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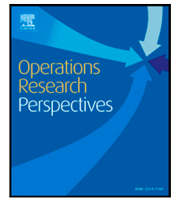
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A stochastic card balance management problem with continuous and batch-type bilateral transactions

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ABSTRACT

We study a stochastic continuous-review card balance management problem with two transaction patterns, namely, continuous and batch-type bilateral transactions, both in a Markovian environment. Motivated by the Autoload program used in public transit systems, the card is managed using a two-parameter band policy. Our cost structure includes activation and loading costs, and a fine for a negative balance. By applying hitting time theory and martingales, we derive the cost functionals and obtain, numerically, the optimal thresholds minimizing the expected discounted total cost. Surprisingly, a numerical study shows that the optimal policy is inherently linked with the outflow patterns, and is more sensitive to changes in withdrawal rates than to changes in batch sizes. We further show that timing is a significant factor in determining the policy: a high discount factor leads to frequent activations with smaller amounts.

1. Introduction

Traditional cash management programs aim to show users how to replenish their balance in order to meet demands over time. This problem can be viewed as a trade-off between the cost of cash holding and the cost induced by insufficient cash balance to meet withdrawals. The model studied here is motivated by the Autoload feature program of the PRESTO card. The PRESTO is a contactless smart-card automated fare collection system used on participating public transit systems in the province of Ontario in Canada, and it was designed to support the use of one common method of fare payment on various participating public transit systems. With the aim of facilitating the tracking of the balance sheet on the card, and minimizing the number of loadings, the PRESTO has the feature of the Autoload program that allows cardholders to automatically load their card when the current balance value drops to some specified threshold. We note that other factors may affect the timing and the size of the activation, e.g., intended usage, timing, salary, the location of charging facilities, etc. However, this is beyond the paper's scope and is left for future research. In this paper, we focus on determining the optimal threshold levels of such a program. To do so, we introduce the card balance process $\tilde{X}(t)$ that behaves as a Markov Arrival Process (MAP) with linear drift and random jumps governed by a continuous-time Markov chain (CTMC) representing the real-life environment. Specifically,

i. There are continuous “small” transactions (manual loadings and withdrawals) with linear state-dependent rates that form a fluid process.

ii. In addition, changing the state may be accompanied by an instantaneous “big” manual load or withdrawal batch, represented by a positive and a negative jump, respectively. We assume that these batches are identical and independently distributed (i.i.d.) random variables (r.v.) having a phase-type (PH) distribution.

In real-world, it is well-established that loadings and withdrawals can exhibit high variability and unpredictably. Thus, it is of practical relevance to present this variability by including a finite-state Markov chain that represents the uncertainty generated by the cash movements; the literature usually use Markov processes due to their versatility in matching key statistical properties to the cash balance process (e.g., Elliott and Swishchuk [1]; Shen and Siu [2]; Fei [3]). Here we assume that the “big” load and withdrawal batches have PH distributions depending on the Markovian environment. In this way, we capture the practice where cardholder decisions are impacted by the background environment. More precisely, whenever the state of the underlying Markov process changes, the model parameters are modulated according to the new state. This regime switching model possesses nice features because it includes the short-run behavior captured by the transactions and the long-run behavior of the market captured by the Markov chain component. Furthermore, this structure remains analytically tractable when one studies the first passage time problem, i.e. the first time a stochastic process crosses a constant upper or lower threshold. It should be noted that the class of phase distributions is a

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generalization of the class of exponential distributions with a matrix algebraic structure, and therefore simplifies computational implementation. Furthermore, when general distributions are appropriate for modeling the amounts, phase distributions can be taken into account in a natural way, as any non-negative continuous distribution of the probability can be approximated with a phase-type distribution (Asmussen and Kella [4]). We note that incorporating continuous rates and instantaneous jumps is well studied also in reliability models (e.g., parts delivered to maintenance service engineers in geographically spread out areas), and in chemical production systems and gas stations.

The card is managed according to a two-parameter band policy, known as the (S, s) -policy; i.e., every time the balance drops to or below threshold s , the card is automatically loaded up to S . Following banking demands, if the card balance becomes negative (due to a batch withdrawal), the account is frozen for L time units (for tracking purposes and pending administration approval to re-activate the card); thereafter the loading is made. We assume that L is a generally distributed r.v. We note that similar programs exist in London's Oyster card, and Hong Kong's Octopus card.

Another application is the management of the bank account of Sharbat Brother. Sharbat Brother is an Israel contracting company that invests in the construction and rental of buildings throughout many countries. The company's cash flow including rental fees and maintenance costs for the privet section (that are relatively small amounts, and can be represented as continuous inflows and outflows), and cash transactions for the public section (that are relatively large and, thus, are represented by batch amounts). The Bank's requirements are to maintain a certain level of financial security, which, when falling below it, money from a liquid source is immediately transferred. In case of deficit, a fee is charged and the account is frozen for approvals.

Band policies have become popular control schemes in cash problems. Since such policies are in continuous time, the underlying stochastic balance level governing the state variables is typically linked to either purely stochastic behavior (Miller and Orr [5]), a compound Poisson process (Chaouch [6]; Azene an Nuler [7]), a Wiener process (Constantinides and Richard [8]; Premachandra [9]; Wand et al. [10]), a superposition of a BM and a compound Poisson process with negative jumps (Bar-Ilan et al. [11]; Yamazaki [12]; Yamazaki [13]), a fluid process (Kawai and Takagi [14]; Barron [15]), or as a number of samples drawn from normal distribution (da Costa Moraes and Naganoda [16]). Further discussion and examples of band policies (in particular the (S, s) -type) are found in the excellent survey paper Zhang [17].

All papers cited above consider a one-dimensional cash process. For the multidimensional cash management systems, Baccarin [18] studies a cash balance that fluctuates as a homogeneous diffusion process in \mathbb{R}^n . Alvarez and Lippi [19] consider a liquid asset with two components: one is frequent and small and one is infrequent and large. An n -dimensional diffusion process with infinite activity is conducted in Davis et al. [20], and later extended in Bayraktar et al. [21]. As we see, most of the literature assumes that the cash balance process allows at most unilateral jumps, hence not allowing lumpy load and withdrawal. Instead, we present a cash balance process that allows for the possibility of, in addition to a continuous component, bilateral lumpy cash movements; accounting for this lumpy nature is realistic and important to characterizing the card management.

We note that the (S, s) policy is well studied in economics, inventory management, production systems, reliability models, queueing models and healthcare management. For example, consider a manufacturer who produces a part needed for few products, as well as a replacement part. The consumption for the products are an aggregate of short-run schedules and can be appropriately modeled by a fluid process, and the consumption for the replacement part can be modeled by a batch-type process compound Poisson process (Sobel and Zhang [22]; Presman and Sethi [23]). For more examples we refer the reader to Bensoussan et al. [24], Benkherouf and Bensoussan [25], Reiman et al. [26], Noba

and Yamazaki [27], Barron [28], Perara and Sethi [29], and the recent review paper Utama et al. [30], references therein.

The cost structure includes a cost for each activation of the Autoload program, a cost proportional to the loaded quantity, and a fine in the case of a negative balance. The objective is to determine the optimal levels S^* and s^* that minimize the sum of expected discounted values of these costs. Our analysis is based on a combination of exit-time theory of Markov arrival processes (Breuer [31]) and multidimensional martingales (Asmussen and Kella [4]); this combination yields simple and easy-to-implement closed-form formulas that may be used for numerical investigation and optimization. While previous studies assume only continuous and unilateral cash movements, the inclusion of bilateral jumps—even though it makes the problem more challenging—is more suitable for interpreting the cash balance process presented here. We further employ a stagnation policy of a random time for administrative procedures, capturing the trade-off between holding extra cash and the risk of a cash shortage. Lastly, we allow state-dependent cost functionals that enable us to obtain a more tractable model for wider applications. To the best of our knowledge, no results are available under the framework developed here.

From our conclusions we numerically glean the following important insights. Firstly, the total cost is convex in S and s . Although we cannot prove it, this enables us to numerically obtain the optimal thresholds by using a linear search over the range $0 \leq s < S$. Secondly, the optimal thresholds S^* and s^* are decreasing in the activation cost and increasing in the fine. Similarly, S^* and s^* are decreasing in the discount factor, such that it is worthwhile to activate the card with smaller amounts and thus achieve a full utilization of the timing. We further show that S^* and s^* are more sensitive to the average frozen time L than to its variability. Finally, the optimal policy is inherently linked with the outflow pattern and, unexpectedly, is more sensible to changes in rates than to changes in batch sizes. Specifically, higher rates yield lower thresholds; thus, when increasing the daily transactions, it is recommended to activate the card more frequently and for smaller amounts.

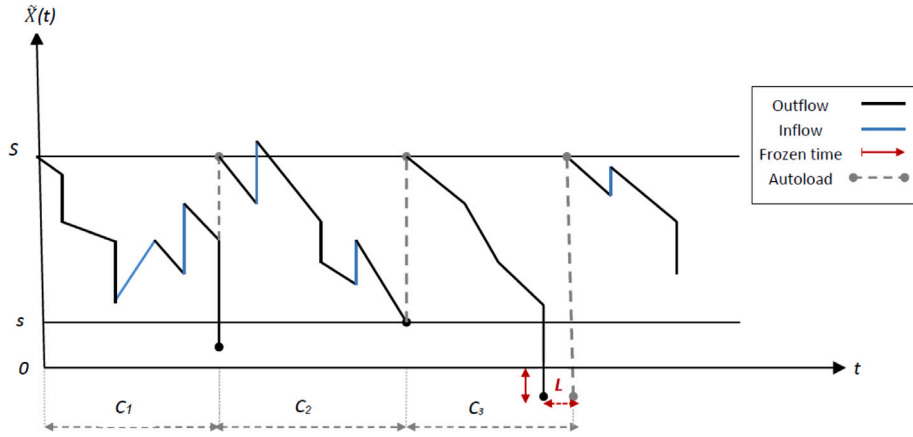
The paper is organized as follows. Section 2 presents the mathematical description of the card balance process and the cost structure. Preliminary results are introduced in Section 3. The closed-form formulas for the costs are derived in Section 4. A summary algorithm is presented. Section 5 is devoted to numerical examples, conclusions and insights. Finally, Section 6 concludes the paper and contains some research suggestions.

