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Article

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Operations Research Perspectives

Provided in Cooperation with:

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Suggested Citation: Nobil, Erfan et al. (2024) : Sustainability inventory management model with warm-up process and shortage, Operations Research Perspectives, ISSN 2214-7160, Elsevier, Amsterdam, Vol. 12, pp. 1-15,
<https://doi.org/10.1016/j.orp.2024.100297>

This Version is available at:

<https://hdl.handle.net/10419/325779>

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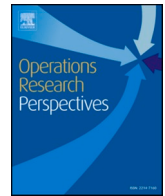
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Sustainability inventory management model with warm-up process and shortage

Erfan Nobil^a, Leopoldo Eduardo Cárdenas-Barrón^{a,b,*}, Dagoberto Garza-Núñez^a, Gerardo Treviño-Garza^c, Armando Céspedes-Mota^a, Imelda de Jesús Loera-Hernández^a, Neale R. Smith^a, Amir Hossein Nobil^d

^a Tecnológico de Monterrey, School of Engineering and Sciences, E. Garza Sada 2501 Sur, C.P. 64849, Monterrey, Nuevo León, México

^b Yonsei Frontier Lab, Yonsei University, Seoul 03722, South Korea

^c Ingram School of Engineering, Texas State University, 601 University Drive San Marcos, TX 78666, USA

^d Departamento de Ingeniería Industrial y Mecánica, Universidad de las Américas Puebla, San Andrés Cholula 72810, Mexico

ARTICLE INFO

Keywords:

Economic production quantity
Imperfect items
Backorder
Warm-up
Karush-kuhn-tucker (KKT)

ABSTRACT

Fast-paced markets require complex interactions from all supply-chain agents to satisfy customer demands and needs. The manufacturing industries face some difficulties in terms of production amounts and smooth delivery rates. Technical experts found that a warm-up period before a production run helps address those challenges and improves the workability of machine tools in the manufacturing process. The use of a warm-up process causes a reduction of faulty products (an adverse production outcome) and improves operational efficiency. Also, a shortage in the supply of commodities creates difficult conditions for inventory management decisions, posing the same production problems as mentioned above. Consideration of the warm-up process has recently been included in the scope of operations research, but it is necessary to study its interaction with the presence of shortage. This study presents a system where a manufacturing environment utilizes the warm-up process in its initial phase and shortages are allowed during the production period, in addition, the study takes into account carbon emissions during manufacturing to integrate environmental concerns. We assume that the company has the capability to trade the surplus carbon capacity it hasn't produced. This study offers a comprehensive framework that incorporates former research that addresses warm-up process, carbon emissions, shortages, and defective items. To solve the proposed non-linear programming problem with inequality constraints, we employ the Karush-Kuhn-Tucker (KKT) conditions method to determine the optimal solutions. Managerial insights are derived, and sensitivity analysis highlights the effects of the system parameters on the decision variables. The sensitivity analysis results indicate that the carbon trading cost has a significant impact on the overall cost, and subsequently, the company's profit.

1. Introduction

Contemporary living requires a diverse range of products designed to enhance our daily routines, fostering greater efficiency and productivity; resulting in escalating demand for manufactured goods within industrial sectors, as an outcome of population growth, urbanization, and changing customers' preferences. Fulfilling these demands relies on companies converting their current manufacturing processes into more productive ones. Another supportive strategy is enhancing efficiency, by which businesses produce goods at a lower cost and increase their revenue and competitiveness in the marketplace. Furthermore, such

efficiency growth leads to better resource utilization thereby reducing waste and conserving natural reserves, which are becoming increasingly scarce and expensive. Additionally, efficient production processes result in shorter response times, allowing companies to meet customer demand more quickly.

There are several approaches to address waste in production processes. One strategy is to rework defective items that do not meet the suitable quality by sending them to repair and afterward reintegrating them into production. From this viewpoint, remanufacturing faulty products helps reduce waste and prevent the loss of resources and energy already invested in them. Another strategy is to discard the items

* Corresponding author.

E-mail addresses: lecarden@tec.mx (L.E. Cárdenas-Barrón), amirhossein.nobil@udlap.mx (A.H. Nobil).

<https://doi.org/10.1016/j.orp.2024.100297>

Received 12 September 2023; Received in revised form 17 December 2023; Accepted 5 February 2024

Available online 8 February 2024

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that don't meet quality standards, which means losing money (and resources) but saving time. Time and cost influence the manufacturer's decision to select one strategy over the other. If the remanufacturing process is more expensive and time-consuming, then the company prefers to scrap or dispose of the defective products in an environmentally responsible manner. This strategy benefits companies as it saves time and money when manufacturing new products, besides, it ensures that the production process only generates high-quality items that meet the marketplace and customers' satisfaction as promised. The strategy has a significant environmental impact, making it crucial to prioritize safe waste disposal. Both approaches seem to have challenges regarding the cost of rework and the environmental impact of waste disposal. In addition, the efficiency of each strategy depends on the availability of resources and materials, in the event of a shortage, manufacturers do not have access to enough supplies to either rework or discard the defective items. Consequently, it is essential for manufacturers to cautiously plan and manage their production processes, considering the probable impacts on the total supply chain.

Resources shortage is a major challenge for production planning and operations, as it directly affects the relationship between manufacturers and customers. If materials shortage is prolonged then customer demand is not fulfilled, either partially or completely, therefore companies miss delivery deadlines and lose sales. In this situation, manufacturers need to take another approach to address this issue. Shortages result in reduced product quality, as manufacturers may be forced to use suboptimal materials or rush production to meet deadlines. Consequently, companies suffer an increase in product returns and warranty claims, which further affect their bottom-line revenues and reputation. In general, shortages have far-reaching penalties, so manufacturers must carefully manage their supply chains and anticipate any potential shortages to minimize the impact on production processes and end customers.

Integrating the negative impact of waste from a production line on the environment, as well as the risk of experiencing a shortage of materials, will allow a system to be more flexible in meeting the requirements of its customers. Production wastes lead to further pollution; in most cases, there is no proper way to discard them, leading to further environmental degradation. Also, production waste contributes to resources shortage in the production process, as a lack of raw materials for manufacturing, and rework process makes fabrication time longer than usual. In addition, when there is a shortage of certain materials, the overuse of other resources is promoted, which causes them to run out faster and creates imbalances in the natural environment. Manufacturers need to consider these issues to correctly address the occurrence of errors and faults that lead to defective production and the possibility of failing to meet customer demand. From the perspective of the administration office, these effects are reflected in the cost and profit of the production system. Therefore, a good solution is necessary to ensure that the production process remains efficient and cost-effective, while also minimizing waste and protecting the environment.

Inventory management uses different schemes to provide the best solution to real-world circumstances. Starting production from the initial state to working condition is one of those assumptions. At the same time, the incorporation of a pre-operational warm-up process stands as an additional assumption frequently employed in practice. Nobil et al., [31] explicitly discussed that this practice is done to prepare machine tools to work more efficiently, as reduced maintenance and longer machine life are typically observed. The warm-up time is assigned to lubricate and wrench the machine tools, which helps the system run more efficiently and produce fewer defects. In general, a system with fewer defective products costs less and fulfills the customer's demand on time.

Diverse manufacturing operations utilize the warm-up process as a necessary part of their process, as it brings machinery up to its optimal operating condition before production begins, thus, improving production rates and consistency, as well as achieving greater efficiency, results in lower production costs. The warm-up process decreases wear and tear in machine tools, leading to fewer breakdowns and longer usage life. In

some cases, production supervisors use the warm-up period to calibrate the system by producing some items, as a result, most scrap occurs at this stage, while main production runs with fewer defective products. Manufacturers can improve the performance of their equipment, reduce energy consumption, and lower production costs by taking the time to warm it up properly. The warm-up process in manufacturing operations helps to ensure efficiency, consistency, and minimal waste production.

There is still limited research on the effectiveness of warm-up processes in manufacturing systems, and there are some gaps that need to be addressed. While several studies have investigated the impact of the warm-up process on the thermal stability and accuracy of CNC machine tools, more research is needed across different manufacturing processes and industries. More studies should focus on assessing the efficacy of the warm-up process with the aim of diminishing energy consumption, reducing material waste, and improving overall product quality.

Additionally, conducting research on optimizing the warm-up process, including (a) identifying the optimal duration, which ideally contributes to reducing production time and cost. (b) studying how the warm-up process affects the amount of defective goods produced, simultaneously, it is crucial to adapt to a demand shortage, and (c) exploring the environmental impact of the warm-up process, including its carbon footprint and overall sustainability.

To ensure maximum efficiency and prevent equipment malfunctions, it is crucial to carry out a proper warm-up process. The warm-up time that is ideal may vary depending on a range of factors, such as the type of equipment being used, the condition of the facility, and the operator's skill level. Previous studies have overlooked the influence of additional factors on obtaining the optimal warm-up length. To address this issue, this study suggests investigating the proper warm-up period by considering specific conditions and implementing an adaptive approach to adjust its length accordingly. Another crucial factor to consider is how the warm-up period affects the decrease in defective products. Finding the optimal warm-up period is crucial for minimizing defects. A longer warm-up period helps to reduce defects, but determining the ideal duration is key. Considering the impact of shortages during the manufacturing process is another key factor to be determined. In such cases, the system's response to shortages should be evaluated to ensure that the equipment heats up adequately even during periods of low resources. When shortages occur, it's important to evaluate the system's response to ensure that the equipment is properly warmed up even during times of low resources.

Furthermore, it is essential to adhere to common environmental practices to reduce the harmful effects of manufacturing processes. The warm-up process plays a role in this effort by reducing waste and energy consumption. By implementing strategies that search for energy-efficient warm-up processes and optimizing machine scheduling, manufacturers can significantly reduce their negative environmental impacts. It is important to address all these issues to ensure that the system's decisions, including the warm-up process, are optimized for production efficiency, defect reduction, and environmental impacts while potential shortages may occur.

