  $N^{(k')}(v_1) = X$  and  $N^{(k')}(v_2) = Y$ . Then,  $V_B^{k'} = X \cup Y = V$ , so  $B$  is completely contagious in  $G$ .

□

Since  $n \geq 5$ ,  $V_B^0$  in Theorem 1 is a minority in  $G$ ,  $q^* \geq \frac{\min\{\lambda_{uv}: uv \in E\}}{k}$  in a magic graph. This theorem describes a best-case scenario for  $n_B^0$  but a worst-case scenario for  $q$  in terms of the optimization questions posed above. By more carefully selecting  $V_B^0$  and  $G$ , the bound on  $q$  can be improved.

Complete and complete bipartite graphs describe a fully regular graph topology; thus, the play of the TPG on magic complete and complete bipartite graphs allows for the influence of the graph labeling on game outcome to be clearly considered. Let  $e' = |\{uv \in E : v \in V_B^0\}| = \sum_{i=1}^{n_B^0} (n - i) = \frac{n_B^0(2n - n_B^0 - 1)}{2}$  denote the number of edges with at least one vertex in  $V_B^0$ . Let  $\{\lambda_{uv_1}, \lambda_{uv_2}, \dots, \lambda_{uv_{e'}}\} = \{\lambda_{uv} : v \in V_B^0\}$  such that  $\lambda_{uv_1} < \lambda_{uv_2} < \dots < \lambda_{uv_{e'}}$ . Finally, let  $m = \sum_{i=1}^{n_B^0-1} \lambda_{uv_i}$ ,  $M = \sum_{i=e'-n_B^0}^{e'} \lambda_{uv_i}$ .

**Theorem 2.** Let  $K_n$  be a complete magic graph.

- (a) If  $n_B^0 = 1$ , with  $V_B^0 = \{v\}$ , and  $q \leq \frac{\min\{\lambda_{uv}: u \in N(v)\}}{k}$ , then  $B$  is completely contagious.
- (b) If  $n_B^0 > 1$ , and  $q \leq \frac{m}{k}$ , then  $B$  is completely contagious.
- (c) If  $q > \frac{M}{k}$ , then  $A$  is completely contagious.

**Proof.** Let  $K_n$  be a complete magic graph.

- (a) Suppose  $n_B^0 = 1$ , with  $V_B^0 = \{v\}$ , and  $q \leq \frac{\min\{\lambda_{uv}: u \in N(v)\}}{k}$ . Since  $N(v) = V_A^0$ ,  $v \in V_A^1$ . For  $u \in V_A^0$ ,  $\frac{\sum_{w \in N_B^0(u)} \lambda_{uw}}{k} = \frac{\lambda_{uv}}{k} > \frac{\min\{\lambda_{uv}: u \in N(v)\}}{k} \geq q$ , so  $u \in V_B^1$ . Then  $V_A^1 = \{v\}$ , so  $N(v) = V_B^1$ , and  $v \in V_B^2$ . For all  $u \in V_B^1$ ,  $\frac{\sum_{w \in N_B^0(u)} \lambda_{uw}}{k} = \frac{k - \lambda_{uv}}{k} \geq \frac{\min\{\lambda_{uv}: u \in N(v)\}}{k} \geq q$  since  $n \geq 5$ , so  $u \in V_B^2$ . Thus,  $B$  is completely contagious after two time steps.
- (b) Suppose  $n_B^0 > 1$ , and  $q \leq \frac{m}{k}$ . For  $v \in V_B^0$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0 \setminus \{v\}} \lambda_{uv}}{k} \geq \frac{m}{k} \geq q$ , so  $v \in V_B^1$ . For  $v \in V_A^0$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0} \lambda_{uv}}{k} \geq \frac{m}{k} > q$ , so  $v \in V_B^1$ . Thus,  $V_B^1 = V$ , and  $B$  is completely contagious after one time step.
- (c) Suppose  $q > \frac{M}{k}$ ,  $M < k$ . For  $v \in V_B^0$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0 \setminus \{v\}} \lambda_{uv}}{k} \leq \frac{M}{k} < q$ , so  $v \in V_A^1$ . For  $v \in V_A^0$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0} \lambda_{uv}}{k} \leq \frac{M}{k} < q$ , so  $v \in V_A^1$ . Thus,  $V_A^1 = V$ , and  $A$  is completely contagious after one time step.

□

Prior to characterizing TPG play on magic complete bipartite graphs, note that a result in [45] states that if a connected bipartite graph  $G = (X, Y)$  is magic, then  $|X| = |Y|$ . This means that only  $K_{p,p}$  is discussed.

**Theorem 3.** Let  $K_{p,p} = (X, Y)$  be a complete bipartite magic graph.

- (a) If  $q \leq \min\left\{\frac{\sum_{v \in V_B^0 \cap X} \min\{\lambda_{uv}:u \in Y\}}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \min\{\lambda_{uv}:u \in X\}}{k}\right\}$ , then  $B$  is completely contagious.
- (b) If  $q > \max\left\{\frac{\sum_{v \in V_B^0 \cap X} \max\{\lambda_{uv}:u \in Y\}}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \max\{\lambda_{uv}:u \in X\}}{k}\right\}$ , then  $A$  is completely contagious.
- (c) If  $\frac{\sum_{v \in V_B^0 \cap Y} \max\{\lambda_{uv}:u \in X\}}{k} < q \leq \frac{\sum_{v \in V_B^0 \cap X} \min\{\lambda_{uv}:u \in Y\}}{k}$  or  $\frac{\sum_{v \in V_B^0 \cap X} \max\{\lambda_{uv}:u \in Y\}}{k} < q \leq \frac{\sum_{v \in V_B^0 \cap Y} \min\{\lambda_{uv}:u \in X\}}{k}$ , then the end behavior of  $K_{p,p}$  is completely 2-periodic.

**Proof.** Let  $K_{p,p} = (X, Y)$  be a magic complete bipartite graph.

- (a) Suppose  $q \leq \min\left\{\frac{\sum_{v \in V_B^0 \cap X} \min\{\lambda_{uv}:u \in Y\}}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \min\{\lambda_{uv}:u \in X\}}{k}\right\}$ . For  $v \in X$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} &= \frac{\sum_{u \in V_B^0 \cap Y} \lambda_{uv}}{k} \\ &\geq \frac{\sum_{w \in V_B^0 \cap Y} \min\{\lambda_{uw} : u \in X\}}{k} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$ . For  $v \in Y$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} &= \frac{\sum_{u \in V_B^0 \cap X} \lambda_{uv}}{k} \\ &\geq \frac{\sum_{w \in V_B^0 \cap X} \min\{\lambda_{uw} : u \in Y\}}{k} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$ , and  $B$  is completely contagious after one time step.

- (b) Suppose  $q > \max\left\{\frac{\sum_{v \in V_B^0 \cap X} \max\{\lambda_{uv}:u \in Y\}}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \max\{\lambda_{uv}:u \in X\}}{k}\right\}$ . For  $v \in X$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} &= \frac{\sum_{u \in V_B^0 \cap Y} \lambda_{uv}}{k} \\ &\leq \frac{\sum_{w \in V_B^0 \cap Y} \max\{\lambda_{uw} : u \in X\}}{k} \\ &< q \end{aligned}$$

so  $v \in V_A^1$ . For  $v \in Y$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} &= \frac{\sum_{u \in V_B^0 \cap X} \lambda_{uv}}{k} \\ &\leq \frac{\sum_{w \in V_B^0 \cap X} \max\{\lambda_{uw} : u \in Y\}}{k} \\ &< q \end{aligned}$$

so  $v \in V_A^1$ , and  $A$  is completely contagious after one time step.

$$(c) \quad \text{Suppose } \frac{\sum_{v \in V_B^0 \cap Y} \max\{\lambda_{uv} : u \in X\}}{k} < q \leq \frac{\sum_{v \in V_B^0 \cap X} \min\{\lambda_{uv} : u \in Y\}}{k}. \text{ For } v \in X,$$

$$\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0 \cap Y} \lambda_{uv}}{k}$$

$$\geq \frac{\sum_{w \in V_B^0 \cap Y} \min\{\lambda_{uw} : u \in X\}}{k}$$

$$\geq q$$

so  $v \in V_B^1$ . For  $v \in Y$ ,

$$\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0 \cap X} \lambda_{uv}}{k}$$

$$\leq \frac{\sum_{w \in V_B^0 \cap X} \max\{\lambda_{uw} : u \in Y\}}{k}$$

$$< q$$

so  $v \in V_A^1$ . Iterating, for all  $i = 1, 2, \dots$ ,  $N(X) = Y = V_A^{2i-1}$  and  $N(Y) = X = V_B^{2i-1}$ , while  $N(X) = V_B^{2i}$  and  $N(Y) = V_A^{2i}$ . Thus,  $X = V_A^{2i}, Y = V_B^{2i}$  and  $X = V_B^{2i-1}, Y = V_A^{2i-1}$ .