Throughout the paper we use blackboard bold symbols for matrices and bold symbols for vectors, $\mathbf{e} = (1, \dots, 1)^T$, \mathbf{e}_k is a row vector with the k th component equal to 1 and all the other components 0, \mathbb{I} is the identity matrix (all of the appropriate sizes), and $\mathbf{1}_{\{A\}}$ is the indicator of an event A .

2. Mathematical description of the card balance problem

2.1. MAP process with PH jumps

Consider a CTMC $\tilde{J} = \{\tilde{J}(t) : t \geq 0\}$ with state space \tilde{E} , initial probability vector $\boldsymbol{\eta} = [\eta_1, \eta_2, \dots, \eta_{|\tilde{E}|}]$, and infinitesimal generator $\tilde{\mathbb{Q}} = (\tilde{q}_{ij})_{i,j \in \tilde{E}}$. As in practice, we assume two types of cash movements: continuous withdrawals and manual loadings, represented by linear outflows and inflows, respectively, and batch-type amounts of money, represented by downward and upward jumps. Specifically, when the state $\tilde{J}(t) = i \in \tilde{E}$, the account is loaded at rate r_i and, independently, cash is withdrawn at rate d_i , and thus, the balance increases at rate $c_i = r_i - d_i$, which can be positive or negative. The state space \tilde{E} can be partitioned into two subsets $\tilde{E} = E_p \cup E_n$, according to increasing and decreasing rates, respectively. Accordingly, define the real-valued process $\tilde{X} = \{\tilde{X}(t) : t \geq 0\}$ as a special case of a Lévy process $\tilde{X}^{(i)}$ with parameters c_i (drift) and \hat{v}_i (Lévy measure) during intervals when

Fig. 1. A typical sample path of $\tilde{X}(t)$.

the phase equals $i \in \tilde{E}$. Whenever \tilde{J} jumps from state $i \in \tilde{E}$ to state $j \in \tilde{E}, i \neq j$, this may be accompanied by a jump of \tilde{X} . We assume that \tilde{v}_i has the form

$$\tilde{v}_i(dx) = \lambda_i^+ 1_{\{x>0\}} \alpha^{(ii)+} \exp(\mathbb{T}^{(ii)+}x) \eta^{(ii)+} dx + \lambda_i^- 1_{\{x<0\}} \alpha^{(ii)-} \exp(-\mathbb{T}^{(ii)-}x) \eta^{(ii)-} dx \quad (1)$$

for all $i \in \tilde{E}$, where $\lambda_i^- \geq 0$ and $\lambda_i^+ \geq 0$ are the rates of cash withdrawals and manual loadings, respectively, and $U^{(ii)\pm} \sim PH(\alpha^{(ii)\pm}, \mathbb{T}^{(ii)\pm})$ are representations of PH distributions without an atom at 0. Denote the amount of money loaded into the card by $U^{(ii)+}$ and the amount of money withdrawn from the card by $U^{(ii)-}$. More details on the PH distribution are given in Section 3.1. Let $\eta^{(ii)\pm} = -\mathbb{T}^{(ii)\pm} \mathbf{e}$ be the exit vectors and denote the order of $U^{(ii)\pm}$ by m_{ii}^{\pm} (i.e., a compound Poisson process with jumps, where jump sizes have a double PH distribution). We further let $\lambda_i = \lambda_i^+ + \lambda_i^-$. Likewise, let p_{ij}^{\pm} ($0 \leq p_{ij}^+ + p_{ij}^- \leq 1, i, j \in \tilde{E}$) be the probability of a positive or negative jump induced by a phase state change from $i \in \tilde{E}$ to $j \in \tilde{E}$; similarly, denote this jump by $U^{(ij)\pm} \sim PH(\alpha^{(ij)\pm}, \mathbb{T}^{(ij)\pm})$ of order m_{ij}^{\pm} , and let $\eta^{(ij)\pm} = -\mathbb{T}^{(ij)\pm} \mathbf{e}$, respectively. The two-dimensional process (\tilde{X}, \tilde{J}) is known as a Markov Arrival Process (MAP).

Next we apply the Autoload program. For $n \geq 1$, define recursively the following stopping times:

$$\mathcal{Z}_0 = 0, \quad \mathcal{Z}_n = \inf\{t > \mathcal{Z}_{n-1} : \tilde{X}(t) \leq s\}, i = 1, 2, \dots, \quad (2)$$

to be the times of dropping to or below level s (called *activation times*). If $\tilde{X}(\mathcal{Z}_n) \geq 0$, then $S - \tilde{X}(\mathcal{Z}_n)$ dollars are loaded to the card, bringing the balance up to level S . Otherwise, if $\tilde{X}(\mathcal{Z}_n) < 0$, the card is frozen for a generally distributed time L , independent of $\tilde{X}(\mathcal{Z}_n)$. We assume that no cash movements may occur during this frozen time, and that, thereafter, the balance is loaded up to level S . (This assumption is also made in Premachandra [9]). Thus, we have $\tilde{X}(t) = \tilde{X}(\mathcal{Z}_n)$ for $t \in [\mathcal{Z}_n, \mathcal{Z}_n + L]$ and $\tilde{X}(\mathcal{Z}_n + L) = S$. Let $f_L(\cdot)$, $F_L(\cdot)$, and $M_L(\cdot)$ be the density probability function, the distribution function, and the Laplace–Stieltjes transform (LST) function of L , respectively.

Let $\mathcal{T}_n, n = 1, 2, \dots$ be the actual loading times; it is easy to see that $\mathcal{T}_n, n = 1, 2, \dots$ satisfy the following recursive equation:

$$\mathcal{T}_0 = 0, \quad \mathcal{T}_n = \left(\mathcal{Z}_n + 1_{\{\tilde{X}(\mathcal{Z}_n) < 0\}} L \right), n = 1, 2, \dots$$

At times $\mathcal{T}_n, n = 1, 2, \dots$, the card is loaded up to level S . The MAP $\{\tilde{X}, \tilde{J}\}$ is a semi-regenerative process with $\mathcal{T}_n, i = 1, 2, \dots$ being the regenerative points. Denote by $C_n = \mathcal{T}_n - \mathcal{T}_{n-1}, n = 1, 2, \dots$ the n th cycle length of $\{\tilde{X}, \tilde{J}\}$. A typical sample path of $\tilde{X}(t)$ is depicted in Fig. 1.

2.2. The cost components

Given the state $\tilde{J}(C_n) \in \tilde{E}$, the processes $(\tilde{X}(t))_{t \leq C_n}$ and $(\tilde{X}(t))_{t > C_n}$ are conditionally independent. Thus, applying renewal theory, we derive

the cost functionals in terms of the first cycle. To that end, we use the generic variables C and Z having the corresponding distributions of C_1 and Z_1 . Let $\beta > 0$ be the discount factor. In the sequel, we use \mathbb{E} to denote a matrix-valued expectation whose (i, j) component is the conditional expectation of the j th entry given initial state. In particular, let $\mathbb{E}(e^{-\beta C})$ denotes the $(|\tilde{E}| \times |\tilde{E}|)$ LST matrix whose (i, j) component is

$$[\mathbb{E}(e^{-\beta C})]_{ij} = E(e^{-\beta C} 1_{\{\tilde{J}(C)=j\}} | \tilde{J}(0)=i, \tilde{X}(0)=S), i, j \in \tilde{E}.$$

We next introduce the cost functionals.

(a) *Activation cost* $AC(\beta)$. We assume that each time the Autoload program is activated, a cost $Y_{\tilde{J}(\mathcal{Z}_n)} S^\rho$ is incurred for some fixed $\rho \geq 1$. This cost can be interpreted as an alternative cost representing the loss of profit for choosing to hold some cash in the card instead of investing it in other profitable assets (we note that power utility functions are widely used in stochastic cash management; see, e.g., Harel *et al.* [32]; Gan *et al.* [33]; Di *et al.* [34]). Denote by Υ the $(|\tilde{E}| \times |\tilde{E}|)$ matrix with $(\Upsilon)_{i,i} = Y_i, i \in \tilde{E}$. We further let $Y_{\tilde{J}(0)} S^\rho$ be the cost of the initial purchasing, activation, and loading of the card.

(b) *Loading cost* $LC(\beta)$. We assume a cost $\gamma_{\tilde{J}(\mathcal{T}_n)}$ is paid for each loaded dollar at time \mathcal{T}_n . Note that the loaded amount, $S - \tilde{X}(\mathcal{T}_n)$, is random. Furthermore, when $\tilde{X}(\mathcal{Z}_n) \geq 0$, the card is immediately loaded up to level S , and thus, $\mathcal{T}_n = \mathcal{Z}_n$; in that case, the activation and loading costs are charged at the same time. Otherwise, when $\tilde{X}(\mathcal{Z}_n) < 0$, the loading is done (and charged) at time $\mathcal{T}_n = \mathcal{Z}_n + L$. Using different times for charging is common and realistic; see, e.g., Lashgari *et al.* [35], Taleizadeh *et al.* [36], Feng *et al.* [37] and Barron [15].

(c) *Negative balance fine* $NC(\beta)$. Whenever the balance becomes negative (this may occur only due to downward jumps), the card is frozen for a period of L units of time, no transactions are allowed, and a fine of $\phi_{\tilde{J}(t)}$ is charged for every dollar in deficit at time t .

The cost functionals are thus given by

$$\begin{aligned} AC(\beta) &= \sum_{n=0}^{\infty} E \left(Y_{\tilde{J}(\mathcal{Z}_n)} S^\rho e^{-\beta \mathcal{Z}_n} \right) \\ &= \eta \left[\Upsilon + (\mathbb{I} - \mathbb{E}(e^{-\beta C}))^{-1} \mathbb{E} \left(Y_{\tilde{J}(Z)} e^{-\beta Z} \right) \right] S^\rho \mathbf{e}, \end{aligned} \quad (3)$$

$$\begin{aligned} LC(\beta) &= \sum_{n=1}^{\infty} E \left(\gamma_{\tilde{J}(\mathcal{T}_n)} e^{-\beta \mathcal{T}_n} (S - \tilde{X}(\mathcal{T}_n)) \right) \\ &= \eta (\mathbb{I} - \mathbb{E}(e^{-\beta C}))^{-1} \mathbb{E} \left(\gamma_{\tilde{J}(C)} e^{-\beta C} (S - \tilde{X}(C)) \right) \mathbf{e}, \end{aligned} \quad (4)$$

$$\begin{aligned} NC(\beta) &= - \sum_{n=1}^{\infty} E \left(\int_{\mathcal{Z}_n}^{\mathcal{Z}_n+L} e^{-\beta t} \tilde{X}(t) \phi_{\tilde{J}(t)} 1_{\{\tilde{X}(t) < 0\}} dt \right) \\ &= -\eta (\mathbb{I} - \mathbb{E}(e^{-\beta C}))^{-1} \mathbb{E} \left(\int_{\mathcal{Z}}^{\mathcal{Z}+L} e^{-\beta t} \phi_{\tilde{J}(t)} \tilde{X}(t) 1_{\{\tilde{X}(t) < 0\}} dt \right) \mathbf{e}. \end{aligned} \quad (5)$$

The total cost of operating the program is therefore

$$TC(\beta) = AC(\beta) + LC(\beta) + NC(\beta). \quad (6)$$

Remark 1. We may also consider a commission fee for managing the card. However, as in practice, this fee is charged at specific dates (usually a fixed time of a month) independently of the number of transactions made or the balance level. Hence, the commission fee does not affect the optimization analysis and, therefore, it is not included.

