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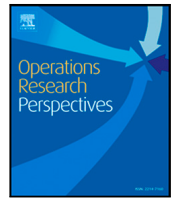
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Lot-sizing and pricing decisions under attraction demand models and multi-channel environment: New efficient formulations

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ABSTRACT

The presented paper considers the pricing and lot-sizing decisions for a manufacturer who produces and sells a single product in different selling channels i.e physical stock, website, mobile, etc. The objective is to find the production plan and prices of each channel to maximize the total profit defined from difference between the revenues and the productions, holding and setups costs. The consumers' demand in each channel is represented by attraction demand models which include the multinomial logit (*MNL*), multiplicative competitive interaction (*MCI*) and linear demand models. The addressed problem is formulated as a non-convex mixed-integer nonlinear program (*MINLP*). Based on properties of attraction functions, an efficient reformulation which transforms the initial non-convex problem into a convex one is presented. Therefore, an optimization approach based on the outer approximation algorithm is presented to solve the problem. Numerical tests based on large benchmark of real inspired instances show the efficiency of the proposed approach to solve the addressed problem compared to the initial non-convex model.

1. Introduction

Lot-sizing problems are classic problems of the industry with objective to determine the quantities to produce or to supply such that the different costs (storage, production, ...) are minimized. Karimi et al. [1] stated that good lot-sizing decisions affect directly the performance of the system and its productivity which are necessary elements that allow companies to remain competitive in the market.

Pricing is one of the critical elements that presents challenges for the companies. The earliest work on the practice of pricing dates back to service industries such as hotels [2] and airlines [3]. Initially, researchers and companies were interested in pricing alone to improve profits. However, in production industries, coordinating pricing decisions with other supply chain decisions such as production is not only useful, but essential. Coordinating these decisions optimizes the whole system rather than individual elements [4]. According to the review of the academic literature, there is a lack of decision support systems that truly integrate production–inventory and pricing decisions, particularly in the multi-product and multi-market context. Tang [5] attributes this gap to lack of efficient algorithms for general, multi-period and, multi-product models.

By the emergence of internet, sellers have embraced the multi-channel selling process using websites, mobile apps, etc. Thus, the same product can be sold on several sales channels with different prices. Consumers for their part have shown different behaviors towards this new mode of interaction with sellers. A portion of consumers still prefer to buy in store, while some of them prefer to buy directly online with cheaper prices. Others prefer to visit physical stocks in order to touch the products, then buy online [6].

Motivated by the challenge of the coordination of pricing and lot-sizing decisions and by the emergence of multi-channel selling, in this paper, joint pricing and lot-sizing in the multi-channel environment is presented. The problem studied considers a manufacturer who produces a single product and sells it in different channels (physical store, online website, ...) over a finite and discrete horizon divided into a set of periods. The demand in each channel is modeled using attraction models. The price is fixed for each period and may change between periods. The problem is formulated as a *MINLP* which is non-convex in the space of prices. The objective is to find the production's quantities and prices in each period such that the profit is maximized for all the horizon.

The contribution of this paper can be summarized as follows:

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- Based on attraction models, a joint pricing and lot-sizing model under multi-channel selling is presented.
- Using the attraction functions properties, a convex reformulation of the problem is proposed by considering demand variables.
- Proposition of an optimization approach based on outer approximation technique to solve the problem.

The reminder of this paper is organized as follows: In Section 2 a literature review of the works that belong to the scope of the studied problem is presented. Section 3 describes the assumptions and the mathematical formulation of the problem. In Section 4, the equivalent and convex reformulation of the problem is detailed. Section 5 describes the outer approximation approach to solve the problem. Section 6 presents the numerical results on real inspired instances. A conclusion is to be found in Section 7.

2. Literature review

The literature review is divided into two sub-sections; in the first section, an overview of the demand functions is presented. Section 2.2 reviews the literature works related to the addressed topic in this paper. Specifically, this section focuses on the joint pricing and lot-sizing with discrete-time and finite horizon since the same assumptions on the horizon of decision are considered in this paper.

2.1. Demand function

Pricing is a critical topic that presents challenges for companies. An important element to consider in a pricing problem, in particular pricing and lot-sizing problem is the demand function which depends on firms' marketing and operational activities. A variety of mathematical functions have been proposed to characterize the demand in different markets. Huang et al. [7] surveyed the commonly used models of demand by considering six factors which include: (i) price (ii) rebate, (iii) lead time, (iv) space, (v) quality, and (vi) advertising. They stated that two of the commonly used functions to model the demand are the linear and iso-elastic. Besides, they noticed that the most of the proposed models of demand belong to the categories (i) and (v).

The linear demand function assumes a linear relation between the demand and market factors such as the price p ($d(p) = a - bp$). Several works on joint pricing and lot-sizing are based on this function [8–20]. One of the advantages of the linear function is its simple form which leads in general to closed-form solution. However, it shows some drawbacks, in particular, it requires an upper bounds on prices to avoid negative demand. Furthermore, it is not applicable for all markets. For instance, Lus and Muriel [21] concluded that applying the linear demand to study the effect of product substitutability leads to highly unrealistic results.

The iso-elastic function also called the constant elasticity demand considers a non-linear relationship between demand and prices. The main advantage of this demand is that it does not require an upper bound on the price since the demand is positive for all prices. However, it represents some drawbacks, for instance the demand approaches infinity when prices are closed to zero which is unrealistic since the number of consumers in the market is finite. Different literature works like [22–26] that coordinate decisions on pricing and lot-sizing are based on this demand.

Another widely used model to estimate the demand in the context of cross-products substitution are the attraction models which are discrete choice models that characterize the market share of products in competitive markets [27]. Based on the attractiveness function, they are used to model consumers' choice in economics, marketing [28] and in operation management [29]. The commonly used function to model the attractiveness are the Multinomial logit *MNL* [30], and Multiplicative Competitive Interaction, *MCI* [31]. Papers of Aydin and Porteus [32], Dong et al. [33], Zambrano-Rey et al. [34], Harsha et al. [35] are an

examples of literature works on pricing that used the attraction models to characterize the demand.

Aydin and Porteus [32] considered joint inventory and pricing decisions of substitute products in a newsvendor model using attraction models. They studied the problem assuming *MNL* and *MCI* attractiveness function, and showed that the profit function is unimodal in the prices' space. Authors in [33] studied dynamic pricing and inventory control of substitute products for a retailer using *MNL* model of consumer choice. They developed a stochastic dynamic programming formulation and derived the optimal dynamic pricing policy. They also compared the performance of three restricted pricing strategies: static, unified dynamic, and mixed dynamic pricing. Zambrano-Rey et al. [34] considered a retail store location and pricing in a competitive environment under constrained multinomial logit (*CMNL*). They presented a particle swarm optimization (*PSO*) that determines the price, store locations and demand segmentation. They showed through their numerical tests the closed relationship between price and location decisions. Recently Harsha et al. [35] studied the omnichannel pricing (*OCP*) problem of nonperishable products using attraction and nested attraction models. They showed that the initial model is non-convex and used linearization technics to obtain a *MIP* which is solved using commercial CPLEX. In the presented paper, a joint pricing and lot-sizing problem in a multi-channel selling environment is studied using attraction models to characterize the substitution between channels.

2.2. Joint pricing and lot-sizing

The single item lot-sizing problem (*SILSP*) is a production planning problem for a single product. The objective is to find the quantities to produce such that the market demand is satisfied and the different costs (productions, setups, inventory holding, etc.) are minimized [36]. The Economic Order Quantity (*EOQ*) which is a continuous time model with infinite horizon is the ancestor of the *SILSP* [37]. When the production quantity is limited, the problem is called capacitated *SILSP*. It has been proven that the capacitated *SILSP* is *NP-Hard* [38,39]. Authors in [36] surveyed the different classes and solutions methods for the *SILSP*.

The first works that deal with joint pricing and lot-sizing decisions date back to the papers of Whitin [40] and Thomas [41]. The first studied the problem in the *EOQ* context, while the second introduced pricing decisions in the *SILSP* model to maximize revenue. Since then, many works have been published to solve different variants of this problem such as shortage-backlogging [42,43], product deterioration [17,44–46] both shortage-backlogging and product deterioration [47–49], integrated promotion [50], inventory age-dependent demand [51], stock dependent demand [52], advance-cash-credit payment [53], rebate [54], quantity discount [11], etc. All these works considered different demand functions that depend on various parameters (price, stock-level, rebate, etc.). However, the attraction models have not been yet explored for the topic of joint pricing and lot-sizing decisions.

The presented work relates to joint pricing and lot-sizing decisions with discrete-time and finite horizon. A variety of production systems have been studied with the same assumptions. From the perspective of production systems with single product, Thomas [41] is one of the earliest paper that incorporated prices in lot-sizing problem. Authors assumed discrete values of prices and demand, and derived an optimal policy to maximize the profit. Shao and Ke [8] investigated a problem in which multiple operation system provides selling a make-to-order product to customer. They developed an analytical model by incorporating time and price sensitive in the demand. The paper of Dao-ming et al. [10] proposed a polynomial algorithm for coordinated pricing and lot-sizing model with backlogging. Lanquepin-Chesnais [13] addressed the profit maximization version of the dynamic economical problem with the presence of cost per price change. They proposed a $O(n^2)$ heuristic to solve the problem. Li and Thorstenson [12] considered a

firm which produces a single product subject to capacity constraints and backlogging. They incorporated a stochastic demand to the problem and proposed a multi-phase heuristic. Liu and Liu [18] proposed a mixed-integer second order conic programming formulation for joint pricing and lot-sizing with limited capacity. Zouadi et al. [19] investigated a joint pricing and lot-sizing problem in a single production line where operation of manufacturing and re-manufacturing are hold. They formulated the problem as a *MIP* and proposed a heuristic and a memetic algorithm to find a near optimal solution.

