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Article

On Hurwicz Preferences in Psychological Games

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Abstract: The literature on *strategic ambiguity* in classical games provides generalized notions of equilibrium in which each player best responds to *ambiguous* or *imprecise* beliefs about his opponents' strategic choices. In a recent paper, strategic ambiguity has been extended to *psychological games*, by taking into account ambiguous *hierarchies of beliefs* and max–min preferences. Given that this kind of preference seems too restrictive as a general method to evaluate decisions, in this paper we extend the analysis by taking into account α -max–min preferences in which decisions are evaluated by a convex combination of the worst-case (with weight α) and the best-case (with weight $1 - \alpha$) scenarios. We define the α -max–min psychological Nash equilibrium; an illustrative example shows that the set of equilibria is affected by the parameter α and the larger the ambiguity, the greater the effect. We also provide a result of stability of the equilibria with respect to perturbations that involve the attitudes toward ambiguity, the structure of ambiguity, and the payoff functions: converging sequences of equilibria of perturbed games converge to equilibria of the unperturbed game as the perturbation vanishes. Surprisingly, a final example shows that the existence of equilibria is not guaranteed for every value of α .

Keywords: psychological games; ambiguous beliefs; α -MEU; equilibrium existence



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1. Introduction

It is well-known that the Nash equilibrium concept for strategic games prescribes the following: (i) each player chooses his best strategy in response to the beliefs he has about his opponents' strategic choices; (ii) each player's beliefs are correct; that is, each player believes with probability 1 that opponents will follow their equilibrium strategies. The evidence arising from decision theory tells us that beliefs cannot always be assumed to be correct. The literature that focuses on the issue of *strategic ambiguity* in classical strategic form games provides generalized notions of equilibrium in which each player best responds to *ambiguous* or *imprecise* beliefs about his opponents' strategic choices, i.e., beliefs may take the form of a capacity or a set of probability distributions (see [1–6] and references therein). There might be many sources of *strategic ambiguity* in a game: for example, Lehrer [3] focuses on the case in which players do not have precise knowledge of the mixed strategy chosen by each of the other players but rather know only the probability of some subsets of pure strategies, not being aware of the precise subdivision of probabilities within those subsets. In [7], the study of strategic ambiguity has been extended to *psychological games* by looking at ambiguous or imprecise *hierarchies of beliefs*. Psychological games provide a generalization of classical games, aiming to explicitly take into account the emotions, opinions, and intentions of the decision-makers in the strategic interaction¹. This class of games is characterized by the assumption that each player's payoff depends on his hierarchy of beliefs, i.e., it depends

not only on what every player does but also on what he thinks every player believes, on what he thinks every player believes the others believe, and so on. The main solution for psychological games is presented in Geanakoplos et al. [14] and is based on the idea that the entire hierarchy of beliefs of each player must be correct in equilibrium.

Since beliefs about opponents' strategic choices can be regarded as first-order beliefs, the literature on strategic ambiguity substantially looks at games in which first-order beliefs are ambiguous. Ref. [7], instead, looks at ambiguity regarding the entire hierarchy of beliefs as, for instance, partial knowledge may appear directly in the second-order (or higher-order) beliefs or strategic ambiguity produces ambiguous higher-order beliefs as a natural consequence. Therefore, the function that maps strategic profiles to the *correct* hierarchies of beliefs, which is used in the classical definition of psychological Nash equilibria, is therein replaced by a set-valued map (called *ambiguous belief correspondence*), which maps strategic profiles to the subsets of those hierarchies of beliefs that players perceive to be consistent with the corresponding strategy profile. In the corresponding equilibrium notion presented in [7], players are assumed to be completely pessimistic as they are endowed with *max–min* preferences (also called *MEU* preferences; see [15]); each player maximizes (with respect to his own strategy) the minimum expected utility computed along the graph of the ambiguous belief correspondence whose values, in turn, depend on the entire strategy profile.

The max–min approach turns out to be analytically convenient; furthermore, it has a clear axiomatic foundation. Nevertheless, it seems to be too restrictive as a general approach because only the “worst-case scenario” is relevant for the evaluation of a decision so the analysis is limited to an extreme form of pessimism. The restrictiveness of the MEU model can be naturally overcome by considering the so-called α -max–min preferences (also called α -MEU or *Hurwicz preferences*), first introduced in [16]. In this model, decisions are evaluated by a convex combination of the worst-case (with weight α) and the best-case (with weight $1 - \alpha$) scenarios. This type of preference was widely analyzed and applied in several different settings in order to include a larger spectrum of ambiguous attitudes (see [17–20] to quote a few).

The literature shows that the generality of the model is affected once preferences are restricted only to the max–min approach; in fact, the behavior of agents with an intermediate attitude toward ambiguity cannot be fully explained by max–min preferences². Hurwicz preferences, instead, are much more general and exploitable when dealing with model uncertainty, as they allow for analysis of the decisions of agents with a larger variety of personal characteristics. This perspective is key when looking at several economic and financial problems, such as establishing the design of optimal insurance contracts or investigating duopolistic competitions (see, for example, [22,23]). In fact, these papers show that the characterization of equilibria explicitly depends on the degree of pessimism/optimism α . Finally, other theoretical papers take into account the Hurwicz preferences, for instance [24,25], where the problems of information processing and awareness, respectively, have been addressed.

In this paper, we extend the analysis of psychological games under ambiguity to α -max–min preferences and provide the notion of α -MEU psychological Nash equilibrium (α -PNE) for situations in which players have Hurwicz preferences. The weights α that characterize the attitudes of the players toward ambiguity turn out to be key to understanding how equilibria change according to the players' degree of pessimism/optimism. We present an illustrative example, showing not only that the set of equilibria depends on the parameter α but also that differences are emphasized by the amount of ambiguity in the game: the *larger* the ambiguity, the greater the differences. The example highlights another relevant feature: equilibria corresponding to a given value of α cannot always be approached by sequences of equilibria of games in which the parameter α is slightly perturbed, meaning that equilibria are unstable with respect to perturbations on the degree of pessimism/optimism. From the mathematical point of view, this implies a lack of *lower semi-continuity* of psychological Nash equilibria under the Hurwicz preferences. The failure of this property is not surprising since

the lack of lower semi-continuity of the equilibrium correspondence is a common feature in most of the game models. We show, instead, that the α -PNE correspondence satisfies upper semi-continuity-like stability: converging sequences of equilibria of perturbed games converge to equilibria of the unperturbed game as the perturbation *vanishes*. The issue of the upper semi-continuity properties of equilibria has been largely investigated in the literature for classical games (see for instance [26–28] and references therein) and is a key property to build refinements of equilibria based on stability with respect to *trembles*. In this paper, we obtain the stability of equilibria under general perturbations that involve, simultaneously, the attitudes toward ambiguity (that is, the parameters α), the structure of ambiguity, and the payoff functions. In particular, this result allows for selection criteria (for PNE under ambiguity) based on stability properties with respect to perturbations on the weights α .

The most surprising feature of α -PNE is, however, a negative result. Although for psychological Nash equilibria and psychological Nash equilibria under max–min preferences an existence result was obtained under standard assumptions, in this paper we provide a counterexample in which a game has no α -PNE. This negative result comes from the fact that the best reply correspondence of the summary utility function (used to obtain equilibrium existence) does not have convex images and therefore fixed points, in general.

The paper is organized as follows: Section 2 defines the game and the equilibrium concept. Section 3 presents the illustrative example while Section 4 is dedicated to the upper-semi-continuity property of equilibria. In Section 5, the issue of the lack of existence of equilibria is studied. All the proofs are relegated to Appendix A. Appendix B, instead, is devoted to complementary results concerning the problem of the (non)-existence of equilibria in games with Hurwicz preferences and non-psychological payoffs.

2. Model and Equilibria

We consider a finite set of players $I = \{1, \dots, n\}$, and, for each player i , we denote with $A_i = \{a_i^1, \dots, a_i^{k(i)}\}$ the (finite) pure strategy set of player i . As usual, the set of strategic profiles A is the Cartesian product of the strategy sets of each player, which is $A = A_1 \times \dots \times A_n = \prod_{i \in I} A_i$, and $A_{-i} = A_1 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n = \prod_{j \neq i} A_j$. Let Σ_i be the set of mixed strategies of player i , where each mixed strategy $\sigma_i \in \Sigma_i$ is a nonnegative vector $\sigma_i = (\sigma_i(a_i))_{a_i \in A_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$. Denote also with $\Sigma = \prod_{i \in I} \Sigma_i$ and with $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$. We use (σ_i, σ_{-i}) with $\sigma_i \in \Sigma_i$ and $\sigma_{-i} \in \Sigma_{-i}$ to represent $\sigma \in \Sigma$.

2.1. Hierarchies of Beliefs

The belief structure is constructed following [14]. Recall that, for any topological space S , $\Delta(S)$ denotes the set of Borel probability measures on S . For every player i and for every $k \in \mathbb{N}$, $k > 1$, the k -th order beliefs set is defined recursively as follows:

$$\begin{aligned} B_i^1 &= \Delta(\Sigma_{-i}), \\ B_i^2 &= \Delta(\Sigma_{-i} \times B_{-i}^1), \\ &\vdots \\ B_i^k &:= \Delta(\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}), \\ &\vdots \end{aligned}$$

where $B_{-i}^k := \prod_{j \neq i} B_j^k$. The set of all *hierarchies of beliefs* of player i is $B_i = \prod_{k=1}^{\infty} B_i^k$. Note that for every k , B_i^k is compact and can be metrized as a separable metric space. Consequently, since B_i is a countable product of separable and compact metric spaces, it is also a separable and compact metric space³.

We will restrict the attention to the subset of *collectively coherent beliefs* $\bar{B}_i \subset B_i$, i.e., the compact set of beliefs of player i in which he is sure that it is common knowledge that beliefs are coherent. Precisely, a belief $b_i = (b_i^1, b_i^2, \dots) \in B_i$ is said to be *coherent* if, for every $k \in \mathbb{N}$, the marginal probability of b_i^{k+1} on $\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}$ coincides with b_i^k , which is

$$\text{marg}(b_i^{k+1}, \Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}) = b_i^k.$$

You can find the construction of the set of collectively coherent beliefs in [14] and the proof of its compactness in [7]. Throughout the remainder of the paper, with an abuse of notation, we will denote with $\bar{B}_i$ the set of collectively coherent beliefs or any of its compact subsets.

As in [7], we allow for ambiguity in the beliefs; therefore, beliefs are compact subsets $K_i \subseteq \bar{B}_i$. We denote with $\mathcal{K}_i$ the set of all compact subsets of $\bar{B}_i$. This choice allows consideration of the ambiguity players encounter during the game due to uncertainty about other players' actions and beliefs: the agent does not have a precise belief b_i but knows that the belief can be any $b_i \in K_i$. If K_i is a singleton, then the belief is not ambiguous, leading the theory back to the standard case.

2.2. Game and Equilibria

Following the model in [14], each agent i is endowed with a utility function of the form

$$u_i : \bar{B}_i \times \Sigma \rightarrow \mathbb{R}, \quad (1)$$

depending not only on the mixed strategy profile but also on the agent's beliefs: $u_i(b_i, \sigma)$ represents the payoff to player i if he believed b_i and the strategy profile σ is actually played. Indeed, fixing b_i , $u_i(b_i, \cdot)$ can be (but not necessarily) the classical expected utility function as it is assumed in [14]. As agents face set-valued beliefs $K_i \in \mathcal{K}_i$, they have a set-valued payoff $\{u_i(b_i, \sigma)\}_{b_i \in K_i}$ for every given ambiguous belief $K_i \in \mathcal{K}_i$ and strategy profile $\sigma \in \Sigma$. There are several ways in which the agents' ambiguity might be solved depending on the agents' attitudes toward ambiguity. In [7], the case was considered where players are ambiguity-averse, modeling the utility functions as max–min preferences. To encompass a broad spectrum of ambiguity attitudes, this paper focuses on the so-called α -max–min preferences, which allow us to range from an ambiguity-seeking attitude (as $\alpha = 0$) to an ambiguity aversion attitude (as $\alpha = 1$). In this framework, each agent i has a utility function of the following form: $U_i^\alpha : \mathcal{K}_i \times \Sigma \rightarrow \mathbb{R}$ defined, for $\alpha_i \in [0, 1]$, by

$$U_i^\alpha(K_i, \sigma) = \alpha_i \left[\inf_{b_i \in K_i} u_i(b_i, \sigma) \right] + (1 - \alpha_i) \left[\sup_{b_i \in K_i} u_i(b_i, \sigma) \right] \quad \forall (K_i, \sigma) \in \mathcal{K}_i \times \Sigma, \quad (2)$$

where α denotes the vector $\alpha = (\alpha_1, \dots, \alpha_n) \in [0, 1]^n$. Now, it is possible to define the game.

