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On the Existence of Recursive Utility

Asen Kochov*

August 24, 2025

Abstract

Recursive utility functions are often defined implicitly, as the solution of a functional equation known as the Koopmans recursion. As conditions for a unique solution can be demanding, a recent literature has tackled the problem of multiplicity, advocating that one focus attention on either the least or greatest solution, which possess a number of attractive properties. While a least solution can be shown to exist, I show that in the most general formulation of the problem, a greatest solution need not. The goal of the paper is to provide a sufficient condition for the existence of a greatest solution, which I term the weakly contractive property and which generalizes the usual contractive property sufficient for uniqueness.

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1 Introduction

Recursive preferences have proven to be a tractable and useful generalization of the standard time-separable model of intertemporal choice. This papers revisits a basic question regarding the existence and uniqueness of such preferences when they are defined implicitly via their time aggregator, as is common in applications. More precisely, I am concerned with the existence of a greatest preference relation when uniqueness fails. The main contribution is to show that such existence is guaranteed for a class of time aggregators which I term *weak contractions*. The class, which nests many common specifications, has appeared in the literature before but has not been studied systematically. Several other contributions and some limitations are outlined further below.

As introduced by Koopmans [8], recursive preferences are represented by a utility function of the form:

$$U(c_0, c_1, c_2, \dots) = W(c_0, U(c_1, c_2, \dots)). \tag{1.1}$$

Thus, the ex ante utility of a consumption sequence is an aggregate of the initial consumption level and the utility of the continuation sequence. The standard time-separable model with exponential discounting arises as the special case in which the *time aggregator* W is both additively separable in its two inputs and linear in its second argument: $W(c, k) = u(c) + \delta k, \delta \in (0, 1)$. Beginning with Lucas and Stokey [10], it has become standard to define recursive preferences via the time aggregator W, in which case the utility function U over sequences – the primary object of interest – is not a primitive but is defined implicitly as the solution to (1.1), viewed as a functional equation in the unknown U. The natural question, of course, is whether a given W admits a solution and whether the solution is unique. Lucas and Stokey [10] answered both questions in the affirmative provided that W is bounded and a contraction in its second argument. More recently, Marinacci and Montrucchio [11] pioneered the study of *Thompson aggregators*, which are concave in their second argument and imply that a utility function U is uniquely defined in the interior of its domain.

As conditions for uniqueness can be sometimes demanding, a number of papers have tackled the problem of multiplicity, advocating that one focus attention on either the greatest or least solution, which turn out to possess a number of desirable properties. Thus, Becker and Rincón-Zapatero [3] emphasize the computational advantages of finding the least solution by means of successive approximations, while Bloise and Vailakis [5] and Bloise, Le Van, and Vailakis [6] highlight the upper semicontinuity of the greatest solution, which ensures the existence of maxima in standard optimization problems.

In this paper, I revisit the existence of a least and greatest solution. For the general version of the problem I study, such existence is claimed in the proof of Proposition 1 in Bloise and Vailakis [5], henceforth BV. The BV proof, however, is not entirely correct. I discuss the technical details in Section 3, where I also show that the existence of a least solution can be recovered after a minor gap in BV's proof is addressed. A greatest solution, on the other hand, need not exist at the level of generality suggested by BV's proof. The problem is twofold. First, Theorem 2 in Section 4 identifies a necessary restriction on W: for every consumption level c, the function $W(c,\cdot)$ must have a greatest fixed point, which I denote by M_c. To derive a greatest utility function U, I strengthen this necessary condition by further requiring that the fixed points M_c be ordered: $c \geq c'$ implies $M_c \geq M_{c'}$. While these conditions may appear technical, I show that both are implied by the conceptually simpler requirement that W be a *weak contraction*, by which I mean that $W(c,k) < k$ for all c and all sufficiently high k. The requirement is naturally interpreted as a minimal form of discounting: if continuation utility is sufficiently high, the agent would rather skip current consumption c and begin consuming the continuation sequence immediately.

The second problem concerning the existence of a greatest solution is that, in addition to restricting W, one must restrict the class of functions U within which one searches for a solution to equation (1.1). The problem has been recognized before, in particular by Streufert [13, 14], who proposes the class of *admissible* functions. I propose a weaker restriction which I term *regularity* and which turns out to be useful in my proofs, as highlighted by Remark 4 in Section 5 and the

discussion in Section 5.3. To establish the necessity of regularity, I show in Example 2 in Section 4.2 that while standard discounted utility is the unique regular solution of the aggregator $W(c,k) = u(c) + \delta k, \delta \in (0,1)$, there is also a continuum of irregular solutions without a supremum. In fact, such irregular U exist whenever $W(c,\cdot)$ is unbounded for some c. As I explain in Section 4.2, this demonstration of the necessity of irregularity is, for the purposes of this paper, somewhat starker than a corresponding one in Streufert [13].

My main result, Theorem 4 in Section 4, shows that if W is weakly contractive, then there exists a greatest regular solution $\overline{U}$. If, in addition, W is lower semicontinuous in its second argument, then $\overline{U}$ is monotone. A further contribution is the systematic study of weakly contractive mappings, with a number of properties derived in lemmas throughout the text. Theorem 2, which delivers new necessary conditions for the *uniqueness* of a solution and bridges useful techniques introduced by Marinacci and Montrucchio [11] with those of Lucas and Stokey [10], might of independent interest too. Finally, I note that inspired by Marinacci and Montrucchio [11, 12], all results in the paper, save one, are purely order-theoretic.

Two limitations must be acknowledged from the outset. First, I restrict attention to aggregators W that have a global minimum, thereby excluding cases in which utility of a consumption sequence may diverge to $-\infty$ as consumption converges to 0. Such cases raise well-understood issues – see the excellent treatise by Becker and Boyd [2, Sect.3.5.4] – and are typically handled separately. The second limitation is more substantive: in solving the recursion in (1.1), I restrict attention to utility functions U defined on the space $l_+^\infty \subset \mathbb{R}_+^\infty$ of *bounded* sequences.[1] While extending the analysis to accommodate larger domains awaits further development, I note that a number of my results, particularly the study of weakly contractive mappings, do not depend on this limitation.

2 Preliminaries

I begin with the formal definition of a time aggregator.

[1] The same holds for Prop.1 of Bloise and Vailakis [5], which prompted the present investigation.

