

Herbon, Avi; David, Israel

Article

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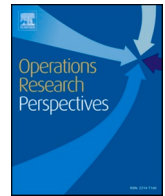
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Optimal manufacturer's cost sharing ratio, shipping policy and production rate – A two-echelon supply chain

Avi Herbon^{*}, Israel David

Department of Management, Bar-Ilan University, Ramat-Gan 52900, Israel

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ABSTRACT

We analyze an integrated inventory supply chain and seek the optimal production lot, optimal production rate, and optimal (integer) number of shipments per production lot. An increasing need for higher operational efficiency, as well as growing competition among multiple products for a limited storage capacity, is driving retailers to require more frequent shipping. This imposes pressure on suppliers to share the shipping cost with retailers. The sharing ratio of the shipment cost has not previously been considered within the context of an integrated supply chain. Therefore, we contribute to the literature by investigating this entirely new parameter, assuming that the shipment cost is shared between a manufacturer and a retailer. We also consider a distributed supply chain in which each party optimizes its own cost. We analyze the problem of finding the optimal sharing ratio of the shipment cost for such a supply chain and show that there exists a specific choice of shipment cost-sharing ratio (set by the manufacturer) that results in total costs similar to those obtained in the integrated inventory model. We develop deterministic models that provides basic insights into the investigated problem. Through mathematical analysis of a nested-designs model, we provide intermediate results (which are of interest in their own right) as well as optimal analytical solutions. We show, through numerical examples, that in the scenario where each party optimizes its own cost, the manufacturer's shipment cost is a central control variable in the sense that it affects the costs of both parties.

1. Introduction

1.1. Motivation and research objective

As industrial environments have become increasingly competitive, effective supply-chain management has become essential [1]. Coordination between supply-chain members is often suggested as a means of facing some of the challenges that arise within the operational management of supplier-retailer chains. The partnership between suppliers and retailers along a supply chain has a substantial impact on supply-chain success. If retailers' expectations are not met on a day-to-day basis, this could result in unsatisfied consumers, which could affect the profits of not only the retailers but also the suppliers.

The ability to reduce cycle times can constitute a powerful competitive advantage, and manufacturers invest considerable efforts in pursuing this goal. In many industries, an increasing need for higher operational efficiency, as well as growing competition among multiple products for a limited storage capacity, are factors that drive retailers to seek more frequent shipping. Accordingly, a quick response to customer

demand is one of the prerequisites for manufacturers to sustain their global 'competitive advantage' [2]. Most manufacturers sell to resellers and distributors, who in turn sell to consumers. The ability to manufacture a product on a tighter schedule enhances the manufacturer's perceived reliability and improves relationships with potential distributors, thus increasing the manufacturer's market share.

Table 1 in Taifa and Vhora [3] presents an overview of the various methods deployed for cycle time reduction (e.g., efficient maintenance, process step reduction, and fine tuning). Many real-life industries (e.g., the manufacture of semi-conductors in Taiwan, the automobile industry in India, and the production of agricultural and construction machinery in Korea) have utilized techniques for cycle time reduction (see Table 2 in Ref. [3]). However, once shorter cycles are achieved, the manufacturer is confronted with the operational problem of how to adapt to the new situation. Our model suggests an optimal solution to this problem. In particular, it determines the optimal production lot, optimal production rate, and optimal (integer) number of shipments per production lot.

When attempting to improve supply-chain management through

^{*} Corresponding author.

E-mail address: her@bezeqint.net (A. Herbon).

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Table 1
Cases for Theorem 1.

| Case | Description |
|------|--------------------------------|
| I | $R(1) \leq R(2)$ |
| II | $R(1) > R(2), R(n_2^*) \geq 0$ |
| III | $R(1) > R(2)$ |

Table 2
Optimal policy for several values of the bound on the cycle length, T .

| T | n_{\min}^{opt} | n_{\max}^T | Case in Theorem 2 | C_T^* | n^* | Q^* | P^* |
|-----|------------------|--------------|-------------------|---------|-------|--------|--------|
| 1 | 3.873 | 1 | f | 6025.0 | 4 | 200 | 266.67 |
| 2 | 7.746 | 1 | c | 3775.0 | 8 | 400 | 266.67 |
| 3 | 11.619 | 4 | c | 3191.7 | 12 | 600 | 266.67 |
| 4 | 15.492 | 11 | c | 3025.0 | 15 | 800 | 266.67 |
| 5 | 19.365 | 26 | b | 3010.8 | 17 | 890.14 | 266.67 |
| 6 | 23.238 | 50 | b | 3010.8 | 17 | 890.14 | 266.67 |

greater coordination between members, it needs to be borne in mind that the perspective of each party is intrinsically different and that optimizing each member's own performance does not necessarily achieve optimal performance of the entire supply chain. For example, it might be more costly to hold a unit of a product for a unit of time on the retailer's shelf than at the supplier's warehouse [4]. The most common assumption in the existing literature (as well as in our paper) is that the manufacturer-buyer chain operates under full cooperation. An example of supply-chain cooperation, and the focus of the current paper, is shared shipment costs between the manufacturer and the buyer. Unfortunately, this is not the case in many real-life supply chains, for numerous reasons. These include: unwillingness of one of the parties to accept external intervention regarding organizational decisions; unwillingness of the manufacturer to share the cost of shipping; unwillingness of the retailer to accept a different shipping policy from that originally planned; and inability to reach an agreement contract that adequately divides the total profit. Within the scope of an integrated supply chain, the ratio between the supplier's and the retailer's share of the shipment cost has no effect on supply-chain performance. However, in a non-cooperative supply chain, lack of willingness on the part of the supplier to share the shipment cost results in a situation where the retailer incurs higher costs. This could decrease the retailer's incentive to work with such suppliers. Thus, we contribute to the literature by investigating an important parameter that has not previously been studied: the ratio between the share of the shipment cost paid by the manufacturer and that paid by the retailer.

None of the aforementioned studies, nor, the vast body of literature on supply-chain coordination has not considered the case where the production rate acts as a decision variable, the optimal value of which has the potential to make a positive contribution to the total performance. This paper analyzes a two-echelon supply chain consisting of a manufacturer who converts raw materials into a final single product, stores the product at his warehouse, and then ships it to a retailer who sells the item to a market with a known fixed demand rate. The manufacturer incurs costs when capacity is limited by slow production, especially when the production-run absorbs almost the entire cycle length. This may result in costs due to, for example, reduced time for preventive maintenance, workers being required to work overtime, and longer usage of manufacturing tools. A production rate that is too high, on the other hand, also incurs costs, due to factors such as faster accumulation of inventory (resulting in greater holding costs) and the need to purchase more expensive equipment.

1.2. The current study and its contribution

In today's hypercompetitive and highly complex business environment, companies are constantly searching for ways of gaining a competitive advantage by improving the speed, efficiency, and quality of the goods and services they deliver to customers through their supply chains. The key to sustained success in supply-chain management is the ability to make optimized business decisions by finding the best possible solution to the company's planning and scheduling problems. Optimization algorithms that generate optimal solutions outperform their heuristic counterparts and enable businesses to maximize both their profit and their operational efficiency. Explicit solutions provide managers with a deeper understanding and suggest practical insights. Such solutions involve determining the best way of synchronizing supply and demand across the supply-chain network, so as to boost customer satisfaction and enhance bottom-line results.

The current research involves mathematical analysis of a nested-designs model. The analysis provides intermediate results that are of interest in their own right as well as explicit solutions. More specifically, our research contributes to the existing literature as follows:

- Most previous studies on the subject of integrated supply chains have analyzed the behavior of the proposed model (or extensions of the model) in an extensive numerical experiment implemented in a commercial solver. In contrast to these studies, the solution approach developed in the current paper is purely analytical.
- Through mathematical analysis of a nested designs model, we provide intermediate results that are of interest in their own right and explicit solutions.
- We generalize the two-echelon supply-chain problem by determining both the optimal production rate, P , and the optimal (integer) number of shipments to the buyer, n , for each production lot Q . Furthermore, we consider bounds on the cycle length (at the manufacturer's warehouse) and on the production rate.
- As Aldurgam et al. [5] remark, variable production rates in the context of an integrated inventory model have been studied very infrequently. We contribute to the scarce literature that considers an integrated inventory combined with a variable production rate.
- In addition to the integrated supply chain, we analyze a partially-coordinated supply chain in which coordination takes place by means of the shared shipment cost, but each party optimizes its own objective. We show that there is a specific value of the shipment cost-sharing ratio (selected by the manufacturer) that results in total costs of the supply chain that are nearly the same as those obtained in the integrated inventory model.

The main result (Theorem 1) pertains to the non-convexity of the optimum cost as a discrete-argument function of n , at the point $n = 2$. In particular, we demonstrate that the speed of production is unimportant (i.e., remains optimal) in the case where the lot size is split into two shipments.

2. Literature review

2.1. Integrated approach

In common with most of the literature on supply-chain inventory management, we assume in this paper that a central decision-maker sets a policy that aims to optimize the total supply-chain performance. This so-called "integrated" approach between the vendor and the buyer (or buyers) for improving performance has been studied for many years (see [6–14]). Utama et al. [15] presented a systematic review of integrated

procurement production. Kim and Ha [16] developed an integrated inventory model with a JIT (just in time) approach and a small lot size with the objective of minimizing the joint total cost. Chou [17] showed that an integrated two-stage inventory model for deteriorating items results in higher profits than a non-integrated approach. Similarly, Klein et al. [18] found that performance improves when parties share strategic information and customize IT. A review of integrated inventory models is provided by Glock [19]. Gharaei et al. [20] employed the null-space method (NSM) for solving a nonlinear programming (NLP) formulation representing an integrated lot-sizing model of a multi-level supply chain. Dolai et al. [21] developed an integrated profit function where the demand function depends on the product's green degree, the advertisement frequency, and the retailer's selling price. Panja and Mondal [22] suggested a game theoretical approach for the case where demand depends not only on the product's green degree and the retailer's selling price but also on the credit period offered by the retailer to customers. Khara et al. [23] investigated a supply chain of used products consisting of a supplier, a manufacturer, a retailer and a "collector" (who collects the used product from consumers). In particular, their model seeks the optimal number of deliveries from supplier to manufacturer, from manufacturer to retailer, and from collector to manufacturer that together maximize the total integrated profit.