Definition 1. An α -MEU normal form psychological game is defined by

$$G^\alpha = \{A_1, \dots, A_n, U_1^\alpha, \dots, U_n^\alpha\}$$

where the utility functions U_i^α are defined as in Formula (2) for every $i \in N$.

In the models of strategic ambiguity where players have partial knowledge of the strategies played by their opponents, players' beliefs depend on the actual strategy and take the form of set-valued maps (correspondences) from the set of strategic profiles to the set of probability distributions over opponents' strategies (see [3,5,29]). In [7], this approach was generalized to hierarchies of beliefs: agent i is endowed with a set-valued

map $\gamma_i : \Sigma \rightsquigarrow \bar{B}_i$ (called *ambiguous belief correspondence* of player i), where each image $\gamma_i(\sigma)$ is a non-empty and compact set, i.e.,

$$\emptyset \neq \gamma_i(\sigma) \in \mathcal{K}_i \quad \forall \sigma \in \Sigma.$$

Each subset $\gamma_i(\sigma) \subseteq \bar{B}_i$ provides the set of hierarchies of beliefs that player i perceives to be consistent given the strategy profile σ . The set-valued maps γ_i are exogenous and have different structures depending on the specific problem; therefore, they can be considered as parameters of the game.

Remark 1. It is well-known that uncertainty and partial ignorance can be modeled with several different tools beyond sets of probability measures, such as the Choquet capacities and belief functions (see [2,17,30–35]). Due to its generality and the exogeneity of the set-valued maps γ_i , our model can embrace several specific cases. For example, one can consider γ_i to be defined as the core of a belief function associated with player i , which is a non-empty and compact set. In the special case of precise probability, the core, and, consequently, the image set of γ_i , reduces to a single probability measure, giving back the non-ambiguous case. In [7], a link with partially specified probabilities [3] was also constructed.

In this paper, we follow the approach in [7]:

Definition 2. An α -MEU psychological Nash equilibrium (henceforth, α -PNE) of the game G^α with belief correspondences $\gamma = (\gamma_1, \dots, \gamma_n)$ is a pair (K^*, σ^*) , where $K^* = (K_1^*, \dots, K_n^*)$ with $K_i^* \subseteq \bar{B}_i$ and $\sigma^* \in \Sigma$ such that, for every player i :

- (i) $K_i^* = \gamma_i(\sigma^*)$;
- (ii) $U_i^\alpha(K_i^*, \sigma^*) \geq U_i^\alpha(K_i^*, (\sigma_i, \sigma_{-i}^*))$ for every $\sigma_i \in \Sigma_i$.

In this case, we can also say that $(\gamma(\sigma^*), \sigma^*)$ is an α -MEU psychological Nash equilibrium.

Remark 2. The definition of α -PNE captures, in a natural way, the main features of the classical equilibrium notions since condition (ii) requires that the equilibrium strategy of each player is optimal given his beliefs and condition (i) requires that beliefs must satisfy a consistency condition with the equilibrium strategy profile that is characterized by the belief correspondences γ_i . However, the nature of this latter consistency condition differentiates it from the concept of psychological Nash equilibrium, which, in turn, inherits from the classical Nash equilibrium the requirement that beliefs must be correct in equilibrium. The α -PNE concept is based on a different perspective that is similar to the one in the definition of self-confirming equilibrium (SCE) in⁴ [5].

As clearly explained by the authors⁵: “... in a SCE, agents best respond to confirmed probabilistic beliefs. Confirmed means that their beliefs are consistent with the evidence they can collect, given the strategies they adopt. ... The key difference between SCE and Nash equilibrium is that, in an SCE, agents may have incorrect beliefs because many possible underlying distributions are consistent with the empirical frequencies they observe.” From the mathematical point of view, in a SCE, beliefs are parametrized by feedback functions that give rise to information about opponents’ strategic profiles, and by a prior belief in these strategic profiles. Such feedback functions and prior beliefs together provide the beliefs that the players perceive to be consistent with the strategy profile, for every profile. In practice, beliefs are represented by a set-valued map (that depends on the strategy profile) that is called identification correspondence. This correspondence can be regarded as a particular case of the belief correspondence used in the definition of α -PNE when we look only at first-order beliefs. Moreover, a version of SCE under uncertainty, namely the MSCE concept (in mixed strategies), can be regarded as a α -PNE (for $\alpha = 1$) in the case in which the opponents’ strategies are replaced by first-order (ambiguous) beliefs in the expected utility of each player.

Naturally, there are differences between α -PNE and SCE. First, in the former, higher-order beliefs enter explicitly the utility function in an arbitrary way, while, in the latter, just the first-order beliefs enter the classical expected utility function in place of opponents’ strategies. Most importantly, the SCE involves a richer structure of beliefs with respect to α -PNE as, in an SCE,

it fully captures two different scenarios: the first one is a repeated game in which there are no intertemporal strategic links between the plays, while the second is the (so-called) large population scenario in which there is a large society of individuals who recurrently play a given game. In the α -PNE, beliefs correspondences represent a generic mathematical tool that can generalize different models like the Identification Correspondence quoted above or like some imprecise perturbations of correct beliefs, as illustrated in the example in Section 3 below, in order to run a robustness analysis on the unperturbed equilibria. Finally, the nature of the definition of equilibrium differs between α -PNE and SCE as the latter always has the classical Nash equilibrium concept as a refinement while the relation between α -PNE and the classical psychological Nash equilibrium depends on the specific model taken into account.

2.3. Summary Utility Functions

Similar to [14], α -PNE has a characterization as Nash equilibria. Let $w_i^\alpha : \Sigma \times \Sigma \rightarrow \mathbb{R}$ be the summary utility function defined by the following:

$$w_i^\alpha(\sigma, \tau) = U_i^\alpha(\gamma_i(\sigma), \tau) = \alpha_i \left[\inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \right] + (1 - \alpha_i) \left[\sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \right] \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma. \quad (3)$$

Then, the following immediately follows from the definition:

Lemma 1. *The profile $(\gamma(\sigma^*), \sigma^*)$ is an α -MEU psychological Nash equilibrium if and only if, for every player i ,*

$$w_i^\alpha(\sigma^*, (\sigma_i^*, \sigma_{-i}^*)) \geq w_i^\alpha(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i. \quad (4)$$

Remark 3. *In [14], the equilibrium beliefs of each agent i are described by the correct beliefs function $\beta_i : \Sigma \rightarrow \overline{B}_i$ which, for every $\sigma \in \Sigma$, specifies the unique hierarchy of beliefs of player i that is correct, given σ . Now, if we replace γ_i with β_i , in Definition 2, we retrieve the definition of classical psychological Nash equilibria. On the other hand, if we replace γ_i with β_i in (3), we obtain the original summary utility function defined in [14].*

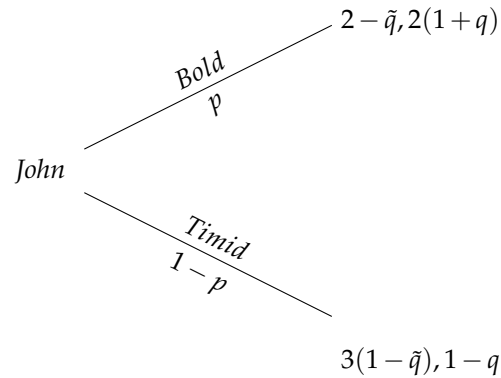
3. An Illustrative Example

In this section, we present an example of a psychological game under ambiguity in which players have Hurwicz preferences. The goal is twofold: on the one hand, we aim to put definitions to work and show how to find psychological Nash equilibria under ambiguity in simple models. On the other hand, the example highlights in which way the equilibria may be sensitive to variations in the amount of the structure of ambiguity in the game and the attitudes of the players toward ambiguity. More precisely, we consider a specific form of ambiguity: players' beliefs are provided by a perturbation of the correct belief function that takes the form of a ball of radius ε around the correct belief. This approach resembles the *contamination model* approach and allows us to analyze the sensitivity of α -PNE with respect to the unique parameter ε . Moreover, as the attitude toward ambiguity of each player i is parametrized by the corresponding value of α_i , we study the sensitivity of equilibria with respect to α_i .

The game considered in the example is the *bravery game* that was first analyzed in the framework of standard psychological games by [14]. In [7], it was shown that allowing for ambiguous hierarchies of beliefs may significantly affect the set of equilibria when players are endowed with max–min preferences. In this work, we study the game with respect to the double parametrization ε and α_i .

Example 1. *The game is described as follows: Player 1 (John) has to publicly make a decision, and he is concerned about what Player 2 (Anne) will think about him. He can either be bold, exposing himself to the possibility of danger, or he can opt for a timid decision; therefore, John's pure strategy set is $A_1 = \{\text{Bold}, \text{Timid}\}$. Anne is inactive during the whole interaction but her beliefs about John have an impact on John's behavior; indeed, his payoff depends not only on what he does but also*

on what he believes Anne thinks he will do. Suppose that John chooses Bold with probability p and Timid with probability $1 - p$. We consider the case in which John cares only about the expectation $\tilde{q}$ of his belief about the expectation q of Anne's first-order belief. Moreover, John would rather be timid, unless he thinks Anne is expecting him to be bold, in which case he prefers not to disappoint her. Anne prefers to think of her friend as bold, and it is better for her if he opts for the bold decision. The game and payoffs are described below:



Equilibria without Ambiguity

Since Anne is a non-active player, the mixed strategy profile is given only by John's mixed strategy p . With the abuse of notation, the correct belief functions are defined as follows: $\beta_2(p) = p$ tells that the expectation of Anne's first-order correct beliefs about John's strategy (p) must be equal to p ; $\beta_1(p) = p$ shows that the expectation of John's correct second-order beliefs about Anne's expectation of the correct first-order belief about John's strategy (p) must be equal to p as well.

The expected utility of John takes the following form:

$$u_1(\tilde{q}, p) = p(2 - \tilde{q}) + 3(1 - p)(1 - \tilde{q}) = p(2\tilde{q} - 1) + 3(1 - \tilde{q}).$$

In the case of non-ambiguous beliefs, the game has three psychological equilibria, as shown in [14]:

- $p = 1 = \tilde{q} = q$: John chooses to be Bold;
- $p = 0 = \tilde{q} = q$: John chooses to be Timid;
- $p = 1/2 = \tilde{q} = q$: John randomizes with probability $p = 1/2$.

The Game in Case of Ambiguity

Now, we assume that John has ambiguous beliefs; in particular, John's belief is represented by the map $\gamma_1^\varepsilon(p) = [p - \varepsilon, p + \varepsilon] \cap [0, 1]$ with $0 < \varepsilon \leq 1$. We look at the equilibria of the game in case players (John, in this case) have α -MEU preferences. In particular, we show in which way the different attitudes toward ambiguity affect the equilibrium behavior. In order to compute John's summary utility function, we first compute, for every pair of John's mixed strategies (p, y) , the following:

$$\arg \min_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \left\{ \tilde{q}' \in [0, 1] \mid u_1(\tilde{q}', y) = \min_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) \right\},$$

$$\arg \max_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \left\{ \tilde{q}' \in [0, 1] \mid u_1(\tilde{q}', y) = \max_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) \right\}.$$

We have the following:

$$\arg \min_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \arg \min_{\tilde{q} \in [p - \varepsilon, p + \varepsilon] \cap [0, 1]} [\tilde{q}(2y - 3) + 3 - y] = \min\{p + \varepsilon, 1\}, \quad \forall y \in [0, 1].$$

Similarly,

$$\arg \max_{\tilde{q} \in \gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \arg \max_{\tilde{q} \in [p-\varepsilon, p+\varepsilon] \cap [0,1]} [\tilde{q}(2y-3) + 3-y] = \max\{p-\varepsilon, 0\}, \quad \forall y \in [0,1].$$

If $p^+ := \min\{p+\varepsilon, 1\}$ and $p^- := \max\{p-\varepsilon, 0\}$, for every pair of John's mixed strategies (p, y) and for every $\alpha \in [0, 1]$, we have the following:

$$\begin{aligned} w_1^\alpha(p, y) &= \alpha \left[\min_{\tilde{q} \in \gamma_1^\varepsilon(p)} \tilde{q}(2y-3) + 3-y \right] + (1-\alpha) \left[\max_{\tilde{q} \in \gamma_1^\varepsilon(p)} \tilde{q}(2y-3) + 3-y \right] = \\ &= \alpha[p^+(2y-3) + 3-y] + (1-\alpha)[p^-(2y-3) + 3-y] = \\ &= y[2\alpha(p^+ - p^-) + 2p^- - 1] - 3\alpha(p^+ - p^-) + 3(1-p^-). \end{aligned}$$

α -PNE

Recall that p gives α -PNE if and only if

$$w_1^\alpha(p, p) \geq w_1^\alpha(p, y) \quad \forall y \in [0, 1], \alpha \in [0, 1].$$

It is clear that equilibria depend on α and ε . Below, we provide a full characterization of all the (α -PNE) equilibria.