This study suggests an inventory management model that takes into account the warm-up process as a part of its operations. Economic production quantity (EPQ) is an area in which inventory management models look specifically at optimal production quantity as well as total system cost. An EPQ model is presented that involves a preliminary warm-up phase before initiating the manufacturing process. This production process will generate both high-quality and defective items. Implementation of an appropriate warm-up period can directly influence the number of defects; therefore, the model represents this connection using a mathematical function that links faulty items to warm-up time. The model considers the possibility of shortages and recognizes that customer demands may not be fully met by the end of the production cycle. In addition, the cap-and-trade approach is utilized to lessen the harmful effects of production on the earth. This method also considers an upper bound for producing carbon emissions; exceeding

this limit can result in negative consequences for manufacturers with excessive costs. On the other hand, emitting less than their allowed limit has benefits such as avoiding fees for carbon emissions and the possibility of selling excess emissions to other companies for profit. Overall, this study incorporates the aforementioned conditions into a single EPQ model and provides a comprehensive approach that accounts for the dynamic nature of manufacturing processes to address real-world challenges.

The frame of this document is ordered in the following manner: [Section 2](#) discusses the linked studies that exist in the inventory management field that deals with the warm-up process. [Section 3](#) defines the suggested production-inventory model and its assumptions and notations. In [Section 4](#) the mathematical solution is presented, and a numerical example is developed in [Section 5](#). Furthermore, the validation of this work with related studies is comprehensively outlined in [Section 6](#). Next, a sensitivity analysis is presented, in [Section 7](#), for the main model parameters. [Section 8](#) suggests some managerial insights derived from the model solution and numerical instance. Finally, [Section 9](#) presents the main conclusions and further research questions.

2. Literature review

The EPQ model is a frequently researched and implemented method to establish the optimal production quantity that either minimizes total production costs or maximizes profit. Nobil et al. [32] mention that the initial inventory model was established in the 1910s by Harris and it was later extended by several authors.

A main proposition of the EPQ model is that product fabrication starts as soon as the operator turns on the machines. Glock and Grosse [19] carried out a literature review on lot-sizing problems, with particular emphasis on the impact of changes in production rates. They discovered that a small number of studies have been conducted based on empirical motivation. So, there is a significant need for research work to be accomplished on real-world applications. Jia and Zhang [22] studied fabrication lines with finite buffers and geometric machines and derived some algorithms and closed-form expressions to calculate performance measurement in both single- and double-machine systems. The conjoint analysis of fabrication and maintenance was proposed by Tian et al., [41] since they were looking for an optimal threshold of lot-sizing and condition-based maintenance under a deteriorating fabrication system. Focusing on smart factories, Gong et al., [20] mentioned that the workforce is mainly in charge of production setup, which allows for low-cost flexibility. They determined the optimum range of setup labor force concerning balances among machine production, setup, and mismatch costs. The coordination of the major concerns raised by the inventory-production system is presented in the study of Wan et al., [43]. As, they proposed a new model with the integration of fabrication, maintenance, and quality control. The study evaluated their approach for an imperfect continuous flow process and obtained more than one percent reduction in total cost compared to the classical model. Nobil et al [30] addressed a complex multi-product multi-machine economic production quantity problem for an imperfect manufacturing system, utilizing a hybrid genetic algorithm to optimize machine utilization, item allocation, and cycle length. Mahata [24] formulated a model accounting for factors such as defective items, rework, learning effects, and fuzzy variables, and proposes an algorithm to find the optimal production and back-order quantities in an imperfect production system. De and Mahata [10] introduced an economic order quantity (EOQ) inventory model for items with imperfect quality and shortage back-ordering, incorporating various managerial leadership styles in a fuzzy game theoretic framework.

Any manufacturing or production process is inherently characterized by uncertainties and possibly impacted by the workforce, machine tools, and other factors. In instances where the items produced do not meet quality standards, they are considered defective products, and the system is called imperfect. To avoid that, Rini et al., [36] drew a scheme in

which a tri-tier supply chain with unreliable consignment and imperfect items was studied to meet customers' satisfaction and not face shortages. The flexibility of their approach comes from the credit-dependent demand to gain the benefit of financial assurance from the wholesaler. Tiwari et al., [42] settled the optimum control policy of defective products under the effect of decaying and trade credit policy. Ali et al., [1] explored the optimal decision in manufacturing green products when several of them are faulty. Niu et al., [28] concluded that it is more advantageous for a retailer to use a third-party remanufacturer to produce remanufactured products rather than manufacturing them in-house. Cheng, [4] proposed that both current sales and future returns are affected by uncertain return amounts. In the case of smart products, Sarkar et al., [38] studied a system within an automation inspection technology to detect imperfect items and remanufacture them in the same production cycle, which resulted in a higher profit when the rate of defects follows a Triangular distribution. If the company is interested in providing a warranty, Ghosh et al., [18] studied the optimum fabrication time and selling price for distinct product warranty times in an imperfect production system. Gautam et al., [17] addressed defective products through an effective remanufacturing strategy that makes them suitable for sale at the original price. Pasandideh et al [35] formulated a mathematical model for an imperfect multi-product single-machine economic production quantity problem with scrap and rework with consideration of budget constraints, service level bounds, and warehouse space constraints. P. Mahata and Mahata [25] addressed an imperfect manufacturing system with credit policies in fuzzy random environments, employing discount cash flow analysis to model an inventory problem and using fuzzy random elapsed time as a parameter to optimize the present value of future cash flows.

Demand is a critical parameter in inventory control models, and significant efforts have been dedicated to satisfying customers' needs and avoiding shortages in the systems. Zhou et al., [44] proposed a two-period dynamic pricing and inventory decision model for perishable items to reduce food waste. They focused on addressing shortages by deriving optimum solutions for order amounts and retail prices under diverse quality situations. Cardona-Valdés et al., [2] studied a two-phase stochastic model using Conditional Value at Risk (CVaR) to minimize system total cost in the fashion industry. Results showed that their proposal is an effective and practical methodology to industrial-scale problems. Ivanov and Dolgui, [21] highlighted the emergence of a shortage economy, characterized by long-term resource shortages and hyperinflation risks, as a new challenge for supply chain and operations management. De et al., [11] presented doubt fuzzy sets in a production inventory model with a disrupted demand function due to shortages and introduced a defuzzification method and solution algorithm for four types of doubt fuzzy sets. De and Mahata [9] explored the application of linguistic dense fuzzy lock sets in a three-layer disruption in supply chain model aimed at cost minimization under partial backorder in demand rates. De and Mahata [8] expanded the decision-making criteria for controlling various cost components under a fuzzy lock environment, highlighting the active role of the decision maker in optimizing inventory processes.

The impact of carbon production on human life has prompted government action to mitigate its effects and incentivize sectors heavily involved in this process, such as manufacturing industries. Carbon taxation and the cap-and-trade approaches are two commonly employed methods for assessing the carbon footprint in inventory systems. Fu et al., [13] explored how carbon emission asymmetry affects firms' decisions to adopt green technology. They demonstrated that a carbon tax strategy may not necessarily drive the acceptance of green technology, but for sure carbon-inefficient companies are more probable to gain from green technology. Feng et al., [12] suggested a noble profit allocation control for retailers under a carbon cap-and-trade policy in a joint replenishment system, where efficient companies transfer surplus carbon margin to non-efficient ones based on their generosity. Das et al., [7] built a model for small and moderate manufacturing industries

under carbon emission regulations and uncertainty, which suggested that adopting emission reduction technologies are more beneficial than paying taxes. Zhou et al., [45] reviewed research on supply chain management under carbon taxes, they categorized studies into low-carbon technology, coordination, supplier selection, production planning, facility location, transportation, and pricing. Khan et al., [23] analyzed the collective decision of replenishment, pricing and prepayment installment for a livestock farm within cap-and-price, carbon tax, and cap-and-trade protocols. Mahato et al. [26] developed a sustainable and smart manufacturing model that reduces waste and pollution, focusing on controlling pollution costs and introducing three pollution control mechanisms, with the model featuring a variable pollution cost showing a 27.76% reduction in pollution levels compared to other models. Choudhury et al. [5] presented a multilayer sustainable production-inventory supply chain model considering deterioration and pollution, analyzing the profit functions of the manufacturer and retailer and optimizing the retailer's profit. Das et al. [6] addressed several factors affecting sales revenue and logistics in the marketing environment, including trade credit, warehouse constraints, volume discounts, and carbon emissions, while determining optimal production and replenishment practices for both the supplier and retailer.

The inventory management literature does not present a large number of studies that directly and/or indirectly discussed the integration of the warm-up period within a production system. Nevertheless, Tan et al., [40] investigated the joint fabrication and energy regulator of a make-to-stock system operating in different power modes such as working, idle, and off, in any case, each mode has a different energy cost, and a warm-up is required to change from one state to another. The optimal control policy is determined by solving an LP formulation of the associated Markov decision process.

To the best of the authors' knowledge, Nobil et al., [31] reported the first study in this field and considered the influences of a machine warm-up time on an imperfect fabrication process with rework. They determined the optimum production quantity to minimize total costs while considering the warm-up time and set-up time to maximize machine efficiency and effectiveness, reduce waste, and make the manufacturing process sustainable. Nobil et al., [29] explored the economic production quantity problem with a warm-up period and proposed mathematical models for single-item and multi-item manufacturing systems, intending to minimize costs, reduce waste generation, improve machine efficiency, and decrease environmental impact. Ganesan and Uthayakumar, [14] proposed two production inventory models to address process warm-up production, hybrid maintenance schedule, anticipated shortages, and rework of imperfect items. They used numerical examples and sensitivity analysis to illustrate the models and suggested a trade-off between the rework and holding costs of defective items. Ganesan and Uthayakumar [15] utilized a genetic algorithm to optimize three production inventory models for an imperfect manufacturing system. These models accounted for warm-up production, hybrid maintenance, and rework. Nobil et al., [33] investigated the calculation of the optimal warm-up time by considering the machine's downtime in an economic production quantity framework. This study is a continuation of the research from E. Nobil et al., [34] on

imperfect production systems that generate defective items, intending to identify an optimal warm-up period that minimizes the production of defective items and, at the same time, considers the carbon footprint. Table 1 provides a comparative analysis of research related to the incorporation of a warm-up process within a production system. As depicted, this study integrates all previously discussed aspects and offers a straightforward solution within the specified conditions.