Joint pricing and lot-sizing decisions for production systems with multiple items has also captured the intention of the researchers. Haugen et al. [9] studied a capacitated lot-sizing problem with pricing decisions. Their model generalizes the work proposed in [41] since it deals with multiple items. Authors in [55] studied the problem assuming a general demand. They proposed an item-by-item heuristic, which they tested on a special case with linear demand. The work of Bajwa and Sox [14] studied a profit maximization problem for a firm that produces and sells multiple items. They proposed a *MINLP* mathematical model that coordinates the prices, advertising money and productions decisions. They demonstrated that by the coordination of marketing and operational decisions, firms can increase their profits. Authors in [23] proposed a joint pricing and lot-sizing model, allowing lost sales for a manufacturer which has a limited production capacity. They showed that determining prices and production plan simultaneously may lead firms to higher profit. Ouazene et al. [16] studied the problem in which the items are sold in multiple markets. They formulated the problem as a *MINLP* and solved it using outer approximation approach. They also compared the dynamic and static pricing strategies. Couzon et al. [24] studied joint pricing and lot-sizing problem with variable capacity. In addition to the prices and productions quantities, they incorporated the capacity into the decisions variables and proposed a heuristic based on Lagrangian multipliers to solve the problem. The same authors in [25] proposed an improved mathematical formulation of the model presented in [23]. They reduced the search space by introducing new lower and upper bounds and they proposed a constructive heuristics to solve the model. The work of Liu et al. [26] formulated a joint pricing and multi-item capacitated lot sizing with backlogging as a *MINLP*. By using some optimality properties of the problem, authors provided a tighter *MINLP* model. Furthermore, they introduced a model based heuristic and a genetic algorithm to solve the large instances. Terzi et al. [20] proposed a model for joint pricing and multi-item lot-sizing that incorporates the effect of substitution and complementarity between products. They provide some theoretical results related to the convexity of the model and they showed through their experimental results that even when the problem is convex, it remains intractable for large instances.

All papers cited above for joint pricing and lot-sizing with discrete-time and finite horizon did not incorporate the multi-channel selling process in their studies. In addition, for the majority of them, the demand is linear or isoelastic with price as a factor. Table 1 resumes the demand function used in these works. Note that the list in the table is not exhaustive; however, it gives an overview of the demand used in the literature when dealing with joint pricing and lot-sizing problem with discrete-time and finite horizon. Column **demand parameters** contains the market parameters that are included in the demand function. *SP elasticity* and *SP sensitivity* measure the changes in demand according to changes in prices. *LT sensitivity* and *ADV sensitivity* measure how lead time and budget spent on advertising impact the demand, respectively. The last column **demand factors** contains the different factors such as (prices, lead time etc.,) which in general are introduced as decisions variables. Table 1 shows the contribution of this paper to the literature. We propose a mathematical model to solve the single item and multi-period joint pricing and lot-sizing problem in the multi-channel context, using attraction models to characterize the demand. To the best of our knowledge, none of the literature works considers joint pricing and lot-sizing with the attraction models.

Table 1

Summary of the used demand functions.

Authors	Demand	Demand parameters	Demand factors	
1. Liu et al. [26] 2. Couzon et al. [25] 3. Couzon et al. [24] 4. Bajwa et al. [23]	Iso-elastic	SP elasticity seasonality	SP	
1. Terzi et al. [20] 2. Zouadi et al. [19] 3. Liu and Liu [18] 4. Ouazene et al. [16] 5. Bajwa and Sox [14] 6. Lanquepin-Chesnais [13] 7. Li and Thorstenson [12] 8. Dao-ming et al. [10] 9. Haugen et al. [9] 10. Shao and Ke [8]	Linear	SP sensitivity seasonality ^{4,5} ADV sensitivity ⁵ LT sensitivity ¹⁰ Maximum demand	SP ADV ⁵ LT ¹⁰	
1.Dai et al. [55] 2. Thomas [41]	General	SP sensitivity	SP	
This paper	Attraction Model	SP sensitivity	SP	

Note: SP: price, ADV: Advertising, LT: Lead time.

3. Problem description and formulation

The current study addresses a joint pricing and single item lot-sizing with capacity constraints and setups costs. Specifically, the addressed problem considers a company that manufactures and sells a single product over a discrete-time and finite horizon divided into T periods. Consumers can interact with the company through multiple channels i.e. they can buy directly from physical stock or use other channels like online website, mobile application, etc. At each period, the firm has to make decisions on: (i) the quantity to be produced, (ii) the quantity to be stocked and (iii) the price in each channel to maximize its total profit over all the horizon. The following assumptions are considered in this study:

- The production capacity is finite for each period and may change between periods.
- Inventories at the beginning and the end of the horizon are equal to zero.
- The price in each channel may change between periods.
- For each period, three costs are incurred: (i) unit production cost, (ii) setup cost and (iii) unit inventory cost. The first two are incurred when production occurs in that period. The last one is the cost of holding a unit of the product in the inventory from period t to $t + 1$. These costs are variable and may change from period to period.
- No shortages are allowed.

3.1. Notations

The following sets and constants are to describe the initial model:

- $H = \{1, 2, \dots, T\}$: The discrete set of periods that composed the time horizon;
- $M = \{1, 2, \dots, |M|\}$: The set of selling channels, and $M \cup \{0\}$ the set of all selling channels and the no-purchase option;
- c_t : Production cost in period t ;
- h_t : Holding cost in period t ;
- a_t : Setup cost in period t ;
- b_t : Available production capacity at period t .
- α^m : A positive value between 0 and 1 and corresponds to the minimum sales per channel.
- $\underline{p}_{mt}$: Lower bound of price in channel m during period t .
- $\bar{p}_{mt}$: Upper bound of price in channel m during period t .

Table 2

Attraction functions used in the current study.

Attraction function	Expression
Multinomial logit (MNL)	$f_{mt} = e^{a_{mt} + b_{mt} p_{mt}}$
Multiplicative Competitive Interaction (MCI)	$f_{mt} = a_{mt} p_{mt}^{b_{mt}}$
Linear	$f_{mt} = a_{mt} + b_{mt} p_{mt}$

- $S = \sum_{t=1}^T b_t$: Available production capacity for all the horizon. It is used in the setup constraints;

The decisions variables are as follows:

- $X_t (\geq 0)$: Production quantity in period t ;
- $I_t (\geq 0)$: Inventory level at the end of period t ;
- $p_{mt} (> 0)$: Price in channel m during period t ;
- $p_t = (p_{1t}, p_{2t}, \dots, p_{Mt})$: The prices vector during period t ;
- $D_{mt} (> 0)$: Demand in channel m during period t ;
- Y_t : A binary variable, equals to one if X_t is greater than zero.

Except for the last variable Y_t , which is binary, the rest of the variables are continuous.

3.2. Attraction demand model

As mentioned before, the attraction models which are discrete choice models are used to represent the customer demand for each channel and period (Eq. (1)). These models characterize the market share of channel $m \in M$ during period $t \in H$, which corresponds to the share of total sales (in quantities sold) for that channel during that period (author in [27] discussed the relation between different attraction functions and the market share). The attraction models in Eq. (1) are adopted from [7]. τ_t is the market size of period t which represents the total number of consumers interested in the product during that period. f_{mt} depends on the channel's attributes (price, lead-time, service, etc.) and measures the attractiveness of the channel m at period t . With the attraction models, the demand is the product of the market size and the probability to choose the channel. This latter is the portion defined from the channel attractiveness and all the alternatives (channels) attractiveness, including the no-purchase alternative for which without loss of generality the attractiveness is normalized to one. Functions $f_{mt}(p_{mt})$ have the property that $f_{mt}(p_{mt}) \geq 0$ and $\frac{\partial f_{mt}}{\partial p_{mt}} < 0$, which means that when the price in a channel increases, the attractiveness in that channel drops, as a result, the channel share of the product reduces, and it is distributed among other channels. These attraction models characterize the cross-channel substitution (i.e., switching) behavior of consumers between all sales channels and the no-purchase option.