First, denote with

$$\tilde{p} = \frac{1}{2} + \varepsilon(1-2\alpha), \quad p^* = \frac{1}{2\alpha} - \varepsilon, \quad \hat{p} = \frac{1-2\alpha}{2-2\alpha} + \varepsilon.$$

It follows that

Lemma 2. Let $0 < \varepsilon \leq \frac{1}{2}$.

- (i) If $\varepsilon < 1/4$, then, for every $\alpha \in [0, 1]$, the α -PNE are: $p = 0$, $p = 1$, $p = \tilde{p}$;
- (ii) If $\varepsilon \geq 1/4$, then:

- for $\alpha \in \left[0, 1 - \frac{1}{4\varepsilon}\right]$, the α -PNE are: $p = 0$, $p = 1$, $p = \hat{p}$;
- for $\alpha \in \left[1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$, the α -PNE are: $p = 0$, $p = 1$, $p = \tilde{p}$;
- for $\alpha \in \left[\frac{1}{4\varepsilon}, 1\right]$, the α -PNE are: $p = 0$, $p = 1$, $p = p^*$;

where $\hat{p} = \tilde{p}$ if $\alpha = 1 - \frac{1}{4\varepsilon}$ and $\tilde{p} = p^*$ if $\alpha = \frac{1}{4\varepsilon}$.

Proof. See Appendix A. $\square$

Lemma 3. Let $\frac{1}{2} < \varepsilon \leq 1$. Then,

- for $\alpha \in \left[0, 1 - \frac{1}{2\varepsilon}\right]$, the unique α -PNE is: $p = 0$;
- for $\alpha \in \left[1 - \frac{1}{2\varepsilon}, \frac{1}{2}\right]$, the α -PNE are: $p = 0$, $p = 1$, $p = \hat{p}$;
- for $\alpha = \frac{1}{2}$, the α -PNE are: $p = 0$, $p = 1$ and every $p \in [1-\varepsilon, \varepsilon]$;
- for $\alpha \in \left[\frac{1}{2}, \frac{1}{2\varepsilon}\right]$, the α -PNE are $p = 0$, $p = 1$, $p = p^*$;
- for $\alpha \in \left[\frac{1}{2\varepsilon}, 1\right]$, the unique α -PNE is: $p = 1$;

where $\hat{p} = 1$ if $\alpha = 1 - \frac{1}{2\varepsilon}$ and $p^* = 0$ if $\alpha = \frac{1}{2\varepsilon}$.

Proof. See Appendix A. $\square$

We can summarize the results in the following table, which is filled with the values of the parameter α , ensuring the existence of the corresponding equilibrium, as follows (Table 1):

Table 1. Values of α for the corresponding equilibrium.

	$p = 0$	$p = 1$	$p = \frac{1}{2} + \varepsilon(1 - 2\alpha)$	$p = \frac{1}{2\alpha} - \varepsilon$	$p = \frac{1-2\alpha}{2-2\alpha} + \varepsilon$	$p \in [1 - \varepsilon, \varepsilon]$
$\varepsilon < \frac{1}{4}$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$\emptyset$	$\emptyset$	$\emptyset$
$\frac{1}{4} \leq \varepsilon \leq \frac{1}{2}$	$[0, 1]$	$[0, 1]$	$\left[1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$	$\left[\frac{1}{4\varepsilon}, 1\right]$	$\left[0, 1 - \frac{1}{4\varepsilon}\right]$	$\emptyset$
$\frac{1}{2} < \varepsilon \leq 1$	$\left[0, \frac{1}{2\varepsilon}\right]$	$\left[1 - \frac{1}{2\varepsilon}, 1\right]$	$\emptyset$	$\left[\frac{1}{2}, \frac{1}{2\varepsilon}\right]$	$\left[1 - \frac{1}{2\varepsilon}, \frac{1}{2}\right]$	$\left\{\frac{1}{2}\right\}$

Note that, for $\varepsilon = 0$, we obtain the three equilibria ($p = 1$, $p = 0$ and $p = \frac{1}{2}$) of the original game in [14], while for $\alpha = 1$, we obtain the same equilibria as the model with max–min preferences computed in [7]. Moreover, as ambiguity increases with ε , the set of equilibria in the two extreme cases $\alpha = 0$ and $\alpha = 1$ shrinks to a unique equilibrium for $\varepsilon > \frac{1}{2}$, but the two equilibria are different (i.e., $p = 0$ and $p = 1$ respectively). More generally, the table above shows that the difference among the different attitudes toward ambiguity becomes sharper as ε increases. In particular, the set of values of α that sustain a given equilibrium generally shrinks as ε converges to 1. There is one exception: for $\varepsilon > \frac{1}{2}$, the value $\alpha = \frac{1}{2}$ is a kind of singularity as it sustains the interval of equilibria $[1 - \varepsilon, \varepsilon]$. As a consequence, when $\alpha = \frac{1}{2}$, for $\varepsilon = 1$, all $p \in [0, 1]$ are α -PNE.

4. A Sensitivity Analysis

The example in the previous section shows some interesting features concerning the sensitivity of equilibria with respect to perturbations on the attitudes toward ambiguity. In particular, we notice that equilibria do not satisfy the *lower semi-continuity-like stability*⁶, i.e., an equilibrium, cannot always be *approached* by a sequence of equilibria of perturbed games if we consider a perturbation on the parameter α .

Example 2. Consider $\varepsilon = \frac{3}{4}$ and $\alpha = \frac{1}{2}$, we have that every $p \in [1 - \varepsilon, \varepsilon] = \left[\frac{1}{4}, \frac{3}{4}\right]$ is an α -PNE. In particular, pick $p = \frac{2}{5} \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Now, fixing $\varepsilon = \frac{3}{4}$, consider a sequence $\{\alpha_v\}_{v \in \mathbb{N}}$ such that $\alpha_v \rightarrow \frac{1}{2}$ as $v \rightarrow \infty$ with $\alpha_v \neq \alpha$ for every v . Since $\varepsilon = \frac{3}{4}$, the only α_v -PNE, for α_v sufficiently close to α are $p = 0$, $p = 1$, $p = \frac{1}{2\alpha_v} - \frac{3}{4}$ and $p = \frac{1-2\alpha_v}{2-2\alpha_v} + \frac{3}{4}$. It follows immediately that any converging sequence of α_v -PNE might converge only to $p = 0$, $p = 1$, $p = \frac{1}{2\frac{1}{2}} - \frac{3}{4} = \frac{1}{4}$ and $p = \frac{1-2\frac{1}{2}}{2-2\frac{1}{2}} + \frac{3}{4} = \frac{3}{4}$. Therefore, $p = \frac{2}{5}$ cannot be approached by any sequence of α_v -PNE.

Lack of lower semi-continuity-like stability is a common feature of equilibria in games. Example 3.4 in [7] shows that, in regard to max–min preferences (i.e., $\alpha = 1$), a psychological Nash equilibrium under ambiguity cannot always be *approached* by a sequence of equilibria of perturbed games if we consider perturbations on the parameter ε ⁷.

The previous example, in turn, shows that the set of equilibria satisfies upper semi-continuity-like stability either if we consider a perturbation on the parameter α or a perturbation on the parameter ε : converging sequences of equilibria of perturbed games converge to equilibria of unperturbed games as the perturbation *vanishes*. The issue of the upper semi-continuity properties of equilibria is a relevant topic in game theory and it has been largely investigated in the literature under many different assumptions and for different solution concepts (for instance, see [26–28] and references therein). Moreover, these properties are key to building refinements of equilibria based on stability, with respect to *trembles* on the strategies or payoffs. In [7], the upper semi-continuity property was investigated for equilibria in psychological games under ambiguity in case of max–min preferences; in particular, the main result therein shows in which way ambiguous belief should converge to correct beliefs so that sequences of psychological equilibria under perturbation converge to psychological equilibria of the unperturbed game. In this paper, we extend that result by looking at the stability with respect to the attitudes toward ambiguity parametrized by the weights α_i .

In order to state the stability problem in a clear way, let us first construct a sequence of perturbed games:

Definition 3. For every player i and for every $v \in \mathbb{N}$, let

- (a) $\{u_{i,v}\}_{v \in \mathbb{N}}$ be a sequence of functions with $u_{i,v} : \bar{B}_i \times \Sigma \rightarrow \mathbb{R}$;
- (b) $\{\gamma_{i,v}\}_{v \in \mathbb{N}}$ be a sequence of set-valued maps $\gamma_{i,v} : \Sigma \rightsquigarrow \bar{B}_i$;
- (c) $\{\alpha_v\}_{v \in \mathbb{N}}$ be a sequence with $\alpha_v = (\alpha_{1,v}, \dots, \alpha_{1,v}) \in [0, 1]^n$;
- (d) $\{U_{i,v}^{\alpha_v}\}_{v \in \mathbb{N}}$ be the sequence of functions $U_{i,v}^{\alpha_v} : \mathcal{X}_i \times \Sigma \rightarrow \mathbb{R}$ defined by the following:

$$U_{i,v}^{\alpha_v}(K_i, \sigma) = \alpha_{i,v} \left[\inf_{b_i \in K_i} u_{i,v}(b_i, \sigma) \right] + (1 - \alpha_{i,v}) \left[\sup_{b_i \in K_i} u_{i,v}(b_i, \sigma) \right] \quad \forall (K_i, \sigma) \in \mathcal{X}_i \times \Sigma.$$

Then the sequence $\{G_v^{\alpha_v}\}_{v \in \mathbb{N}}$, with $G_v^{\alpha_v} = \{A_1, \dots, A_n, U_{1,v}^{\alpha_v}, \dots, U_{n,v}^{\alpha_v}\}$ for every $v \in \mathbb{N}$, is a sequence of α -MEU psychological games.

Therefore, we have the following:

Problem 1. Find conditions under which the sequence $\{G_v^{\alpha_v}\}_{v \in \mathbb{N}}$ converges to the game G^α so that any converging sequence $\{\sigma_v^*\}_{v \in \mathbb{N}}$ of α_v -PNE of $G_v^{\alpha_v}$ has a limit σ^* that is an α -PNE of G^α .

In order to state and prove this limit result, we first recall the definitions of variational sequence convergences for functions and set-valued maps.

4.1. Preliminary Definitions

We refer mainly to reference [39] for the following definitions and results.

Definition 4. Let X be a topological space. Consider a sequence of functions⁸. $\{g_v\}_{v \in \mathbb{N}}$ with $g_v : X \subset \mathbb{R}^k \rightarrow \bar{\mathbb{R}}$ for every $v \in \mathbb{N}$ and a function $g : X \subset \mathbb{R}^k \rightarrow \bar{\mathbb{R}}$.

Then, the sequence $\{g_v\}_{v \in \mathbb{N}}$ sequentially converges (or continuously converges) to the function g if for every $x \in X$ and for every sequence $\{x_v\}_{v \in \mathbb{N}} \subset X$ converging to x in X we have the following:

$$g(x) = \lim_{v \rightarrow \infty} g_v(x_v) = \limsup_{v \rightarrow \infty} g_v(x_v) = \liminf_{v \rightarrow \infty} g_v(x_v). \quad (5)$$

The next definition is devoted to set-valued maps.