Alternatively, suppose  $\frac{\sum_{v \in V_B^0 \cap X} \max\{\lambda_{uv} : u \in Y\}}{k} < q \leq \frac{\sum_{v \in V_B^0 \cap Y} \min\{\lambda_{uv} : u \in X\}}{k}$ . For  $v \in X$ ,

$$\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0 \cap Y} \lambda_{uv}}{k}$$

$$\leq \frac{\sum_{w \in V_B^0 \cap Y} \max\{\lambda_{uw} : u \in X\}}{k}$$

$$< q$$

so  $v \in V_A^1$ . For  $v \in Y$ ,

$$\frac{\sum_{u \in N_B^0(v)} \lambda_{uv}}{k} = \frac{\sum_{u \in V_B^0 \cap X} \lambda_{uv}}{k}$$

$$\geq \frac{\sum_{w \in V_B^0 \cap X} \min\{\lambda_{uw} : u \in Y\}}{k}$$

$$\geq q$$

so  $v \in V_B^1$ . Iterating, for all  $i = 1, 2, \dots$ ,  $N(X) = V_B^{2i-1}$  and  $N(Y) = V_A^{2i-1}$ , while  $N(X) = V_A^{2i}$  and  $N(Y) = V_B^{2i}$ . Thus,  $X = V_B^{2i}, Y = V_A^{2i}$  and  $X = V_A^{2i-1}, Y = V_B^{2i-1}$ . Hence, all vertices have 2-periodic end behavior.

□

Magic labelings are often studied on strongly regular graphs, which generalize complete/complete bipartite connectivity. Thus, consider the TPG on strongly regular graphs. Let  $\Lambda = \{\lambda_{uv} : uv \in E\}$  and  $m = \text{median}(\Lambda)$ . Let  $\Lambda_1 = \{\lambda_{uv} \in \Lambda : \lambda_{uv} < m\}$  and  $\Lambda_2 = \{\lambda_{uv} \in \Lambda : \lambda_{uv} \geq m\}$ . Additionally, Prim’s algorithm is applied in the following results. Prim’s algorithm is used to find a minimum spanning tree,  $T$ , of a graph, with the input a weighted, undirected, connected graph [46]. The steps in Prim’s algorithm are as follows:

- (i) Select an arbitrary starting vertex,  $v_1$ , and add it to  $T$ .
- (ii) Select the edge of smallest weight incident to  $T$  that does not form a cycle with  $T$ .
- (iii) Add the corresponding vertex to  $T$ .
- (iv) Repeat (ii) and (iii) until  $V(T) = V(G)$ .

**Theorem 4.** Let  $G$  be a strongly regular magic graph,  $q \leq \frac{\min \Lambda_2}{k}$ , and  $V_B^0 = \{v_1, v_2\}$ , such that  $v_1 \sim v_2$  and  $\lambda_{v_1 v_2} \in \Lambda_2$ . Then,  $B$  is completely contagious in  $G$ .

**Proof.** Part I: Notice that since  $G$  is strongly regular, for all  $v \in V$ , there exists  $u \in N(v)$  such that  $\lambda_{uv} \in \Lambda_2$ , and the diameter of  $G$  is either 1 or 2. Additionally,  $G$  regular and  $|\Lambda_2| = |\Lambda_1|$  or  $|\Lambda_2| = |\Lambda_1| + 1$  implies that there exists, at most, one  $v \in V$  such that  $|\{u \in N(v) : \lambda_{uv} \in \Lambda_2\}| = 1$ . Hence, there does not exist an edge cut  $E' \subseteq E$  such that  $\{\lambda_{uv} : uv \in E'\} \subseteq \Lambda_1$ . Select an arbitrary vertex and apply Prim’s algorithm on negated edge weights to find a maximum spanning tree  $T$  of  $G$ . Then, for all  $uv \in E(T)$ ,  $\lambda_{uv} \in \Lambda_2$ .

Part II: Let  $v \in V$  and  $u \in N_B^i(v)$  such that  $\lambda_{uv} \in \Lambda_2$ . Then  $q \leq \frac{\min \Lambda_2}{k} \leq \frac{\lambda_{uv}}{k} \leq \frac{\sum_{u' \in N_B^i(v)} \lambda_{u'v}}{k}$ . Thus, if  $\{uv \in E : u \in N_B^i(v), \lambda_{uv} \in \Lambda_2\} \neq \emptyset$ , then  $v \in V_B^{i+1}$ .

Combining I and II, if  $V_B^0 = \{v_1, v_2\}$  such that  $\lambda_{v_1 v_2} \in \Lambda_2$ , then  $v_1, v_2 \in V_B^i$  for all  $i = 0, 1, \dots$ , and either  $v_1 v_2 \in E(T)$  for some maximum spanning tree  $T$  or  $v_1 v_2 \sim v_1 v_3 \in E(T)$ . Let  $P_v$  denote the shortest path in  $T$  between  $v$  and  $\{v_1, v_2\}$ . If for all  $u \in P_v \setminus \{v\}$ ,  $u$  is of state  $B$  at  $t_i$ ,  $v$  takes state  $B$  at  $t_j$  for all  $j > i$ . Since  $T$  is a spanning tree of  $G$ , there exists  $i'$  such that  $V_B^{i'} = V$ , and thus  $B$  is completely contagious in  $G$ .  $\square$

Since  $n \geq 5$ ,  $n_B^0 = 2$  is a minority  $V_B^0$ . Thus,  $q^* \geq \frac{\min \Lambda_2}{k}$  in a strongly regular magic graph. With an additional condition on  $G$ ,  $n_B^0$  can be improved for the same bound on  $q$ .

**Corollary 1.** Let  $G$  be a strongly regular non-bipartite magic graph, and  $q \leq \frac{\min \Lambda_2}{k}$ . If there exists an odd cycle  $C$  in  $G$  such that, for all  $uv \in E(C)$ ,  $\lambda_{uv} \in \Lambda_2$ , then  $V_B^0 = \{v\}$  is contagious in  $G$  for all  $v \in V$ .

**Proof.** As shown in the proof of Theorem 4, every strongly regular magic graph has a spanning tree  $T$  whose edges have weights in  $\Lambda_2$ . Thus,  $V_B^0 = \{v\}$  spreads state  $B$  through  $T$ , and there exists time  $t_i$  such that for  $u \in V(C) \cap T = V(C)$ ,  $u \in V_B^i$ . Since  $C$  is an odd cycle, as described in the proof of Theorem 1,  $B$  takes hold in  $C$  and spreads through  $G$ . Hence,  $B$  is completely contagious in  $G$ .  $\square$

Theorem 4 and Corollary 1 are good for game outcome since they balance the values of  $n_B^0$  and  $q$  for a contagious  $V_B^0$  that does not require careful choosing. However, this is only guaranteed when  $G$  is strongly regular. Figure 5 shows that not all strongly regular non-bipartite magic graphs contain such a cycle. More broadly, Figure 6 provides an example of an irregular magic graph in which there is no spanning tree with all edge weights in  $\Lambda_2$ . Instead, to achieve spanning with edge weights in  $\Lambda_2$ , a forest must be used. An example of a three-tree forest is provided. However, for an irregular magic graph, there may exist  $v \in V$  such that, for all  $u \in N(v)$ ,  $\lambda_{uv} \in \Lambda_1$ . Thus, Corollary 2 generalizes the approach provided in Theorem 4 to all magic graphs. In fact, if there exists a spanning tree  $T$  in a strongly regular magic graph such that, for all  $uv \in E(T)$ ,  $\lambda_{uv} > \min \Lambda_2$ , Corollary 2 provides a better bound on  $q$  than Theorem 4. An example of such a graph is shown in Figure 5.

**Corollary 2.** Let  $G$  be a magic graph and  $T$  be a maximum spanning tree of  $G$ . Let  $q \leq \frac{\min\{\lambda_{uv}:uv \in E(T)\}}{k}$ , and  $V_B^0 = \{v_1, v_2\}$  such that  $v_1 v_2 \in E(T)$ . Then,  $B$  is completely contagious in  $G$ .

**Proof.** Let  $G$  be a magic graph,  $T$  a maximum spanning tree of  $G$ ,  $q \leq \frac{\min\{\lambda_{uv}:uv \in E(T)\}}{k}$ , and  $V_B^0 = \{v_1, v_2\}$ , such that  $v_1 v_2 \in E(T)$ . Then,  $v_1, v_2 \in V_B^i$  for all  $i = 0, 1, \dots$ . Let  $P_v$  denote the shortest path in  $T$  between  $v$  and  $\{v_1, v_2\}$ . If for all  $u \in P_v \setminus \{v\}$ ,  $u$  is of state  $B$  at  $t_i$ ,  $v$  takes state  $B$  at  $t_j$  for all  $j > i$ . Since  $T$  is a spanning tree of  $G$ , there exists  $i'$  such that  $V_B^{i'} = V$ , and thus  $B$  is completely contagious in  $G$ .  $\square$

Since  $n \geq 5$ , Corollary 2 indicates that  $q^* \geq \frac{\min\{\lambda_{uv}:uv \in E(T)\}}{k}$  for the TPG on a magic graph. Both complete and complete bipartite graphs are strongly regular graphs, so the results obtained for strongly regular magic graphs may also be compared with the already presented results regarding end behavior for magic complete and complete bipartite graphs.