3. Problem definition

The focal point of this study is to integrate the warm-up process into the EPQ model by addressing shortages in response to their absence in prior works. The warm-up process is an important factor that affects the production rate and the quality of the produced batches. Additionally, shortages significantly influence the EPQ model cost structure and the optimal order quantity. Therefore, there is a need to incorporate the warm-up process into an EPQ model with shortages and develop an efficient optimization algorithm that addresses the complexity of the problem.

Manufacturing systems are characterized by imperfections, resulting in the production of some items that fail to meet the required quality standards, thereby requiring reworking or scrapping. In this study, the relationship between the number of defective products and the duration of the warm-up is expressed as a mathematical function, which indicates that the number of defects decreases while warm-up time increases until reaching a limit beyond which no additional benefit is gained. In addition, shortages occur in the system due to the inability to meet customer demand, resulting in a full accumulation of shortages. Carbon emissions are also considered, and cap-and-trade policies are implemented to mitigate their negative impact. An optimal inventory cycle time solution to the model is presented, which is determined through a nonlinear optimization problem that involves three decision variables.

The cap-and-trade policy is an effective approach that was established by lawmakers that aim to mitigate pollution and reduce greenhouse gas emissions through market-based mechanisms. In this policy, the government legislates a cap on the total amount of emissions permissible to a specific industry or sector. The cap is then distributed among tradable permits known as allowances, which grant the permit holder the right to produce a certain number of pollutants. If corporations reduce their emissions beyond the specific cap, they have the option to exchange or trade their allowances and reap some profits. The cap-and-trade policy incentivizes companies to reduce their emissions by offering them economic benefits while providing flexibility in how to meet their environmental goals.

3.1. Assumptions and notation

The assumptions in this study are as follows:

- 1- In each production cycle, the imperfect manufacturing system generates several defective items, which are contingent on the warm-up time period, as depicted in Figure 1.
- 2- The horizon line of time is infinite.

Table 1
Comparing the relevant research with the elements proposed.

Paper	Objective Function		Warm-up process		Solution method		Shortage	Carbon Emission	
	Profit	Cost	Constant	Variable	Analytical	Numerical		Carbon tax	Cap-and-trade
Tan et al [40]	*		*		*				
Nobil et al [31]		*	*		*				
Nobil et al [29]		*	*		*				
Ganesan and Uthayakumar [14]		*	*		*				
Ganesan and Uthayakumar [15]		*	*	*	*				
Nobil et al [33]		*		*		*		*	
Nobil et al [34]		*		*		*		*	
Current Study		*		*	*		*		*

- 3- The system manufactures a single product in the cycle time.
- 4- The demand is not totally satisfied and results in an accumulation of shortages, which are fully backlogged.
- 5- During the warm-up process no product is manufactured.
- 6- The manufacturing system, by its inherent imperfections, makes it essential to implement a warm-up process. Therefore, failing to employ this warm-up process results in a portion of low-quality items, denoted as a . Additionally, pushing the warm-up process beyond a certain threshold offers no advantages; instead, it disrupts the manufacturing operation. Therefore, it is important to note that reducing the proportion of defects below b is an unachievable goal in this context.

The notation used in this research is:

Parameters:

- c : Production cost (\$/unit)
- v : Cost of scrapped items (\$/unit)
- A : Setup cost (\$/setup)
- r : Warm-up cost (\$/unit of time)
- h : Holding cost (\$/unit/unit of time)
- s : Backordering cost (\$ /unit/unit of time)
- d : Demand rate (units/unit of time)
- p : Production rate (units /unit of time); $p > d$
- a : Maximum permissible portion of poor-quality items
- b : Minimum inevitable portion of poor-quality items; ($0 \leq b \leq a \leq 1$)
- m : Maximum permissible time for the warm-up period (unit of time)
- y_1 : Carbon emission linked with each setup process (unit of weight/ setup)
- y_2 : Carbon emission of the warm-up time (unit of weight/unit of time)
- y_3 : Carbon emission of production (unit of weight/unit)
- y_4 : Carbon emission of disposal scrap (unit of weight/unit)
- y_5 : Carbon emission of inventory (unit of weight/unit/unit of time)
- G : Carbon tax (\$/unit of weight)
- θ : Maximum permissible carbon emission (unit of weight/unit of time)
- α : Carbon trading cost (\$/unit of weight)

Dependent decision variables:

- I : Maximum inventory level (units)
- Q : Production quantity per cycle (units)
- t_1 : Production time per cycle when the system has no shortage (unit of time)
- e_1 : Consumption time per cycle when the system has no shortage (unit of time)

- t_2 : Production time per cycle when the system is facing shortages (unit of time)
- e_2 : Consumption period per cycle when the system is facing shortages (unit of time)
- y : Total carbon emission (unit of weight/unit of time)
- Y : Total emission cost of carbon (\$/unit of time)
- X : Total cost without emission cost of carbon (\$/unit of time)
- K : Total carbon trading cost (\$/unit of time)
- Z : Integrated total cost (\$/unit of time)

Decision variables:

- T : Cycle length (unit of time)
- w : Warm-up period (unit of time)
- B : Shortage quantity (units)

The inventory behavior through time of the production-inventory model is shown in Fig. 1. Production rate (p) comprises two categories of items: those of good quality and those of poor quality. The poor-quality items are manufactured at a rate of $f(w)p$. The $f(w)$ is a function of the portion of the poor-quality items, which rest on the length of the warm-up time. This function is linear and associates the portion of the poor-quality items with the warm-up time, see Fig. 2.

Fig. 1 illustrates the variation in inventory levels over time in a system that employs a warm-up process prior to launching regular production. Within this system, multiple items are produced with varying degrees of defectiveness, and $f(w)$ represents the rate of their production in time. As shown in Fig. 2, its value depends on the duration of the warm-up process. If it increases, then the number of defective items is lessened. Regarding the operation of the production line during a period of interruption, neglecting the warm-up process implies that the system generates the default number of imperfect items (a). Obviously, the usage of the warm-up process is limited to a possible range that should be lower than the whole production time. On the other hand, surpassing the warm-up period beyond the upper range (m) is not feasible and there is no benefit in doing so.

As shown in Fig. 1, once the fabrication is stopped, the maximum inventory level (I) is reached; after that, the consumption period (e_1) begins at this inventory level and moves to zero. The maximum inventory level, production, and consumption periods are calculated in the following manner:

$$I = ((1 - f(w))p - d) \frac{Q}{p} - B \tag{1}$$

$$t_1 = \frac{I}{((1 - f(w))p - d)} = \frac{Q}{p} - \frac{B}{((1 - f(w))p - d)} \tag{2}$$

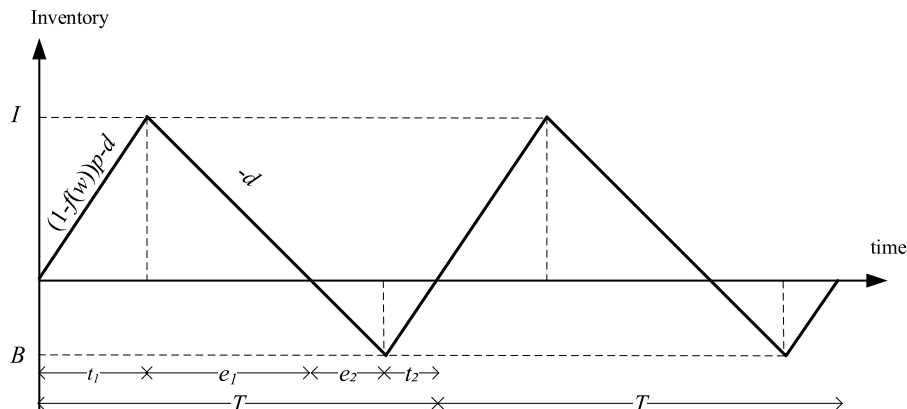


Fig. 1. The behavior inventory level of the production-inventory model.

$$e_1 = \frac{I}{d} = ((1 - f(w))p - d) \frac{Q}{pd} - \frac{B}{d} \tag{3}$$

$$e_2 = \frac{B}{d} \tag{4}$$

$$t_2 = \frac{B}{((1 - f(w))p - d)} \tag{5}$$

So, based on Eqs. (2) and (3), the cycle length is calculated with:

$$T = t_1 + e_1 + e_2 + t_2 = \frac{(1 - f(w))Q}{d} \tag{6}$$

and

$$Q = \frac{dT}{1 - f(w)} \tag{7}$$

As represented in Fig. 2, when the length of the warm-up time is zero, there is a maximum portion of poor-quality items, while if the warm-up time takes up to the maximum time (m) the minimum portion of poor-quality articles quality is reached. Thus, the function representing the proportion of poor-quality items in relation to the duration of the warm-up period is expressed as follows:

$$f(w) = a - \left(\frac{a - b}{m}\right)w; \quad 0 < w \leq m \tag{8}$$

The total cost of the production-inventory system incorporates setup, warm-up, production, disposal, and holding costs. The computation details of each one of them are presented below:

- The setup cost per cycle is A , hence, the setup cost per unit time converts to A/T .
- The cost of unit warm-up time is r and the length of the warm-up period (w), therefore, the warm-up cost per cycle is determined by multiplying the warm-up cost by the length of the warm-up period (rw). And so, the warm-up cost per unit of time is equal to rw / T .
- The production amount in each cycle is Q and c is the fabrication cost of each item, subsequently, the production cost per cycle is cQ . According to Eq. (7), the production cost per unit of time equals to $dc / 1 - f(w)$.
- During the production period, the amount of the poor-quality items is $f(w)Q$, from Eq. (7), then the disposal cost per unit time is $dvf(w)/1 - f(w)$, here v is the unit scrapped cost of the poor-quality product.
- In addition, the holding cost per unit of time is equal to $hI(t_1 + e_1)/2]/T$. Substituting I , t_1 , and e_1 from Eq. (1) to Eq. (3) into this expression, then the holding cost can be computed as (the details of the calculation of the holding cost is shown in Appendix A):

$$\begin{aligned} \text{holding cost} &= \frac{hd((1 - f(w))p - d)}{2(1 - f(w))p} (T) - hB \\ &+ \frac{h(1 - f(w))p}{2d((1 - f(w))p - d)} \left(\frac{B^2}{T}\right) \end{aligned} \tag{9}$$

to $(s(1 - f(w))p / 2d((1 - f(w))p - d))(B^2 / T)$. We provide the details of the calculation of the shortage cost in Appendix B.