Definition 1 *A function $W : \mathbb{R}_+^2 \to \mathbb{R}_+$ said to be a **time aggregator** if it is (i) nondecreasing in each component and (ii) for each $c \in \mathbb{R}_+$, there exists k such that $W(c, k) = k$.*

Property (ii) is necessary for the existence of recursive utility: a quick look at equation (1.1) reveals that the utility $U(c, c, ..)$ of the constant sequence $(c, c, \dots)$ is a fixed point of $W(c, \cdot)$. Property (i), while not necessary for the existence of recursive utility per se, is virtually always assumed as it is necessary for the existence of *monotone* recursive utility. Save for the implication that W have a global minimum, which was discussed in the introduction, the nonnegativity of W is a normalization. See Section 5.1 for details.

Elements of l_+^∞ are denoted by capital letters C, C', etc., or as sequences $(c_0, c_1, \dots)$, with c_t interpreted as the level of consumption in period t. Unless otherwise specified, t is always understood to vary in the set $\{0, 1, 2, \dots\}$. The space is l_+^∞ is endowed with the usual pointwise order. A function $U : l_+^\infty \to \mathbb{R}_+$ is **monotone** if $C \geq C'$ implies $U(C) \geq U(C')$. In the context of one-dimensional functions, I also use **nondecreasing** to mean monotone and **increasing** to mean nondecreasing and invertible.

Given t and $C \in l_+^\infty$, ${}_tC$ denotes the sequence $(c_t, c_{t+1}, \dots)$. Note that ${}_0C = C$. With this notation, recursion (1.1) can be more compactly written as $U(C) = W(c_0, U({}_1C))$, where I take it as understood that c_0 is the first element of C. If a function $U : l_+^\infty \to \mathbb{R}$ satisfies the recursion for all C, I say that U is **recursive**, omitting reference to the time aggregator W, which I treat as given and fixed throughout.[2] Given a set $B \subset l_+^\infty$ of consumption sequences, a function U is **recursive on** B if the recursion holds for all $C \in B$. A recursive function U is **least** if for any other recursive function U', we have $U'(C) \geq U'(C)$ for all C. A **greatest** recursive function is defined analogously. Given $c \in \mathbb{R}_+$ and $C \in l_+^\infty$, (c, C) denotes the sequence $(c'_0, c'_1, \dots)$ such that $c'_0 = c$ and for all $t > 0$, $c'_t = c_{t-1}$. Given a finite sequence $a \in \mathbb{R}_+^t, t > 0$ and $C \in l_+^\infty$, the sequence (a, C) is defined analogously. On occasion, it will also be helpful to allow a to be "empty" and write

[2]The only exception occurs in Section 5.1.

(a, C) for the sequence C.

Let $\mathcal{U}$ be the set of all functions $U : l_+^\infty \to \mathbb{R}_+$. Define the **Koopmans operator** $T : \mathcal{U} \to \mathcal{U}$ as

$$[TU](C) = W(c_0, U({}_1C)) \quad \forall C.$$

Observe that U is recursive if and only if $TU = U$. Finding a recursive utility is thus equivalent to finding a fixed point of T.

Marinacci and Montrucchio [11] show that it is sometimes convenient to study a pointwise version of the above fixed-point problem. First, given a consumption sequence C, say that $v = (v_0, v_1, \dots) \in \mathbb{R}_+^\infty$ is a **recursive utility sequence (RUS) for** C if $v_t = W(c_t, v_{t+1})$ for all t. An RUS v is **bounded** if $v \in l_+^\infty$. It is clear that a recursive utility function $U : l_+^\infty \to \mathbb{R}$ induces an RUS for *any* consumption sequence C. Theorem 2 will show that the converse is true as well: given any C, any RUS for C arises from some recursive function U. Now, for each $C \in l_+^\infty$, define the **MM operator** $T_C : l_+^\infty \to l_+^\infty$ as

$$[T_C v]_t = W(c_t, v_{t+1}) \quad \forall t.$$

Then, a bounded v is an RUS for C if and only if it is a fixed point of T_C.

3 Existence of a Least Recursive Utility

I begin by showing the existence of a least recursive function. First, a definition.

Definition 2 *A function $U : l_+^\infty \to \mathbb{R}_+$ is **minimally monotone** if $U(0, 0, \dots) \leq U(C) \leq U(\|C\|, \|C\|, \dots)$ for every C.*

Minimal monotonicity is Streufert's [13] notion of admissibility, adapted to the present setting. I opt for a different, more descriptive name so as to avoid confusion with the term regularity introduced in Section 4.2.

Theorem 1 *There exists a least recursive utility $\underline{U}$. It is unique and minimally monotone.*[3]

The finding that $\underline{U}$ is minimally monotone is new to this paper. It is noteworthy that while Streufert [13, 14] uses minimal monotonicity to restrict the class of functions within which one searches for a unique or greatest recursive utility – an issue I expand on in Section 5.3 – here minimal monotonicity emerges as implication, not a restriction.

Turn to existence. As noted in the introduction, a version of the result is found in the proof of Proposition 1 in BV, which in turn relies extensively on ideas of Marinacci and Montrucchio [11]. I make some clarifying remarks. First, Prop.1 in BV is stated for the smaller class of Thompson aggregators W. However, their proof of the existence of a least and greatest solution does not rely on the "Thompson-ness" of W. Instead, the proof invokes Proposition 1 in Marinacci and Montrucchio [11], which concerns arbitrary time aggregators W and asserts the existence of a fixed point of the operator T_C defined in Section 2. More precisely, BV invoke a remark following Proposition 1 in Marinacci and Montrucchio [11], which, in lieu of a proof, asserts that by Tarski's fixed point theorem the operator T_C admits a least and greatest fixed point. This passing remark in Marinacci and Montrucchio [11] is incorrect. The problem is that the set l_+^∞ is not a complete lattice as it has no supremum: each sequence in l_+^∞ is bounded, but there is no uniform bound across sequences. Accordingly, T_C need not have a greatest fixed point, as Example 1 in Section 4 illustrates. The correct statement is that T_C has a least fixed point, since l_+^∞ is an "inf-complete" lattice. I omit the details which mimic the proof of Lemma 1 below and related arguments in Marinacci and Montrucchio [11]. The existence proof can then be completed as in BV. Specifically, if $(\underline{U}_0(C), \underline{U}_1(C), \dots) \in l_+^\infty$ is the least fixed point of T_C, they show that the function $C \mapsto \underline{U}_0(C)$ is recursive and, by construction, the least recursive function.