The presented study considers a Multinomial logit (MNL), Multiplicative Competitive Interaction (MCI) and a linear attraction functions that depend on the price in the considering channel (Table 2). Parameters $a_{mt} (> 0)$ and b_{mt} can be interpreted respectively as the constant influence and the price sensitivity of the channel m at period t . Note that $b_{mt} < 0$ to ensure that the attractiveness of the channel drops when its price increases. Furthermore, for the MCI model, $b_{mt} < -1$. Parameters a_{mt} and b_{mt} may vary from channel to channel since they represent the heterogeneous behaviors of consumers towards the channels attributes (in particular, price for the presented study).

$$D_{mt}(p_t) = \tau_t \frac{f_{mt}(p_{mt})}{1 + \sum_{m' \in M} f_{m't}(p_{m't})} \quad (1)$$

3.3. Initial model

The initial model denoted P_0 is detailed as follows:

$$\max \pi = \sum_{t=1}^T \sum_{m \in M} p_{mt} D_{mt} - \sum_{t=1}^T c_t X_t + h_t I_t + a_t Y_t \quad (2)$$

such that:

$$D_{mt}(p_t) = \tau_t \frac{f_{mt}(p_{mt})}{1 + \sum_{m' \in M} f_{m't}(p_{m't})}, \quad \forall t \in H, m \in M \quad (3)$$

$$D_{mt} \geq \alpha^m \left(\sum_{m' \in M} D_{m't} \right), \quad \forall t \in H, m \in M \quad (4)$$

$$X_t \leq b_t, \quad \forall t \in H \quad (5)$$

$$\sum_{m \in M} D_{mt} + I_t - I_{t-1} - X_t = 0, \quad t \in \{2, \dots, T-1\} \quad (6)$$

$$\sum_{m \in M} D_{m1} + I_1 - X_1 = 0 \quad (7)$$

$$\sum_{m \in M} D_{mT} - I_{T-1} - X_T = 0 \quad (8)$$

$$X_t - S Y_t \leq 0, \quad \forall t \in H \quad (9)$$

$$\underline{p}_{mt} \leq p_{mt} \leq \overline{p}_{mt}, \quad \forall t \in H, m \in M \quad (10)$$

$$X_t, I_t \geq 0, \quad \forall t \in H \quad (11)$$

$$p_{mt}, D_{mt} \geq 0, \quad \forall t \in H, m \in M \quad (12)$$

$$Y_t \in \{0, 1\}, \quad \forall t \in H \quad (13)$$

Eq. (2) is the objective function to maximize. Eq. (3) define the attraction demand for each channel and each period. Constraints (4) bound the minimum presence per channel. Constraints (5) limit the quantity to produce in each period by the production capacity. Constraints (6), (7), and (8) are the inventory balance equations. Constraints (9) consider the setup cost for the production periods. Constraints (10) bound the values of the price in each channel and period. Constraints (11) and (12) ensure the non-negativity of the continuous variables. Finally, constraints (13) are the binary variables constraints.

3.4. Convexity of the problem

P_0 is a Mixed Integer Non-Linear Programming (MINLP) owing to the objective function, constraints (3) and (13). In addition, binary variables are involved only in linear constraints. If P_0 is convex i.e. the objective function is concave in the continuous and relaxed binary variables and constraints (3) are convex, then the production planning and prices' policy that maximize the profit function in Eq. (2) can be returned efficiently by MINLP approaches like generalized benders decomposition [56,57], branch and bound [58] and outer approximation [59]. However, in general, the objective function as defined in (2) is not concave (Example 1), which implies that P_0 is a non-convex MINLP. As a result, classical MINLP approaches do not guarantee to reach the global optimum of P_0 . In the next section, an equivalent reformulation which transforms P_0 into a convex MINLP in the demand space is presented.

Example 1. In the case where there is one period and two channels i.e. ($H = \{1\}$ and $M = \{1, 2\}$) with the MNL attraction function. Production, holding, and setup costs are fixed to 1 and $\tau_1 = 5$. Attraction functions are given as follows: $f_{11}(p_{11}) = e^{2-p_{11}}$ and $f_{21}(p_{21}) = e^{1-p_{21}}$. Prices bounds are $0 \leq p_{11} \leq 30$ and $0 \leq p_{21} \leq 30$.

The objective function is equal to:

$$\pi = p_{11} D_{11} + p_{21} D_{21} - X_1 - I_1 - Y_1 \quad (14)$$

After replacing the demand by its expression from Eq. (1), π is equal to:

$$\pi = \frac{5(p_{11} e^{2-p_{11}} + p_{21} e^{1-p_{21}})}{1 + e^{2-p_{11}} + e^{1-p_{21}}} - X_1 - I_1 - Y_1 \quad (15)$$

$$\pi = R(p_{11}, p_{21}) + Costs(X_1, I_1, Y_1) \quad (16)$$

The function $Costs(X_1, I_1, Y_1)$ is concave in variables X_1, I_1 and the relaxed Y_1 since it is the sum of linear terms. However, the revenue function $R(p_{11}, p_{21})$ is not concave in prices space (see [Appendix for proof](#)). As a result, π is not concave in the continuous and relaxed binary variables.

4. Mathematical reformulation

In this section, the equivalent model of P_0 denoted P_1 will be presented. This model is called the *Demand – based formulation model* and its convexity will be proved based on properties of the attraction functions.

4.1. Demand-based formulation model

As the initial model P_0 is a non-convex problem in prices space, a new equivalent model P_1 in the demand space is presented, i.e. decisions variables are demands and prices are a function of demands. This equivalent reformulation is possible since functions f_{mt} are strictly decreasing.

First, from constraints (3), a new constraints is defined as follows:

$$\tau_t = D_{0t} + \sum_{m \in M} D_{mt}, \quad t \in H \quad (17)$$

As the demand D_{mt} captures the number of consumers that decide to purchase from the channel m during period t . The new variable D_{0t} corresponds to the no-purchase option i.e. the number of consumers that decide to not purchase from any channel during period t and it is equal to:

$$D_{0t} = \tau_t - \sum_{m \in M} D_{mt} = \tau_t - \tau_t \sum_{m \in M} \frac{f_{mt}(p_{mt})}{1 + \sum_{m' \in M} f_{m't}(p_{m't})} \quad (18)$$

$$D_{0t} = \frac{\tau_t + \tau_t \sum_{m \in M} f_{mt}(p_{mt}) - \tau_t \sum_{m' \in M} f_{m't}(p_{m't})}{1 + \sum_{m' \in M} f_{m't}(p_{m't})} \quad (19)$$

Proposition 1. By considering constraints (17) and (18), the model P_0 can be formulated and is equivalent to the following *MINLP* noted P_1 .

$$\pi_{dem} = \sum_{t=1}^T \sum_{m \in M} D_{mt} g_{mt} \left(\frac{D_{mt}}{D_{0t}} \right) - \sum_{t=1}^T c_t X_t + h_t I_t + a_t Y_t \quad (20)$$

such that:

$$D_{0t} + \sum_{m \in M} D_{mt} = \tau_t, \quad \forall t \in H \quad (21)$$

$$D_{mt} \geq \alpha^m \left(\sum_{m' \in M} D_{m't} \right), \quad \forall t \in H, m \in M \quad (22)$$

$$X_t \leq b_t, \quad \forall t \in H \quad (23)$$

$$\sum_{m \in M} D_{mt} + I_t - I_{t-1} - X_t = 0, \quad t \in \{2, \dots, T-1\} \quad (24)$$

$$\sum_{m \in M} D_{m1} + I_1 - X_1 = 0 \quad (25)$$

$$\sum_{m \in M} D_{mT} - I_{T-1} - X_T = 0 \quad (26)$$

$$X_t - S.Y_t \leq 0, \quad \forall t \in H \quad (27)$$

$$f_{mt}(\overline{p_{mt}}) \left(\tau_t - \sum_{m' \in M} D_{m't} \right) \leq D_{mt}, \quad \forall t \in H, m \in M \quad (28)$$

$$D_{mt} \leq f_{mt}(\underline{p_{mt}}) \left(\tau_t - \sum_{m' \in M} D_{m't} \right), \quad \forall t \in H, m \in M \quad (29)$$

$$X_t, I_t \geq 0, \quad \forall t \in H \quad (30)$$

$$D_{mt} \geq 0, \quad \forall t \in H, m \in M \quad (31)$$

$$D_{0t} \geq 0, \quad \forall t \in H \quad (32)$$

$$Y_t \in \{0, 1\}, \quad \forall t \in H \quad (33)$$

Proof. The objective behind the reformulation is to explore the demand D_{mt} and the new variable D_{0t} as defined respectively in (1) and (18) to replace prices by demands. Since constraints ((4)–(9), (11), and (13)) do not incorporate prices variables, they remain the same and they are expressed in P_1 by constraints ((22)–(27), (30), and (33)). Constraints (31) and (32) ensure the positivity of D_{mt} and D_{0t} , respectively. Finally, constraints (21) correspond to (17).

Now, the relationship between prices and demands variables is considered. From Eq. (19), $D_{0t} > 0$, as a result:

$$\frac{D_{mt}}{D_{0t}} = f_{mt}(p_{mt}) \quad (34)$$

As mentioned before, f_{mt} can take the *MNL*, *MCI* and linear form with $b_{mt} < 0$. Under these assumptions, f_{mt} is strictly decreasing. This means that f_{mt} is invertible and the expression of p_{mt} is given as follows:

$$p_{mt} = f_{mt}^{-1} \left(\frac{D_{mt}}{D_{0t}} \right) = g_{mt} \left(\frac{D_{mt}}{D_{0t}} \right) \quad (35)$$

A direct outcome from Eq. (35) is that the objective function defined in (2) will be equal in the demand space to:

$$\max \pi_{dem} = \sum_{t=1}^T \sum_{m \in M} D_{mt} g_{mt} \left(\frac{D_{mt}}{D_{0t}} \right) - \sum_{t=1}^T c_t X_t + h_t I_t + a_t Y_t \quad (36)$$

Now, prices bounds constraints are considered. They are expressed as:

$$\underline{p_{mt}} \leq p_{mt} \leq \overline{p_m} \quad (37)$$

Since functions f_{mt} are strictly decreasing:

$$f_{mt}(\overline{p_{mt}}) \leq f_{mt}(p_{mt}) \leq f_{mt}(\underline{p_{mt}}) \quad (38)$$

As $1 + \sum_{m' \in M} f_{m't}(p_{m't}) > 0$ and $\tau_t > 0$:

$$\frac{\tau_t f_{mt}(\overline{p_{mt}})}{1 + \sum_{m' \in M} f_{m't}(p_{m't})} \leq \frac{\tau_t f_{mt}(p_{mt})}{1 + \sum_{m' \in M} f_{m't}(p_{m't})} \leq \frac{\tau_t f_{mt}(\underline{p_{mt}})}{1 + \sum_{m' \in M} f_{m't}(p_{m't})} \quad (39)$$

As a result, prices bounds constraints in P_0 are expressed in the demand space as follows:

$$f_{mt}(\overline{p_{mt}}) D_{0t} \leq D_{mt} \leq f_{mt}(\underline{p_{mt}}) D_{0t} \quad (40)$$

Finally, by replacing D_{0t} from its expression in (18), prices bounds constraints are expressed as (28) and (29).

