

Escobar, Marcos; Spies, Ben; Zagst, Rudi

Article

Do jumps matter in discrete-time portfolio optimization?

Operations Research Perspectives

Provided in Cooperation with:

Elsevier

Suggested Citation: Escobar, Marcos; Spies, Ben; Zagst, Rudi (2024) : Do jumps matter in discrete-time portfolio optimization?, Operations Research Perspectives, ISSN 2214-7160, Elsevier, Amsterdam, Vol. 13, pp. 1-13,
<https://doi.org/10.1016/j.orp.2024.100312>

This Version is available at:

<https://hdl.handle.net/10419/325792>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

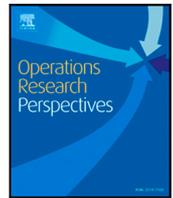
Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



<https://creativecommons.org/licenses/by-nc/4.0/>



Do jumps matter in discrete-time portfolio optimization?

Marcos Escobar-Anel^a, Ben Spies^{b,*}, Rudi Zagst^b

^a Department of Statistical and Actuarial Sciences, University of Western Ontario, London, Ontario, Canada N6A5B7

^b Department of Mathematics, Technical University of Munich, Parkring 11/II, D-85748 Garching b. München, Germany

ARTICLE INFO

JEL classification:

G11
C58
C61

Keywords:

Dynamic portfolio optimization
Lévy GARCH models
Jumps
Wealth-equivalent loss

ABSTRACT

This paper studies a discrete-time portfolio optimization problem, wherein the underlying risky asset follows a Lévy GARCH model. Besides a Gaussian noise, the framework allows for various jump increments, including infinite-activity jumps. Using a dynamic programming approach and exploiting the affine nature of the model, we derive a single equation satisfied by the optimal strategy, and we show numerically that this equation leads to a unique solution in all special cases. In our numerical study, we focus on the impact of jumps and evaluate the difference to investors employing a Gaussian HN-GARCH model without jumps or a homoscedastic variant. We find that both jump-free models yield insignificant values for the wealth-equivalent loss when recalibrated to simulated returns from the jump models. The low wealth-equivalent loss values remain consistent for modified parameters in the jump models, indicating extreme market situations. We therefore conclude, in support of practitioners' preferences, that simpler models can successfully mimic the strategy and performance of discrete-time conditional heteroscedastic jump models.

1. Introduction

Jumps have been a topic of immense interest in portfolio optimization for decades, particularly in a continuous-time setting thanks to analytical solutions, see for instance Merton [1]. Since empirical evidence suggests that financial markets exhibit occasional large price movements, considering jump components allows investors to model the underlying asset price movements more adequately with the corresponding improvement in financial decisions, from derivative pricing to risk analysis and portfolio management. On the other hand, models are desired to remain mathematically tractable and in the best case deliver closed-form expressions for pricing and portfolio optimization problems, a property that is easily imperiled.

Within the area of jump-free GARCH models, the lack of analytical solutions was overcome only when Heston and Nandi [2] introduced their HN-GARCH model, allowing for closed-form solutions in derivative pricing via the affine nature of the cumulant generating function. In the corresponding class of affine GARCH models, still jump-free, the focus of the following years was on extensions towards non-Gaussian innovations, see Christoffersen et al. [3] and Badescu et al. [4] for a formalized framework. The most common approach to extending these models to incorporate jumps consists of adding a (dynamic) Compound Poisson component in addition to the Gaussian noise. In this manner, Christoffersen et al. [5] introduced a model of GARCH type with stochastic volatility and separate dynamics for jumps, related to

the non-affine work by Maheu and McCurdy [6] and Duan et al. [7,8]. In their version, however, Christoffersen et al. [5] allow for jumps not only in the dynamics of the log price process, but also in the variance of the normal innovation and in the jump intensity, jeopardizing the affine form of the cumulant generating function. Ornathanalai [9] formulated the corresponding model with jumps incorporated in the return dynamics only, preserving the affine nature of the model. The model includes two contemporaneously independent innovations: A Gaussian innovation and a Lévy jump component. The general framework for the latter allows for finite and for infinite-activity jumps and in this way extends the range of jump innovations beyond the Compound Poisson approach. This framework has been generalized recently and forms a special case of the affine multi-factor model introduced in Augustyniak et al. [10].

Both the HN-GARCH and the Lévy GARCH framework were originally introduced for option pricing; Escobar-Anel et al. [11,12] first used the form of the generating function to derive the solution to a discrete-time portfolio optimization problem based on the jump-free model formulation by Badescu et al. [4]. The present paper extends this approach to the Lévy GARCH framework and thus provides the opportunity to study the relevance of modeling jumps for portfolio optimization problems in the affine GARCH environment.

The impact of capturing extreme events on investment strategies has been studied extensively, although mostly in continuous time.¹

* Corresponding author.

E-mail addresses: marcos.escobar@uwo.ca (M. Escobar-Anel), ben.spies@tum.de (B. Spies), zagst@tum.de (R. Zagst).

¹ We are not aware of any work investigating the impact of jumps in discrete-time portfolio optimization.

The analysis by Liu et al. [13] suggests that price jumps have a larger effect than jumps in volatility. Ascheberg et al. [14] claimed that jumps matter more if either the expected jump size or the jump intensity is large. These lines serve as a guideline for creating extreme jump settings to challenge the capability of jump-free models to deliver comparable investment strategies. Our paper differs from the existing literature on one key aspect. We take advantage of the capacity of the HN-GARCH model to deliver accurate estimates of the parameters – see Escobar-Anel et al. [15] – in order to assess if the portfolio of an investor with a simple model could perform as well as the investor with an advanced Lévy GARCH model. In other words, given that the true model of reality is unknown, we act as a practical investor, working with a simple model to make decisions. This allows to check in a control experiment if a simple, albeit wrong model could match the performance of a correct advanced model. Our conclusions are eye-opening concerning the power of simple models in a discrete-time portfolio optimization setting.

Many authors have investigated the role of non-Gaussianity in asset returns in the context of portfolio selection. One prominent example is the work by Bekaert et al. [16] suggesting that it is rather immaterial for an investor considering emerging market allocations to accommodate skewness and excess kurtosis. The crucial point that a wide range of expected utility problems can be well approximated by a mean-variance approach was summarized by Markowitz [17].

Our results are not contradictory with the continuous-time literature for three main reasons. Firstly, a continuous-time investor, although hypothetical, could in principle realize small benefits between a true and a wrong model more frequently. Such dynamics could be very profitable in the long term, and a detailed analysis in this regard could certainly be subject to further research. Secondly, most studies of complex models rely on sensitivity analyses for the parameters. However, the same model with different parameters does not describe the same reality, so results could be misleading. Lastly, even when two models are estimated on the same data, small sample sizes may incorporate estimation errors into the analysis, possibly creating a distorted conclusion, since the estimation error could exacerbate a small difference in performance. In conclusion, our analyses provide a fresh perspective into the old question of the importance of jumps for portfolio decisions.

Our main contributions are:

- We fully develop a discrete-time dynamic portfolio optimization problem in an affine GARCH framework with jumps and derive a single equation satisfied by the optimal strategy, based on an established and reliable approximation of the self-financing condition.
- We apply the general result to three different types of jump structures: the Merton jump, the Normal Inverse Gaussian and the Variance Gamma model. Furthermore, we derive the well-known HN-GARCH model as a jump-free special case of our setting. In all cases, our numerical studies show that there is a unique solution to the optimality equation.
- We study the impact of jumps on investment strategies via a comparison to the HN-GARCH model and a homoscedastic variant. We find that correctly calibrated jump-free models can imitate all three investigated jump models very well. This result is reflected in a low wealth-equivalent loss for investors following suboptimal jump-free strategies, and it is robust with respect to more extreme jump environments.

With these contributions, we overcome three key challenges in discrete-time portfolio optimization. Firstly, we accommodate a wide range of stylized facts of asset returns by using a Lévy GARCH framework, which allows for jumps in the asset price process. Secondly, we provide a single equation for the optimal solution and demonstrate in our numerical studies that this is easy to solve in practice. Lastly, our method of finding the best possible jump-free analogue for a Lévy GARCH

model eliminates the risk of estimation errors due to small sample sizes and allows us to make a fair comparison between jump-free and jump models.

The remainder of this paper is organized as follows: In Section 2, we introduce the framework for the portfolio optimization problem, which is solved in Section 3, providing optimality conditions in general form and explicitly for three different types of jump increments. The numerical analysis follows in Section 4, targeting specifically the impact of jumps and a comparison to jump-free HN-GARCH and homoscedastic models. We conclude in Section 5.

2. Problem description

2.1. The Lévy GARCH model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t=0,1,\dots,T}$. We start by introducing the dynamics of a Lévy GARCH model for the underlying asset price process in our portfolio optimization problem. Adopting the setting proposed by Ornathanalai [9], we use two contemporaneously independent random variables z_t and y_t to model the log return $X_t - X_{t-1} = \log(S_t/S_{t-1})$ of the single risky asset in the market. While z_t follows a normal $\mathcal{N}(0, h_{z,t})$ distribution, y_t is a pure Lévy jump innovation with generalized conditional Fourier transform

$$F_y(u; t-1, t) = \mathbb{E}_{t-1} [e^{uy_t}] = \exp\{\psi_y(u; t-1, t)\},$$

where ψ_y denotes the cumulant exponent (or cumulant generating function) of y_t . The cumulant exponent of the Lévy jump innovation y_t is assumed to be multiplicatively separable with respect to h_y , i.e., it admits the representation

$$\psi_y(u; t-1, t) = h_{y,t} \cdot \xi_y(u), \tag{1}$$

for some quadratic differentiable $\xi_y(u)$, which is independent of $h_{y,t}$. The explicit distribution of y_t remains unspecified at this point, potential choices include the Merton jump, a Normal Inverse Gaussian (NIG) or a Variance Gamma (VG) innovation. Note that the cumulant exponent of the Gaussian innovation z_t is multiplicatively separable with respect to h_z :

$$\psi_z(u; t-1, t) = h_{z,t} \cdot \frac{u^2}{2}, \quad \text{i.e.,} \quad \xi_z(u) = \frac{u^2}{2}.$$

According to Ornathanalai [9], let the log return process follow the dynamics given by

$$\begin{aligned} R_t &\equiv \log\left(\frac{S_t}{S_{t-1}}\right) = X_t - X_{t-1} \\ &= r_t + (\lambda_z - \xi_z(1)) \cdot h_{z,t} + (\lambda_y - \xi_y(1)) \cdot h_{y,t} + z_t + y_t, \end{aligned} \tag{2}$$

where λ_z and λ_y are the market prices of risk for the Lévy innovations z_t and y_t , respectively. $\xi_z(1) \cdot h_{z,t}$ and $\xi_y(1) \cdot h_{y,t}$ are the so-called convexity adjustments, implying that

$$\mathbb{E}_{t-1} [S_t] = S_{t-1} \exp\{r_t + \lambda_z h_{z,t} + \lambda_y h_{y,t}\}. \tag{3}$$

r_t is time-deterministic, and $\lambda_z h_{z,t} + \lambda_y h_{y,t}$ is referred to as the conditional equity premium (rate of return in excess of the risk-free rate). Note that, by the properties of the cumulant generating function, the conditional variance of the log-asset return is given by

$$Var_{t-1} [X_t - X_{t-1}] \equiv \sigma_t^2 = h_{z,t} + h_{y,t} \cdot \xi_y''(0). \tag{4}$$

Concerning the evolution of the parameters $h_{z,t}$ and $h_{y,t}$, we follow Ornathanalai [9] and Heston and Nandi [2], respectively, and assume the following dynamics for a set of parameters $\omega_z, \omega_y, a_z, a_y, b_z, b_y, c_z, c_y$:

$$h_{z,t+1} = \omega_z + b_z \cdot h_{z,t} + \frac{a_z}{h_{z,t}} (z_t - c_z h_{z,t})^2, \tag{5a}$$

$$h_{y,t+1} = \omega_y + b_y \cdot h_{y,t} + \frac{a_y}{h_{z,t}} (z_t - c_y h_{z,t})^2. \tag{5b}$$

This is equivalent to the evolution of the conditional variance in the HN-GARCH model by Heston and Nandi [2], which was formulated for a standard normal innovation. Ornathanalai [9] adapted the dynamics in (5) for non-standardized innovations, $z_t \sim \mathcal{N}(0, h_{z,t})$, but the convergence to a Cox–Ingersoll–Ross process if the length of the time step shrinks to zero [cf.4] is preserved.