I provide an alternative proof that focuses on the Koopmans operator T originally used in Lucas and Stokey [10] and others since then. In the process, I state some

[3]From now on, I will suppress the well-known fact that the least and greatest elements of a subset of an ordered space are unique whenever they exist.

auxiliary results, such as Lemma 2 below, which reveal useful properties of recursive functions and may be of independent interest.

Before turning to the proof, a final clarification is needed. Since BV focus on the aggregator T_C, which, as in Marinacci and Montrucchio [11], is defined to deliver *bounded* RUS's, it may appear that BV are implicitly restricting attention to recursive functions U that are **pointwise bounded** (my language) in the sense that for every C, the sequence $(U({}_tC))_t$ is bounded. Such a restriction is not needed, as my proof explicitly demonstrates: a least recursive function exists in the space of all recursive functions, and this function happens to be pointwise bounded. This conclusion should be contrasted with the analysis in Section 5, where I establish the existence of a greatest recursive function in the space of all *regular* recursive functions, a subset of the set of pointwise bounded recursive functions. The distinction between regularity and pointwise boundedness is discussed further in Remarks 1 and 4.

3.1 Some useful lemmas

I begin the proof of Theorem 1 with a simple lemma, an immediate consequence of Tarski's fixed-point theorem. Its role is to highlight the specific features of a time aggregator W.

Lemma 1 *For each c, $W(c,\cdot)$ has a least fixed point m_c. If $c \geq c'$, then $m_c \geq m_{c'}$.*

Proof. Let $X \subset \mathbb{R}_+$ be the set of fixed points of $W(c,\cdot)$, which is nonempty by the definition of a time aggregator, and let m_c be the infimum of X. The infimum is finite since W is nonnegative and thus bounded below. Since $W(c,\cdot)$ is nondecreasing, $W(c,m_c) \leq W(c,x) = x$ for all $x \in X$. Conclude that $W(c,\cdot)$ is a self-map on $[0,m_c]$, which, by Tarski's fixed-point theorem, implies that $W(c,\cdot)$ has a fixed point in $[0,m_c]$, which can be only m_c. To prove the second assertion, observe that if $c' \leq c$, then $W(c',m_c) \leq W(c,m_c) = m_c$. Thus, $W(c',\cdot)$ is a self map on $[0,m_c]$ and possesses a fixed point in $[0,m_c]$, which implies that $m_{c'} \leq m_c$. ■

It is obvious that for some or all c, $W(c,\cdot)$ may not have a greatest fixed point. As will be seen in Section 4, this implies that there does no exist a greatest recursive utility.

The next lemma is key to my proof and might be of independent interest. Given k, let $B_k = \{C \in l_+^\infty : \|C\| \leq k\}$ be the set of consumption sequences bounded above by k. Take $k' > k$. The lemma shows that a recursive function on B_k can be extended to the superset $B_{k'}$, *provided that* W admits some recursive function U' on $B_{k'}$. Furthermore, under suitable conditions, the extension of U can be chosen to inherit an upper bound on U'.

Lemma 2 (Extension Lemma) *Take $0 \leq k < k'$ and suppose U_k is a recursive utility function on B_k, while $U_{k'}$ is a recursive utility function on $B_{k'}$. Then U_k can be extended to a recursive utility function V on $B_{k'}$. Furthermore, if $U_k \leq b_k$ and $U_{k'} \leq b_{k'}$, where b_k and $b_{k'}$ are respectively fixed points of $W(k,\cdot)$ and $W(k',\cdot)$ such that $b_k \leq b'_k$, then V can be chosen so that $V \leq b_{k'}$.*

Proof. Let $B_k^e \subset B_{k'}$ be the set of $C \in B_{k'}$ s.t. $\|_t C\| \leq k$ for some t. Define V on $B_{k'}$ as follows. For each $C \in B_k^e$, let τ_C be the least t such that $\|_t C\| \leq k$. If $\tau_C = 0$ (i.e., $C \in B_k$), let $V(C) = U_k(C)$. If $\tau_C > 0$, let $a := (c_0, \ldots, c_{\tau_C - 1})$ and $V(C) := W(a, U_k(_{\tau_C} C))$. For $C \notin B_k^e$, let $V(C) := U_{k'}(C)$. By construction, $V = U_k$ on B_k. Thus, V is recursive on B_k. Next, observe that if $C \notin B_k^e$, then $_1 C \notin B_k^e$. Hence,

$$V(C) = U_{k'}(C) = W(c_0, U_{k'}(_1 C)) = W(c_0, V(_1 C)).$$

Finally, suppose $C \in B_k^e \setminus B_k$. Note that $\tau_C > 0$ since $C \notin B_k$. Then,

$$V(_{\tau_C - 1} C) = W(c_{\tau_C - 1}, U_k(_{\tau_C} C)) = W(c_{\tau_C - 1}, V(_{\tau_C} C))$$

Similarly, $V(_{\tau_C - 2} C) = W(c_{\tau_C - 2}, V(_{\tau_C - 1} C)$. Working backwards, conclude that $V(C) = W(c_0, V(_1 C))$.

Turn to regularity. If $C \in B_k \cup [B_{k'} \setminus B_k^e]$, then V is equal to either U_k or $U_{k'}$, and so $V(C) \leq \max\{b_k, b_{k'}\} = b_{k'}$. Next, let (c, C) be such that $\|C\| \leq k < c \leq k'$, which

means that $(c, C) \in B_k^e \setminus B_k$. Then,

$$V(c, C) = W(c, U_k(C)) \leq W(c, b_k) \leq W(k', b_{k'}) = b_{k'}.$$

Proceeding inductively, take any $C \in B_k^e \setminus B_k$ such that $V(C) \leq b_{k'}$. For any $c \leq k'$, it follows that $V(c, C) = W(c, V(C)) \leq W(k', b_{k'}) = b_{k'}$. ■

3.2 Proof of Theorem 1

The proof of Theorem 1 can now be completed. In addition to the preceding two lemmas, it uses the obvious fact that the set of recursive functions (solving a particular W) is a lattice.