Lastly, based on Eq. (7), the setup, production, warm-up, disposal, holding, and shortage costs of the system without carbon emission considerations are summarized as follows:

$$\begin{aligned} X &= \frac{A + rw}{T} + \frac{hd\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T) \\ &+ \frac{(h + s)\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right) - hB + \frac{dc}{1 - a + \left(\frac{a-b}{m}\right)w} \\ &+ \frac{dv\left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} \end{aligned} \tag{10}$$

The impacts of the emission costs of carbon added to Eq. (10) are: the total carbon emissions are calculated based on the carbon quantity emitted from the setup, warm-up production, disposal, and holding processes. The carbon emission generated in each setup is y_1 , so, the carbon emission associated with the setup process per year would be y_1/T . Also, the carbon emission per warm-up unit time is y_2 , hence, the warm-up process carbon emission per year is calculated as y_2w/T . The carbon emission released per manufactured item is y_3 , hence, the total carbon emissions caused for all fabricated products per unit of time is $y_3Q/T = y_3d/(1 - f(w))$. Furthermore, the carbon emission released by disposing of each poor-quality item is y_4 , then, the total carbon emissions created for the disposal process per year is equal to $f(w)y_4Q/T = y_4f(w)d/(1 - f(w))$. Also, the carbon emissions secreted from the holding process is $y_5[I(t_1 + e_1)/2]/T$. During shortage periods, there is no holding process for carbon emission, as there are no items in the inventory. Therefore, the total carbon emission amount for setup, warm-up, production, disposal, and holding processes is:

$$\begin{aligned} y &= \frac{y_1 + y_2w}{T} + \frac{dy_5\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T) - y_5B \\ &+ \frac{y_5\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right) + \frac{dy_3}{1 - a + \left(\frac{a-b}{m}\right)w} \\ &+ \frac{dy_4\left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} \end{aligned} \tag{11}$$

Consequently, the total emission cost of carbon is found by multiplying the carbon tax by the total carbon emissions, Eq. (11), thus:

$$Y = Gy = G \left(\frac{y_1 + y_2w}{T} + \frac{dy_3}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{dy_4\left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{dy_5\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T) - y_5B + \frac{y_5\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right) \right) \tag{12}$$

- In similar fashion, the shortage cost is calculated as $s[B(e_2 + t_2)/2]/T$. Substituting e_2 and t_2 from Eq. (4) and (5), the shortage cost becomes

As mentioned before, the total carbon emissions amount is equal to y , and the maximum carbon emissions that the government allows to

produce per unit of time equals to θ . If the total carbon emissions are greater than the maximum limit, the company should buy any excess capacity from another company at a carbon trading cost per unit (α). At the same time, if the company has any excess capacity, it can sell it to another one at the trading cost. As a result, the carbon trading cost is formulated as follows:

$$K = \alpha(y - \theta)$$

$$= \alpha \left(\frac{y_1 + y_2 w}{T} + \frac{dy_3}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{dy_4 \left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{dy_5 \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2 \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T) - y_5 B + \frac{y_5 \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right) - \theta \right) \quad (13)$$

The system's total cost is integra by the total inventory cost in Eq. (10), the total carbon emissions cost in Eq. (12), and the total carbon trading cost in Eq. (13) as follows:

$$Z = X + Y + K$$

$$= \frac{A + rw + G(y_1 + y_2 w)}{T} + \frac{d(c + Gy_3)}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{d(v + Gy_4) \left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w}$$

$$- (h + Gy_5)B + \frac{d(h + Gy_5) \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2 \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T)$$

$$+ \frac{(h + s + Gy_5) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right)$$

$$+ \alpha \left(\frac{y_1 + y_2 w}{T} + \frac{dy_3}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{dy_4 \left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{dy_5 \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2 \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T) \right.$$

$$\left. - y_5 B + \frac{y_5 \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right) - \theta \right)$$

$$= -\alpha\theta + \frac{A + rw + (G + \alpha)(y_1 + y_2 w)}{T} + \frac{d(c + (G + \alpha)y_3)}{1 - a + \left(\frac{a-b}{m}\right)w}$$

$$+ \frac{d(v + (G + \alpha)y_4) \left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} - (h + (G + \alpha)y_5)B$$

$$+ \frac{d(h + (G + \alpha)y_5) \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2 \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T)$$

$$+ \frac{(h + s + (G + \alpha)y_5) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right)$$

Here, the warm-up period occurs at the end of the consumption time, followed by the fabrication time. Therefore, the warm-up time should be smaller than the consumption time, hence:

$$w \leq e_1 + e_2 \leq ((1 - f(w))p - d) \frac{Q}{pd} \leq ((1 - f(w))p - d) \frac{T}{(1 - f(w))p} \quad (15)$$

Substituting e_1 and e_2 from Eqs. (3) and (4), respectively, leads to $w \leq e_1 + e_2 \leq ((1 - f(w))p - d) \frac{Q}{pd}$, then, using Eq. (7), the constraint is $((1 - f(w))p - d) \frac{T}{(1 - f(w))p}$. Finally, by substituting $f(w)$ from Eq. (8), the close form of the constraint is specified below:

$$T \geq \frac{\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p\right)w}{\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d} \quad (16)$$

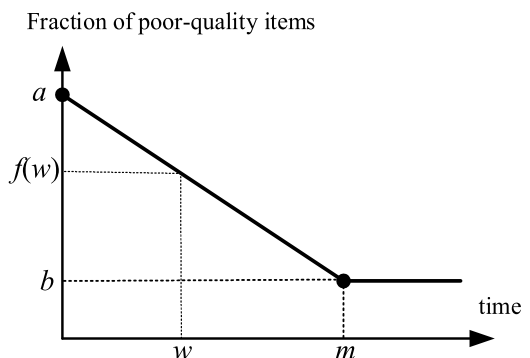
Conversely, if the warm-up period is larger than the maximum permissible time, it is not practical. Thus, the warm-up period happens between its limit, i.e., $0 \leq w \leq m$. Taking into account the objective function and the constraints, the optimization problem is constructed as:

$$\text{Min } Z(w, T, B) = -\alpha\theta + \frac{A + rw + (G + \alpha)(y_1 + y_2 w)}{T} + \frac{d(c + (G + \alpha)y_3)}{1 - a + \left(\frac{a-b}{m}\right)w}$$

$$+ \frac{d(v + (G + \alpha)y_4) \left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} - (h + (G + \alpha)y_5)B$$

$$+ \frac{d(h + (G + \alpha)y_5) \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)}{2 \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} (T)$$

$$+ \frac{(h + s + (G + \alpha)y_5) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{2d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B^2}{T}\right) \quad (17)$$



(14) Fig. 2. The behavior of the proportion of the poor-quality items (Nobil et al. [34]).

$$T \geq T_{Min} \tag{17-1}$$

$$T \geq 0 \tag{17-2}$$

$$B \geq 0 \tag{17-3}$$

$$0 \leq w \leq m \tag{17-4}$$

where,

$$T_{Min} = \frac{\left(\left(1 - a + \left(\frac{a-b}{m} \right) w \right) p \right) w}{\left(1 - a + \left(\frac{a-b}{m} \right) w \right) p - d} \tag{18}$$

4. Solution method

This part establishes the convexity of the objective function and the optimization model, providing a solid foundation for any optimization problem derived from it. As it is widely known, the convexity of the objective function is a critical factor in determining the effectiveness of any optimization model. The Hessian matrix is one of the essential tools to establish the convexity of an objective function. Provides a way to calculate the second-order partial derivatives of the objective function, which are used to determine the curvature of the function, Eq. (17). Because of the complexity of the problem, we cannot establish that the value of total cost, denoted as $Z(w, T, B)$, exhibits joint convexity with respect to the variables w, T , and B . However, under certain conditions, we can demonstrate the convexity of the objective function. Given that our current model involves three decision variables, we have employed Theorem 1 to establish the convexity of the objective function under certain conditions. This theorem, as referenced from the work of San-José et al. [37] and Chang et al. [3], provides a rigorous foundation for proving the convexity of the objective function. It is based on specific assumptions that permit a limited number of variables to have fixed values, which further strengthens the validity of our approach.

Theorem 1. (1) For any given B , if the Hessian Matrix determinant is positive, then $Z(w, T, B)$ is joint convex in w and T .

(2) For any given w and T , $Z(w, T, B)$ is strictly pseudo-convex in B , and hence has a unique global optimal solution B .

Proof.

For any given B , taking the first and second order partial derivatives of $Z(w, T, B)$ with respect to w and T , are represented by:

$$\begin{aligned} \frac{\partial Z}{\partial T} &= \frac{-A - rw - (G + \alpha)(y_1 + y_2 w)}{T^2} \\ &- \frac{(h + s + (G + \alpha)y_5) \left(1 - a + \left(\frac{a-b}{m} \right) w \right) p \left(\frac{B^2}{T^2} \right)}{2d \left(\left(1 - a + \left(\frac{a-b}{m} \right) w \right) p - d \right)} \\ &+ \frac{(h + (G + \alpha)y_5)d \left(\left(1 - a + \left(\frac{a-b}{m} \right) w \right) p - d \right)}{2 \left(1 - a + \frac{w(a-b)}{m} \right) p} \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial^2 Z}{\partial T^2} &= \frac{2(A + rw + (G + \alpha)(y_1 + y_2 w))}{T^3} \\ &+ \frac{(h + s + (G + \alpha)y_5) \left(1 - a + \frac{w(a-b)}{m} \right) p \left(\frac{B^2}{T^3} \right)}{d \left(\left(1 - a + \frac{w(a-b)}{m} \right) p - d \right)} \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{\partial Z}{\partial w} &= \frac{r + (G + \alpha)y_2}{T} - \frac{(c + (G + \alpha)y_3)d(a-b)}{m \left(1 - a + \frac{w(a-b)}{m} \right)^2} \\ &- \frac{d(v + (G + \alpha)y_4)(a-b)}{m \left(1 - a + \frac{w(a-b)}{m} \right)} \\ &- \frac{d(v + (G + \alpha)y_4)(a-b) \left(a - \frac{w(a-b)}{m} \right)}{m \left(1 - a + \frac{w(a-b)}{m} \right)^2} \\ &+ \frac{d(h + (G + \alpha)y_5)(a-b)T}{2m \left(1 - a + \frac{w(a-b)}{m} \right)} \\ &+ \frac{d(h + (G + \alpha)y_5)(a-b) \left(d - p \left(1 - a + \frac{w(a-b)}{m} \right) \right) T}{2mp \left(1 - a + \frac{w(a-b)}{m} \right)^2} \\ &- \frac{p(h + (G + \alpha)y_5 + s)(a-b) \left(\frac{B^2}{T} \right)}{2dm \left(d - p \left(1 - a + \frac{w(a-b)}{m} \right) \right)} \\ &- \frac{p^2(h + (G + \alpha)y_5 + s)(a-b) \left(1 - a + \frac{w(a-b)}{m} \right) \left(\frac{B^2}{T} \right)}{2dm \left(d - p \left(1 - a + \frac{w(a-b)}{m} \right) \right)^2} \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\partial^2 Z}{\partial w^2} &= \frac{2(c + (G + \alpha)y_3)d(a-b)^2}{m^2 \left(1 - a + \frac{w(a-b)}{m} \right)^3} + \frac{2d(v + (G + \alpha)y_4)(a-b)^2}{m^2 \left(1 - a + \frac{w(a-b)}{m} \right)^2} \\ &+ \frac{2d(v + (G + \alpha)y_4)(a-b)^2 \left(a - \frac{w(a-b)}{m} \right)}{m^2 \left(1 - a + \frac{w(a-b)}{m} \right)^3} - \frac{d(h + (G + \alpha)y_5)(a-b)^2 T}{m^2 \left(1 - a + \frac{w(a-b)}{m} \right)^2} \\ &- \frac{d(h + (G + \alpha)y_5)(a-b)^2 \left(d - p \left(1 - a + \frac{w(a-b)}{m} \right) \right) T}{m^2 p \left(1 - a + \frac{w(a-b)}{m} \right)^3} \\ &- \frac{p^2(h + (G + \alpha)y_5 + s)(a-b)^2 \left(\frac{B^2}{T} \right)}{dm^2 \left(d - p \left(1 - a + \frac{w(a-b)}{m} \right) \right)^2} \\ &- \frac{p^3(h + (G + \alpha)y_5 + s)(a-b)^2 \left(1 - a + \frac{w(a-b)}{m} \right) \left(\frac{B^2}{T} \right)}{dm^2 \left(d - p \left(1 - a + \frac{w(a-b)}{m} \right) \right)^3} \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial^2 Z}{\partial w \partial T} &= \frac{\partial^2 Z}{\partial T \partial w} \\ &= \frac{d(h + (G + \alpha)y_5)(a - b)}{2m \left(1 - a + \frac{w(a-b)}{m}\right)} - \frac{r + (G + \alpha)y_2}{T^2} \\ &\quad + \frac{d(h + (G + \alpha)y_5)(a - b) \left(d - p \left(1 - a + \frac{w(a-b)}{m}\right)\right)}{2mp \left(1 - a + \frac{w(a-b)}{m}\right)^2} \\ &\quad + \frac{p(h + (G + \alpha)y_5 + s)(a - b) \left(\frac{B^2}{T^2}\right)}{2dm \left(d - p \left(1 - a + \frac{w(a-b)}{m}\right)\right)} \\ &\quad + \frac{p^2(h + (G + \alpha)y_5 + s)(a - b) \left(1 - a + \frac{w(a-b)}{m}\right)}{2dm \left(d - p \left(1 - a + \frac{w(a-b)}{m}\right)\right)^2} \left(\frac{B^2}{T^2}\right) \end{aligned} \quad (23)$$

The Hessian matrix determinant is computed as:

$$\begin{aligned} \text{Hessian}(w, T|B) &= [w, T] \begin{bmatrix} \frac{\partial^2 Z}{\partial w^2} & \frac{\partial^2 Z}{\partial w \partial T} \\ \frac{\partial^2 Z}{\partial T \partial w} & \frac{\partial^2 Z}{\partial T^2} \end{bmatrix} \begin{bmatrix} w \\ T \end{bmatrix} \\ &= \frac{d^2(h + (G + \alpha)y_5)(a - b)(1 - a)wT}{mp \left(1 - a + \frac{w(a-b)}{m}\right)^3} + \frac{p^2(h + (G + \alpha)y_5 + s)(a - b)w \left(\frac{B^2}{T}\right)}{m \left(\left(1 - a + \frac{w(a-b)}{m}\right)p - d\right)^2} \\ &\quad + \frac{2(c + (G + \alpha)y_3)d(a - b)^2w^2}{m^2 \left(1 - a + \frac{w(a-b)}{m}\right)^3} + \frac{2d(v + (G + \alpha)y_4)(a - b)^2w^2}{m^2 \left(1 - a + \frac{w(a-b)}{m}\right)^3} \\ &\quad + \frac{p^3(h + (G + \alpha)y_5 + s)(a - b)^2w^2 \left(\frac{B^2}{T}\right) + 2(A + (G + \alpha)(y_1))}{m^2 \left(\left(1 - a + \frac{w(a-b)}{m}\right)p - d\right)^3} \\ &\quad + \frac{(h + s + (G + \alpha)y_5) \left(1 - a + \frac{w(a-b)}{m}\right)p \left(\frac{B^2}{T}\right)}{d \left(\left(1 - a + \frac{w(a-b)}{m}\right)p - d\right)} \geq 0 \end{aligned} \quad (24)$$

Based on the model assumptions, it is evident that $1 - a + \frac{w(a-b)}{m} \geq 0$, $1 \geq a$, $a \geq b$, and $\left(1 - a + \frac{w(a-b)}{m}\right)p \geq d$, therefore, for any given B , the Hessian Matrix determinant in (24) is positive.

Similarly, for any given w and T , the first and second order partial derivatives of $Z(w, T, B)$ with respect to B , are:

$$\frac{\partial Z}{\partial B} = \frac{(h + (G + \alpha)y_5 + s) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B}{T}\right) - h - (G + \alpha)y_5 \quad (25)$$

$$\frac{\partial^2 Z}{\partial B^2} = \frac{(h + (G + \alpha)y_5 + s) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{1}{T}\right) \geq 0 \quad (26)$$

Given that $1 - a + \frac{w(a-b)}{m} \geq 0$ and $\left(1 - a + \frac{w(a-b)}{m}\right)p \geq d$, then Eq. (26) is positive ($\partial^2 Z(B|T, w) / \partial B^2 \geq 0$), and subsequently, it is proved that the objective function with respect to B is convex for any given w and T .

Theorem 1 is a fundamental tool to establish the convexity of the optimization problems under certain conditions. Utilizing Theorem 1 as a foundation, we propose the following algorithm for determining the optimal decision variables and the optimal value of total cost. Therefore, if we assume that we know the values of T and w , it is proved that the

objective function with respect to B is convex, and subsequently, we can determine the optimal value of B using the first derivative of decision variable B in the objective function is as follows:

$$\frac{\partial Z}{\partial B} = \frac{(h + (G + \alpha)y_5 + s) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}{d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)} \left(\frac{B}{T}\right) - h - (G + \alpha)y_5 = 0 \quad (27)$$

Then, the value of shortage is:

$$B = \frac{(h + (G + \alpha)y_5)d \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)(T)}{(h + (G + \alpha)y_5 + s) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} \quad (28)$$

As the proposed model lacks any constraints related to shortages, we can revise it in accordance with Eq. (17) by replacing the shortage value with the expression from Eq. (28) as follows:

$$\begin{aligned} \text{Min } Z(w, T) &= -\alpha\theta + \frac{A + rw + (G + \alpha)(y_1 + y_2w)}{T} + \frac{d(c + (G + \alpha)y_3)}{1 - a + \left(\frac{a-b}{m}\right)w} \\ &\quad + \frac{d(v + (G + \alpha)y_4) \left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} \\ &\quad + \left(\frac{d(h + (G + \alpha)y_5) \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)s}{2(h + (G + \alpha)y_5 + s) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}\right)(T) \end{aligned} \quad (29)$$

$$\left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p\right)wT - \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p + d \leq 0 \quad (29-1)$$

$$w - m \leq 0 \quad (29-2)$$

$$T, w \geq 0 \quad (29-3)$$

We are now dealing with a non-linear programming model featuring two decision variables, T and w . Given the convex nature of the objective function for these variables, as established by Theorem 1, and the presence of two inequality constraints, we employ the Karush-Kuhn-Tucker (KKT) conditions to identify the optimal solutions. This method is widely recognized for addressing such problems, exemplified by Nayak et al. [27] who employed a game theoretic approach with Lagrange multiplier to resolve an EPQ model.

If "Z" is the objective function in Eq. (29), "g₁" and "g₂" are inequality constraints in Eqs. (29-1) and (29-2), and "λ₁" and "λ₂" are Lagrange multipliers, the Lagrangian function "k" is typically defined as: $k = Z - \lambda_1 g_1 - \lambda_2 g_2$, where Z , g_1 , and g_2 are as follows:

$$\begin{aligned} Z &= -\alpha\theta + \frac{A + rw + (G + \alpha)(y_1 + y_2w)}{T} + \frac{d(c + (G + \alpha)y_3)}{1 - a + \left(\frac{a-b}{m}\right)w} \\ &\quad + \frac{d(v + (G + \alpha)y_4) \left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} \\ &\quad + \left(\frac{d(h + (G + \alpha)y_5) \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p - d\right)s}{2(h + (G + \alpha)y_5 + s) \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p}\right)(T) \end{aligned} \quad (30)$$

$$g_1 = \left(\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p\right)wT - \left(1 - a + \left(\frac{a-b}{m}\right)w\right)p + d \quad (31)$$

$$g_2 = w - m \tag{32}$$

The values of T , w , λ_1 , and λ_2 can be determined by solving the following equations: $\frac{\partial k}{\partial T} = 0$, $\frac{\partial k}{\partial w} = 0$, $\frac{\partial k}{\partial \lambda_1} = 0$, and $\frac{\partial k}{\partial \lambda_2} = 0$. Therefore, we can determine the optimal solution to optimize the proposed NLP model with two inequality constraints using the following KKT conditions.