The crucial property of the Lévy GARCH setting is the availability of the multivariate generating function in exponentially affine form, which we will present in Proposition 2.1. For enhanced readability, the proof is postponed to Appendix.

Proposition 2.1. *Let T be some finite time horizon and let $t < T-1$. In the setting of a Lévy GARCH model as introduced above, the one-step multivariate generating function of the log return R_{t+1} and the parameters $h_{y,t+2}$ and $h_{z,t+2}$, under the physical measure, conditioned on the information available at time t , is given by*

$$\begin{aligned} \Psi_{(R_{t+1}, h_{z,t+2}, h_{y,t+2})}(u, v_z, v_y | \mathcal{F}_t) &:= \mathbb{E}_t \left[\exp \left\{ u R_{t+1} + v_z h_{z,t+2} + v_y h_{y,t+2} \right\} \right] \\ &= \exp \left\{ A(u, v_z, v_y; t, t+1) \right. \\ &\quad + B(u, v_z, v_y; t, t+1) \cdot h_{z,t+1} \\ &\quad \left. + C(u, v_y; t, t+1) \cdot h_{y,t+1} \right\}, \end{aligned} \quad (6)$$

where we define

$$A(u, v_z, v_y; t, t+1) = u r_{t+1} + v_z \omega_z + v_y \omega_y - \frac{1}{2} \log(1 - 2(v_z a_z + v_y a_y)), \quad (7a)$$

$$B(u, v_z, v_y; t, t+1) = u(\lambda_z - \xi_z(1)) + v_z(b_z + a_z c_z^2) + v_y a_y c_y^2 + \frac{(u - 2v_z a_z c_z - 2v_y a_y c_y)^2}{2(1 - 2(v_z a_z + v_y a_y))}, \quad (7b)$$

$$C(u, v_y; t, t+1) = u(\lambda_y - \xi_y(1)) + v_y b_y + \xi_y(u). \quad (7c)$$

Together with the ingredients presented in Section 2.2, this property will be essential in the process of solving the portfolio optimization problem later in Section 3.

2.2. Optimization problem and self-financing condition

Our goal is to solve an expected utility problem in a market with one risky asset and the bank account (with time-deterministic interest rate r_t). We assume that the investor derives utility from terminal wealth via a power utility function $U(v) = \frac{1}{\gamma} v^\gamma$, with parameter $\gamma < 1$, $\gamma \neq 0$, and we denote the investors wealth process by $\{V_t\}_{t=0, \dots, T}$. The fraction of wealth invested in the risky asset at time $[t, t+1)$ is π_t . That is, we intend to solve the problem

$$\max_{\{\pi_s\}_t^{T-1}} \mathbb{E}_t[U(V_T)]. \quad (8)$$

In order to exploit the affine nature of the underlying model, we work with an approximation W_t of $\log V_t$ to solve (8). Following the approach by Campbell and Viceira [18], an approximation of the original self-financing condition

$$\frac{V_t}{V_{t-1}} = \pi_{t-1} \frac{S_t}{S_{t-1}} + (1 - \pi_{t-1}) \frac{B_t}{B_{t-1}}, \quad (9)$$

where B_t is the value of the riskless asset² at time t , leads to

$$\begin{aligned} W_t &= W_{t-1} + \pi_{t-1} R_t + \frac{1}{2} (\pi_{t-1} - \pi_{t-1}^2) \sigma_t^2 + (1 - \pi_{t-1}) r_t \\ &= W_{t-1} + \pi_{t-1} R_t + \frac{1}{2} (\pi_{t-1} - \pi_{t-1}^2) (h_{z,t} + h_{y,t} \xi_y''(0)) + (1 - \pi_{t-1}) r_t, \end{aligned} \quad (10)$$

² B_t is not to be confused with the coefficient B of the generating function in Proposition 2.1, which does not carry an index.

which is a linear expression in R_t , $h_{z,t}$ and $h_{y,t}$. This approach was proved to be accurate in the context of affine GARCH models by Escobar-Anel et al. [11,12] and furthermore used in a similar context by Escobar-Anel et al. [19,20]. We refer to this extensive amount of preceding work for more details. Getting back to our original optimization problem (8), we use the approximation of the log wealth in (10) to rewrite our problem as

$$\max_{\{\pi_s\}_t^{T-1}} \frac{1}{\gamma} \mathbb{E}_t \left[\exp \left\{ \gamma W_T \right\} \right]. \quad (11)$$

3. Optimal investment maximizing CRRA utility

3.1. Maximum utility representation

The solution to the dynamic portfolio optimization (11) is presented in Theorem 3.1. Again, for the sake of readability, the proofs in this chapter are postponed to Appendix.

Theorem 3.1 (Maximum Utility Representation). *Assume that the decision maker maximizes a power utility function with corresponding parameter $\gamma < 0$. Furthermore, let the log price of the risky asset be described by a Lévy GARCH model, satisfying the additional condition that $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$ holds for some $\Lambda \in \mathbb{R}$, for all $z_t \in \mathbb{R}$ and $t \in \{0, \dots, T-1\}$. Then, the optimal value function at time t can be written as*

$$\begin{aligned} \phi_t(W_t, h_{z,t+1}, h_{y,t+1}) &:= \max_{\{\pi_s\}_t^{T-1}} \frac{1}{\gamma} \mathbb{E}_t \left[\exp \left\{ \gamma W_T \right\} \right] \\ &= \frac{1}{\gamma} \exp \left\{ D_{t,T}(\pi_t^*) + \gamma W_t \right. \\ &\quad \left. + E_{t,T}^z(\pi_t^*) \cdot h_{z,t+1} + E_{t,T}^y(\pi_t^*) \cdot h_{y,t+1} \right\} \\ &= \frac{1}{\gamma} \exp \left\{ D_{t,T}(\pi_t^*) + \gamma W_t \right. \\ &\quad \left. + \left[E_{t,T}^z(\pi_t^*) + \Lambda E_{t,T}^y(\pi_t^*) \right] \cdot h_{z,t+1} \right\}, \end{aligned} \quad (12)$$

with $D_{t,T}$, $E_{t,T}^z$ and $E_{t,T}^y$ given via $D_{T,T} = E_{T,T}^z = E_{T,T}^y = 0$ and the recursive equations

$$D_{t,T}(\pi_t^*) = D_{t+1,T}(\pi_{t+1}^*) + (1 - \pi_t^*) \gamma r_{t+1} + A(\gamma \pi_t^*, E_{t+1,T}^z(\pi_{t+1}^*), E_{t+1,T}^y(\pi_{t+1}^*); t, t+1), \quad (13a)$$

$$E_{t,T}^z(\pi_t^*) = \frac{\gamma}{2} (\pi_t^* - (\pi_t^*)^2) + B(\gamma \pi_t^*, E_{t+1,T}^z(\pi_{t+1}^*), E_{t+1,T}^y(\pi_{t+1}^*); t, t+1), \quad (13b)$$

$$E_{t,T}^y(\pi_t^*) = \frac{\gamma}{2} (\pi_t^* - (\pi_t^*)^2) \cdot \xi_y''(0) + C(\gamma \pi_t^*, E_{t+1,T}^y(\pi_{t+1}^*); t, t+1), \quad (13c)$$

relying on the coefficients A , B , and C defined in Proposition 2.1. The optimal solution π_t^* satisfies

$$\begin{aligned} \left[\lambda_z - \pi_t^* + \frac{\gamma \pi_t^* - 2E_{t+1,T}^z(\pi_{t+1}^*) a_z c_z - 2E_{t+1,T}^y(\pi_{t+1}^*) a_y c_y}{1 - 2E_{t+1,T}^z(\pi_{t+1}^*) a_z - 2E_{t+1,T}^y(\pi_{t+1}^*) a_y} \right] \\ + A \cdot \left[\xi_y''(0) \left(\frac{1}{2} - \pi_t^* \right) + (\lambda_y - \xi_y(1)) + \xi_y'(\gamma \pi_t^*) \right] = 0, \end{aligned} \quad (14)$$

together with the second-order condition

$$A \cdot \xi_y''(\gamma \pi_t^*) > \frac{1 + \Lambda \xi_y''(0)}{\gamma} - \frac{1}{1 - 2E_{t+1,T}^z(\pi_{t+1}^*) a_z - 2E_{t+1,T}^y(\pi_{t+1}^*) a_y}. \quad (15)$$

Theorem 3.1 solves (8) in approximate form up to the solution of (14), applying a recursive approach via Bellman's value iteration. This result generalizes existing work in the realm of affine GARCH models with only one source of risk [see, e.g., 11,19,20] to the significantly more complex dynamics of Lévy GARCH models. The exact shape of Eq. (14), which remains to be solved for the optimal solution π_t^* , still

depends on the choice for the distribution of the Lévy innovation.³ This entails difficulties with respect to a general statement about existence and uniqueness of the optimal solution. Corollaries 3.3 and 3.4 show that it is possible to obtain the desired result, i.e., existence and uniqueness, in some special cases. In the other cases, we rely on numerical methods for root searching in the context of univariate equations. We numerically obtain unique solutions in all applications considered in Section 4.