For each k, let $\underline{\mathcal{U}}_k$ be the set of functions $U : B_k \to [0, m_k]$. The Koopmans operator T is a self-map on $\underline{\mathcal{U}}_k$, since:

$$[TU](C) = W(c_0, U({}_1C)) \leq W(\|C\|, U({}_1C))) \leq W(k, m_k) = m_k.$$

Since $\underline{\mathcal{U}}_k$ is a complete lattice, T has a least fixed point $\underline{U}_k$ in $\underline{\mathcal{U}}_k$. For each C, let $\underline{U}(C) = \underline{U}_{\|C\|}(C)$. To show that U is recursive, we need to show that $\underline{U}_{\|{}_1C\|}({}_1C) = \underline{U}_{\|C\|}({}_1C)$, since then

$$U(C) = U_{\|C\|}(C) = W(c_0, U_{\|C\|}({}_1C)) = W(c_0, U_{\|{}_1C\|}({}_1C)) = W(c_0, U({}_1C)).$$

Note that $\|{}_1C\| \leq \|C\|$ and hence $m_{\|{}_1C\|} \leq m_{\|C\|}$. By Lemma 2, $\underline{U}_{\|{}_1C\|}$ can be extended to a recursive utility in $\underline{\mathcal{U}}_{\|C\|}$. Conclude that $\underline{U}_{\|{}_1C\|}({}_1C) \geq \underline{U}_{\|C\|}({}_1C)$. Suppose $\underline{U}_{\|{}_1C\|}({}_1C) > \underline{U}_{\|C\|}({}_1C)$. On $B_{\|{}_1C\|}$, define the function $V = \underline{U}_{\|{}_1C\|} \wedge \underline{U}_{\|C\|}$. V is recursive, belongs to $\underline{\mathcal{U}}_{\|{}_1C\|}$, and is such that $V \lneq \underline{U}_{\|{}_1C\|}$. This contradicts the minimality of $\underline{U}_{\|{}_1C\|}$.

The minimality of $\underline{U}$ follows from similar arguments. I first claim that for each k, $\underline{U}$ coincides with $\underline{U}_k$ on B_k. Take any $C \in B_k$. Then, $\underline{U}(C) = \underline{U}_{\|C\|}(C) \leq m_{\|C\|} \leq m_k$. It follows that $\underline{U}|_{B_k} \in \underline{\mathcal{U}}_k$ and, hence, $\underline{U}_k \leq \underline{U}|_{B_k}$. If the inequality is strict for some $C \in B_k$, then $[\underline{U}_k|_{B_{\|C\|}} \wedge \underline{U}_{\|C\|}] \lneq \underline{U}_{\|C\|}$, contradicting the minimality of $\underline{U}_{\|C\|}$. Now,

suppose U is recursive and such that $U(C) < \underline{U}(C)$ for some C. Defining $V = U \wedge \underline{U}$ on $B_{\|C\|}$, we see that $V = [U \wedge \underline{U}_{\|C\|}] \lneq \underline{U}_{\|C\|}$ on $B_{\|C\|}$. Since V is recursive and $V \in \mathcal{U}_{\|C\|}$, this contradicts the minimality of $\underline{U}_{\|C\|}$.

It remains to show that $\underline{U}$ is minimally monotone. We have already seen that $\underline{U}(C) \leq m_{\|C\|}$. Applied to constant sequences, we see that $\underline{U}(c, c, \dots) \leq m_c$. Since $\underline{U}(c, c, \dots)$ is a fixed point of $(W(c, \cdot)$, the latter inequality is in fact an equality. It follows that $\underline{U}(C) \leq m_{\|C\|} = \underline{U}(\|C\|, \|C\|, \dots)$. Next, suppose that for some C, $\underline{U}(C) < m_0$. Note that this requires that $m_0 > 0$ and, hence, that $W(0, 0) > 0$. It follows that $W(c_0, \underline{U}({}_1C) < W(0, m_0) \leq W(c_0, m_0)$ and, in particular, that $\underline{U}({}_1C) < m_0$. The latter implies that $W(0, \underline{U}({}_1C)) > \underline{U}({}_1C)$. Otherwise, $W(0, \cdot)$ will be a self-map on $[0, \underline{U}({}_1C)]$ and, hence, have a fixed point strictly lower than m_0. Conclude that $\underline{U}(C) = W(c_0, \underline{U}({}_1C)) > \underline{U}({}_1C)$. Iterating the argument, we see that $(\underline{U}({}_tC))_t$ is a decreasing sequence. Let k^* be its limit. By similar arguments, deduce that $W(0, k^*) > k^*$. Then, by the monotonicity of W, $\underline{U}({}_tC) = W(c_{t-1}, \underline{U}({}_tC) > W(0, k^*) > k^*$ for all t, which contradicts k^* being the limit of $(\underline{U}({}_tC))_t$.

4 Necessary Conditions for a Greatest Recursive Utility

In this section, I study necessary conditions for the existence of a greatest recursive function. Since such conditions are automatically necessary for the existence of a unique recursive function, the analysis delivers some novel insights on that count as well.

4.1 Weak Contractions

I begin with the definition of a weak contraction that is central to my analysis.

Definition 3 *A time-aggregator W is a **weak contraction** if for every c, there exists k such that $W(c, k') < k'$ for all $k' \geq k$.*

As noted in the introduction, the property imposes a minimal form of discounting:

for sufficiently high continuation utility, the agent would rather skip current consumption and begin the continuation sequence immediately. The property is automatically satisfied if W is a Blackwell aggregator, that is, a contraction in its second argument. It is also satisfied by Thompson aggregators, as shown by Marinacci and Montrucchio [11, Lemma 1].

The next lemma deduces two key properties of weak contractions.

Lemma 3 *If W is a weak contraction, then for each c: (i) the function $W(c,\cdot)$ has a greatest fixed point M_c and, (ii) $M_c \geq M_{c'}$ whenever $c \geq c'$.*

Proof. Take some c and let k be such that $W(c,k') < k'$ for all $k' \geq k$. Then, $W(c,\cdot)$ is a self-map on $[0,k]$ and, by Tarski's fixed point theorem, has a greatest fixed point in $[0,k]$, which is in fact the greatest fixed point of the function. To prove the second statement, suppose $c \geq c'$ and take some $k > M_{c'}$ such that $W(c,k) < k$. Observe that $W(c,\cdot)$ is a self-map on $[M_{c'},k]$ and, by Tarski again, must have a fixed point $k^* \geq M_{c'}$. To prove the converse, suppose W is increasing. ■

The next example illustrates the necessity of $W(c,\cdot)$ having a greatest fixed point for the existence of a greatest recursive function.

