

Safa, Benghebrid; Touffik, Bouremani; Djamel, Benterki

Article

On Isaac's war game of attrition and attack using dynamic programming approach

Games

Provided in Cooperation with:

MDPI – Multidisciplinary Digital Publishing Institute, Basel

Suggested Citation: Safa, Benghebrid; Touffik, Bouremani; Djamel, Benterki (2024) : On Isaac's war game of attrition and attack using dynamic programming approach, Games, ISSN 2073-4336, MDPI, Basel, Vol. 15, Iss. 6, pp. 1-18,
<https://doi.org/10.3390/g15060035>

This Version is available at:

<https://hdl.handle.net/10419/330104>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



<https://creativecommons.org/licenses/by/4.0/>

Article

On Isaac's War Game of Attrition and Attack Using Dynamic Programming Approach

Benghebrid Safa ^{1,*}, Bouremani Touffik ^{2,†} and Benterki Djamel ^{1,†}

¹ Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, University of Ferhat Abbas Setif-1, Setif 19000, Algeria; djbenterki@univ-setif.dz

² Laboratory of Applied Mathematics, Faculty of Technology, University of Ferhat Abbas Setif-1, Setif 19000, Algeria; touffik.bouremani@univ-setif.dz

* Correspondence: safabenghebrid@univ-setif.dz; Tel.: +213-781318405

† These authors contributed equally to this work.

Abstract: In this study, we use the dynamic programming method introduced by Mirică (2004) to solve the well-known war game of attrition and attack as formulated by Isaacs (1965). By using this modern approach, we extend the classical framework to explore optimal strategies within the differential game setting, offering a complete, comprehensive and theoretically robust solution. Additionally, the study identifies and analyzes feedback strategies, which represent a significant advancement over other strategy types in game theory. These strategies dynamically adapt to the evolving state of the system, providing more robust solutions for real-time decision-making in conflict scenarios. This novel contribution enhances the application of game theory, particularly in the context of warfare models, and illustrates the practical advantages of incorporating feedback mechanisms into strategic decision-making. The admissible feedback strategies and the corresponding value function are constructed through a refined application of Cauchy's Method of characteristics for stratified Hamilton–Jacobi equations. Their optimality is proved using a suitable Elementary Verification Theorem for the associated value function as an argument for sufficient optimality conditions.

Keywords: differential game; differential inclusion; feedback strategies; dynamic programming; Hamiltonian flow; value function



Citation: Safa, B.; Touffik, B.; Djamel, B. On Isaac's War Game of Attrition and Attack Using Dynamic Programming Approach. *Games* **2024**, *15*, 35. <https://doi.org/10.3390/g15060035>

Academic Editor: Ulrich Berger

Received: 14 July 2024

Revised: 28 September 2024

Accepted: 22 October 2024

Published: 24 October 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Differential games, a branch of static game theory, provide a framework for analyzing strategic interactions between two or more players, in which each player's decisions, in turn, influence the dynamics of a system. Rufus Isaac's seminal book *Differential Games* [1], published in 1965, laid the foundation for the study of pursuit-evasion games and offered a comprehensive framework for analyzing differential games involving multiple players. This work laid the groundwork for addressing a diverse array of conflict scenarios and inspired researchers to delve further into this field. This exploration brought numerous criticisms of Isaac's heuristic approach to the forefront, including those highlighted in [2–8]. Ultimately, the research prompted by Isaac's missteps has left a lasting legacy, manifesting in improved methodologies, interdisciplinary collaborations, and a richer understanding of the nuances of the problem. Also, the divide between theoretical rigor and practical applicability has prompted researchers to seek a middle ground. Efforts have been made to develop theoretical frameworks that can accommodate the intricacies of concrete examples (e.g., [9–12]). This involved refining existing mathematical methods and, in some cases, introducing novel approaches that maintain a balance between theoretical soundness and practical relevance. Since 1982, significant developments have occurred in the theory of viscosity solutions, initiated by [13], which have provided various characterizations of the value function as a solution to the Hamilton–Jacobi–Isaac's equation. However, the elaboration of the theory of viscosity solutions has not effectively contributed to the

accurate and complete resolution of any concrete problems proposed in the literature. This prevailing anomaly in the study of differential games is about to be corrected, due to the recent contributions presented in Mirică's work [14–16]. The fundamental content of this new Dynamic Programming approach consists of sufficient optimality conditions, illustrated by a main theoretical support consisting of seven verification theorems. It essentially extends to much more realistic cases, the elementary verification theorem is from Isaac's (1965) in [1], the only known previous one (applied, unjustifiably, to problems in which the value function is not differentiable). The constructive aspect of this approach also encompasses significant extensions and generalizations of the characteristics' method for non-smooth Hamilton–Jacobi equations, providing a rigorous foundation for the heuristic procedures proposed by Isaac's (1965) and various other related works. Unlike previous methodologies, optimality is defined not within the framework of saddle points [3,17], but rather in the more convenient though ostensibly equivalent class of relatively optimal feedback strategies. As noted in [15], the only realistic approach to engaging in a differential game (particularly in the context of an optimal control problem) is to employ feedback strategies that are calculated in advance. This perspective is, in fact, shared by [1,11]. One of the classic problems in differential games is the War of Attrition, a compelling concept with applications across various fields, including military strategy and resource management. It is regarded as a well-known example within the domain of differential games. It models a scenario in which two or more players are engaged in a contest to capture a valuable resource, such as territory or prey. However, this contest incurs costs. Players must determine the duration of their involvement in the struggle, carefully weighing the potential benefits of victory against the cumulative costs of sustained conflict.

The aim of this paper is to apply step by step manner the theoretical Dynamic Programming algorithm, described in [15,16], and to integrate these results with numerical procedures in order to achieve a more rigorous and theoretically complete solution to the War of Attrition game, which was formulated and studied heuristically in [1] (Section 5.4, page 96). This model may be considered as a two-player zero-sum game, where the players have completely opposite interests namely, a player's gain is an equivalent loss to the opposing player. Our approach was first employed to address problems of exceptionally high complexity. Among these challenges was the well-known Homicidal chauffeur game in [18]. Additionally, in the explicit model illustrated in [19], it was shown that, only the maximal value function is admissible and is associated with certain feedback strategies. Furthermore, within the realm of conflict problems, the model presented in [20] is particularly relevant as it provides a framework that can be directly compared with the results in [21]. The primary distinction between the two studies lies in the methodological approaches employed to analyze warfare dynamics: dynamic programming in our case, contrasted with the Lanchester equation in theirs. The conclusions drawn from this comparison suggest that our approach offers a broader and more realistic framework for modeling warfare dynamics, particularly in real-world scenarios where strategies and conditions evolve over time. While the Lanchester equation is useful, it is constrained by its static nature and limited scope, making our approach more applicable to complex, dynamic conflicts.

Utilizing the Dynamic Programming method to solve this problem presents the advantage of allowing us to determine all admissible trajectories associated with the problem. Moreover, the hypotheses that need to be verified are significantly more natural and easier to establish, drawing upon elements of Hamilton–Jacobi theory as well as recent findings in Non-Smooth Analysis as referenced in [15,22–24].

The paper is organised as follows: after the introduction, we present in Section 2 the formulation of the problem, its Dynamic Programming formulation, and the characterization of the Hamiltonian. Section 3 gives the generalized stratified Hamiltonian field. In Section 4, we describe the partial Hamiltonian flow whose trajectories have terminal segments on each of the strata. Section 5 shows the existence of the corresponding value function which defines a certain pair of admissible and possibly optimal feedback strate-

gies of the considered game problem. Finally, some concluding remarks are provided in Section 6.

2. Formulation of the Problem

In [1], we consider a warfare game model between two nations \mathbb{U} and \mathbb{V} , engaged in a protracted war, that consists of optimizing the cost function given by the following:

$$\mathcal{C}(u(\cdot), v(\cdot)) = \int_0^T [(1 - v(t))x_2(t) - (1 - u(t))x_1(t)]dt, \quad (1)$$

and defined by the warfare dynamic system:

$$\begin{cases} x' = (m_1 - c_1 v(t)x_2, m_2 - c_2 u(t)x_1, -1), & x(0) = x_0, \\ (u(t), v(t)) \in [0, 1] \times [0, 1], & t \in [0, T], \\ x_0 \in \mathbb{R}_+^3, & c_1 > c_2 > 0, \end{cases} \quad (2)$$

the involved functions have the following significance:

- $x_1(t), x_2(t)$: represent the force of the nations \mathbb{U} and \mathbb{V} , respectively, at $t \in [0, T]$;
- m_1, m_2 : the weapon production rate of the nations \mathbb{U} and \mathbb{V} , respectively;
- c_1, c_2 : are the measure of weapon effectiveness of \mathbb{V} versus \mathbb{U} and \mathbb{U} versus \mathbb{V} , respectively;
- $u(t)$ and $v(t)$: represent the strategies of the two nations (or players) involved in the game. Specifically, $u(t)$ is the strategy chosen by the nation \mathbb{U} (the attacker), which determines the intensity or allocation of its military efforts over time t . The value of $u(t)$ is constrained within the interval $[0, 1]$, where $u(t) = 0$ represents no attack, and $u(t) = 1$ represents the maximum possible attack effort. In relation to $v(t)$ it is the strategy chosen by the nation \mathbb{V} (the defender), representing how much effort it allocates to defending itself at any time t . Similar to $u(t)$, $v(t)$ lies within $[0, 1]$, with $v(t) = 0$ indicating no defense effort and $v(t) = 1$ representing the maximum defense effort.
- The first and second equation of system (2) represent, the rate of change in nation \mathbb{U} 's military forces over time (respectively, the rate of change in nation \mathbb{V} 's military forces). While, the third equation models the time evolution within the game. It implies that time is decreasing uniformly, as the conflict proceeds, from T to 0. This negative time progression is a standard feature in differential games to reflect the countdown toward the end of the game.

From the intuitive formulation of the problem in (1) and (2), it is understood that there are two nations (players), \mathbb{U} and, \mathbb{V} and they can choose, an optimal strategy $\tilde{u}(\cdot)$, respectively, $\tilde{v}(\cdot)$ for which the dynamic system in (2) generates a trajectory $\tilde{x}(\cdot) = x_{\tilde{u}, \tilde{v}}(\cdot)$ and such that, the player \mathbb{U} tries to minimize the cost functional $\mathcal{C}(\cdot, \tilde{v}(\cdot))$, while the player \mathbb{V} tries to maximize the cost functional $\mathcal{C}(\tilde{u}(\cdot), \cdot)$.

