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## Article

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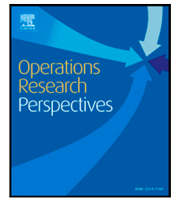
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# Originating multiple-objective portfolio selection by counter-COVID measures and analytically instigating robust optimization by mean-parameterized nondominated paths

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## ABSTRACT

The COVID-19 pandemic is unleashing crises of humanity, economy, and finance. Portfolio selection is widely recognized as the foundation of modern financial economics. Therefore, it is naturally crucial and inviting to utilize portfolio selection in order to counter COVID-19 in stock markets. We originate a counter-COVID measure for stocks, extend portfolio selection, and construct multiple-objective portfolio selection. Because of the uncertainty in measuring counter-COVID, we perform robust optimization. Specifically, we analytically compute the optimal solutions as a trail of an optimal portfolio due to the change of counter-COVID. We call the trail as *mean-parameterized nondominated path*. Moreover, the path is a continuous function of the change, so the portfolio relatively mildly varies for the change. In contrast, researchers typically still focus on 2-objective robust illustrations and infrequently explicitly compute the optimal solutions for multiple-objective portfolio optimization.

To the best of our knowledge, there is limited research for multiple-objective portfolio selection of COVID and for the robust optimization of multiple-objective portfolio selection. In such an area, this paper contributes to the literature. The implications to fight COVID are that investors minimize risk, maximize return, and maximize counter-COVID in stock markets and that investors ascertain the multiple-objective portfolio selection as relatively robust.

## 1. Introduction

In this section, we briefly review portfolio selection and robust optimization and accordingly highlight our originality.<sup>2</sup> Specifically, we present models to clarify the review and our originality. Moreover, we clearly separate this inclusive introduction into concise subsections. We will further review the literature in Section 2.

### 1.1. Portfolio selection and the robust optimization

Nobel Laureate Markowitz [2, p. 6] emphasizes both risk and return. As the birth of modern finance (as emphasized by Rubinstein

[3, p. 1041]), Markowitz [4, p. 83] formulates portfolio selection as the following 2-objective optimization:

$$\begin{aligned} \min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\ \max z_2 &= \mathbf{x}^T \boldsymbol{\mu}_2 \\ \text{subject to } \mathbf{x} &\in S, \end{aligned} \quad (1)$$

where for  $n$  stocks,  $\Sigma$  is a covariance matrix of stock returns,  $\boldsymbol{\mu}_2$  is a vector of stock expected returns, investors determine a portfolio by its weight vector  $\mathbf{x}$ ,  $z_1$  and  $z_2$  respectively measure the variance and expectation of the portfolio return, and  $S$  is a feasible region in  $\mathbb{R}^n$ .<sup>3</sup> The optimal solutions of (1) are called *efficient frontier*.

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<sup>2</sup> Ehrgott et al. [1, p. 17] define robust optimization as generalized optimization under the presence of uncertain parameters.

<sup>3</sup> For notations, normal symbols (e.g.,  $n$ ) denote scalars; bold-face symbols (e.g.,  $\Sigma$  or  $\boldsymbol{\mu}_2$ ) denote matrices or vectors. Some bold-face symbols bear subscripts. For instance,  $\boldsymbol{\mu}_2$  and  $\boldsymbol{\mu}_3$  denote vectors; subscripts 2 and 3 specify which vector. Overall, we designate  $\mathbf{x}$  for a portfolio-weight vector, designate  $\mathbf{z}$  for a portfolio in  $(z_1, z_2, z_3)$  space, and match  $\mathbf{x}$ -symbols with  $\mathbf{z}$ -symbols (e.g.,  $x_i$  with  $z_i$ ).