4.1.1. Convexity of the demand-based reformulation model

The problem P_1 is a *MINLP* since the objective function (20) is nonlinear and constraints (27) include binary variables. As for P_0 , the convexity of P_1 is considered. Since P_1 is a maximization problem and all its constraints are linear, its convexity depends only on the concavity of the objective function (20) in the continuous and relaxed binary variables.

Proposition 2. The objective function as defined by Eq. (20) is concave in the continuous and relaxed binary variables and P_1 is a convex *MINLP*.

Proof. From Eq. (36), the objective function of P_1 is equal to $\pi_{dem} = R + \text{Costs}$ such that:

$$R = \sum_{t=1}^T \sum_{m \in M} R_{mt}(D_{0t}, D_{mt}) \quad (41)$$

$$\text{with : } R_{mt}(D_{0t}, D_{mt}) = D_{mt} g_{mt} \left(\frac{D_{mt}}{D_{0t}} \right)$$

and:

$$\text{Costs} = - \sum_{t=1}^T c_t X_t + h_t I_t + a_t Y_t \quad (42)$$

To prove the concavity of π_{dem} , one can prove the concavity of R and Costs separately. As Costs is the sum of linear terms, it is concave in variables X_t , I_t and the relaxed Y_t . Now, the first double sum R which corresponds to the total revenue is considered. One way to prove its concavity is by proving that each term $R_{mt}(D_{0t}, D_{mt})$ in the sum is concave. To do that, it is sufficient to show that the Hessian matrix $H_{R_{mt}}$ of each $R_{mt}(D_{0t}, D_{mt})$ is semi-definite negative. i.e the quadratic form $V^T H_{R_{mt}} V \leq 0$ for all V in $\mathbb{R}^2$. In the following, we demonstrate that $H_{R_{mt}}$ is semi-definite negative for the *MNL*, *MCI*, and *linear* cases. This means that each term $R_{mt}(D_{0t}, D_{mt})$ is concave.

• **Case 1: MNL attraction function**

From Table 2, the expression of f_{mt} in the *MNL* case is $f_{mt}(p_{mt}) = e^{a_{mt} + b_{mt} p_{mt}}$. By replacing f_{mt} by its *MNL* expression, the equality in (34) becomes:

$$\frac{D_{mt}}{D_{0t}} = e^{a_{mt} + b_{mt} p_{mt}} \quad (43)$$

As stated before, f_{mt} is invertible and the expression of p_{mt} in the *MNL* case is equal to:

$$p_{mt} = \frac{1}{b_{mt}} \left[\log \left(\frac{D_{mt}}{D_{0t}} \right) - a_{mt} \right] = g_{mt} \left(\frac{D_{mt}}{D_{0t}} \right) \quad (44)$$

As a result, R_{mt} is expressed as:

$$R_{mt}(D_{0t}, D_{mt}) = \frac{D_{mt}}{b_{mt}} \left[\log \left(\frac{D_{mt}}{D_{0t}} \right) - a_{mt} \right] \quad (45)$$

To compute $H_{R_{mt}}$, it is necessary to compute at first the gradient ∇R_{mt} which is equal to:

$$\nabla R_{mt} = \frac{1}{b_{mt}} \left(\frac{-D_{mt}}{D_{0t}}, \log \left(\frac{D_{mt}}{D_{0t}} \right) - a_{mt} + 1 \right) \quad (46)$$

$H_{R_{mt}}$ is computed from ∇R_{mt} and it has the following expression:

$$H_{R_{mt}} = \frac{D_{0t}}{D_{mt} b_{mt}} \begin{pmatrix} \frac{D_{mt}^2}{D_{0t}^3} & \frac{-D_{mt}}{D_{0t}^2} \\ \frac{-D_{mt}}{D_{0t}^2} & \frac{1}{D_{0t}} \end{pmatrix} \quad (47)$$

Let $V = (x, y)^T \in \mathbb{R}^2$:

$$V^T H_{R_{mt}} V = \frac{D_{0t}}{D_{mt} b_{mt}} (x \ y) \begin{pmatrix} \frac{D_{mt}^2}{D_{0t}^3} & \frac{-D_{mt}}{D_{0t}^2} \\ \frac{-D_{mt}}{D_{0t}^2} & \frac{1}{D_{0t}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (48)$$

$$= \frac{D_{0t}}{D_{mt} b_{mt}} \left(x^2 \frac{D_{mt}^2}{D_{0t}^3} - 2xy \frac{D_{mt}}{D_{0t}^2} + \frac{y^2}{D_{0t}} \right) \\ = \frac{1}{D_{mt} b_{mt}} \left(x \frac{D_{mt}}{D_{0t}} - y \right)^2 \quad (49)$$

Since $D_{mt} \geq 0$, $b_{mt} < 0$, and $\left(x \frac{D_{mt}}{D_{0t}} - y \right)^2 \geq 0$, the quadratic form $V^T H_{R_{mt}} V \leq 0$, $\forall V \in \mathbb{R}^2$. Consequently, $H_{R_{mt}}$ is semi-definite negative and R_{mt} is concave for the *MNL* attraction function.

• **Case 2: MCI attraction function**

After replacing the attraction function f_{mt} by its *MCI* expression in Table 2, the equality in (34) is expressed as:

$$\frac{D_{mt}}{D_{0t}} = a_{mt} (p_{mt})^{b_{mt}} \quad (50)$$

Since f_{mt} is invertible:

$$p_{mt} = \left(\frac{D_{mt}}{a_{mt} D_{0t}} \right)^{\frac{1}{b_{mt}}} = g_{mt} \left(\frac{D_{mt}}{D_{0t}} \right) \quad (51)$$

As a result:

$$R_{mt}(D_{0t}, D_{mt}) = D_{mt} \left(\frac{D_{mt}}{a_{mt} D_{0t}} \right)^{\frac{1}{b_{mt}}} \quad (52)$$

As for the *MNL* case, the gradient ∇R_{mt} is computed initially. Then, the Hessian matrix $H_{R_{mt}}$ is computed. They are given by the two following equations.

$$\nabla R_{mt} = \left(\frac{-a_{mt}}{b_{mt}} \left(\frac{D_{mt}}{a_{mt} D_{0t}} \right)^{\frac{1}{b_{mt}} + 1}, \left(1 + \frac{1}{b_{mt}} \right) \left(\frac{D_{mt}}{a_{mt} D_{0t}} \right)^{\frac{1}{b_{mt}}} \right) \quad (53)$$

and:

$$H_{R_{mt}} = \frac{a_{mt}}{b_{mt}} \frac{D_{0t}^2}{D_{mt}^2} \left(1 + \frac{1}{b_{mt}} \right)^{\frac{1}{b_{mt}}} \begin{pmatrix} \frac{D_{mt}^2}{D_{0t}^3} & \frac{-D_{mt}}{D_{0t}^2} \\ \frac{-D_{mt}}{D_{0t}^2} & \frac{1}{D_{0t}} \end{pmatrix} \quad (54)$$

Let $V = (x, y)^T \in \mathbb{R}^2$. The quadratic form $V^T H_{R_{mt}} V$ is equal to:

$$V^T H_{R_{mt}} V = \frac{a_{mt}}{b_{mt}} \frac{D_{0t}^2}{D_{mt}^2} \left(1 + \frac{1}{b_{mt}} \right)^{\frac{1}{b_{mt}}} (x \ y) \begin{pmatrix} \frac{D_{mt}^2}{D_{0t}^3} & \frac{-D_{mt}}{D_{0t}^2} \\ \frac{-D_{mt}}{D_{0t}^2} & \frac{1}{D_{0t}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (55)$$

$$= \frac{a_{mt}}{b_{mt}} \frac{D_{0t}^2}{D_{mt}^2} \left(1 + \frac{1}{b_{mt}} \right)^{\frac{1}{b_{mt}}} \left(x^2 \frac{D_{mt}^2}{D_{0t}^3} - 2xy \frac{D_{mt}}{D_{0t}^2} + \frac{y^2}{D_{0t}} \right) \\ = \frac{a_{mt}}{b_{mt}} \frac{D_{0t}^2}{D_{mt}^2} \left(1 + \frac{1}{b_{mt}} \right)^{\frac{1}{b_{mt}}} \left(x \frac{D_{mt}}{D_{0t}} - y \right)^2 \quad (56)$$

Terms $\left(1 + \frac{1}{b_{mt}} \right)^2$, a_{mt} , $\frac{D_{0t}}{D_{mt}^2}$, and $\left(x \frac{D_{mt}}{D_{0t}} - y \right)^2$ are positive. However, the term $\frac{1}{b_{mt}} < 0$ which means that $V^T H_{R_{mt}} V \leq 0$, $\forall V \in \mathbb{R}^2$. As a result, $H_{R_{mt}}$ is semi-definite negative and R_{mt} is concave.