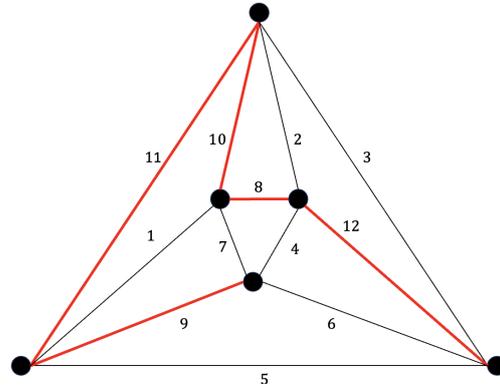


Figure 5. Maximum spanning tree of a strongly regular supermagic graph,  $k = 26$ . The maximum spanning tree is outlined in red.

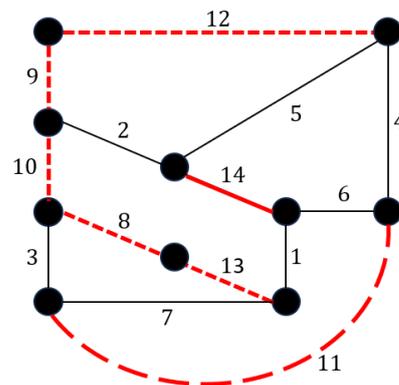
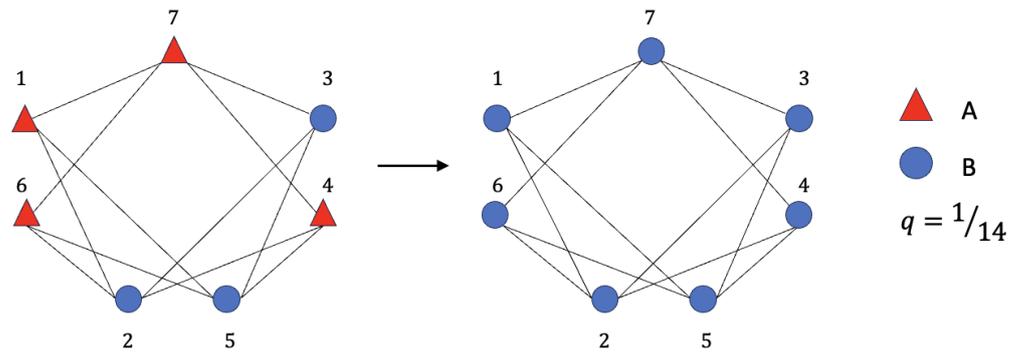


Figure 6. Irregular magic graph,  $k = 21$ . A forest with edge weights at least  $\min \Lambda_2$  is given in red. There are three trees in the forest, which are differentiated by pattern.

### 6.2. Threshold Protocol Game on $\Sigma$ -Graphs

Consider a vertex-labeling of a graph, where  $\lambda_v$  denotes the label of  $v \in V$ . When the TPG is played on such a graph, the vertex labels will be considered as a measure of the influence of each vertex. Then, a vertex  $v \in V$  will take state  $B$  at time  $t_{i+1}$  if  $q \sum_{u \in N_A^i(v)} \lambda_u \leq (1 - q) \sum_{u \in N_B^i(v)} \lambda_u$ , or  $q \leq \frac{\sum_{u \in N_B^i(v)} \lambda_u}{\sum_{u \in N(v)} \lambda_u}$ , and  $A$  otherwise. In a  $\Sigma$ -graph, this means  $v \in V$  selects state  $B$  at  $t_{i+1}$  if  $q \leq \frac{\sum_{u \in N_B^i(v)} \lambda_u}{k}$ . An example of TPG play on a  $\Sigma$ -graph is provided in Figure 7. The  $\Sigma$ -graph has  $k = 14$  and seven players, three of whom begin in state  $B$ . Then, with  $q = \frac{1}{14}$ ,  $B$  is completely contagious after one time step.



**Figure 7.** Example of TPG play on a  $\Sigma$ -graph with  $q = \frac{1}{14}$  and  $k = 14$ .

As with magic graphs, first consider a sufficiently small  $q$ , such that all minimum  $V_B^0$  are contagious in  $G$ .

**Theorem 5.** Let  $G$  be a  $\Sigma$ -labeled graph with vertex sum  $k$ . Let  $q \leq \frac{1}{k}$ . Then

- (a) If  $G$  is non-bipartite,  $B$  is completely contagious for  $V_B^0 = \{v\}$  for all  $v \in V$ .
- (b) If  $G = (X, Y)$  is bipartite,  $B$  is completely contagious for  $V_B^0 = \{v_1, v_2\}$  for all  $v_1 \in X, v_2 \in Y$ .

**Proof.** Note that  $q \leq \frac{1}{k} \leq \frac{\sum_{v \in W} \lambda_v}{k}$  for all  $W \subseteq V$ . Thus, if  $v \in N(V_B^i)$ , then  $v \in V_B^{i+1}$ , so the proof of (a) and (b) are the same as in the proof of Theorem 1.  $\square$

Thus, Theorem 5 indicates that  $q^* \geq \frac{1}{k}$  in a  $\Sigma$ -labeled graph. Recall that  $\Sigma$ -partitions are used to construct  $\Sigma$ -labeled graphs and let  $V_1, \dots, V_m$  denote the  $m$  parts of a  $\Sigma$ -partition of  $V$ . Let  $H$  be the hypergraph formed from the parts of a  $\Sigma$ -partition, so the corresponding graph  $G$  is  $\Sigma$ -labeled if  $H$  is  $r$ -regular and part  $V_i, i = 1, 2, \dots, m$ , forms a complete bipartite graph with each of the  $r$  parts it is adjacent to in  $H$ .

**Theorem 6.** Let  $G$  be a  $\Sigma$ -labeled graph formed from a  $\Sigma$ -partition of  $V$  into  $V_1, V_2, \dots, V_m$ . Let the corresponding hypergraph  $H$  be  $r$ -regular.

- (a) If  $G$  is non-bipartite, let  $V_B^0 = V_i, i \in \{1, 2, \dots, m\}$ .
  - (b) If  $G = (X, Y)$  is bipartite, let  $V_B^0 = V_i \cup V_j, i, j \in \{1, 2, \dots, m\}$  such that  $V_i \subseteq X, V_j \subseteq Y$ .
- Then  $B$  is completely contagious in  $G$  if and only if and  $q \leq \frac{1}{r}$ .

**Proof.** Let  $H$  be the hypergraph whose vertices are the parts of the  $\Sigma$ -partition of  $G$ . Then, for all  $V_i, i = 1, 2, \dots, m, V_i$  has  $r$  neighbors in  $H$ , given by  $N_H(V_i) = \{V_{i_1}, \dots, V_{i_r}\}$ . Suppose  $V_i = V_B^j$  for some  $j \geq 0$ . Then, for all  $v \in V(G)$  such that  $v \in N_H(V_i), \frac{\sum_{u \in N_B^i(v)} \lambda_u}{\sum_{u \in N(v)} \lambda_u} = \frac{k'}{k} = \frac{k'}{rk'} = \frac{1}{r}$ .

Suppose  $q > \frac{1}{r}$ , Then for all  $v \in V \setminus N_H(V_i), N(v) = V_A^0$ , so  $v \in V_A^1$ . For all  $v \in N_H(V_i), \frac{\sum_{u \in N_B^i(v)} \lambda_u}{\sum_{u \in N(v)} \lambda_u} = \frac{1}{r} < q$ , so  $v \in V_A^1$ . Thus,  $A$  is completely contagious after one time step.

Suppose  $q \leq \frac{1}{r}$ . Note that if  $V_i \subseteq V_B^j$  for  $j \geq 0, N_H(V_i) \subseteq V_B^{j+1}$ .  $H$  is bipartite if and only if  $G$  is bipartite, so the proof of (a) and (b) is the same as the proof of Theorem 1.  $\square$

Theorem 6 is clearly analogous to Theorem 5, exploiting a  $\Sigma$ -partition of the graph. Note that  $r$  is generally smaller than  $k$ , meaning  $q^* \geq \frac{1}{r}$  is generally a better bound on  $q^*$  than  $q^* \geq \frac{1}{k}$ . However,  $q^* \geq \frac{1}{r}$  from Theorem 6 is achieved by a  $V_B^0$  that is not necessarily minimal, while  $V_B^0$  in Theorem 5 is always minimal.

There is no  $\Sigma$ -labeling for a complete graph ( $m = 1$ ). However, the end behavior of the TPG can be further specified on complete bipartite graphs, corresponding to  $m = 2$ .