3. Preliminaries

3.1. PH distribution

A phase-type (PH) distribution with representation $PH(\alpha, \mathbb{T})$ of order m is the distribution of the time until absorption into state $m+1$, of a continuous-parameter Markov chain $\mathcal{J}(t)$ with infinitesimal generator $Q = \begin{pmatrix} \mathbb{T} & \mathbf{T}^0 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where \mathbb{T} is an $(m \times m)$ transition rate matrix ($(\mathbb{T})_{ii} < 0$ for $1 \leq i \leq m$, $(\mathbb{T})_{ij} > 0$ for $i \neq j$); thus, the time spent at each state $i, i = 1, \dots, m$, is $\exp(-(\mathbb{T})_{ii}t)$. The m -column vector \mathbf{T}^0 describes the absorption rate into the state $m+1$, and η is the initial probability vector. For $X \sim PH(\eta, \mathbb{T})$, we have $f_X(x) = \eta \exp(\mathbb{T}x) \mathbf{T}^0$, $F_X(x) = 1 - \eta \exp(\mathbb{T}x) \mathbf{e}$, $x \geq 0$, expectation $E(X) = -\eta \mathbb{T}^{-1} \mathbf{e}$, and LST $M_X(\beta) = \eta(\beta \mathbb{I} - \mathbb{T})^{-1} \mathbf{T}^0$. It is easy to verify that the truncated expectation is given by

$$E(X \mathbf{1}_{\{X \leq x\}}) = \mathbf{e}^T (-\mathbb{T}(\mathbb{I} - e^{x\mathbb{T}}) - x e^{x\mathbb{T}}) \mathbf{e}, x \geq 0. \quad (7)$$

3.2. The multidimensional martingale

Let $\{\tilde{\mathcal{X}}(t), t \geq 0\}$ be a right-continuous Markov-modulated Lévy process with a modulating process $\{\tilde{\mathcal{J}}(t), t \geq 0\}$ that is a right-continuous irreducible finite state space CTMC. Let $\{Y(t), t \geq 0\}$ be an adapted continuous process with a finite expected variation on finite intervals and let $W(t) = \tilde{\mathcal{X}}(t) + Y(t)$. It is shown in Asmussen [4] that the matrix with elements $E_i \left[e^{a\tilde{\mathcal{X}}(t)}; \tilde{\mathcal{J}}(t) = j \right]$ has the form of $e^{t\mathbb{K}(\alpha)}$ for some matrix $\mathbb{K}(\alpha)$. Theorem 2.1 in Asmussen and Kella [4] yields that under certain mild conditions on $\{W(t), t \geq 0\}$, the multidimensional process

$$\mathbf{M}(\alpha, t) = \int_0^t e^{aW(v)} \mathbf{1}_{\tilde{\mathcal{J}}(v)} d v \mathbb{K}(\alpha) + e^{aW(0)} \mathbf{1}_{\tilde{\mathcal{J}}(0)} - e^{aW(t)} \mathbf{1}_{\tilde{\mathcal{J}}(t)} + \alpha \int_0^t e^{aW(v)} \mathbf{1}_{\tilde{\mathcal{J}}(v)} d Y(v) \quad (8)$$

is a (row) vector-valued zero-mean martingale (here, $\mathbf{1}_i$ denotes the i th row of the identity matrix). In Section 4.3, we apply the optional stopping theorem (OST) to the multidimensional process (8).

3.3. The joint LST matrix at a random time

Consider a CTMC process $\tilde{\mathcal{J}} = \{\tilde{\mathcal{J}}(t) : t \geq 0\}$ with state space \tilde{E} and a generator matrix \tilde{Q} . Let L be a nonnegative r.v. with LST $M_L(\beta)$. Let $\mathbb{M}_L(\beta)$ be the joint LST matrix of order $|\tilde{E}| \times |\tilde{E}|$, whose (i, j) component is $(\mathbb{M}_L(\beta))_{i,j} = E(e^{-\beta L} \mathbf{1}_{\{\tilde{\mathcal{J}}(L)=j\}} | \tilde{\mathcal{J}}(0) = i)$. Applying $(\exp(\tilde{Q}x))_{i,j} = E(\tilde{\mathcal{J}}(x) = j | \tilde{\mathcal{J}}(0) = i)$ for CTMC, the total probability theorem yields that

$$\mathbb{M}_L(\beta) = M_L(\beta \mathbb{I} - \tilde{Q}). \quad (9)$$

3.4. The MMFF process

We apply similar parameters to those in Breuer [31]. Central to our analysis is the determination of the joint distribution of the first time level s is hit and the state at that time; clearly, hitting s must occur in a descending environment, either continuously or by a downward jump. For that, we apply the Markov-modulated fluid flow process (MMFF) $\{\mathcal{X}, \mathcal{J}\}$, which is a modified form of the MAP $\{\tilde{\mathcal{X}}, \tilde{\mathcal{J}}\}$.

Specifically, the main advantage of PH jumps is the possibility of lengthening the jumps into a succession of linear periods of exponential

duration (having slopes 1 and -1 for the positive and negative jumps, respectively), and retrieving the original process through a simple time change. This transformation is done as follows. Let Ω contains all pairs $(i, j), i, j \in \tilde{E}$ that are accompanied by a jump (we note that some transitions from state $i \in \tilde{E}$ to state $j \in \tilde{E}$ may not be accompanied by a jump). For $(i, j) \in \Omega$, we introduce two new phase spaces,

$$E_{\pm} := \{(i, j, k, \pm) : (i, j) \in \Omega, 1 \leq k \leq m_{ij}^{\pm}\}, \quad (10)$$

to model the jumps. Next, we define the enlarged state space $E \equiv \tilde{E} \cup E_{-} \cup E_{+}$. The modified phase process \mathcal{J} with state space E is determined by the generator matrix $\mathbb{Q} = (q_{ij})_{i,j \in E}$. The elements $(q_{ij})_{i,j \in E}$ are given by

$$q_{ih} = \begin{cases} \tilde{q}_{ii} - \lambda_i, & h = i \in \tilde{E}, \\ \tilde{q}_{ih}(1 - p_{ih}^{+} - p_{ih}^{-}), & h \neq i, h \in \tilde{E}, \\ \lambda_i^{\pm} \alpha_k^{(ii)\pm}, & h = (i, i, k, \pm) \in E_{\pm}, \\ \tilde{q}_{ij} p_{ij}^{\pm} \alpha_k^{(ij)\pm}, & h = (i, j, k, \pm) \in E_{\pm}, \end{cases} \quad (11)$$

for $i \in \tilde{E}$, as well as

$$q_{(i,j,k,\pm)(i,j,l,\pm)} = T_{kl}^{(ij)\pm} \text{ and } q_{(i,j,k,\pm)j} = \eta_k^{(ij)\pm} \quad (12)$$

for $(i, j) \in \Omega$; the rest of the elements in \mathbb{Q} are 0. Let $q_i = -q_{ii}, i \in E$.

Doing so, \mathcal{J} tracks the phases in the PH jumps in addition to the original states in $\tilde{\mathcal{J}}$. The modified process constructed using such a technique now becomes MMFF without jumps and is denoted by $\{\mathcal{X}, \mathcal{J}\}$. Accordingly, the infinitesimal generator $\mathbb{Q} = \begin{pmatrix} \mathbb{Q}_{11} & \mathbb{Q}_{12} \\ \mathbb{Q}_{21} & \mathbb{Q}_{22} \end{pmatrix}$ can be written in block form with respect to the subspaces $E_1 = E_p \cup E_{+}$ (ascending phases) and $E_2 = E_n \cup E_{-}$ (descending phases).

First hitting times

We next obtain the LST of the first passage times in MAPs. Define $\tilde{\tau}(x) = \inf\{t \geq 0, \tilde{\mathcal{X}}(t) > x\}$ for $x \geq 0$ and assume that $\tilde{\mathcal{X}}(0) = 0$. (Note that $\tilde{\tau}(x)$ is the first passage time over the level x for the original MAP $\tilde{\mathcal{X}}$; i.e., we do not count the time spent in the jump states E_{\pm}). Let $\mathbb{E}(e^{-\beta \tilde{\tau}(x)})$ be the $|E| \times |E|$ LST matrix of $\tilde{\tau}(x)$ whose (i, j) component is

$$E_{ij}(e^{-\beta \tilde{\tau}(x)}) = E(e^{-\beta \tilde{\tau}(x)}, J(\tau(x)) = j | X(0) = 0, J(0) = i), \quad i, j \in E. \quad (13)$$