Example 1 *Let $W(c,k) = \max\{c,k\}$. It is easy to check $U(c_0,c_1,\dots) = \sup_t c_t$ is a recursive utility function. It is indeed the least recursive function. There is on the other hand no greatest recursive function. To see this, define for any constant $m > 0$ the function:*

$$V_m(C) = \begin{cases} U(C) & \text{if } C \text{ attains its supremum} \\ U(C) + m & \text{otherwise} \end{cases}$$

It is easy to check that each V_m is recursive. Hence, W does not admit a greatest recursive utility. Likewise, given any consumption sequence C, any constant sequence $(k,k,...)$ such that $k \geq \|C\|$ is an RUS for C. Thus, there is no greatest (bounded) RUS either.

The example may appear contrived in that W, while nondecreasing, is not *increasing*. The next theorem shows that this is not essential. The theorem also establishes the previously mentioned "bridge" between recursive functions and RUS's: just as a recursive function induces an RUS for any consumption sequence C, an RUS for any given C can be extended to a recursive function. To state the theorem, which will be used repeatedly in subsequent sections, let $f_c = W(c, \cdot)$ and for each $a = (c_0, \dots, c_t)$, let $W(a, \cdot) = f_{c_0} \circ f_{c_1} \cdots \circ f_{c_t}$.

Theorem 2 *If $v = (v_t)_t$ is an RUS for C, then there exists a recursive function V such that $V({}_tC) = v_t$ for all t. In particular, if k is fixed point of $W(a, \cdot)$, there is a recursive utility V such that $U(a, a, \dots) = k$.*

Proof. By Theorem 1, there exists a recursive function U. Given an RUS $(v_t)_t$ for some C, define a function:

$$V(C') = \begin{cases} W(a, v_t) & \text{if } C' = (a, {}_tC) \text{ for some } a \text{ and } t \\ U(C') & \text{otherwise} \end{cases}$$

In the above, I allow $t = 0$ and a to be "empty," so that, in particular, $C' = C$ and $V(C) = v_0$. To see that V is well-defined, suppose $(a, {}_tC) = (a', {}_{t'}C)$ and $t > t'$. Then, $a = (a', a'')$ for some a''. It follows that ${}_{t'}C = (a'', {}_tC)$ and, since $(v_t)_t$ is an RUS for C, that $v_{t'} = W(a'', v_t)$. Thus,

$$V(a, {}_tC) = W(a, v_t) = W(a', W(a'', v_t)) = W(a', v_{t'}) = V(a', {}_{t'}C). \tag{4.1}$$

To show recursivity, let $E^*(C)$ be the set of C' such that $C' = (a, {}_tC)$ for some a and t, once again allowing $t = 0$ and a to be "empty" so that $C' = C$. If $C' \in E^*(C)$, then ${}_1C' \in E^*(C)$. A sequence of equalities analogous to (4.1) show that $V(C') = W(c'_0, V({}_1C'))$. If $C' \notin E^*(C)$, then ${}_1C' \notin E^*(C)$ and $V(C') = W(c'_0, V({}_1C'))$ holds immediately from the definition of V.

The second assertion of the theorem can be deduced as an application of the first. I give a direct proof, highlighting the simpler construction of the function V that is

possible in this case. In particular, suppose k is a fixed point of $W(a,\cdot)$ and let V be the function:

$$V(C) = \begin{cases} W(a',k) & \text{if } C = (a', a, a, \dots) \\ U(C) & \text{otherwise} \end{cases} \tag{4.2}$$

V is well-defined since $W(a', W(a'', k)) = W((a', a''), k)$. Taking $a' = a$, we see that $V(a, a, \dots) = W(a, k) = k$. To show that V is recursive, first given a sequence C', let $E(C')$ be the set of C'' such that $C'' = (a', C')$ for some a'. Observe that for general C', $E(C')$ is a strict subset of the set $E^*(C')$ defined in the previous part of the proof. However, a sequence C' is repeating if and only if $E(C') = E^*(C')$. In particular, if C' is repeating, then $C'' \in E(C')$ if and only if ${}_1C'' \in E(C')$. We can thus replicate the arguments used in the first part of the theorem. Specifically, if $C \notin E(a, a, \dots)$, then ${}_1C \notin E(a, a, \dots)$ and so

$$V(C) = U(C) = W(c, U(C)) = W(c, V(C)).$$

Alternatively, if $C \in E(a, a, \dots)$, then ${}_1C \in E(a, a, \dots)$. Thus, ${}_1C = (a', a, a, \dots)$ for some a', and

$$V(C) = V(c_0, a', a, a, \dots) = W(c_0, W(a', k)) = W(c_0, V({}_1C)).$$

■

I leave open the problem whether the second property of weak contractions, that $c \geq c'$ implies $M_c \geq M_{c'}$, is necessary for the existence of a greatest utility function. On the other hand, observe that by Lemma 1 and Theorem 2, the property is necessary for the existence of a *unique* recursive function. Another necessary condition for uniqueness, which emerges as a corollary of Theorem 2, is that any $a \in \cup_{t>0}\mathbb{R}^t_+$, the function $W(a,\cdot)$ must have a unique fixed point. This is stronger than the requirement that for any c, the function $W(c,\cdot)$ must have a unique fixed point.[4]

[4] An example is available upon request.

4.2 Regular Utility Functions

As remarked in the introduction, to obtain a greatest recursive utility function U, one must not only restrict W but also the class of functions U that are considered a solution to (1.1). To that end, let M_k be the greatest fixed point of $W(k,\cdot)$, which exists whenever W is a weak contraction.

Definition 4 *A function $U : l_+^\infty \to \mathbb{R}_+$ is **regular** if $U(C) \leq M_{\|C\|}$ for every C.*

The next example demonstrates the necessity of regularity. I note the use of Theorem 2 in the construction.