**Theorem 7.** Let  $K_{p_1, p_2}$  be a  $\Sigma$ -labeled complete bipartite graph.

- (a) If  $q \leq \min\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\}$ , then  $B$  is completely contagious.
- (b) If  $q > \max\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\}$ , then  $A$  is completely contagious.
- (c) If  $\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k} \neq \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}$  and

$$\min\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\} < q \leq \max\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\},$$

then the end behavior on  $K_{p_1, p_2}$  is all 2-periodic.

**Proof.**  $K_{p_1, p_2}$  be a  $\Sigma$ -labeled complete bipartite graph.

- (a) Let  $q \leq \min\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\}$ . Then, for all  $v \in V$ ,  $q \leq \frac{\sum_{u \in N_B^0(v)} \lambda_u}{k}$ , so  $v \in V_B^1$ , and  $B$  is completely contagious after one time step. Thus, the end behavior on  $K_{p_1, p_2}$  is all static  $B$ .
- (b) Let  $q > \max\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\}$ . Then, for all  $v \in V$ ,  $q > \frac{\sum_{u \in N_B^0(v)} \lambda_u}{k}$ , so  $v \in V_A^1$ , and  $A$  is completely contagious after one time step. Thus, the end behavior on  $K_{p_1, p_2}$  is all static  $A$ .
- (c) Suppose  $\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k} \neq \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}$  and let

$$\min\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\} < q \leq \max\left\{\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}, \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}\right\}.$$

Suppose  $\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k} < \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}$ , so  $\frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k} < q \leq \frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k}$ . Then for  $v \in X$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_u}{k} = \frac{\sum_{u \in V_B^0 \cap Y} \lambda_u}{k} \geq q$ , so  $v \in V_B^1$ , and for  $v \in Y$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_u}{k} = \frac{\sum_{u \in V_B^0 \cap X} \lambda_u}{k} < q$ , so  $v \in V_A^1$ . Hence,  $V_B^1 = X$  and  $V_A^1 = Y$ , so  $N(V_B^1) = V_A^1$  and  $N(V_A^1) = V_B^1$ . Iterating,  $X = V_A^{2i}$  and  $Y = V_B^{2i}$ , and  $X = V_B^{2i+1}$  and  $Y = V_A^{2i+1}$ ,  $i = 1, 2, \dots$ . Thus, the end behavior on  $K_{p_1, p_2}$  is all 2-periodic.

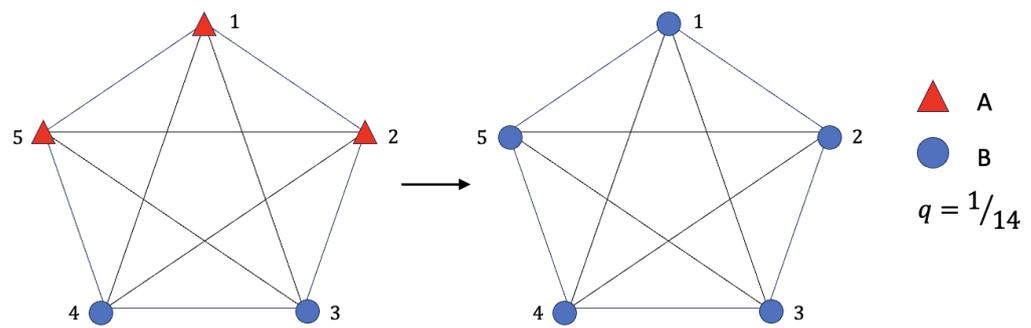
Similarly, suppose  $\frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k} < \frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}$ , so  $\frac{\sum_{v \in V_B^0 \cap Y} \lambda_v}{k} < q \leq \frac{\sum_{v \in V_B^0 \cap X} \lambda_v}{k}$ . Then for  $v \in X$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_u}{k} = \frac{\sum_{u \in V_B^0 \cap Y} \lambda_u}{k} < q$ , so  $v \in V_A^1$ , and for  $v \in Y$ ,  $\frac{\sum_{u \in N_B^0(v)} \lambda_u}{k} = \frac{\sum_{u \in V_B^0 \cap X} \lambda_u}{k} \geq q$ , so  $v \in V_B^1$ . Hence,  $V_B^1 = Y$  and  $V_A^1 = X$ , so  $N(V_B^1) = V_A^1$  and  $N(V_A^1) = V_B^1$ . Iterating,  $X = V_A^{2i}$  and  $Y = V_B^{2i}$ , and  $X = V_B^{2i+1}$  and  $Y = V_A^{2i+1}$ ,  $i = 1, 2, \dots$ . Thus, the end behavior on  $K_{p_1, p_2}$  is all 2-periodic.

□

Theorem 7 shows that on  $\Sigma$ -labeled  $K_{p_1, p_2}$ ,  $q^* \geq \frac{\min\{\max\{\lambda_u : u \in X\}, \max\{\lambda_u : u \in Y\}\}}{k}$ .

### 6.3. Threshold Protocol Game on $\Sigma'$ -Graphs

Consider the TPG on a  $\Sigma'$ -labeled graph. Since a  $\Sigma'$ -labeling is a vertex labeling, a vertex  $v \in V$  in a  $\Sigma'$ -labeled graph takes state  $B$  at  $t_{i+1}$  if  $q \leq \frac{\sum_{u \in N_B^i(v)} \lambda_u}{\sum_{u \in N(v)} \lambda_u} = \frac{\sum_{u \in N_B^i(v)} \lambda_u}{k - \lambda_v}$  and state  $A$  otherwise. Since  $n \geq 5$  and  $G$  is connected,  $k > \lambda_v$  for all  $v \in V$ . An example of TPG play on a  $\Sigma'$ -graph is provided in Figure 8. The  $\Sigma'$ -graph has  $k = 15$  and five players, two of whom begin in state  $B$ . Then, with  $q = \frac{1}{14}$ ,  $B$  is completely contagious after one time step.



**Figure 8.** Example of TPG play on a  $\Sigma'$ -graph with  $q = \frac{1}{14}$  and  $k = 15$ .

As previously, first consider the bound on  $q$  for the minimum contagious  $V_B^0$  in a  $\Sigma'$ -labeled graph.

**Theorem 8.** Let  $G$  be a  $\Sigma$ -labeled graph with vertex sum  $k$ . Let  $q \leq \frac{1}{k-1}$ .

- (a) If  $G$  is non-bipartite,  $B$  is completely contagious for  $V_B^0 = \{v\}$  for all  $v \in V$ .
- (b) If  $G = (X, Y)$  is bipartite,  $B$  is completely contagious for  $V_B^0 = \{v_1, v_2\}$  for all  $v_1 \in X, v_2 \in Y$ .

**Proof.** Note that  $q \leq \frac{1}{k-1} \leq \frac{\sum_{v \in W} \lambda_v}{k-1} \leq \frac{\sum_{v \in W} \lambda_v}{k-\lambda_v}$  for all  $W \subseteq V, v \in V$ . Thus, if  $v \in N(V_B^i)$ , then  $v \in V_B^{i+1}$ , so the proofs of (a) and (b) are the same as in the proof of Theorem 1.  $\square$

Theorem 8 indicates that  $q^* \geq \frac{1}{k-1}$  in a  $\Sigma'$ -labeled graph. Theorem 13 in Ref. [3] states that a complete  $m$ -partite graph  $K_{p_1, p_2, \dots, p_m}$  is a  $\Sigma'$ -labeled graph, if and only if,  $p_i = 1$  for all  $i = 1, 2, \dots, m$ , i.e., if  $K_{p_1, p_2, \dots, p_m} = K_n$ . Clearly, on  $K_n, k = \frac{n(n+1)}{2}$ . Consider the TPG on a  $\Sigma'$ -labeled  $K_n$ .

**Theorem 9.** Let  $K_n$  be a  $\Sigma'$ -labeled complete graph.

- (a) If  $n_B^0 = 1$  such that  $V_B^0 = \{v\}$ , then  $B$  is completely contagious in  $G$  if  $q \leq \frac{\lambda_v}{k-1}$ .
- (b) If  $n_B^0 > 1$ , then  $B$  is completely contagious in  $G$  if  $q \leq \frac{\sum_{u \in V_B^0} \lambda_u - m}{k-1}$ , where  $m = \max\{\lambda_v : v \in V_B^0\}$ .
- (c)  $A$  is completely contagious in  $G$  if  $q > \frac{\sum_{u \in V_B^0} \lambda_u}{k-n}$ .