Definition 5. Let X and Y be metric spaces. Let $\{\Gamma_v\}_{v \in \mathbb{N}}$ be a sequence of set-valued maps with $\Gamma_v : X \rightsquigarrow Y$ for every $v \in \mathbb{N}$ and let $\Gamma : X \rightsquigarrow Y$ be a set-valued map. Let $S(y, \varepsilon)$ be the ball in Y with the center in y and radius ε and

$$\liminf_{v \rightarrow \infty} \Gamma_v(x_v) = \{y \in Y \mid \forall \varepsilon > 0, \exists \bar{v} \text{ s.t. for all } v \geq \bar{v}, S(y, \varepsilon) \cap \Gamma_v(x_v) \neq \emptyset\},$$

$$\limsup_{v \rightarrow \infty} \Gamma_v(x_v) = \{y \in Y \mid \forall \varepsilon > 0, \forall \bar{v}, \exists v \geq \bar{v} \text{ s.t. } S(y, \varepsilon) \cap \Gamma_v(x_v) \neq \emptyset\}.$$

Then, the sequence $\{\Gamma_v\}_{v \in \mathbb{N}}$ is sequentially convergent to Γ if, for every $x \in X$ and for every sequence $\{x_v\}_{v \in \mathbb{N}} \subset X$ converging to x in X , we have the following:

$$\limsup_{v \rightarrow \infty} \Gamma_v(x_v) \subseteq \Gamma(x) \subseteq \liminf_{v \rightarrow \infty} \Gamma_v(x_v).$$

4.2. The Stability Result

Now, we can state the limit theorem. The proof is contained in Appendix A.

Theorem 1. Let $G^\alpha = \{A_1, \dots, A_n, U_1^\alpha, \dots, U_n^\alpha\}$ be an α -MEU psychological game and $\{G_v^{\alpha_v}\}_{v \in \mathbb{N}}$ be a sequence of α -MEU psychological games, as constructed in Definition 3. Assume that, for every player i ,

- (i) The sequence $\{u_{i,v}\}_{v \in \mathbb{N}}$ sequentially converges to the function u_i ;⁹
- (ii) each function $u_{i,v}$ and function u_i are continuous in $\bar{B}_i \times \Sigma$;
- (iii) the sequence $\{\alpha_v\}_{v \in \mathbb{N}}$ converges to $\alpha = (\alpha_1, \dots, \alpha_n)$;
- (iv) the sequence $\{\gamma_{i,v}\}_{v \in \mathbb{N}}$ sequentially converges to the set-valued map γ_i . Suppose additionally that each $\gamma_{i,v}$ and γ_i have compact and non-empty values for every $\sigma \in \Sigma$.

If the sequence $\{\sigma_v^*\}_{v \in \mathbb{N}} \subset \Sigma$ converges to $\sigma^* \in \Sigma$ and $(\gamma_v(\sigma_v^*), \sigma_v^*)$ is a α -MEU psychological Nash equilibrium of $G_v^{\alpha_v}$ for every $v \in \mathbb{N}$, then it follows that $(\gamma(\sigma^*), \sigma^*)$ is an α -MEU psychological Nash equilibrium of G^α .

4.3. A Remark on Equilibrium Selection

The connection between the upper-semi-continuity property and the lack of lower-semi-continuity property of the equilibrium correspondence is key to building equilibrium selection devices based on stability properties with respect to perturbations (see [40] for an extensive survey on this topic in the classical framework). The idea behind the (so-called) *equilibrium refinements* is that, in the case of games with multiple equilibria, some may not be robust with respect to perturbations on the strategies or the payoffs of the players; so, it is possible to restrict properly the set of equilibria by picking only the ones that are limits of specific sequences of equilibria of perturbed games, when the perturbation converges to zero (this is actually the idea behind the seminal concept of the *trembling hand perfect equilibrium* introduced in Selten [41]). With respect to this issue, the upper semi-continuity property guarantees that limits of sequences of equilibria of perturbed games are equilibria of the unperturbed one, while, the lack of the lower-semi-continuity property makes it possible that some equilibria are not limit points. In the framework of the present paper, the example in Section 3 immediately shows that not every α -PNE is *stable* with respect to perturbations on preferences, particularly when there are *trembles* on the degree of optimism/pessimism. In fact, as noticed in Example 2, within the subset of equilibria $p \in [1/4, 3/4]$, only the equilibria $p = 1/4$ and $p = 3/4$ are stable with respect to perturbations on the parameter α . Therefore, the property of stability with respect to trembles on the degree of pessimism/optimism provides an effective selection device for α -PNE.

5. Existence of Equilibria: A Counterexample

Different from [7,14], in which an existence theorem was proved, respectively, for psychological Nash equilibria and psychological Nash equilibria under ambiguity (in the case of max–min preferences), in our framework, equilibrium existence fails in very simple examples as the one shown below. This example also highlights the fact that equilibrium existence might depend on the parameter α : equilibria exist if and only if α belongs to a proper subset of $[0, 1]$. Finally, for the sake of simplicity, the example focuses on an extreme form of ambiguity, given by full ignorance.

Example 3. We consider a two-player game: the pure strategy set of Player 1 (Anne) is $A_1 = \{\text{Accept}, \text{Reject}\}$ and the pure strategy set of Player 2 (John) is $A_2 = \{\text{Accept}, \text{Reject}\}$. We denote with p the mixed strategy of Player 1, where, with an abuse of notation, p is the probability of Accept and $1 - p$ is the probability of Reject. Similarly, r is the mixed strategy of Player 2; again, with an abuse of notation, r is the probability of Accept and $1 - r$ is the probability of Reject. It is assumed that John's utility does not depend on beliefs while Anne's utility depends on her second-order beliefs. Moreover, as conducted in the previous example, it is considered the case in which only the expectations of beliefs play a role in Anne's utility function. We denote with $q \in [0, 1]$ the expectation of John's first-order beliefs about Anne's mixed strategy (p) and $\bar{q} \in [0, 1]$ the expectation of Anne's second-order beliefs about the expectation q of John's first-order beliefs. The game is represented as follows (Table 2):

Table 2. Game form for Example 3.

Anne	John	Accept	Reject
Accept		$-2\tilde{q} - 1, 1$	$2, 0$
Reject		$2, 0$	$2\tilde{q} - 3, 1$

A mixed strategy profile is identified by the pair (p, r) . The correct belief functions simply map the strategic profiles (p, r) to the correct expectation of beliefs; more precisely, $\beta_1(p, r) = p$ tells that the expectation of John's correct first-order beliefs about Anne's strategy (p) must be equal to p , and $\beta_2(p, r) = p$ shows that the expectation of Anne's correct second-order beliefs about John's first-order beliefs about Anne's strategy (p) must be equal to p as well.

The best reply of Player 2 can be easily computed, as there are no psychological effects:

$$BR_2(p) = \begin{cases} 0 & \text{if } p \in [0, 1/2[, \\ [0, 1] & \text{if } p = 1/2, \\ 1 & \text{if } p \in]1/2, 1]. \end{cases}$$

The expected utility for Anne (Player 1) having a second-order belief $\tilde{q}$ and given that the mixed strategy profile (p, r) is as follows:

$$u_1(\tilde{q}, (p, r)) = -8pr + (5 - 2\tilde{q})p + (5 - 2\tilde{q})r + 2\tilde{q} - 3 = 2\tilde{q}(1 - p - r) - 8pr + 5p + 5r - 3.$$

We consider the case in which there is full ambiguity in Anne's second-order beliefs. More precisely, Anne's second-order belief is given by $\gamma_1(p, r) = [0, 1]$ for every strategy profile (p, r) . Let $\alpha = \alpha_1$ denote Anne's ambiguity attitude parameter. Then,

Lemma 4. Anne's best reply correspondence is given by the following:

- for $\alpha \in [0, 1/2[$,

$$BR_1^\alpha(r) = \begin{cases} 1 & \text{if } r \in [0, 1/2[, \\ \{0, 1\} & \text{if } r = 1/2, \\ 0 & \text{if } r \in]1/2, 1]; \end{cases}$$

- for $\alpha = 1/2$,

$$BR_1^\alpha(r) = \begin{cases} 1 & \text{if } r \in [0, 1/2[, \\ [0, 1] & \text{if } r = 1/2, \\ 0 & \text{if } r \in]1/2, 1]; \end{cases}$$

- for $\alpha \in]1/2, 1]$,

$$BR_1^\alpha(r) = \begin{cases} 1 & \text{if } r \in [0, \frac{5-2\alpha}{8}[, \\ [\frac{3+2\alpha}{8}, 1] & \text{if } r = \frac{5-2\alpha}{8}, \\ 1-r & \text{if } r \in [\frac{5-2\alpha}{8}, \frac{3+2\alpha}{8}], \\ [0, \frac{5-2\alpha}{8}] & \text{if } r = \frac{3+2\alpha}{8}, \\ 0 & \text{if } r \in]\frac{3+2\alpha}{8}, 1]. \end{cases}$$

Therefore,

- for $\alpha \in [0, 1/2[$ there are no α -PNE;
- for $\alpha \in [1/2, 1]$ there is only one α -PNE which is given by $(p, r) = (1/2, 1/2)$.

Proof. See Appendix A $\square$

Remark 4. It is natural to imagine that the lack of a general existence theorem depends on a general lack of convexity of the images of the best reply correspondences. This is actually true, but it is useful to understand what kind of best reply we refer to. To this purpose, let $w_i(\sigma, \tau)$ be the summary utility function of player i (it can be the one in [14], the one in [7], or the general w_i^α considered in this work). Then the two possible best replies can be defined for player i :

(1) $\overline{BR}_i : \Sigma_{-i} \rightsquigarrow \Sigma_i$, where

$$\overline{BR}_i(\sigma_{-i}^*) = \{\sigma_i \in \Sigma_i \mid w_i((\sigma_i, \sigma_{-i}^*), (\sigma_i, \sigma_{-i}^*)) \geq w_i((\sigma_i, \sigma_{-i}^*), (\tau_i, \sigma_{-i}^*)) \forall \tau_i \in \Sigma_i\},$$

(2) $BR_i : \Sigma \rightsquigarrow \Sigma_i$, where

$$BR_i(\sigma^*) = \{\sigma_i \in \Sigma_i \mid w_i((\sigma_i^*, \sigma_{-i}^*), (\sigma_i, \sigma_{-i}^*)) \geq w_i((\sigma_i^*, \sigma_{-i}^*), (\tau_i, \sigma_{-i}^*)) \forall \tau_i \in \Sigma_i\}.$$

It follows from the definition that σ^* is a psychological Nash equilibrium if and only if it is a fixed point for the set-valued map (1) $\overline{BR}_1 \times \cdots \times \overline{BR}_n$ or for the set-valued map (2) $BR_1 \times \cdots \times BR_n$ without any distinction. However, the interpretation is different because in BR_i the hierarchy of beliefs depends on the entire equilibrium profile, while in $\overline{BR}_i$, the hierarchy depends only on opponents' equilibrium strategies. Now, in the examples in [14] or [7], the set-valued maps $\overline{BR}_i$ do not have convex images even if, for these games, an equilibrium existence theorem holds. In fact, the existence theorem follows from the convexity of the images of the set-valued maps BR_i , which is guaranteed in the models by [14] or by [7]. On the contrary, in the example above, also the set-valued map BR_i does not have convex images, leading to the nonexistence of equilibria.

6. Conclusions

Max–min preferences are a common and analytically convenient approach to solving decision problems under ambiguity. Nevertheless, the Hurwicz preferences seem to be more flexible and general tools to handle ambiguity as the degree of pessimism is parametrized by a real-valued weight α . In this work, we look at the effects of this kind of preference in static psychological games where the source of ambiguity is the entire hierarchy of beliefs. Our aim is to provide methodological tools to run a sensitivity analysis of the equilibria based on the pessimism's coefficient α . An illustrative example shows that the analysis might be easily done and is effective in simple applications. On the other hand, the theoretical results on the semi-continuity properties of the equilibrium correspondence make it possible to refine equilibria based on the *stability* with respect to perturbations on α . Finally, the existence of equilibria might be lost when we deviate from max–min preferences: an existence result is obtained only for $\alpha = 1$, and counterexamples with no equilibria are provided for $\alpha < 1/2$. However, Example 3 shows that for $\alpha \geq 1/2$, an equilibrium does exist, suggesting that conditions on α could be found to obtain equilibrium existence. Further research will be conducted in this direction. Moreover, the game model presented in this work has some limitations, as it covers only the static case. Future research might also focus on dynamic psychological games as presented in [8].