2.1. Dynamic Programming Formulation

In order to use the Dynamic Programming approach in [15,16], we reformulate the problem (1) and (2) using standard notations in game theory and embedding this problem in a set of problems associated with each initial point in the phase space as in [18–20]. We obtain the following standard Lagrangian autonomous differential game problem which, in a rather vague formulation, may be stated as follows:

Problem 1. Given $m_1, m_2 > 0, c_1 > c_2 > 0$. Find:

$$\inf_{u(\cdot)} \sup_{v(\cdot)} \mathcal{C}(y; u(\cdot), v(\cdot)), \quad \forall y \in Y_0, \quad (3)$$

subject to the following:

$$\begin{aligned}
\mathcal{C}(y; u(\cdot), v(\cdot)) &= g(x(T)) + \int_0^T f_0(x(t), u(t), v(t)) dt, \quad y \in Y_0, \\
x'(t) &= f(x(t), u(t), v(t)) \text{ a.e. } (0, T), \quad x(0) = y, \\
u(t) &\in U(x(t)), \quad v(t) \in V(x(t)) \text{ a.e. } (0, T), \\
x(\cdot) &\in \Omega, \quad (u(\cdot), v(\cdot)) \in \mathcal{P}, \quad f_0(x(\cdot), u(\cdot), v(\cdot)) \in L^1([0, T], \mathbb{R}), \\
x(t) &\in Y_0, \quad \forall t \in [0, T], \quad x(T) \in Y_1,
\end{aligned} \tag{4}$$

defined by the following data:

$$\begin{aligned}
f(x, u, v) &= (m_1 - c_1 vx_2, m_2 - c_2 ux_1, -1), \\
f_0(x, u, v) &= (1 - v)x_2 - (1 - u)x_1, \\
U(x) &= U = [0, 1], \quad V(x) = V = [0, 1], \quad g(\xi) = 0, \quad \forall \xi \in Y_1, \\
Y_0 &= \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T], \quad Y_1 = \mathbb{R}^+ \times \mathbb{R}^+ \times \{0\}.
\end{aligned} \tag{5}$$

where $\mathcal{P} = \mathcal{U} \times \mathcal{V}$ is the (largest) class of measurable admissible control functions $(u(\cdot), v(\cdot))$ and Ω is the corresponding class of absolutely continuous admissible trajectories.

2.2. The Hamiltonian and the Set of Transversely Terminal Points

The pseudo-Hamiltonian $\mathcal{H}(x, p, u, v) = \langle p, f(x, u, v) \rangle + f_0(x, u, v)$ is given in our case by the following:

$$\mathcal{H}(x, p, u, v) = p_1 m_1 + p_2 m_2 - p_3 + x_2 - x_1 + x_1(1 - c_2 p_2)u - x_2(1 + c_1 p_1)v, \tag{6}$$

where, p represents Lagrange multipliers; using the fact that:

$$\begin{aligned}
\min_{u \in U} [(1 - c_2 p_2)u] &= \begin{cases} 0 & \text{if } p_2 \leq \frac{1}{c_2}, \\ 1 - c_2 p_2 & \text{if } p_2 > \frac{1}{c_2}. \end{cases} \\
\max_{v \in V} [-(1 + c_1 p_1)v] &= \begin{cases} 0 & \text{if } p_1 \geq -\frac{1}{c_1}, \\ -(1 + c_1 p_1) & \text{if } p_1 < -\frac{1}{c_1}. \end{cases}
\end{aligned}$$

hence, the corresponding extreme value of the control parameters is given by the formulas:

$$\begin{aligned}
\hat{U}(x, p) = \hat{U}(p) &= \begin{cases} \{0\} & \text{if } p_2 < \frac{1}{c_2}, \\ \{1\} & \text{if } p_2 > \frac{1}{c_2}, \\ U = [0, 1] & \text{if } p_2 = \frac{1}{c_2}. \end{cases} \\
\hat{V}(x, p) = \hat{V}(p) &= \begin{cases} \{0\} & \text{if } p_1 > -\frac{1}{c_1}, \\ \{1\} & \text{if } p_1 < -\frac{1}{c_1}, \\ V = [0, 1] & \text{if } p_1 = -\frac{1}{c_1}. \end{cases}
\end{aligned} \tag{7}$$

The Isaac's Hamiltonian:

$$H(x, p) = \min_{u \in U} \max_{v \in V} \mathcal{H}(x, p, u, v) = \max_{v \in V} \min_{u \in U} \mathcal{H}(x, p, u, v), \quad (x, p) \in Z = \text{dom}(H(\cdot, \cdot)),$$

as well as its domain Z are stratified by the stratification $S_H = \{Z^{\pm, \pm}, Z^{\pm, \mp}, Z^{0, \pm}, Z^{\pm, 0}, Z^{0, 0}\}$ defined by the following:

$$\begin{aligned}
Z^{+,+} &= \{(x, p) \in Z : p_1 > -\frac{1}{c_1}, p_2 > \frac{1}{c_2}\}, \\
Z^{+,-} &= \{(x, p) \in Z : p_1 > -\frac{1}{c_1}, p_2 < \frac{1}{c_2}\}, \\
Z^{+,0} &= \{(x, p) \in Z : p_1 > -\frac{1}{c_1}, p_2 = \frac{1}{c_2}\}, \\
Z^{-,+} &= \{(x, p) \in Z : p_1 < -\frac{1}{c_1}, p_2 > \frac{1}{c_2}\}, \\
Z^{-,-} &= \{(x, p) \in Z : p_1 < -\frac{1}{c_1}, p_2 < \frac{1}{c_2}\}, \\
Z^{-,0} &= \{(x, p) \in Z : p_1 < -\frac{1}{c_1}, p_2 = \frac{1}{c_2}\}, \\
Z^{0,+} &= \{(x, p) \in Z : p_1 = -\frac{1}{c_1}, p_2 > \frac{1}{c_2}\}, \\
Z^{0,-} &= \{(x, p) \in Z : p_1 = -\frac{1}{c_1}, p_2 < \frac{1}{c_2}\}, \\
Z^{0,0} &= \{(x, p) \in Z : p_1 = -\frac{1}{c_1}, p_2 = \frac{1}{c_2}\}.
\end{aligned} \tag{8}$$

If we denote by $H^{\pm,\pm}(\cdot, \cdot) = H(\cdot, \cdot) |_{Z^{\pm,\pm}}$, $H^{\pm,\mp}(\cdot, \cdot) = H(\cdot, \cdot) |_{Z^{\pm,\mp}}$, $H^{\pm,0}(\cdot, \cdot) = H(\cdot, \cdot) |_{Z^{\pm,0}}$, $H^{0,\pm}(\cdot, \cdot) = H(\cdot, \cdot) |_{Z^{0,\pm}}$, $H^{0,0}(\cdot, \cdot) = H(\cdot, \cdot) |_{Z^{0,0}}$ we obtain:

$$\begin{aligned}
H^{+,+}(x, p) &= p_1 m_1 + p_2 (m_2 - c_2 x_1) - p_3 + x_2, \\
H^{+,-}(x, p) &= p_1 m_1 + p_2 m_2 - p_3 + x_2 - x_1, \\
H^{+,0}(x, p) &= p_1 m_1 + \frac{m_2}{c_2} - p_3 + x_2 - x_1, \\
H^{-,+}(x, p) &= p_1 (m_1 - c_1 x_2) + p_2 (m_2 - c_2 x_1) - p_3, \\
H^{-,-}(x, p) &= p_1 (m_1 - c_1 x_2) + p_2 m_2 - p_3 - x_1, \\
H^{-,0}(x, p) &= p_1 (m_1 - c_1 x_2) + \frac{m_2}{c_2} - p_3 - x_1, \\
H^{0,+}(x, p) &= -\frac{m_1}{c_1} + p_2 (m_2 - c_2 x_1) - p_3 + x_2, \\
H^{0,-}(x, p) &= -\frac{m_1}{c_1} + p_2 m_2 - p_3 + x_2 - x_1, \\
H^{0,0}(x, p) &= -\frac{m_1}{c_1} + \frac{m_2}{c_2} - p_3 + x_2 - x_1.
\end{aligned} \tag{9}$$

Next, we need to compute the set of terminal transversality values defined in the general case by the following:

$$\begin{aligned}
Z_{+,-}^* &= \{(\xi, q) \in Y_1 \times \mathbb{R}^3 : H(\xi, q) = 0, \langle q, \bar{\xi} \rangle = Dg(\xi) \bar{\xi}, \forall \bar{\xi} \in T_{\xi} Y_1\}, \\
Dg(\xi) \bar{\xi} &= \frac{\partial g}{\partial \xi}(\xi) \bar{\xi}.
\end{aligned} \tag{10}$$

Lemma 1. The set of terminal transversality values, Z^* , in our case is given by the following:

$$Z^* = \{((s_1, s_2, 0), (0, 0, s_2 - s_1)); s_1, s_2 \geq 0\} \subset Z^{+,-}. \tag{11}$$

Proof of Lemma 1. Since, $g(\xi) = 0$ and the tangent space $T_{\xi} Y_1 = \mathbb{R} \times \mathbb{R} \times \{0\}$ then, it follows from (10) that, $q_1 \bar{\xi}_1 + q_2 \bar{\xi}_2 + q_3 \bar{\xi}_3 = 0, \forall \bar{\xi}_1, \bar{\xi}_2 \in \mathbb{R}, \bar{\xi}_3 = 0$ and, therefore

$$q_1 = q_2 = 0, q_3 \in \mathbb{R}.$$

Starting from the fact that, for $\xi = (s_1, s_2, 0) \in Y_1, q = (0, 0, q_3), q_3 \in \mathbb{R}$. If $z = (\xi, q) \in Z^{+,+} \cup Z^{-,\pm}$ then, we obtain the following contradictions, $q_2 = 0 \not> \frac{1}{c_2}, q_1 = 0 \not< -\frac{1}{c_1}$, and if $z = (\xi, q) \in Z^{\pm,0} \cup Z^{0,\pm} \cup Z^{0,0}$ we obtain $q_2 = \frac{1}{c_2} \neq 0$ and $q_1 = -\frac{1}{c_1} \neq 0$. Therefore, the only admissible trajectories are the ones which have segments on the stratum $Z^{+,-}$ because, $q_1 = 0 > -\frac{1}{c_1}, q_2 = 0 < \frac{1}{c_2}$. Besides, using the fact that, $H^{+,-}(\xi, q) = -q_3 + s_2 - s_1 = 0$ hence, $q_3 = s_2 - s_1$. \square