It is shown in Breuer [31] that the matrix $\mathbb{E}(e^{-\beta \tilde{\tau}(x)})$ can be written in block form according to E_1 and E_2 as

$$\mathbb{E}(e^{-\beta \tilde{\tau}(x)}) = \begin{pmatrix} e^{\mathbb{U}(\beta)x} & \mathbf{0} \\ \mathbb{A}(\beta)e^{\mathbb{U}(\beta)x} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_a \\ \mathbb{A}(\beta) \end{pmatrix} \begin{pmatrix} e^{\mathbb{U}(\beta)x} & \mathbf{0} \end{pmatrix}, \quad (14)$$

where \mathbb{I}_a denotes the identity matrix of dimensions $|E_1| \times |E_1|$ and $\mathbb{U}(\beta)$ and $\mathbb{A}(\beta)$ are matrices of dimensions $|E_1| \times |E_1|$ and $|E_2| \times |E_1|$, respectively. Let \mathbf{e}_i^T denote the i th canonical row base vector, with appropriate dimension according to the context. Section 2.2 of Breuer [31] shows that the matrices $\mathbb{U}(\beta)$ and $\mathbb{A}(\beta)$ can be determined by successive approximation as the limit of the sequence $(\mathbb{A}_n(\beta), \mathbb{U}_n(\beta)) : n \geq 0$ with initial values

$$\mathbb{A}_0(\beta) = \mathbf{0}, \quad \mathbb{U}_0(\beta) = -\text{diag}\left(\frac{q_i + \beta}{c_i} \mathbf{1}_{\{i \in E_p\}} + q_i \mathbf{1}_{\{i \in E_{+}\}}\right)_{i \in E_1}, \quad (15)$$

and the following iteration:

$$\begin{aligned} \mathbf{e}_h^T \mathbb{U}_{n+1}(\beta) &= \sum_{l=1}^{m_{ij}^{+}} \mathbb{T}_{kl}^{(ij)+} \mathbf{e}_l^T + \eta_k^{(ij)+} \mathbf{e}_j^T \begin{pmatrix} \mathbb{I}_a \\ \mathbb{A}_n(\beta) \end{pmatrix} \text{ for } h = (i, j, k, +) \in E_{+}, \\ \mathbf{e}_i^T \mathbb{U}_{n+1}(\beta) &= -\frac{q_i + \beta}{c_i} \mathbf{e}_i^T + \frac{1}{c_i} \sum_{j \in E_{+}, j \neq i} q_{ij} \mathbf{e}_j^T \begin{pmatrix} \mathbb{I}_a \\ \mathbb{A}_n(\beta) \end{pmatrix} \text{ for } i \in E_p, \\ \mathbf{e}_i^T \mathbb{A}_{n+1}(\beta) &= \sum_{j \in E_{+}, j \neq i} q_{ij} \mathbf{e}_j^T \begin{pmatrix} \mathbb{I}_a \\ \mathbb{A}_n(\beta) \end{pmatrix} ((q_i + \beta) \mathbb{I} + c_i \mathbb{U}_n(\beta))^{-1} \text{ for } i \in E_n, \\ \mathbf{e}_i^T \mathbb{A}_{n+1}(\beta) &= \sum_{j \in E_{+}, j \neq i} q_{ij} \mathbf{e}_j^T \begin{pmatrix} \mathbb{I}_a \\ \mathbb{A}_n(\beta) \end{pmatrix} (q_i \mathbb{I} - \mathbb{U}_n(\beta))^{-1} \text{ for } i \in E_{-}. \end{aligned} \quad (16)$$

For more details, see [31].

In our model we need to derive the LST matrix of the first hitting (dropping) time to or below level s , given an initial level S . To achieve this, we build a *mirror-image* process $\mathcal{X}^R \equiv -\mathcal{X} + S$ by reversing the roles of the up and down movements with the corresponding $E^R, \tilde{\tau}^R(x), \mathbb{A}^R(\beta)$, and $\mathbb{U}^R(\beta)$. (Note that changing the sign of c_i leads to $E_{\pm}^R = E_{\mp}$, $E_p^R = E_n$, and $E_n^R = E_p$.) It is easy to verify that the process $\{\mathcal{X}^R(t)\}_{0 \leq t \leq \tilde{\tau}^R(S-s)}$ with $\mathcal{X}^R(0) = 0$ has the same distribution as the process $\{\mathcal{X}(t)\}_{0 \leq t \leq \tilde{\tau}(S-s)}$ with $\mathcal{X}(0) = S$. In the sequence, we use $\mathbb{M}_{\mathcal{Z}}(\beta) \equiv \mathbb{E}(e^{-\beta \mathcal{Z}}) = \mathbb{E}(e^{-\beta \tilde{\tau}^R(S-s)}) = \begin{pmatrix} \mathbb{I}_a \\ \mathbb{A}^R(\beta) \end{pmatrix} \begin{pmatrix} e^{\mathbb{U}^R(\beta)(S-s)} & \mathbf{0} \end{pmatrix}$.

4. Cost derivation

4.1. Cycle length

Although the state space is expanded to E , the cost functionals in (3)–(5) are of size $(|\tilde{E}| \times |\tilde{E}|)$. Specifically, we use the $(|\tilde{E}| \times |\tilde{E}|)$ matrix $\mathbb{E}(e^{-\beta C})$, associated with $\{\tilde{X}, \tilde{J}\}$, whereas the matrix $\mathbb{E}(e^{-\beta \mathcal{Z}})$ is associated with $\{X, J\}$ and, thus, of order $(|E| \times |E|)$. Hence, appropriate adjustments should be made. To that end, we build two transformation matrices: an $(|\tilde{E}| \times |E|)$ matrix $\tilde{\Sigma}$ whose $(\tilde{\Sigma})_{i,j} = 1$, with $i \in \tilde{E}$, and 0 otherwise, and an $(|E| \times |\tilde{E}|)$ matrix Σ whose $(\Sigma)_{(i,j,k,-),j} = 1$, with $(i,j,k,-) \in E_-$, $j \in \tilde{E}$, and 0 otherwise. The multiplications $\mathbb{E}(e^{-\beta C})\tilde{\Sigma}$ and $\mathbb{E}(e^{-\beta \mathcal{Z}})\Sigma$ adjust the state space from \tilde{E} to E and vice versa, respectively.

Consider the MAP process $\{\tilde{X}, \tilde{J}\}$. At time \mathcal{Z} , level s is hit either continuously (with $\tilde{X}(\mathcal{Z}) = s$ and $C = \mathcal{Z}$) or by a downward jump (with $\tilde{X}(\mathcal{Z}) < s$). In the latter case, we need to distinguish between $0 \leq \tilde{X}(\mathcal{Z}) < s$, where $C = \mathcal{Z}$, and $\tilde{X}(\mathcal{Z}) < 0$, where $C = \mathcal{Z} + L$. To do so, we consider the modified process $\{X, J\}$, and assume that it hits level s by a downward jump at state $J(\mathcal{Z}) = (i, j, k, -)$.

Corollary 4.1. Let $\Lambda_{(i,j,k,-)}$ be the withdrawal amount above level s (i.e., $\Lambda_{(i,j,k,-)} = s - \tilde{X}(\mathcal{Z})$). Then, by applying the PH properties and the memoryless property of the exponential distribution we have $\Lambda_{(i,j,k,-)} \sim PH(\mathbf{e}_k, \mathbb{T}^{(ij)-})$ of order m_{ij}^- (see also [38,39]).

As a result, we have

$$P(0 \leq \tilde{X}(\mathcal{Z}) \leq s \mid J(\mathcal{Z}) = (i, j, k, -)) = F_{\Lambda_{(i,j,k,-)}}(s) = 1 - \mathbf{e}_k \text{Exp}(\mathbb{T}^{(ij)-})s \mathbf{e}. \quad (17)$$

Let $\mathbb{F}(s)$ be a diagonal matrix of order $|E|$ whose (l, l) -th entry is

$$(\mathbb{F}(s))_{l,l} = \begin{cases} F_{\Lambda_{(i,j,k,-)}}(s) & l = (i, j, k, -) \in E_-, \\ 1 & l \in E_n, \\ 0 & l \in E_1. \end{cases} \quad (18)$$

(Note that when $l \in E_n$, level s is passed continuously and $\tilde{X}(\mathcal{Z}) = s$.) We further let $\hat{\mathbb{F}}(s)$ be an $(|E| \times |E|)$ matrix with $(\hat{\mathbb{F}}(s))_{l,l} = 1 - (\mathbb{F}(s))_{l,l}$, $l \in E_2$, and 0 otherwise.

Lemma 4.1. The expected discounted cycle-length matrix of order $(|\tilde{E}| \times |\tilde{E}|)$ is given by

$$\mathbb{E}(e^{-\beta C}) = \tilde{\Sigma} \mathbb{M}_{\mathcal{Z}}(\beta) (\mathbb{F}(s) \Sigma + \hat{\mathbb{F}}(s) \Sigma \mathbb{M}_L(\beta)). \quad (19)$$

Proof. We have the next decomposition:

$$\begin{aligned} \mathbb{E}(e^{-\beta C}) &= \mathbb{E}(e^{-\beta \mathcal{Z}} \mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}) + \mathbb{E}(e^{-\beta(\mathcal{Z}+L)} \mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) \\ &= \mathbb{E}(e^{-\beta \mathcal{Z}}) \mathbb{E}(\mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}) + \mathbb{E}(e^{-\beta \mathcal{Z}}) \mathbb{E}(\mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) \mathbb{E}(e^{-\beta L}). \end{aligned} \quad (20)$$

Applying (9), we have $\mathbb{E}(e^{-\beta L}) = \mathbb{M}_L(\beta) = M_L(\beta \mathbb{I} - \tilde{\mathbb{Q}})$; the matrix $\tilde{\mathbb{Q}}$ arising since we need to follow the initial and final states of \tilde{J}

for the continuity of the process. Substituting $\mathbb{E}(\mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}) = \mathbb{F}(s)$, $\mathbb{E}(\mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) = \hat{\mathbb{F}}(s)$, and $\mathbb{E}(e^{-\beta \mathcal{Z}}) = \mathbb{M}_{\mathcal{Z}}(\beta)$ completes the proof. The multiplication by the matrix $\tilde{\Sigma}$ in (19) expands all the terms to include states in E_- ; similarly, the multiplication by the matrix Σ collects all terms corresponding to states in E_- into state \tilde{E} . ■

Corollary 4.2 (Activation Cost). The expected discounted activation cost, $AC(\beta)$, is obtained immediately by substituting (19) into (3) (with the appropriate state adjustments),

$$AC(\beta) = \eta S^p \left[\mathbb{I} + (\mathbb{I} - \mathbb{E}(e^{-\beta C}))^{-1} \tilde{\Sigma} \mathbb{M}_{\mathcal{Z}}(\beta) \Sigma \right] \mathbf{Y} \mathbf{e}. \quad (21)$$

4.2. Loading cost

Denote by $\boldsymbol{\gamma}$ the $|\tilde{E}|$ -column vector with $(\boldsymbol{\gamma}_i) = \gamma_i, i \in \tilde{E}$. Arranging the right hand-side of (4), we have:

$$\mathbb{E}(e^{-\beta C} \gamma_{\tilde{J}(\mathcal{Z})} (S - \tilde{X}(\mathcal{Z}))) \mathbf{e} = S \mathbb{E}(e^{-\beta C}) \boldsymbol{\gamma} - \mathbb{E}(e^{-\beta C} \gamma_{\tilde{J}(\mathcal{Z})} \tilde{X}(\mathcal{Z})) \mathbf{e}. \quad (22)$$