2.2. Variable production rate as an important managerial decision

Numerous studies that aim to determine inventory and production policies allow for the production rate P to be a decision variable. This can affect the problem in a number of ways, as P can influence the rate of defects, the stochastic time period until failure or repair, inventory levels, and ultimately, the total cost. Glock [24] studied the impact of production rates on the buildup of inventory in a two-stage production system and showed that deviating from the 'design production rate' of the system may reduce the system's total costs. Khouja [25], Giri et al. [26], Larsen [27], Sana [28], Manna et al. [29], Dolai and Mondal [30], and Askari et al. [31] all chose to study the well-known EPQ model – the EOQ model with production (see, e.g., [32,33]). Thus, they investigated the scenario in which a manufacturer optimizes the batch size given the parameters of the EOQ model and an additional parameter P – the production rate. A comprehensive and systematic overview of EPQ-type lot-sizing models that consider controllable production rates is given in Glock and Grosse [34].

Relatively few studies incorporate both an integrated inventory model and a variable production rate. Glock [35] considered a single-vendor single-buyer JELS (Joint Economic Lot Size) model with stochastic demand and variable lead-time. In this model, lead-time can be shortened in a number of ways: by reducing the lot size, by increasing the production rate, or by crashing a constant delay time. Jauhari and Pujawan [36], in analyzing the JELS model, employed an iterative procedure to determine simultaneously the following parameters: the safety factor, delivery lot size, delivery frequency, production batch, raw material lot size, and production rate. The objective was to minimize the total cost under stochastic demand. Following the work of Glock [35], Aldurgam et al. [37] analyzed a generalization that considers shipment constraints and raw material purchases. They considered a two-echelon single-vendor single-manufacturer supply chain where a vendor produces an intermediate product that is shipped to a manufacturer. The manufacturer transforms the intermediate product into a final product and has the opportunity to vary its production rate, which in turn affects the speed of completion of the lot size. The unit production cost $C_p(P)$ is parametrically dependent on the (expected) cycle length, which is influenced by P , a decision variable. In Aldurgam et al. [37], the manufacturer is the buyer, while in our model, the manufacturer is the vendor. In spite of the relatively general setting, they did not provide analytical solutions. Kim and Glock [38] assumed that the production rates of the manufacturer's machines may be varied within given limits. This allows the manufacturer to adjust his total production capacity

according to the demand and to exploit different cost structures of the available machines. The authors analyzed the behavior of the proposed model in an extensive numerical experiment where the model was implemented using a commercial solver. In recent work by Jauhari et al. [39], in addition to utilizing regular production as a production base, the vendor also uses green production, which is costlier than regular production, but generates lower emissions. In a follow-up paper [40], this model was generalized by assuming imperfect production. In particular, the authors assumed that the production rate is adjustable and that it influences both the production cost and the emissions that result from the production and reworking processes. In contrast to the above-mentioned studies, the solution approach used in the current research is purely analytical. We further contribute to the existing literature by assuming that the shipment cost is shared between the manufacturer and the retailer, and then investigating the optimal value of this cost-sharing ratio.

2.3. Closely related studies

In this subsection, we describe the previous work that is most relevant to the current study and we identify the gap in the literature addressed by this paper. Firstly, we note that, in common with previous publications that examine JELS models (e.g., [41–43]), we assume that the shipments are all of equal size. Our study extends the approach of David and Eben-Haim [44], who analyzed an integrated inventory system in which the production rate is fixed and the cycle length is unrestricted. Very long cycle lengths inevitably mean very large lots, which, in turn, give rise to a large and inefficient warehouse. In addition, very long cycle lengths indirectly imply a long production-cycle time. To prevent these unwelcome scenarios, we restrict the cycle length and denote the upper bound on it by T .

The present paper studies a similar problem setting to that investigated by Herbon [45]. Yet, there are several key differences between the two studies, as described in the following, which are crucial for understanding our current contribution:

- The problem analyzed in Herbon [45] addresses the issue of bounding the production cycle length, while the current manuscript assumes a bound on the entire cycle length. Although these two parameters are related, they remain fundamentally different.
- A fundamental difference is that Herbon [45] develops a heuristic to solve the complex problem, which is classified as an integer programming problem, while in the current paper, we develop a mathematical method to optimally solve the general problem.
- Under small bounds on the production cycle length (see first line in Table 3 in Ref. [45]), non-feasible solutions (i.e., production rates) are obtained. Our current approach, on the other hand, rules out such scenarios by definition and is therefore applicable to all given bounds.
- The current study analyzes the problem of finding the optimal sharing ratio of the shipment cost. This was not addressed in Herbon [45], nor has it been tackled by any previous literature on the topic of manufacturer-retailer chains. Furthermore, although both articles analyze an integrated inventory supply chain (i.e., cooperative mode), in the current manuscript, we also mathematically analyze two further scenarios. The first is a non-cooperative game between the manufacturer and the retailer in which each party optimizes its own cost. We analytically determine the best response of each party given the opponent's decisions. Second, we analyze a partially-coordinated supply chain, in which coordination takes place by means of the shared shipment cost, but each party optimizes its own objective.

In Section 3, the integrated inventory supply chain is described and the problem is formulated. In Section 4, we fully solve the mixed integer program for an unrestricted cycle length, while in Section 5, the

generally constrained problem is addressed iteratively (namely, P given Q and n , then Q given n using optimal P , and finally optimizing over n). Section 6 analyzes the case of partial cooperation between the parties, where each party optimizes its own costs. Section 7 concludes.

3. Supply-chain description and problem formulation

In this section we introduce the two-echelon supply chain analyzed in this paper. We present the notations, formulate the mathematical model of the integrated inventory supply chain, and state the key assumptions.

3.1. Supply chain description

A single retailer (hereafter, buyer), who does not permit shortages, observes a market demand rate of D for a single product. All items are purchased from a single manufacturer (hereafter, vendor) who supplies them periodically. We assume that the manufacturer and the retailer fully cooperate and that the relevant information is available to both parties. The manufacturer produces the product with rate $P \geq D$ (to be determined) and, in common with the buyer, does not permit shortages. Each commencement of a production cycle (which results in a production lot of size Q) is associated with a setup cost of K . The lot is delivered in several shipments, n (to be determined), each of size q (i.e., $q = Q/n$), from the manufacturer's storage facility to the retailer's warehouse. Fig. 1, adapted from Herbon [45], presents the integrated inventory level, the buyers' inventory level, and the vendor's inventory level. At this juncture, it is useful to define the demand-to-production rate ratio $r = D/P$, such that r replaces P as a decision variable. Interestingly, the combined inventory never empties, except in the special case where the production rate is infinite, $P = \infty$ (i.e., purchasing). For the common scenario in which $P > D$, i.e., the scenario presented in Fig. 1, the vendor's inventory

increases step-wise over time for the entire duration of production, after which it decreases step-wise to 0. The sudden "drops" in inventory level along the cycle length represent shipments to the buyer.

In Figs. 2a–2c, we illustrate inventory levels for three special cases, where we denote the maximal production rate by U and the cycle length (for the manufacturer) by $L = Q/D$. Setting $P = D$, as presented in Fig. 2a, indicates that the manufacturer never stops production. This, of course, is impracticable. At the other extreme, Fig. 2b presents the case where $P = U = \infty$. This indicates that the manufacturer purchases raw materials, but does not produce the item. Fig. 2c presents a more common case in which there are exactly two equal shipments per lot. In this figure, we denote half the vendor's idle time in a cycle by $\Delta = \Delta_{q,P,D} = q/D - q/P$. To complete the set of notations, we let h_V and h_B be the vendor's holding cost and the buyer's holding cost (respectively), for each unit per unit time. Similarly, we let k_V and k_B denote the shipment cost associated with the vendor (i.e., manufacturer) and the buyer (i.e., retailer), respectively.

3.2. Notations and assumptions

Parameters

- D – demand rate for a single product (expressed in units per unit time)
- r – demand-to-production rate ratio, $r = D/P$ (dimensionless)
- Q – production lot size (units)
- U – maximal production rate (units per unit time)
- h_V – vendor's holding cost for each unit per unit time
- h_B – buyer's holding cost for each unit per unit time
- k_V – shipment cost (\$) associated with the vendor (i.e., manufacturer)
- k_B – shipment cost (\$) associated with the buyer (i.e., retailer)