3. Generalized Hamiltonian and Characteristic Flow

The first main computational operation consists of the backward integration for $t \leq 0$, of the Hamiltonian inclusion:

$$(\dot{x}, \dot{p}) \in d_S^{\sharp} H(x, p), (x(0), p(0)) = z = (\xi, q) \in Z^*, \tag{12}$$

defined by the generalized Hamiltonian orientor field $d_S^{\sharp} H(\cdot, \cdot)$:

$$d_S^\# H(x, p) = \left\{ (\dot{x}, \dot{p}) \in T_{(x,p)} Z; \dot{x} \in f(x, \hat{U}(x, p), \hat{V}(x, p)), \right. \\ \left. \langle \dot{x}, \dot{p} \rangle - \langle \dot{p}, \dot{x} \rangle = DH(x, p)(\bar{x}, \bar{p}), \forall (\bar{x}, \bar{p}) \in T_{(x,p)} Z \right\}, \quad (13)$$

where, $DH(x, p)(\bar{x}, \bar{p})$ denotes the directional derivative of Hamiltonian function $H(.,.)$ at the point $(x, p) \in Z$ in the direction $(\bar{x}, \bar{p}) \in T_{(x,p)} Z$ and is described as follows:

$$DH(x, p)(\bar{x}, \bar{p}) = \frac{\partial H}{\partial x}(x, p)\bar{x} + \frac{\partial H}{\partial p}(x, p)\bar{p}.$$

As specified in the Algorithm in [15,16], for each terminal point $z = (\zeta, q) \in Z^*$ one should identify the maximal solutions: $X^*(.) = (X(.,.), P(.,.)) : I(z) = (t^-(z), 0] \rightarrow Z$, of the Hamiltonian inclusion in (12) that satisfies the following conditions:

$$\begin{aligned} X(t) &\in Y_0 \forall t \in I_0(z) = (t^-(z), 0), \\ H(X(t), P(t)) &= 0, \forall t \in I(z), \\ X'(t) &= f(X(t), u(t), v(t)) \text{ a.e. } I_0(z), \\ u(t) &\in \hat{U}(X^*(t)), v(t) \in \hat{V}(X^*(t)), \text{ a.e. } I_0(z). \end{aligned} \quad (14)$$

If there are several solutions for the same terminal point $z = (\zeta, q) \in Z^*$, it is necessary to parameterize all these solutions by $\lambda \in \Lambda(z)$ in order to obtain the generalized Hamiltonian flow $X^*(.,.) = (X(.,.), P(.,.)) : B = \{(t, a), t \in I(a) \text{ } a \in A\} \rightarrow Z; A = \text{graph}(\Lambda(.)), a = (z, \lambda)$. We recall also, the fact that, for each $(t, a) \in B_0 = \{(t, a) \in B, t \neq 0\}$ the Hamiltonian flow $X^*(.,.)$ defines the controls and, respectively, the trajectories:

$$\begin{aligned} u_{t,a}(s) &= u_a(t+s), \quad v_{t,a}(s) = v_a(t+s), \quad s \in [0, -t], \\ x_{t,a}(s) &= X(t+s, a), \end{aligned} \quad (15)$$

which are admissible with respect to the initial point $y = X(t, a) \in Y_0$, and for which the value of the cost functional in (4) is given by the function $V(.,.)$ defined by the following:

$$V(t, a) = g(\zeta) + \int_0^t \langle P(\sigma, a), X'(\sigma, a) \rangle d\sigma, \quad a = (z, \lambda), \quad (16)$$

and which, together with the Hamiltonian flow $X^*(.,.) = (X(.,.), P(.,.))$ defines the generalized characteristic flow $C^*(.,.) = (X^*(.,.), V(.,.))$; using the definition of the Hamiltonian $H(.,.)$ and the second condition in (14) one has $\langle P(\sigma, a), X'(\sigma, a) \rangle = -f_0(X(\sigma, a), \hat{u}(X^*(\sigma, a)), \hat{v}(X^*(\sigma, a)))$, it follows from (3) that, the function $V(.,.)$ having as formula:

$$V(t, a) = \int_0^t ((1 - \hat{u}(X^*(\sigma, a)))X_1(\sigma, a) - (1 - \hat{v}(X^*(\sigma, a)))X_2(\sigma, a))d\sigma, \quad (17)$$

therefore, it follows from (8) and (9) that, the Hamiltonian oriented field $d_S^\# H(.,.)$ is given by the formulas:

$$d_S^\# H(x, p) = \begin{cases} d_S^\# H^{\pm, \pm}(x, p) & \text{if } (x, p) \in Z^{\pm, \pm}, \\ d_S^\# H^{\pm, \mp}(x, p) & \text{if } (x, p) \in Z^{\pm, \mp}, \\ d_S^\# H^{\pm, 0}(x, p) & \text{if } (x, p) \in Z^{\pm, 0}, \\ d_S^\# H^{0, \pm}(x, p) & \text{if } (x, p) \in Z^{0, \pm}, \\ d_S^\# H^{0, 0}(x, p) & \text{if } (x, p) \in Z^{0, 0}. \end{cases} \quad (18)$$

Since the manifolds $Z^{\pm,\pm}, Z^{\pm,\mp} \subset Z$ are open subsets, the Hamiltonian oriented fields $d_S^\# H^{\pm,\pm}(\cdot, \cdot)$ and $d_S^\# H^{\pm,\mp}(\cdot, \cdot)$ in (13) coincide with classical Hamiltonian vector fields:

$$\begin{aligned} d_S^\# H^{\pm,\pm}(x, p) &= \left\{ \left(\frac{\partial H^{\pm,\pm}}{\partial p}(x, p), -\frac{\partial H^{\pm,\pm}}{\partial x}(x, p) \right) \right\}, \\ d_S^\# H^{\pm,\mp}(x, p) &= \left\{ \left(\frac{\partial H^{\pm,\mp}}{\partial p}(x, p), -\frac{\partial H^{\pm,\mp}}{\partial x}(x, p) \right) \right\}, \end{aligned} \quad (19)$$

which are easy to calculate and will be described and studied later. While, on the singular stratum $\tilde{Z} \in \{Z^{\pm,0}, Z^{0,\pm}, Z^{0,0}\}$ the corresponding Hamiltonian field $d_S^\# \tilde{H}(\cdot, \cdot) \in \{d_S^\# H^{\pm,0}(\cdot, \cdot), d_S^\# H^{0,\pm}(\cdot, \cdot), d_S^\# H^{0,0}(\cdot, \cdot)\}$ is characterized by the following result.

Lemma 2. For any $(x, p) \in \tilde{Z}$ one has the following:

$$d_S^\# \tilde{H}(x, p) = \emptyset. \quad (20)$$

Proof of Lemma 2. If $(x, p) \in Z^{\pm,0}$, in order to compute the generalized Hamiltonian field $d_S^\# H^{\pm,0}(\cdot, \cdot)$, we note first that, according to some classical results as in [15], the tangent space to the five-dimensional manifolds $Z^{\pm,0}$ is given by the following:

$$T_{(x,p)} Z^{\pm,0} = \{(\bar{x}, \bar{p}) \in \mathbb{R}^3 \times \mathbb{R}^3; \bar{p}_2 = 0\}, \quad (21)$$

and $DH^{+,0}(x, p)(\bar{x}, \bar{p}) = -\bar{x}_1 + \bar{x}_2 + m_1 \bar{p}_1 - \bar{p}_3$. Therefore, the condition $\langle \dot{x}, \bar{p} \rangle - \langle \dot{p}, \bar{x} \rangle = DH^{+,0}(x, p)(\bar{x}, \bar{p})$ is fully characterized by the expression:

$$\begin{aligned} (p'_1 - 1)\bar{x}_1 + (1 + p'_2)\bar{x}_2 + p'_3 \bar{x}_3 + (m_1 - x'_1)\bar{p}_1 - x'_2 \bar{p}_2 - (x'_3 + 1)\bar{p}_3 &= 0, \\ \forall \bar{x}_i, \bar{p}_i \in \mathbb{R} \ i = 1, 2, 3. \end{aligned} \quad (22)$$

It follows that, at each point $(x, p) \in Z^{+,0}$ one has the following:

$$x'_1 = m_1, x'_2 = 0, x'_3 = -1, p'_1 = 1, p'_2 = -1, p'_3 = 0, \quad (23)$$

since $(x', p') \in T_{(x,p)} Z^{+,0}$ then, $p'_2 = 0$, this contradicts the fact that $p'_2 = -1$.

Symmetrically, on the stratum $Z^{-,0}$ working as in the previous case we obtain:

$$x'_1 = m_1 - c_1 x_2, x'_2 = 0, x'_3 = -1, p'_1 = 1, p'_2 = c_1 p_1, p'_3 = 0, \quad (24)$$

since $(x', p') \in T_{(x,p)} Z^{-,0}$ then, $p'_2 = 0$. While, from (24) it follows that, $p_1 = 0$ that contradicts the fact that $(x, p) \in Z^{-,0}$. Concerning the strata $Z^{0,\pm}$, the proof is conducted in the same way as in the previous cases.

Next, on the stratum $Z^{0,0}$, using the same type of computations and arguments as in above, we obtain:

$$\begin{aligned} T_{(x,p)} Z^{0,0} &= \{(\bar{x}, \bar{p}) \in \mathbb{R}^2 \times \mathbb{R}^2; (\bar{p}_1, \bar{p}_2) = (0, 0)\}, \\ DH^{0,0}(x, p)(\bar{x}, \bar{p}) &= -\bar{x}_1 + \bar{x}_2 - \bar{p}_3. \end{aligned} \quad (25)$$

While the condition $\langle x', \bar{p} \rangle - \langle p', \bar{x} \rangle = DH^{0,0}(x, p)(\bar{x}, \bar{p})$ is characterized by the expression:

$$(p'_1 - 1)\bar{x}_1 + (p'_2 + 1)\bar{x}_2 - (x'_3 + 1)\bar{p}_3 + p'_3 \bar{x}_3 = 0, \forall \bar{x}_i, \bar{p}_3 \in \mathbb{R},$$

from here we deduce that at each point $(x, p) \in Z^{0,0}$ one has the following:

$$(x'_1, x'_2) \in \mathbb{R}^2, x'_3 = -1, p'_1 = 1, p'_2 = -1, p'_3 = 0,$$

the fact that $(x', p') \in T_{(x,p)} Z^{0,0}$ gives $p'_1 = 0 \neq 1$ and $p'_2 = 0 \neq -1$, which leads to a contradiction. \square

3.1. The Hamiltonian System on the Open Stratum $Z^{+,+}$

On the open stratum $Z^{+,+}$ for which, $p_1 > -\frac{1}{c_1}$ and $p_2 > \frac{1}{c_2}$ the differential inclusion in (19) coincides with the smooth Hamiltonian system:

$$\begin{cases} x' = (m_1, m_2 - c_2 x_1, -1), \\ p' = (p_2 c_2, -1, 0). \end{cases} \quad (26)$$