To derive $\mathbb{E}(e^{-\beta C} \gamma_{\tilde{J}(\mathcal{Z})} \tilde{X}(\mathcal{Z}))$, recall that when the level drops below 0, the card is frozen for L time units, and the load and charge are performed at time $\mathcal{Z} + L$. We further note that both processes, the MAP and its modified MMFF process, should be simultaneously addressed. We have:

$$\begin{aligned} \mathbb{E}(e^{-\beta C} \gamma_{\tilde{J}(\mathcal{Z})} \tilde{X}(\mathcal{Z})) \mathbf{e} &= \\ \tilde{\Sigma} \left[\mathbb{E}(e^{-\beta \mathcal{Z}} \gamma_{\tilde{J}(\mathcal{Z})} \tilde{X}(\mathcal{Z}) \mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}) \mathbf{e} + \mathbb{E}(e^{-\beta(\mathcal{Z}+L)} \gamma_{\tilde{J}(\mathcal{Z}+L)} \tilde{X}(\mathcal{Z}) \mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) \mathbf{e} \right]. \end{aligned} \quad (23)$$

Given $J(\mathcal{Z})$, the processes $\{\tilde{X}(t), t \geq \mathcal{Z}\}$ and $\{\tilde{X}(t), t < \mathcal{Z}\}$ are stochastically independent. Thus, the Markov property yields:

$$\begin{aligned} \mathbb{E}(e^{-\beta \mathcal{Z}} \gamma_{\tilde{J}(\mathcal{Z})} \tilde{X}(\mathcal{Z}) \mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}) \mathbf{e} &= \\ = \mathbb{E}(e^{-\beta \mathcal{Z}}) \mathbb{E}(\gamma_{\tilde{J}(\mathcal{Z})} \tilde{X}(\mathcal{Z}) \mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}), \end{aligned} \quad (24)$$

$$\begin{aligned} \mathbb{E}(e^{-\beta(\mathcal{Z}+L)} \gamma_{\tilde{J}(\mathcal{Z}+L)} \tilde{X}(\mathcal{Z}) \mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) \mathbf{e} &= \\ = \mathbb{E}(e^{-\beta \mathcal{Z}}) \mathbb{E}(\tilde{X}(\mathcal{Z}) \mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) \Sigma \mathbb{E}(e^{-\beta L} \gamma_{\tilde{J}(L)}). \end{aligned} \quad (25)$$

Assume that $J(\mathcal{Z}) = \zeta \in E_2$. We distinguish between two cases:

(i) When $\zeta \in E_n$, level s is passed continuously, $\tilde{X}(\mathcal{Z}) = s$, and $\mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}} = 0$. In this case,

$$\begin{aligned} E(\gamma_{\zeta} \tilde{X}(\mathcal{Z}) \mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}) &= \gamma_{\zeta} s, \\ E(\tilde{X}(\mathcal{Z}) \mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) &= 0. \end{aligned}$$

(ii) When $\zeta = (i, j, k, -) \in E_-$, level s is hit by an (i, j) downward jump at phase k and, thus, $\tilde{X}(\mathcal{Z}) = s - \Lambda_{\zeta}$. Here, (24) and (25) become

$$\begin{aligned} E(\gamma_{\zeta} \tilde{X}(\mathcal{Z}) \mathbf{1}_{\{0 \leq \tilde{X}(\mathcal{Z}) \leq s\}}) &= \gamma_i E((s - \Lambda_{\zeta}) \mathbf{1}_{\{\Lambda_{\zeta} < s\}}), \\ E(\tilde{X}(\mathcal{Z}) \mathbf{1}_{\{\tilde{X}(\mathcal{Z}) < 0\}}) &= E((s - \Lambda_{\zeta}) \mathbf{1}_{\{\Lambda_{\zeta} > s\}}). \end{aligned} \quad (26)$$

Applying Corollary 4.1, and the phase-type properties, it is easy to verify that

$$E(\Lambda_{\zeta} \mathbf{1}_{\{\Lambda_{\zeta} < s\}}) = \mathbf{e}_k \left(-(\mathbb{T}^{(ij)-})^{-1} (\mathbb{I} - e^{s \mathbb{T}^{(ij)-}}) - s e^{s \mathbb{T}^{(ij)-}} \right) \mathbf{e}. \quad (27)$$

Substituting (27) into (26), and noting that $E(\Lambda_{\zeta}) = -\mathbf{e}_k (\mathbb{T}^{(ij)-})^{-1} \mathbf{e}$ and $E(\mathbf{1}_{\{\Lambda_{\zeta} < s\}}) = F_{\Lambda_{(i,j,k,-)}}(s)$, leads to:

$$\begin{aligned} E((s - \Lambda_{\zeta}) \mathbf{1}_{\{\Lambda_{\zeta} < s\}}) &= s + \mathbf{e}_k (\mathbb{T}^{(ij)-})^{-1} (\mathbb{I} - e^{s \mathbb{T}^{(ij)-}}) \mathbf{e}, \\ E((s - \Lambda_{\zeta}) \mathbf{1}_{\{\Lambda_{\zeta} > s\}}) &= \mathbf{e}_k (\mathbb{T}^{(ij)-})^{-1} e^{s \mathbb{T}^{(ij)-}} \mathbf{e}. \end{aligned}$$

Let Λ_{pos} and Λ_{neg} be the $(|E| \times |E|)$ diagonal matrices representing the conditional positive and negative balance at time \mathcal{Z} , respectively; their

(l, l) entry is given by:

$$(\Lambda_{pos})_{l,l} = \begin{cases} s + \mathbf{e}_k (\mathbb{T}^{(ij)-})^{-1} (\mathbb{I} - e^{s\mathbb{T}^{(ij)-}}) \mathbf{e} & l = (i, j, k, -) \in E_-, \\ s & l \in E_n \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

$$(\Lambda_{neg})_{l,l} = \begin{cases} \mathbf{e}_k (\mathbb{T}^{(ij)-})^{-1} e^{s\mathbb{T}^{(ij)-}} \mathbf{e}, & l = (i, j, k, -) \in E_-, \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

Substituting (28) and (29) into (22), and noting that $\mathbb{E}(e^{-\beta L} \gamma_{\tilde{J}(L)}) = \mathbb{M}_L(\beta) \gamma$, we have

$$\mathbb{E}(e^{-\beta C} \gamma_{\tilde{J}(C)} \tilde{X}(C)) \mathbf{e} = \tilde{\Sigma} \mathbb{M}_Z(\beta) (\Lambda_{pos} \Sigma + \Lambda_{neg} \Sigma \mathbb{M}_L(\beta)) \gamma. \quad (30)$$

4.3. Negative balance fine

Here, we derive the right hand-side of (5). Let Θ be an $|\tilde{E}|$ -column vector whose i th entry $(\Theta)_i = \phi_i, i \in \tilde{E}$.

Proposition 4.1. *The fine on the negative balance per cycle is given by*

$$\begin{aligned} & -\mathbb{E} \left(\int_{t=Z}^{Z+L} e^{-\beta t} \tilde{X}(t) \phi_{\tilde{J}(t)} \mathbf{1}_{\{\tilde{X}(t) < 0\}} dt \right) \mathbf{e} \\ & = -\tilde{\Sigma} \mathbb{M}_Z(\beta) \Lambda_{neg} \Sigma (\mathbb{M}_L(\beta) - \mathbb{I}) (\tilde{\mathbb{Q}} - \beta \mathbb{I})^{-1} \Theta. \end{aligned} \quad (31)$$

Proof. Applying the Markov property, we have

$$\begin{aligned} & \mathbb{E} \left(\int_{t=Z}^{Z+L} e^{-\beta t} \tilde{X}(t) \phi_{\tilde{J}(t)} \mathbf{1}_{\{\tilde{X}(t) < 0\}} dt \right) \mathbf{e} \\ & = \mathbb{E} \left(e^{-\beta Z} \tilde{X}(Z) \mathbf{1}_{\{\tilde{X}(Z) < 0\}} \right) \Sigma \mathbb{E} \left(\int_{t=0}^L e^{-\beta t} \phi_{\tilde{J}(t)} dt \right) \mathbf{e}. \end{aligned} \quad (32)$$

The left-hand-side of (32) is obtained by applying (26). It is easy to verify that

$$\mathbb{E} \left(\int_{t=0}^L e^{-\beta t} \phi_{\tilde{J}(t)} dt \right) \mathbf{e} = \mathbb{E} \left(\int_{t=0}^L e^{-\beta t} dt \right) \Theta. \quad (33)$$

To derive the $(|\tilde{E}| \times |\tilde{E}|)$ term $\mathbb{E} \left(\int_{t=0}^L e^{-\beta t} dt \right)$, we apply the OST to the Asmussen–Kella multidimensional martingale as follows. Consider the process

$$\hat{X}(t) = \int_{v=0}^t c_{\tilde{J}(v)} dv, \quad 0 \leq t \leq L, \quad \text{with } \hat{X}(0) = 0, \quad (34)$$

i.e., the balance level with no jumps. Let $C = \text{diag}\{c_i, i \in \tilde{E}\}$ and $\mathbb{K}(\alpha) = \tilde{\mathbb{Q}} + \alpha C$. Chapter XI, Prop. 2.2 of Asmussen [40] yields that

$$\mathbb{E} \left(e^{\alpha \hat{X}(t)}, \tilde{J}(t) = j \mid \tilde{J}(0) = i, \hat{X}(0) = 0 \right) = (e^{t\mathbb{K}(\alpha)})_{ij}.$$