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Appendix A

Appendix A.1. Proof of Lemma 2

Suppose $0 < \varepsilon \leq \frac{1}{2}$. In this case, the summary utility function takes the following form:

$$w_\varepsilon^\alpha(p, y) = \begin{cases} y[2\alpha(p + \varepsilon) - 1] - 3\alpha(p + \varepsilon) + 3 & \text{if } 0 \leq p \leq \varepsilon, \\ y[4\alpha\varepsilon + 2(p - \varepsilon) - 1] - 6\alpha\varepsilon + 3(1 - p + \varepsilon) & \text{if } \varepsilon < p < 1 - \varepsilon \\ y[2\alpha(1 - p + \varepsilon) + 2(p - \varepsilon) - 1] + (3 - 3\alpha)(1 - p + \varepsilon) & \text{if } 1 - \varepsilon \leq p \leq 1. \end{cases}$$

Let¹⁰ $\tilde{p} = \frac{1}{2} + \varepsilon(1 - 2\alpha)$, $p^* = \frac{1}{2\alpha} - \varepsilon$, and $\hat{p} = \frac{1-2\alpha}{2-2\alpha} + \varepsilon$.

- (a) Denote with $h_1(y) := y[2\alpha(p + \varepsilon) - 1] - 3\alpha(p + \varepsilon) + 3$. If $\alpha = 0$, the function $h_1(y)$ decreases throughout the entire interval $[0, 1]$. If $\alpha > 0$, $h_1(y)$ decreases throughout the entire interval $[0, 1]$ for $p < p^*$, is constant on $[0, 1]$ for $p = p^*$, and increases on $[0, 1]$ for $p > p^*$. Moreover, $p^* \geq 0$ for every $\alpha \in]0, 1]$, while $p^* \leq \varepsilon$ if and only if $\alpha \in \left[\frac{1}{4\varepsilon}, 1\right]$.
- (b) Denote with $h_2(y) := y[4\alpha\varepsilon + 2(p - \varepsilon) - 1] - 6\alpha\varepsilon + 3(1 - p + \varepsilon)$. The function $h_2(y)$ decreases throughout the entire interval $[0, 1]$ for $p < \tilde{p}$, is constant on $[0, 1]$ for $p = \tilde{p}$, and increases on $[0, 1]$ for $p > \tilde{p}$. Moreover, $\tilde{p} > \varepsilon$ if and only if $\alpha \in \left[0, \frac{1}{4\varepsilon}\right]$, while $\tilde{p} < 1 - \varepsilon$ if and only if $\alpha \in \left]1 - \frac{1}{4\varepsilon}, 1\right]$.
- (c) Denote with $h_3(y) := y[2\alpha(1 - p + \varepsilon) + 2(p - \varepsilon) - 1] + (3 - 3\alpha)(1 - p + \varepsilon)$. If $\alpha = 1$ the function $h_3(y)$ increases throughout the entire interval $[0, 1]$. If $\alpha < 1$, the function $h_3(y)$ decreases on $[0, 1]$ for $p < \hat{p}$, is constant on $[0, 1]$ for $p = \hat{p}$ and increases throughout the entire interval $[0, 1]$ if $p > \hat{p}$. Moreover, $\hat{p} \leq 1$ for every $\alpha \in [0, 1]$, while $\hat{p} \geq 1 - \varepsilon$ if and only if $\alpha \in \left[0, 1 - \frac{1}{4\varepsilon}\right]$.

Note that, if $\varepsilon < 1/4$, then $[0, 1] \subset \left[1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$, if $\varepsilon = 1/4$, then $\left[1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right] = [0, 1]$, while if $\varepsilon > 1/4$, then $\left[1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right] \subset [0, 1]$. Therefore,

- If $\varepsilon < 1/4$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ decreases on $[0, 1]$ for $p < \tilde{p}$, is constant for $p = \tilde{p}$, and increases for $p > \tilde{p}$.
- If $\varepsilon \geq 1/4$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ decreases on $[0, 1]$ for $p < P$, is constant for $p = P$, and increases for $p > P$, where we have the following:

$$P = \begin{cases} \hat{p} & \text{if } \alpha \in \left[0, 1 - \frac{1}{4\varepsilon}\right], \\ \tilde{p} & \text{if } \alpha \in \left[1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right], \\ p^* & \text{if } \alpha \in \left[\frac{1}{4\varepsilon}, 1\right]. \end{cases}$$

So, equilibria are computed as follows:

Subcase 1.1: If $\varepsilon < \frac{1}{4}$, for every $\alpha \in [0, 1]$, we have the following:

$$\begin{aligned} w_\varepsilon^\alpha(p, 0) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1], \quad \text{if } p < \tilde{p}; \\ w_\varepsilon^\alpha(p, 1) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{if } p > \tilde{p}; \\ w_\varepsilon^\alpha(p, y) &= 3/2 \quad \forall y \in [0, 1], \quad \text{if } p = \tilde{p}. \end{aligned}$$

Therefore, for every $\alpha \in [0, 1]$, we have the following equilibria: $p = 0$, $p = 1$, and $p = \tilde{p}$.

Subcase 1.2: If $\varepsilon \geq \frac{1}{4}$,

(i) For $\alpha \in \left[0, 1 - \frac{1}{4\varepsilon}\right]$,

$$\begin{aligned} w_\varepsilon^\alpha(p, 0) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1[, \quad \text{if } p < \hat{p}; \\ w_\varepsilon^\alpha(p, 1) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{if } p > \hat{p}; \\ w_\varepsilon^\alpha(p, y) &= 3/2 \quad \forall y \in [0, 1], \quad \text{if } p = \hat{p}. \end{aligned}$$

Therefore, we obtain the following equilibria: $p = 0$, $p = 1$, and $p = \hat{p}$.

(ii) For $\alpha \in \left]1 - \frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$,

$$\begin{aligned} w_\varepsilon^\alpha(p, 0) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1[, \quad \text{if } p < \tilde{p}; \\ w_\varepsilon^\alpha(p, 1) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{if } p > \tilde{p}; \\ w_\varepsilon^\alpha(p, y) &= 3/2 \quad \forall y \in [0, 1], \quad \text{if } p = \tilde{p}. \end{aligned}$$

Therefore, we obtain the following equilibria: $p = 0$, $p = 1$, and $p = \tilde{p}$.

(iii) For $\alpha \in \left[\frac{1}{4\varepsilon}, 1\right]$,

$$\begin{aligned} w_\varepsilon^\alpha(p, 0) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1[, \quad \text{if } p < p^*; \\ w_\varepsilon^\alpha(p, 1) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{if } p > p^*; \\ w_\varepsilon^\alpha(p, y) &= 3/2 \quad \forall y \in [0, 1], \quad \text{if } p = p^*. \end{aligned}$$

Therefore, we obtain the following equilibria: $p = 0$, $p = 1$, and $p = p^*$.

Appendix A.2. Proof of Lemma 3

Suppose $\frac{1}{2} < \varepsilon \leq 1$. In this case, the summary utility function takes the following form:

$$w_\varepsilon^\alpha(p, y) = \begin{cases} y[2\alpha(p + \varepsilon) - 1] - 3\alpha(p + \varepsilon) + 3 & \text{if } 0 \leq p \leq 1 - \varepsilon \\ y[2\alpha - 1] - 3\alpha + 3 & \text{if } 1 - \varepsilon < p < \varepsilon. \\ y[(p - \varepsilon)(2 - 2\alpha) + 2\alpha - 1] + (3 - 3\alpha)(1 - p + \varepsilon) & \text{if } \varepsilon \leq p \leq 1, \end{cases}$$

Note that $0 < 1 - \frac{1}{2\varepsilon} \leq \frac{1}{2} \leq \frac{1}{2\varepsilon} < 1$; consider again $p^* = \frac{1}{2\alpha} - \varepsilon$ and $\hat{p} = \frac{1-2\alpha}{2-2\alpha} + \varepsilon$.

- (a) Denote with $h_1(y) := y[2\alpha(p + \varepsilon) - 1] - 3\alpha(p + \varepsilon) + 3$. If $\alpha = 0$, the function $h_1(y)$ decreases throughout the entire interval $[0, 1]$. If $\alpha > 0$, $h_1(y)$ decreases on $[0, 1]$ if $p < p^*$, is constant on $[0, 1]$ if $p = p^*$, and increases on $[0, 1]$ if $p > p^*$. Moreover, $p^* \geq 0$ if and only if $\alpha \in \left]0, \frac{1}{2\varepsilon}\right]$, while $p^* \leq 1 - \varepsilon$ if and only if $\alpha \in \left[\frac{1}{2}, 1\right]$.
- (b) Denote with $h_2(y) := y[2\alpha - 1] - 3\alpha + 3$; the function $h_2(y)$ decreases throughout the entire interval $[0, 1]$ if $\alpha < \frac{1}{2}$, is constant on the interval $[0, 1]$ if $\alpha = \frac{1}{2}$, and increases throughout the entire interval $[0, 1]$ if $\alpha > \frac{1}{2}$.
- (c) Denote with $h_3(y) := y[(p - \varepsilon)(2 - 2\alpha) + 2\alpha - 1] + (3 - 3\alpha)(1 - p + \varepsilon)$. If $\alpha = 1$, the function $h_3(y)$ increases throughout the entire interval $[0, 1]$. If $\alpha < 1$, the function $h_3(y)$ decreases on $[0, 1]$ for $p < \hat{p}$, is constant on $[0, 1]$ for $p = \hat{p}$, and increases throughout the entire interval $[0, 1]$ if $p > \hat{p}$. Moreover, $\hat{p} \leq 1$ if and only if $\alpha \in \left[1 - \frac{1}{2\varepsilon}, 1\right]$, while $\hat{p} \geq \varepsilon$ if and only if $\alpha \in \left[0, \frac{1}{2}\right]$.

It follows that

- If $\alpha \in \left[0, 1 - \frac{1}{2\varepsilon}\right]$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ decreases on $[0, 1]$ for every $p \in [0, 1]$.
- If $\alpha = 1 - \frac{1}{2\varepsilon}$, then the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ decreases on $[0, 1]$ for $p < \hat{p} = 1$, and is constant in $[0, 1]$ for $p = \hat{p} = 1$.
- If $\alpha \in \left]1 - \frac{1}{2\varepsilon}, \frac{1}{2}\right]$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ decreases on $[0, 1]$ for $p < \hat{p}$, is constant on $[0, 1]$ for $p = \hat{p}$, and increases on $[0, 1]$ for $p > \hat{p}$.

- If $\alpha = \frac{1}{2}$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ decreases on $[0, 1]$ for $p < 1 - \varepsilon$, is constant in $[0, 1]$ for $1 - \varepsilon \leq p \leq \varepsilon$, and increases on $[0, 1]$ for $p > \varepsilon$.
- If $\alpha \in \left] \frac{1}{2}, \frac{1}{2\varepsilon} \right[$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ decreases on $[0, 1]$ for $p < p^*$, is constant on $[0, 1]$ for $p = p^*$, increases on $[0, 1]$ for $p > p^*$.
- If $\alpha = \frac{1}{2\varepsilon}$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ increases on $[0, 1]$ for $p > p^* = 0$, and is constant for $p = p^* = 0$.
- If $\alpha \in \left] \frac{1}{2\varepsilon}, 1 \right]$, the function $y \rightarrow w_\varepsilon^\alpha(p, y)$ increases on $[0, 1]$ for every $p \in [0, 1]$.

The equilibria are computed as follows:

- (i) For $\alpha \in \left[0, 1 - \frac{1}{2\varepsilon} \right[$,

$$w_\varepsilon^\alpha(p, 0) > w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1[, \quad \text{for all } p \in [0, 1].$$

Therefore, we only have the following equilibrium: $p = 0$.

- (ii) For $\alpha \in \left[1 - \frac{1}{2\varepsilon}, \frac{1}{2} \right[$,

$$\begin{aligned} w_\varepsilon^\alpha(p, 0) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1[, \quad \text{if } p < \hat{p}; \\ w_\varepsilon^\alpha(p, 1) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{if } p > \hat{p}; \\ w_\varepsilon^\alpha(p, y) &= 3/2 \quad \forall y \in [0, 1], \quad \text{if } p = \hat{p}. \end{aligned}$$

Therefore, we have the following equilibria: $p = 0$, $p = 1$, and $p = \hat{p}$. Note that for $\alpha = 1 - \frac{1}{2\varepsilon}$ we obtain the following: $\hat{p} = 1$.