Example 2 (Irregular solutions of the standard recursion) *Consider the standard time aggregator $W(c,k) = u(c_t) + \delta k$, with $\delta < 1$ and $u(\cdot)$ nonnegative, which admits the classical discounted utility function $U(c_0, c_1, \dots) = \sum_t \delta^t u(c_t)$ as a solution. Take any C and any $v_0 > U(C)$. For $t > 0$, define $v_t = \delta^{-1}(v_{t-1} - u(c_{t-1}))$, so that for all t:*

$$v_t = \frac{1}{\delta^t}(v_0 - \sum_{\tau=0}^{t-1} \delta^\tau u(c_\tau)) > 0, \quad \textit{and}$$
$$v_t = u(c_t) + \delta v_{t+1}$$

By construction, $(v_t)_t$ is an RUS for C. It follows from Theorem 2 that there exists a recursive function $V : l_+^\infty \to \mathbb{R}_+$ such that $V({}_tC) = v_t$ for all t. Since v_0 can be chosen arbitrarily, there is no greatest recursive function. On the other hand, since $(v_t)_t$ diverges, none of the functions V constructed in this matter are regular, which requires, in particular, that $v_t = V({}_tC) \leq M_{\|{}_tC\|} \leq M_{\|C\|}$ for all t.[5]

Remark 1 *A useful distinction between Examples 1 and 2 can be drawn if one considers a notion weaker than regularity. From Section 3, recall that a function*

[5]The multiplicity of functions V does not stem from "cardinal" transformations of utility: cardinality is already encoded in the aggregator W. Indeed, the functions V will typically represent distinct preferences. For a sharp demonstration of this, see Becker and Rincon-Zapatero [4].

$U : l_+^\infty \to \mathbb{R}_+$ is pointwise bounded if $\sup_t U({}_tC) < \infty$ for all $C \in l_+^\infty$. Clearly, the functions constructed in Example 2 are not simply irregular; they fail to be pointwise bounded. On the other hand, the functions V constructed in Example 1 are all pointwise bounded, highlighting the need to restrict W, not only the space of functions U. Regularity cannot be put to such use since it cannot be defined in the context of the max aggregator in Example 1: the greatest fixed points M_c do not exist. While the latter might suggest that poinwise boundedness rather than regularity is the appropriate restriction to impose on the functions U, I have been unable to establish the existence of a greatest U within this broader class, as I explain further in Remark 4 in Section 5. For this reason, the rest of the analysis is couched in the context of regular, rather than pointwise bounded, functions.

The construction behind Example 1 is not limited to the standard recursion but applies to any W for which we can find a sequence C such that for each t, the function $W(c_t, \cdot)$ is unbounded. The latter ensures that starting from any $v_0 > 0$, one can solve the forward recursion $v_{t+1} = W^{-1}(c_t, v_t)$. Picking $v_0 > M_{\|C\|}$ further ensures that the constructed function V fails regularity.

Proposition 3 *Suppose W is such that for some $C \in l_+^\infty$, $W(c_t, \cdot)$ is unbounded for every t. Then W does not admit a greatest recursive function in the space of all functions $U : l_+^\infty \to \mathbb{R}_+$.*

As noted in the introduction, the need to restrict the space of functions U is recognized in Streufert [13, 14]. In direct relation to Example 2, Streufert [13, p.84] shows that the standard aggregator W admits a continuum of "inadmissible general solutions." Translated to the present setting, admissibility coincides with minimal monotonicity, which is stronger than regularity. More substantively, the "general solutions" constructed by Streufert [13] are not recursive functions in the sense of this paper, which is what Example 2 delivers by leveraging Theorem 2.

5 Greatest Regular Recursive Utility

The next theorem shows that the weak-contraction property is sufficient for the existence of a greatest *regular* recursive function. Furthermore, the function is minimally monotone.

Theorem 4 *If W is a weak contraction, then there is a greatest regular recursive utility $\overline{U}$. Moreover, $\overline{U}$ is minimally monotone and such that $\overline{U}(c, c, \dots) = M_c$ for all c.*

To prove the theorem, I need the following lemma, an elaboration on Theorem 2.

Lemma 4 *If W is a weak contraction and $v = (v_t)_t$ is an RUS for C such that $\|v\| \leq M_{\|C\|}$, then there exists a regular recursive function V such that $V(C) = v_0$.*

Proof. By Theorem 1, there exists a regular recursive function U. Define a recursive function V as in the proof of Theorem 2. To check that V is regular, take $c' > \|C\|$. Observe that $M_{\|(c',C)\|} = M_{c'}$ and that, by Lemma 3, $M_{c'} \geq M_{\|C\|}$. It follows that

$$W(c', C) \leq W(c', M_{\|C\|}) \leq W(c', M_{c'}) = M_{c'} = M_{\|(c',C)\|}.$$

If $c' \leq \|C\|$, one shows analogously (without the need to invoke Lemma 3) that $W(c', C) \leq M_{\|C\|} = M_{\|(c',C)\|}$. Using the definition of V and iterating the preceding argument shows that V is regular. ■

The proof of Theorem 4 can now be completed.

Proof of Theorem 4. For each k, let $\overline{\mathcal{U}}_k^r$ be the set of regular functions on B_k. The range of all such functions is $[0, M_k]$. Also, $\overline{\mathcal{U}}_k^r$ is a complete lattice and T is a self-map on $\overline{\mathcal{U}}_k^r$, the latter since

$$[TU](C) = W(c_0, U({}_1C)) \leq W(k, M_k) = M_k.$$

Conclude that for each k, there is a greatest regular recursive function $\overline{U}_k$ on B_k.

Define $\overline{U}$ on l^∞_+ by letting $\overline{U}(C) := \overline{U}_{\|C\|}(C)$. Clearly, $\overline{U}$ is regular. I claim that $\overline{U}$ is recursive. Take any C and, to ease notation, let $k := \|_1 C\|, k' := \|C\|$, and $M := M_k$ and $M' := M_{k'}$. Observe that $k' \geq k$ and, by Lemma 3, $M' \geq M$. The restriction of $\overline{U}_{k'}$ to B_k is both regular and recursive on B_k. Thus, $\overline{U}_{k'} \leq \overline{U}_k$ on B_k. By Lemma 2, $\overline{U}_k$ can be extended to a regular and recursive function on $B_{k'}$. It follows that $\overline{U}_k \leq \overline{U}_{k'}$ on B_k. Conclude that $\overline{U}_{k'} = \overline{U}_k$ on B_k. But then

$$\overline{U}(C) = \overline{U}_{k'}(C) = W(c_0, \overline{U}_{k'}({}_1C)) = W(c_0, \overline{U}_k({}_1C)) = W(c_0, \overline{U}({}_1C)).$$

In addition to recursivity, the above argument shows that $\overline{U}$ is equal to $\overline{U}_k$ on each set B_k, which shows that $\overline{U}$ is the greatest regular recursive function.