Define $Y(t) = -(\beta/\alpha)t$ and let $W(t) = \hat{X}(t) + Y(t)$. Theorem 2.1 of Asmussen and Kella [4] yields that the process

$$\begin{aligned} \mathbf{M}(\alpha, t) &= \int_0^t e^{\alpha W(v)} \mathbf{1}_{\tilde{J}(v)} dv \mathbb{K}(\alpha) + e^{\alpha W(0)} \mathbf{1}_{\tilde{J}(0)} - e^{\alpha W(t)} \mathbf{1}_{\tilde{J}(t)} \\ &\quad + \alpha \left(\int_0^t e^{\alpha W(v)} \mathbf{1}_{\tilde{J}(v)} dY(v) \right) \\ &= \int_0^t e^{\alpha \hat{X}(v) - \beta v} \mathbf{1}_{\tilde{J}(v)} dv \left(\mathbb{K}(\alpha) - \beta \mathbb{I} \right) + e^{\alpha \hat{X}(0)} \mathbf{1}_{\tilde{J}(0)} - e^{\alpha \hat{X}(t) - \beta t} \mathbf{1}_{\tilde{J}(t)} \end{aligned} \quad (35)$$

is an $|\tilde{E}|$ -row vector-valued zero-mean martingale. Applying the OST at time L yields $\mathbb{E}(\mathbf{M}(\alpha, L)) = \mathbb{E}(\mathbf{M}(\alpha, 0)) = 0$, i.e.,

$$\mathbb{E} \left(\int_0^L e^{\alpha \hat{X}(t) - \beta t} dt \right) = [\mathbb{E}(e^{\alpha \hat{X}(L) - \beta L}) - \mathbb{E}(e^{\alpha \hat{X}(0)})](\mathbb{K}(\alpha) - \beta \mathbb{I})^{-1}. \quad (36)$$

Substituting $\alpha = 0$ yields

$$\mathbb{E} \left(\int_0^L e^{-\beta t} dt \right) = [\mathbb{E}(e^{-\beta L}) - \mathbb{I}](\mathbb{K}(0) - \beta \mathbb{I})^{-1}. \quad (37)$$

(Note that $\mathbb{E}(e^{\alpha \hat{X}(0)})|_{\alpha=0} = \mathbb{I}$.) Substituting $\mathbb{E}(e^{-\beta L}) = \mathbb{M}_L(\beta)$ (see Eq. (9)) arrives at

$$\mathbb{E} \left(\int_0^L e^{-\beta t} dt \right) = (\mathbb{M}_L(\beta) - \mathbb{I}) (\tilde{\mathbb{Q}} - \beta \mathbb{I})^{-1}. \quad (38)$$

Substituting (26), (33), and (38) into (32) completes the proof. ■

Remark 2. The discounted expected loaded amount (LA_{cyc}) and deficit (DA_{cyc}) of the first cycle can be easily derived by applying (22), (30), and (31), i.e.,

$$LA_{cyc} = \mathbb{E} \left(e^{-\beta Z} (S - \tilde{X}(Z)) \right) = \eta \tilde{\Sigma} \mathbb{M}_Z(\beta) (S \mathbb{I} - (\Lambda_{pos} + \Lambda_{neg})) \mathbf{e}, \quad (39)$$

$$DA_{cyc} = -\mathbb{E} \left(e^{-\beta Z} \tilde{X}(Z) \mathbf{1}_{\{\tilde{X}(Z) < 0\}} \right) = -\eta \tilde{\Sigma} \mathbb{M}_Z(\beta) \Lambda_{neg} \mathbf{e}. \quad (40)$$

4.4. A summary of the cost' derivation

For clarification, we summarize the algorithm for the derivation of the cost components. The algorithm includes two main steps.

Step 1 (build the modified process)

- Define the enlarged state space $E \equiv \tilde{E} \cup E_- \cup E_+$ where $E_{\pm} := (i, j, k, \pm) : (i, j) \in \Omega, 1 \leq k \leq m_{ij}^{\pm}$.
- Built the generator matrix \mathbb{Q} of the modified MMFF process; use Eqs. (11) and (12).
- Built the process \mathcal{X}^R by reversing the roles of the up and down movements.
- Initial $\mathbb{A}_0^R(\beta)$ and $\mathbb{U}_0^R(\beta)$; compute the recursive algorithm for $((\mathbb{A}_n^R(\beta), \mathbb{U}_n^R(\beta)) : n \geq 0)$ to obtain $\mathbb{A}^R(\beta)$ and $\mathbb{U}^R(\beta)$.
- Built the $(|E| \times |\tilde{E}|)$ matrix Σ with $(\Sigma)_{(i,j,k,-),j} = \begin{cases} 1 & (i,j,k,-) \in E_- \text{ and } j \in \tilde{E} \\ 0 & \text{else} \end{cases}$, and the $(|\tilde{E}| \times |E|)$ matrix $\tilde{\Sigma}$ with $(\tilde{\Sigma})_{i,i} = \begin{cases} 1 & i \in \tilde{E} \\ 0 & \text{else} \end{cases}$.

Step 2 (derive the cost components)

For S and $0 \leq s \leq S-1$ do:

- Calculate the $\text{LST } \mathbb{M}_Z(\beta) \equiv \mathbb{E}(e^{-\beta Z}) = \mathbb{E}(e^{-\beta \tilde{r}^R(S-s)}) = \left(e^{\mathbb{I}_d} \right)_{\mathbb{A}^R(\beta)} \left(e^{\mathbb{U}^R(\beta)(S-s)} \quad \mathbf{0} \right)$.
- Derive the expected discounted cycle-length matrix cycle length $\mathbb{E}(e^{-\beta C})$; use Eq. (19).
- Apply (21) to obtain the expected discounted activation cost $AC(\beta)$.
- Substitute (19) and (30) into (22); then use (4) to obtain the expected discounted loading cost $LC(\beta)$.
- Calculate the expected discounted fine on the negative balance $NC(\beta)$; use Eq. (31).
- The discounted total cost $TC(\beta)$ is obtained by applying Eq. (6).
- Additionally, LA_{cyc} and DA_{cyc} may be obtained by applying (39) and (40), respectively.

5. Numerical examples

Our base case includes CTMC with state space $\tilde{E} = \{1, 2\}$, (i.e., two states), initial probability vector $\eta = (0.4 \ 0.6)$, and infinitesimal matrix

$$\tilde{\mathbb{Q}} = \begin{pmatrix} -0.03 & 0.03 \\ 0.05 & -0.05 \end{pmatrix}.$$

During state 1, the rate of outflow (demand) is $d_1 = 1$, and the rate of inflow (return) is $r_1 = 1.5$, thus, the net rate is $c_1 = 0.5$. During state 2, we have demand at rate $d_2 = 2$, and return at rate $r_2 = 0.5$, thus $c_2 = -1.5$. Hence, state 1 is an ascending state and state 2 is a descending state; $E_p = \{1\}$, $E_n = \{2\}$, and $n_1 = n_2 = 1$. In addition, during the states, we assume batches (positive and negative) arrive at rate $\lambda_1 = 0$, $\lambda_2^+ = 0.1$ (a batch of return), $\lambda_2^- = 0.2$ (a batch of demand); thus, we have $\lambda_2 = \lambda_2^+ + \lambda_2^- = 0.3$. Changing from state i to state $j \neq i$ is accompanied by a (positive and negative) batch with probabilities $p_{12}^+ = 0.1$, $p_{12}^- = 0.3$ (i.e., when changing from state 1 to state 2, a return occurs with probability 0.1, and a demand occurs with probability 0.3); similarly, $p_{21}^+ = 0.25$, and $p_{21}^- = 0.25$. Thus, $\Omega = \{(1, 2), (2, 1), (2, 2)\}$. The batch sizes are PH distributed r.v. with parameters (all with $m_{ij}^{\pm} = 2$):

$$\alpha^{(12)+} = (0.7 \ 0.3), \quad \mathbb{T}^{(12)+} = \begin{pmatrix} -5 & 2 \\ 1 & -3 \end{pmatrix},$$

Table 1
The impact of S and s .

The impact of S and s	$S \uparrow$	$s \uparrow$
$E(e^{-\beta C})$	\downarrow	\uparrow
LA_{cyc}	\uparrow	\downarrow
DA_{cyc}	\star	\downarrow

$$\alpha^{(12)-} = (0.2 \ 0.8), \mathbb{T}^{(12)-} = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.25 \end{pmatrix},$$

$$\alpha^{(21)+} = (0.5 \ 0.5), \mathbb{T}^{(21)+} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix},$$

$$\alpha^{(21)-} = (0.5 \ 0.5), \mathbb{T}^{(21)-} = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix},$$

$$\alpha^{(22)+} = (0.9 \ 0.1), \mathbb{T}^{(22)+} = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix},$$

$$\alpha^{(22)-} = (0.5 \ 0.5), \mathbb{T}^{(22)-} = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix},$$

where $E(U^{(12)+}) = 0.408$, $E(U^{(12)-}) = 3.867$, $E(U^{(21)+}) = 0.5$, $E(U^{(21)-}) = 2$, $E(U^{(22)+}) = 0.33$, and $E(U^{(22)-}) = 10$. Let $E(\Xi^+)$ and $E(\Xi^-)$ be the average amount of inflows and outflows per time unit, respectively, i.e.,

$$E(\Xi^+) = \sum_{i \in E_p} \pi_i c_i + \sum_{(i,j) \in \Omega} \pi_i \lambda_i^+ E(U^{(ii)+}) + \sum_{\substack{(i,j) \in \Omega \\ j \neq i}} \pi_i \tilde{q}_{ij} p_{ij}^+ E(U^{(ij)+}),$$

$$E(\Xi^-) = \sum_{i \in E_n} \pi_i |c_i| + \sum_{(i,j) \in \Omega} \pi_i \lambda_i^- E(U^{(ii)-}) + \sum_{\substack{(i,j) \in \Omega \\ j \neq i}} \pi_i \tilde{q}_{ij} p_{ij}^- E(U^{(ij)-}), \quad (41)$$

where $\pi = (\pi_1, \dots, \pi_{|\bar{E}|})$ is the stationary probability distribution of the Markov chain \tilde{J} . Here, $\pi = (\frac{5}{8}, \frac{3}{8})$, $E(\Xi^+) = 0.328$ and $E(\Xi^-) = 1.344$ (as in common, $E(\Xi^+) < E(\Xi^-)$). Next, we consider the expanded $\{\mathcal{X}, \mathcal{J}\}$ process. We use the lexicographic order $E_1 = 1, 121^+, 122^+, 211^+, 212^+, 221^+, 222^+$ and $E_2 = 2, 121^-, 122^-, 211^-, 212^-, 221^-, 222^-$; thus, we have a total number of $|E| = |E_1| \cup |E_2| = 14$ states.

The impact of changing S and s

We start by studying the impact of S and s on the system's measures. Table 1 indicates the changes in the cycle length, the loaded amount, and the deficit as functions of S and s . The symbol " \uparrow " (" \downarrow ") presents an increase (decrease) of the measurement as a function of the thresholds; the symbol " \star " indicates a special behavior. We see that increasing (decreasing) s (S) decreases (increases) the cycle length, and the loaded amount. However, while increasing s decreases the deficit, the change in DA_{cyc} as a function of S is more complicated. To study this change, Fig. 2 curves $DA_{cyc}(S)$ for $s \in \{1, 5, 9, 15\}$. Contrary to expectations, DA_{cyc} is concave in S ; this probably because increasing S (particularly when S is low) at first allows more downward jumps, and hence increases the deficit, but as S continues to increase, this deficit decreases.