- (iii) For $\alpha = \frac{1}{2}$,

$$\begin{aligned} w_\varepsilon^\alpha(p, 0) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1[, \quad \text{if } p < 1 - \varepsilon; \\ w_\varepsilon^\alpha(p, 1) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{if } p > \varepsilon; \\ w_\varepsilon^\alpha(p, y) &= 3/2 \quad \forall y \in [0, 1], \quad \text{if } 1 - \varepsilon \leq p \leq \varepsilon. \end{aligned}$$

In this case, we have an infinite number of equilibria: $p = 0$, $p = 1$, and every $p \in [1 - \varepsilon, \varepsilon]$.

- (iv) For $\alpha \in \left] \frac{1}{2}, \frac{1}{2\varepsilon} \right]$,

$$\begin{aligned} w_\varepsilon^\alpha(p, 0) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in]0, 1[, \quad \text{if } p < p^*; \\ w_\varepsilon^\alpha(p, 1) &> w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{if } p > p^*; \\ w_\varepsilon^\alpha(p, y) &= 3/2 \quad \forall y \in [0, 1], \quad \text{if } p = p^*. \end{aligned}$$

Therefore, we have three equilibria: $p = 0$, $p = 1$, and $p = p^*$. Note that for $\alpha = \frac{1}{2\varepsilon}$, $p^* = 0$.

- (v) For $\alpha \in \left] \frac{1}{2\varepsilon}, 1 \right]$,

$$w_\varepsilon^\alpha(p, 1) > w_\varepsilon^\alpha(p, y) \quad \forall y \in [0, 1[, \quad \text{for all } p \in [0, 1].$$

Therefore, we have a unique equilibrium: $p = 1$.

Appendix A.3. Proof of Theorem 1

For every player i and every $\nu \in \mathbb{N}$, let $w_{i,\nu}^{\alpha_\nu}$ be the summary utility function of the game $G_\nu^{\alpha_\nu}$, i.e.,

$$w_{i,\nu}^{\alpha_\nu}(\sigma, \tau) := \alpha_{i,\nu} \left[\inf_{b_i \in \gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i, \tau) \right] + (1 - \alpha_{i,\nu}) \left[\sup_{b_i \in \gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i, \tau) \right] \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma,$$

and w_i^α be the summary utility function of the game G^α , i.e.,

$$w_i^\alpha(\sigma, \tau) := \alpha_i \left[\inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \right] + (1 - \alpha_i) \left[\sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \right] \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma.$$

The continuous convergence of the sequence of functions $\{w_{i,v}^{\alpha_v}\}_{v \in \mathbb{N}}$ to the function w_i^α , for every $i \in I$, guarantees the result. In fact, if $\{\sigma_v^*\}_{v \in \mathbb{N}} \subset \Sigma$ is a sequence converging to $\sigma^* \in \Sigma$ such that, for every $v \in \mathbb{N}$, $(\gamma_v(\sigma_v^*), \sigma_v^*)$ is an α -MEU psychological Nash equilibrium of $G_v^{\alpha_v}$, then it follows that, for every player i ,

$$w_{i,v}^{\alpha_v}(\sigma_v^*, \sigma_v^*) \geq w_{i,v}^{\alpha_v}(\sigma_v^*, (y_i, \sigma_{-i,v}^*)) \quad \forall y_i \in \Sigma_i.$$

Applying the continuous convergence of $\{w_{i,v}^{\alpha_v}\}_{v \in \mathbb{N}}$ to w_i^α , we obtain the following:

$$w_i^\alpha(\sigma^*, \sigma^*) = \lim_{v \rightarrow \infty} w_{i,v}^{\alpha_v}(\sigma_v^*, \sigma_v^*) \geq \lim_{v \rightarrow \infty} w_{i,v}^{\alpha_v}(\sigma_v^*, (y_i, \sigma_{-i,v}^*)) = w_i^\alpha(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i.$$

This latter inequality implies that $(\gamma(\sigma^*), \sigma^*)$ is an α -MEU psychological Nash equilibrium of G^α . Therefore, the proof reduces when verifying the continuous convergence of $\{w_{i,v}^{\alpha_v}\}_{v \in \mathbb{N}}$ to w_i^α . That is, we need to check that for every $(\sigma, \tau) \in \Sigma \times \Sigma$ and for every sequence $\{(\sigma_v, \tau_v)\}_{v \in \mathbb{N}}$ converging to (σ, τ) we obtain the following inequalities:

$$\limsup_{v \rightarrow \infty} w_{i,v}^{\alpha_v}(\sigma_v, \tau_v) \leq w_i^\alpha(\sigma, \tau) \leq \liminf_{v \rightarrow \infty} w_{i,v}^{\alpha_v}(\sigma_v, \tau_v). \quad (\text{A1})$$

Denote with

$$w_{i,v}^m(\sigma, \tau) = \inf_{b_i \in \gamma_{i,v}(\sigma)} u_{i,v}(b_i, \tau), \quad w_{i,v}^M(\sigma, \tau) = \sup_{b_i \in \gamma_{i,v}(\sigma)} u_{i,v}(b_i, \tau),$$

and

$$w_i^m(\sigma, \tau) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau), \quad w_i^M(\sigma, \tau) = \sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau).$$

Consider $(\sigma, \tau) \in \Sigma \times \Sigma$ and take a sequence $\{(\sigma_v, \tau_v)\}_{v \in \mathbb{N}}$ converging to (σ, τ) . Now we prove the following:

$$\limsup_{v \rightarrow \infty} w_{i,v}^m(\sigma_v, \tau_v) \leq w_i^m(\sigma, \tau) \leq \liminf_{v \rightarrow \infty} w_{i,v}^m(\sigma_v, \tau_v),$$

and

$$\limsup_{v \rightarrow \infty} w_{i,v}^M(\sigma_v, \tau_v) \leq w_i^M(\sigma, \tau) \leq \liminf_{v \rightarrow \infty} w_{i,v}^M(\sigma_v, \tau_v).$$

First, we show that

$$w_i^m(\sigma, \tau) \leq \liminf_{v \rightarrow \infty} w_{i,v}^m(\sigma_v, \tau_v) \quad \left(\text{resp. } w_i^M(\sigma, \tau) \geq \limsup_{v \rightarrow \infty} w_{i,v}^M(\sigma_v, \tau_v) \right).$$

Suppose by contradiction that we have the following:

$$w_i^m(\sigma, \tau) > \liminf_{v \rightarrow \infty} w_{i,v}^m(\sigma_v, \tau_v) \quad \left(\text{resp. } w_i^M(\sigma, \tau) < \limsup_{v \rightarrow \infty} w_{i,v}^M(\sigma_v, \tau_v) \right). \quad (\text{A2})$$

This means that along a subsequence $\{(\sigma_{v_k}, \tau_{v_k})\}_{k \in \mathbb{N}}$, we have the following:

$$\lim_{k \rightarrow \infty} w_{i,v_k}^m(\sigma_{v_k}, \tau_{v_k}) < w_i^m(\sigma, \tau) \quad \left(\text{resp. } \lim_{k \rightarrow \infty} w_{i,v_k}^M(\sigma_{v_k}, \tau_{v_k}) > w_i^M(\sigma, \tau) \right). \quad (\text{A3})$$

Additionally, the continuity of u_i and $u_{i,\nu}$ for every ν and the compactness of the images of γ_i and $\gamma_{i,\nu}$, for every ν , guarantee the following: exist $b_i^m \in \gamma_i(\sigma)$ and $b_{i,\nu}^m \in \gamma_{i,\nu}(\sigma_\nu)$, (resp. $b_i^M \in \gamma_i(\sigma)$ and $b_{i,\nu}^M \in \gamma_{i,\nu}(\sigma_\nu)$), for every ν , such that we have the following:

$$w_i^m(\sigma, \tau) = u_i(b_i^m, \tau) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau), \quad \left(\text{resp. } w_i^M(\sigma, \tau) = u_i(b_i^M, \tau) = \sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \right)$$

and

$$w_{i,\nu}^m(\sigma_\nu, \tau_\nu) = u_{i,\nu}(b_{i,\nu}^m, \tau_\nu) = \inf_{b_{i,\nu} \in \gamma_{i,\nu}(\sigma_\nu)} u_{i,\nu}(b_{i,\nu}, \tau_\nu),$$

$$\left(\text{resp. } w_{i,\nu}^M(\sigma_\nu, \tau_\nu) = u_{i,\nu}(b_{i,\nu}^M, \tau_\nu) = \sup_{b_{i,\nu} \in \gamma_{i,\nu}(\sigma_\nu)} u_{i,\nu}(b_{i,\nu}, \tau_\nu) \right).$$

Consider the sequence of beliefs $\{b_{i,\nu_k}^m\}_{k \in \mathbb{N}}$, (resp. $\{b_{i,\nu_k}^M\}_{k \in \mathbb{N}}$), obtained along the subsequence $\{(\sigma_{\nu_k}, \tau_{\nu_k})\}_{k \in \mathbb{N}}$, as in (A3). The sequence $\{b_{i,\nu_k}^m\}_{k \in \mathbb{N}}$, (resp. $\{b_{i,\nu_k}^M\}_{k \in \mathbb{N}}$), has a subsequence $\{b_{i,\nu_h}^m\}_{h \in \mathbb{N}}$, (resp. $\{b_{i,\nu_h}^M\}_{h \in \mathbb{N}}$), which converges to a point $\hat{b}_i^m \in \bar{B}_i$, (resp. $\hat{b}_i^M \in \bar{B}_i$), since $\bar{B}_i$ is compact. The point $\hat{b}_i^m$, (resp. $\hat{b}_i^M$), actually belongs to $\gamma_i(\sigma)$. In fact, by definition, the upper limit $\limsup_{\nu \rightarrow \infty} \gamma_{i,\nu}(\sigma_\nu)$ contains the limits of every converging subsequence of $\{b_{i,\nu_k}^m\}_{k \in \mathbb{N}}$, (resp. $\{b_{i,\nu_k}^M\}_{k \in \mathbb{N}}$); that is,

$$\hat{b}_i^m, \hat{b}_i^M \in \limsup_{\nu \rightarrow \infty} \gamma_{i,\nu}(\sigma_\nu).$$

Moreover, $\{\gamma_{i,\nu}\}_{\nu \in \mathbb{N}}$ is sequentially upper-convergent to γ_i , meaning that $\limsup_{\nu \rightarrow \infty} \gamma_{i,\nu}(\sigma_\nu) \subseteq \gamma_i(\sigma)$; therefore, $\hat{b}_i^m, \hat{b}_i^M \in \gamma_i(\sigma)$. By construction $u_i(b_i^m, \tau) \leq u_i(\hat{b}_i^m, \tau)$ (resp. $u_i(b_i^M, \tau) \geq u_i(\hat{b}_i^M, \tau)$). The sequence $\{u_{i,\nu}\}_{\nu \in \mathbb{N}}$ sequentially converges to u_i ; since $(b_{i,\nu_h}^m, \tau_{\nu_h}) \rightarrow (\hat{b}_i^m, \tau)$, (resp. $(b_{i,\nu_h}^M, \tau_{\nu_h}) \rightarrow (\hat{b}_i^M, \tau)$), we obtain the following:

$$u_i(\hat{b}_i^m, \tau) = \lim_{h \rightarrow \infty} u_{i,\nu_h}(b_{i,\nu_h}^m, \tau_{\nu_h}), \quad \left(\text{resp. } u_i(\hat{b}_i^M, \tau) = \lim_{h \rightarrow \infty} u_{i,\nu_h}(b_{i,\nu_h}^M, \tau_{\nu_h}) \right).$$

Hence,

$$w_i^m(\sigma, \tau) = u_i(b_i^m, \tau) \leq u_i(\hat{b}_i^m, \tau) = \lim_{h \rightarrow \infty} u_{i,\nu_h}(b_{i,\nu_h}^m, \tau_{\nu_h}) = \lim_{h \rightarrow \infty} w_{i,\nu_h}^m(\sigma_{\nu_h}, \tau_{\nu_h}),$$

$$\left(\text{resp. } w_i^M(\sigma, \tau) = u_i(b_i^M, \tau) \geq u_i(\hat{b}_i^M, \tau) = \lim_{h \rightarrow \infty} u_{i,\nu_h}(b_{i,\nu_h}^M, \tau_{\nu_h}) = \lim_{h \rightarrow \infty} w_{i,\nu_h}^M(\sigma_{\nu_h}, \tau_{\nu_h}) \right).$$