Condition 1) $\frac{\partial k}{\partial T} = 0 \rightarrow \frac{-(A+rw+(G+\alpha)(y_1+y_2w))}{T^2} + \frac{d(h+(G+\alpha)y_5)((1-a+(\frac{a-b}{m})w)p-d)s}{2(h+(G+\alpha)y_5+(1-a+(\frac{a-b}{m})w)p)} - \lambda_1((1-a+(\frac{a-b}{m})w)p)w = 0$

Condition 2) $\frac{\partial k}{\partial w} = 0 \rightarrow$

$$\frac{r + y_2(G + \alpha)}{T} - \frac{d(a - b)(c + y_3(G + \alpha))}{m\left(\frac{w(a - b)}{m} - a + 1\right)^2} - \frac{d(a - b)(v + y_4(G + \alpha))}{m\left(\frac{w(a - b)}{m} - a + 1\right)} - \frac{d(a - b)(v + y_4(G + \alpha))\left(a - \frac{w(a - b)}{m}\right)}{m\left(\frac{w(a - b)}{m} - a + 1\right)^2} + \frac{Tds(a - b)(h + y_5(G + \alpha))}{m\left(\frac{w(a - b)}{m} - a + 1\right)(2h + 2s + 2y_5(G + \alpha))} + \frac{Tds(a - b)(h + y_5(G + \alpha))\left(d - p\left(\frac{w(a - b)}{m} - a + 1\right)\right)}{mp\left(\frac{w(a - b)}{m} - a + 1\right)^2(2h + 2s + 2y_5(G + \alpha))} - \lambda_1 p \left(\frac{w(a - b)}{m} - a + 1\right) T + \frac{\lambda_1 p(a - b)}{m} - \frac{\lambda_1 T p w(a - b)}{m} - \lambda_2 = 0$$

- Condition 3) $\lambda_1 g_1 = 0 \rightarrow \lambda_1((1 - a + (\frac{a-b}{m})w)p)wT - \lambda_1(1 - a + (\frac{a-b}{m})w)p + \lambda_1 d = 0$
- Condition 4) $\lambda_2 g_2 = 0 \rightarrow \lambda_2(w - m) = 0$
- Condition 5) $g_1 \leq 0 \rightarrow ((1 - a + (\frac{a-b}{m})w)p)wT - (1 - a + (\frac{a-b}{m})w)p + d \leq 0$
- Condition 6) $g_2 \leq 0 \rightarrow w - m \leq 0$
- Condition 7) $\lambda_1, \lambda_2 \leq 0$ (Because the problem is a minimize function)
- Condition 8) $T, w \geq 0$

Following KKT principles, in conditions 3 and 4, we have $\lambda=0$ or $g=0$ either. This leads to the existence of four distinct cases:

- Case I) $\lambda_1 = 0, \lambda_2 = 0$;
- Case II) $\lambda_1 = 0, \lambda_2 \neq 0$;
- Case III) $\lambda_1 \neq 0, \lambda_2 = 0$;
- Case IV) $\lambda_1 \neq 0, \lambda_2 \neq 0$.

Solving these four cases should ensure the fulfillment of all the eight KKT conditions. If each of these cases meets all the necessary conditions, it represents the optimal solution.

Hence, the proposed solution algorithm for this problem involves the following steps:

1. Check the feasibility of the model, as follows $(1 - b)p \geq (1 - a)p \geq d$. If the problem has feasible conditions, go to Step 2.
2. Compute the shortage quantity, Eq (28), and substitute it, into Eq. (17), then, formulate the model in (29)

3. Apply the KKT conditions to obtain the optimum lengths of the warm-up period (w^*) and the cycle length (T^*).
4. Once the optimum warm-up period and cycle length are determined, use Eq. (28) to calculate the optimal value of B^* .
5. Based on the optimum values, then the lowest total cost is calculated using Eq. (17).

5. Numerical example

The solution approach presented in the previous section is applied to solve the following numerical example: Let $A = \$; 500/\text{setup}$, $c = \$; 20/\text{unit}$, $v = \$; 10/\text{scrapped unit}$, $d = 300 \text{ units/shift}$, $p = 2000 \text{ units/shift}$, $h = \$; 5/\text{unit/shift}$, $s = \$; 20/\text{unit/shift}$, $r = \$; 8000 \text{ shift}$, $m = 0.3 \text{ shift}$, $G = \$; 0.4/\text{kg}$, $y_1 = 100 \text{ kg/setup}$, $y_2 = 3000 \text{ kg/shift}$, $y_3 = 2 \text{ kg/unit}$, $y_4 = 5 \text{ kg/unit}$, $y_5 = 5 \text{ kg/unit/ shift}$, $\alpha = \$; 10/\text{kg}$, $\theta = 400 \text{ kg/shift}$, $a = 0.314$, and $b = 0.2$.

The warm-up function is $f(w) = 0.314 - (0.38)w$; $0 \leq w \leq 0.3$. Initially, the feasible solution condition is checked as follows: $(1 - b)p \geq (1 - a)p \geq d$. Then, substituting Eqs. (28) into (17) and formulating model (30), and applying the solution procedure under KKT conditions. Among the four cases of KKT conditions, only Case I, where $\lambda_1 = 0$ and $\lambda_2 = 0$, fulfills all the required KKT conditions. Consequently, the optimal values for the warm-up period and cycle length are found to be $w^* = 0$ shifts and $T^* = 0.9420790$ shifts. According to the above values, using Eq. (28), the optimum shortage quantity is $B^* \cong 163.4$ units. Finally, the optimum total cost is $Z^* \cong \$; 25,625.5$, based on Eq. (17) and the optimal values obtained above. Also, the total carbon emission produced in a shift is approximately equal to 1704.6 kg, i.e., $y^* \cong 1704.6 \text{ kg}$. Subsequently, the company should purchase $1704.6 - 400 = 1304.6 \text{ kg}$, as 400 is the maximum allowable carbon emission limit in a shift, from another company at price 10 \$ per carbon kilogram. The optimum solutions of the numerical instance are presented in Table 2.

This study highlights the contribution of different aspects of the inventory model and allows the researchers to investigate them separately in the system. Carbon emissions is one of the key topics covered by the model, as it admits compliance with environmental concerns. Nonetheless, it is possible to omit this factor and allow the system to find an optimal solution without it. The results endorse the solution method and provide a rough comparison among different strategies. To do so, the elements that represent the carbon consideration Y^* and K^* are set equal to zero to ignore the environmental policy from the model.

The model's solutions without carbon emissions consideration are presented in Table 3. The results show a decrease in the total cost compared to the original instance, which is an environmentally friendly approach and imposes excessive cost for polluting the environment as a penalty. Another consideration in this study is shortage, if the system is allowed to meet all the demands of the customers, then the system reports an increase in total cost, as well as in carbon emissions (y^*) and environmental costs (Y^* and K^*), numerical solutions are summarized in Table 4.

6. Validation

In this context, we establish certain parameters of the proposed model as null to assess the compatibility of this study with prior research. Initially, we posit that a carbon tax is imposed on all carbon

Table 2
General information about the optimal solutions of the numerical example.

Notations	Values	Notations	Values
w^*	0 shift	X^*	11,897.6 \$
T^*	0.942 shifts	Y^*	681.8 \$
B^*	163.468 units	K^*	13,046.1 \$
y^*	1704.618 Kg	Z^*	25,625.5 \$

production, and the company cannot buy and sell the carbon among themselves. In mathematical terms, we assign a value of zero to the parameter α . Furthermore, we mandate that no shortages are permissible within the inventory system, signifying that decision variable B must also be set to zero. Hence, the objective function depicted in Eq. (17) in this study is transformed into Eq. (33), as presented by Nobil et al. [34], who addressed an EPQ problem incorporating warm-up, absence of shortages, and compliance with carbon tax regulations. Following this, we applied the solution method proposed by Nobil et al. [34] to solve the numerical example with $B = \alpha = 0$, yielding the optimal solutions as presented in Table 5.

$$\begin{aligned} \text{Min } Z(w, T) = & \frac{A + Gy_1 + (r + Gy_2)w}{T} + \frac{d(c + Gy_3)}{1 - a + \left(\frac{a-b}{m}\right)w} \\ & + \frac{d(v + Gy_4)\left(a - \left(\frac{a-b}{m}\right)w\right)}{1 - a + \left(\frac{a-b}{m}\right)w} + \frac{(h + Gy_5)dT}{2} \\ & - \frac{(h + Gy_5)d^2T}{2\left(1 - a + \left(\frac{a-b}{m}\right)w\right)p} \end{aligned} \quad (33)$$

After that, we also make the assumption that the warm-up process has no impact on the production system, and the government does not levy taxes on carbon emissions. In mathematical terms, this can be expressed as $w = G = 0$. Consequently, by taking into account the conditions $B = \alpha = w = G = 0$, the objective function of the proposed model is transformed, resulting in Eq. (34). Eq. (34) represents the classic EPQ model with scrap, as outlined by Taleizadeh [39]. Following this, we employed the derivation method expressed as $T^* = \sqrt{2A(1-a)p/hd((1-a)p-d)}$ to solve the numerical example under the condition of $B = \alpha = w = G = 0$. The result yielded the optimal solutions as $T^* = 0.923$ shifts and $Z^* = 11202.1$ \$.

$$\text{Min } Z(T) = \frac{A}{T} + \frac{hd}{2} \left(1 - \frac{d}{(1-a)p}\right)T + \frac{dc}{1-a} + \frac{dva}{1-a} \quad (34)$$

In the final step, we assume that the production system operates perfectly, with no scrap items produced, meaning $a = 0$. Therefore, the objective function of the proposed model is modified to Eqs. (35). (35) represents the classic EPQ model, as described by García-Laguna et al. [16]. Subsequently, we applied the derivation method expressed as $T^* = \sqrt{2Ap/hd(p-d)}$ to solve the numerical example with all parameters set to $B = \alpha = w = G = a = 0$. This resulted in the optimal solutions: $T^* = 0.885$ shifts and $Z^* = 7129.1$ \$.

$$\text{Min } Z = \frac{A}{T} + \frac{hd}{2} \left(1 - \frac{d}{p}\right)T + dc = \frac{DA}{Q} + \frac{hQ}{2} \left(1 - \frac{d}{p}\right) + dc \quad (35)$$

7. Sensitivity Analysis

This section takes into consideration all system parameters of the proposed model as relevant variables to achieve a sensitivity analysis of the problem. Each analyzed parameter's value changes within a range from -40% up to 40% of its original value. Table 6 includes parameters that are related to the system's natural characteristics. Additionally, the

Table 3
General information about the optimal solutions of the numerical example without the carbon cost.