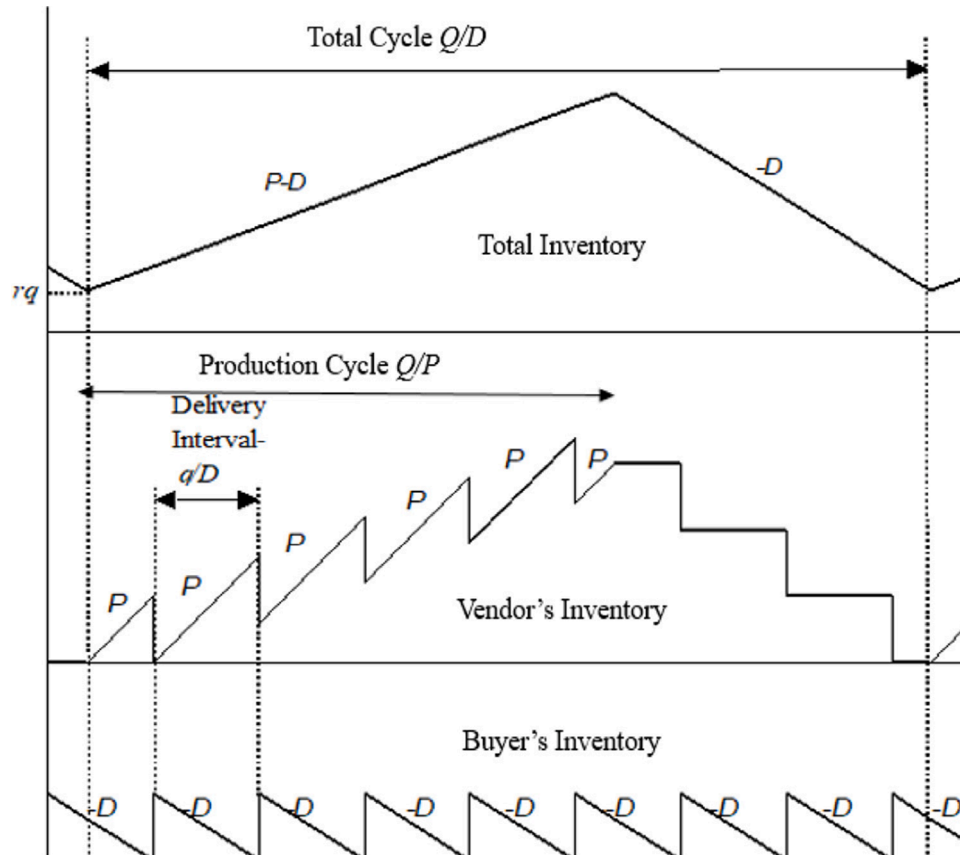


Fig. 1. Inventory levels at the vendor and the buyer, along with the total inventory level

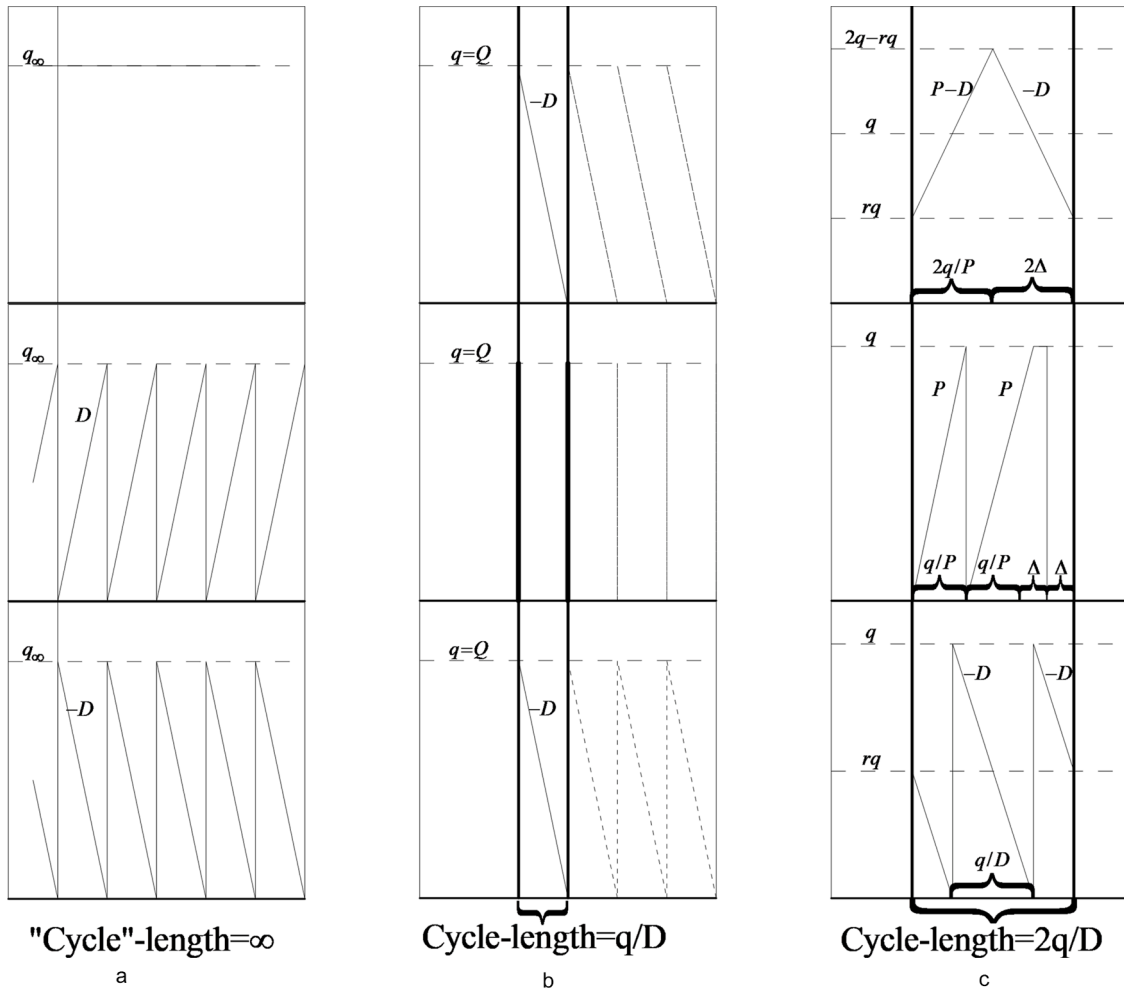


Fig. 2. a. $P=D$ ($n = \infty$). b. "lot-for-lot" ($n = 1$). c. Any P ($n = 2$)

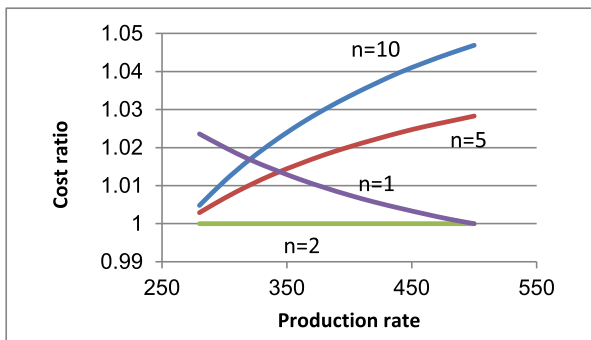


Fig. 3. Total cost as a function of the production rate, relative to the optimal cost solution

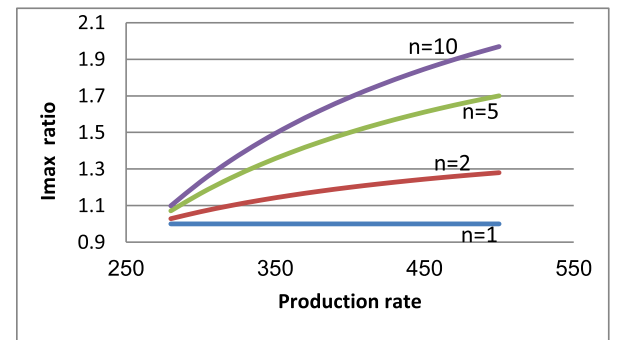


Fig. 4. Peak inventory versus production rate, relative to the peak inventory for the optimal cost solution

K – setup cost (\$) associated with the commencement of a production cycle (i.e., a production lot of size Q)

r_{\max} – upper bound on r (dimensionless)

Decision variables

n – number of shipments (integer) per lot

q – shipment size, i.e., $q=Q/n$ (units)

P – production rate (units per unit time)

Variables

$C_t(q, r, n)$ – total cost per unit time (\$ per unit time)

The key assumptions of the proposed model are as follows:

A1. The manufacturer and the retailer cooperate through an integrated inventory system.

A2. Market and operational information (i.e., information about demand and information about costs) is fully shared.

A3. Both parties prohibit shortages.

A4. The supply-chain profit in the integrated inventory system is divided between the two parties according to a pre-arranged agreement between them.

A5. Once production has started, it is uninterrupted until the lot is completed (see Ref. [8]).

A6. Production setup is required only in the case where there is a finite interval of time during which an inventory level of 0 is observed at the manufacturer (see Fig. 1).

3.3. Problem formulation

The average total inventory level over time is $\bar{I}_t = rq + (1-r)q/2$ and the average inventory level for the buyer is $\bar{I}_B = Q/(2n)$. Thus, after excluding fixed costs associated with purchasing (in the case of the buyer) and production (in the case of the manufacturer), the total cost per unit time is

$$C_t(Q, P, n) = \frac{DK}{Q} + \frac{k_v D \cdot n}{Q} + h_v \left(\frac{D}{P} \frac{Q}{n} + \frac{Q}{2} \left(1 - \frac{D}{P} \right) - \frac{Q}{2n} \right) + \frac{k_B D \cdot n}{Q} + h_B \frac{Q}{2n}. \quad (1)$$

The suggested formulation considers two additional factors that are likely to apply in real-life systems. Firstly, the formulation prohibits scenarios in which the production rate approaches the demand rate, that is, scenarios in which production takes place almost without breaks. Thus, we denote by r_{\max} an upper bound on r , and accordingly, there is a minimal production rate $P = P_{\min}$. Secondly, we avoid scenarios in which the cycle length (at the manufacturer's warehouse) is unbounded. The problem we address in this paper is formally defined as

Problem (P1):

$$\begin{aligned} \min_{q, r, n} C_t(q, r, n) \\ \text{s.t.} \\ D/U \leq r \leq r_{\max} \\ nq \leq DT \\ q > 0 \\ n \geq 1, \text{ integer} \end{aligned} \quad (2)$$

$C_t(q, r, n)$, or equivalently $C_t(Q, P, n)$, is given in (1). Model formulation (2) is classified as a mixed integer non-linear optimization problem and the analysis begins in the following section.