Then, inequality (A3) implies that

$$w_i^m(\sigma, \tau) \leq \lim_{h \rightarrow \infty} w_{i,\nu_h}^m(\sigma_{\nu_h}, \tau_{\nu_h}) < w_i^m(\sigma, \tau),$$

$$\left(\text{resp. } w_i^M(\sigma, \tau) \geq \lim_{h \rightarrow \infty} w_{i,\nu_h}^M(\sigma_{\nu_h}, \tau_{\nu_h}) > w_i^M(\sigma, \tau) \right),$$

which results in a contradiction. So,

$$w_i^m(\sigma, \tau) \leq \liminf_{\nu \rightarrow \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu), \quad \left(\text{resp. } w_i^M(\sigma, \tau) \geq \limsup_{\nu \rightarrow \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \right).$$

Now we show the following:

$$w_i^m(\sigma, \tau) \geq \limsup_{\nu \rightarrow \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu), \quad \left(\text{resp. } w_i^M(\sigma, \tau) \leq \liminf_{\nu \rightarrow \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \right).$$

Let $b_i^m \in \gamma_i(\sigma)$ (resp. $b_i^M \in \gamma_i(\sigma)$) be such that

$$u_i(b_i^m, \tau) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) = w_i^m(\sigma, \tau) \quad \left(\text{resp. } u_i(b_i^M, \tau) = \sup_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) = w_i^M(\sigma, \tau) \right).$$

The points b_i^m and b_i^M exist because of the continuity of u_i and the compactness of $\gamma_i(\sigma)$ for every $\sigma \in \Sigma$. Since the sequence $\{\gamma_{i,\nu}\}_{\nu \in \mathbb{N}}$ is sequentially convergent to γ_i , i.e.,

$$\gamma_i(\sigma) \subseteq \liminf_{\nu \rightarrow \infty} \gamma_{i,\nu}(\sigma_\nu),$$

then, by definition, there exists a sequence $\{\hat{b}_{i,\nu}^m\}_{\nu \in \mathbb{N}}$ converging to b_i^m , (resp. $\{\hat{b}_{i,\nu}^M\}_{\nu \in \mathbb{N}}$ converging to b_i^M), such that, for every ν , $\hat{b}_{i,\nu}^m \in \gamma_{i,\nu}(\sigma_\nu)$ (resp. $\hat{b}_{i,\nu}^M \in \gamma_{i,\nu}(\sigma_\nu)$).

The sequence $\{u_{i,\nu}\}_{\nu \in \mathbb{N}}$ sequentially converges to u_i ; it follows that

$$\limsup_{\nu \rightarrow \infty} u_{i,\nu}(\hat{b}_{i,\nu}^m, \tau_\nu) \leq u_i(b_i^m, \tau), \quad \left(\text{resp. } \liminf_{\nu \rightarrow \infty} u_{i,\nu}(\hat{b}_{i,\nu}^M, \tau_\nu) \geq u_i(b_i^M, \tau) \right).$$

By construction, for every $\nu \in \mathbb{N}$, we have the following:

$$w_{i,\nu}^m(\sigma_\nu, \tau_\nu) \leq u_{i,\nu}(\hat{b}_{i,\nu}^m, \tau_\nu), \quad \left(\text{resp. } w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \geq u_{i,\nu}(\hat{b}_{i,\nu}^M, \tau_\nu) \right).$$

This finally implies the following:

$$\limsup_{\nu \rightarrow \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) \leq \limsup_{\nu \rightarrow \infty} u_{i,\nu}(\hat{b}_{i,\nu}^m, \tau_\nu) \leq u_i(b_i^m, \tau) = w_i^m(\sigma, \tau),$$

$$\left(\text{resp. } \liminf_{\nu \rightarrow \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \geq \limsup_{\nu \rightarrow \infty} u_{i,\nu}(\hat{b}_{i,\nu}^M, \tau_\nu) \geq u_i(b_i^M, \tau) = w_i^M(\sigma, \tau) \right).$$

So, we obtain the following:

$$\limsup_{\nu \rightarrow \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) \leq w_i^m(\sigma, \tau) \leq \liminf_{\nu \rightarrow \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu)$$

and

$$\limsup_{\nu \rightarrow \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \leq w_i^M(\sigma, \tau) \leq \liminf_{\nu \rightarrow \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu).$$

Hence, from the properties of the upper and lower limits, we obtain the following:

$$\limsup_{\nu \rightarrow \infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu, \tau_\nu) = \limsup_{\nu \rightarrow \infty} [\alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) + (1 - \alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu, \tau_\nu)] \leq$$

$$\limsup_{\nu \rightarrow \infty} \alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) + \limsup_{\nu \rightarrow \infty} (1 - \alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \leq$$

$$\left(\limsup_{\nu \rightarrow \infty} \alpha_{i,\nu} \right) \left(\limsup_{\nu \rightarrow \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) \right) + \left(\limsup_{\nu \rightarrow \infty} (1 - \alpha_{i,\nu}) \right) \left(\limsup_{\nu \rightarrow \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \right) \leq$$

$$\alpha_i w_i^m(\sigma, \tau) + (1 - \alpha_i) w_i^M(\sigma, \tau),$$

and

$$\alpha_i w_i^m(\sigma, \tau) + (1 - \alpha_i) w_i^M(\sigma, \tau) \leq$$

$$\left(\liminf_{\nu \rightarrow \infty} \alpha_{i,\nu} \right) \left(\liminf_{\nu \rightarrow \infty} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) \right) + \left(\liminf_{\nu \rightarrow \infty} (1 - \alpha_{i,\nu}) \right) \left(\liminf_{\nu \rightarrow \infty} w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \right) \leq$$

$$\liminf_{\nu \rightarrow \infty} \alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) + \liminf_{\nu \rightarrow \infty} (1 - \alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu, \tau_\nu) \leq$$

$$\liminf_{\nu \rightarrow \infty} [\alpha_{i,\nu} w_{i,\nu}^m(\sigma_\nu, \tau_\nu) + (1 - \alpha_{i,\nu}) w_{i,\nu}^M(\sigma_\nu, \tau_\nu)] = \liminf_{\nu \rightarrow \infty} w_{i,\nu}^{\alpha_\nu}(\sigma_\nu, \tau_\nu).$$

Condition (A1) is satisfied and $\{w_{i,\nu}^{\alpha_\nu}\}_{\nu \in \mathbb{N}}$ continuously converges to w_i^α .

Appendix A.4. Proof of Lemma 4

For every pair of strategic profiles (p, r) and (x, y) , we have the following:

$$U_1^\alpha(\gamma_1(x, y), (p, r)) = \alpha \left[\min_{\tilde{q} \in \gamma_1(x, y)} u_1(\tilde{q}, (p, r)) \right] + (1 - \alpha) \left[\max_{\tilde{q} \in \gamma_1(x, y)} u_1(\tilde{q}, (p, r)) \right].$$

Recalling the form of γ_1 , we obtain the following:

$$\arg \min_{\tilde{q} \in \gamma_1(x, y)} u_1(\tilde{q}, (p, r)) = \arg \min_{\tilde{q} \in [0, 1]} u_1(\tilde{q}, (p, r)) =$$

$$\arg \min_{\tilde{q} \in [0, 1]} [2\tilde{q}(1 - p - r) - 8pr + 5p + 5r - 3] = \begin{cases} 0 & \text{if } p < 1 - r, \\ [0, 1] & \text{if } p = 1 - r, \\ 1 & \text{if } p > 1 - r, \end{cases}$$

and

$$\arg \max_{\tilde{q} \in \gamma_1(x, y)} u_1(\tilde{q}, (p, r)) = \arg \max_{\tilde{q} \in [0, 1]} u_1(\tilde{q}, (p, r)) =$$

$$\arg \max_{\tilde{q} \in [0, 1]} [2\tilde{q}(1 - p - r) - 8pr + 5p + 5r - 3] = \begin{cases} 1 & \text{if } p < 1 - r, \\ [0, 1] & \text{if } p = 1 - r, \\ 0 & \text{if } p > 1 - r. \end{cases}$$

Therefore, given the two strategic profiles (x, y) and (p, r) ,

$$\min_{\tilde{q} \in \gamma_1(x, y)} u_1(\tilde{q}, (p, r)) = \begin{cases} -8pr + 5p + 5r - 3 & \text{if } p \leq 1 - r, \\ -8pr + 3p + 3r - 1 & \text{if } p > 1 - r, \end{cases}$$

and

$$\max_{\tilde{q} \in \gamma_1(x, y)} u_1(\tilde{q}, (p, r)) = \begin{cases} -8pr + 5p + 5r - 3 & \text{if } p \geq 1 - r, \\ -8pr + 3p + 3r - 1 & \text{if } p < 1 - r. \end{cases}$$

Hence, the summary utility function has the following form:

$$w_1^\alpha((x, y), (p, r)) = U_1^\alpha(\gamma_1(x, y), (p, r)) =$$

$$\begin{cases} p(5 - 8r - 2\alpha) + 5r - 3 - 2\alpha r + 2\alpha & \text{if } p > 1 - r, \\ p(3 - 8r + 2\alpha) + 3r - 1 + 2\alpha r - 2\alpha & \text{if } p < 1 - r, \\ p(5 - 8r) + 5r - 3 & \text{if } p = 1 - r. \end{cases}$$

Recall that

$$BR_1^\alpha(r) = \{p \in \Sigma_1 \mid w_1^\alpha((p, r), (p, r)) \geq w_1^\alpha((p, r), (x, r)), \forall x \in \Sigma_1\}.$$

In order to construct $BR_1^\alpha(r)$, note the following:

- If $p > 1 - r$, the function $w_1^\alpha((x, y), (p, r))$ increases on p for $r < \frac{5-2\alpha}{8}$, is constant for $r = \frac{5-2\alpha}{8}$, decreasing for $r > \frac{5-2\alpha}{8}$;
- If $p < 1 - r$, the function $w_1^\alpha((x, y), (p, r))$ increases on p for $r < \frac{3+2\alpha}{8}$, is constant for $r = \frac{3+2\alpha}{8}$, decreasing for $r > \frac{3+2\alpha}{8}$;
- If $p = 1 - r$, the function $w_1^\alpha((x, y), (p, r))$ is constant on p since $w_1^\alpha((x, y), (p, r)) = 8r^2 - 8r + 2$.

We need to distinguish three cases:

- Suppose $\alpha < 1/2$; in this case, we have that

$$\frac{3+2\alpha}{8} < \frac{5-2\alpha}{8}.$$

Therefore, for every (x, y) , we have the following:

- If $r < \frac{3+2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ increases on $[0, 1]$ and attains the maximum for $p = 1$.
- If $r = \frac{3+2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ is constant on $[0, 1-r]$, increases on $[1-r, 1]$, and attains the maximum for $p = 1$.
- If $\frac{3+2\alpha}{8} < r < \frac{1}{2}$, then $w_1^\alpha((x, y), (\cdot, r))$ decreases on $[0, 1-r]$, increases on $[1-r, 1]$, and attains the maximum for $p = 1$.
- If $r = \frac{1}{2}$, then $w_1^\alpha((x, y), (\cdot, r))$ decreases on $[0, 1-r]$, increases on $[1-r, 1]$, and attains the maximum for $p = 1$ and $p = 0$.
- If $1/2 < r < \frac{5-2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ decreases on $[0, 1-r]$, increases on $[1-r, 1]$, and attains the maximum for $p = 0$.
- If $r = \frac{5-2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ decreases on $[0, 1-r]$, is constant on $[1-r, 1]$, and attains the maximum for $p = 0$.
- If $r > \frac{5-2\alpha}{8}$, then $w_1^\alpha((x, y), (p, r))$ decreases on $[0, 1]$ and attains the maximum for $p = 0$.