Due to the uncertainty in measuring  $\mu_2$ , Best and Grauer [5, pp. 322–326] dissect the robust optimization for (1) in the following parametric-quadratic-programming model:

$$\begin{aligned} \min \mathbf{x}^T \Sigma \mathbf{x} - \lambda_2 \mathbf{x}^T (\mu_2 + t_2 \Delta_2), \lambda_2 \geq 0, \\ \text{subject to } \mathbf{x}^T \mathbf{1} = 1, \end{aligned} \quad (2)$$

where  $\mathbf{1}$  is a vector of ones,  $\lambda_2$  is a (scalar) parameter and can be interpreted as investors' risk-tolerance parameter to expected return,  $t_2$  is a (scalar) parameter for the uncertain amount,  $\Delta_2$  is a fixed vector for a specific direction, and  $t_2 \Delta_2$  dictates the uncertainty in measuring  $\mu_2$ . For given  $\lambda_2$  and with  $t_2$  varying, Best and Grauer [5] analytically compute a trail of an optimal portfolio with the varying of  $t_2$ . The trail stands as the solution of (2) and describes the portfolio's sensitivity with respect to  $t_2$ . Best and Grauer [5, p. 324] call the trail as *mean-parameterized efficient frontier*.

### 1.2. Multiple-objective portfolio selection

Markowitz [6, pp. 471 & 476] detects extra objectives in addition to  $z_1$  and  $z_2$  of (1). Sharpe [7] also realizes the objectives and incorporates them into utility functions. Fama [8, pp. 445–447] and Cochrane [9, pp. 1081–1082] emphasize multiple factors for asset pricing and further propose the factors' risks as objectives.

Fama [8]; Steuer et al. [10]; Dorfleitner et al. [11]; Utz et al. [12]; Qi et al. [13]; Qi and Steuer [14]; and Qi [15] extend (1) and formulate the extra objectives by multiple-objective portfolio selection as follows:

$$\begin{aligned} \min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\ \max z_2 &= \mathbf{x}^T \mu_2 \\ \max z_3 &= \mathbf{x}^T \mu_3 \\ &\vdots \\ \max z_k &= \mathbf{x}^T \mu_k \\ \text{subject to } \mathbf{x} &\in S, \end{aligned} \quad (3)$$

where  $\mu_3 \dots \mu_k$  are vectors of general stock expected objectives (e.g., corporate social responsibility);  $z_3 \dots z_k$  measure the general portfolio expected objectives.

Overall, Spronk and Hallerbach [16], Bana e Costa and Soares [17], Steuer and Na [18], Zopounidis et al. [19], Masmoudi and Abdelaziz [20], and Aouni et al. [21] offer surveys. In this paper, multiple objectives mean three or more objectives.

### 1.3. Research limitation of robust optimization for multiple-objective portfolio selection

Many researchers investigate multiple-objective robust optimization (as surveyed by Ide and Schöbel [22], Georgantass et al. [23], and Ghahtarani et al. [24]). However, the researchers infrequently explicitly compute the optimal solutions for multiple-objective portfolio optimization. For example, Fliege and Werner [25, pp. 426–428 & 430–432], Georgantass et al. [23], and Ghahtarani et al. [24] still adopt 2-objective models. Pavlou et al. [26] still empirically focus on 2-objective models and one multiple-objective model. Xidonas et al. [27] still review typically 2-objective models. Ehrgott et al. [1, pp. 20–30], Bokrantz and Fredriksson [28, pp. 684–691], and Schöbel and Zhou-Kangas [29, pp. 786–792] still exploit 2-objective-model illustrations.

### 1.4. Theoretical originality and practical contribution: multiple-objective robust optimization with analytical solutions and portfolio selection for countering COVID

The COVID-19 pandemic is unleashing crises of humanity, economy, and finance. With industry shutdown, loss of employment, and unimaginable death toll, the global economy is falling into recession. To make things worse, the COVID-19 virus is still mutating and evolving, so the humanity suffering is far from being over. Consequently, the suffering brings huge volatility and uncertainty to financial markets and puts financial stability at risk.