Standard results from differential equations theory show that the general solution of the system (26) is described by the formulas:

$$\begin{cases} x^{+,+}(t) = (m_1 t + k_1, -\frac{c_2 m_1}{2} t^2 + (m_2 - c_2 k_1) t + k_2, -t + k_3), & t < 0, \\ p^{+,+}(t) = (-\frac{c_2}{2} t^2 + k_4 c_2 t + k_5, -t + k_4, k_6), & k_i \in \mathbb{R}, i = 1, \dots, 6. \end{cases} \quad (27)$$

3.2. The Hamiltonian System on the Open Stratum $Z^{+,-}$

On the open stratum $Z^{+,-}$ for which $p_1 > -\frac{1}{c_1}$ and $p_2 < \frac{1}{c_2}$ the differential inclusion in (19) coincides with the Hamiltonian system:

$$\begin{cases} x' = (m_1, m_2, -1), \\ p' = (1, -1, 0), \end{cases} \quad (28)$$

its general solution is described by the following:

$$\begin{cases} x^{+,-}(t) = (m_1 t + k_1, m_2 t + k_2, -t + k_3), & t < 0, \\ p^{+,-}(t) = (t + k_4, -t + k_5, k_6), & k_i \in \mathbb{R}, i = 1, \dots, 6. \end{cases} \quad (29)$$

3.3. The Hamiltonian System on the Open Stratum $Z^{-,+}$

On the stratum $Z^{-,+}$ for which $p_1 < -\frac{1}{c_1}$ and $p_2 > \frac{1}{c_2}$ differential inclusion (19) coincides with the Hamiltonian system:

$$\begin{cases} x' = (-c_1 x_2 + m_1, -c_2 x_1 + m_2, -1), \\ p' = (c_2 p_2, c_1 p_1, 0), \end{cases} \quad (30)$$

which has as a general solution:

$$\begin{cases} x_1^{-,+}(t) = k_1 e^{-\sqrt{c_1 c_2} t} + k_2 e^{\sqrt{c_1 c_2} t} + \frac{m_2}{c_2}, & t < 0, \\ x_2^{-,+}(t) = k_1 \sqrt{\frac{c_2}{c_1}} e^{-\sqrt{c_1 c_2} t} - k_2 \sqrt{\frac{c_2}{c_1}} e^{\sqrt{c_1 c_2} t} + \frac{m_1}{c_1}, \\ x_3^{-,+}(t) = -t + k_3, \\ p_1^{-,+}(t) = k_4 e^{-\sqrt{c_1 c_2} t} + k_5 e^{\sqrt{c_1 c_2} t}, \\ p_2^{-,+}(t) = -k_4 \sqrt{\frac{c_1}{c_2}} e^{-\sqrt{c_1 c_2} t} + k_5 \sqrt{\frac{c_1}{c_2}} e^{\sqrt{c_1 c_2} t}, \\ p_3^{-,+}(t) = k_6, & k_i \in \mathbb{R}, i = 1, \dots, 6. \end{cases} \quad (31)$$

3.4. The Hamiltonian System on the Open Stratum $Z^{-,-}$

On the open stratum $Z^{-,-}$ for which $p_1 < -\frac{1}{c_1}$ and $p_2 < \frac{1}{c_2}$ the differential inclusion in (19) coincides with the smooth Hamiltonian system:

$$\begin{cases} x' = (m_1 - c_1 x_2, m_2, -1), \\ p' = (1, c_1 p_1, 0), \end{cases} \quad (32)$$

which, in turn, has the general solution:

$$\begin{cases} x^{-,-}(t) = (-\frac{1}{2} c_1 m_2 t^2 + (m_1 - c_1 k_1) t + k_2, m_2 t + k_1, -t + k_3), & t < 0, \\ p^{-,-}(t) = (t + k_4, \frac{1}{2} c_1 t^2 + c_1 k_4 t + k_5, k_6), & k_i \in \mathbb{R}, i = 1, \dots, 6. \end{cases} \quad (33)$$

4. Construction of Hamiltonian Flows

4.1. The Hamiltonian Flow Ending on the Stratum $Z^{+,-}$

In this section, we describe the partial Hamiltonian flow in which his trajectories have terminal segments on the stratum $Z^{+,-}$. Consider the general solution in (29) an admissible trajectory $X_{+,-}^*(.,z) = (X^{+,-}(.,z), P^{+,-}(.,z)), z \in Z^*$ of system (28). This trajectory should satisfy the terminal conditions from the set of transversality terminal points Z^* and $X_{+,-}^*(t,z) \in Z^{+,-}, \forall t < 0$. From the terminal condition in (11) it follows that, $k_1 = s_1, k_2 = s_2, k_3 = k_4 = k_5 = 0$ and $k_6 = s_2 - s_1$. Therefore, we obtain the solution of the differential system in (28) in the form of maximal flow, with its components are given by the formulas:

$$\begin{cases} X^{+,-}(t, s_1, s_2) = (m_1 t + s_1, m_2 t + s_2, -t), & t < 0, \\ P^{+,-}(t, s_1, s_2) = (t, -t, s_2 - s_1), & s_1, s_2 \geq 0. \end{cases} \quad (34)$$

From the Dynamic Programming algorithm in [15,16], it follows that we must retain only the trajectories $X_{+,-}^*(., s_1, s_2), s_1, s_2 \geq 0$ that satisfy the conditions in (14). We note that the second condition in (14) is automatically satisfied since $H^{+,-}(.,.)$ defined in (9) is a first integral of the differential system in (28), hence:

$$\begin{aligned} P_1^{+,-}(t) &> -\frac{1}{c_1}, P_2^{+,-}(t) < \frac{1}{c_2}, \\ H^{+,-}(X_{+,-}^*(t, s_1, s_2)) &= 0, \forall t < 0. \end{aligned} \quad (35)$$

The admissible trajectories $X_{+,-}^*(., s_1, s_2), s_1, s_2 \geq 0$ must satisfy the conditions:

$$\begin{aligned} X_{+,-}^*(t, s_1, s_2) &= (X^{+,-}(t, s_1, s_2), P^{+,-}(t, s_1, s_2)) \in Z^{+,-}, \\ X_{+,-}^*(t, s_1, s_2) &\in Y_0 = \mathbb{R}^+ \times \mathbb{R}^+ \times (0, T), \end{aligned} \quad (36)$$

on the maximal intervals $I^{+,-}(s_1, s_2) = (\tau^{+,-}(s_1, s_2), 0), s_1, s_2 \geq 0$, hence, the extremity $\tau^{+,-}(.,.)$ is defined by the following:

$$\begin{aligned} \tau^{+,-}(s_1, s_2) &= \max\{\tau_1^{+,-}(s_1, s_2), \tau_2^{+,-}\}, \\ \tau_1^{+,-}(s_1, s_2) &= \inf\{\tau < 0; X^{+,-}(t, s_1, s_2) \in Y_0, \forall t \in (\tau, 0)\}, \\ \tau_2^{+,-} &= \inf\{\tau < 0; P_1^{+,-}(t) > -\frac{1}{c_1}, P_2^{+,-}(t) < \frac{1}{c_2}, \forall t \in (\tau, 0)\}. \end{aligned} \quad (37)$$

Trying to obtain an explicit formula for the extremity $\tau^{+,-}(.,.)$. To this end, it follows from (34) as well as from the fact that, $c_1 > c_2$ then, $P_1^{+,-}(t) = t > -\frac{1}{c_1} > -\frac{1}{c_2}$ and, therefore:

$$\tau_2^{+,-} = -\frac{1}{c_1}. \quad (38)$$

From here together with (34) and (37), we deduce that the extremity $\tau^{+,-}(.,.)$ may be characterized as follows:

$$\tau^{+,-}(s_1, s_2) = \max\left\{-\frac{s_1}{m_1}, -\frac{s_2}{m_2}, -\frac{1}{c_1}\right\}, s_1, s_2 \geq 0, \quad (39)$$

which, in turn, can be expressed as follows:

$$\tau^{+,-}(s_1, s_2) = \begin{cases} \tau_2^{+,-} = -\frac{1}{c_1} & \text{if } s_1 > \frac{m_1}{c_1}, s_2 > \frac{m_2}{c_1}, \\ -\frac{s_1}{m_1} & \text{if } s_1 < \frac{m_1}{c_1}, s_2 > \frac{s_1 m_2}{c_1 m_1}, \\ -\frac{s_2}{m_2} & \text{if } s_1 > \frac{s_2 m_1}{m_2}, s_2 < \frac{m_2}{c_1}. \end{cases} \quad (40)$$

Geometrically, the trajectories $X^{+,-}(.,.)$ are the curves in Figure 1 and cover the domain $Y^{+,-} = Y_0^{+,-} \cup Y_1^{+,-}$ defined by the following:

$$\begin{aligned}
Y_0^{+,-} &= X^{+,-}(B^{+,-}) = \{X^{+,-}(t, s); (t, s) \in B^{+,-}\}, \\
Y_1^{+,-} &= (0, +\infty) \times (0, +\infty) \times \{0\}, \\
B^{+,-} &= \begin{cases} (-\frac{1}{c_1}, 0) \times (\frac{m_1}{c_1}, +\infty) \times (\frac{m_2}{c_1}, +\infty), \\ \{(t, s_1, s_2); t \in (-\frac{s_1}{m_1}, 0), s_1 < \frac{m_1}{c_1}, s_2 > \frac{m_2}{m_1} s_1\}, \\ \{(t, s_1, s_2); t \in (-\frac{s_2}{m_2}, 0), s_1 > \frac{m_1}{m_2} s_2, s_2 < \frac{m_2}{c_1}\}. \end{cases}
\end{aligned} \quad (41)$$

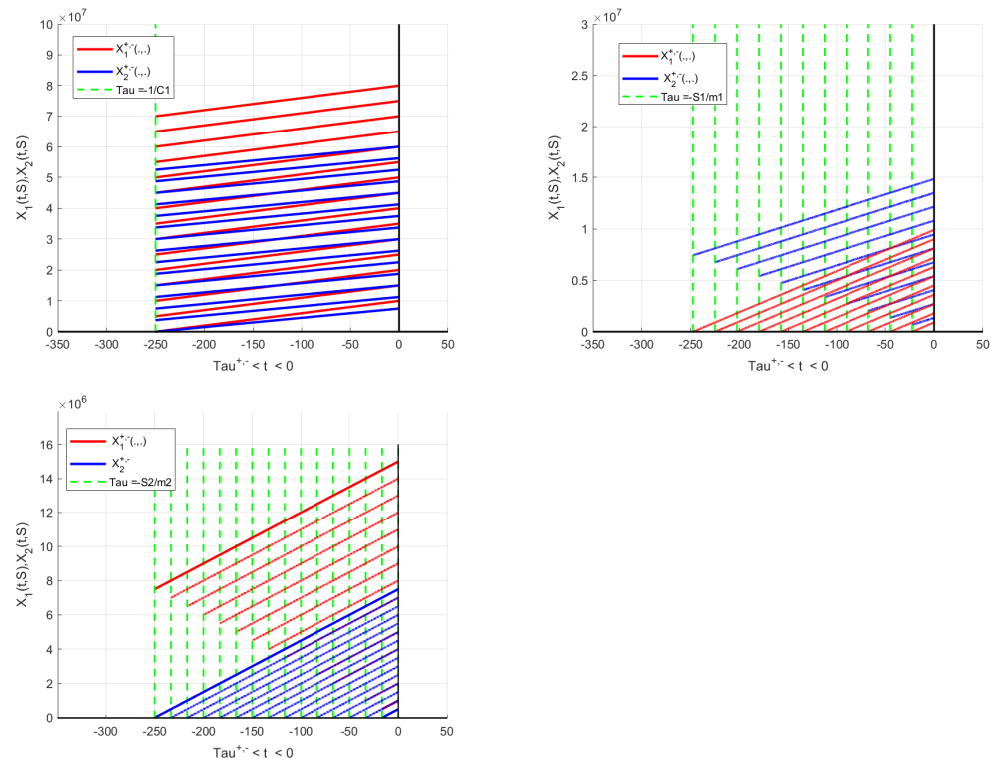


Figure 1. Admissible trajectories $X^{+,-}(\cdot, \cdot)$.