Remark 3.2. It is remarkable that an additional assumption concerning the dependence of h_y and h_z is necessary in Theorem 3.1, originating in the nature of the first-order condition for the optimal portfolio strategy, which is of the form $[\dots] \cdot h_{z,t+1} + [\dots] \cdot h_{y,t+1} = 0$ (see Eq. (A.5) in the proof of Theorem 3.1). If $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$, this equation boils down to (14). However, in the case of $h_{y,t+1} \neq \Lambda \cdot h_{z,t+1}$, this condition can be satisfied for any values of $h_{z,t+1}$ and $h_{y,t+1}$ only if both coefficients in square brackets are set to zero, implying two equations for only one variable π_t . Since the coefficient corresponding to $h_{z,t+1}$ is linear in π_t , we obtain an explicit expression for the optimal strategy dependent on the other parameters. This solution candidate – name it $\pi_t^{(z)}$ – is time-dependent because of $E_{t+1,T}^{z,*}$ and $E_{t+1,T}^{y,*}$.⁴ Concerning the coefficient corresponding to $h_{y,t+1}$, this yields a condition for ξ_y , stating that

$$\xi_y''(0) \left(\frac{1}{2} - \pi_t^{(z)} \right) + (\lambda_y - \xi_y(1)) + \xi_y' \left(\gamma \pi_t^{(z)} \right) = 0, \tag{16}$$

for all $t \in \{0, \dots, T - 1\}$. These T different conditions, however, need to be satisfied for all investors, i.e., for all possible choices for γ and the corresponding values of $\pi_t^{(z)}$. Thus, we need to look at (16) as a differential equation for ξ_y , which after the substitution $u = \gamma \pi_t^{(z)}$ has the form

$$\xi_y'(u) = \frac{k_2}{\gamma} u + k_1 - \frac{k_2}{2} - \lambda_y, \tag{17}$$

subject to the two conditions $\xi_y(1) = k_1$ and $\xi_y''(0) = k_2$. This still – counter-intuitively – imposes a condition on the statistical properties of log asset returns related to the investor’s risk aversion parameter γ . Furthermore, integrating yields

$$\xi_y(u) = \frac{k_2}{2\gamma} u^2 + \left(k_1 - \frac{k_2}{2} - \lambda_y \right) u + C \tag{18}$$

for some constant $C \in \mathbb{R}$, with

$$\xi_y(1) = \frac{k_2}{2\gamma} + k_1 - \frac{k_2}{2} - \lambda_y + C = k_1, \tag{19a}$$

$$\xi_y''(0) = \frac{k_2}{\gamma} = k_2. \tag{19b}$$

It is easy to see that (19b) requires $\gamma = 1$ for the risk aversion parameter, which implies $C = \lambda_y$ from (19a). Hence, this restricts the analysis to risk-neutral investors and contradicts our assumption that $\gamma < 1$. This shows the necessity of the additional assumption $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$ in Theorem 3.1.

An analogue to our condition $h_{y,t} = \Lambda h_{z,t}$ can be found, e.g., in the setting by Liu and Pan [21], where the stochastic jump intensity is connected to the instantaneous variance process. In practice, we ensure the linear dependence of $h_{z,t}$ and $h_{y,t}$ by connecting the parameters in (5b) to the ones in (5a), i.e., by assuming the initial condition $h_{y,1} = \Lambda h_{z,1}$ together with

$$c_y = c_z, \quad b_y = b_z, \quad \omega_y = \Lambda \omega_z, \quad \text{and} \quad a_y = \Lambda a_z, \tag{20}$$

implying a slightly simplified expression for (14).

We emphasize that a simple reduction of our model delivers a well-known special case without jumps. Given parameters $\lambda_z, \omega_z, a_z, b_z, c_z$ and a sequence of innovations $z_t \sim \mathcal{N}(0, h_{z,t})$ for all $t = 1, \dots, T$, the log-asset return in the HN-GARCH model⁵ follows the dynamics

$$R_t = r_t + \left(\lambda_z - \frac{1}{2} \right) h_{z,t} + z_t, \tag{21a}$$

$$h_{z,t+1} = \omega_z + b_z \cdot h_{z,t} + \frac{a_z}{h_{z,t}} (z_t - c_z h_{z,t})^2. \tag{21b}$$

It is easy to see that the HN-GARCH model is obtained as a special case of a Lévy GARCH model by disregarding the jump innovation, i.e., by setting $\xi_y(u) = 0$, $\Lambda = 0$, and $\lambda_y = 0$. Note that due to the convexity adjustment present in the Lévy GARCH formulation, the market price of risk is given by $\lambda_z - \frac{1}{2}$. This connection allows us to derive the optimal solution to the expected utility problem in an HN-GARCH environment as a special case of Theorem 3.1.

Corollary 3.3 (HN-GARCH). *The optimal portfolio strategy for (11), given that the underlying asset follows an HN-GARCH model, can be obtained from Theorem 3.1 by setting $\xi_y(u) = 0$, $\Lambda = 0$, and $\lambda_y = 0$. In particular,*

$$\pi_t^* = \frac{\lambda_z \left(1 - 2E_{t+1,T}^{z,*} a_z \right) - 2E_{t+1,T}^{z,*} a_z c_z}{1 - 2E_{t+1,T}^{z,*} a_z - \gamma}, \tag{22}$$

and $1 - 2E_{t+1,T}^{z,*} a_z > 0$ ensures (15).

Note that it can be easily shown that the second-order condition (15) is trivially satisfied for all π_t on the real line in the HN-GARCH model. That is, we know that the objective function is concave and the solution to the first-order condition is a global maximum.

3.2. Examples of jump innovations

In this section, we consider several choices for the Lévy jump innovation and its distribution, respectively. First, the above optimality problem is investigated under the well-known Merton jump model. Subsequently, we also tackle Normal Inverse Gaussian (NIG) and Variance Gamma (VG) innovations.

Merton jump model

In the Merton jump model [cf.9], each jump size is drawn independently from $\mathcal{N}(\theta, \delta^2)$. The number of jumps η_{t+1} occurring in the interval $(t, t + 1]$ is a counting process with predictable intensity $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$, i.e., this intensity serves as the multiplicatively separable parameter in our Lévy GARCH model. Thus, let $y_{t+1} = \sum_{k=1}^{\eta_{t+1}} Y_k$, with $\{Y_k\}_k$ i.i.d. according to $\mathcal{N}(\theta, \delta^2)$. Note that

$$\mathbb{E}_t \left[\exp \{ u \cdot y_{t+1} \} \mid \eta_{t+1} = n \right] = \exp \left\{ \left(u\theta + \frac{1}{2} \delta^2 u^2 \right) \cdot n \right\},$$

which allows us to deduce from the tower property that

$$\mathbb{E}_t \left[\exp \{ u \cdot y_{t+1} \} \right] = \exp \left\{ \left(\exp \left\{ u\theta + \frac{1}{2} \delta^2 u^2 \right\} - 1 \right) \cdot h_{y,t+1} \right\}.$$

Having performed the above analysis conditioned on the information at time t , we obtain the cumulant exponent from the last equation and state that in the Merton jump model,

$$\xi_y(u) = \exp \left\{ u\theta + \frac{1}{2} \delta^2 u^2 \right\} - 1. \tag{23}$$

³ Eq. (14) involves $E_{t+1,T}^z(\pi_{t+1}^*)$ and $E_{t+1,T}^y(\pi_{t+1}^*)$, which again involve B and C as of Eq. (7). Eq. (7) shows that B and C depend on the distribution of the Lévy innovation since ξ_y is involved.

⁴ $E_{t+1,T}^{x,*}$ is short for $E_{t+1,T}^x(\pi_{t+1}^*)$, with $x \in \{y, z\}$.

⁵ The HN-GARCH model was originally introduced by Heston and Nandi [2]. The dynamic portfolio optimization solution for the HN-GARCH model was first presented by Escobar-Anel et al. [12].

Transferring this representation back to our optimality Eq. (14), we now need to solve

$$\left[\lambda_z - \pi_t + \frac{\gamma \pi_t - 2 \left(E_{t+1,T}^{z,*} - \Lambda E_{t+1,T}^{y,*} \right) a_z c_z}{1 - 2a_z \left(E_{t+1,T}^{z,*} + \Lambda E_{t+1,T}^{y,*} \right)} \right] + \Lambda \cdot \left[\theta^2 \left(\frac{1}{2} - \pi_t \right) + \left(\lambda_y - \exp \left\{ \left(\theta + \frac{1}{2} \delta^2 \right) \right\} + 1 \right) + \left(\theta + \delta^2 \gamma \pi_t \right) \cdot \exp \left\{ \theta \gamma \pi_t + \frac{1}{2} \delta^2 \gamma^2 \pi_t^2 \right\} \right] = 0. \tag{24}$$

The optimality equation above yields the following statement on existence and uniqueness of an optimal solution in the prevailing model:

Corollary 3.4. Assume a Merton jump model for the dynamics of the risky asset. If (15) holds for all π_t on the real line, for $t \in \{0, \dots, T - 1\}$, then there exists a unique solution for the optimal relative portfolio strategy π_t^* .

However, the strategy π_t is involved both in linear and in exponential terms, making (24) difficult to solve for π_t explicitly. One straightforward way to an approximate solution uses that the values for the parameters θ and δ are very small (referring to the MLEs reported by Ornthalalai [9]), which implies that the approximation $\exp\{x\} \approx 1 + x$ is sufficiently good in this case.

NIG model

A Normal inverse Gaussian innovation [9,22] is a normal variance-mean mixture [cf. 23, Def. 2.1] where the mixing density is the inverse Gaussian distribution. In particular, we introduce $\alpha > 0$ and β with $|\beta| < \alpha$. Conditional on $x \sim \text{IG} \left(h_{y,t+1}, \sqrt{\alpha^2 - \beta^2} \right)$, the distribution of the Lévy jump innovation is $y_{t+1} \sim \mathcal{N}(\beta x, x)$.⁶ According to Barndorff-Nielsen [22] and Ornthalalai [9], the cumulant exponent of this innovation is given by

$$\psi_y(u; t, t + 1) = -h_{y,t+1} \cdot \left(\sqrt{\alpha^2 - (\beta + u)^2} - \sqrt{\alpha^2 - \beta^2} \right),$$

with the part excluding $h_{y,t+1}$ (note the minus in front!) in the role of $\xi_y(u)$ as of Eq. (1). In this setting too, we rely on the condition $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$, which then leads to the following optimality equation:

$$\left[\lambda_z - \pi_t + \frac{\gamma \pi_t - 2 \left(E_{t+1,T}^{z,*} - \Lambda E_{t+1,T}^{y,*} \right) a_z c_z}{1 - 2a_z \left(E_{t+1,T}^{z,*} + \Lambda E_{t+1,T}^{y,*} \right)} \right] + \Lambda \cdot \left[\frac{\alpha^2 - 2\beta^2}{(\alpha^2 - \beta^2)^{\frac{3}{2}}} \left(\frac{1}{2} - \pi_t \right) + \lambda_y + \sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2} + \frac{\beta + \gamma \pi_t}{\sqrt{\alpha^2 - \beta^2 - 2\beta \gamma \pi_t - \gamma^2 \pi_t^2}} \right] = 0. \tag{25}$$

VG model

Another potential choice concerning the model for the jump innovation is the increment of a Variance Gamma process [24], generated as a normal variance-mean mixture with the Gamma distribution as the mixing density. That is, the innovation is distributed according to $y_{t+1} \sim \mathcal{N}(\theta x, x)$, conditional on $x \sim \text{Gamma}(\alpha, \beta)$, for $\alpha, \beta > 0$. Again referring to Ornthalalai [9], we obtain via conditioning on x :

$$\mathbb{E}_t \left[\exp \{ u \cdot y_{t+1} \} \right] = \left(\frac{\beta}{\beta - \frac{1}{2} u^2 - \theta u} \right)^\alpha$$

⁶ In order to make the cumulant exponent of this innovation multiplicatively separable in one of its parameters, we set the location parameter μ in the original formulation of the normal variance-mean mixture $\mathcal{N}(\mu + \beta x, x)$ to zero.

Table 1

Standard values for the investment parameters used for the plots in Section 4.