• **Case 3: Linear attraction function**

With the linear attraction function f_{mt} is equal to $a_{mt} + b_{mt} p_{mt}$. As a consequence, the equality (34) is expressed as:

$$\frac{D_{mt}}{D_{0t}} = a_{mt} + b_{mt} p_{mt} \quad (57)$$

The expressions of g_{mt} and R_{mt} are given as follows:

$$p_{mt} = \frac{1}{b_{mt}} \left(\frac{D_{mt}}{D_{0t}} - a_{mt} \right) = g_{mt} \left(\frac{D_{mt}}{D_{0t}} \right) \quad (58)$$

and:

$$R_{mt}(D_{0t}, D_{mt}) = \frac{D_{mt}}{b_{mt}} \left(\frac{D_{mt}}{D_{0t}} - a_{mt} \right) \quad (59)$$

The gradient ∇R_{mt} is equal to:

$$\nabla R_{mt} = \frac{1}{b_{mt}} \left(- \left(\frac{D_{mt}}{D_{0t}} \right)^2, \frac{2D_{mt}}{D_{0t}} - a_{mt} \right) \quad (60)$$

$H_{R_{mt}}$ has the following expression:

$$H_{R_{mt}} = \frac{2}{b_{mt}} \begin{pmatrix} \frac{D_{mt}^2}{D_{0t}^3} & \frac{-D_{mt}}{D_{0t}^2} \\ \frac{-D_{mt}}{D_{0t}^2} & \frac{1}{D_{0t}} \end{pmatrix} \quad (61)$$

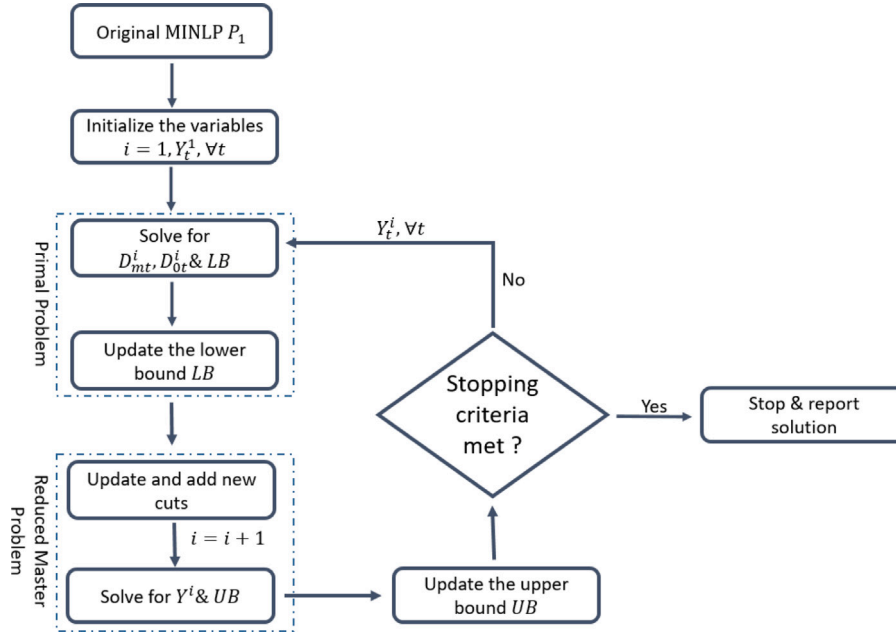


Fig. 1. Flow chart of the outer approximation approach.

Let $V = (x, y)^T \in \mathbb{R}^2$ and the quadratic form $V^T H_{R_{mt}} V$ given by:

$$V^T H_{R_{mt}} V = \frac{2}{b_{mt}} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \frac{D_{mt}^2}{D_{0t}^3} & -\frac{D_{mt}}{D_{0t}^2} \\ -\frac{D_{mt}}{D_{0t}^2} & \frac{1}{D_{0t}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (62)$$

$$= \frac{2}{b_{mt}} \left(x^2 \frac{D_{mt}^2}{D_{0t}^3} - 2xy \frac{D_{mt}}{D_{0t}^2} + \frac{y^2}{D_{0t}} \right) \quad (63)$$

$$= \frac{2}{b_{mt} D_{0t}} \left(x \frac{D_{mt}}{D_{0t}} - y \right)^2$$

From the last equation $V^T H_{R_{mt}} V \leq 0$ since $\frac{2}{b_{mt} D_{0t}} < 0$ and $\left(\frac{D_{mt}}{D_{0t}} - y \right)^2 \geq 0$. As a result, $H_{R_{mt}}$ is semi-definite negative and R_{mt} is concave for the linear attraction function.

Finally, as R_{mt} is concave for the *MNL*, *MCI*, and linear attraction functions. The total revenue function R as defined in (41) is concave. Since the objective function of P_1 on its part is equal to $\pi_{dem} = R + Costs$, it is concave. As a result, the equivalent problem P_1 is a convex *MINLP* for the three attraction functions.

5. Solution approach

This section describes the optimization approach for the resolution of P_1 . Before proceeding further, some key features of P_1 are underlined: (i) the continuous and the binary variables are separable in the objective function and constraints, (ii) the binary variables affect the objective function linearly, (iii) the revenue function is concave and (iv) all constraints are linear. Duran and Grossmann [59] compared different resolution approaches for this class of problems. They showed that the outer approximation algorithm yields a tighter formulation and has better convergence properties when compared to the Generalized Benders Decomposition algorithm. In the following, the outer approximation algorithm will be detailed to solve P_1 .

The outer approximation algorithm is an iterative process where at each iteration a non-linear programming (*NLP*) and a mixed-integer linear programming (*MILP*) sub-problems are solved. The *NLP* sub-problem is called a primal problem and it is a restriction of P_1 . The *MILP* is called a master problem and it is a relaxation of P_1 . Solutions obtained by the primal and master problems present a lower bound

(*LB*) and upper bound (*UB*) of the solution of P_1 , respectively. These two sub-problems are solved iteratively where the solution of one sub-problem is an input for solving the other. The algorithm stops when the two bounds converge and returns the optimal solution for P_1 . Duran and Grossmann [59], Fletcher and Leyffer [60] showed that the algorithm does not cycle and converges in a finite iterations number. Fig. 1 presents the flow chart of the solution approach.

5.1. Primal problem

The primal problem at iteration i is obtained by fixing the setup variables to 0 and 1 which corresponds to a specific production setups configuration $Y^i = (Y_1^i, \dots, Y_T^i)$. The resulting problem is a convex *NLP* which is a restriction of the original *MINLP*. The resolution of the primal problem gives a lower bound of the problem P_1 . At each iteration i , a new primal problem is solved by fixing the setup variables to a new configuration Y^i . The current lower bound is compared to the returned solution. If this latter gives a higher profit, the lower bound is updated. This process gives rise to a set of non-decreasing lower bounds. Next, the master problem for P_1 is formulated by considering the *MNL*, *MCI* and linear attraction functions.

5.2. Master problem

The master problem is a *MILP* obtained by the linearization of the objective function (20) in suitable continuous points for outer approximation. This linearization provides an overestimation of the original objective function of P_1 . More specifically, for all feasible setups Y^i , the related primal problem is solved and the solution returned is used as outer approximation points in the master problem. Authors in [59] showed that the resulting master problem and the problem P_1 have the same optimal solution. Instead of solving the master problem which includes a very large number of outer approximations, the reduced master problem which corresponds to the relaxation of the master problem is solved. This reduced master problem accumulates the different linear approximations of previous iterations by introducing at each iteration i a new additional cuts. These new linear cuts are added at the point that corresponds to the solution returned by the primal problem. Note that the reduced master problem is a relaxation of P_1 which means that its solution gives an upper bound for the solution of P_1 . As the execution

proceeds, more cuts are added to the reduced master problem, which tightens the formulation and generates a set of non-increasing upper bounds for P_1 . Before the presentation of the reduced master problem, some new notations are introduced:

- $D = (D_{01}, \dots, D_{0T}, D_{11}, \dots, D_{|M|1}, \dots, D_{1T}, \dots, D_{|M|T})$: The vector of lost sales for each period and all demands in each channel.
- $D^k = (D_{01}^k, \dots, D_{0T}^k, D_{11}^k, \dots, D_{|M|1}^k, \dots, D_{1T}^k, \dots, D_{|M|T}^k)$: The vector of lost sales for each period and all demands in each channel returned after resolving the primal problem of iteration k in outer approximation.