4.2. Continuation of the Trajectories on the Stratum $Z^{-,-}$

The trajectories $X_{+,-}^*(t, s_1, s_2)$, $(t, s_1, s_2) \in B^{+,-}$ in (24) may be continued for $t < \tau_2^{+,-} = -\frac{1}{c_1}$, for $s_1 > \frac{m_1}{c_1}$, $s_2 > \frac{m_2}{c_1}$ since, the extremity:

$$\begin{aligned}
z^{+,-}(s_1, s_2) &= X_{+,-}^*(\tau_2^{+,-}, s_1, s_2) \\
&= \left(-\frac{m_1}{c_1} + s_1, -\frac{m_2}{c_1} + s_2, \frac{1}{c_1} \right), \left(-\frac{1}{c_1}, \frac{1}{c_1}, s_2 - s_1 \right),
\end{aligned} \quad (42)$$

belongs to the open stratum $Z^{+,-}$ and also to the boundary of the stratum $Z^{-,-}$; therefore, the continuation of the trajectories $X_{+,-}^*(\cdot, s_1, s_2)$ for $s_1 > \frac{m_1}{c_1}$, $s_2 > \frac{m_2}{c_1}$, is possible only on the open stratum $Z^{-,-}$.

Starting from the fact that, $P_1^{-,-}(\tau_2^{+,-}) = -\frac{1}{c_1}$, the possibility of continuation of the trajectories $X_{+,-}^*(\cdot, s_1, s_2)$ in (34) for $t < \tau_2^{+,-}$, on the stratum $Z^{-,-} \subset Z$ (for which, $p_1 < -\frac{1}{c_1}$, $p_2 < \frac{1}{c_2}$) is guaranteed firstly by the condition $\frac{d}{dt}(P_1^{-,-}(\tau_2^{+,-})) = 1$ since in this case, the component $P_1^{-,-}(\cdot)$ is strictly increasing on an interval of the form $(\tau_2^{+,-} - \delta, \tau_2^{+,-})$, $\delta > 0$. Hence, for any $t < \tau_2^{+,-}$, $P_1^{-,-}(t) < P_1^{-,-}(\tau_2^{+,-}) = -\frac{1}{c_1}$.

In this case the trajectories in (34) may be continued by the trajectories $X_{-,-}^*(\cdot, s_1, s_2) = (X^{-,-}(\cdot, s_1, s_2), P^{-,-}(\cdot, s_1, s_2))$ which are solution of the Hamiltonian system in (32) with

the property $X_{-,-}^*(\tau_2^{+,-}, s_1, s_2) = z^{+,-}(s_1, s_2)$ and there exists $\tau^{-,-}(s_1, s_2) < \tau_2^{+,-}$ for $s_1 > \frac{m_1}{c_1}, s_2 > \frac{m_2}{c_1}$ such that:

$$P_1^{-,-}(t) < -\frac{1}{c_1}, P_2^{-,-}(t) < \frac{1}{c_2}, H^{-,-}(X_{-,-}^*(t, s_1, s_2)) = 0, t < \tau_2^{+,-}. \quad (43)$$

From the general solution in (33) on the open stratum $Z^{-,-}$ together with the terminal condition in (42), the trajectories $X_{-,-}^*(., s_1, s_2)$ having as components:

$$\begin{cases} X_1^{-,-}(t, s_1, s_2) = -\frac{1}{2}c_1m_2t^2 + (m_1 - c_1s_2)t + \frac{m_2}{2c_1} + s_1 - s_2, \\ X_2^{-,-}(t, s_1, s_2) = m_2t + s_2, \\ X_3^{-,-}(t, s_1, s_2) = -t, \\ P^{-,-}(t) = \left(t, \frac{1}{2}c_1t^2 + \frac{1}{2c_1}, s_2 - s_1\right), \end{cases} \quad (44)$$

at which one must also satisfy the following admissibility conditions:

$$\begin{cases} X_{-,-}^*(t, s_1, s_2) = (X^{-,-}(t, s_1, s_2), P^{-,-}(t, s_1, s_2)) \in Z^{-,-}, \\ X^{-,-}(t, s_1, s_2) \in Y_0 = \mathbb{R}^+ \times \mathbb{R}^+ \times (0, T), \end{cases} \quad (45)$$

on the maximal intervals $I^{-,-}(s_1, s_2) = (\tau^{-,-}(s_1, s_2), \tau^{+,-}(s_1, s_2))$, $s_1 > \frac{m_1}{c_1}, s_2 > \frac{m_2}{c_1}$. Hence, the extremity $\tau^{-,-}(.)$ is defined by the following:

$$\begin{aligned} \tau^{-,-}(s_1, s_2) &= \max\{\tau_1^{-,-}(s_1, s_2), \tau_2^{-,-}\}, s_1 > \frac{m_1}{c_1}, s_2 > \frac{m_2}{c_1}, \\ \tau_1^{-,-}(s_1, s_2) &= \inf\{\tau < \tau^{+,-}; X^{-,-}(t, s_1, s_2) \in Y_0, \forall t \in (\tau, \tau^{+,-})\}, \\ \tau_2^{-,-} &= \inf\{\tau < \tau^{+,-}; P_1^{-,-}(t) < -\frac{1}{c_1}, P_2^{-,-}(t) < \frac{1}{c_2}, \forall t \in (\tau, \tau^{+,-})\}. \end{aligned} \quad (46)$$

Trying to obtain an explicit formula for the extremity $\tau^{-,-}(.,.)$. To this end, it follows from (44) that:

$$P_2^{-,-}(t) - \frac{1}{c_2} = \frac{c_1}{2} \left[t^2 - \frac{1}{c_1^2} \left(\frac{2c_1}{c_2} - 1 \right) \right],$$

since, $\tau_2^{-,-} < 0$ then:

$$\tau_2^{-,-} = -\frac{1}{c_1} \sqrt{\frac{2c_1}{c_2} - 1}, \quad (47)$$

and therefore, $P_2^{-,-}(t) < \frac{1}{c_2} \forall t \in (\tau_2^{-,-}, 0) \supset (\tau_2^{-,-}, \tau_2^{+,-})$. Next, let $\tau_0(.)$ denotes the extremity defined by the following:

$$\tau_0(s_2) = -\frac{s_2}{m_2}, \quad (48)$$

for which, $X_2^{-,-}(t, s_1, s_2) > 0, \forall t \in (\tau_0(s_2), 0)$ then, from (47) we deduce that:

$$\tau_2^{-,-} - \tau_0(s_2) = \frac{c_1}{m_2^2} \frac{\left[s_2^2 - \left(\frac{m_2}{c_1} \right)^2 \left(\frac{2c_1}{c_2} - 1 \right) \right]}{\left[\frac{c_1 s_2}{m_2} + \sqrt{\frac{2c_1}{c_2} - 1} \right]}, \quad (49)$$

and we obtain:

$$\begin{cases} \tau_2^{-,-} = \tau_0(s_2), \text{ for } s_2 = \frac{m_2}{c_1} \sqrt{\frac{2c_1}{c_2} - 1}, \\ \tau_2^{-,-} > \tau_0(s_2), \forall s_2 \in \left(\frac{m_2}{c_1} \sqrt{\frac{2c_1}{c_2} - 1}, +\infty \right), \\ \tau_2^{-,-} < \tau_0(s_2), \forall s_2 \in \left(\frac{m_2}{c_1}, \frac{m_2}{c_1} \sqrt{\frac{2c_1}{c_2} - 1} \right). \end{cases} \quad (50)$$

In order to characterize the extremity $\tau_1^{-,-}(\cdot, \cdot)$, the second-order equation, $X_1^{-,-}(t, s_1, s_2) = 0$ is characterized by $\Delta(s_1, s_2) = (m_1 - c_1 s_2)^2 + m_2[m_2 + 2c_1(s_1 - s_2)]$. If we assume that, $\Delta(s_1, s_2) = 0$ then, $\Delta'_{\Delta(s_1, s_2)} = 2m_2 c_1^2(m_1 - c_1 s_1) < 0, \forall s_1 > \frac{m_1}{c_1}$ and, therefore, $\Delta(s_1, s_2) > 0, \forall s_1 > \frac{m_1}{c_1}, s_2 > \frac{m_2}{c_1}$. From here, we deduce that the second order equation has as a negative root:

$$t_1(s_1, s_2) = \frac{m_1 - c_1 s_2 - \sqrt{\Delta(s_1, s_2)}}{c_1 m_2}. \quad (51)$$

To describe $\tau^{-,-}(\cdot, \cdot)$, we use the same type of computations as in the previous case and we obtain:

$$\begin{aligned} t_1(s_1, s_2) - \tau_0(s_2) &= -\frac{\Delta(s_1, s_2) - m_1^2}{c_1 [m_1 + \sqrt{\Delta(s_1, s_2)}]}, \\ \Delta(s_1, s_2) - m_1^2 &= c_1^2 s_2^2 - 2c_1(m_1 + m_2)s_2 + m_2^2 + 2m_2 c_1 s_1, \\ \Delta'_{\Delta(s_1, s_2)}(s_2) &= c_1^2 [m_1^2 + 2m_2(m_1 - c_1 s_1)], \end{aligned} \quad (52)$$

and we can extract two cases:

Case 1: $\Delta'_{\Delta(s_1, s_2)}(s_2) < 0, \forall s_1 > \frac{m_1}{c_1} \left[1 + \frac{m_1}{2m_2}\right]$ hence, $\Delta(s_1, s_2) - m_1^2 > 0$, which leads that, $t_1(s_1, s_2) < \tau_0(s_2) \forall s_1 > \frac{m_1}{c_1} \left[1 + \frac{m_1}{2m_2}\right]$. It follows from (50) that:

$$\tau_2^{-,-} > \tau_0(s_2) > t_1(s_1, s_2), \forall s_1 > \frac{m_1}{c_1} \left[1 + \frac{m_1}{2m_2}\right], s_2 > \frac{m_2}{c_1} \sqrt{\frac{2c_1}{c_2} - 1}, \quad (53)$$

and, therefore:

$$\tau^{-,-}(s_1, s_2) = \tau_2^{-,-} = -\frac{1}{c_1} \sqrt{\frac{2c_1}{c_2} - 1}. \quad (54)$$

Case 2: $\Delta'_{\Delta(s_1, s_2)}(s_2) > 0, \forall s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1} \left[1 + \frac{m_1}{2m_2}\right]\right)$ in this case, the second-order equation $\Delta(s_1, s_2) - m_1^2 = 0$ has also two roots:

$$\begin{aligned} \tilde{s}_2 &= \frac{1}{c_1} \left[m_1 + m_2 - \sqrt{m_1^2 + 2m_2(m_1 - c_1 s_1)} \right] > \frac{m_2}{c_1}, \\ \tilde{\tilde{s}}_2 &= \frac{1}{c_1} \left[m_1 + m_2 + \sqrt{m_1^2 + 2m_2(m_1 - c_1 s_1)} \right] > \tilde{s}_2. \end{aligned} \quad (55)$$

Therefore, $\Delta(s_1, s_2) - m_1^2 < 0, \forall s_2 \in (\tilde{s}_2, \tilde{\tilde{s}}_2)$ hence, $t_1(s_1, s_2) > \tau_0(s_2)$. From here as well as from (50), we deduce that:

$$t_1(s_1, s_2) > \tau_0(s_2) > \tau_2^{-,-}, \forall s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1} \left[1 + \frac{m_1}{2m_2}\right]\right), s_2 \in (\tilde{s}_2, \tilde{\tilde{s}}_2), \quad (56)$$

and also:

$$\tau^{-,-}(s_1, s_2) = t_1(s_1, s_2) = \frac{m_1 - c_1 s_2 - \sqrt{\Delta(s_1, s_2)}}{c_1 m_2}, \quad (57)$$

while, in the case $s_2 \in \left(\frac{m_2}{c_1}, \tilde{s}_2\right) \cup (\tilde{\tilde{s}}_2, +\infty)$ then, $\Delta(s_1, s_2) - m_1^2 > 0$ which implies that, $t_1(s_1, s_2) < \tau_0(s_2)$.

We mention here that, condition (54) is the only case studied in [1]. On the other hand, the complexity of the extremities $t_1(\cdot, \cdot)$ and $\tau_2^{-,-}$ does not allow an explicit expression for the extremity $\tau^{-,-}(s_1, s_2)$ in the case $s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1} \left[1 + \frac{m_1}{2m_2}\right]\right), s_2 \in \left(\frac{m_2}{c_1}, \tilde{s}_2\right) \cup (\tilde{\tilde{s}}_2, +\infty)$. However, certain information on the admissible conditions may be obtained from numerical tests and the images of the trajectories $X^{-,-}(\cdot, \cdot)$. We develop an implementation using Matlab 2018 language and we present some simulations on the graphs of these trajectories for different parameter values.

Geometrically, the trajectories $X^{-,-}(\cdot, \cdot)$ given in (44) are the curves in Figure 2 and cover the domain $Y^{-,-} = Y_0^{-,-} \cup Y_1^{-,-}$, $Y_0^{-,-} = X^{-,-}(B^{-,-})$ defined by the following:

$$\begin{aligned} B^{-,-} &= \{(t, s_1, s_2), t \in I^{-,-}(s_1, s_2), s_1 > \frac{m_1}{c_1}, s_2 > \frac{m_2}{c_1}\}, \\ Y_1^{-,-} &= (0, +\infty) \times (0, +\infty) \times \{0\}. \end{aligned} \quad (58)$$

Finally, the trajectories $X_{+,-}^*(\cdot, \cdot)$ in (34) together with $X_{-,-}^*(\cdot, \cdot)$ in (44) may be concatenated to obtain a new extended Hamiltonian flow, described by the formula:

$$X_{\oplus, \ominus}^*(t, s_1, s_2) = \begin{cases} X_{+,-}^*(t, s_1, s_2), & t \in [\tau_2^{+,-}, 0), \\ X_{-,-}^*(t, s_1, s_2), & t \in (\tau^{-,-}(s_1, s_2), \tau_2^{+,-}), \end{cases}$$

where its trajectories are illustrated below in Figure 2.

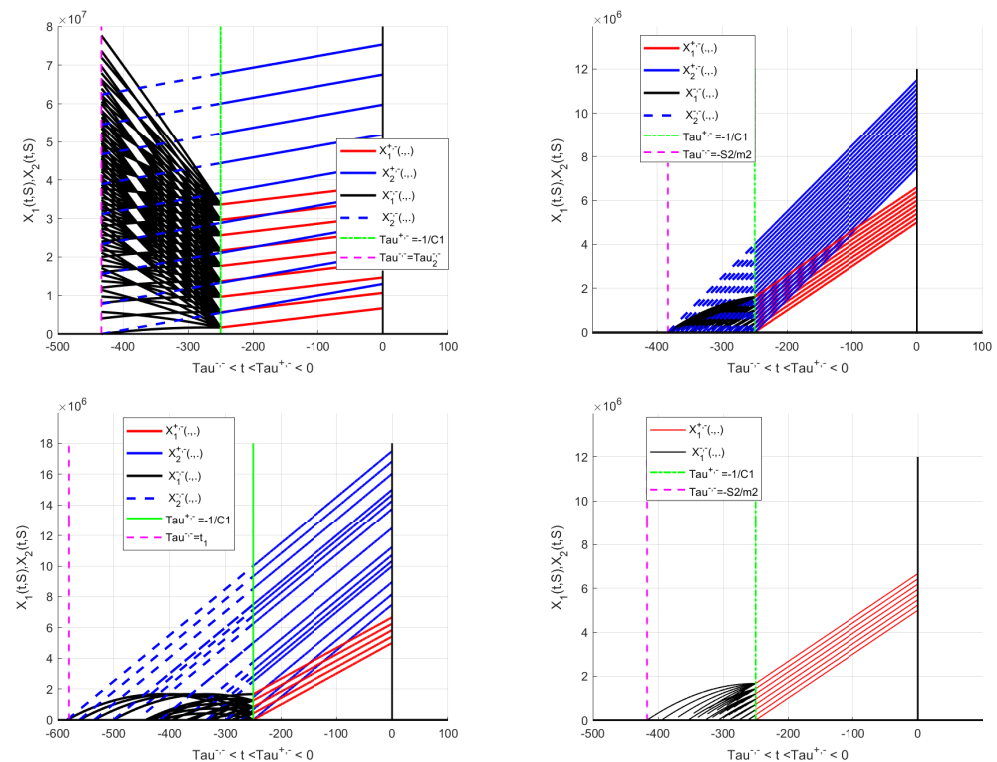


Figure 2. Admissible trajectories $X^{\oplus, \ominus}(\cdot, \cdot)$.

Thus, the Hamiltonian systems in (28) and (32) generate the generalized characteristic flows $C_{+,-}^*(\cdot, \cdot) = (X_{+,-}^*(\cdot, \cdot), V(\cdot, \cdot))$ and $C_{-,-}^*(\cdot, \cdot) = (X_{-,-}^*(\cdot, \cdot), V(\cdot, \cdot))$ described explicitly in (34) and (44) and which, according to the Well-known classical results as in [15] satisfy the basic differential relation:

$$DV(t, s)(\bar{t}, \bar{s}) = \langle P^{+,-}(t, s), DX^*(t, s)(\bar{t}, \bar{s}) \rangle \quad \forall (\bar{t}, \bar{s}) \in T_{(t,s)}B^{+,-}, \quad (59)$$

where, $T_{(t,s)}B^{+,-}$ denotes the tangent space at the point $(t, s) \in B^{+,-}$.

5. Partial Value Functions and Feedback Strategies

As indicated in the theoretical algorithm in [15,16], the natural candidates for value functions and optimal strategies of Problem 1 are the extreme ones, defined by the following:

$$\begin{aligned}
W^m(x) &= \begin{cases} g(x), & \text{if } x \in Y_1 \\ W_0^m(x) = \inf_{X(t,s)=x} V(t,s), & \text{if } x \in Y_0 \end{cases} \\
W^M(x) &= \begin{cases} g(x), & \text{if } x \in Y_1 \\ W_0^M(x) = \sup_{X(t,s)=x} V(t,s), & \text{if } x \in Y_0 \end{cases} \\
\hat{B}_m(x) &= \{(t,s) \in B : X(t,s) = x, W_0^m(x) = V(t,s)\}, \\
\hat{B}_M(x) &= \{(t,s) \in B : X(t,s) = x, W_0^M(x) = V(t,s)\}, \\
\tilde{U}_m(x) &= \bar{U}(\hat{B}_m(x)), \quad \tilde{V}_m(x) = \bar{V}(\hat{B}_m(x)), \\
\tilde{U}_M(x) &= \bar{U}(\hat{B}_M(x)), \quad \tilde{V}_M(x) = \bar{V}(\hat{B}_M(x)), \\
\bar{U}(t,s) &= \{u_s(t), u_s(t) \in \bar{U}(s)\}, \quad \bar{V}(t,s) = \{v_s(t), v_s(t) \in \bar{V}(s)\},
\end{aligned} \tag{60}$$

where, $\bar{U}(s) = \{u_s(\cdot)\}$, $\bar{V}(s) = \{v_s(\cdot)\}$, denote the sets of control mapping that satisfy (14), one may note that:

$$\bar{U}(t,s) \in \hat{U}(X^*(t)), \quad \bar{V}(t,s) \in \hat{V}(X^*(t)), \quad \forall (t,s) \in B, \tag{61}$$

and also that, if the component $X(\cdot, \cdot)$ is invertible at $(t,s) \in B$ with inverse:

$$\hat{B}(x) = (X(\cdot, \cdot))^{-1}(x),$$

then one has the following:

$$W_0^M(x) = W_0^m(x) = V(\hat{B}(x)) = W_0(x), \quad \hat{B}_M(x) = \hat{B}_m(x) = \hat{B}(x). \tag{62}$$