In order to counter COVID-19 in stock markets and thus stabilize financial markets, investors naturally expect utilizing portfolio selection but discover few academic clues. Consequently, we originate a counter-COVID measure for stocks and integrate the measure into the following model:

$$\begin{aligned} \min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\ \max z_2 &= \mathbf{x}^T \mu_2 \\ \max z_3 &= \mathbf{x}^T \mu_3 \text{ counter-COVID} \\ \text{subject to } \mathbf{x}^T \mathbf{1} &= 1, \end{aligned} \quad (4)$$

where  $\mu_3$  is a vector of stock expected counter-COVID;  $z_3$  measures portfolio expected counter-COVID.

Ordinary optimization is based on only one objective function. However, multiple objective functions are frequently involved in daily decision making. Therefore, multiple-objective optimization extends ordinary optimization. For (4), investors simultaneously try to minimize risk (measured by variance), maximize expected return, and maximize expected counter-COVID. For risk alone, investors can minimize risk. For risk, expected return, and expected counter-COVID simultaneously, investors need to balance the three objective functions altogether and consider the trade-off. The optimal solutions of (4) form an efficient surface. The efficient frontier of (1) extends into the surface. We depict the extension by a green arrow with text “extend to” in Fig. 1. For (4), investors thus enjoy more freedom of choices by balancing and selecting objectives (i.e., variance, expected return, and expected counter-COVID).

Because of the inherent uncertainty in measuring counter-COVID, we dissect the robust optimization for (4) in the following parametric-quadratic-programming model:

$$\begin{aligned} \min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\ \max z_2 &= \mathbf{x}^T \mu_2 \\ \max z_3 &= \mathbf{x}^T (\mu_3 + t_3 \Delta_3) \\ \text{subject to } \mathbf{x}^T \mathbf{1} &= 1, \end{aligned} \quad (5)$$

where  $t_3$  is a (scalar) parameter for the uncertain amount,  $\Delta_3$  is a fixed vector for a specific direction, and  $t_3 \Delta_3$  dictates the uncertainty in measuring  $\mu_3$ .

For a given optimal portfolio and with  $t_3$  varying, we analytically compute a trail of the portfolio with the varying of  $t_3$ . The trail stands as the solution of (5) and describes the portfolio's sensitivity with respect to  $t_3$ . We call the trail as mean-parameterized nondominated path.

For instance in Fig. 1, investors begin with  $t_3 = 0$  for (5) and obtain an efficient surface with label  $t_3 = 0$ . Investors pinpoint a portfolio  $\mathbf{z}_e$  on the surface. Due to the uncertainty in measuring counter-COVID, investors substitute  $t_3 = 0.05$  and  $t_3 = 0.1$  into (5) and obtain two corresponding efficient surfaces with labels  $t_3 = 0.05$  and  $t_3 = 0.1$ . Investors detect the transformation from the surface with label  $t_3 = 0$  to the surface with label  $t_3 = 0.05$  and then to the surface with label  $t_3 = 0.1$ . Particularly during the transformation, investors witness the shift of  $\mathbf{z}_e$  as a mean-parameterized nondominated path which is depicted as a thick blue curve. The path's equation is listed in Fig. 1.

Although the scenario is for countering COVID, our computation functions for general multiple-objective portfolio selection and “general” means that the computation works for any  $\mu_3$ .