To show that $\overline{U}$ is minimally monotone, suppose first that $\overline{U}(C) < M_0$. Observe that the MM operator T_C is a self-map on $[M_0, M_{\|C\|}]^\infty \subset l^\infty_+$. Thus, C has an RUS v such that $v_0 \geq M_0$ and $\|v\| \leq M_{\|C\|}$. By Lemma 4, there is a regular recursive function V such that $V(C) \geq M_0 > \overline{U}(C)$, which contradicts $\overline{U}$ being the greatest regular recursive function. Likewise, suppose $\overline{U}(C) > \overline{U}(\|C\|, \|C\|, \dots)$ for some C. Since $(M_{\|C\|}, M_{\|C\|}, \dots)$ is an RUS for $(\|C\|, \|C\|, \dots)$, there is by Lemma 4 a regular recursive utility function V such that $V(\|C\|, \|C\|, \dots) = M_{\|C\|}$. Since $\overline{U} \geq V$, conclude that $\overline{U}(\|C\|, \|C\|, \dots) = M_{\|C\|}$. But then $M_{\|C\|} < \overline{U}(C)$, which contradicts the regularity of $\overline{U}$. ■

Remark 2 *The last step in the proof of Theorem 4 establishes that $\overline{U}(c, c, \dots) = M_c$ for every c. This parallels the fact that $\underline{U}(c, c, \dots) = m_c$ established in the proof of Theorem 1.*

Remark 3 *The full force of the weak-contraction property is not needed for the result; the two implications derived in Lemma 3 suffice. I require the full property only in the proof of Theorem 5 below. See Lemma 5 in particular.*

Remark 4 *A key step in the proof uses the fact, by Lemma 2, a regular and recursive function on B_k can be extended to a regular and recursive function on $B_{k'}$. Returning to the discussion in Remark 1, I note I have been unable to establish an*

analogous statement for pointwise-bounded functions. Intuitively, the proof of Lemma 2 uses the fact for a regular function U, the bounds $M_{\|C\|}$ scale up with the sequence C. The same need not be true of pointwise-bounded functions.

Theorem 4 should be compared with a result in Streufert [14, Thm 2.2]. The latter shows that if W admits an upper-convergent function U, then U is the greatest admissible (minimally monotone) recursive function. Upper-convergence – see (5.1) for a definition adapted to the present setting – is a natural property to impose on a utility function U: it says that U can be obtained via successive approximations from above. The main contribution of Theorem 4 relative to Streufert [14, Thm 2.2] is to identify a sufficient condition – the weakly contractive property – not in terms of U but in terms of W, which is necessary when, as in Lucas and Stokey [10], W is the primitive, not U.

5.1 Ordinality of the Results

It is important to note that both regularity and the weak-contraction property are *ordinal* conditions. To understand what I mean by this, suppose U is recursive (given W) and let $f : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function. As is well understood, the function $V = f \circ U$, which induces the same preference relation as U, satisfies the recursion $V(C) = \hat{W}(c_0, V({}_1C))$, where $\hat{W}(c, k) = f(W(c, f^{-1}(k))$ is a time aggregator no longer normalized to be nonnegative. Notably, U is a solution to the recursion defined by W if and only if $V = f \circ U$ is a solution to the recursion defined by $\hat{W}$. By the same token, k is a fixed point of $W(c, \cdot)$ if and only if $f(k)$ is a fixed point of $\hat{W}(c, \cdot)$. It follows that U is regular if and only if V is regular, and the same is true for minimal monotonicity. Hence, while W itself is a cardinal object, the greatest regular recursive utility $\overline{U}$, just as the least recursive function $\underline{U}$, reflects purely ordinal information embedded in W. Finally, observe that W is a weak contraction if and only if $\hat{W}$ is.

5.2 Continuous aggregators

The next theorem shows that if W is lower semicontinuous in its second argument, then the greatest minimally monotone recursive utility $\overline{U}$ is in fact monotone.[6] Furthermore, if W is continuous (jointly, in both arguments), then $\overline{U}$ is upper semicontinuous in the sup norm on l^∞_+. I note that the proof of upper semicontinuity invokes two additional properties of the weakly contractive aggregators detailed in Lemma 5 below. By comparison, the analogous result that the least recursive utility is monotone and lower semicontinuous does not require W to be weakly contractive; I omit the arguments, which can be found in Bloise and Vailakis [5, Prop. 1].

Theorem 5 *Suppose W is a weak contraction. If W is lower semicontinuous in its second argument, then $\overline{U}$ is monotone and such that*

$$\overline{U}(C) = \lim_t W(c_0, W(c_1, \dots, W(c_t, M_{\|C\|}) \dots) \quad \forall C. \tag{5.1}$$

If W is continuous, then $\overline{U}$ is, in addition, upper semicontinuous in the sup norm on l^∞_+.

I begin with a lemma, highlighting additional properties of weak contractions.

Lemma 5 *Suppose W is a weak contraction. For every $C \in l^\infty_+$, $W(c_0, M_{\|_1 C\|}) \leq M_{\|C\|}$. Moreover, if W is continuous and $c_n \to c$, then $\limsup_n M_{c_n} \leq M_c$.*

Proof. To prove the first assertion, suppose $M_{c_0} \geq M_{\|_1 C\|}$. Then $W(c_0, M_{\|_1 C\|}) \leq W(c_0, M_{c_0}) = M_{c_0} = M_{\|C\|}$. Alternatively, if $M_{c_0} < M_{\|_1 C\|}$, then, by Lemma 3, $c_0 < \|_1 C\|$ and $W(c_0, M_{\|_1 C\|}) \leq W(\|_1 C\|, M_{\|_1 C\|}) = M_{\|_1 C\|} = M_{\|C\|}$. To prove the second assertion, I first claim that the sequence $(M_{c_n})_n$ is bounded. Take

[6] The use of lower semicontinuity to deduce monotonicity may appear odd but recall that in the context of a nondecreasing function $f : \mathbb{R} \to \mathbb{R}$, lower semicontinuity is equivalent to the purely order-theoretic assumption that $f(x_n) \downarrow x$ for all $x_n \downarrow x$. In the context of Thompson aggregators, a thorough analysis of order continuity and its implications can be found in Becker and Rincón-Zapatero [3]

$\delta > 0$. Since W is continuous, it must be that for all n sufficiently large, $c_n \leq c + \delta$ and, hence, $M_{c_n} \leq W(c + \delta, M_{c_n})$. Conclude that $(M_{c_n})_n$ cannot be unbounded, as this would contradict the weak-contraction property. Finally, as $M_{c_n} = W(c_n, M_{c_n})$ for every n, one deduces that every limit point of $(M_{c_n})_n$ is a fixed point of $W(c, \cdot)$. The assertion follows. ■

I can now complete the proof of Theorem 5.