4. Optimization for a given integer n – unbounded cycle length

This section addresses a reduced version of the problem in formulation (2). This solution strategy enables us to obtain intermediate insights, while the optimal solution acts as a lower bound on the original problem. The reduced problem that we address in this section is formally defined as

Problem (P2):

$$\begin{aligned} \min_{q, r, n} C_t(q, r, n) \\ \text{s.t.} \\ D/U \leq r \leq r_{\max} \\ q > 0 \\ n \geq 1, \text{ integer} \end{aligned} \quad (3)$$

4.1. Optimization of the production rate P given n and q

We begin by stating the following insightful conclusion:

Proposition 1. For a given number of shipments n (integer) and shipment quantity q , the optimal production rate is

$$P^*(q, n) = P^*(n) = \begin{cases} U & n = 1 \\ P \in [P_{\min}, U] & n = 2. \\ P_{\min} & n \geq 3 \end{cases} \quad (4)$$

Proof. Considering that $q = Q/n$ and $r = D/P$, cost function (1) is identical to

$$C_t(q, r, n) = \frac{DK}{nq} + \frac{D(k_B + k_v)}{q} + h_v \left(rq + \frac{(n-1)q}{2} - \frac{rnq}{2} \right) + h_B \frac{q}{2}.$$

By grouping terms in the above expression, it can be seen that the cost function is linear in r , with slope $h_v q(1 - \frac{n}{2})$. Therefore, when seeking the minimum cost under the condition that $D/U \leq r \leq r_{\max}$, we obtain the following three outcomes: for $n=1$, the optimum r is the endpoint $r=D/U$, which means that $P=U$; for $n=2$ (zero slope), r is arbitrary; and for $n \geq 3$, $r = r_{\max}$, or $P = P_{\min}$.

Since solution (4) is feasible under the first constraint of problem (P2), it is also optimal for the general problem (P1) for any feasible set (q, n) , i.e., for the problem that assumes a cycle-length bound. In particular, it is optimal for the optimal set (q^*, n^*) of problem (P2). The optimal value of P is independent of lot size q . Interestingly, unless a production run consists of exactly two equal shipments, the optimal strategy is either to produce at the maximum rate (if the whole batch is shipped at once) or to produce at the minimal allowable rate (if there are at least three shipments). Similar adequate production regimes (e.g., $n=2$) are also found to be optimal for controlling simple dynamic production systems with a single machine (see Maimon et al. [46]).

Intuitively, if $n=1$, then the manufacturer ships the entire lot immediately upon completion, and has the opportunity to completely eliminate all inventory holding costs by making $P = \infty$ (if feasible). This is the case in Fig. 2b. If, on the other hand, $n \geq 2$, the vendor (manufacturer) necessarily holds some inventory, but the system can minimize the holding costs by producing at the demand rate while incurring no production setup costs and only shipping costs. This is the case in Fig. 2a. In the special case where $n = 2$, the vendor's average inventory level over time is shown to be equal to $q/2$, regardless of P (see Fig. 2c). The average inventory of the buyer is also $q/2$. Since P enters the cost expressions only through the vendor's holding cost, the optimal total cost per unit time does not depend on the chosen production rate in the case where it is optimal to ship exactly twice per lot.

4.2. Optimizing the shipment size q given n

Using (1), setting $Q=qn$, and defining $k = k_B + k_v$,

$$C_t(q, r, n) = \frac{KD}{nq} + \frac{kD}{q} + h_B \frac{q}{2} + h_v \left((1-r) \frac{nq}{2} + rq - \frac{q}{2} \right). \quad (5)$$

For the sake of brevity, assume herein that $P^*(2) = D$ in (4), so we may re-write (4) as

$$r^*(q, n) = r^*(n) = \begin{cases} D/U & n = 1 \\ r_{\max} & n \geq 2 \end{cases} \quad (6)$$

Consequently, $r^*(n)$ is applied; thus,

$$\begin{aligned} C_t(q, n) &\equiv C_t(q, r^*(n), n) \\ &= \frac{DK}{nq} + \frac{Dk}{q} + h_v \left(r^*(n) \cdot q + \frac{(n-1)q}{2} - \frac{r^*(n) \cdot n \cdot q}{2} \right) + h_B \frac{q}{2}. \end{aligned} \quad (7)$$

Noting that $\min_{q, r, n} C_t(q, r, n)$ may be written as $\min_n \left\{ \min_{q>0} C_t(q, n) \right\}$ and defining

$$e_n^{par} = D \cdot K + k \cdot D \cdot n,$$

the FOC for the problem $\min_{q>0} C_t(q, n)$, in the case where $n = 1$, is

$$\frac{h_v D}{2U} - \frac{e_1^{par}}{q^2} + \frac{h_B}{2} = 0 \quad (8)$$

thus, $q^2(1) = \frac{2UD \cdot (K+k)}{h_v D + h_B U}$. For $n \geq 2$, the FOC takes on the form

$$h_v r_{\max} \frac{2-n}{2n} + h_v \frac{n-1}{2n} - \frac{e_n^{par}}{n^2 q^2} + \frac{h_B}{2n} = 0.$$

Thus,

$$q^*(n) = \sqrt{\frac{2e_n^{par}}{h_V(n(n-1) - r_{\max}n(n-2)) + h_B n}} \quad (9)$$

and

$$Q^*(n) = \sqrt{\frac{2ne_n^{par}}{h_V(n-1 - r_{\max}(n-2)) + h_B}}. \quad (10)$$

Note that $q^*(n)$ holds for any set of parameters where $h_V + h_B > 0$. In particular, $q^*(2) = \sqrt{\frac{e_2^{par}}{h_V + h_B}}$ and $Q^*(2) = \sqrt{\frac{4e_2^{par}}{h_V + h_B}}$ (regardless of r_{\max} , as shown geometrically in Fig. 2c). In addition,

$$q_{\infty}^* \equiv \lim_{n \rightarrow \infty} q^*(n) = \begin{cases} \sqrt{\frac{2Dk}{h_V + h_B}} & r_{\max} = 1 \\ 0 & r_{\max} < 1 \end{cases}. \quad (11)$$

Again, regardless of r_{\max} , $Q^*(n) \rightarrow \infty$ as n tends to infinity. If $r_{\max} = 1$, this limiting-infinite batch is comprised of an infinite number of equal shipments of finite, positive size q_{∞}^* (see Fig. 2a). Intuitively, this case is warranted if the setup cost is high (this argument is expounded upon in the following section). If, however, $r_{\max} < 1$, the limiting-infinite batch is comprised of an infinite number of shipments of infinitesimal size. However, this scenario seems to lack any practical significance.

We summarize the above results in the following proposition:

Proposition 2. The problem $\min_{C_t(q,n)}$ has a minimal solution $q^*(n)$ for any $n \geq 1$. This shipment size is given by

$$q^*(n) = \begin{cases} \sqrt{\frac{2UD(K+k)}{h_V D + h_B U}} & n = 1 \\ \sqrt{\frac{2e_n^{par}}{h_V(n(n-1) - r_{\max}n(n-2)) + h_B n}} & n \geq 2 \end{cases}. \quad (12)$$

4.3. A numerical illustration of the effect of production rate

Defining $C_t(n) = C_t(r^*(n), q^*(n), n)$ ($q^*(n)$ is inserted into (7)), we are left with the problem

$$\min_{n \geq 1, \text{integer}} C_t(n), \quad (13)$$

which solves Problem (P2) – the version that disregards the cycle length. In this case,

$$C_t(n) = \frac{e_n^{par}}{nq^*(n)} + \frac{h_V q^*(n)}{2} (n - (n-2)r^*(n)) + \frac{(h_B - h_V)q^*(n)}{2}. \quad (14)$$

A general conclusion from (14) is that P clearly does not affect the buyer's costs. This reduces the inter-dependency between the two parties, thus increasing the likelihood that they will cooperate. Applying (6) and (12), we get

$$C_t(1) = \sqrt{2(h_V D/U + h_B) e_1^{par}}, \text{ for } n = 1$$

$$C_t(n) = \sqrt{2D(K/n + k)(h_V(n-1) - h_V r_{\max}(n-2) + h_B)} \text{ for } n \geq 2. \quad (15)$$

For the given problem setting (see Fig. 1), the maximal total inventory is given by $I_t^{\max}(n) = I_t^{\max}(r^*(n), q^*(n), n) = r^*(n)q^*(n) + (1-r)nq^*(n)$. Let us pause here to illustrate the results so far by means of a numerical example.

Example 1. The effect of production rate

To highlight the effect of the production rate on the integrated inventory system and on the maximal mutual inventory, Fig. 3 depicts the

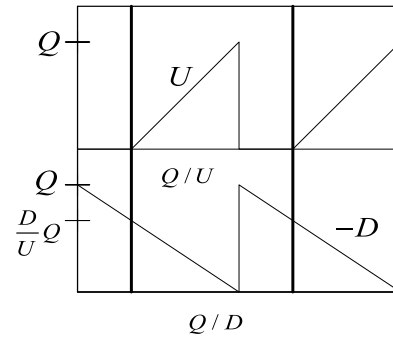


Fig. 5. Lot-for-lot inventory patterns for the vendor (above) and buyer (below)

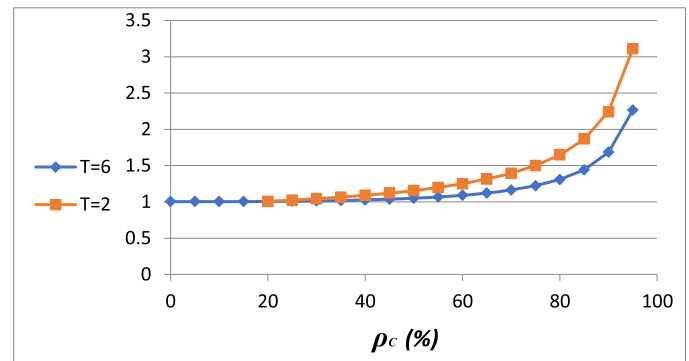


Fig. 6. The ratio between the total optimal costs of the partially-coordinated supply chain and the integrated supply chain, as a function of the level of coordination, where $h_V = h_B$.

ratio between $C_t(n)$ (see (15)) and $C_t(q^*(r), r, n)$ (see (14), where $r^*(n)$ is replaced by the given r), as a function of the production rate P and the number of shipments n , for the following choice of parameter values: $D=200$ (daily), $U=500$ (daily), $K=5000$, $k_V=50$, $k_B=50$, $h_V=10$, $h_B=10$, and $r_{\max}=0.75$. Figs. 3 and 4 illustrate the case of a fixed unit-production cost of $c_V = 100$ ($D \cdot c_V$ is added to $C_t(Q, P, n)$). Four curves are shown, for four different n values (1, 2, 5 and 10). Similarly, Fig. 4 depicts the ratio of $I_t^{\max}(q^*(r), r, n)$ to $I_t^{\max}(n)$ as a function of production rate.