Moreover, it follows from (59) that if, in addition, the function $W_0(\cdot)$ is differentiable at the point $x \in \text{Int}(Y_0)$ then its derivative is given by the following:

$$DW_0(x) = \tilde{P}(x) = P(\hat{B}(x)), \tag{63}$$

and also satisfies the following relations:

$$\begin{aligned}
DW_0(x) \cdot f(x, \bar{u}, \bar{v}) + f_0(x, \bar{u}, \bar{v}) &= 0 \quad \forall \bar{u} \in \tilde{U}(x), \quad \bar{v} \in \tilde{V}(x), \\
\tilde{U}(x) &= \hat{U}(x, \tilde{P}(x)), \quad \tilde{V}(x) = \hat{V}(x, \tilde{P}(x)),
\end{aligned} \tag{64}$$

where, $\tilde{U}(\cdot)$, $\tilde{V}(\cdot)$, are the corresponding candidates for optimal feedback strategies, and also from (9) and (14) it follows that, $W_0(\cdot)$ verifies Isaac's basic equations:

$$\begin{aligned}
\min_{u \in U(x)} \max_{v \in V(x)} [DW_0(x) \cdot f(x, u, v) + f_0(x, u, v)] &= \\
\max_{v \in V(x)} \min_{u \in U(x)} [DW_0(x) \cdot f(x, u, v) + f_0(x, u, v)] &= 0.
\end{aligned} \tag{65}$$

Next, we shall prove that, the extreme solutions in (60) may be expressed as in (62). The main ingredient is the following quasi-elementary result:

Lemma 3. *The following statements are true:*

- (1) *The mapping $X^{+,-}(\cdot, \cdot) : B^{+,-} \rightarrow Y_0^{+,-}$ defined in (34) is a C^1 -stratified diffeomorphism with its inverse $\hat{B}^{+,-}(\cdot)$ is described by the following:*

$$\begin{aligned}
\hat{B}^{+,-}(x) &= (\hat{t}^{+,-}(x), \hat{s}_1^{+,-}(x), \hat{s}_2^{+,-}(x)), \quad x \in Y_0^{+,-}, \\
\hat{t}^{+,-}(x) &= -x_3, \\
\hat{s}_1^{+,-}(x) &= x_1 + m_1 x_3, \\
\hat{s}_2^{+,-}(x) &= x_2 + m_2 x_3.
\end{aligned} \tag{66}$$

- (2) *Symmetrically, the mapping $X^{-,-}(\cdot, \cdot) : B^{-,-} \rightarrow Y_0^{-,-}$ defined in (44) is a C^1 -stratified diffeomorphism, with its inverse $\hat{B}^{-,-}(\cdot)$ described by the following:*

$$\begin{aligned}
\widehat{B}^{-,-}(x) &= (\widehat{t}^{-,-}(x), \widehat{s}_1^{-,-}(x), \widehat{s}_2^{-,-}(x)), \quad x \in Y_0^{-,-}, \\
\widehat{t}^{-,-}(x) &= -x_3, \\
\widehat{s}_1^{-,-}(x) &= x_1 + x_2 + (m_1 + m_2)x_3 - c_1x_2x_3 - \frac{1}{2}c_1m_2x_3^2 - \frac{m_2}{2c_1}, \\
\widehat{s}_2^{-,-}(x) &= x_2 + m_2x_3.
\end{aligned} \tag{67}$$

Proof of Lemma 3. (1) If $x = (x_1, x_2, x_3) \in Y_0^{+,-}$ then, it follows from (34) that, a point $(t, s_1, s_2) \in B^{+,-}$ for which $X^{+,-}(t, s_1, s_2) = x$ is characterized by the expressions, $x_1 = m_1t + s_1$, $x_2 = m_2t + s_2$, $x_3 = -t$. Hence, the existence and uniqueness of the functions $t = \widehat{t}^{+,-}(x) < 0$, $s_1 = \widehat{s}_1^{+,-}(x)$ and $s_2 = \widehat{s}_2^{+,-}(x)$ having the formulas as in (66).

(2) In order to prove the second statement, use the same type of computation and arguments as in the previous case. Thus, it follows easily from (44) that, there exist $t = \widehat{t}^{-,-}(x)$, $s_1 = \widehat{s}_1^{-,-}(x)$ and $s_2 = \widehat{s}_2^{-,-}(x)$ of the form as in (67). \square

The results in Lemma 3 show that the characteristic flows $C_{+,-}^*(\cdot, \cdot) = (X_{+,-}^*(\cdot, \cdot), V(\cdot, \cdot))$ and $C_{-,-}^*(\cdot, \cdot) = (X_{-,-}^*(\cdot, \cdot), V(\cdot, \cdot))$ described, respectively, in (17), (34) and (44) are invertible in the sense of (62) and generate the smooth partial proper value function:

$$W_0(x) = \begin{cases} W_0^{+,-}(x) = \frac{1}{2}(m_2 - m_1)x_3^2 + (x_2 - x_1)x_3, & x \in Y_0^{+,-}, \\ W_0^{-,-}(x) = \frac{1}{6}c_1m_2x_3^3 - \frac{1}{2}m_1x_3^2 + \frac{1}{2}c_1x_2x_3^2 - x_1x_3, & x \in Y_0^{-,-}, \end{cases} \tag{68}$$

which is of class C^1 and may be naturally extended by $W(\xi) = g(\xi) = 0$, $\forall \xi \in Y_1$.

While, from (7) and (64), we deduce that the corresponding admissible feedback strategies are given by the following:

$$\widetilde{U}(x) \times \widetilde{V}(x) = \begin{cases} \{(\widetilde{u}^{+,-}(x), \widetilde{v}^{+,-}(x))\} = \{(0, 0)\}, & x \in Y_0^{+,-}, \\ \{(\widetilde{u}^{-,-}(x), \widetilde{v}^{-,-}(x))\} = \{(0, 1)\}, & x \in Y_0^{-,-}. \end{cases} \tag{69}$$

The main result in this section is the following.

Theorem 1. *The following statements are true:*

1. The function $W_0(\cdot)$ defined in (68) is a solution of Isaac's equation defined in (65) on the corresponding open domain $Y_0^{+,-} \cup Y_0^{-,-}$. Moreover, each of them is the value function in the sense of (60) of the corresponding admissible feedback strategies given in (69).
2. The feedback strategies $(\widetilde{U}(\cdot), \widetilde{V}(\cdot))$ described in (69) are optimal for the restriction on their open domain $Y_0^{+,-} \cup Y_0^{-,-}$.

Proof of Theorem 1. For (1), from (9), (14), (63), (65) and (69) it follows that, if $x \in Y_0^{+,-}$ then:

$$\begin{aligned}
& \min_{u \in U(x)} \max_{v \in V(x)} [DW_0^{+,-}(x) \cdot f(x, u, v) + f_0(x, u, v)] \\
&= \min_{u \in U(x)} \max_{v \in V(x)} \mathcal{H}(x, \widetilde{P}^{+,-}(x), u, v) = \mathcal{H}(x, \widetilde{P}^{+,-}(x), \widetilde{u}^{+,-}(x), \widetilde{v}^{+,-}(x)) \\
&= H^{+,-}(X^{+,-}(\widehat{B}^{+,-}(x)), \widetilde{P}^{+,-}(x)) = 0,
\end{aligned}$$

while, if $x \in Y_0^{-,-}$ we obtain:

$$\begin{aligned}
& \min_{u \in U(x)} \max_{v \in V(x)} [DW_0^{-,-}(x) \cdot f(x, u, v) + f_0(x, u, v)] \\
&= \min_{u \in U(x)} \max_{v \in V(x)} \mathcal{H}(x, \widetilde{P}^{-,-}(x), u, v) = \mathcal{H}(x, \widetilde{P}^{-,-}(x), \widetilde{u}^{-,-}(x), \widetilde{v}^{-,-}(x)) \\
&= H^{-,-}(X^{-,-}(\widehat{B}^{-,-}(x)), \widetilde{P}^{-,-}(x)) = 0,
\end{aligned}$$

hence, $W_0(\cdot)$ defined in (68) is a solution of Isaac's Equation (65).

(2). Since the value function $W_0(\cdot)$ in (68) is of class C^1 then, in order to prove the optimal of the pair of feedback strategies in (69), we use the well-known *Elementary Verifica-*

tion Theorem [1,12,15] according to which, a sufficient optimal condition for the admissible feedback strategies $(\tilde{U}(\cdot), \tilde{V}(\cdot))$ is to check the following differential inequalities:

$$\begin{aligned} \inf_{u \in U, \bar{v} \in \tilde{V}(x)} [DW_0(x)f(x, u, \bar{v}) + f_0(x, u, \bar{v})] &\geq 0, \\ \sup_{\bar{u} \in \tilde{U}(x), v \in V} [DW_0(x)f(x, \bar{u}, v) + f_0(x, \bar{u}, v)] &\leq 0. \end{aligned} \quad (70)$$

First, if $x \in Y_0^{+,-}$ then, it follows from (5), (68) and (69) that:

$$\begin{aligned} DW_0^{+,-}(x) &= (-x_3, x_3, -x_1 + x_2 + m_2x_3 - m_1x_3) \\ f(x, u, \bar{v}) &= (m_1, m_2 - c_2ux_1, -1), f_0(x, u, \bar{v}) = x_2 - (1 - u)x_1 \\ f(x, \bar{u}, v) &= (m_1 - c_1vx_2, m_2, -1), f_0(x, \bar{u}, v) = (1 - v)x_2 - x_1 \\ \bar{u} &= \tilde{u}^{+,-}(x) = 0, \bar{v} = \tilde{v}^{+,-}(x) = 0, \end{aligned} \quad (71)$$

and, therefore:

$$\inf_{u \in U, \bar{v} \in \tilde{V}^{+,-}(x)} [DW_0^{+,-}(x)f(x, u, \bar{v}) + f_0(x, u, \bar{v})] = \inf_{u \in U} [(1 - c_2x_3)ux_1],$$

since $c_1 > c_2$ and $x_3 \in [0, \frac{1}{c_1})$ we deduce that:

$$\inf_{u \in U} [(1 - c_2x_3)ux_1] = 0,$$

for the second inequality, it follows from (71) that:

$$\sup_{\bar{u} \in \tilde{U}^{+,-}(x), v \in V} [DW_0^{+,-}(x)f(x, \bar{u}, v) + f_0(x, \bar{u}, v)] = \sup_{v \in V} [(-1 + c_1x_3)x_2v] = 0.$$