**Proof.** (a) Suppose  $n_B^0 = 1$  with  $V_B^0 = \{v\}$ . Then,  $v \in V_A^1$  since  $N(v) = V_A^0$ . For  $w \in V_A^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(w)} \lambda_u}{k - \lambda_w} &= \frac{\lambda_v}{k - \lambda_w} \\ &\geq \frac{\lambda_v}{k - 1} \\ &\geq q \end{aligned}$$

so  $w \in V_B^1$  and  $V_A^0 = V_B^1$ . Consider  $V_A^1 = \{v\}$ . Then, for  $v$ ,  $N(v) = V_B^1$ , so  $v \in V_B^2$ . For all  $w \in V_B^1$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(w)} \lambda_u}{k - \lambda_w} &= \frac{\sum_{u \in V \setminus \{v,w\}} \lambda_u}{k - \lambda_w} \\ &= \frac{k - \lambda_v - \lambda_w}{k - \lambda_w} \\ &\geq \frac{k - \lambda_v - \lambda_w}{k - 1} \\ &\geq \frac{\lambda_v}{k - 1} \text{ since } n \geq 5 \\ &\geq q \end{aligned}$$

so  $w \in V_B^2$ . Thus,  $V_B^2 = V$ , and  $B$  is completely contagious in  $K_n$  after two time steps.

(b) Suppose  $n_B^0 > 1$ , and let  $m = \max\{\lambda_v : v \in V_B^0\}$ . Let  $q \leq \frac{\sum_{u \in V_B^0} \lambda_u - m}{k - 1}$ . For all  $v \in V_A^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_u}{k - \lambda_v} &= \frac{\sum_{u \in V_B^0} \lambda_u}{k - \lambda_v} \\ &\geq \frac{\sum_{u \in V_B^0} \lambda_u}{k - 1} \\ &> \frac{\sum_{u \in V_B^0} \lambda_u - m}{k - 1} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$  and  $V_B^0 \subseteq V_B^1$ . For all  $v \in V_B^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_u}{k - \lambda_v} &= \frac{\sum_{u \in V_B^0 \setminus \{v\}} \lambda_u}{k - \lambda_v} \\ &= \frac{\sum_{u \in V_B^0} \lambda_u - \lambda_v}{k - \lambda_v} \\ &\geq \frac{\sum_{u \in V_B^0} \lambda_u - m}{k - 1} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$  and  $V_A^0 \subseteq V_B^1$ . Thus,  $V_B^1 = V$ , and  $B$  is completely contagious after one time step.

(c) Let  $q > \frac{\sum_{u \in V_B^0} \lambda_u}{k - n}$ . For all  $v \in V_A^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_u}{k - \lambda_v} &= \frac{\sum_{u \in V_B^0} \lambda_u}{k - \lambda_v} \\ &\leq \frac{\sum_{u \in V_B^0} \lambda_u}{k - n} \\ &< q \end{aligned}$$

so  $v \in V_A^1$  and  $V_B^0 \subseteq V_A^1$ .  
 For all  $v \in V_B^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} \lambda_u}{k - \lambda_v} &= \frac{\sum_{u \in V_B^0 \setminus \{v\}} \lambda_u}{k - \lambda_v} \\ &= \frac{\sum_{u \in V_B^0} \lambda_u - \lambda_v}{k - \lambda_v} \\ &< \frac{\sum_{u \in V_B^0} \lambda_u}{k - n} \\ &< q \end{aligned}$$

so  $v \in V_A^1$  and  $V_A^0 \subseteq V_A^1$ . Thus,  $V_A^1 = V$ , and  $A$  is completely contagious in one time step.  
 $\square$

Since the magic constant is determined by the sum over each closed neighborhood, the results for the  $\Sigma'$ -labeled graphs are less conclusive than those of  $\Sigma$ -labeled graphs. However, these two labelings are also supported by different graph topologies. For example, there does not exist a  $\Sigma$ -labeled complete graph, while the only  $m$ -partite  $\Sigma'$ -labeled graphs that exist are complete.

#### 6.4. Threshold Protocol Game on Vertex-Magic Total Graphs

So far, the TPG on graphs with edge or vertex labelings has been considered. The final common type of graph labeling is on the vertices and edges of  $G$ . When the TPG is played using a graph with vertex and edge labels,  $v \in V$  takes state  $B$  in  $t_{i+1}$  if  $q \sum_{u \in N_A^i(v)} (\lambda_u + \lambda_{uv}) \leq (1 - q) \sum_{u \in N_B^i(v)} (\lambda_u + \lambda_{uv})$ , or  $q \leq \frac{\sum_{u \in N_B^i(v)} (\lambda_u + \lambda_{uv})}{\sum_{u \in N(v)} (\lambda_u + \lambda_{uv})}$ , and  $A$  otherwise. Then, on a vertex-magic total graph,  $v \in V$  takes state  $B$  in  $t_{i+1}$  if  $q \leq \frac{\sum_{u \in N_B^i(v)} (\lambda_u + \lambda_{uv})}{\sum_{u \in N(v)} (\lambda_u + \lambda_{uv})} = \frac{\sum_{u \in N_B^i(v)} (\lambda_u + \lambda_{uv})}{(k - \lambda_v) + \sum_{u \in N(v)} \lambda_u}$ , and  $A$  otherwise. An example of TPG play with a vertex-magic total graph is presented in Figure 9. This is a particularly nice vertex-magic total graph as it is also a 0-edge consecutive magic graph and 4-vertex consecutive magic graph. The vertex-magic total graph has  $k = 11$  and five players, two of whom begin in state  $B$ . Then, with  $q = \frac{1}{4}$ ,  $B$  is completely contagious after two time steps.

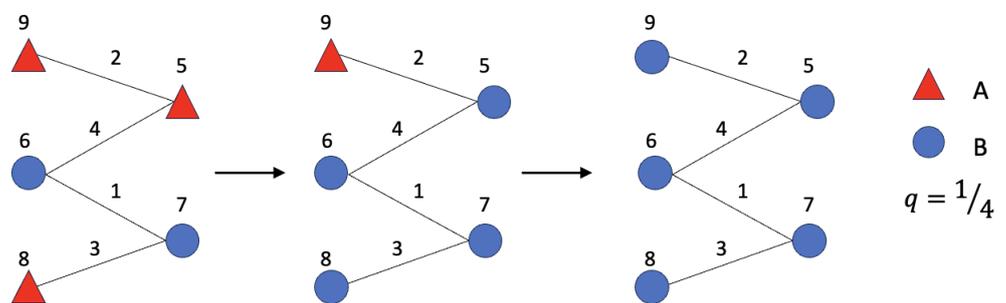


Figure 9. Example of TPG play on a vertex-magic total graph with  $q = \frac{1}{4}$  and  $k = 11$ .

Consider a minimum contagious  $V_B^0$  for a small  $q$ .

**Theorem 10.** Let  $G$  be a vertex-magic total graph with magic constant  $k$ . Denote the maximum degree of  $G$  using  $\Delta$  and  $m = \min\{\lambda_u + \lambda_{uv} : uv \in E\}$ . Let  $q \leq \frac{m}{k - 1 + \Delta(n + e)}$ .

- (a) If  $G$  is non-bipartite,  $B$  is completely contagious for  $V_B^0 = \{v\}$  for all  $v \in V$ .

- (b) If  $G = (X, Y)$  is bipartite,  $B$  is completely contagious for  $V_B^0 = \{v_1, v_2\}$  for all  $v_1 \in X, v_2 \in Y$ .

**Proof.** Note that for all  $v \in V, q \leq \frac{m}{k-1+\Delta(n+e)} < \frac{\lambda_{v'}+\lambda_{vv'}}{(k-\lambda_v)+\sum_{u \in N(v)} \lambda_u}$ , such that  $vv' \in E$ . Thus, if  $v \in N(V_B^i)$ , then  $v \in V_B^{i+1}$ , so the proof of (a) and (b) are the same as in the proof of Theorem 1.  $\square$

Since  $n \geq 5$ , Theorem 10 indicates that  $q^* \geq \frac{\min\{\lambda_u+\lambda_{uv}:uv \in E\}}{k-1+\Delta(n+e)}$  in a vertex-magic total graph. In Ref. [47], it is proved that there exists a vertex-magic total  $K_n$  for all  $n \equiv 0 \pmod{4}$ . Thus, there exist vertex-magic total complete graphs and the TPG can be considered on vertex-magic total complete graphs.