A comparison of Figs. 3 and 4 reveals that the peak inventory is strongly sensitive to the production rate, while the total cost shows much lower sensitivity. In both cases, the sensitivity increases with n , $n \geq 2$. The peak inventory is particularly sensitive to a deviation from the optimal P . For high values of n , the lion's share of the total system inventory is allocated to the vendor. Interestingly, exceptional behavior is observed for the case of a single shipment ($n=1$), namely, a decrease in the C_t ratio with production rate. This behavior originates from (4) and

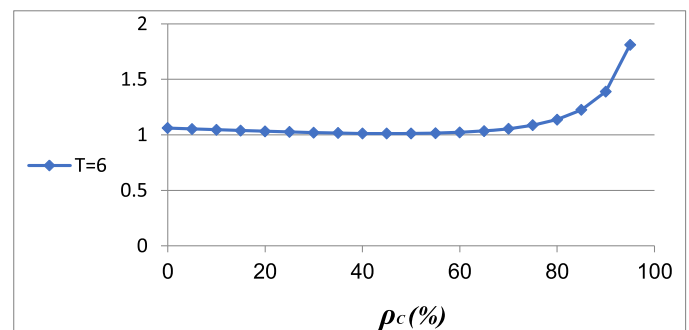


Fig. 7. The ratio between the optimal costs of the partially-coordinated supply chain and the integrated supply chain, as a function of the level of coordination, when $h_V > h_B$.

(6), under which the form of the optimal solution differs for a single shipment ($n=1$). For such a case, the vendor holds a total of $Q^2/2P$ inventory units per cycle (see Fig. 5 below), according to which the cost is minimized by $P = U$, regardless of Q .

4.4. Overall optimization

We now seek the optimal number of shipments, n^* (an integer). Let us denote by $\lfloor x \rfloor$ either the left-hand side integer neighbor of a positive real x , or its right-hand side integer neighbor, whichever gives the smaller value of $C_t(n)$ (see (15)). If x is an integer, the left-hand side neighbor is x . We shall also need the abbreviation

$$n_2^* = \begin{cases} \lfloor n^*(r_{\max}) \rfloor_{C_t} & n^*(r_{\max}) \geq 2 \\ 2 & n^*(r_{\max}) < 2 \end{cases} \quad (16)$$

where, $n^*(r_{\max})$ is the real quantity that minimizes the second row in (15)). Let us further define $R_h = h_B/h_V$ and $R_k = k/K$. Writing down the inequality $C_t(n) < C_t(1)$ for $n \geq 2$, we see (by (15)) that it is equivalent to $R(n) < 0$, where $R(n)$ is the quadratic $an^2 + bn + d$ and

$$\begin{aligned} a &= R_k(1 - r_{\max}) \\ b &= 1 - R_h - (r_{\min} + r_{\max}) - (1 - 2r_{\max} + r_{\min})R_k \\ d &= R_h - (1 - 2r_{\max}) \end{aligned} \quad (17)$$

Substituting and cancelling terms, we find that $R(1) = (r_{\max} - r_{\min})(1 + R_k)$, which is a positive number. We are now ready to summarize the state of affairs when the constraint on L (the cycle length for the manufacturer) is ignored:

Theorem 1. Consider the classification of three mutually exclusive possible cases, as shown in Table 1. If $h_B \geq h_V(1 - 2r_{\max})$ then

$$n^* = \begin{cases} 1 & \text{Case I and Case II} \\ \lfloor m \rfloor_{C_t} & \text{Case III} \end{cases} \quad (18)$$

where,

$$m = \max \left\{ n \mid R_h \geq (1 - 2r_{\max}) + n^2 \frac{k}{K} (1 - r_{\max}) \right\} \quad (19)$$

Proof.

See Appendix A.

Note that if $r_{\max} = 1$, we formally get $n^*(r_{\max}) = \infty$. Thus, it follows from Theorem 1 that n^* is either 1 or infinity (Fig. 2a or 2b).

Example 1. (continued). Optimal solutions for various parameter settings

For the parameters in Example 1, Case III (see Table 1) applies. The optimal solution is characterized by $n^* = 17, Q^* = 890.4, P^* = 266.67$, and $q^* = 52.36, q^* = 52.36$, with $I_t^{\max} = 261.80$ and a minimal cost of $C_t^* = 3010.8\$$. If we change some of the parameters, specifically, if we set $r_{\max} = 0.5, k_V = 1250$, and $k_B = 1250$, Case III still applies, but with an optimal solution of $n^* = 2$ (such that Fig. 2c applies). Further, we find that the minimal cost is $C_t^* = 6324.6\$$ and the optimal lot size is $Q^* = 632.45$. Thus, for such a combination of parameter values, it is optimal to produce Q^* units per lot, irrespective of the production rate, and to split them into two equal shipments. A solution with $n^* = 1$ would only hold for a small setup cost (relative to the delivery cost) and a high vendor-to-buyer ratio for the holding cost parameter, as can be gleaned from (14). Theorem 1 specifies this qualification: if, for instance, $k_V = k_B = 4000$, and $r_{\max} = 0.5$, then Case I applies with an optimal solution of $n^* = 1, Q^* = 609.45$, and $C_t^* = 8532.29\$$ (by (15)). Fig. 5 shows a schematic representation of such a lot-for-lot inventory pattern. In this case, $(Q_1^*)^2 \rightarrow 2D(K+k)/h_B$ as $U \rightarrow \infty$. $U = \infty$ may be interpreted as outside

procurement (with zero lead-time). The picture is then one of immediate outsourcing, shipping the quantity $q = \sqrt{2D(K+k)/h_B}$ to the buyer and then resting for the entire duration of the q/D cycle length. Example 1 and Figs. 3 and 4 effectively represent the scenario where there is no constraint on the cycle length.

5. An exact, complete solution of Problem (P1)

We return to the original problem (P1), including the constraint on the cycle length $L \leq T$, or $Q \leq DT$, alongside the existing constraint on the proportional production time $r \leq r_{\max}$.

5.1. The maximal possible number of shipments for each production lot

The optimal production rate (6) is not affected by the constraint on the cycle length; thus,

$$r_L^*(n) = \begin{cases} D/U & n = 1 \\ r_{\max} & n \geq 2 \end{cases} \quad (20)$$

where, $r_L^*(n)$ is the doubly constrained minimum. Since objective (5) is strictly convex in q ,

$$q_L^*(n) = \min \{ DT/n, q^*(n) \} \quad (21)$$

For $n \geq 1$, where $q^*(n)$ is given in (9) and $q_L^*(n)$ is the doubly constrained minimum, given n . For $n \geq 2$, $Q^*(n)$ in (10) increases in n . Denote by n_{\max}^T the maximal possible number of shipments for each production lot when $n \geq 2$, that is

$$n_{\max}^T = \max \left\{ \text{integer } n, n \geq 2 \mid \sqrt{\frac{2ne_n^{\text{par}}}{h_V(n-1) - h_V r_{\max}(n-2) + h_B}} \leq DT \right\} \quad (22)$$

If $Q^*(2) > DT$, that is, if $\sqrt{4e_2^{\text{par}}/(h_V + h_B)} > DT$, we let $n_{\max}^T = 1$.

The following proposition summarizes the conclusion that the maximal possible number of shipments for each production lot is either $n_{\max}^T = 1$ or, when $n_{\max}^T \geq 2$, it is given by the "integer part" (i.e., rounding down) of the positive root of a quadratic equation in n (denoted below by $p(n) = 0$).

Proposition 3. Let $p(n)$ be the convex parabola $p(n) = An^2 + Bn + C$, where

$$A = 2Dk, B = 2DK - h_V(1 - r_{\max})D^2T^2, \text{ and } C = D^2T^2(h_V(1 - 2r_{\max}) - h_B) \quad (23)$$

If $n_{\max}^T \geq 2$ then

$$n_{\max}^T = \left\lfloor \frac{-B + \sqrt{B^2 - 4AC}}{2A} \right\rfloor \quad (24)$$

where, $\lfloor x \rfloor$ signifies the "integer part" (i.e., the rounding down) of x , and A, B and C are given in (23). Otherwise, $n_{\max}^T = 1$.