Parameter	T	r	γ
Value	252	$0.01/252$	-1

$$= \exp \left\{ -\alpha \cdot \log \left(1 - \frac{\theta u + \frac{1}{2} u^2}{\beta} \right) \right\}.$$

Thus, choosing $h_{y,t+1}$ to take the role of α , we arrive at

$$\xi_y(u) = \log \left(\frac{\beta}{\beta - \frac{1}{2} u^2 - \theta u} \right). \tag{26}$$

Using this in (14) yields the optimality equation

$$\left[\lambda_z - \pi_t + \frac{\gamma \pi_t - 2 \left(E_{t+1,T}^{z,*} - \Lambda E_{t+1,T}^{y,*} \right) a_z c_z}{1 - 2a_z \left(E_{t+1,T}^{z,*} + \Lambda E_{t+1,T}^{y,*} \right)} \right] + \Lambda \cdot \left[\frac{\beta + \theta^2}{\beta^2} \left(\frac{1}{2} - \pi_t \right) + \lambda_y - \log \left(\frac{\beta}{\beta - \theta - \frac{1}{2}} \right) + \frac{\theta + \gamma \pi_t}{\beta - \theta \gamma \pi_t - \frac{1}{2} \gamma^2 \pi_t^2} \right] = 0. \tag{27}$$

Multiplying both sides with the denominator of the last term leads to a third-order polynomial.

Remark 3.5. Although Eqs. (24), (25) and (27) seem to be tedious to solve for π_t , we can check numerically that the second-order condition is satisfied for the entire real line at all time points (for reasonable parameter values), i.e., that the objective function is concave in π_t and that a solution will be a unique maximum.

4. Numerical analysis

This section is devoted to the numerical study of the prevailing Lévy GARCH model. In particular, the sensitivity of the optimal portfolio allocation to the central parameters of the model is investigated as well as the wealth-equivalent loss (WEL) occurring when the investor follows suboptimal strategies. We are interested specifically in the impact of jumps on the optimal strategy and on potential losses and want to compare the Lévy GARCH models to the nested HN-GARCH (which does not incorporate jumps) and a homoscedastic variant. All analyses are going to be presented for the Merton jump model (MJ) as well as for NIG and VG innovations.

If not specified otherwise, the choices for the model's investment parameters are given in Table 1.

4.1. Parameter estimates

Our analysis is based on the maximum-likelihood estimates provided by Ornthalalai [9]. Since in the estimation process for these MLEs the linear dependence $h_{y,t} = \Lambda h_{z,t}$ for $t \in \{1, \dots, T\}$, was not assumed, we need to modify the original parameter set. The central question in this regard is how to derive the value for the parameter Λ , since this, by construction, will govern the behavior of the process $\{h_{y,t}\}_t$. The approach presented here uses the unconditional mean of the process $\{h_{y,t}\}_t$ and derives a value for Λ such that the unconditional means of $\Lambda h_{z,t}$ and $h_{y,t}$ are matched. Assuming that both h_z and h_y satisfy the stationarity conditions $b_z + a_z c_z^2 < 1$ and $b_y < 1$, the unconditional first moments of h_z and h_y , respectively, are given by

$$\bar{h}_z^{(1)} = \frac{\omega_z + a_z}{1 - b_z - a_z c_z^2}, \quad \bar{h}_y^{(1)} = \frac{\omega_y + a_y + a_y c_y^2 \bar{h}_z^{(1)}}{1 - b_y}. \tag{28}$$

We replace the parameters a_y , b_y , c_y and ω_y in (28) according to the relations in (20) to obtain the Λ -dependent version $\hat{h}_y^{(1)}$. Subsequently, we choose the value for Λ such that $\hat{h}_y^{(1)}$ and $\bar{h}_y^{(1)}$ are equal, leading to the equation

$$\frac{\Lambda\omega_z + \Lambda a_z + \Lambda a_z c_z^2 \bar{h}_z^{(1)}}{1 - b_z} = \frac{\omega_y + a_y + a_y c_y^2 \bar{h}_z^{(1)}}{1 - b_y}, \quad (29)$$

which is equivalent to

$$\Lambda = \frac{(1 - b_z) (\omega_y + a_y + a_y c_y^2 \bar{h}_z^{(1)})}{(1 - b_y) (\omega_z + a_z + a_z c_z^2 \bar{h}_z^{(1)})}. \quad (30)$$

Note that one could also include higher unconditional moments in the above approach and then derive the best value for Λ , e.g., based on the method of least squares. This, however, would sacrifice an exact match of the long-term mean of h_y , forming the reason why we remain with the exact choice in (30).

4.2. Methodology for comparisons and WEL

In order to correctly investigate the impact of jumps on the solution to our portfolio optimization problem, we want to compare the prevailing jump model to the closest HN-GARCH model, which is nested in our setup as described in Corollary 3.3 and does not incorporate jumps. In order to avoid estimation errors from small samples, for every jump model, we simulate 100 paths with a long sample of $N = 2 \times 10^4$ steps of log returns for the prevailing jump model and then derive maximum-likelihood estimates for the HN-GARCH parameters based on this dataset. To this end, we follow the lines of the returns-only estimation of Escobar-Anel et al. [15], exploiting the normal distribution of the HN-GARCH innovations to obtain the log-likelihood function l . Given the model dynamics (21) with parameter vector $\theta = (\lambda_z, \omega_z, a_z, b_z, c_z)$ and the log return x_{t-1} , the law for X_t is

$$X_t | (x_{t-1}, \theta) \sim \mathcal{N} \left(x_{t-1} + r_t + \left(\lambda_z - \frac{1}{2} \right) h_{z,t}, h_{z,t} \right). \quad (31)$$

The joint density function f_{X_1, \dots, X_T} of the log returns can be expressed via conditioning on the previous values:

$$f_{X_1, \dots, X_T} (x_1, \dots, x_T) = f_{X_1} (x_1) \cdot f_{X_2 | X_1} (x_2) \cdot \dots \cdot f_{X_T | X_{T-1}} (x_T). \quad (32)$$

Combining (31) and (32) with the density of a normally distributed random variable and subsequently taking the logarithm leads to the log-likelihood function

$$l_T (x_1, \dots, x_T | \theta) = -\frac{1}{2} \sum_{t=1}^T (\log(2\pi) + \log(h_{z,t}) + \epsilon_t^2), \quad (33a)$$

where

$$\epsilon_t = \frac{x_t - x_{t-1} - r_t - \left(\lambda_z - \frac{1}{2} \right) h_{z,t}}{\sqrt{h_{z,t}}}. \quad (33b)$$

The estimates $\hat{\theta}$ for θ are given by $\hat{\theta} = \arg \max_{\theta} l_T (x_1, \dots, x_T | \theta)$. After finding estimates $\hat{\theta}_1, \dots, \hat{\theta}_{100}$ for all paths, we take the median as the best proxy for the HN parametrization⁷, which can be understood as the best HN-GARCH approximating a given Lévy GARCH. The same dataset is used to obtain the corresponding homoscedastic Gaussian model (HS), i.e., an HN-GARCH model with $a_z = b_z = c_z = 0$. In the following, the jump models – we use the MJ, the NIG and the VG model as examples – are always compared to their re-calibrated versions of the HN-GARCH and the homoscedastic variant.

⁷ We use the median, rather than the average, due to the presence of outliers in the estimates (i.e., local minimum on some samples). We also report the standard error from the 100 samples to confirm the accuracy of the estimate.

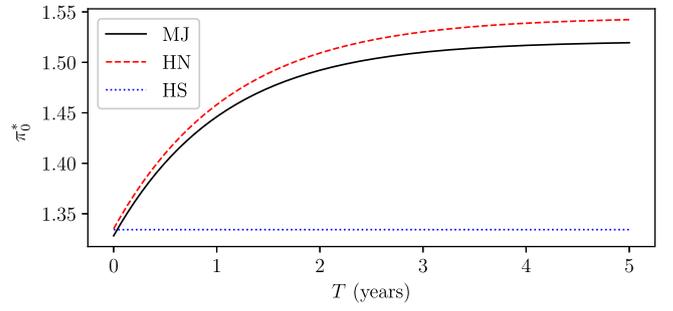


Fig. 1. Values of the optimal strategy π_0^* in the MJ model and the corresponding re-calibrated HN-GARCH and homoscedastic model, dependent on the length of the time horizon, for $T \in [1, 5 \cdot 252]$. The parameters are given in Table 1 (investment parameters) and in Table 2 (MJ, HN, HS).

For the purpose of measuring the losses that investors incur when following suboptimal strategies, we make use of the exponentially affine value function from Theorem 3.1. In particular, we compare the optimal value function at time t , ϕ_t , which is obtained via using the optimal strategy $\{\pi_t^*\}_t$ in the recursive parameters, to the expected utility corresponding to the suboptimal strategy $\{\pi_t^s\}_t$, denoted by ϕ_t^s . The wealth-equivalent loss (WEL) L_t^s is the percentage of wealth at time t that an investor following the optimal strategy can forgo in order to obtain the same expected utility as another investor following the suboptimal strategy $\{\pi_t^s\}_t$. Thus, L_t^s satisfies the equation

$$\phi_t^s (\log V_t, h_{z,t+1}, h_{y,t+1}) = \phi_t (\log (V_t (1 - L_t^s)), h_{z,t+1}, h_{y,t+1}). \quad (34)$$

Using (12) and exploiting $h_{y,t+1} = \Lambda h_{z,t+1}$ yields the following closed-form representation of the WEL:

$$L_t^s = 1 - \exp \left\{ \frac{1}{\gamma} \cdot \left[D_{t,T}^s (\pi_t^s) - D_{t,T} (\pi_t^*) + \left(E_{t,T}^{z,s} (\pi_t^s) - E_{t,T}^z (\pi_t^*) + \Lambda E_{t,T}^{y,s} (\pi_t^s) - \Lambda E_{t,T}^y (\pi_t^*) \right) h_{z,t+1} \right] \right\}. \quad (35)$$

4.3. Results for the Merton jump model

This subsection presents an analysis of the sensitivity of the optimal solution to the most relevant investment parameters and aims at comparing the Merton jump model to the closest HN-GARCH and homoscedastic solution.

To this end, we derived maximum likelihood parameter estimates for the latter two jump-free models according to the methodology presented in Section 4.2, based on the MJ parameters from Ornthanalai [9], adapted to our setting as described in Section 4.1. We add that, compared to the sample statistics for the original S&P 500 dataset used to estimate the MJ parameters in Table 2, namely a sample skewness of -0.194 and a kurtosis of 11.15 [9], the first 15 years of observations in our artificial dataset created with these MJ parameters yield a sample skewness of 0.010 and a kurtosis of 3.47 .⁸ For all three investigated models, similar sample statistics are obtained when simulating with or without modifying the parameter set according to Section 4.1 to achieve the dependence in (20).

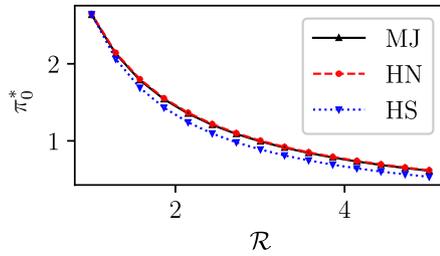
Comparing the solutions to the portfolio optimization problem for MJ to its no-jump analogues as of Table 2, Fig. 1 shows that the re-calibrated HN-GARCH model without jumps replicates the MJ very well, with the difference in the optimal initial allocation decreasing

⁸ While these numbers for the artificial set of MJ returns are less extreme than the actual sample statistics of the original dataset, this picture changes when looking at different Lévy GARCH models, see Section 4.4 for the NIG model.