Next, if $x \in Y_0^{-,-}$ we use the same type of computation and arguments as in previous case; thus, it follows from (68) and (69) that:

$$\begin{aligned} DW_0^{-,-}(x) &= (-x_3, \frac{1}{2}c_1x_3^2, -x_1 + c_1x_2x_3 - m_1x_3 + \frac{1}{2}c_1m_2x_3^2), \\ f(x, u, \bar{v}) &= (m_1 - c_1x_2, m_2 - c_2ux_1, -1), f_0(x, u, \bar{v}) = (u - 1)x_1, \\ f(x, \bar{u}, v) &= (m_1 - c_1vx_2, m_2, -1), f_0(x, \bar{u}, v) = (1 - v)x_2 - x_1, \\ \bar{u} &= \tilde{u}^{-,-}(x) = 0, \bar{v} = \tilde{v}^{-,-}(x) = 1, \end{aligned} \quad (72)$$

and we also find:

$$\inf_{u \in U, \bar{v} \in \tilde{V}^{-,-}(x)} [DW_0^{-,-}(x)f(x, u, \bar{v}) + f_0(x, u, \bar{v})] = \inf_{u \in U} \left[\left(1 - \frac{1}{2}c_1c_2x_3^2\right)x_1u \right],$$

from here, we can extract two cases:

Case 1 : If $x_3 \in \left(\frac{1}{c_1}, \frac{1}{c_1}\sqrt{\frac{2c_1}{c_2} - 1}\right)$ then, $1 - \frac{1}{2}c_1c_2x_3^2 > 0$ hence:

$$\inf_{u \in U} \left[\left(1 - \frac{1}{2}c_1c_2x_3^2\right)x_1u \right] = 0.$$

Case 2 : If $x_3 \in \left(\frac{1}{c_1}, -t_1(s_1, s_2)\right)$, $s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1}\left[1 + \frac{m_1}{2m_2}\right]\right)$, $s_2 \in (\tilde{s}_2, \tilde{\tilde{s}}_2)$ then, from (56) we obtain, $x_3 < -t_1(s_1, s_2) < \frac{1}{c_1}\sqrt{\frac{2c_1}{c_2} - 1}$ and the rest of the proof is conducted in the same way as in the previous case; therefore, the first inequality in (70) is verified in both cases.

For the second inequality, one has the following:

$$\sup_{\bar{u} \in \tilde{U}^{-,-}(x), v \in V} [DW_0^{-,-}(x)f(x, \bar{u}, v) + f_0(x, \bar{u}, v)] = \sup_{v \in V} [(1 - c_1x_3)x_2(1 - v)] = 0,$$

which proves inequalities (70) and hence the optimality of the admissible feedback strategies $(\tilde{U}(\cdot), \tilde{V}(\cdot))$ holds. \square

6. Conclusions

Finally, we are now in a position to derive several key findings, of which the following are noteworthy:

1. Our analysis builds upon Isaacs' foundational framework for the war game of attrition and attack, thereby enhancing the understanding of the strategic interactions between nations. We demonstrate that the extremities of the maximal interval of trajectories provide a more nuanced perspective than previously articulated, revealing multiple pathways for conflict resolution that extend beyond Isaacs' original single-path analysis.
2. The introduction of newly extended trajectories substantially modifies the dynamics of the attrition model. In contrast to Isaacs' initial considerations, our findings suggest that nations can employ strategies designed to deplete the opponent's resources gradually, thereby prolonging the duration of the conflict. This shift underscores the necessity of considering a range of strategic responses in extended engagements.
3. Our results underscore that specific selections regarding the extremities of trajectories, particularly those articulated in equations (40), can yield fixed durations of warfare. This insight highlights the critical importance of strategic decision-making in shaping conflict outcomes, whereby the absence of attrition may arise under certain conditions, ultimately influencing the war's trajectory.
4. The analysis reveals significant differences in how nations, represented by \mathbb{U} and \mathbb{V} , conduct their military engagements. Our findings indicate that while one nation may encounter diminishing returns in strength, the other can achieve a steady increase in power. This asymmetry suggests that strategic advantages may fluctuate over time, favoring the nation that effectively manages attrition.
5. We demonstrate that while nation \mathbb{V} is capable of sustaining prolonged military operations with a gradual increase in strength, nation \mathbb{U} faces a significant decline in resources, ultimately resulting in its disadvantage. This asymmetry suggests that the balance of power progressively shifts in favor of \mathbb{V} , with substantial implications for optimal strategies and conflict outcomes. By contrasting our findings with Isaac's original analyses, we illuminate previously unexplored possibilities within the structure of the maximal interval of trajectories and their influence on the dynamics of warfare. The conclusions provide a nuanced perspective on the impact of strategic choices on attrition and warfare duration, indicating the likelihood of \mathbb{V} 's eventual triumph over \mathbb{U} .

However, working heuristically, Isaacs in [1] tries to identify certain geometric concepts such as, the dispersal line, equivocal line and singular surface, . . . , etc. Unfortunately, the significance of optimality was not specified. To address this aspect, in works like [3,9,12,17], optimality is examined through the saddle point condition for the cost function $\mathcal{C}(\cdot, \cdot)$, in the sense that:

$$\mathcal{C}(\tilde{U}(\cdot), V(\cdot)) \leq \mathcal{C}(\tilde{U}(\cdot), \tilde{V}(\cdot)) \leq \mathcal{C}(U(\cdot), \tilde{V}(\cdot)), \forall (U(\cdot), V(\cdot)) \in \mathcal{P}. \quad (73)$$

In relation to our approach, the optimality of a pair of admissible feedback strategies $(\tilde{U}(\cdot), \tilde{V}(\cdot))$ is confirmed through the verification of the weaker conditions in (70) which are easier to verify, and much more efficient because do not require the presence of all pairs of admissible strategies, $(\tilde{U}(\cdot), V(\cdot))$ and $(U(\cdot), \tilde{V}(\cdot))$ which in return, are necessary when checking the saddle point optimality condition in (73).

In summary, the current study encompasses contributions from the authors in the following directions:

1. The use of some recent concepts and results from *Non-Smooth Analysis* and relevant applications in the differential games theory, as well as employing the synthesis of

the very recent theory in [14–16] regarding the rigorous approach and constructive of differential game problems;

2. The identification of a pair of feedback strategies, as well as the corresponding complete solution and the rigorous demonstration of its optimality;
3. The development of the implementation with *MATLAB 2018-software*, has traced the evolutions of the state's constraints considered in the problem. The results found show that, Dynamic Programming is the most effective tool for the complete resolution of concrete problems and provides accurate results.

Author Contributions: Conceptualization, B.S. and B.T.; methodology, B.S. and B.T.; software, B.S. and B.D.; validation, B.S., B.T. and B.D.; formal analysis, B.S. and B.D.; writing—original draft preparation, B.S. and B.T.; writing—review and editing, B.S., B.T. and B.D.; supervision, B.T. and B.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data presented in the paper are available upon request.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

1. Isaacs, R. *Differential Games*; Wiley: New York, NY, USA, 1965.
2. Bardi, M.; Capuzzo Dolcetta, I. *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*; Birkhuser: Berlin, Germany, 1997.
3. Başar, T.; Olsder, G. *Dynamic Noncooperative Game Theory*, 2nd ed.; Academic Press: London, UK, 1995.
4. Blaquière, A.; Leitmann, G. *Jeux Quantitatifs*; Gauthier-Villars: Paris, France, 1969.
5. Chigir, S.A. The game problem on the Dolichobrachistochrone. *J. Appl. Math. Mech.* **1976**, *40*, 1003–1013. [\[CrossRef\]](#)
6. Kamneva, L.V.; Patsko, V.S.; Turova, V.L. Analytical and numerical study of the Dolichobrachistochrone problem. In Proceedings of the Analysis and Control of Deterministic and Stochastic Evolution Equation, Bressanone-Brixen, Italy, 3–7 July 2000.
7. Lidov, M.L. On a differential game problem. *Avtomat Telemekhan* **1971**, *4*, 173–175.
8. Patsko, V.S.; Turova, V.L. Level sets of the value function in differential games with the Homicidal chauffeur dynamics. *Int. Games Theory Rev.* **2001**, *3*, 67–112. [\[CrossRef\]](#)
9. Berkovitz, L.D. *Optimal Control Theory*; Springer: New York, NY, USA, 1974.
10. Breakwell, J.V. Zero-sum differential games with terminal payoff. *Differ. Games Appl.* **1977**, *3*, 70–95.
11. Krassovskii, N.N.; Subbotin, A.I. *Positional Differential Games*; Springer: New York, NY, USA, 1988.
12. Subbotin, A.I. Generalization of the main equation of differential games. *J. Opt. Theory Appl.* **1984**, *43*, 103–134. [\[CrossRef\]](#)
13. Crandall, M.G.; Evans, L.C.; Lions, P.L. Some properties of viscosity solutions of Hamilton–Jacobi–Bellman equations. *Trans. AMS* **1984**, *282*, 487–502. [\[CrossRef\]](#)
14. Mirică, Ș. Verification theorems for optimal feedback strategies in differential games. *Int. Game Theory Rev.* **2003**, *5*, 167–189. [\[CrossRef\]](#)
15. Mirică, Ș. *Constructive Dynamic Programming in Optimal Control Autonomous Problems*; Editura Academiei Române: Bucharest, Romania, 2004.
16. Mirică, Ș. User's Guide on Dynamic Programming for autonomous differential games and optimal control problems. *Rev. Roumaine Math. Pures Appl.* **2004**, *49*, 501–529.
17. Elliot, R.J. *Viscosity Solutions and Optimal Control*; Pitman Research Notes in Mathematics; Longman: Harlow, UK, 1987.
18. Mirică, Ș.; Bouremani, T. On the solution of the homicidal chauffeur differential game. *Math. Rep.* **2006**, *8*, 53–81.
19. Bouremani, T.; Mirică, Ș. On the solution of a simple differential game with a singular focal line. *Bull. Math. Soc. Sci. Math. Roum.* **2006**, *49*, 113–139.
20. Bouremani, T.; Slimani, Y. Study of a warfare differential game via Dynamic Programming approach. *Dyn. Games Appl.* **2024**, *14*, 733–750. [\[CrossRef\]](#)
21. Chen, X.; Qiu, J. Differential Game for a Class of Warfare Dynamic Systems with Reinforcement Based on Lanchester Equation. *Abstr. Appl. Anal.* **2014**, *2014*, 1–8. [\[CrossRef\]](#)
22. Aubin, J.P.; Cellina, A. *Differential Inclusions*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1984.
23. Cesari, L. *Optimization-Theory and Applications*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1983.
24. Mirică, Ș. Hamilton–Jacobi equations on possibly non-symplectic differentiable manifolds. *Bull. Math. Soc. Sci. Math. Roumanie* **1998**, *41*, 23–69.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.