The reduced master problem at iteration i is formulated as follows:

$$\max \pi_{dem}^i = \eta - \sum_{t=1}^T c_t X_t + h_t I_t + a_t Y_t \quad (64)$$

subject to constraints (21)–(33)

And:

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} R_{mt}(D_{0t}^k, D_{mt}^k) + \nabla R_{mt}(D_{0t}^k, D_{mt}^k) \begin{pmatrix} D_{0t} - D_{0t}^k \\ D_{mt} - D_{mt}^k \end{pmatrix}; \quad (65)$$

$$\forall k \in \{1, 2, \dots, i-1\}$$

Eq. (64) is the objective function of the reduced master problem of iteration i . Constraints (65) correspond to a collection of hyperplanes defined from the primal solutions of iterations $\{1, \dots, i-1\}$. The right hand of these constraints is the linearization of the revenue function defined by Eq. (41) at the point D^k that corresponds to k th primal problem solution. This linearization is equal to the total revenue for $D = D^k$ and is greater than the total revenue for $D \neq D^k$. Note that the term η can be considered as an upper bound of the total revenue function. The rest of this sections details the constraints (65) for the *MNL*, *MCI* and *linear* attraction functions.

• Case 1: *MNL* attraction function

With the *MNL* attraction function, $R_{mt}(D_{0t}, D_{mt})$ and ∇R_{mt} are defined by Eqs. (45) and (46) respectively. As a results, constraints (65) are explained as follows:

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} \frac{D_{mt}^k}{b_{mt}} \left[\log \left(\frac{D_{mt}^k}{D_{0t}^k} \right) - a_{mt} \right] + \frac{(D_{0t} - D_{0t}^k)}{b_{mt}} \frac{D_{mt}^k}{D_{0t}^k} + \frac{(D_{mt} - D_{mt}^k)}{b_{mt}} \left[\log \left(\frac{D_{mt}^k}{D_{0t}^k} \right) - a_{mt} + 1 \right] \quad (66)$$

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} \frac{(D_{0t}^k - D_{0t})}{b_{mt}} \frac{D_{mt}^k}{D_{0t}^k} + \frac{D_{mt}}{b_{mt}} \left[\log \left(\frac{D_{mt}^k}{D_{0t}^k} \right) - a_{mt} + 1 \right] - \frac{D_{mt}^k}{b_{mt}}$$

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} \frac{1}{b_{mt}} \left[(D_{0t}^k - D_{0t}) \frac{D_{mt}^k}{D_{0t}^k} + D_{mt} \left[\log \left(\frac{D_{mt}^k}{D_{0t}^k} \right) - a_{mt} + 1 \right] - D_{mt}^k \right] \quad (67)$$

• Case 2: *MCI* attraction function

From Eqs. (52) and (53), constraints (65) with the *MCI* attraction function are equal to:

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} D_{mt}^k \left(\frac{D_{mt}^k}{a_{mt} D_{0t}^k} \right)^{\frac{1}{b_{mt}}} - \frac{a_{mt}}{b_{mt}} \left(\frac{D_{mt}^k}{a_{mt} D_{0t}^k} \right)^{\frac{1}{b_{mt}}+1} (D_{0t} - D_{0t}^k) + \left(1 + \frac{1}{b_{mt}} \right) \left(\frac{D_{mt}^k}{a_{mt} D_{0t}^k} \right)^{\frac{1}{b_{mt}}} (D_{mt} - D_{mt}^k)$$

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} - \frac{a_{mt}}{b_{mt}} \left(\frac{D_{mt}^k}{a_{mt} D_{0t}^k} \right)^{\frac{1}{b_{mt}}+1} (D_{0t} - D_{0t}^k) + \left(1 + \frac{1}{b_{mt}} \right) \left(\frac{D_{mt}^k}{a_{mt} D_{0t}^k} \right)^{\frac{1}{b_{mt}}} D_{mt}$$

$$- \frac{D_{mt}^k}{b_{mt}} \left(\frac{D_{mt}^k}{a_{mt} D_{0t}^k} \right)^{\frac{1}{b_{mt}}} \quad (68)$$

• Case 3: *Linear* attraction function

According to Eqs. (58) and (60), constraints (65) under the *linear* attraction function is equal to:

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} \frac{D_{mt}^k}{b_{mt}} \left(\frac{D_{mt}^k}{D_{0t}^k} - a_{mt} \right) - \frac{1}{b_{mt}} \left(\frac{D_{mt}^k}{D_{0t}^k} \right)^2 (D_{0t} - D_{0t}^k) + \frac{1}{b_{mt}} \left[\frac{2D_{mt}^k}{D_{0t}^k} - a_{mt} \right] (D_{mt} - D_{mt}^k) \quad (69)$$

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} \frac{D_{mt}^k}{b_{mt}} \left(\frac{D_{mt}^k}{D_{0t}^k} - a_{mt} \right) - \frac{1}{b_{mt}} \left(\frac{D_{mt}^k}{D_{0t}^k} \right)^2 (D_{0t} - D_{0t}^k) + \frac{1}{b_{mt}} \left[\frac{2D_{mt}^k}{D_{0t}^k} - a_{mt} \right] D_{mt} - \frac{D_{mt}^k}{b_{mt}} \left[\frac{D_{mt}^k}{D_{0t}^k} - a_{mt} \right] - \frac{D_{mt}^k}{b_{mt}} \left[\frac{D_{mt}^k}{D_{0t}^k} \right]$$

$$\eta \leq \sum_{t=1}^T \sum_{m \in M} \frac{1}{b_{mt}} \left[- \left(\frac{D_{mt}^k}{D_{0t}^k} \right)^2 (D_{0t} - D_{0t}^k) + \left[\frac{2D_{mt}^k}{D_{0t}^k} - a_{mt} \right] D_{mt} - D_{mt}^k \left[\frac{D_{mt}^k}{D_{0t}^k} \right] \right] \quad (70)$$

6. Experimental study

In this section, the numerical tests carried out to evaluate the proposed approach are presented. Models P_0 and P_1 are implemented on **Pyomo** which is a Python-based open-source software package that supports a diverse set of optimization capabilities for formulating, solving, and analyzing optimization models [61,62]. They are then solved using the **MindtPy** toolbox which uses the decomposition methods to solve the convex and non-convex *MINLP* problems. All numerical tests are performed by setting the strategy to ‘**OA**’ which corresponds to the **Outer Approximation** algorithm. A computer with an Intel(R) Core(TM) i7-10810U CPU, 1.1 GHz processor, a RAM of 16 Go and Ubuntu 20.04 environment was used for all the conducted numerical tests.

6.1. Benchmark design

To analyze the performance of the proposed approach, a benchmark composed of different instances inspired from the literature is addressed. As the studied problem considers the pricing and lot-sizing, instances data can be divided into two sets called *Logistics data* and *Market data*. The first characterizes the production system (production capacity, holding costs, etc.). The second set describes the market (market length, demand parameter, etc.).

The *Logistic data* are generated based on the same pattern as in [63]. Two parameters denoted sc and cap are used for the setup costs and production capacity, respectively. First, they are fixed to ($sc = 100, cap = 250$) and ($sc = 900, cap = 250$). Then, for each tuple (sc, cap), the setup cost a_t and the production capacity b_t are generated uniformly from $[0.5sc, 1.5sc]$ and $[0.5cap, 1.5cap]$, respectively. Finally, the production cost c_t and the holding cost h_t , are sampled uniformly from $[4, 6]$ and $[0.5, 1.5]$, respectively.

The *Market data* contains prices bounds p_{mt} and $\overline{p_{mt}}$, the market length τ_t and the attraction function parameters a_{mt} and b_{mt} . Prices bounds are distributed uniformly from $[15, 50]$ and $[75, 110]$ while the market length follows the distribution $U[0.5b_t, 1.5b_t]$. Regarding the attraction function f_{mt} , only the *MNL* model is considered in our tests. Parameters a_{mt} and b_{mt} are generated using the same method as in [64]. The *MNL* demand is a discrete choice model based on the utility theory. The utility derived from each choice (channel in our case) is equal to $U_{mt} = V_{mt} + \epsilon_{mt}$ and the utility of no-purchase option is given by $U_{0t} = \epsilon_{0t}$. V_{mt} is deterministic and depends on factors that characterize channels. The current study considers only the price

Table 3Numerical results on ($T = 32, sc = 100, cap = 250$) instances.

$ M = 2$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 4$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	305086	1	330987	1	-8.00%	Instance 1	372142	2	375539	3	-1.00%
Instance 2	285771	1	316603	1	-10.00%	Instance 2	361838	2	372299	3	-3.00%
Instance 3	261304	2	304085	7	-14.00%	Instance 3	347839	2	361058	3	-4.00%
Instance 4	294293	1	317653	2	-7.00%	Instance 4	337310	2	346349	3	-3.00%
Instance 5	274557	1	303327	2	-9.00%	Instance 5	374098	2	376504	4	-1.00%
Instance 6	316201	1	346164	2	-9.00%	Instance 6	370097	2	381770	2	-3.00%
Instance 7	310014	1	339471	1	-9.00%	Instance 7	343385	2	355912	6	-4.00%
Instance 8	316100	1	356936	2	-11.00%	Instance 8	357634	2	375281	2	-5.00%
Instance 9	302690	1	336810	1	-10.00%	Instance 9	353524	2	365618	2	-3.00%
Instance 10	325646	2	341922	1	-5.00%	Instance 10	346712	2	354148	3	-2.00%
avg	-	1	-	2	-9.20%	avg	-	2	-	3	-2.90%
$ M = 3$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 5$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	364691	2	380394	2	-4.00%	Instance 1	348597	4	348684	28	-0.03%
Instance 2	352236	2	363747	2	-3.00%	Instance 2	374840	3	374840	3	0.00%
Instance 3	374995	3	385318	2	-3.00%	Instance 3	363887	3	363887	2	-0.01%
Instance 4	370600	2	381203	2	-3.00%	Instance 4	384405	3	384405	3	0.00%
Instance 5	349593	1	361644	2	-3.00%	Instance 5	335912	3	335942	5	0.00%
Instance 6	344797	3	350061	2	-2.00%	Instance 6	334912	3	334912	3	0.00%
Instance 7	335081	2	341702	2	-2.00%	Instance 7	375258	3	375258	2	0.00%
Instance 8	333358	1	343843	2	-3.00%	Instance 8	364533	3	364533	3	0.00%
Instance 9	326040	1	336022	2	-3.00%	Instance 9	367277	3	367277	3	0.00%
Instance 10	308245	1	327280	2	-6.00%	Instance 10	343515	3	343515	3	0.00%
avg	-	2	-	2	-3.20%	avg	-	3	-	6	0.00%

Table 4Numerical results on ($T = 32, sc = 900, cap = 250$) instances.