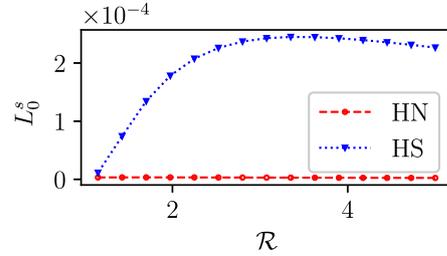
Table 2

Maximum likelihood estimates for the MJ model and the recalibrated HN-GARCH and homoscedastic model, respectively. Standard errors for the recalibrated models in parentheses. The MJ parameters are based on Ornathanalai [9] and the methodology presented in Section 4.1, the two jump-free models have been estimated according to Section 4.2.

Par.	MJ	HN	HS
λ_z	1.47	2.67	(2.70×10^{-1})
ω_z	-2.39×10^{-6}	-1.23×10^{-6}	(5.67×10^{-7})
a_z	2.98×10^{-6}	4.45×10^{-6}	(5.80×10^{-7})
b_z	9.41×10^{-1}	9.59×10^{-1}	(5.59×10^{-3})
c_z	1.35×10^2	9.04×10^1	(1.19×10^1)
λ_y	6.18×10^{-2}		
Λ	2.31×10^2		
θ	-2.39×10^{-3}		
δ	2.85×10^{-2}		



(a) Values of π_0^* dependent on \mathcal{R} . The lines for the MJ and the HN are overlapping.



(b) WEL dependent on \mathcal{R} .

Fig. 2. Values of the optimal initial allocation π_0^* and of wealth-equivalent losses, dependent on the investor's level of relative risk aversion $\mathcal{R} = 1 - \gamma \in [1.1, 5]$. The parameters are given in Table 1 (investment parameters) and in Table 2 (MJ, HN, HS).

as shorter time horizons are considered. In contrast to this, the homoscedastic solution is not sensitive to the length of the time horizon and stays constant, also causing an increasing difference to the MJ as the time horizon increases.

Fig. 2(a) addresses the sensitivity of the optimal solution with respect to the investor's level of relative risk aversion, which in the prevailing setting of a power utility function with parameter $\gamma < 1$ is given via $\mathcal{R} = 1 - \gamma$. Clearly, the fraction of wealth invested in the risky asset increases in all three models as the level of relative risk aversion decreases. This can lead to major differences between investors using the same model, but having different risk preferences. Following the MJ, for instance, with a relative risk aversion of $\mathcal{R} \approx 1.3$ yields an initial risky allocation of around 200%, whereas $\mathcal{R} \approx 4$ leads to an allocation below 100%. For the entire plotted range, the HN-GARCH strategy remains very close to the optimal strategy in the MJ, while investors following the homoscedastic variant have a slightly lower risky allocation. Fig. 2(b) uses the same parameter range and considers the wealth-equivalent loss that investors incur when following the HN-GARCH or the homoscedastic strategy, while log returns actually evolve according to an MJ. In general, the losses remain at a very low level for a one-year time horizon, yielding values below 5 basis points for the homoscedastic model and even less for the HN-GARCH. This confirms the impression from Figs. 1 and 2(a) that the jump-free models lead to very similar investment strategies.

In Fig. 3, we consider more extreme jump settings, which are obtained via increasing the absolute mean of the normally distributed jump size in the MJ up to a factor of 10 and via increasing the standard deviation of the jump size by the same factor. For each grid point between the original parameter estimate as of Table 2 and the most extreme setting, we re-calibrate the jump-free models to a set of simulated log return according to Section 4.2.⁹ Both plots suggest

that the capability of the HN-GARCH and the homoscedastic model to imitate the MJ is not significantly weakened, i.e., losses remain negligible for all considered parameter settings.

Besides modifications concerning the jump size in the MJ, the jump intensity can of course play a huge role too, bearing in mind, e.g., the analysis by Ascheberg et al. [14], suggesting that jumps matter more if the intensity is large. In the Merton jump model, the intensity can easily be modified via the parameter Λ , governing the jump intensity given by $\Lambda h_{z,t}$ in the interval $(t, t + 1]$. One has to take into account, however, that the long-run equity premium, obtained via

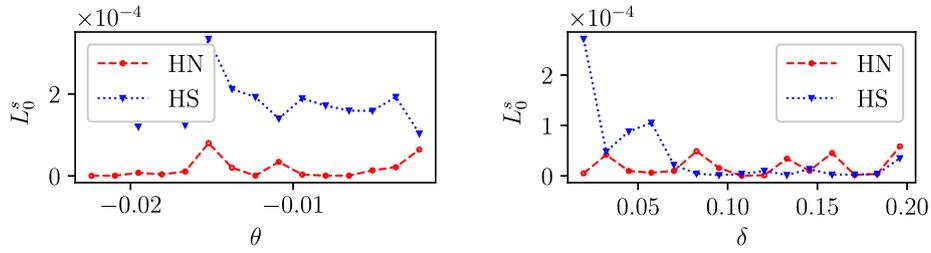
$$\bar{\lambda} := \lambda_z \bar{h}_z^{(1)} + \lambda_y \bar{h}_y^{(1)} = \lambda_z \bar{h}_z^{(1)} + \lambda_y \Lambda \bar{h}_z^{(1)}, \tag{36}$$

changes accordingly, which will have a major impact on the optimal solution. Since we wish to isolate the effect of modifications to the jump size, we compensate for changes in the long-run equity premium by adapting λ_z such that $\bar{\lambda}$ remains constant.

Fig. 4 uses this approach and increases Λ up to one order of magnitude from the original value in the MJ model (see Table 2). Again, both jump-free models are re-calibrated for every grid point. As seen for the parameters affecting the jump size, the parameter Λ does not have a significant impact on the wealth-equivalent losses occurring for the closest HN-GARCH and homoscedastic models, as all values stay below or close to 2 basis points, with the HN-GARCH outperforming the constant solution and yielding even lower values.

To conclude our numerical investigation of the Merton jump model, we modify both the mean jump size and the intensity at the same time. In particular, we increase the parameters θ and Λ by a factor of 10. Concerning changes in the jump intensity, implied by an increase in Λ , we again use the approach described above to avoid changes in the long-run equity premium. Fig. 5(a) shows the values of the initial optimal risky allocation for investors with different risk aversion levels, for the MJ as well as for the two jump-free models. The plot once more suggests that the strategies obtained from the HN-GARCH and the homoscedastic model are similar to the one in the MJ. Concerning the wealth-equivalent losses investors incur when following the former

⁹ The recalibration exercise leads to natural inaccuracies in the estimates, which explains the minor variation in the calculation of WEL among grid points.



(a) WEL dependent on θ , up to an absolute increase by a factor of 10. (b) WEL dependent on δ , up to an increase by a factor of 10.

Fig. 3. Values of the wealth-equivalent losses, dependent on the jump parameters in the MJ model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

Table 3

Maximum likelihood estimates for the NIG model and the recalibrated HN-GARCH and homoscedastic model, respectively. Standard errors for the recalibrated models in parentheses. The NIG parameters are based on Ornthalalai [9] and the methodology presented in Section 4.1, the two jump-free models have been estimated according to Section 4.2.

Par.	NIG	HN		HS	
λ_z	8.00×10^{-1}	1.83	(4.30×10^{-1})	1.83	(3.97×10^{-1})
ω_z	-1.65×10^{-6}	3.40×10^{-6}	(8.81×10^{-8})	3.21×10^{-4}	(2.96×10^{-5})
a_z	2.41×10^{-6}	4.21×10^{-7}	(4.62×10^{-8})		
b_z	9.40×10^{-1}	7.04×10^{-1}	(2.25×10^{-2})		
c_z	1.43×10^2	6.70×10^2	(6.14×10^1)		
λ_y	6.88×10^{-1}				
Λ	2.62				
α	1.16×10^1				
β	-6.86				

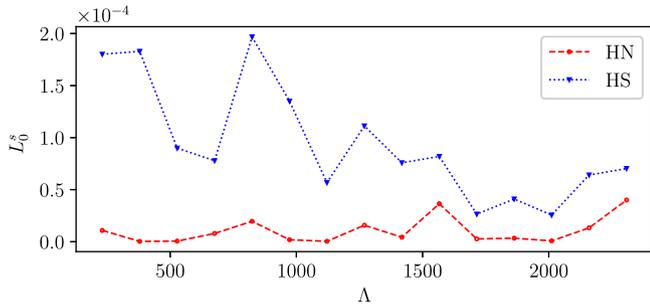


Fig. 4. Values of the wealth-equivalent losses for the HN-GARCH and the homoscedastic model, dependent on the jump intensity in the MJ model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

two suboptimal strategies, Fig. 5(b) suggests a very good performance of the re-calibrated models, yielding values below one basis point for the entire range of relative risk aversion.

4.4. Results for the Normal Inverse Gaussian model

In this subsection, we address the Normal Inverse Gaussian model in terms of the sensitivity of the optimal solution to the main model parameters and compare the model to the Gaussian HN-GARCH model without jumps and to a homoscedastic variant.

The parameters for these three models are displayed in Table 3. The estimates for the NIG model are based on Ornthalalai [9], adapted to our setting according to Section 4.1. As described in 4.2, the HN-GARCH and the homoscedastic model parameters are re-estimated based on a set of simulated log returns from the NIG model. The sample

skewness for the first 15 years of the simulated log returns is -2.99 , the kurtosis amounts to 73.90 .¹⁰

Concerning the evolution over time, Fig. 6 shows that an investor with a time horizon of more than 1.5 years will invest 2.5 percentage points more than an investor with a very short time horizon. While the HN-GARCH solution only shows slight differences in the initial allocation for very small values of T , the homoscedastic solution does not change for different time horizons.

Plotting the values of the initial optimal portfolio allocation for a one-year time horizon for investors with different levels of relative risk aversion, Fig. 7(a) shows that the small differences observed in Fig. 6 for the special case of $\mathcal{R} = 2$ are confirmed for other reasonable values, too. This is also reflected in Fig. 7(b), suggesting extremely low values for the wealth-equivalent loss occurring when following the HN-GARCH or the homoscedastic solution, while returns actually evolve according to the NIG model.

The interpretation of the NIG innovation as a normal variance-mean mixture $y_t \sim \mathcal{N}(\beta x, x)$, with $x \sim \text{IG}(h_{y,t}, \sqrt{\alpha^2 - \beta^2})$, helps to get an intuition about the role of the different parameters in this setting. As for the MJ model in Section 4.3, we intend to increase the average jump size, which can be achieved via modifications to β . As we change β , we correct the inverse Gaussian distribution to remain the same by adapting α accordingly. Increasing the absolute value of β up to one order of magnitude in this manner, and then re-calibrating the HN-GARCH and the homoscedastic model leads to the wealth-equivalent losses plotted in Fig. 8(a). The loss increases towards higher absolute jump sizes for both suboptimal strategies, but the numbers remain on a very low level in general. It is remarkable that in this case, the homoscedastic model could produce a lower WEL than the heteroscedastic HN-GARCH for some very extreme settings (see Fig. 9).

¹⁰ These values are much more extreme than the numbers observed in the original dataset from the S&P 500 (see Section 4.3), similar to the picture for implied average conditional skewness and kurtosis observed by Ornthalalai [9].