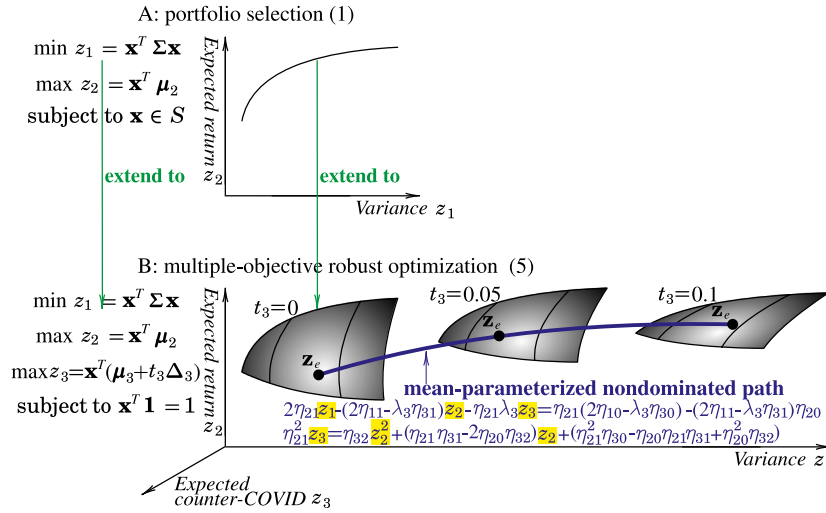


Fig. 1. The extension from an efficient frontier of (1) to an efficient surface of (4), three efficient surfaces for  $t_3 = 0$  and  $t_3 = 0.05$  and  $t_3 = 0.1$  of (5), and a mean-parameterized nondominated path for a surface portfolio  $\mathbf{z}_e$ .

To the best of our knowledge, there is quite limited research for multiple-objective portfolio selection for COVID. There is also limited research for the robust optimization of multiple-objective portfolio selection. In such an area, this paper contributes to the literature.

For operations research for pandemic, Corbett and Van Wassenhove [30] propose natural-drift hypotheses and discuss important issues of operations research for crisis management. Altay and Green [31] review the research of operations research in disaster management and suggest future directions. Gupta et al. [32] review disaster-management research from a Production-and-Operations-Management perspective. Queiroz et al. [33] evaluate the role of supply chain for COVID by a structured literature review. Choi [34] examines operations-research literature for pandemics and proposes a future research agenda. Ghorui et al. [35] locate COVID-spread risk by fuzzy MCDM methodology. Kazancoglu et al. [36] appraise stakeholder engagement for COVID-vaccine supply chains. Jain et al. [37] predict COVID spread by machine learning methods.

The rest of this paper is organized as follows: We review multiple-objective optimization, portfolio optimization, and robust optimization in Section 2. We perform robust optimization for (5) in Section 3. We incorporate interest rates as important economic factors into (5), treat interest-rate uncertainty, and dissect the robust optimization in Section 4. We originate a counter-COVID measure and justify it by stakeholder theory in Section 5. We illustrate our analysis by the 30 component stocks of Dow Jones Industrial Average and obtain affirmative results in Section 6. We conclude this paper in Section 7.

## 2. Multiple-objective optimization, portfolio optimization, and portfolio selection

We review multiple-objective optimization, portfolio optimization, and multiple-objective robust optimization and appraise portfolio selection in this section.

### 2.1. Multiple-objective optimization

Scholars (e.g., Steuer [38], Miettinen [39], and Ehrgott [40]) formulate multiple-objective optimization as follows:

$$\begin{aligned} \max z_1 &= f_1(\mathbf{x}) \\ &\vdots \\ \max z_k &= f_k(\mathbf{x}) \\ \text{subject to } \mathbf{x} &\in S, \end{aligned} \quad (6)$$

where  $\mathbf{x} \in \mathbb{R}^n$  is a decision vector in decision space,  $k$  is the number of objectives,  $f_1(\mathbf{x}) \dots f_k(\mathbf{x})$  are objective functions,  $\mathbf{z} = [z_1 \dots z_k]^T$  is an objective vector in objective space,  $S \subset \mathbb{R}^n$  is the feasible region in decision space, and  $Z = \{\mathbf{z} \mid \mathbf{x} \in S\}$  is the feasible region in objective space.

**Definition 1.** That  $\bar{\mathbf{z}} = [\bar{z}_1 \dots \bar{z}_k]^T \in Z$  dominates  $\mathbf{z} = [z_1 \dots z_k]^T \in Z$  is defined as  $\bar{z}_1 \geq z_1, \dots, \bar{z}_k \geq z_k$  with at least one strict inequality.

**Definition 2.** That  $\bar{\mathbf{z}} \in Z$  is *nondominated* is defined as that there does not exist a  $\mathbf{z} \in Z$  such that  $\mathbf{z}$  dominates  $\bar{\mathbf{z}}$ .