The optimal policy

Our aim is to find the optimal thresholds S^* and s^* that minimize the expected total cost TC . However, in practice this aim can be achieved only numerically, since the expressions for the cost functionals are too cumbersome for mathematical analysis. Our results show that TC is convex in S and s . Although we do not prove this convexity, it appears in all our numerical results. Thus, the search for S^* and s^* can be easily accomplished by substituting integers for S and varying $s < S$. Let $TC^* = TC(S^*, s^*)$. In Table 2, we vary the system's parameters one at a time to see the effect of this variation on S^* and s^* . We see that the activation cost Υ and the fine Θ have an opposite impact on S^* and s^* . While increasing Υ decreases the thresholds in order to postpone the time of activation, S^* and s^* are increasing in Θ , clearly to avoid a negative balance. Similarly, our results show that increasing

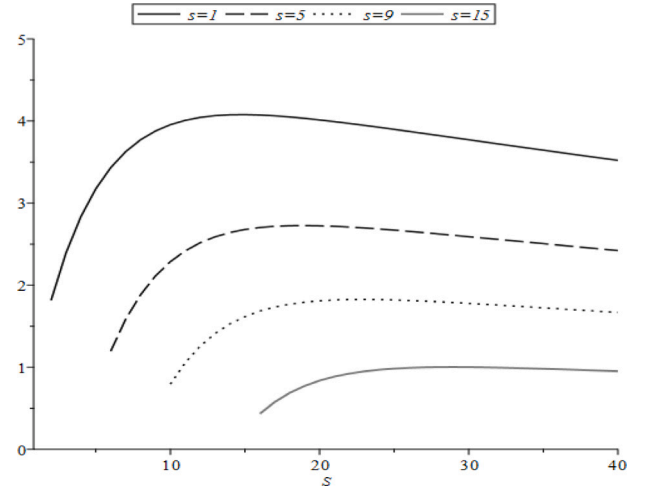


Fig. 2. $DA_{cyc}(S)$ for $s \in \{1, 5, 9, 15\}$.

Table 2
The impact of the parameters on S^* , s^* .

The impact of the parameters	$\Upsilon \uparrow$	$\gamma \uparrow$	$\Theta \uparrow$	$\beta \uparrow$
S^*	\downarrow	\uparrow	\uparrow	\downarrow
s^*	\downarrow	\downarrow	\uparrow	\downarrow

β significantly decreases S^* and s^* and $S^* - s^*$. When β is low, it is worthwhile for the cardholder to postpone the activation with a larger amount in order to utilize the timing. However, when β is high, the difference between the optimal thresholds, $S^* - s^*$, decreases, resulting in frequent activations with smaller amounts. Thus, even though the risk of a negative balance increases (due to a low s^*), the discount factor is dominant in reducing the cost. Finally, Table 2 shows that increasing γ increases the difference $S^* - s^*$, which is, in general, not intuitive. Here, too, a larger $S^* - s^*$ postpones the loadings and, together with the impact of the inflows and the discount factor, eventually reduces the cost.

The impact of ρ and L on the optimal policy

Next, we focus on the power factor ρ and the time L . To this end, let $\beta = 0.01$, $\Upsilon = (4, 4)$, $\gamma = (1, 1)$, and $\phi = (10, 10)$. Starting with a fixed L , Fig. 3 plots (S^*, s^*) for $L \in \{1, 3, 5, 7, 10\}$ and $\rho = \{1, 1.1, 1.2\}$. To study the effect of the variation, Figs. 4 further plots (S^*, s^*) as a function of ρ for distributions (all with $E(L) = 5$) (i) $L = 5$, $cv_L = 0$, (ii) $L \sim \text{Gamma}(\frac{1}{80}, \frac{1}{16})$, $cv_L = 4$, and (iii) $L \sim \text{Gamma}(\frac{1}{320}, \frac{1}{64})$, $cv_L = 8$, represented by black, gray, and blue curves, respectively. (We note that similar curves are obtained for $L \sim \exp(1/5)$ and $L \sim U(1, 9)$.) The solid and dashed curves refer to S^* and s^* , respectively. Figs. 3 and 4 imply that both S^* and s^* are increasing in $E(L)$ and decreasing in ρ and cv_L . However, the growth rate of S^* is faster than that of s^* , resulting in the increase of $(S^* - s^*)$ and LA_{cyc} . Moreover, Fig. 5 curves $DA_{cyc}(L)$ for different values of ρ (here, $cv_L = 0$). We see that for a relatively low ρ ($\rho = 1, 1.1$), the impact of L dominates by increasing S^* and s^* and decreasing DA_{cyc} . Setting ρ higher decreases S^* and increases DA_{cyc} ; however, increasing L overweights the effect of ρ , increases S^* and s^* , and eventually decreases DA_{cyc} . Overall, the (S, s) policy is more sensitive under parameters $E(L)$ and ρ than under the parameter cv_L .

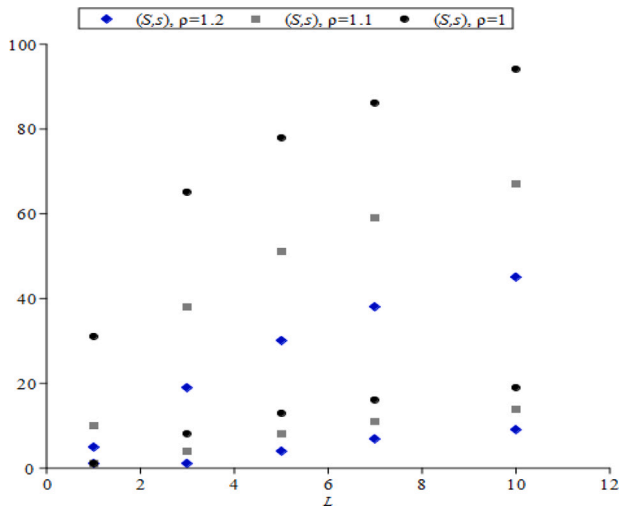
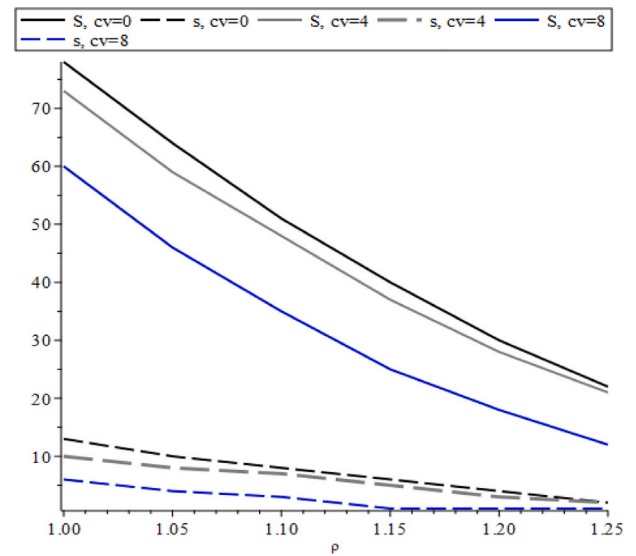
The impact of β and outflow patterns on the optimal policy

In the sequel, we present the results on the effect of the discount factor and different outflows on the optimal policy. In order to obtain closely-related systems as much as possible, we construct two scenarios that have the same average outflows $E(\Xi^-)$ but differ in their outflow

Table 3

 (S^*, s^*) and TC^* for Scenarios (i) and (ii) and different values of β and $E(\Xi^-)$.

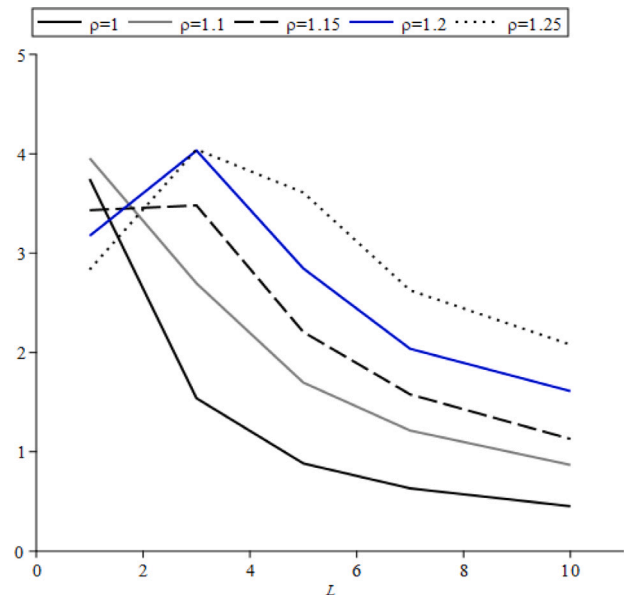
$(S^*, s^*)TC^*$	$\beta=0.01$		$\beta=0.03$		$\beta=0.05$		$\beta=0.1$	
	Sc. (i)	Sc. (ii)	Sc. (i)	Sc. (ii)	Sc. (ii)	Sc. (ii)	Sc. (i)	Sc. (ii)
1.147	51, 10 1119	47, 8 975.784	30, 7 530.03	30, 5 541.31	22, 5 378.78	22, 4 395.61	11, 2 230.71	10, 1 241.54
* 1.344	51, 8 1271	51, 8 1271	30, 5 577.37	30, 5 577.37	22, 4 410.75	22, 4 410.75	10, 1 245.16	10, 1 245.16
1.73	51, 6 1563	52, 8 1292	29, 3 676.69	31, 6 593.04	21, 2 469.11	23, 4 418.51	8, 1 270.21	11, 1 251.02
2.725	42, 2 2223	54, 9 1344	22, 1 892.76	32, 6 615.25	14, 1 590.28	24, 5 433.65	7, 1 327.85	11, 2 259.59
4.69	15, 1 3302	76, 10 1676	9, 1 1241	38, 6 764.64	8, 1 799.38	25, 5 530.89	6, 1 440.14	9, 2 306.43

Fig. 3. $S^*(L)$, $s^*(L)$ for $\rho = \{1, 1.1, 1.2\}$.Fig. 4. $S^*(\rho)$, $s^*(\rho)$ for $cv_L = \{0, 4, 8\}$.