$ M = 2$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 4$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	249172	1	324884	2	-23.00%	Instance 1	270765	2	343075	3	-21.00%
Instance 2	162364	1	301378	2	-46.00%	Instance 2	263752	2	334364	6	-21.00%
Instance 3	174905	1	297069	2	-41.00%	Instance 3	279105	2	344390	4	-19.00%
Instance 4	190378	1	294252	3	-35.00%	Instance 4	249716	2	338725	5	-26.00%
Instance 5	204101	1	313906	3	-35.00%	Instance 5	369232	2	370633	3	0.00%
Instance 6	178633	1	322473	5	-45.00%	Instance 6	237959	2	333639	5	-29.00%
Instance 7	203989	1	325821	4	-37.00%	Instance 7	359644	3	363387	3	-1.00%
Instance 8	221327	1	299897	2	-26.00%	Instance 8	342621	3	345361	4	-1.00%
Instance 9	206146	1	305447	2	-33.00%	Instance 9	249594	3	351732	21	-29.00%
Instance 10	187358	1	287847	3	-35.00%	Instance 10	307738	2	366544	4	-16.00%
avg	-	1	-	3	-35.60%	avg	-	2	-	6	-16.30%
$ M = 3$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 5$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	260049	2	332441	6	-22.00%	Instance 1	334778	4	336511	6	-1.00%
Instance 2	209982	2	309235	10	-32.00%	Instance 2	332817	5	335679	16	-1.00%
Instance 3	253969	2	305414	3	-17.00%	Instance 3	259534	2	331221	5	-22.00%
Instance 4	238682	2	315874	3	-24.00%	Instance 4	288252	2	365298	8	-21.00%
Instance 5	241513	1	342190	4	-29.00%	Instance 5	290121	2	336472	8	-14.00%
Instance 6	247838	2	339157	2	-27.00%	Instance 6	318507	4	319658	7	0.00%
Instance 7	241507	2	304865	2	-21.00%	Instance 7	326112	3	327557	5	0.00%
Instance 8	245314	2	316059	4	-22.00%	Instance 8	264882	2	325959	7	-19.00%
Instance 9	220296	2	318034	3	-31.00%	Instance 9	299873	2	352025	4	-15.00%
Instance 10	236870	1	334349	3	-29.00%	Instance 10	279009	2	332358	6	-16.00%
avg	-	2	-	4	-25.40%	avg	-	3	-	7	-10.90%

as factor. The random components ϵ_{mt} are independent and follow a standard Gumbel distribution. The linear form of $V_{mt}(p_{mt}) = a_{mt} + b_{mt}p_{mt}$ is used to generate parameters a_{mt} and b_{mt} . For each channel, at each period, $V_{mt}(p_{mt})$ and $V_{mt}(\overline{p_{mt}})$ are chosen uniformly from $[2\sigma, 4\sigma]$ and $[-4\sigma, -2\sigma]$, respectively, with $\sigma = \frac{\pi}{\sqrt{6}}$ is the standard deviation of the standard Gumbel distribution. Then, a_{mt} and b_{mt} are computed by solving the linear system $V(p_{mt}) = a_{mt} + b_{mt}p_{mt}$ and $V(\overline{p_{mt}}) = a_{mt} + b_{mt}\overline{p_{mt}}$.

The last generated data are the number of periods and channels denoted T and $|M|$, respectively. Two values for the former, $T \in \{32, 52\}$ and for values for the latter, $|M| \in \{2, 3, 4, 5\}$ are considered. For the periods number, if one period presents a week, then $T = 32$ presents a production plan for about two-thirds of a year and $T = 52$ corresponds to a production plan of one year.

For each combination (T, M, sc, cap), 10 instances are generated. As a result, the total number of the generated instances is $2 \times 4 \times 2 \times 1 \times 10 = 160$ which are grouped according to (T, M, sc, cap).

6.2. Numerical results

The objective of the conducted tests is to compare the performances of the initial model P_0 and the reformulation P_1 solved with the outer approximation approach. To do that, the *gap* (Eq. (71)) is defined in percentage and corresponds to the relative difference between the objective functions returned by P_0 and P_1 which are denoted P_{0-obj} and P_{1-obj} , respectively. In addition to the objective functions and the *gap*, columns T_{P_0} and T_{P_1} are given in seconds and correspond to the computational times of P_0 and P_1 , respectively. The following tables show the obtained results on all the generated instances.

$$gap = \frac{P_{0-obj} - P_{1-obj}}{P_{1-obj}} \quad (71)$$

Tables 3 and 4 correspond to the results on instances with $T = 32$ periods. Analyzing the column *gap* of the Table 3, one can observe that

Table 5Numerical results on ($T = 52, sc = 100, cap = 250$) instances.

$ M = 2$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 4$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	471764	4	530838	24	-11.00%	Instance 1	602505	3	617514	6	-2.00%
Instance 2	479695	2	549108	3	-13.00%	Instance 2	559195	3	579916	20	-4.00%
Instance 3	512447	2	562103	3	-9.00%	Instance 3	622561	3	654545	11	-5.00%
Instance 4	513710	2	539619	2	-5.00%	Instance 4	581711	4	586020	6	-1.00%
Instance 5	549825	2	586787	3	-6.00%	Instance 5	571469	6	577361	20	-1.00%
Instance 6	505346	2	534604	3	-5.00%	Instance 6	607316	3	616771	4	-2.00%
Instance 7	488204	2	527788	8	-7.00%	Instance 7	580938	3	594650	6	-2.00%
Instance 8	489662	2	517900	2	-5.00%	Instance 8	590561	3	610254	6	-3.00%
Instance 9	487934	4	532622	17	-8.00%	Instance 9	607391	4	618594	5	-2.00%
Instance 10	556446	1	581355	2	-4.00%	Instance 10	571657	3	591453	8	-3.00%
avg	-	2	-	7	-7.30%	avg	-	4	-	9	-2.50%
$ M = 3$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 5$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	566271	2	584832	3	-3.00%	Instance 1	630315	5	630315	5	0.00%
Instance 2	562249	4	571335	17	-2.00%	Instance 2	595898	5	595898	4	0.00%
Instance 3	618553	2	629727	5	-2.00%	Instance 3	572711	5	572711	15	0.00%
Instance 4	576050	2	584381	4	-1.00%	Instance 4	636065	5	636065	6	0.00%
Instance 5	553291	3	557045	17	-1.00%	Instance 5	626996	4	626996	4	0.00%
Instance 6	559286	6	575960	14	-3.00%	Instance 6	651608	5	651608	4	0.00%
Instance 7	554967	7	578483	4	-4.00%	Instance 7	585876	5	588069	5	-0.37%
Instance 8	579977	3	587200	3	-1.00%	Instance 8	631693	4	631693	5	0.00%
Instance 9	553245	6	568215	9	-3.00%	Instance 9	591798	5	591842	12	-0.01%
Instance 10	519875	4	562037	9	-8.00%	Instance 10	645248	5	645248	4	0.00%
avg	-	4	-	9	-2.80%	avg	-	5	-	7	-0.04%

Table 6Numerical results on ($T = 52, sc = 900, cap = 250$) instances.

$ M = 2$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 4$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	309364	6	503176	52	-39.00%	Instance 1	519358	3	571551	7	-9.00%
Instance 2	277187	4	483479	9	-43.00%	Instance 2	519676	3	608177	8	-15.00%
Instance 3	355134	2	493090	9	-28.00%	Instance 3	551773	14	555934	17	-1.00%
Instance 4	319336	5	520540	24	-39.00%	Instance 4	544089	3	612008	6	-11.00%
Instance 5	354132	2	539637	7	-34.00%	Instance 5	541745	3	589632	7	-8.00%
Instance 6	393759	5	533361	6	-26.00%	Instance 6	483295	3	584406	12	-17.00%
Instance 7	317209	6	523901	67	-39.00%	Instance 7	408713	5	540253	24	-24.00%
Instance 8	314462	21	505898	60	-38.00%	Instance 8	481432	3	558620	7	-14.00%
Instance 9	349400	3	516430	17	-32.00%	Instance 9	475656	4	592869	6	-20.00%
Instance 10	315060	3	505404	22	-38.00%	Instance 10	431037	67	545903	18	-21.00%
avg	-	6	-	27	-35.60%	avg	-	11	-	11	-14.00%
$ M = 3$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)	$ M = 5$	P_{0-obj}	T_{P_0} (s)	P_{1-obj}	T_{P_1} (s)	gap (%)
Instance 1	427232	4	531924	9	-20.00%	Instance 1	542357	47	543320	108	0.00%
Instance 2	389882	6	533053	44	-27.00%	Instance 2	447325	11	538249	37	-17.00%
Instance 3	382277	5	532254	25	-28.00%	Instance 3	571623	19	578848	43	-1.00%
Instance 4	463323	3	564398	7	-18.00%	Instance 4	545823	51	547869	27	0.00%
Instance 5	433338	5	554791	10	-22.00%	Instance 5	570057	5	571089	8	0.00%
Instance 6	460374	3	563251	7	-18.00%	Instance 6	540363	7	543446	43	-1.00%
Instance 7	429109	3	566247	7	-24.00%	Instance 7	551664	33	555445	45	-1.00%
Instance 8	413027	5	516455	9	-20.00%	Instance 8	511013	4	574467	14	-11.00%
Instance 9	459004	3	556806	9	-18.00%	Instance 9	469690	10	555175	27	-15.00%
Instance 10	407670	11	545605	10	-25.00%	Instance 10	490978	22	585177	30	-16.00%
avg	-	5	-	14	-22.00%	avg	-	26	-	37	-6.20%