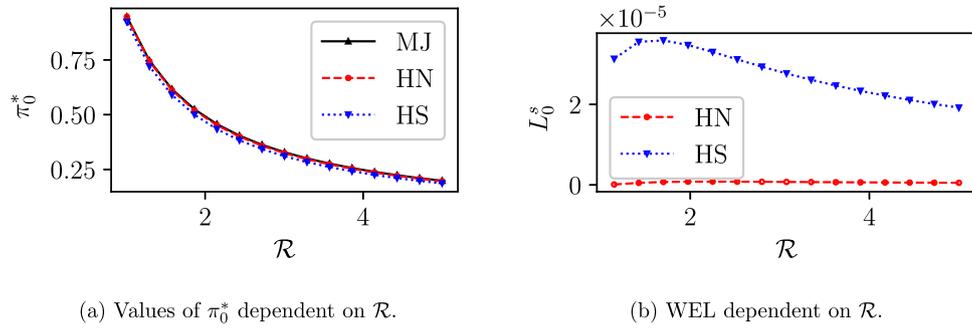


Fig. 5. Values of the optimal initial allocation π_0^* and of wealth-equivalent losses, dependent on the investor's level of relative risk aversion $\mathcal{R} = 1 - \gamma \in [1, 1.5]$. These plots are based on a modified, extreme parameter setting for the MJ with θ and Λ one order of magnitude larger than the original estimates in Table 2. The HN-GARCH and the homoscedastic model have been re-calibrated according to Section 4.2. The time horizon is one year with 252 trading days.

Table 4
Maximum likelihood estimates for the VG model and the recalibrated HN-GARCH and homoscedastic model, respectively. Standard errors for the recalibrated models in parentheses. The VG parameters are based on Ornathanalai [9] and the methodology presented in Section 4.1, the two jump-free models have been estimated according to Section 4.2.

Par.	VG	HN	HS
λ_z	6.82×10^{-1}	2.33	(3.34×10^{-1})
ω_z	-1.75×10^{-6}	1.25×10^{-7}	(3.18×10^{-7})
a_z	2.57×10^{-6}	2.85×10^{-6}	(3.23×10^{-7})
b_z	9.38×10^{-1}	9.44×10^{-1}	(6.50×10^{-3})
c_z	1.47×10^2	1.31×10^2	(1.50×10^1)
λ_y	1.68×10^{-2}		
Λ	1.17×10^2		
θ	-8.19		
β	9.49×10^2		

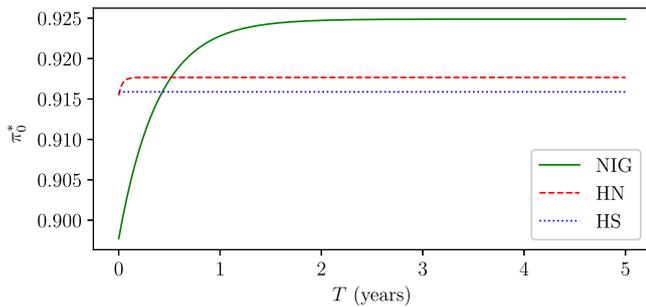


Fig. 6. Values of the optimal strategy π_0^* in the NIG model and the corresponding re-calibrated HN-GARCH and homoscedastic model, dependent on the length of the time horizon, for $T \in [1, 5 \cdot 252]$. The parameters are given in Table 3.

We add that the third parameter that can influence the jumps, Λ , plays a different role in the NIG as compared to the MJ model. While we were looking at an increase of said parameter in the MJ to raise the jump intensity, we may now increase Λ to obtain a larger mean of the inverse Gaussian random variable in the normal variance-mean mixture. Looking at an increase of up to one order of magnitude, and re-calibrating the HN-GARCH and the homoscedastic model at each of the grid points also shows insignificant values for the wealth-equivalent loss. The WEL evaluated for an investor with $\mathcal{R} = 2$ does not exceed one basis point.

4.5. Results for the Variance Gamma model

This subsection is devoted to the numerical results for the special case of the Variance Gamma model, see Corollary 3.4.

We follow the lines of Sections 4.3 and 4.4 and investigate the impact of more extreme jump settings in the VG model on the capability of the HN-GARCH and the homoscedastic model when it comes to dynamic investment strategies.

Table 4 shows the estimates for the VG model, derived from Ornathanalai [9] and adapted according to Section 4.1. The HN-GARCH and the homoscedastic parameter estimates are obtained according to the methodology described in Section 4.2. In this case, the sample skewness for the first 15 years of simulated log returns is -0.13 , the sample kurtosis is 4.29 .

Fig. 10 shows that, the re-calibrated HN-GARCH model, based on a simulated dataset of VG returns, produces an initial value for the risky allocation which is quite close to the one obtained directly from the VG model for all time horizons up to five years. The initial allocation resulting from the homoscedastic model is constant over different time horizons and almost identical to the HN-GARCH and the VG strategy for very small T .

This is also reflected in the plots in Fig. 11, showing the difference of the initial risky allocations for a one-year time horizon across different levels of relative risk aversion. The HN-GARCH yields a value almost identical to the one from the VG model, the homoscedastic model seems to be quite close, too. Concerning the wealth-equivalent losses, the picture observed for the MJ and the NIG models in Sections 4.3 and 4.4 is confirmed once more, since the values for the losses when following the HN-GARCH are very low, exceeded by the losses produced with the homoscedastic solution, which, however, also remain small — around one basis point for \mathcal{R} between 2 and 4.

Concerning the jump parameters in the VG model, we again look at the normal variance-mean mixture interpretation of the underlying distribution. In particular, considering $y_t \sim \mathcal{N}(\theta x, x)$, based on $x \sim \text{Gamma}(h_{y,t}, \beta)$, we quickly identify θ as the key to modifications in the jump size. Since the mean of x is given by $h_{y,t}/\beta$, and this again is reflected in both the mean and the variance of y_t , we can modify these two measures via decreasing β and leaving the behavior of the process $h_{y,t}$ unchanged.

Repeating the same procedure as for the other WEL evaluations here yields a very familiar picture, as shown in Fig. 12: Re-calibrated using simulations from the jump model for different parameter values, the losses for the HN-GARCH and the homoscedastic model remain very low, barely exceeding one basis point for a one-year time horizon in the VG setting.

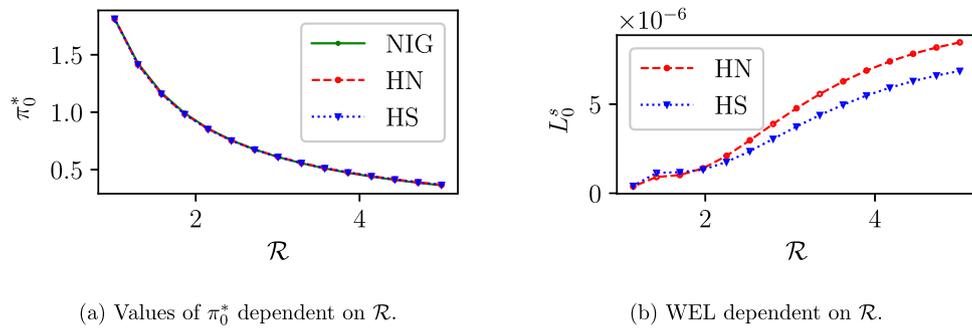


Fig. 7. Values of the optimal initial allocation π_0^* and of wealth-equivalent losses in the NIG model, dependent on the investor's level of relative risk aversion $R = 1 - \gamma \in [1.1, 5]$. The time horizon is one year with 252 trading days, the parameters are given in Table 3.

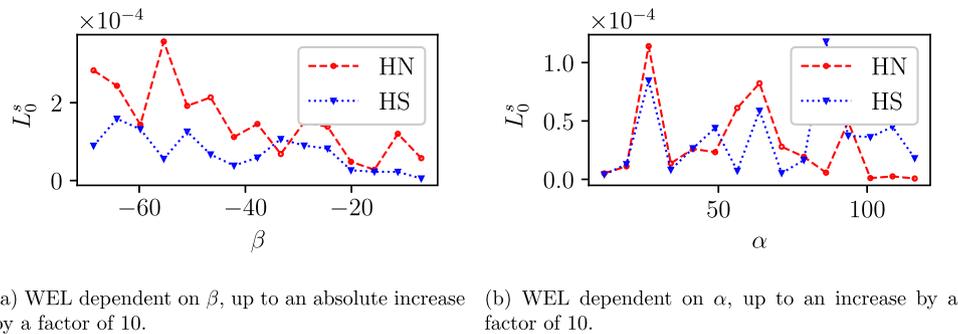


Fig. 8. Values of the wealth-equivalent losses, dependent on the jump parameters in the NIG model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

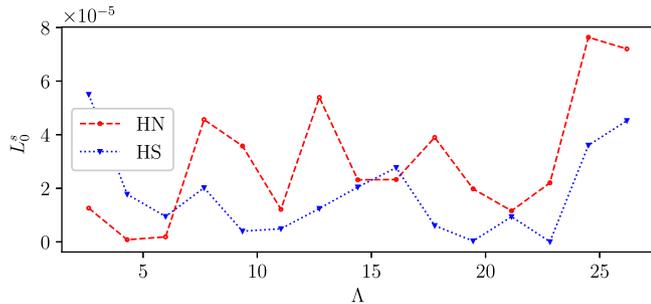


Fig. 9. Values of wealth-equivalent losses for HN-GARCH and homoscedastic model, compared to the NIG model and dependent on the parameter λ . For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

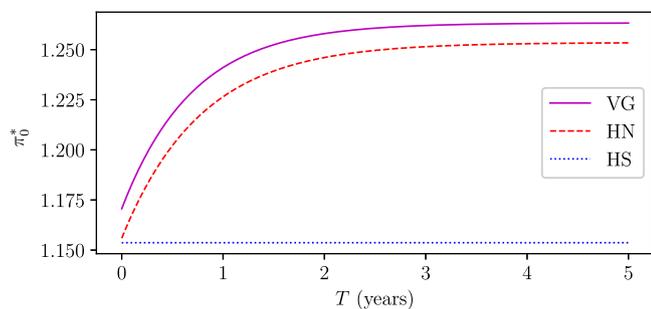


Fig. 10. Values of the optimal strategy π_0^* in the VG model and the corresponding re-calibrated HN-GARCH and homoscedastic model, dependent on the length of the time horizon, for $T \in [1.5 \cdot 252]$. The parameters are given in Table 4.

The evidence in this section indicates that Gaussian models excel at replicating the simulated data of jump processes across different jump models and parameter scenarios. In this context, we note that our Gaussian models are fitted based on a relatively large data set, see Section 4.2. Other results, e.g., in the context of regime switches [25–27, among others], are based on comparatively short time series. This suggests that the large amount of data points we use helps to eliminate the estimation error and significantly improves the performance of the Gaussian models.