**Theorem 11.** Let  $K_n$  be a vertex-magic total complete graph. Let  $\mathcal{K} = \sum_{u \in V} \lambda_u$ .

- (a) If  $n_B^0 = 1$  with  $V_B^0 = \{v\}$ , then  $B$  is completely contagious if  $q \leq \frac{\min\{\lambda_u+\lambda_{uv}:uv \in E\}}{k-2+\mathcal{K}}$   
 (b) If  $n_B^0 > 1$ , then  $B$  is completely contagious if  $q \leq \frac{\sum_{u \in V_B^0}(\lambda_u+\min\{\lambda_{uv}:v \in N(u)\})-\max\{\lambda_u:u \in V_B^0\}}{k-2+\mathcal{K}}$   
 (c)  $A$  is completely contagious if  $q > \frac{\sum_{u \in V_B^0}(\lambda_u+\max\{\lambda_{uv}:v \in N(u)\})}{k-2\max\{\lambda_u:u \in V\}+\mathcal{K}}$

**Proof.** (a) Suppose  $n_B^0 = 1, V_B^0 = \{v\}$ , and  $q \leq \frac{\min\{\lambda_u+\lambda_{uv}:uv \in E\}}{k-2+\mathcal{K}}$ . For  $v$  such that  $V_B^0 = \{v\}, N(v) = V_A^0$ , so  $v \in V_A^1$ . For  $w \in V_A^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(w)}(\lambda_u + \lambda_{uw})}{k - \lambda_w + \sum_{u \in N(w)} \lambda_u} &= \frac{\lambda_v + \lambda_{vw}}{k - 2\lambda_w + \mathcal{K}} \\ &\geq \frac{\min\{\lambda_u + \lambda_{uw} : uw \in E\}}{k - 2 + \mathcal{K}} \\ &\geq q \end{aligned}$$

so  $w \in V_B^1$ . Thus,  $V_A^0 = V_B^1$  and  $V_A^1 = \{v\}$ . Thus,  $N(v) = V_B^1$ , so  $v \in V_B^2$ . For all  $w \in V_B^1$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(w)}(\lambda_u + \lambda_{uw})}{k - \lambda_w + \sum_{u \in N(w)} \lambda_u} &= \frac{\sum_{u \in V}(\lambda_u + \lambda_{uw}) - (\lambda_v + \lambda_{vw}) - \lambda_w}{k - 2\lambda_w + \mathcal{K}} \\ &\geq \frac{\min\{\lambda_u + \lambda_{uw} : uw \in E\}}{k - 2 + \mathcal{K}} \text{ since } n \geq 5 \\ &\geq q \end{aligned}$$

so  $w \in V_B^2$ . Thus,  $V_B^2 = V$ , so  $B$  is completely contagious after two time steps.

- (b) Suppose  $n_B^0 > 1$  and  $q \leq \frac{\sum_{u \in V_B^0}(\lambda_u+\min\{\lambda_{uv}:v \in N(u)\})-\max\{\lambda_u:u \in V_B^0\}}{k-2+\mathcal{K}}$ . For all  $v \in V_A^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)}(\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0}(\lambda_u + \lambda_{uv})}{k - 2\lambda_v + \mathcal{K}} \\ &\geq \frac{\sum_{u \in V_B^0}(\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - 2 + \mathcal{K}} \\ &\geq \frac{\sum_{u \in V_B^0}(\lambda_u + \min\{\lambda_{uv} : v \in N(u)\}) - \max\{\lambda_u : u \in V_B^0\}}{k - 2 + \mathcal{K}} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$  and  $V_A^0 \subseteq V_B^1$ . For all  $v \in V_B^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0} (\lambda_u + \lambda_{uv}) - \lambda_v}{k - 2\lambda_v + \mathcal{K}} \\ &\geq \frac{\sum_{u \in V_B^0} (\lambda_u + \lambda_{uv}) - \max\{\lambda_u : u \in V_B^0\}}{k - 2 + \mathcal{K}} \\ &\geq \frac{\sum_{u \in V_B^0} (\lambda_u + \min\{\lambda_{uw} : w \in N(u)\}) - \max\{\lambda_u : u \in V_B^0\}}{k - 2 + \mathcal{K}} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$  and  $V_B^0 \subseteq V_B^1$ . Thus,  $V_B^1 = V$ , so  $B$  is completely contagious after one time step.

(c) Suppose  $n_B^0 > 1$  and  $q > \frac{\sum_{u \in V_B^0} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - 2 \max\{\lambda_u : u \in V\} + \mathcal{K}}$ . For all  $v \in V_A^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0} (\lambda_u + \lambda_{uv})}{k - 2\lambda_v + \mathcal{K}} \\ &\leq \frac{\sum_{u \in V_B^0} (\lambda_u + \max\{\lambda_{uw} : w \in N(u)\})}{k - 2 \max\{\lambda_u : u \in V\} + \mathcal{K}} \\ &< q \end{aligned}$$

so  $v \in V_A^1$  and  $V_A^0 \subseteq V_A^1$ . For all  $v \in V_B^0$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0} (\lambda_u + \lambda_{uv}) - \lambda_v}{k - 2\lambda_v + \mathcal{K}} \\ &\leq \frac{\sum_{u \in V_B^0} (\lambda_u + \max\{\lambda_{uw} : w \in N(u)\}) - \lambda_v}{k - 2 \max\{\lambda_u : u \in V\} + \mathcal{K}} \\ &\leq \frac{\sum_{u \in V_B^0} (\lambda_u + \max\{\lambda_{uw} : w \in N(u)\})}{k - 2 \max\{\lambda_u : u \in V\} + \mathcal{K}} \\ &< q \end{aligned}$$

so  $v \in V_A^1$  and  $V_A^0 \subseteq V_A^1$ . Thus,  $V_A^1 = V$ , and  $A$  is completely contagious after one time step.

□

Along with the results obtained for vertex-magic total complete graphs, Ref. [47] contains a proof that  $K_{p,p}$  is the vertex-magic total for all  $p > 1$ . In a refinement of this result, Ref. [4] concludes that  $K_{m,p}$  is not vertex-magic total if  $|m - p| > 1$ . That said, there exists a vertex-magic total  $K_{m,p}$  on which the TPG can be considered.

**Theorem 12.** Let  $K_{m,p} = (X, Y)$  be a vertex-magic total complete bipartite graph. Let  $\mathcal{X} = \sum_{u \in X} \lambda_u$  and  $\mathcal{Y} = \sum_{u \in Y} \lambda_u$ .

(a)  $B$  is completely contagious if

$$q \leq \min\left\{ \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in Y\} + \mathcal{X}}, \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in X\} + \mathcal{Y}} \right\}.$$

(b)  $A$  is completely contagious if

$$q > \max\left\{ \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in Y\} + \mathcal{X}}, \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in X\} + \mathcal{Y}} \right\}.$$

(c) The end behavior on  $K_{m,p}$  is 2-periodic if

$$\left. \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in X\} + y} \right\} < q \leq \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in Y\} + x}$$

or

$$\frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in Y\} + x} < q \leq \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in X\} + y}.$$

**Proof.** (a) Suppose

$$q \leq \min \left\{ \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in Y\} + x}, \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in X\} + y} \right\}.$$

Then, for all  $v \in X$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \lambda_{uv})}{k - \lambda_v + y} \\ &\geq \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \min\{\lambda_{uw} : w \in N(u)\})}{k - \min\{\lambda_w : w \in X\} + y} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$ . For all  $v \in Y$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \lambda_{uv})}{k - \lambda_v + x} \\ &\geq \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \min\{\lambda_{uw} : w \in N(u)\})}{k - \min\{\lambda_w : w \in Y\} + x} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$ . Thus,  $V_B^1 = V$ , and  $B$  is completely contagious after one time step.

(b) Suppose

$$q > \max \left\{ \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in Y\} + x}, \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in X\} + y} \right\}.$$

Then, for all  $v \in X$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \lambda_{uv})}{k - \lambda_v + y} \\ &\leq \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \max\{\lambda_{uw} : w \in N(u)\})}{k - \max\{\lambda_w : w \in X\} + y} \\ &< q \end{aligned}$$

so  $v \in V_A^1$ . For all  $v \in Y$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \lambda_{uv})}{k - \lambda_v + x} \\ &\leq \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \max\{\lambda_{uw} : w \in N(u)\})}{k - \max\{\lambda_w : w \in Y\} + x} \\ &< q \end{aligned}$$

so  $v \in V_A^1$ . Thus,  $V_A^1 = V$ , and  $A$  is completely contagious after one time step.

(c) Suppose

$$\left. \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in X\} + y} \right\} < q \leq \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in Y\} + x}.$$

Then, for all  $v \in X$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \lambda_{uv})}{k - \lambda_v + y} \\ &\leq \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \max\{\lambda_{uw} : w \in N(u)\})}{k - \max\{\lambda_w : w \in X\} + y} \\ &< q \end{aligned}$$

so  $v \in V_A^1$ . For all  $v \in Y$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \lambda_{uv})}{k - \lambda_v + x} \\ &\geq \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \min\{\lambda_{uw} : w \in N(u)\})}{k - \min\{\lambda_w : w \in Y\} + x} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$ . Hence,  $X = V_A^1$  and  $Y = V_B^1$ . For all  $v \in X$ ,  $N(v) = V_B^1$ , and for all  $v \in Y$ ,  $N(v) = V_A^1$ . Iterating,  $X = V_B^{2i-1}$  and  $Y = V_A^{2i-1}$ , while  $X = V_A^{2i}$  and  $Y = V_B^{2i}$ ,  $i = 1, 2, \dots$ . Thus, the end behavior of the TPG is 2-periodic.