Notations	Values	Notations	Values
w^*	0.3 shifts	X^*	10,628 \$
T^*	2.438 shifts	Y^*	0 \$
B^*	110.901 units	K^*	0 \$
y^*	2486.214 Kg	Z^*	10,628 \$

Table 4
General information about the optimal solutions of the numerical example without shortage.

Notations	Values	Notations	Values
w^*	0 shift	X^*	11,442.2 \$
T^*	0.480 shifts	Y^*	820.3 \$
B^*	0 units	K^*	16,508.6 \$
y^*	2050.860 Kg	Z^*	28,771.1 \$

Table 5
General information about the optimal solutions of the numerical example with $B = \alpha = 0$.

Notations	Values	Notations	Values
w^*	0.3 shifts	X^*	10,923.0 \$
T^*	1.966 shifts	Y^*	1132.7 \$
B^*	0 units	K^*	0 \$
y^*	2831.944 Kg	Z^*	12,055.7 \$

impact of variations in system parameters on the total cost, as shown in Table 6, is visually depicted in Fig. 3. The results show that the warm-up period has a neutral response to the production and demand rate, while variations in maximum and minimum allowable fraction of low-quality items and in maximum permissible time for the warm-up period produce some changes in the warm-up period length. The total carbon emission (y) and the integrated total cost (Z) remain almost unchanged as response to any change in the parameter's values. Contrary to the high sensitivity of the shortage quantity per cycle (B) and the cycle length (T).

Table 7 investigates the response to changes in cost parameters. Moreover, we graphically illustrated the impact of variations in cost parameters on the total cost in Fig. 4. The warm-up period (w), total carbon emission (y) and integrated total cost (Z) are less sensitive to changes in cost levels than the shortage quantity per cycle (B) and the cycle length (T).

Finally, Table 8 presents the system's response to changes in carbon emissions related parameters. Similarly, the warm-up period (w), the total carbon emission (y), and the integrated total cost (Z) slightly change as these parameters vary. In addition, the shortage quantity per cycle (B) and the cycle length (T) have tangible reactions to changes in the studied parameters. In conclusion, the influence of changes in

Table 6
Sensitivity Analysis for the production system parameters.

Parameter	% Change in	% Change in				
		w	T	B	y	Z
Initial	0	0	0	0	0	0
p	-40	0	0.1088	-0.0980	-0.0082	-0.0125
	-20	0	0.0370	-0.0356	-0.0029	-0.0045
	20	0	-0.0224	0.0230	0.0019	0.0029
d	40	0	-0.0376	0.0392	0.0032	0.0050
	-40	0	0.2243	-0.1831	-0.3817	-0.4347
	-20	0	0.0880	-0.0808	-0.1899	-0.2160
m	20	0	-0.0603	0.0643	0.1885	0.2139
	40	0	-0.1030	0.1150	0.3760	0.4260
	-40	0.18	1.3170	1.4092	-0.1153	-0.0702
a	-20	0	0	0	0	0
	20	0.36	2.1265	2.2509	-0.0339	0.0371
	40	0.42	2.3532	2.4867	-0.0112	0.0672
b	-40	Infeasible				
	-20	0	-0.0114	0.0116	-0.1496	-0.1456
	20	0.30	1.8821	1.9967	-0.0584	0.0047
	40	0.30	1.8821	1.9967	-0.0584	0.0047
	40	0.30	1.8523	2.0280	-0.1955	-0.1280
	-20	0	0	0	0	0
	20	0	0	0	0	0
	40	0	0	0	0	0

* The relative change of the all the variables with respect to the initial solution are determined.

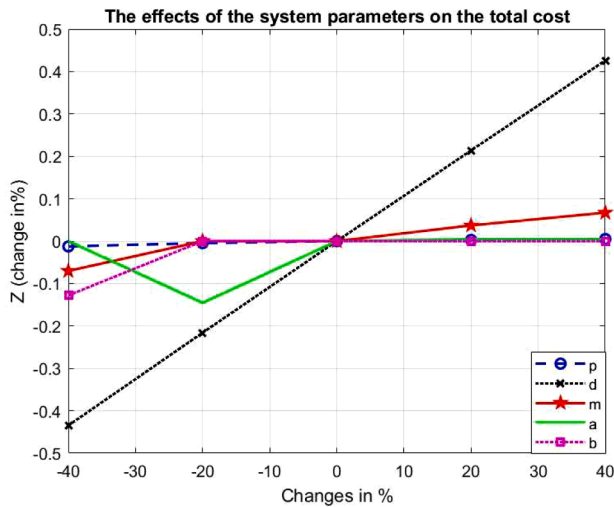


Fig. 3. The effects of the system parameters variations on the total cost.

carbon emission parameters on the total cost is visually represented in Fig. 5.

8. Managerial insights

The use of machine tools is growing, and more and more manufacturers are interested in using them for their production process. Several machine tools, like CNC units, need to have a warm-up process to improve working efficiency and obtain a longer lifespan. This study proposes a new model that integrates warm-up period in production lines, shortages, and carbon emission restrictions. Defective items have a direct correlation with the warm-up process, since their number decreases as the duration of the warm-up process increases. Shortage is a common issue in almost every production line, while carbon emissions issues are significant enough to be constantly on the minds of manufacturers. The solution process proposed for this situation is simple and straightforward enough that it can be used to obtain the optimal solution for similar problems in this field of research. The proposed model incorporates former research that addresses carbon emissions, shortages, and defective items. Consequently, this study offers a comprehensive framework that empowers readers to arrive at their own solutions. This approach saves valuable time for industries grappling with warm-up processes, shortages, and imperfect systems, providing a unified solution.

Managers have greater insight into the concept of the warm-up process and how to select its length to minimize the number of faulty items. On the other hand, the inclusion of shortage in the model allows manufacturers to make decisions during periods without products available in the inventory. Furthermore, one of the negative results of the manufacturing process, carbon emission, is also considered in the model.

The numerical examples highlight the importance of applying carbon reduction policy (i.e., cap-and-trade method) to companies, which face the decision to reduce carbon emission or buy extra allowances consistent with their assigned cap. Overall, the effectiveness of the solution proposed in this study provides a simple approach to answer difficult challenges in inventory management. The outcome of this study facilitates manufacturers to take advantage of their business by selecting the system's optimal values.

The sensitivity analysis highlights the significant influence of the maximum permissible warm-up period on both the total cost and other decision variables. This underscores the critical importance of precisely tuning the warm-up period. Additionally, the cost associated with backorders emerges as a significant factor among the other expenses. The findings strongly recommend careful attention to carbon emissions

Table 7 Sensitivity Analysis for the cost parameters.

Parameter	% Change in	% Change in				
		w	T	B	y	Z
Initial	0	0	0	0	0	0
A	-40	0	-0.0671	-0.0671	0.0030	-0.0085
	-20	0	-0.0329	-0.0330	0.0014	-0.0042
	20	0	0.0320	0.0319	-0.0012	0.0040
	40	0	0.0630	0.0629	-0.0023	0.0080
c	-40	0	0	0	0	-0.1365
	-20	0	0	0	0	-0.0682
	20	0.30	1.8821	1.9967	-0.0584	0.0632
	40	0.30	1.8821	1.9967	-0.0584	0.1218
v	-40	0.30	1.8821	1.9967	-0.0584	-0.0069
	-20	0.30	1.8821	1.9967	-0.0584	-0.0010
	20	0.30	1.8821	1.9967	-0.0584	0.0106
	40	0.30	1.8821	1.9967	-0.0584	0.0164
h	-40	0.30	1.8956	1.9827	-0.0556	0.0029
	-20	0.30	1.8887	1.9898	-0.0570	0.0038
	20	0	-0.0021	0.0022	-0.0004	0.0002
	40	0	-0.0043	0.0044	-0.0009	0.0005
s	-40	0.30	2.5222	3.0869	-0.1273	-0.0647
	-20	0.30	2.1374	2.4410	-0.0907	-0.0263
	20	0	-0.0636	-0.1099	0.0090	0.0086
	40	0	-0.1119	-0.1955	0.0172	0.0160
r	-40	0	0	0	0	0
	-20	0.30	1.8296	1.9422	-0.0556	-0.0022
	20	0	0	0	0	0
	40	0	0	0	0	0
G	-40	0.30	1.8700	1.9732	-0.0564	-0.0052
	-20	0	-0.0016	-0.0035	0.0002	-0.0053
	20	0.30	1.8881	2.0085	-0.0594	0.0097
	40	0	0.0034	0.0069	-0.0005	0.0106
α	-40	0	-0.0811	-0.1943	0.0202	-0.2060
	-20	0.30	1.7290	1.6889	-0.0299	-0.0910
	20	0	0.0422	0.0841	-0.0065	0.1013
	40	0.30	2.1742	2.5393	-0.098	0.1871

*The relative change of the all the variables with respect to the initial solution are determined.

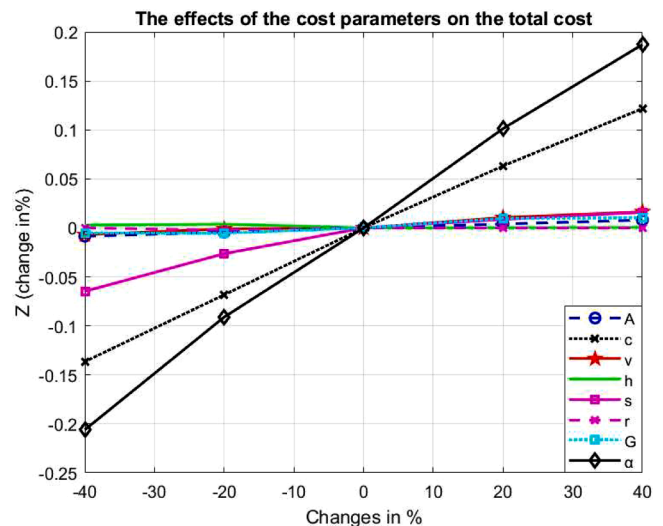


Fig. 4. The effects of the cost parameters variations on the total cost.

throughout the production process, as increasing taxes in this domain ultimately adds to the overall system's costs.