5. Conclusion

This paper provides an approximate solution for a discrete-time portfolio optimization problem in a Lévy GARCH framework. We lay out optimality criteria for different special cases of the framework, including both finite-activity and infinite-activity jump components. Our numerical study focuses on the impact of jumps on the optimal strategy and compares three different jump models to the jump-free Gaussian HN-GARCH model and a homoscedastic Gaussian model. Across all considered jump models, the results demonstrate that investors following the closest possible jump-free variants obtain very similar strategies and incur only insignificant wealth-equivalent losses. Our finding remains true for modified parameter settings indicating extreme jump behavior in the Lévy GARCH models. This surprising performance of simple models might be explained by two factors. One is the connection between expected utility maximization and mean-variance theory, stating that expected utility maximization can be well approximated using mean-variance theory, i.e., by the first two moments [17]. The quality of this approximation in other settings, e.g., with a disappointment-averse investor [28] could be subject to further research. This highlights the importance of matching the first two moments of the wealth distribution. Secondly, the frequency of re-balancing in a discrete-time setting leaves little room to take advantage of sudden jumps. This latter factor might be crucial for our results, considering the many studies of the impact of jumps on optimal portfolios

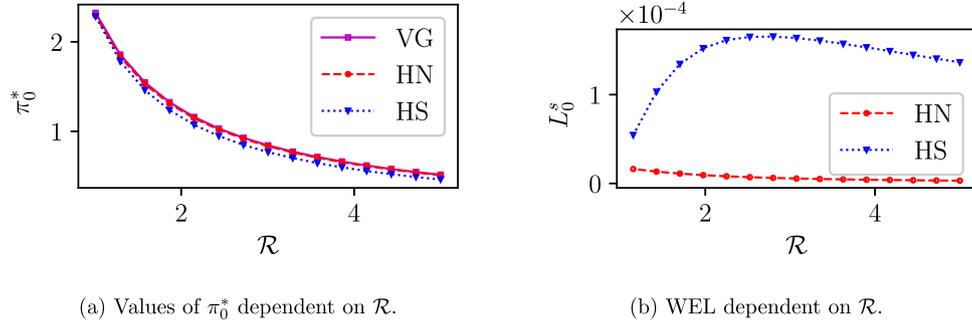


Fig. 11. Values of the optimal initial allocation π_0^* and of wealth-equivalent losses in the VG model, dependent on the investor's level of relative risk aversion $R = 1 - \gamma \in [1.1, 5]$. The time horizon is one year with 252 trading days, the parameters are given in Table 4.

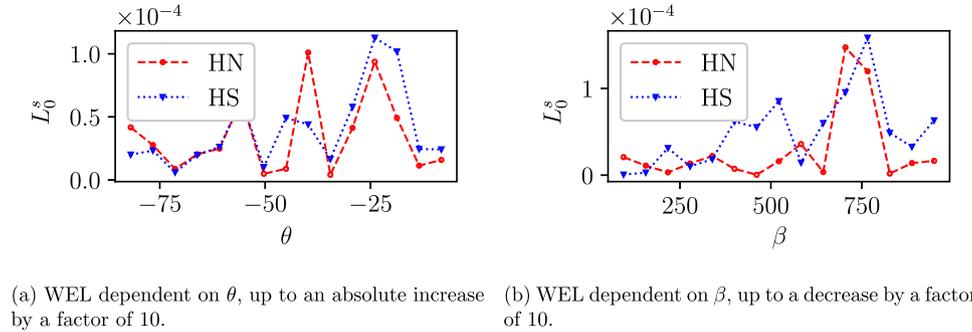


Fig. 12. Values of the wealth-equivalent losses, dependent on the jump parameters in the VG model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

in a continuous-time context, for instance by Das and Uppal [29] or, more recently, Yang et al. [30]. In the discrete-time GARCH framework, our analysis therefore shows that correctly calibrated jump-free models could diminish the necessity of modeling jumps for investors.

CRedit authorship contribution statement

Marcos Escobar-Anel: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Project administration, Methodology, Investigation, Conceptualization. **Ben Spies:** Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Conceptualization. **Rudi Zagst:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors report that they have no conflict of interest to declare.

Data availability

No data was used for the research described in the article.

Appendix. Proofs from Sections 2 and 3

Proof of Proposition 2.1. We know that the innovations z_{t+1} and y_{t+1} are contemporaneously independent and that for $z_{t+1} \sim \mathcal{N}(0, h_{z,t+1})$, we have

$$\begin{aligned} & \mathbb{E}_t \left[\exp \left\{ \alpha z_{t+1} + \beta z_{t+1}^2 \right\} \right] \\ &= \exp \left\{ \frac{\alpha^2 h_{z,t+1}}{2(1 - 2\beta h_{z,t+1})} - \frac{1}{2} \log(1 - 2\beta h_{z,t+1}) \right\}. \end{aligned} \tag{A.1}$$

Using (A.1) in the last step, we prove Proposition 2.1:

$$\begin{aligned} & \Psi_{(R_{t+1}, h_{z,t+2}, h_{y,t+2})} (u, v_z, v_y | \mathcal{F}_t) \\ &= \mathbb{E}_t \left[\exp \left\{ u \cdot R_{t+1} + v_z \cdot h_{z,t+2} + v_y \cdot h_{y,t+2} \right\} \right] \\ &= \mathbb{E}_t \left[\exp \left\{ u \left[r_{t+1} + (\lambda_z - \xi_z(1)) h_{z,t+1} + (\lambda_y - \xi_y(1)) h_{y,t+1} + z_{t+1} + y_{t+1} \right] \right. \right. \\ & \quad \left. \left. + v_z \cdot h_{z,t+2} + v_y \cdot h_{y,t+2} \right\} \right] \\ &= \exp \left\{ u \left[r_{t+1} + (\lambda_z - \xi_z(1)) h_{z,t+1} + (\lambda_y - \xi_y(1)) h_{y,t+1} \right] \right\} \\ & \quad \times \mathbb{E}_t \left[\exp \left\{ u \cdot (z_{t+1} + y_{t+1}) \right. \right. \\ & \quad \left. \left. + v_z \left[\omega_z + b_z h_{z,t+1} + \frac{a_z}{h_{z,t+1}} (z_{t+1} - c_z h_{z,t+1})^2 \right] \right. \right. \\ & \quad \left. \left. + v_y \left[\omega_y + b_y h_{y,t+1} + \frac{a_y}{h_{z,t+1}} (z_{t+1} - c_y h_{z,t+1})^2 \right] \right\} \right] \\ &= \exp \left\{ u \cdot r_{t+1} + v_z \cdot \omega_z + v_y \cdot \omega_y \right. \\ & \quad \left. + \left[u \cdot (\lambda_z - \xi_z(1)) + v_z \cdot (b_z + a_z c_z^2) + v_y a_y c_y^2 \right] \cdot h_{z,t+1} \right. \\ & \quad \left. + \left[u \cdot (\lambda_y - \xi_y(1)) + v_y \cdot b_y \right] \cdot h_{y,t+1} \right\} \\ & \quad \times \mathbb{E}_t \left[\exp \left\{ u \cdot y_{t+1} + \left[u - 2v_z a_z c_z - 2v_y a_y c_y \right] \cdot z_{t+1} \right. \right. \\ & \quad \left. \left. + \left[v_z \cdot \frac{a_z}{h_{z,t+1}} + v_y \cdot \frac{a_y}{h_{z,t+1}} \right] \cdot z_{t+1}^2 \right\} \right] \\ &= \exp \left\{ u r_{t+1} + v_z \omega_z + v_y \omega_y \right. \\ & \quad \left. + \left[u \cdot (\lambda_z - \xi_z(1)) + v_z \cdot (b_z + a_z c_z^2) + v_y a_y c_y^2 \right] \cdot h_{z,t+1} \right. \\ & \quad \left. + \left[u \cdot (\lambda_y - \xi_y(1)) + v_y b_y + \xi_y(u) \right] \cdot h_{y,t+1} \right. \\ & \quad \left. + \frac{(u - 2v_z a_z c_z - 2v_y a_y c_y)^2}{2[1 - 2(v_z a_z + v_y a_y)]} \cdot h_{z,t+1} - \frac{1}{2} \log(1 - 2(v_z a_z + v_y a_y)) \right\} \end{aligned}$$

justifying (6) together with the definitions in (7). This completes the proof of Proposition 2.1. \square

Proof of Theorem 3.1. We use Bellman's principle in order to maximize via backwards induction and use the optimal solution at time $t + 1$

for our approach at time t . This results in the calculations below, where for the terminal step at time $t = T - 1$ we just work with

$$\phi_T(W_T) = U(V_T) = \frac{1}{\gamma} \exp\{\gamma W_T\},$$

resulting in $D_{T,T} = E_{T,T}^z = E_{T,T}^y = 0$. We obtain:

$$\begin{aligned} & \max_{\pi_t} \mathbb{E}_t [\phi_{t+1}(W_{t+1}, h_{z,t+2}, h_{y,t+2})] \\ &= \max_{\pi_t} \mathbb{E}_t \left[\frac{1}{\gamma} \exp\{D_{t+1,T}(\pi_{t+1}^*) + \gamma W_{t+1} \right. \\ & \quad \left. + E_{t+1,T}^z(\pi_{t+1}^*) \cdot h_{z,t+2} + E_{t+1,T}^y(\pi_{t+1}^*) \cdot h_{y,t+2}\} \right] \\ &= \max_{\pi_t} \frac{1}{\gamma} \exp\{D_{t+1,T}(\pi_{t+1}^*)\} \\ & \quad \times \mathbb{E}_t \left[\exp\{\gamma \cdot [W_t + \pi_t \cdot R_{t+1} \right. \\ & \quad \left. + \frac{1}{2}(\pi_t - \pi_t^2) \cdot (h_{z,t+1} + h_{y,t+1} \cdot \xi_y''(0)) + (1 - \pi_t) r_{t+1}] \right. \\ & \quad \left. + E_{t+1,T}^z(\pi_{t+1}^*) \cdot h_{z,t+2} + E_{t+1,T}^y(\pi_{t+1}^*) \cdot h_{y,t+2}\} \right] \\ &= \max_{\pi_t} \frac{1}{\gamma} \exp\{D_{t+1,T}(\pi_{t+1}^*) + (1 - \pi_t) \gamma r_{t+1} + \gamma W_t \\ & \quad + \frac{\gamma}{2}(\pi_t - \pi_t^2) \cdot (h_{z,t+1} + h_{y,t+1} \cdot \xi_y''(0))\} \\ & \quad \times \mathbb{E}_t \left[\exp\{\gamma \pi_t R_{t+1} + E_{t+1,T}^z(\pi_{t+1}^*) \cdot h_{z,t+2} + E_{t+1,T}^y(\pi_{t+1}^*) \cdot h_{y,t+2}\} \right] \\ &= \max_{\pi_t} \frac{1}{\gamma} \exp\{D_{t+1,T}(\pi_{t+1}^*) + (1 - \pi_t) \gamma r_{t+1} + \gamma W_t \\ & \quad + \frac{\gamma}{2}(\pi_t - \pi_t^2) \cdot (h_{z,t+1} + h_{y,t+1} \cdot \xi_y''(0))\} \\ & \quad \times \Psi_{(R_{t+1}, h_{z,t+2}, h_{y,t+2})}(\gamma \pi_t; E_{t+1,T}^z(\pi_{t+1}^*), E_{t+1,T}^y(\pi_{t+1}^*) | \mathcal{F}_t) \\ &= \max_{\pi_t} \frac{1}{\gamma} \exp\{D_{t,T}(\pi_t) + \gamma W_t + E_{t,T}^z(\pi_t) \cdot h_{z,t+1} + E_{t,T}^y(\pi_t) \cdot h_{y,t+1}\}, \end{aligned} \tag{A.2}$$

where we used the multivariate generating function from the proof of Proposition 2.1 in the Lévy GARCH setting and the definitions from Eqs. (13) in the last step. We seek to maximize (A.2) with respect to the single variable π_t . Thus, we approach this step by taking derivatives, setting the first derivative to zero while having a negative second derivative at this point (or globally in case of concave functions). Concerning the first of the above-mentioned steps, note that, given some $\gamma \neq 0$, the first derivative is equal to zero if and only if