pattern—such that one is continuous and the other is batch-type. (i) In the first scenario we change the continuous pattern of the outflows by varying d_2 in 1.5, 2, 3, 5.5, 10.5 so that c_2 varies in $-1, -1.5, -2.5, -5, -10$. (ii) In the first scenario we change the batch-type pattern of the outflows by varying the average size $E(U^{(21)-})$ in 0, 2, 6, 16.6, 36. To do so, we use the following parameters:

$$\begin{aligned}
 U^{(21)-} &\sim PH\left((0 \ 0), \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}\right), & E(U^{(21)-}) &= 0, \\
 U^{(21)-} &\sim PH\left((0.5 \ 0.5), \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}\right), & E(U^{(21)-}) &= 2, \\
 U^{(21)-} &\sim PH\left((0.5 \ 0.5), \begin{pmatrix} -0.1 & 0 \\ 0 & -0.5 \end{pmatrix}\right), & E(U^{(21)-}) &= 6, \\
 U^{(21)-} &\sim PH\left((0.5 \ 0.5), \begin{pmatrix} -0.835 & 0 \\ 0 & -0.03125 \end{pmatrix}\right), & E(U^{(21)-}) &= 16.6, \\
 U^{(21)-} &\sim PH\left((0.5 \ 0.5), \begin{pmatrix} -0.025 & 0 \\ 0 & -0.03125 \end{pmatrix}\right), & E(U^{(21)-}) &= 36.
 \end{aligned}$$

Under both scenarios, the average outflow is $E(\Xi^-) \in \{1.147, 1.344, 1.73, 2.725, 4.69\}$. Note that the values $c_2 = 2$ and $E(U^{(21)-}) = 2$ (with $E(\Xi^-) = 1.344$) are used in the base case. From a

Fig. 5. $DA_{cyc}(L)$ for $\rho = \{1, \dots, 1.25\}$.

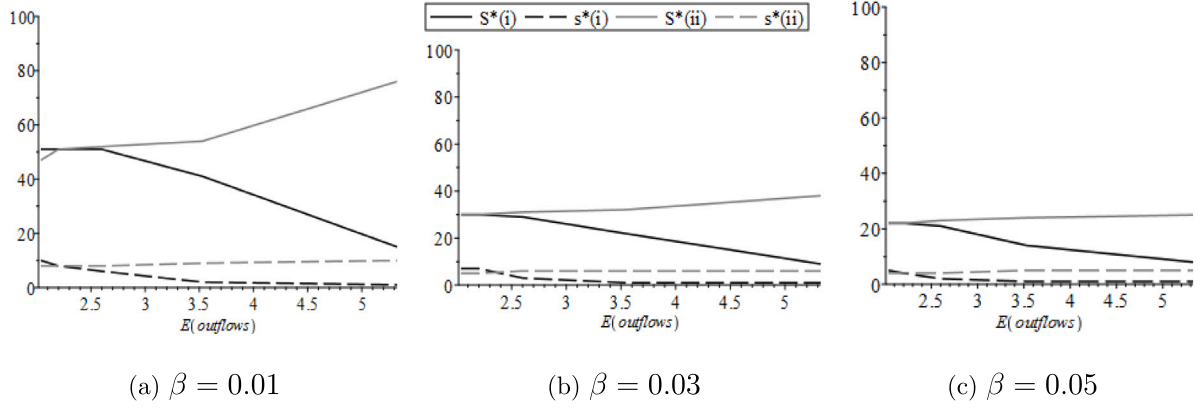


Fig. 6. The optimal (S^* , s^*) as a function of $E(\Xi^-)$ for (a) $\beta = 0.01$, (b) $\beta = 0.03$, and (c) $\beta = 0.05$.

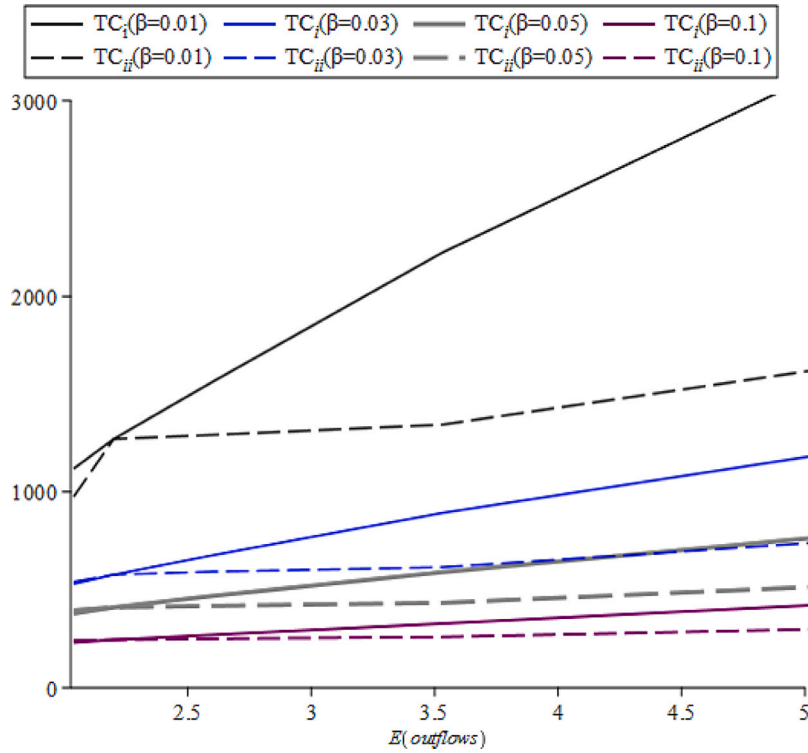


Fig. 7. TC^* under Scenarios (i) and (ii) for $\beta \in \{0.01, 0.03, 0.05, 0.1\}$.

practical point of view, it is reasonable to assume that outflows occur more frequently with a significantly higher average relative to inflows, and therefore our study focuses on different outflow patterns.

Table 3 tabulates the optimal policy (S^* , s^*) and TC^* as a function of $\beta \in \{0.01, 0.03, 0.05, 0.1\}$ and $E(\Xi^-)$ under scenarios (i) and (ii). To emphasize the distance between the optimal thresholds, Fig. 6(a–c) depicts S^* and s^* as a function of $E(\Xi^-)$ for $\beta \in \{0.01, 0.03, 0.05\}$. In each figure, the black solid (dashed) line curves S^* (s^*) under Scenario (i); similarly, the gray solid (dashed) line curves S^* (s^*) under Scenario (ii). Finally, Fig. 7 plots TC^* under Scenarios (i) and (ii) as a function of β and $E(\Xi^-)$.

In Table 3 and Figs. 6 and 7, we observe the following:

The optimal total cost TC^* . In general, we observe a similar behavior of TC^* with respect to the output pattern; i.e., TC^* is decreasing in β and increasing in $E(\Xi^-)$. However, the rate of that increase is significantly higher under Scenario (i), especially for lower values of β . For example, when $\beta = 0.03$, under Scenario (i), increasing $E(\Xi^-)$ from 1.147 to 4.69 increases TC^* by 134% (from 530 to 1241), compared

to only 41% under Scenario (ii) (from 541 to 764). Similarly, when $\beta = 0.1$, under Scenario (i) we see that TC^* increases by 195%, compared to 72% under Scenario (ii). This behavior can be explained by the fact that Scenario (ii) is driven by instantaneous changes in the withdrawal amount that blur the effect of timing, and therefore slow down the rate of growth. Comparing the costs, we see that $TC_i^* - TC_{ii}^*$ increases in $E(\Xi^-)$, such that the total cost of handling the card is more sensitive to continuous changes than to batch-type ones.

The optimal thresholds S^* and s^* . Contrary to expectations, Table 3 and Fig. 6 show an opposite behavior of the optimal thresholds with respect to the output pattern in the two scenarios. While S^* , s^* , and $S^* - s^*$ are decreasing in $E(\Xi^-)$ under Scenario (i), they are increasing in it under Scenario (ii). When withdrawals occur more frequently and in small amounts (and thus can be represented by a linear rate), it is economically profitable to load more frequently and with smaller amounts (by decreasing $S^* - s^*$). In such cases, cash shortages (and the resulting fees) are relatively rare and given minor consideration; increasing β even intensifies this by reducing the optimal thresholds and

the loaded amounts in order to achieve a full utilization of the timing. By contrast, larger outflow amounts quickly prompt the cardholder to raise the optimal thresholds to avoid a cash shortage. Here, too, the growth rate declines by β , reflecting the trade-off between the fee for a negative balance and the advantage of utilizing the timing.

To summarize, the optimal policy is clearly dependent on the arrival outflows, especially when β is low. When the continuous outflows increase, it is worthwhile for the cardholder to apply frequent activations with smaller amounts. However, when the amount sizes increase, the thresholds are set higher for optimality. Thus, when regular and daily transactions are expected, we recommend that cardholders activate the card more frequently with smaller amounts. By contrast, when occasional large transactions are expected, the card needs to be activated less frequently with larger amounts. Nevertheless, the optimal policy is less sensitive to batch-type changes than to changes in the continuous ones.

6. Summary and future research

This paper considers a cash balance process with continuous and batch-type bilateral transactions in a Markovian environment under an (S, s) -type policy. Using hitting time analysis, martingales, and renewal theory, we construct an easy tool for deriving the cost functionals and obtain, albeit numerically, the decision variables. Numerical results show that timing has a significant impact on the optimal policy, and a high discount factor yields frequent activations with smaller amounts. We further see that the thresholds S^* and s^* are more sensitive under the parameters $E(L)$ and ρ than under the parameter cv_L . Comparing changes in rates and batch sizes reveals that the optimal policy is inherently linked with the outflow patterns, and is more sensible to changes in rates than to changes in batch sizes. Specifically, we show that higher rates yield lower thresholds, which are decreasing in β ; here, frequent activations with small amounts are recommended. Overall, capturing the specific parameters is required to determine the best policy for managing the card.

We suggest three natural extensions for future research. Our model assumes a fine for a negative balance. It would be interesting to compare this policy to other shortage policies that allow a limited negative balance, for an additional cost. In this case, the cardholder can choose among different levels of a cash shortage, each of which incurs with a different fine. Also, it would be useful to investigate other band policies, such as loading the card with a fixed amount selected from among several options, similar to the well-known (Q, r) policy. Finally, here we assume that the card is automatically loaded when the current balance value drops to some specified threshold. It is worth studying the impact of other factors, such as, intended usage, dates of salary arrival and location of charging facilities.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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