P_1 solved to optimality with outer approximation reaches better results compared to the initial model P_0 . However, the same column shows that for some instances, the $gap = 0\%$, which means that the optimal solution is returned for these instances by P_0 even if, in general P_0 cannot theoretically guarantee the convergence to the optimal solution since it is a non-convex *MINLP*. Now, looking to the *gap* values in Table 4, the same observation can be made regarding the two models. Results obtained by P_1 which returns the optimal solutions are better than the ones returned by P_0 on all instances with ($T = 32, sc = 900$). The *gap* between these two models increases to reach in average -35.60%, -25.40%, -16.30% and -10.90% on instances with $|M| = 2$, $|M| = 3$, $|M| = 4$ and $|M| = 5$, respectively. Finally, regarding the computational times, both P_0 and P_1 take less than 30 s on all instances with $T = 32$ periods.

Tables 5 and 6 show the results on instances with $T = 52$ periods. As in Tables 3 and 6, the *gap* values in Tables 5 and 6 show the superiority

of the model P_1 which returns the optimal solution compared to the model P_0 . Except for some instances that belong to the groups ($T = 52, M = 5, sc = 100, cap = 250$) and ($T = 52, M = 5, sc = 900, cap = 250$) for which both P_0 and P_1 returns the optimal solution, for the rest of instances with $T = 52$ periods, the *gap* between the two models increases to reach in average -35% and -22% on ($T = 52, M = 2, sc = 900, cap = 250$) and ($T = 52, M = 3, sc = 900, cap = 250$) groups, respectively. Regarding the computational times, from Table 5, one can notice that P_0 and P_1 are solved in less than 25 s for all instances with $sc = 100$. However, computational times from Table 6 show that P_1 needs more time than P_0 on instances with $sc = 900$. Nevertheless, this computational time does not exceed 2 min.

Results in Tables 3–6 show the efficiency of the model P_1 solved to optimality with outer approximation compared to the initial model P_0 . The same tables reveal that, if one period of production planning corresponds to one week, then the best production plan and the best

pricing policy for of about one year can be obtained in one minute with the model P_1 .

7. Conclusion

The presented work studied the problem of joint pricing and lot-sizing decisions in multi-channel context. Selling through multiple sales channels is the approach that sellers have adopted with the emergence of internet to target as many customers as possible in the market.

As the first contribution, the paper used the attraction models to characterize the demand in each channel and formulated the profit maximizing problem as non-convex $MINLP$ in prices space. The use of attraction models to deal with pricing and lot-sizing decisions is rarely studied in the literature when compared to other models that characterize the demand.

As the second contribution, an equivalent and convex reformulation of the initial problem is proposed to get the optimal production plan and the pricing policy. This reformulation is denoted P_1 and its convexity has been proved based on properties of attraction models. Then, as the third contribution, an optimization method based on outer approximation is proposed to solve P_1 . The optimality of the returned solution is theoretically guaranteed since P_1 is a convex $MINLP$. Finally, the conducted numerical tests on a set of real inspired instances showed the efficiency of the proposed approach compared to the initial non-convex model. Numerical results reveal that under the assumption that one period is equal to one week, the proposed approach is able to find the production plan and the pricing policy that maximize the profit of about one year in only one minute of computational time.

As the presented study considers both pricing and lot-sizing, its extension can be viewed through two different perspectives. The first concerns the pricing side, a possible extension is the incorporation of other factors such as (lead time, advertising, etc.) in attraction functions to achieve a more accurate representation of the market. A second extension is the addition of the assumptions concerning the segmentation of the market by considering the different behaviors of consumers [6]. From the lot-sizing point of view, the problem can be extended to deal with the case of multiple products. Other hypotheses can be added, in particular the authorization of backlogging and the presence of manufacturing and re-manufacturing operations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix. Proof of non-concavity in Example 1

The revenue function $R(p_{11}, p_{21})$ in Example 1 is equal to:

$$R(p_{11}, p_{21}) = \frac{5(p_{11}e^{2-p_{11}} + p_{21}e^{1-p_{21}})}{1 + e^{2-p_{11}} + e^{1-p_{21}}} \quad (A.1)$$

To proof that $R(p_{11}, p_{21})$ is not concave in the prices space, it is sufficient to show that its Hessian matrix noted H_R is not negative semi-definite matrix for a couple (p_{11}, p_{21}) such that (p_{11}, p_{21}) satisfies the prices bounds constraints (10).

First, the gradient ∇R is calculated, which is equal to $\nabla R = \left(\frac{\partial R}{\partial p_{11}}, \frac{\partial R}{\partial p_{21}} \right)$ with:

$$\begin{aligned} \frac{\partial R}{\partial p_{11}} &= \frac{5e^{p_{21}+2} \left((p_{11} - 1)(-e^{p_{11}+p_{21}}) + e^{p_{11}+1}(-p_{11} + p_{21} + 1) + e^{p_{21}+2} \right)}{(e^{p_{11}+p_{21}} + e^{p_{11}+1} + e^{p_{21}+2})^2} \\ \frac{\partial R}{\partial p_{21}} &= \frac{5e^{p_{11}+1} \left(e^{p_{21}+2}(p_{11} - p_{21} + 1) - (p_{21} - 1)e^{p_{11}+p_{21}} + e^{p_{11}+1} \right)}{(e^{p_{11}+p_{21}} + e^{p_{11}+1} + e^{p_{21}+2})^2} \end{aligned}$$

Then, H_R is calculated from ∇R and it is equal to:

$$H_R = \begin{pmatrix} \frac{\partial^2 R}{\partial p_{11}^2} & \frac{\partial^2 R}{\partial p_{11} \partial p_{21}} \\ \frac{\partial^2 R}{\partial p_{21} \partial p_{11}} & \frac{\partial^2 R}{\partial p_{21}^2} \end{pmatrix} \quad (A.2)$$

With:

$$\begin{aligned} \frac{\partial^2 R}{\partial p_{11}^2} &= \frac{10e^{2-p_{11}}(e^{2-p_{11}} - p_{11}e^{2-p_{11}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^2} + \frac{5(-2e^{2-p_{11}} + p_{11}e^{2-p_{11}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})} \\ &+ \frac{10e^{4-2p_{11}}(p_{11}e^{2-p_{11}} + p_{21}e^{1-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^3} - \frac{5e^{2-p_{11}}(p_{11}e^{2-p_{11}} + p_{21}e^{2-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^2} \\ \frac{\partial^2 R}{\partial p_{11} \partial p_{21}} &= \frac{\partial^2 R}{\partial p_{21} \partial p_{11}} = \frac{5e^{1-p_{21}}(e^{2-p_{11}} - p_{11}e^{2-p_{11}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^2} \\ &+ \frac{5e^{2-p_{11}}(e^{1-p_{21}} - p_{21}e^{1-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^2} \\ &+ \frac{10e^{3-p_{11}-p_{21}}(p_{11}e^{2-p_{11}} + p_{21}e^{1-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^3} \\ \frac{\partial^2 R}{\partial p_{21}^2} &= \frac{10e^{1-p_{21}}(e^{1-p_{21}} - p_{21}e^{1-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^2} + \frac{5(-2e^{1-p_{21}} + p_{21}e^{1-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})} \\ &+ \frac{10e^{2-2p_{21}}(p_{11}e^{2-p_{11}} + p_{21}e^{1-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^3} - \frac{5e^{1-p_{21}}(p_{11}e^{2-p_{11}} + p_{21}e^{1-p_{21}})}{(1 + e^{2-p_{11}} + e^{1-p_{21}})^2} \end{aligned}$$

Let the couple $(p_{11} = 2, p_{21} = 3.5)$ which satisfies the prices constraints of Example 1. Then, $H_R(2, 3.5)$ is equal to:

$$H_R(2, 3.5) = \begin{pmatrix} -2.41077 & 0.12335 \\ 0.12335 & 0.0573501 \end{pmatrix} \quad (A.3)$$

The $\frac{\partial^2 R}{\partial p_{21}^2} = 0.0573501 > 0$ and the determinant of $H_R(2, 3.5)$ is equal to: $-2.41077 \times 0.0573501 - 0.12335 \times 0.12335 = -0.153 < 0$, which means that $H_R(2, 3.5)$ is not negative semi-definite, as a result $R(p_{11}, p_{21})$ is not concave in the prices space.

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