5.2. The optimal number of shipments for each production lot

The rest of this section is devoted to finding the optimal number of shipments in a single lot, n^* , when we consider two key intermediate variables, n_{\max}^T (the maximal possible number of shipments per production lot) and n_{\min}^{act} (defined below). Below, in (25), is the expression for the total cost. Consider the case where the constraint representing an upper bound on the cycle length becomes active (i.e., $Q = DT$ is inserted into (7)). Using the superscript *act* to denote this case, we obtain:

$$C_T^{act}(1) = \frac{e_1^{par}}{DT} + \frac{h_V D^2 T}{2U} + \frac{h_B D T}{2}$$

$$C_T^{act}(n) = \frac{K + kn}{T} + \frac{h_V D T}{2n} (n(1 - r_{\max}) + 2r_{\max}) + \frac{(h_B - h_V) D T}{2n}. \quad (25)$$

Treating n as a continuous variable, and following (25), we denote n_{\min}^{act} as

$$n_{\min}^{act} = \sqrt{\frac{DT^2(h_B - h_V(1 - 2r_{\max}))}{2k}}, \text{ if } h_B \geq h_V(1 - 2r_{\max}); \text{ otherwise } n_{\min}^{act} = n_{\max}^T \quad (26)$$

Below we generalize Theorem 1 to consider the case where there is an upper bound on the cycle length. Define

$$Q_1^{act} = \sqrt{\frac{2UD(K + k)}{h_V D + h_B U}} \text{ and } n_{\min}$$

$$= \begin{cases} \sqrt{\frac{K(h_B - h_V(1 - 2r_{\max}))}{kh_V(1 - r_{\max})}} & h_B \geq h_V(1 - 2r_{\max}) \\ 2 & \text{otherwise} \end{cases}$$

Theorem 2. The optimal number of shipments for each production lot n^* (Problem (P1)) is specified below. The specification is arranged according to six cases:

| | |
|---|---|
| Case a. $Q_1^{par} \leq DT$ and $0 < n_{\min} < 1$ | |
| $0 < n_{\min}^{act} < n_{\max}^T + 1$ | $n^* = \argmin\{C_T^{act}(n_{\max}^T + 1), C_t(1)\}$ |
| $n_{\max}^T + 1 \leq n_{\min}^{act}$ | $\argmin\{C_T^{act}(\lfloor n_{\min}^{act} \rfloor), C_T^{act}(\lceil n_{\min}^{act} \rceil), C_t(1)\}$ |
| Case b. $Q_1^{par} \leq DT$ and $1 \leq n_{\min} < n_{\max}^T$ | |
| $0 < n_{\min}^{act} < n_{\max}^T + 1$ | $n^* = \argmin\{C_t(\lfloor n_{\min} \rfloor), C_t(\lceil n_{\min} \rceil), C_t(1), C_T^{act}(n_{\max}^T + 1)\}$ |
| $n_{\max}^T + 1 \leq n_{\min}^{act}$ | $\argmin\{C_t(\lfloor n_{\min} \rfloor), C_t(\lceil n_{\min} \rceil), C_t(1), C_T^{act}(\lfloor n_{\min}^{act} \rfloor), C_T^{act}(\lceil n_{\min}^{act} \rceil)\}$ |
| Case c. $Q_1^{par} \leq DT$ and $n_{\max}^T \leq n_{\min}$ | |
| $0 < n_{\min}^{act} < n_{\max}^T + 1$ | $n^* = \argmin\{C_t(n_{\max}^T), C_t(1), C_T^{act}(n_{\max}^T + 1)\}$ |
| $n_{\max}^T + 1 \leq n_{\min}^{act}$ | $\argmin\{C_t(n_{\max}^T), C_t(1), C_T^{act}(\lfloor n_{\min}^{act} \rfloor), C_T^{act}(\lceil n_{\min}^{act} \rceil)\}$ |
| Case d. $Q_1^{par} > DT$ and $0 < n_{\min} < 1$ | |
| $0 < n_{\min}^{act} < n_{\max}^T + 1$ | $n^* = \argmin\{C_T^{act}(n_{\max}^T + 1), C_T^{act}(1)\}$ |
| $n_{\max}^T + 1 \leq n_{\min}^{act}$ | $\argmin\{C_T^{act}(\lfloor n_{\min}^{act} \rfloor), C_T^{act}(\lceil n_{\min}^{act} \rceil), C_T^{act}(1)\}$ |
| Case e. $Q_1^{par} > DT$ and $1 \leq n_{\min} < n_{\max}^T$ | |
| $0 < n_{\min}^{act} < n_{\max}^T + 1$ | $n^* = \argmin\{C_t(\lfloor n_{\min} \rfloor), C_t(\lceil n_{\min} \rceil), C_T^{act}(1), C_T^{act}(n_{\max}^T + 1)\}$ |
| $n_{\max}^T + 1 \leq n_{\min}^{act}$ | $\argmin\{C_t(\lfloor n_{\min} \rfloor), C_t(\lceil n_{\min} \rceil), C_T^{act}(1), C_T^{act}(\lfloor n_{\min}^{act} \rfloor), C_T^{act}(\lceil n_{\min}^{act} \rceil)\}$ |
| Case f. $Q_1^{par} > DT$ and $n_{\max}^T \leq n_{\min}$ | |
| $0 < n_{\min}^{act} < n_{\max}^T + 1$ | $n^* = \argmin\{C_t(n_{\max}^T), C_T^{act}(1), C_T^{act}(n_{\max}^T + 1)\}$ |
| $n_{\max}^T + 1 \leq n_{\min}^{act}$ | $\argmin\{C_t(n_{\max}^T), C_T^{act}(1), C_T^{act}(\lfloor n_{\min}^{act} \rfloor), C_T^{act}(\lceil n_{\min}^{act} \rceil)\}$ |

Proof. The theorem results from (i) the convexity of $C_t(n)$ and of $C_T^{act}(n)$, (ii) the relative position of n_{\max}^T to n_{\min} and of n_{\max}^T to n_{\min}^{act} , and (iii) the satisfiability of constraint (22) at $n = 1$.

Note that n_{\max}^T is independent of the maximal production rate U (Eq.

(24)). Nevertheless, n^* depends on U through the conditioning on Q_1^{par} and the values of $C_t(1)$ and $C_T^{act}(1)$.

5.3. A numerical illustration of the effect of the bound on the cycle length, T

Example 2. The effect of the bound on the cycle length, T

Using the parameter values of Example 1 above, let us employ Theorem 2 and Proposition 3 to calculate n^* , together with the other two decision variables P^* and q^* . We set $r_{\max} = 0.75$. Table 2 shows the characteristics of the overall vendor-buyer optimal policy as a function of the constraint T (days) on the cycle length. We note that the unconstrained value of n^* (given in Theorem 1) is 17, $n_{\min} = 17.32$, and $Q_1^{par} = 381.725$. These values of n_{\min} and Q_1^{par} allow us to obtain the list of cases in Table 2. For example, if $T = 1$, Case *f* applies (since $n_{\max}^T = 1$ and $n_{\min}^{act} = 3.88$), while if $T = 5$, Case *b* applies (since $n_{\max}^T = 26$ and $n_{\min}^{act} = 19.36$).

For values of T of up to 4 days, it is T that binds the optimization problem: the cycle length is T , and Q is equal to the total demand during the cycle time. A bound on the cycle length that is too low may be very costly, as shown by the case $T = 1$, where the optimal cost is almost twice the value of the optimal cost for $T = 2$. For higher values of T ($T = 5, 6$), it is r_{\max} that is binding. It can be seen that Q for these cases is less than 0.5% smaller than the value of 894.4 that is yielded by the well-known "EOQ-with-P" formula, $Q^2 = 2DK/h_V(1 - r)$ for $r = r_{\max}$. P is set to the same value for all cases, which is the minimum value determined from r_{\max} . Similarly, q does not seem to change significantly as a function of T . Indeed, comparing the optimal solution of the continuous model with the present solution indicates that the differences between them are merely artifacts of the integrality of n^* . n^* is determined in line with the specification in Theorem 2. The pattern observed in the last two rows of Table 2 (the triple n^*, P^*, Q^*) is essentially unchanged for $T \geq 5$. Of course, by applying different parameter settings of the example, it would be possible to obtain the remainder of the cases specified in Theorem 2 (i.e., Cases *a*, *d* and *e*).

6. Comparison between the integrated and the partially-coordinated supply chains

In this section we compare the integrated supply-chain mode with the self-optimization mode, where the latter implies that each of the parties makes its own decisions with the goal of minimizing self-expenditure.

6.1. The self-optimization mode

Recalling (1), and avoiding repetition of the analysis detailed in previous sections for each of the parties, we denote by $C_t^B(Q, P, n)$ and $C_t^V(Q, P, n)$ the objectives (costs per unit time) of the buyer and the vendor, respectively. From (1),

$$C_t^B(Q, P, n) = \lim_{\substack{K \rightarrow 0 \\ h_V \rightarrow 0 \\ k_V \rightarrow 0}} C_t(Q, P, n) = \frac{k_B D n}{Q} + h_B \frac{Q}{2n} \quad (27)$$

and

$$C_t^V(Q, P, n) = \lim_{\substack{h_B \rightarrow 0 \\ k_B \rightarrow 0}} C_t(Q, P, n)$$

$$= \frac{DK}{Q} + \frac{k_V D \cdot n}{Q} + h_V \left(\frac{D}{P} \frac{Q}{n} + \frac{Q}{2} \left(1 - \frac{D}{P} \right) - \frac{Q}{2n} \right). \quad (28)$$

Note that the limits in (27) and (28) are used to obtain the objectives of the parties. Interactions between the parties still exist in the partially-coordinated supply-chain model. For example, it is assumed that the retailer provides full information about the market demand. Moreover,

the vendor takes on part of the shipping cost, which obviously reduces the costs on the buyer's side, but may also be in vendor's interests (as is demonstrated in Section 6.5). We define $\rho_C \equiv k_V/k$, where $0 \leq \rho_C < 1$, as the level of coordination between the parties. A higher value of ρ_C signifies a greater level of coordination between the parties.

6.2. The optimal decision from the buyer's perspective

We use (27) to introduce the optimal decision made by the buyer. The retailer decides upon the shipment size q , while the manufacturer's key decisions concern the production rate P and the lot size Q . We start with the retailer, who seeks the optimal q . The problem is

$$\min_{0 < q < \infty} C_t^B(Q, P, n) = \min_{0 < q < \infty} \left\{ \frac{k_B D}{q} + h_B \frac{q}{2} \right\}. \quad (29)$$

Assigning $k_B = k - k_V$, the optimal solution is the familiar EOQ, that is,

$$q^* = q^*(k_V) = \sqrt{\frac{2D(k - k_V)}{h_B}}. \quad (30)$$

Unlike the classical EOQ solution, the optimal shipment size (30) depends on the partial shipment cost k_V , which is determined solely by the manufacturer. This decision variable plays a key role in the following subsections.