Alternatively, suppose

$$\left. \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k - \max\{\lambda_v : v \in Y\} + x} \right\} < q \leq \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\})}{k - \min\{\lambda_v : v \in X\} + y}.$$

Then, for all  $v \in X$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \lambda_{uv})}{k - \lambda_v + y} \\ &\geq \frac{\sum_{u \in V_B^0 \cap Y} (\lambda_u + \min\{\lambda_{uw} : w \in N(u)\})}{k - \min\{\lambda_w : w \in X\} + y} \\ &\geq q \end{aligned}$$

so  $v \in V_B^1$ . For all  $v \in Y$ ,

$$\begin{aligned} \frac{\sum_{u \in N_B^0(v)} (\lambda_u + \lambda_{uv})}{k - \lambda_v + \sum_{u \in N(v)} \lambda_u} &= \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \lambda_{uv})}{k - \lambda_v + x} \\ &\leq \frac{\sum_{u \in V_B^0 \cap X} (\lambda_u + \max\{\lambda_{uw} : w \in N(u)\})}{k - \max\{\lambda_w : w \in Y\} + x} \\ &< q \end{aligned}$$

so  $v \in V_A^1$ . Hence,  $X = V_B^1$  and  $Y = V_A^1$ . For all  $v \in X$ ,  $N(v) = V_A^1$ , and for all  $v \in Y$ ,  $N(v) = V_B^1$ . Iterating,  $X = V_A^{2i-1}$  and  $Y = V_B^{2i-1}$ , while  $X = V_B^{2i}$  and  $Y = V_A^{2i}$ ,  $i = 1, 2, \dots$ . Thus, the end behavior of the TPG is 2-periodic.

□

The analysis of a TPG using vertex-magic total graphs is unique in this work because of the influence of labels on both the edges and vertices of  $G$ . That said, results analogous

to those found for edge labelings and vertex labelings are achievable, although complete characterization is not possible for an arbitrary  $V_B^0$ .

### 6.5. Equivalent and near Equivalent Labelings

There are several other magic square-generalization labelings on the edges and vertices of graphs that should be mentioned. However, TPG gameplay on graphs with these labelings is equivalent to systems already discussed, so they will not be studied independently.

#### 6.5.1. Threshold Protocol Game on a-Vertex Consecutive and b-Edge Consecutive Magic Graphs

An a-vertex consecutive magic labeling is a type of vertex-magic total labeling. Thus, results obtained for vertex-magic total graphs can be refined on a-vertex consecutive magic graphs, and Theorems 10 and 11 can be restated in terms of a-vertex consecutive magic graphs. Note that particular vertex labels referenced in Theorem 12 cannot be further specified, so the results are not refined on a-vertex consecutive magic graphs.

**Theorem 13.** *Let  $G$  be an a-vertex consecutive magic graph with magic constant  $k$ . Let  $m = \min\{\lambda_u + \lambda_{uv} : uv \in E\}$  and  $q \leq \frac{m}{k-a-1+\Delta(a+n)}$ .*

- (a) *If  $G$  is non-bipartite,  $B$  is completely contagious for  $V_B^0 = \{v\}$  for all  $v \in V$ .*
- (b) *If  $G = (X, Y)$  is bipartite,  $B$  is completely contagious for  $V_B^0 = \{v_1, v_2\}$  for all  $v_1 \in X, v_2 \in Y$ .*

**Theorem 14.** *Let  $K_n$  be an a-vertex consecutive magic graph. Let  $\mathcal{K} = \sum_{u \in V} \lambda_u = \frac{n(2a+n+1)}{2}$ .*

- (a) *If  $n_B^0 = 1$  with  $V_B^0 = \{w\}$ , then  $B$  is completely contagious if  $q \leq \frac{\min\{\lambda_u + \lambda_{uv} : uv \in E\}}{k-2(a+1)+\mathcal{K}}$ .*
- (b) *If  $n_B^0 > 1$ , then  $B$  is completely contagious if  $q \leq \frac{\sum_{u \in V_B^0} (\lambda_u + \min\{\lambda_{uv} : v \in N(u)\}) - \max\{\lambda_u : u \in V_B^0\}}{k-2(a+1)+\mathcal{K}}$ .*
- (c) *If  $n_B^0 > 1$  then  $A$  is completely contagious if  $q > \frac{\sum_{u \in V_B^0} (\lambda_u + \max\{\lambda_{uv} : v \in N(u)\})}{k-2(a+n)+\mathcal{K}}$ .*

b-edge consecutive magic labelings are also a type of vertex-magic total labeling. Thus, results for vertex-magic total graphs also apply to b-edge consecutive magic graphs. However, Theorems 10–12 do not rely on the use of particular possible edge labels, so the results cannot be refined using b-edge consecutive magic graphs.

#### 6.5.2. Threshold Protocol Game on Edge-Magic Total Graphs

Consider the play of a TPG on an edge-magic total graph. Since an edge-magic total labeling is a labeling on both the vertices and edges of  $G$ , a vertex  $v \in V$  takes state  $B$  at  $t_{i+1}$  if  $q \leq \frac{\sum_{u \in N_B^i(v)} (\lambda_u + \lambda_{uv})}{\sum_{u \in N(v)} (\lambda_u + \lambda_{uv})}$ , which can be refined on an edge-magic total graph as

$$\begin{aligned} q &\leq \frac{\sum_{u \in N_B^i(v)} (\lambda_u + \lambda_{uv})}{\sum_{u \in N(v)} (\lambda_u + \lambda_{uv})} \\ &= \frac{d_B^i(v)(k - \lambda_v)}{d(v)(k - \lambda_v)} \\ &= \frac{d_B^i(v)}{d(v)}. \end{aligned}$$

Thus, the threshold of the TPG in an edge-magic total graph is the same as the threshold in a simple graph, which means that vertices update states as in the classic TPG. Hence, the play of the TPG is equivalent between edge-magic total graphs and unlabeled graphs.

## 7. Discussion

The update rule of the classic TPG states that  $v \in V$  takes state  $B$  at  $t_{i+1}$  if  $q \leq \frac{d_B^i(v)}{d(v)}$  and  $A$  otherwise. On a labeled graph, the update rule changes based on the type of labeling. In an edge labeled graph,  $v$  takes state  $B$  if  $q \leq \frac{\sum_{u \in N_B^i(v)} \lambda_{uv}}{\sum_{u \in N(v)} \lambda_{uv}}$ , in a vertex labeled graph if  $q \leq \frac{\sum_{u \in N_B^i(v)} \lambda_u}{\sum_{u \in N(v)} \lambda_u}$ , and in a graph with vertex and edge labels if  $q \leq \frac{\sum_{u \in N_B^i(v)} (\lambda_u + \lambda_{uv})}{\sum_{u \in N(v)} (\lambda_u + \lambda_{uv})}$ . There may be restrictions on graph topology based on the labelings used, but properties of the TPG that do not depend on  $q$  hold regardless of labeling.

Unlike randomly weighted graphs or arbitrary influence graphs, a graph labeling provides a fully specified scheme of the weights and/or influence values. This work demonstrates that the outcome of the TPG may be characterized in terms of a graph labeling with a few caveats. Since a-vertex consecutive and b-edge consecutive magic labelings are examples of vertex-magic total labelings, they will not currently be considered separately. There are four labelings of primary interest. A magic labeling is an edge labeling,  $\Sigma$ - and  $\Sigma'$ -labelings are vertex labelings, and a vertex-magic total labeling is a labeling on both vertices and edges, but all follow a similar principle in which the labels of some subsets of elements sum to a constant.

Two types of results are provided for each labeling. First, a low (worst-case) bound on  $q$  is provided, which allows for a minimum  $V_B^0$  to be contagious in an arbitrary  $G$ , where restrictions on  $V_B^0$  are determined only by whether  $G$  is non-bipartite or bipartite. This  $V_B^0$  is advantageous because it requires no particular selection of vertices when  $G$  is non-bipartite, and requires only that a vertex is selected from each partite set when  $G$  is bipartite. A version of this result is achievable for all the labelings considered here. Hence, for each of these four labelings, there exist completely contagious  $V_B^0$ 's, and a lower bound on  $q^*$  is provided by Theorem 1 on magic graphs, Theorem 5 on  $\Sigma$ -labeled graphs, Theorem 8 on  $\Sigma'$ -labeled graphs, and Theorem 10 on vertex-magic total graphs. A similar result can be achieved for  $\Sigma$ -labeled graphs in terms of  $\Sigma$ -partition hypergraphs, which is shown in Theorem 6.

Second, a characterization is provided of variations on complete and complete bipartite graphs, as appropriate. It is typical to consider strongly regular magic graphs, which include complete and complete bipartite graphs. Theorem 4 and Corollary 1 describe a better bound on  $q$  than Theorem 1 for a similar  $n_B^0$ , but some care must be taken in selecting  $V_B^0$ . Additionally, since they depend on  $G$  being strongly regular, these results do not hold on all magic graphs. Corollary 2 generalizes the maximum spanning tree approach to all magic graphs, with  $n_B^0 = 2$ . This result provides a same or slightly worse  $n_B^0$  than Theorem 1, but the bound on  $q$  may be better. That said, there are also restrictions on  $V_B^0$ , making the result of Corollary 2 more restrictive than that of Theorem 1.