It follows that

$$BR_1^\alpha(r) = \begin{cases} 1 & \text{if } r \in [0, 1/2[, \\ \{0, 1\} & \text{if } r = 1/2, \\ 0 & \text{if } r \in]1/2, 1]. \end{cases}$$

- Suppose $\alpha = 1/2$; in this case, we have that

$$\frac{3+2\alpha}{8} = \frac{5-2\alpha}{8} = \frac{1}{2}.$$

Therefore, for every (x, y) , we have the following:

- If $r < \frac{1}{2}$, then $w_1^\alpha((x, y), (\cdot, r))$, increases on $[0, 1]$, and attains the maximum for $p = 1$.
- If $r = \frac{1}{2}$, then $w_1^\alpha((x, y), (\cdot, r))$ is constant on $[0, 1]$; therefore, every $p \in [0, 1]$ is a maximum point.
- If $r > \frac{1}{2}$, then $w_1^\alpha((x, y), (\cdot, r))$ decreases on $[0, 1]$ and attains the maximum for $p = 0$.

In this case,

$$BR_1^\alpha(r) = \begin{cases} 1 & \text{if } r \in [0, 1/2[, \\ [0, 1] & \text{if } r = 1/2, \\ 0 & \text{if } r \in]1/2, 1]. \end{cases}$$

- Suppose $\alpha > 1/2$; in this case, we have that

$$\frac{3+2\alpha}{8} > \frac{5-2\alpha}{8}.$$

Therefore, for every (x, y) , we have the following:

- If $r < \frac{5-2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ increases on $[0, 1]$ and attains the maximum for $p = 1$.
- If $r = \frac{5-2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ increases on $[0, 1-r]$ and is constant on $[1-r, 1]$; therefore, every $p \in [1-r, 1]$ is a maximum point.
- If $\frac{5-2\alpha}{8} < r < \frac{3+2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ increases on $[0, 1-r]$, decreases in $[1-r, 1]$, and attains the maximum for $p = 1-r$.

- If $r = \frac{3+2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ is constant on $[0, 1 - r]$ and decreases on $[1 - r, 1]$; therefore, every $p \in [0, 1 - r]$ is a maximum point.
- If $r > \frac{3+2\alpha}{8}$, then $w_1^\alpha((x, y), (\cdot, r))$ decreases on $[0, 1]$ and attains the maximum for $p = 0$.

In this case,

$$BR_1^\alpha(r) = \begin{cases} 1 & \text{if } r \in [0, \frac{5-2\alpha}{8}[, \\ [\frac{3+2\alpha}{8}, 1] & \text{if } r = \frac{5-2\alpha}{8}, \\ 1 - r & \text{if } r \in [\frac{5-2\alpha}{8}, \frac{3+2\alpha}{8}[, \\ [0, \frac{5-2\alpha}{8}] & \text{if } r = \frac{3+2\alpha}{8}, \\ 0 & \text{if } r \in]\frac{3+2\alpha}{8}, 1]. \end{cases}$$

The equilibria computation follows immediately from the fixed points of the best reply correspondences.

Appendix B

The counterexample in Section 5 highlights that the lack of existence of equilibria depends on the lack of convexity of the best reply correspondence, which, in turn, depends on the lack of quasi-concavity of the player's utility with respect to their own strategy. Now, it is clear that the lack of quasi-concavity is due to the presence (in the utility function) of the *max* operator, with respect to beliefs. In the quoted example, only second-order beliefs play a role in a player's utility. Now, since second-order beliefs directly depend on the player's strategy, the natural question is whether the presence of the *max* operator with respect only to first-order beliefs still affects the quasi-concavity of the utility functions in the classical models under strategic ambiguity (i.e., without psychological utilities). Below, we analyze this issue.

In order to focus on the mathematical problem, we consider the following simple form of strategic ambiguity: for every mixed strategy profile (σ_i, σ_{-i}) , agent i does not observe precisely σ_{-i} but considers as consistent any belief (q) in a given subset $\gamma_i(\sigma_i, \sigma_{-i}) \subseteq \Sigma_i$. For each belief $q \in \gamma_i(\sigma_i, \sigma_{-i})$, his expected utility would be¹¹

$$u_i(\sigma_i, q) = \sum_{a_i \in A_i} \sigma_i(a_i) u_i(a_i, q) = \sum_{a_i \in A_i} \sigma_i(a_i) \left[\sum_{a_{-i} \in A_{-i}} q(a_{-i}) \hat{u}_i(a_i, a_{-i}) \right]$$

so that an *optimistic* player. player i (i.e. $\alpha_i = 0$), has the following utility function $U_i : \Sigma \rightarrow \mathbb{R}$ defined by the following:

$$U_i(\sigma_i, \sigma_{-i}) = \max_{q \in \gamma_i(\sigma_i, \sigma_{-i})} u_i(\sigma_i, q) \quad \forall (\sigma_i, \sigma_{-i}) \in \Sigma,$$

where $\gamma_i : \Sigma \rightsquigarrow \Sigma_{-i}$ is the belief correspondence.

Now, it follows immediately that the function U_i is concave with respect to σ_i if the set-valued map $M : \Sigma \rightsquigarrow \Sigma_{-i}$, defined by

$$M_i(\sigma_i, \sigma_{-i}) = \arg \max_{q \in \gamma_i(\sigma_i, \sigma_{-i})} u_i(\sigma_i, q) \quad \forall (\sigma_i, \sigma_{-i}) \in \Sigma$$

depends only on σ_{-i} , i.e.,

$$M_i(\sigma'_i, \sigma_{-i}) = M_i(\sigma''_i, \sigma_{-i}) \quad \forall \sigma'_i, \sigma''_i \in \Sigma_i. \quad (\text{A4})$$

In fact, in this case, we immediately have that—for any arbitrary σ_{-i} —there exists $\hat{q}$, such that, for every $t \in [0, 1]$,

$$\hat{q} \in M_i(t\sigma'_i + (1-t)\sigma''_i, \sigma_{-i});$$

so

$$U_i(\sigma'_i, \sigma_{-i}) = u_i(\sigma'_i, \hat{q}), \quad U_i(\sigma''_i, \sigma_{-i}) = u_i(\sigma''_i, \hat{q}), \quad U_i(t\sigma'_i + (1-t)\sigma''_i, \sigma_{-i}) = u_i(t\sigma'_i + (1-t)\sigma''_i, \hat{q})$$

and

$$U_i(t\sigma'_i + (1-t)\sigma''_i, \sigma_{-i}) = tU_i(\sigma'_i, \sigma_{-i}) + (1-t)U_i(\sigma''_i, \sigma_{-i}),$$

implying that U_i is concave with respect to σ_i .

When condition (A4) does not hold, utilities are not always quasi-concave. So, the best reply correspondences do not have convex values and equilibria may not exist, as shown in the example below.

Example A1. We consider a two-player game: the pure strategy set of Player 1 (Anne) is $A_1 = \{\text{Accept}, \text{Reject}\}$ and the pure strategy set of Player 2 (John) is $A_2 = \{\text{Accept}, \text{Reject}\}$. The game is represented as follows (Table A1):

Table A1. Game form for Example A1.

Anne	John	Accept	Reject
Accept		1, 0	0, 2
Reject		0, 1	2, 0

As usual, p is the mixed strategy of Player 1, where, with an abuse of notation, p is the probability of Accept and $1 - p$ is the probability of Reject. Similarly, r is the mixed strategy of Player 2, where, with an abuse of notation, r is the probability of Accept and $1 - r$ is the probability of Reject. John's utility is assumed to be the classical expected utility without ambiguity. Anne, on the other hand, faces ambiguity: she does not observe r but, if the chosen strategy profile is (p, r) , she perceives as consistent all the beliefs q (where q is the probability of John's Accept and $1 - r$ is the probability of John's Reject) that belong to $\gamma_1(p, r)$, where γ_1 is Anne's belief correspondence defined by the following¹²:

$$\gamma_1(p, r) = \begin{cases} r & \text{if } p \in [0, 1/2[, \\ [0, 1] & \text{if } p \in [1/2, 1]. \end{cases}$$

For every strategy (p) and belief (q) , the expected utility of Anne is as follows:

$$u_1(p, q) = 3pq - 2p - 2q + 2.$$

Now, let $\alpha_1 = 0$ be Anne's ambiguity attitude parameter. We have the following:

$$\arg \max_{q \in \gamma_1(p, r)} u_1(q, (p, r)) = \arg \max_{q \in \gamma_1(p, r)} [3pq - 2p - 2q + 2] = \begin{cases} r & \text{if } p \in [0, 1/2[, \\ 0 & \text{if } p \in [1/2, 2/3[, \\ [0, 1] & \text{if } p = 2/3, \\ 1 & \text{if } p \in]2/3, 1]. \end{cases}$$

Therefore,

$$U_1(p, r) = \max_{q \in \gamma_1(p, r)} u_1(p, q) = \begin{cases} 3pr - 2p - 2r + 2 & \text{if } p \in [0, 1/2[, \\ -2p + 2 & \text{if } p \in [1/2, 2/3[, \\ p & \text{if } p \in]2/3, 1] \end{cases}$$

Now, we compute Anne's best reply:

$$BR_1(r) = \{p \in [0, 1] \mid U_1(p, r) \geq U_1(x, r), \forall x \in [0, 1]\}.$$

Note that we have the following:

- CASE $p \in [1/2, 1]$, the function U_1 attains the maximum value 1 in the two maximum points $p = 1/2$, $p = 1$, for every $r \in [0, 1]$, i.e.,

$$\max_{p \in [1/2, 1]} U_1(p, r) = U_1(1/2, r) = U_1(1, r) = 1, \quad \forall r \in [0, 1].$$

- CASE $p \in [0, 1/2[$ the function

- (a) strictly decreases on p for $r \in [0, 2/3[$ and attains the maximum for $p = 0$, so

$$\max_{p \in [0, 1/2[} U_1(p, r) = 2 - 2r.$$

Note that

$$2 - 2r > 1 \iff r \in [0, 1/2[$$

- (b) is constant on p for $r = 2/3$:

$$U_1(p, 2/3) = 2/3 \quad \forall p \in [0, 1]$$

- (c) strictly increases on p for $r \in]2/3, 1]$ and attains no maximum points because the domain is not closed, but, in this case, we have the following:

$$U_1(p, r) = 3pr - 2p - 2r + 2 < \frac{3}{2}r - 1 - 2r + 2 = 1 - \frac{1}{2}r < 1.$$

We have the following:

$$BR_1(r) = \begin{cases} \{0\} & \text{if } r \in [0, 1/2[, \\ \{0, 1/2, 1\} & \text{if } r = 1/2, \\ \{1/2, 1\} & \text{if } r \in]1/2, 1]. \end{cases}$$

The best reply of Player 2 can be easily computed, as there are no psychological effects:

$$BR_2(p) = \begin{cases} 1 & \text{if } p \in [0, 1/3[, \\ [0, 1] & \text{if } p = 1/3, \\ 0 & \text{if } p \in]1/3, 1]. \end{cases}$$

It immediately follows that there are no equilibria.

Notes

- ¹ The literature on psychological games has increased considerably in the past decades; we recall [8] for further theoretical findings, refs. [9–11] for some applications, and [12,13] for surveys on psychological games and references.
- ² Optimistic and intermediate attitudes actually have strong empirical support (see for example [21]).
- ³ See [7] for additional details on the topological and metric structure of the beliefs space.
- ⁴ The self-confirming (or conjectural) equilibrium was first studied in [36,37]. In [5], the definition was extended by taking into account different attitudes toward ambiguity or model uncertainty.
- ⁵ See the Introduction section in [5].
- ⁶ Note that, lower and upper semi-continuous set-valued maps (or correspondences) are also often denoted in the literature, respectively, as lower and upper hemi-continuous set-valued maps. However, we follow the notation in the book [38].
- ⁷ The game considered in Example 3.4 in [7] is different from the one presented in the present paper; however, ambiguous hierarchies of beliefs have the same structure.
- ⁸ For technical reasons, we consider the case where functions take values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

- ⁹ The function u_i is the one appearing in the construction of U_i (see Equation (2)).
- ¹⁰ Note that, in this section, the value of α , such that p^* and $\hat{p}$ are not defined, is studied separately.
- ¹¹ In this section, with the abuse of notation, we denote with $\hat{u}_i : A \rightarrow \mathbb{R}$ the classical utility function defined over the set of pure strategic profiles, and with $u_i : \Sigma \rightarrow \mathbb{R}$ the classical expected utility.
- ¹² In this particular example, the belief correspondence depends only on p . This is the simplest model to show the mathematical property we look at.

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