6.3. Problem formulation from the manufacturer's perspective

Keeping in mind that the manufacturer sets the partial shipment cost k_V , he may use (6) to obtain the optimal production rate ratio $r^*(n)$, given the number of shipments n (an integer) and the shipment size q , for the case where the cycle length is bounded:

$$r^*(q, n, k_V) = r^*(n) = \begin{cases} D/U & n = 1 \\ r_{\max} & n \geq 2 \end{cases}. \quad (31.1)$$

The optimal lot-size $Q^*(n, k_V)$ given n and k_V (continuous), for the case where the cycle length is restricted, is

$$Q^*(n, k_V) = nq^*(k_V), \quad \forall (n, k_V) \in \{nq^*(k_V) \leq DT\}. \quad (31.2)$$

The above expressions represent the results of the optimization when the cycle length is restricted. From (31.2), it can be seen that $Q^*(n, k_V)$ increases with n and decreases with k_V , and that not all pairs (n, k_V) are feasible. We use the optimal solution (31.1) and (31.2) and the limits in ((28) to introduce the optimal decisions made by the manufacturer. It remains for the manufacturer to make decisions about the number of shipments n (integer) in each production lot and the sharing ratio ρ_C of the shipment cost (k). Thus, the optimization problem is formulated as

$$\min_{\substack{1 \leq n < \infty \\ 0 \leq k_V < k}} C_t^V(q^*(k_V), r^*(n), n) = \min_{\substack{1 \leq n < \infty \\ 0 \leq k_V < k}} \left\{ K \frac{D}{Q^*(n, k_V)} + k_V \frac{D}{q^*(k_V)} + h_V \left[(1 - r^*(n)) \frac{Q^*(n, k_V)}{2} + r^*(n) q^*(k_V) - \frac{q^*(k_V)}{2} \right] \right\}.$$

We address this problem in a two-stage nested design. Feasible solutions for a nested multinomial logit model are derived by Roohnavazfar et al. [47]. The first stage finds the optimal k_V when n is assumed

to be given, while the second stage seeks to determine the optimal n . When n is given, $Q^*(k_V) = n\sqrt{\frac{2D(k - k_V)}{h_B}} \quad \forall k_V \in \left\{ n\sqrt{\frac{2D(k - k_V)}{h_B}} \leq DT \right\}$, meaning that k_V is bounded from below by $k_V^{\text{lower}}(n)$, $\frac{2n^2 k - DT^2 h_B}{2n^2} = k - \frac{DT^2 h_B}{2n^2}$, that is,

$$k_V^{\text{lower}}(n) = \max \left\{ 0, k - \frac{DT^2 h_B}{2n^2} \right\}. \quad (32)$$

Expression (32) indicates that $k_V^{\text{lower}}(n)$ is not always differentiable and that the greater the value of n , the larger the minimal share of the shipment cost taken on by the manufacturer in order to keep the cycle length bounded. Thus, under $k_V^{\text{lower}}(n) \leq k_V < k$, we obtain $Q^*(n, k_V) = n\sqrt{\frac{2D(k - k_V)}{h_B}}$. The inner optimization (when n is given) becomes

$$\min_{k_V^{\text{lower}}(n) \leq k_V < k} \left\{ K \frac{D}{n\sqrt{\frac{2D(k - k_V)}{h_B}}} + k_V \frac{D}{\sqrt{\frac{2D(k - k_V)}{h_B}}} + \frac{h_V}{2} \left[((n - 1)(1 - r^*(n)) + r^*(n)) \sqrt{\frac{2D(k - k_V)}{h_B}} \right] \right\}. \quad (33)$$

6.4. The optimal decisions from the manufacturer's perspective

The following lemma states the unimodality of the objective $C_t^V(k_V, r(n), n)$ with respect to the variable k_V . It allows $k_V^*(n)$ to be easily found by any standard numerical search over a single dimension.

Lemma 1. The objective (given n) $C_t^V(k_V, r(n), n)$ has at most a single local minimum over $k_V \in [k_V^{\text{lower}}(n), k]$.

Proof.

See Appendix B.

We summarize by stating the following proposition:

Proposition 4. The optimal shipment cost (given n) associated with the manufacturer, $k_V^*(n)$ is

$$k_V^*(n) = \begin{cases} k_V^{\text{lower}}(n) & \text{if } k_V^{\text{min}}(n) \leq k_V^{\text{lower}}(n) \\ \min\{k, k_V^{\text{min}}(n)\} & \text{if } k_V^{\text{min}}(n) > k_V^{\text{lower}}(n) \end{cases}, \quad (34)$$

where,

$$k_V^{\text{min}}(n) = \frac{\frac{Kh_B}{n} + k[2h_B - h_V((n - 1)(1 - r^*(n)) + r^*(n))]}{h_B - h_V((n - 1)(1 - r^*(n)) + r^*(n))}, \quad \forall n \geq 1. \quad (35)$$

Proof.

The result can be straightforwardly derived from Lemma 1. \square

It remains for the manufacturer to seek the optimal n, n^* , according to:

$$\min_{1 \leq n < \infty} C_t^V(r(n), n) = \min_{1 \leq n < \infty} \left\{ K \frac{D}{n \sqrt{\frac{2D(k - k_V^*(n))}{h_B}}} + k_V^*(n) \frac{D}{\sqrt{\frac{2D(k - k_V^*(n))}{h_B}}} + \frac{h_V}{2} \left[((n-1)(1-r^*(n)) + r^*(n)) \sqrt{\frac{2D(k - k_V^*(n))}{h_B}} \right] \right\}. \quad (36)$$

The following lemma assists in seeking the optimal n^* , while [Theorem 3](#) (below) provides the expression for determining n^* .

Lemma 2. The objective $C_t^V(r(n), n)$ has at least a single local minimum over $n \in [2, \infty)$.

Proof.

See [Appendix C](#).

Theorem 3. Let $\{n_i^*, i = 1, 2, \dots, M\}$ denote the set of all local minimum points over $n \in [2, \infty)$. The optimal number of shipments for each production lot n^* is $n^* = \operatorname{argmin}_{i=1,2,\dots,M} \{C_t^V(\lfloor n_i^* \rfloor), C_t^V(\lceil n_i^* \rceil), C_t^V(1)\}$.

Proof.

Can be immediately derived from [Lemma 2](#) and from the discontinuity of $r^*(n)$ at $n=1$. \square

6.5. Numerical illustration

[Fig. 6](#) illustrates, for $T=6$ (r_{\max} is binding) and for $T=2$ (T is binding), the ratio between the total optimal cost yielded by the partially-coordinated model and the optimal cost yielded by the integrated model. The horizontal axis represents the level of coordination, ρ_C , expressed as a percentage. The results show that for both bounds, the optimal cost ratio increases with the level of coordination, ρ_C , where the rate of increase is faster for $T=2$ than for $T=6$.

For the case where $T=6$ (r_{\max} is binding), a relatively small sharing ratio (i.e., proportion of the shipping costs paid by the manufacturer), of the order of 7%, is sufficient to achieve a loss (i.e., increase in costs) relative to the optimal integrated mode (although note that the difference in optimal costs is less than 0.5%). A further increase in the sharing ratio significantly increases the ratio between the costs. For example, for the case of a fifty-fifty sharing ratio (e.g., $k_B = k_V = 50$ \$), the optimal total cost is 5% higher than the optimal cost of the integrated model. It is worth noting that this loss of 5% is not evenly distributed between the parties. In fact, the manufacturer's cost increases by nearly 13.5% while the buyer's cost decreases by nearly 30%. This, of course, further discourages the manufacturer from sharing the shipment cost with the buyer. The case where $T=2$ illustrates the infeasibility of some of the (theoretically) possible manufacturer's decisions. In particular, insisting on a low participation level (lower than 18.37%) yields $Q^*(k_V) = DT = 400$ and $n = 400/q^*(k_V)$. The obtained value of n , following (30), might not be an integer, which is infeasible according to the problem's setting.

The previous examples show that under this setting, the manufacturer would select a relatively low level of participation in the shipping costs, that is, approximately 7% (for $T=6$) and 18.37% (for $T=2$), respectively. Furthermore, these participation levels result in the lowest possible total costs (i.e., close to those of the integrated system). Yet, this is not always the case. Consider, for example, the case where $T=6$ and $h_V = 50$ \$ (see [Fig. 7](#)), i.e., the case where the holding costs at the manufacturer's warehouse are considerably higher than those of the buyer.

A comparison between [Figs. 6](#) and [7](#) shows that the ratio between the holding costs (for a unit, per unit of time) of the two parties is a crucial parameter in determining the manufacturer's preferred level of coordination. The optimal cost of the integrated system for the setting illustrated in [Fig. 7](#) is 6183.31\$. In the case where the manufacturer does

not coordinate with the buyer at all, an increase of more than 6% in the total costs is observed (optimal costs of 5929.27\$ for the vendor and 632.45\$ for the buyer). However, increasing the coordination level from 0 to nearly 45% reduces the loss such that the optimal cost of the partially-coordinated system is only 1.2% higher than that of the integrated system (optimal costs of 5788.40\$ for the vendor and 469.042\$ for the buyer in the partially-coordinated system). This outcome is profitable for both sides, although the buyer does not play as active a role as in the integrated model.

7. Conclusions

7.1. Summary

Two supply-chain settings, i.e., an integrated inventory mode and a self-optimization mode, are analyzed. We derive intermediate results, which are of interest in their own right. We also provide analytical solutions. Through numerical examples, we highlight the effect of the production rate on the integrated inventory system, including on the maximal mutual inventory. Further, we examine the effect of the shipment cost-sharing ratio on performance when the parties act independently.

7.2. Conclusions

The mathematical analysis leads to relatively general theoretical conclusions. The optimal production rate P is insensitive to the lot size. Interestingly, unless a production run consists of exactly two equal shipments, the optimal strategy is to produce either at the maximum rate (if the whole batch is shipped at once) or at the minimal allowable rate (if there are three shipments or more). A general conclusion from (14) is that P does not affect the buyer's costs. This reduces the interdependency between the two parties, thus increasing the likelihood that they will cooperate.

The vendor's participation is expressed through his contribution to the shipment costs. This obviously reduces the costs on the buyer's side, as the buyer initially pays for shipping (EOQ setting), but it may also be beneficial for the vendor. The numerical analysis demonstrates that the manufacturer's share of the shipment cost, k_V , is a central control variable in affecting the total cost, under the assumption that each of the players acts independently. When both parties have similar holding costs (per unit, per unit time), the performance of the manufacturer suffers when he increases his participation in the shipping costs.