Proof of Theorem 5. For any C, the sequence

$$W(c_0, M_{\|C\|}),\ W(c_0, W(c_1, M_{\|C\|})),\ W(c_0, W(c_1, W(c_2, M_{\|C\|}))), \dots$$

is decreasing and bounded from below by 0. Let $V(C)$ be its limit. If W is lower semicontinuous in its second argument, then

$$\begin{aligned} V(C) &= \lim_t W(c_0, W(c_1, \dots, W(c_t, M_{\|C\|}) \dots) \\ &= W(c_0, \lim_t W(c_1, \dots, W(c_t, M_{\|C\|}) \dots)) = W(c_0, V({}_1C)) \end{aligned}$$

Thus, V is a recursive utility. By construction, V is monotone. Since V is monotone and recursive, $V \leq \overline{U}$. To prove the opposite inequality, it is enough to show that for every t,

$$\overline{U}(C) \leq \overline{U}(c_0, \dots, c_t, \|C\|, \|C\|, \dots) = W(c_0, W(c_1, \dots, W(c_t, M_{\|C\|})).$$

The inequality follows since $\overline{U}$ is recursive and minimally monotone; the equality follows from recursivity and since $\overline{U}(\|C\|, \|C\|, \dots) = M_{\|C\|}$, as noted in Remark 2.

To prove the final assertion, suppose W is continuous and take C^n converging to C in the sup-norm on l^∞_+. For each t, let $v_t = \limsup_n \overline{U}({}_tC^n)$ and deduce that

$$v_t = \limsup_n W(c^n_t, \overline{U}({}_{t+1}C^n)) = W(c_t, \limsup_n \overline{U}({}_{t+1}C^n)) = W(c_t, v_{t+1}).$$

Thus, $(v_t)_t$ is an RUS for C. By Lemmas 5 and 3 in turn, deduce that $v_t \leq M_{\|{}_tC\|} \leq M_{\|C\|}$ for every t. By Lemma 4, there is a regular recursive utility $\hat{U}$ such that $\hat{U}(C) = \limsup_n \overline{U}(C^n)$. Conclude that $\limsup_n \overline{U}(C^n) \leq \overline{U}(C)$. ■

5.3 A Uniqueness Result for Contractions

As an application of Theorems 1 and 4, I now show that every contraction W admits a *unique* regular recursive utility. The point is to dispense with the non-trivial assumption that W be bounded made in Lucas and Stokey [10]. This generalization is not entirely new: with some care, the result can be deduced as a special case of Theorem F in Streufert [13] or Theorem 1 in Le Van and Vailakis [9]. (These results establish uniqueness within different classes of functions, though this difference is not substantive.) I state the result so as to provide a novel proof via Boyd's [7] celebrated weighted contraction theorem, using the fixed points M_c of W to construct a suitable weighting function. We have already seen that their existence is necessary for uniqueness; the present argument provides another illustration of their usefulness in the study of recursive utility. To avoid introducing Boyd's machinery in full, I only outline the argument at the end of the section and first present a formal proof using the methods behind Theorems 1 and 4.

Definition 5 *A time-aggregator W is a **contraction**, or a **Blackwell aggregator**, if there is $\delta < 1$ such that $|W(c,k) - W(c,k')| \leq \delta|k - k'|$ for all $c, k, k' \in \mathbb{R}_+$.*

Theorem 6 *If W is a contraction, there exists a unique regular recursive utility U, which is furthermore monotone. If, in addition, W is continuous, then U is continuous in the product topology on l_+^∞.*

Proof. Recall the definitions of the sets $\overline{\mathcal{U}}_k^r$ and $\underline{\mathcal{U}}_k$ from the proofs of Theorems 4 and Theorem 1 respectively. Since W is a contraction, $W(c,\cdot)$ has a unique fixed point for each c. Conclude that $\overline{\mathcal{U}}_k^r \subset \underline{\mathcal{U}}_k$. Each set $\overline{\mathcal{U}}_k^r$ is a Banach space when endowed with the sup norm. As T is a self map on $\overline{\mathcal{U}}_k^r$ and a contraction itself, T has a unique fixed point in $\overline{\mathcal{U}}_k^r$, which coincides with the greatest recursive function $\overline{U}$ derived in Theorem 4. Since the least recursive utility function is regular itself, its restriction to B_k belongs to $\overline{\mathcal{U}}_k^r$ as well. Since k is arbitrary, one concludes that $\underline{U} = \overline{U} =: U$. Since W is a contraction, it continuous in its second argument. Monotonicity follows from Theorem 5.

The final assertion is a corollary of results in Becker and Boyd [2]. Specifically, Theorem 9 in Becker and Boyd [2, p.45] shows that continuity in the product topology is equivalent to order continuity. Sufficient conditions for the latter are given in Proposition 3 in Becker and Boyd [2, p.89]. Namely, U must be monotone, continuous in c_t for each t, and biconvergent. Monotonicity was already established; continuity in each component follows from continuity of W in its first argument and recursivity. Biconvergence stipulates that for every C and every $k \geq \|C\|$, we have

$$U(C) = \lim_t U(c_0, \dots, c_t, k, k, \dots) = \lim_t U(c_0, \dots, c_t, 0, 0, \dots).$$

These equalities can be established as in the proof of Theorem 5, after taking into account that $U = \overline{U} = \underline{U}$ and that each $W(c, \cdot)$ has a unique fixed point. ■

To deduce Theorem 6 from Boyd's [7] weighted contraction theorem, consider the function $\phi : C \mapsto 1 + M_{\|C\|}$ and observe that it meets all the requirements of a *weighting function* in Boyd's theorem. It follows that there exists a unique recursive utility in the space of all functions $V : l_+^\infty \to \mathbb{R}_+$ such that $\sup_C [V(C)/\phi(C)] < +\infty$.[7] Since this space is strictly larger than the space of regular functions, the approach delivers a (slightly) stronger conclusion than Theorem 6.

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