While the main goal of the TPG is to motivate  $B$  to be completely contagious, there are sets of parameters for which  $B$  will not be completely contagious.  $G$ ,  $V_B^0$ , and  $q$  each play a key role in the end behavior the TPG will exhibit. To study the influence of labelings on game outcome, which impact feasible bounds on  $q$ ,  $G$  is fixed as a complete or complete bipartite graph, and  $V_B^0$  is arbitrary. Theorems 2 and 3 characterize the end behavior of a TPG on magic complete and complete bipartite graphs. There are no  $\Sigma$ -labeled complete graphs, but Theorem 7 characterizes the play of the TPG on  $\Sigma$ -labeled complete bipartite graphs. In contrast, the other vertex labeled graphs that were considered,  $\Sigma'$ -labeled graphs, do not exist for complete bipartite graphs, but do exist for complete graphs. Thus, Theorem 9 characterizes the play of the TPG on  $\Sigma'$ -labeled complete graphs. Theorems 11 and 12 describe the outcomes of the TPG when using vertex-magic total complete and complete bipartite graphs, respectively. The bounds on  $q$  in these results are ideal, since they are provided in terms of  $V_B^0$  and the graph labeling, which are parts of the game definition. For all complete labeled graphs, an upper bound can be found on  $q$  such that  $B$  is completely contagious and a lower bound can be found on  $q$  such that  $A$  is completely contagious. This is also true of complete bipartite graphs, although there

may also exist an interval  $I \subseteq (0, 1)$  such that  $q \in I$  causes a 2-periodic end behavior in the TPG. Note, however, that these statements hold provided the bound(s) fall in  $(0, 1)$ . While general characterizations are provided, the only labeling considered here for which the characterization on  $K_n$  or  $K_{m,p}$  is complete is the  $\Sigma$ -labeling. A complete characterization of the other labelings would depend on each vertex in  $V_B^0$ , which is neither feasible nor general enough to be of interest.

There are a few additional comparisons that can be drawn between the labelings based on these results. First, the labeling on the edges and vertices of a graph results in the most complex update rule for the TPG. The update rules for edge and vertex labelings are similar in terms of complexity. This suggests that when considering other labelings, characterizations of graphs with labeled vertices and edges may not be possible to the extent that characterizations on graphs with vertex labelings or edge labelings can be found. However, in the magic-square generalization labelings considered in this work, the type of labeling is not the primary factor in the level of characterization that can be achieved. Note that the constant sum at each  $v \in V$  is independent of  $\lambda_v$  for magic and  $\Sigma$ -graphs, but dependent on  $\lambda_v$  for  $\Sigma'$ - and vertex-magic total graphs. Stronger and more varied characterizations of the TPG are possible when using magic and  $\Sigma$ -graphs than  $\Sigma'$ - and vertex-magic total graphs, since the analysis is simplified for these labelings.

## 8. Conclusions

The TPG is a valuable and simple economic model with interesting dynamics. The primary aim of the TPG is to study the circumstances under which a state that begins in the minority will overtake the population. Historically, this game has been studied using unweighted graphs and unlabeled players. This means that the outcome of the TPG is studied when all players and the relationships between them have equal value in the game. In contrast, playing the game with a labeled graph requires accounting for the particular scheme of values applied to each player and/or relationship. A magic square-generalization labeling describes the distribution of weights when there is some constant level of influence felt by each player.

Here, labelings were considered on the edges, vertices, and edges and vertices of a graph, as these translate directly to the properties of the game. A magic labeling served as a viable edge labeling,  $\Sigma$ - and  $\Sigma'$ -labelings as vertex labelings, and a vertex-magic total labeling as a labeling on both edges and vertices. In all cases, it was possible to find a minimum contagious  $V_B^0$  for a sufficiently small  $q$  in terms of the labeling. To understand the extent to which characterization in terms of a graph labeling is possible, complete and complete bipartite graphs were used to fix  $G$ , and the selection of  $V_B^0$  was arbitrary. Then the end behavior of the TPG can be characterized by bounds on  $q$  in terms of the labeling of  $G$ , the topology of  $G$ , and  $V_B^0$ .

The playing of graphical games using labeled graphs has not been considered previously and presents a new avenue to explore the union between graph features and graphical game outcome. In practice, magic square-generalization labelings are a highly regulated representation of a normalized level of influence experienced by each player. Through generalizing this concept on a social network, players could experience a constant level of influence from relationships or other players, but a perfect graph labeling is unlikely to arise. This work indicates that such an approach to a social network would allow for the game outcome to be characterized more specifically than under general weighting and perhaps more realistically than on a simple graph. However, the level of characterization that graph labelings allow for would likely not be possible.

The impact of graph labels on the game appears in the update rules, where players weight the contribution of each neighbor differently. However, this is merely a linear weighting, meaning that determining the outcome of the TPG on labeled graphs has the same computational complexity as on unlabeled graphs.

### Future Work

There are a few main avenues to continue this research. The first is based on results regarding the topological properties of graphs with the mentioned labelings. These provide a basis to begin exploring the union between the influence of labelings and graph topology on game outcome. One such feature occurs on magic graphs. Consider the following three theorems:

1. Hall's Marriage Theorem [46,48]: A bipartite graph  $G = (X, Y)$  has a matching that saturates  $X$  if and only if  $|N(W)| \geq |W|$  for all  $W \subseteq X$ .  
Corollary: For  $r > 0$ , every  $r$ -regular bipartite graph has a perfect matching.
2. Jeurissen Theorem 1 [45]: A connected bipartite graph  $G = (X, Y)$  is magic if and only if the following statements hold:
  - (a)  $|X| = |Y|$ .
  - (b)  $|N(W)| > |W|$  for all  $W \subset X, W \neq \emptyset$ .
  - (c)  $G$  does not consist of two disjoint balanced bipartite graphs connected by a cross-bridge.
3. Jeurissen Theorem 2 [45]: A connected non-bipartite graph  $G =$  with vertex set  $V$  is magic if and only if the following statements hold:
  - (a)  $|N(W)| > |W|$  for all  $W \subset V, W \neq \emptyset$ .
  - (b)  $V(G)$  cannot be partitioned into subsets  $W_1, W_2$  containing one handle at  $W_1$ , one handle at  $W_2$ , and  $|W_1| = |W_2|$ .
  - (c)  $G$  is not a graph consisting of a balanced graph  $W_1, W_2$  connected by one edge from  $W_1$  and one from  $W_2$  to another (possibly disconnected) graph.

Hence, let  $W \subset V$  in a nonbipartite magic graph or  $W \subset V$  in a bipartite magic graph such that  $W$  has a nonempty intersection with both partite sets. Then,  $|N(W)| > |W|$ . As indicated above,  $V_B^0$  is not contagious in a bipartite graph if this is a subset of one partite set. Then, these three theorems show that, for each possibly contagious  $V_B^0$ ,  $|N(V_B^0)| > |V_B^0|$ .

Another feature to consider arises in complete bipartite graphs. It has been shown that  $K_{m,p}$  can be magic only if  $m = p$  and can be vertex-magic total only if the difference between  $m$  and  $p$  is, at most, 1. Whether these restrictions can be used to clarify the outcome of the TPG in magic and vertex-magic total  $K_{m,p}$  should be considered.

An alternative continuation of this work is to examine the outcome of a TPG on graphs with different labelings. While the most common and direct magic square-generalization labelings have been considered in this work, there are other types of magic square-generalization labelings for which the use of a TPG could be considered. These include H-magic labelings and inner magic labelings, although H-magic graphs appear to be a more promising avenue of study. A broader extension of this work is to study the outcome of the TPG on graphs with a variety of labelings. Other popular graph labelings include graceful labelings, antimagic labelings, and harmonious labelings. For a survey of graph labelings, see [5]. The results observed for the TPG with magic square-generalization labelings rely heavily on the fact that each labeling corresponds to a constant sum of the labels of some set of features at each vertex. It is reasonable to expect, then, that irregular labelings such as graceful or antimagic labelings will not strongly facilitate TPG play, while the update rules for edge labelings, vertex labelings, and labelings on vertices and edges hold regardless of labeling. This means that in the case of a particular labeled  $G$ , a  $V_B^0$ , and a given  $q$ , the computations for the TPG would be similar across all edge labelings, all vertex labelings, etc. Characterizing the outcomes of the game, however, relies on the use of particular labeling properties. Since the update rules cannot be greatly simplified for irregular labelings, an analysis of the TPG is expected to be limited with these labelings.

The influence of labeled graphs was not considered for graphical games beyond the TPG. By beginning with a different stage game, questions similar to those asked here about the TPG can be addressed. It would be reasonable to begin with popular two-player stage games such as the prisoner's dilemma or a hawk-dove game.

An additional layer of complexity and interest can be added by considering dynamic graph features, such as changing the values of  $q$  or shifting labels. While dynamic game elements may make the proposed scheme more widely applicable, they introduce considerable uncertainty in the game. It is unlikely the conclusions drawn would be as strong as those possible for the TPG.

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