$$D'_{t,T}(\pi_t) + \left(E_{t,T}^z\right)'(\pi_t) \cdot h_{z,t+1} + \left(E_{t,T}^y\right)'(\pi_t) \cdot h_{y,t+1} = 0, \tag{A.3}$$

which, according to (13), is equivalent to

$$\begin{aligned} & -\gamma r_{t+1} + \frac{\partial}{\partial u} A(\gamma \pi_t, E_{t+1,T}^{z,*}, E_{t+1,T}^{y,*}; t, t + 1) \cdot \gamma \\ & + \left[\frac{\gamma}{2} - \gamma \pi_t + \frac{\partial}{\partial u} B(\gamma \pi_t, E_{t+1,T}^{z,*}, E_{t+1,T}^{y,*}; t, t + 1) \cdot \gamma\right] \cdot h_{z,t+1} \\ & + \left[\xi_y''(0) \cdot \left(\frac{\gamma}{2} - \gamma \pi_t\right) + \frac{\partial}{\partial u} C(\gamma \pi_t, E_{t+1,T}^{z,*}, E_{t+1,T}^{y,*}; t, t + 1) \cdot \gamma\right] \cdot h_{y,t+1} = 0, \end{aligned} \tag{A.4}$$

where $E_{t+1,T}^{x,*}$ is short for $E_{t+1,T}^x(\pi_{t+1}^*)$ with $x \in \{y, z\}$. Now, using Definitions (7), we note that the first line reduces to zero. Dividing by γ yields

$$\begin{aligned} & \left[\frac{1}{2} - \pi_t + \lambda_z - \xi_z(1) + \frac{\gamma \pi_t - 2E_{t+1,T}^{z,*} a_z c_z - 2E_{t+1,T}^{y,*} a_y c_y}{1 - 2E_{t+1,T}^{z,*} a_z - 2E_{t+1,T}^{y,*} a_y}\right] \cdot h_{z,t+1} \\ & + \left[\xi_y''(0) \left(\frac{1}{2} - \pi_t\right) + (\lambda_y - \xi_y(1)) + \xi_y'(\gamma \pi_t)\right] \cdot h_{y,t+1} = 0. \end{aligned}$$

Exploiting the Gaussian distribution of z , we further use $\xi_z(1) = 1/2$ to obtain

$$\begin{aligned} & \left[\lambda_z - \pi_t + \frac{\gamma \pi_t - 2E_{t+1,T}^{z,*} a_z c_z - 2E_{t+1,T}^{y,*} a_y c_y}{1 - 2E_{t+1,T}^{z,*} a_z - 2E_{t+1,T}^{y,*} a_y}\right] \cdot h_{z,t+1} \\ & + \left[\xi_y''(0) \left(\frac{1}{2} - \pi_t\right) + (\lambda_y - \xi_y(1)) + \xi_y'(\gamma \pi_t)\right] \cdot h_{y,t+1} = 0. \end{aligned} \tag{A.5}$$

Remember that the two multiplicatively separable parameters of the cumulant exponents are assumed to be linearly dependent, i.e.,

$$h_{y,t+1} = \Lambda \cdot h_{z,t+1} \tag{A.6}$$

holds for some $\Lambda \in \mathbb{R}$. This yields the single equation

$$\begin{aligned} & \left[\lambda_z - \pi_t + \frac{\gamma \pi_t - 2E_{t+1,T}^{z,*} a_z c_z - 2E_{t+1,T}^{y,*} a_y c_y}{1 - 2E_{t+1,T}^{z,*} a_z - 2E_{t+1,T}^{y,*} a_y}\right] \\ & + \Lambda \cdot \left[\xi_y''(0) \left(\frac{1}{2} - \pi_t\right) + (\lambda_y - \xi_y(1)) + \xi_y'(\gamma \pi_t)\right] = 0. \end{aligned} \tag{A.7}$$

How easily the resulting equation can be solved for π_t , now largely depends on the shape of ξ_y , i.e. on the explicit distribution of the Lévy jump innovation y . In order to ensure that the solution to (A.7) is actually a maximum, we require (15). This concludes the proof. \square

Proof of Corollary 3.4. If we require (15) with strict inequality for all π_t , then the continuous function on the left-hand side in the optimality Eq. (24) is strictly decreasing in π_t . Furthermore, using that $\gamma < 0$, we obtain that the left-hand side goes to $\pm\infty$ if $\pi_t \rightarrow \mp\infty$. Using the intermediate value theorem, the existence of a unique solution follows. \square

References

- [1] Merton RC. Optimum consumption and portfolio rules in a continuous-time model. *J Econom Theory* 1971;3(4):373–413. [http://dx.doi.org/10.1016/0022-0531\(71\)90038-x](http://dx.doi.org/10.1016/0022-0531(71)90038-x).
- [2] Heston SL, Nandi S. A closed-form GARCH option valuation model. *Rev Financ Stud* 2000;13(3):585–625. <http://dx.doi.org/10.1093/rfs/13.3.585>.
- [3] Christoffersen P, Heston S, Jacobs K. Option valuation with conditional skewness. *J Econometrics* 2006;131(1–2):253–84. <http://dx.doi.org/10.1016/j.jeconom.2005.01.010>.
- [4] Badescu A, Cui Z, Ortega J-P. Closed-form variance swap prices under general affine GARCH models and their continuous-time limits. *Ann Oper Res* 2019;282(1–2):27–57. <http://dx.doi.org/10.1007/s10479-018-2941-9>.
- [5] Christoffersen P, Jacobs K, Ornathanalai C. Dynamic jump intensities and risk premiums: Evidence from S&P500 returns and options. *J Financ Econ* 2012;106(3):447–72. <http://dx.doi.org/10.1016/j.jfineco.2012.05.017>.
- [6] Maheu JM, McCurdy TH. News arrival, jump dynamics, and volatility components for individual stock returns. *J Finance* 2004;59(2):755–93. <http://dx.doi.org/10.1111/j.1540-6261.2004.00648.x>.
- [7] Duan J-C, Ritchken P, Sun Z. Approximating GARCH-jump models, jump-diffusion processes, and option pricing. *Math Finance* 2006;16(1):21–52. <http://dx.doi.org/10.1111/j.1467-9965.2006.00259.x>.
- [8] Duan J-C, Ritchken P, Sun Z. Jump starting GARCH: Pricing and hedging options with jumps in returns and volatilities. Federal Reserve Bank of Cleveland; 2006. <http://dx.doi.org/10.26509/frbc-wp-200619>.
- [9] Ornathanalai C. Lévy jump risk: Evidence from options and returns. *J Financ Econ* 2014;112(1):69–90. <http://dx.doi.org/10.1016/j.jfineco.2013.11.009>.
- [10] Augustyniak M, Badescu A, Bégin J-F. A discrete-time hedging framework with multiple factors and fat tails: On what matters. *J Econometrics* 2023;232(2):416–44. <http://dx.doi.org/10.1016/j.jeconom.2021.08.002>.
- [11] Escobar-Anel M, Spies B, Zagst R. Expected utility theory on general affine GARCH models. *Appl Math Finance* 2021;28(6):477–507. <http://dx.doi.org/10.1080/1350486x.2022.2101010>.
- [12] Escobar-Anel M, Gollart M, Zagst R. Closed-form portfolio optimization under GARCH models. *Oper Res Perspect* 2022;9:100216. <http://dx.doi.org/10.1016/j.orp.2021.100216>.
- [13] Liu J, Longstaff FA, Pan J. Dynamic asset allocation with event risk. *J Finance* 2003;58(1):231–59. <http://dx.doi.org/10.1111/1540-6261.00523>.
- [14] Ascheberg M, Branger N, Kraft H, Seifried FT. When do jumps matter for portfolio optimization? *Quant Finance* 2016;16(8):1297–311. <http://dx.doi.org/10.1080/14697688.2015.1131844>.
- [15] Escobar-Anel M, Stentoft L, Ye X. The benefits of returns and options in the estimation of GARCH models. A Heston-Nandi GARCH insight. *Econometrics Stat* 2022. <http://dx.doi.org/10.1016/j.ecosta.2022.12.001>.
- [16] Bekaert G, Erb CB, Harvey CR, Viskanta TE. Distributional characteristics of emerging market returns and asset allocation. *J Portfolio Manag* 1998;24(2):102–16. <http://dx.doi.org/10.3905/jpm.24.2.102>.
- [17] Markowitz HM. Mean-variance approximations to expected utility. *European J Oper Res* 2012;234(2):346–55. <http://dx.doi.org/10.1016/j.ejor.2012.08.023>.
- [18] Campbell JY, Viceira LM. Consumption and portfolio decisions when expected returns are time varying. *Q J Econ* 1999;114(2):433–95. <http://dx.doi.org/10.1162/003355399556043>.

- [19] Escobar-Anel M, Spies B, Zagst R. Mean–variance optimization under affine GARCH: A utility-based solution. *Finance Res Lett* 2024;59(104749). <http://dx.doi.org/10.1016/j.frl.2023.104749>.
- [20] Escobar-Anel M, Spies B, Zagst R. Optimal consumption and investment in general affine GARCH models. *OR Spectrum* 2024. <http://dx.doi.org/10.1007/s00291-024-00749-z>.
- [21] Liu J, Pan J. Dynamic derivative strategies. *J Financ Econ* 2003;69(3):401–30. [http://dx.doi.org/10.1016/s0304-405x\(03\)00118-1](http://dx.doi.org/10.1016/s0304-405x(03)00118-1).
- [22] Barndorff-Nielsen OE. Processes of normal inverse Gaussian type. *Finance Stoch* 1997;2(1):41–68. <http://dx.doi.org/10.1007/s007800050032>.
- [23] Barndorff-Nielsen O, Kent J, Sørensen M. Normal variance-mean mixtures and z distributions. *Int Stat Rev / Rev Int Stat* 1982;50(2):145. <http://dx.doi.org/10.2307/1402598>.
- [24] Madan DB, Seneta E. The variance Gamma (V.G.) model for share market returns. *J Bus* 1990;63(4):511. <http://dx.doi.org/10.1086/296519>.
- [25] Ang A, Bekaert G. Regime switches in interest rates. *J Bus Econom Statist* 2002;20(2):163–82. <http://dx.doi.org/10.1198/073500102317351930>.
- [26] Guidolin M, Timmermann A. Asset allocation under multivariate regime switching. *J Econom Dynam Control* 2007;31(11):3503–44. <http://dx.doi.org/10.1016/j.jedc.2006.12.004>.
- [27] Campani CH, Garcia R, Lewin M. Optimal portfolio strategies in the presence of regimes in asset returns. *J Bank Financ* 2021;123:106030. <http://dx.doi.org/10.1016/j.jbankfin.2020.106030>.
- [28] Dahlquist M, Farago A, Tédongap R. Asymmetries and portfolio choice. *Rev Financ Stud* 2017;30(2):667–702. <http://dx.doi.org/10.1093/rfs/hhw091>.
- [29] Das SR, Uppal R. Systemic risk and international portfolio choice. *J Finance* 2004;59(6):2809–34. <http://dx.doi.org/10.1111/j.1540-6261.2004.00717.x>.
- [30] Yang B-Z, Lu X, Ma G, Zhu S-P. Robust portfolio optimization with multi-factor stochastic volatility. *J Optim Theory Appl* 2020;186(1):264–98. <http://dx.doi.org/10.1007/s10957-020-01687-w>.