9. Conclusion

This study investigates an inventory model that describes an imperfect manufacturing system and allows the use of a warm-up process prior to production and fully backlogged demand shortages. The

Table 8
Sensitivity Analysis for the carbon emission parameters.

Parameter	% Change in	% Change in				
		w	T	B	y	Z
Initial	0	0	0	0	0	0
y ₁	-40	0	-0.1456	-0.1456	-0.0217	-0.0185
	-20	0	-0.0699	-0.0699	-0.0102	-0.0089
	20	0	0.0654	0.0653	0.0092	0.0083
	40	0	0.1270	0.1270	0.0178	0.0162
y ₂	-40	0	0	0	0	0
	-20	0	0	0	0	0
	20	0	0	0	0	0
	40	0	0	0	0	0
y ₃	-40	0	0	0	-0.2052	-0.1419
	-20	0	0	0	-0.1026	-0.0709
	20	0	0	0	0.1026	0.0709
	40	0.3	1.8821	1.9967	0.1175	0.1265
y ₄	-40	0	0	0	-0.1611	-0.1114
	-20	0	0	0	-0.0805	-0.0557
	20	0.3	1.8821	1.9967	-0.0144	0.0351
	40	0	0	0	0.1611	0.1114
y ₅	-40	0	0.0721	-0.0671	0.0003	-0.0085
	-20	0	0.0286	-0.0277	0.0004	-0.0035
	20	0	-0.0201	0.0206	-0.0006	0.0026
	40	0	-0.0352	0.0366	-0.0012	0.0046
θ	-40	0.3	1.8821	1.9967	-0.0584	0.0671
	-20	0.3	1.8821	1.9967	-0.0584	0.0359
	20	0	0	0	0	-0.0312
	40	0.3	1.8821	1.9967	-0.0584	-0.0576

* The relative change of the all the variables with respect to the initial solution are determined.

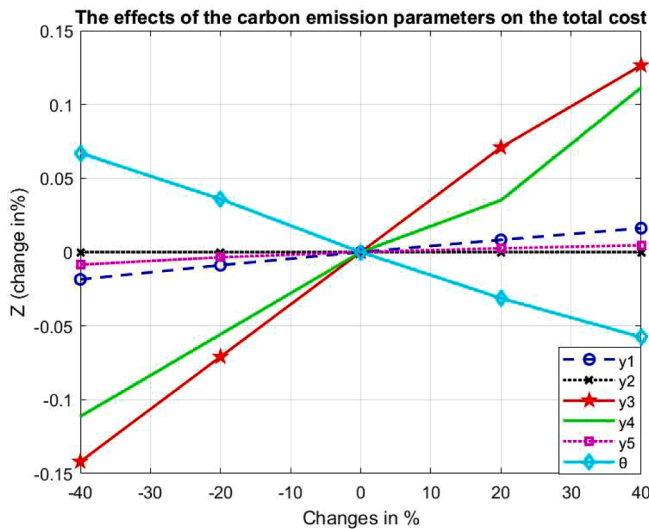


Fig. 5. The effects of the carbon emission parameters variations on the total cost.

solution approach obtains the optimal inventory cycle time (T), the optimal warm-up period (w), and the optimal amount of backlogged (B). The model explains the actual condition when a system utilizes the warm-up period in its process and faces shortages during manufacturing. Although the concept of the warm-up process is widespread and used in everyday life, its benefits have not been sufficiently explored within the field of inventory control, and even less has it been combined with production problems, such as demand shortages or impact of carbon emissions, as this study proposes. In addition, the application of the cap-and-trade policy to the model represents the environmental concerns in the manufacturing sector. Companies benefit by following the scheme according to the policy or balancing their tradeoffs between item production and carbon emissions.

A mathematical method is developed as a step-by-step solution approach to get the optimum value of the inventory cycle time, the warm-up period, and the amount of backlogged. The proposed solution ensures that the optimum warm-up period is applied when the problem has a feasible result. Overall, the model is minimized to obtain the lowest total cost as the three mentioned variables reach their optimal. The nature of this system is a nonlinear optimization problem or NP-hard problem, so the optimal solution is obtained by applying a numerical method like KKT conditions method to find the answers of all variables and parameters.

For the first time in the literature of inventory management, the concept of the warm-up process is integrated into a system where the shortage is allowed and fully backlogged. The manufacturing system in this study produces a fraction of defective items, so manufacturers gain the benefits of increased machine tools efficiency by using a warm-up period prior to main production. The warm-up process allows machine tools to operate at higher performance levels and reduce scrap production. In addition, the choice of the optimal length of the warm-up period provides the fewest possible number of defective items. The model presents the number of defective items as a decreasing function of the warm-up period, with a physical upper bound. The solution method can readily converge towards a more streamlined solution, which has been previously explored by other researchers. Hence, this study serves as a foundational reference point for prior research in this field.

In the manufacturing system, every activity, including the setup, warm-up, production, disposal, and holding process, participates in emitting carbon into the environment. To lessen the harmful impact of carbon emissions a cap-and-trade policy is used in this study. It allows manufacturers the option to produce a regulated amount of carbon emissions without any penalization. If any of them exceeds the established limit, the company need to pay a penalty related to the excess produced. Instead, if the emissions are below the allowed amount, it is possible to sell the unused capacity to another company. The sensitivity analysis graphically depicted the changes in the maximum permissible carbon emissions on decision variables. As these emissions decrease, companies are compelled to reduce their production. To expand their output and reap greater benefits, they should consider implementing new strategies to mitigate their environmental impact. This could involve initiatives such as afforestation or the adoption of more environmentally friendly equipment.

The proposed solution approach and results provide useful information for a system dealing with defective items, warm-up process, shortages, and carbon footprint implications. Both easily allow any manager (and reader) to get initial answers and new insights for systems in the same category. Future research aimed at investigating relevant concepts in this area such as consideration of the effect of pricing on the system, whether shortage is regarded as a partial backordering, the existence of perishable products in the manufacturing line, among others.

CRedit authorship contribution statement

Erfan Nobil: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Software, Visualization, Writing – original draft. **Leopoldo Eduardo Cárdenas-Barrón:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Supervision, Validation, Visualization, Writing – review & editing. **Dagoberto Garza-Núñez:** Conceptualization, Formal analysis, Investigation, Methodology. **Gerardo Treviño-Garza:** Conceptualization, Formal analysis, Investigation, Methodology. **Armando Céspedes-Mota:** Conceptualization, Formal analysis, Investigation, Methodology. **Imelda de Jesús Loera-Hernández:** Conceptualization, Formal analysis, Investigation, Methodology. **Neale R. Smith:** Conceptualization, Formal analysis, Investigation, Methodology. **Amir Hossein Nobil:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Validation, Writing – original draft.

Declaration of competing interest

We wish to draw the attention of the Editor to the following facts which may be considered as potential conflicts of interest and to significant financial contributions to this work. We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but

are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

Data availability

The data is included in the paper.

Appendix A. The details of the calculation of the holding cost

The holding cost is determined by computing the area in the positive region of Fig. 1, corresponding to the inventory plot. This area forms a triangular shape with a height of T and a base of $(t_1 + e_1)$. As a result, the holding cost per unit of time can be expressed as $h[(t_1 + e_1)/2]/T$. To find the specific value of the holding cost, we can substitute the values of I , t_1 , and e_1 from Eqs. (1), (2), and (3), respectively, resulting in the following holding cost equation:

$$\begin{aligned} \text{holding cost} &= \left(\frac{h}{2T}\right)(I)(t_1 + e_1) = \left(\frac{h}{2T}\right)\left(\left((1-f(w))p-d\right)\frac{Q}{p}-B\right)\left(\frac{Q}{p}-\frac{B}{((1-f(w))p-d)}+\left((1-f(w))p-d\right)\frac{Q}{pd}-\frac{B}{d}\right) \\ &= \left(\frac{h}{2T}\right)\left(\left((1-f(w))p-d\right)\frac{Q}{p}-B\right)\left(\frac{(1-f(w))Q}{d}-\frac{(1-f(w))pB}{d((1-f(w))p-d)}\right) \\ &= \left(\frac{h}{2T}\right)\left(\left((1-f(w))((1-f(w))p-d)\right)\frac{Q^2}{pd}-\frac{2(1-f(w))BQ}{d}+\frac{(1-f(w))pB^2}{d((1-f(w))p-d)}\right) \end{aligned} \quad (\text{A-4})$$

By substituting the value of Q from Eq. (7) into Eq. (A-1), the holding cost is determined as follows:

$$\text{holding cost} = \left(\frac{h}{2T}\right)\left(\left((1-f(w))p-d\right)\frac{dT^2}{(1-f(w))p}-2BT+\frac{(1-f(w))pB^2}{d((1-f(w))p-d)}\right) = \frac{hd((1-f(w))p-d)}{2(1-f(w))p}(T) - hb + \frac{h(1-f(w))p}{2d((1-f(w))p-d)}\left(\frac{B^2}{T}\right) \quad (\text{A-2})$$

Appendix B. The details of the calculation of the shortage cost

We calculate the shortage cost based on the area of the inventory plot located in the negative part of Fig. 1. Therefore, this area is a triangle that its height and base are B and $(e_2 + t_2)$, respectively. As a result, the shortage cost per unit of time can be expressed as: $s[B(e_2 + t_2)/2]/T$. To find the specific value of the shortage cost, we can substitute the values of e_2 and t_2 from Eqs. (4) and (5), respectively, resulting in the following shortage cost equation:

$$\text{shortage cost} = \left(\frac{s}{2T}\right)B(e_2 + t_2) = \left(\frac{s}{2T}\right)\left(\frac{B}{d}+\frac{B}{((1-f(w))p-d)}\right)B = \left(\frac{s(1-f(w))p}{2d((1-f(w))p-d)}\right)\left(\frac{B^2}{T}\right) \quad (\text{B-5})$$

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