7.3. Managerial aspects

This research mainly generates theoretical findings. However, it also offers several managerial insights, as outlined below:

- In the partially-coordinated mode (i.e., each of the players makes its own decisions), and in the case where the buyer initially pays the shipping cost (EOQ setting), the manufacturer can set the shipment cost-sharing ratio, without the buyer's cooperation, such that the total costs are very close to the lower bound (i.e., close to the optimal costs of the integrated chain). In comparison

to the integrated supply chain, the advantage of coordinating the supply chain through sharing the shipment cost is that the manufacturer does not require the agreement of the buyer. A simple way of implementing this strategy in practice would be for the manufacturer to offer the buyer to participate in her shipping cost. Since the buyer can only benefit from such an offer (by experiencing lower costs), an objection is not expected.

- (b) By offering to share in the shipment cost, the manufacturer may either increase or reduce his own expenditure (relative to the integrated chain). This means that the strategy of offering to contribute to the shipping cost should be exercised with caution. The key indicator for the manufacturer to initiate cooperation is when his holding cost is high relative to that of the buyer.
- (c) In the case where the number of shipments n is greater than 1 (i.e., the production lot is split into multiple shipments), the manufacturer may choose $P = P_{\min}$ without losing optimality. This, of course, makes the decision-making process easier, but it also grants a longer shelf-life to the machinery and tools used in the production process.

7.4. Limitations and future research directions

The proposed model suffers from several limitations, some of which may suggest avenues for future research:

- (a) The proposed model is not intended to be the most general model ever published on the integrated manufacturer-buyer supply problem. Real-life systems may include many other elements that are not considered in our model (e.g., stochastic demand, variable lead-time, environmental aspects and partial information sharing). Some of these elements have been addressed in the existing literature by means of iterative methods and efficient algorithms. We leave it to future research to seek explicit analytical solutions to such generalizations.
- (b) In some real-life situations, the parties may not wish to cooperate fully. Our model, in this case, suggests the partially-coordinated

supply chain, in which each of the parties makes its own decisions undisturbed. However, the assumption that the partial shipment cost k_V will be determined solely by the manufacturer is likely to be questionable in some real-life situations (for example, the situation where the initial mode dictates that the manufacturer pays for shipping), as the actions of the manufacturer (changing his share of the shipment cost) might lead to an increase in the buyer's expenditure. Consequently, the buyer may respond either by initiating an integrated system or by stopping her dealings with the manufacturer. This possibility is not considered in our paper and it is left for future research.

- (c) It is known that the integrated chain always leads to the lowest total cost. Yet, the assumption of full cooperation depends on several factors, including the simple condition that the cost each player incurs in the integrated chain is lower than that incurred in the non-cooperative mode. Future models could incorporate a revenue-sharing agreement that guarantees this condition is met.
- (d) Finally, we suggest that future research should determine the exact conditions (i.e., the minimal ratio between the players' holding costs) under which the manufacturer would prefer the self-optimization mode to the integrated supply-chain mode.

CRediT authorship contribution statement

Avi Herbon: Conceptualization, Funding acquisition, Writing – original draft, Writing – review & editing, Visualization. **Israel David:** Conceptualization, Writing – original draft, Writing – review & editing, Visualization.

Declaration of Competing Interest

None.

Data availability

No data was used for the research described in the article.

Appendix A. Proof of Theorem 1

Under the condition cited in the theorem, $C_t(n)$ of (15) is convex over $n \geq 2$; it is not necessarily convex over $n \geq 1$. By this convexity, the set $N(1) \equiv \{n | n \geq 2, C_t(n) < C_t(1)\}$ is either empty or comprised of a finite number of consecutive integers. Since $C_t(n)$ is unbounded, $N(1)$ is bounded and (if nonempty) it possesses an integer maximum. Note now that $R(1) = (r_{\max} - r_{\min})(1 + R_k)$, which is positive. Thus, in Case I, $R(n)$ for $n \geq 2$ consists of an ascending branch of a convex parabola, such that $C_t(n) > C_t(1)$ for all $n \geq 2$. It follows that $N(1)$ is empty and that $n = 1$ shipment per batch is a global minimum. Case II implies again that the set $N(1)$ is empty, with the same conclusion as before. In Case III, $N(1)$ is not empty, so $n^* \geq 2$. We know, as established at the outset (Proposition 1, Eq. (6)), that r_{\max} is the optimal r for any $n \geq 2$. Thus, by continuous optimization, $n^*(r_{\max})$ should be used to minimize n in $N(1)$, i.e., to determine n^* . However, $n^*(r_{\max})$ is mostly non-integer. Thus, in letting $m = \lfloor n^*(r_{\max}) \rfloor$ (integer value), it follows from continuous optimization that

$$m^2 \leq \frac{K(h_B - h_V(1 - 2r_{\max}))}{kh_V(1 - r_{\max})} < (m + 1)^2. \quad (\text{A.1})$$

This double inequality reduces to

$$(1 - 2r_{\max}) + m^2 \frac{k}{K} (1 - r_{\max}) \leq R_h < (1 - 2r_{\max}) + (m + 1)^2 \frac{k}{K} (1 - r_{\max}), \text{ which implies that (19) holds.}$$

Appendix B. Proof of Lemma 1

Taking the first derivative with respect to variable k_V results in

$$\frac{\partial}{\partial k_V} C_t^V(q^*(k_V), r^*(n), n) = \left\{ \begin{array}{l} \frac{D^2 K \sqrt{h_B} [2D(k - k_V)]^{-3/2}}{n} + \\ \frac{[2D(k - k_V)]^{1/2} D \sqrt{h_B} + k_V D^2 \sqrt{h_B} [2D(k - k_V)]^{-1/2}}{2D(k - k_V)} \\ \frac{h_V}{2\sqrt{h_B}} ((n - 1)(1 - r^*(n)) + r^*(n)) D [2D(k - k_V)]^{-1/2} \end{array} \right\}$$

The FOC is

$$\begin{aligned} \frac{D^2 K \sqrt{h_B}}{n} + [2D(k - k_V)]D\sqrt{h_B} + k_V D^2 \sqrt{h_B} - \frac{h_V}{2\sqrt{h_B}}((n-1)(1-r^*(n)) + r^*(n))D[2D(k - k_V)] &= 0 \\ \frac{D^2 K \sqrt{h_B}}{n} + 2D^2 k \sqrt{h_B} - \frac{h_V}{\sqrt{h_B}}((n-1)(1-r^*(n)) + r^*(n))D^2 k &= \\ k_V \left[2D^2 \sqrt{h_B} - D^2 \sqrt{h_B} - \frac{h_V}{\sqrt{h_B}}((n-1)(1-r^*(n)) + r^*(n))D^2 \right] & \end{aligned} \quad (B.1)$$

with the solution

$$k_V^{\min}(n) = \frac{\frac{Kh_B}{n} + k[2h_B - h_V((n-1)(1-r^*(n)) + r^*(n))]}{h_B - h_V((n-1)(1-r^*(n)) + r^*(n))}, \forall n \geq 1 \quad (B.2)$$

A unique solution (B.2) always exists. Thus, in every finite domain, such as $k_V \in [k_V^{\text{lower}}(n), k]$, there can be no more than a single extreme point. It is left to show that this point must be a minimal point. Rearranging the first derivative gives

$$[2D(k - k_V)]^{-3/2} \left\{ \frac{D^2 K \sqrt{h_B}}{n} + D^2 k \sqrt{h_B} + (k - k_V) \left[D^2 \sqrt{h_B} - \frac{h_V}{\sqrt{h_B}}((n-1)(1-r^*(n)) + r^*(n))D^2 \right] \right\}.$$

As $k_V \rightarrow k$, the derivative function is positive (since $k_V < k$). At $k_V = k_V^{\text{lower}}(n)$, there are two possibilities: either the derivative is positive (thus the derivative function remains non-negative, never crossing the horizontal axis) or there is an outcome in which there are no local points. In the case where the derivative as $k_V \rightarrow k_V^{\text{lower}}(n)$ is negative, the derivative function must cross the horizontal axis one time at the value $k_V = k_V^{\min}(n)$, altering its sign from negative to positive, implying that the local point is a minimum.

Appendix C. Proof of Lemma 2

Since, for $n = 1$, a discontinuity in $C_t^V(r(n), n)$ is expected, we separately compute $C_t^V(r(1), 1)$ and proceed by assuming $n \geq 2$; thus

$$\min_{2 \leq n < \infty} C_t^V(r(n), n) = \left\{ \begin{aligned} & n \sqrt{\frac{DK}{2D(k - k_V^*(n))h_B}} + k_V^*(n) \frac{D}{n \sqrt{2D(k - k_V^*(n))h_B}} + \\ & \frac{h_V}{2} \left[((n-1)(1-r_{\max}) + r_{\max}) \sqrt{\frac{2D(k - k_V^*(n))}{h_B}} \right] \end{aligned} \right\}.$$

Examining $k_V^*(n)$ for continuous n , this is a continuous function and is not necessarily differentiable ($k_V^{\text{lower}}(n)$ starts at the value of 0 for $n = 0$ and then at some point starts to increase, although it never reaches k ; $\min\{k, k_V^{\min}(n)\}$ starts at the value of k for $n = 1$, decreases with n and becomes negative, and then increases again; however, it never reaches k). As $n \rightarrow 0$, the function $C_t^V(r(n), n)$ approaches infinity. As $n \rightarrow \infty$, the function $\sqrt{\frac{2D(k - k_V^*(n))}{h_B}} \rightarrow 0$ (in order to keep $n \sqrt{\frac{2D(k - k_V^*(n))}{h_B}} \leq DT$). Thus, as $n \rightarrow \infty$, $C_t^V(r(n), n)$ approaches infinity. We conclude that since $C_t^V(r(n), n)$ is continuous, it must have at least one local minimum. □

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