**Definition 3.** That  $\bar{\mathbf{x}} \in S$  is *efficient* is defined as that its objective vector  $\bar{\mathbf{z}}$  is nondominated.

One common approach to solve (6) is a weighted-sum method. By the method, the following model with a weighting vector  $\boldsymbol{\lambda} = [\lambda_1 \dots \lambda_k]^T \succeq \mathbf{0}$  is optimized:

$$\begin{aligned} \max \lambda_1 f_1(\mathbf{x}) + \dots + \lambda_k f_k(\mathbf{x}) \\ \text{subject to } \mathbf{x} \in S, \end{aligned} \quad (7)$$

where  $\mathbf{0}$  is a vector of zeros. Geoffrion [41] describes the relationship between (6) and (7) in the following theorem:

**Theorem 1.** If  $\mathbf{x}$  is the optimal solution of (7) with a  $\boldsymbol{\lambda} \succeq \mathbf{0}$ ,  $\mathbf{x}$  is weakly efficient of (6). For (6) with  $S$  as a convex set and with  $f_1(\mathbf{x}) \dots f_k(\mathbf{x})$  as concave functions, if  $\mathbf{x}$  is weakly efficient of (6), there exists a  $\boldsymbol{\lambda} \succeq \mathbf{0}$  such that  $\mathbf{x}$  is the optimal solution of (7) with the  $\boldsymbol{\lambda}$ .

### 2.2. Portfolio optimization

Markowitz [2] calls the nondominated set of (1) as an efficient frontier, while we instead utilize the term nondominated set and respectively reserve “nondominated” and “efficient” for objective space and decision space.

Nobel laureate Sharpe [42], Merton [43] inspect the following model:

$$\begin{aligned} \min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\ \max z_2 &= \mathbf{x}^T \boldsymbol{\mu}_2 \\ \text{subject to } \mathbf{x}^T \mathbf{1} &= 1. \end{aligned} \quad (8)$$

Merton [43] analytically computes the minimum-variance frontier as the boundary of the feasible region  $Z$ . Although unlimited weights are

allowed in (8), most results can be analytically computed. Sharpe [42] further derives capital asset pricing models.

Qi et al. [13] explore (4) with general  $\mu_3$  and apply an  $\epsilon$ -constraint method to (4) as follows<sup>4</sup>:

$$\begin{aligned} \min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\ \text{subject to } \mathbf{x}^T \mu_2 &= e_2 \\ \mathbf{x}^T \mu_3 &= e_3 \\ \mathbf{x}^T \mathbf{1} &= 1, \end{aligned} \quad (9)$$

where  $e_2$  and  $e_3$  are the parameters. With varying  $e_2$  and  $e_3$ , the objective-vector  $[z_1 \ z_2 \ z_3]^T$  set of all optimal solutions of (9) is called the minimum-variance surface of (4). The surface is the boundary of the feasible region  $Z$  of (4). Qi et al. [13] analytically compute the surface and prove it as a paraboloid as follows:

$$\begin{aligned} z_1 &= \mathbf{d}_2^T \Sigma \mathbf{d}_2 z_2^2 + \mathbf{d}_3^T \Sigma \mathbf{d}_3 z_3^2 + 2\mathbf{d}_2^T \Sigma \mathbf{d}_3 z_2 z_3 + 2\mathbf{d}_2^T \Sigma \mathbf{x}_0 z_2 \\ &\quad + 2\mathbf{d}_3^T \Sigma \mathbf{x}_0 z_3 + \mathbf{x}_0^T \Sigma \mathbf{x}_0, \end{aligned} \quad (10)$$

where  $\mathbf{x}_0$ ,  $\mathbf{d}_2$ , and  $\mathbf{d}_3$  are calculated and  $a \dots f$  are introduced as follows:

$$\begin{aligned} \mathbf{x}_0 &= \frac{1}{|\mathbf{C}|} ((be - cd)\Sigma^{-1}\mu_2 + (bc - ae)\Sigma^{-1}\mu_3 + (ad - bb)\Sigma^{-1}\mathbf{1}) \\ \mathbf{d}_2 &= \frac{1}{|\mathbf{C}|} ((df - ee)\Sigma^{-1}\mu_2 + (ce - bf)\Sigma^{-1}\mu_3 + (be - cd)\Sigma^{-1}\mathbf{1}) \\ \mathbf{d}_3 &= \frac{1}{|\mathbf{C}|} ((ce - bf)\Sigma^{-1}\mu_2 + (af - cc)\Sigma^{-1}\mu_3 + (bc - ae)\Sigma^{-1}\mathbf{1}) \\ \mathbf{C} &\equiv \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \equiv \begin{bmatrix} \mu_2^T \Sigma^{-1} \mu_2 & \mu_2^T \Sigma^{-1} \mu_3 & \mathbf{1}^T \Sigma^{-1} \mu_2 \\ \mu_2^T \Sigma^{-1} \mu_3 & \mu_3^T \Sigma^{-1} \mu_3 & \mathbf{1}^T \Sigma^{-1} \mu_3 \\ \mathbf{1}^T \Sigma^{-1} \mu_2 & \mathbf{1}^T \Sigma^{-1} \mu_3 & \mathbf{1}^T \Sigma^{-1} \mathbf{1} \end{bmatrix}_{3 \times 3} \end{aligned} \quad (11)$$

$$a > 0 \quad d > 0 \quad f > 0. \quad (12)$$

Qi et al. [13] thus prove the nondominated set as a paraboloidal segment.

Due to the complexity of robust optimization, we introduce numerous symbols and clearly define them in this paper. This paper's set of symbols is distinct from the set of symbols of Qi et al. [13].

### 2.3. Multiple-objective robust optimization

For the concept, researchers extend robust optimization from 1-objective optimization to multiple-objective optimization. For example, Ehrgott et al. [1] extend the concept of minmax robustness and call the extension robust efficiency for uncertain multiple-objective optimization. Ide and Schöbel [22] survey and inspect different concepts of robustness for uncertain multiple-objective optimization and broaden the concepts of robust efficiency. Botte and Schöbel [44] suggest multiple-scenario efficiency for multiple-objective robust optimization and compare several robust concepts. Groetzner and Werner [45] enrich the concept of regret and launch multivariate relative regret.

Then, researchers scrutinize methods and conditions for detecting robust solutions of multiple-objective optimization. For example, Bokrantz and Fredriksson [28] prove necessary and sufficient conditions for multiple-objective robust efficiency and investigate the necessary conditions for convex functions. Schmidt et al. [46] employ min-ordering methods and max-ordering methods to probe multiple-objective robust optimization and extend the analysis to multiple-objective uncertain combinatorial optimization. Schöbel and Zhou-Kangas [29] quantitatively compare minmax and lightly robust and nominal efficient solutions, demonstrate the effectiveness of lightly robust efficient solutions, and suggest two strategies to support decision making. Shavazipour et al. [47] modulate classic shallow lake problems and secure feasibility and robust efficiency for the solutions.

Inspired by Markowitz [4], researchers naturally deploy robust optimization for multiple-objective portfolio selection. For example, Fliege and Werner [25] consider uncertainty, perform robust optimization for general convex parametric multiple-objective optimization, and illustrate the analysis by portfolio optimization. Pavlou et al. [26] compare the robustness of three 2-objective models and one multiple-objective model and detect that (1) provides relatively robust results in out-of-sample performance. Xidonas et al. [27] categorically review robust portfolio optimization and emphasize future directions.

The researchers substantially expedite the research for robust optimization but infrequently explicitly compute the optimal solutions for multiple-objective portfolio optimization. For example, Fliege and Werner [25, pp. 426–428 & 430–432], Georgantas et al. [23], and Ghahtarani et al. [24] still adopt 2-objective models. Pavlou et al. [26] still empirically focus on 2-objective models and one multiple-objective model and verify (1) as quite robust in comparison to other 2-objective models. Xidonas et al. [27] still review typically 2-objective models and offer a categorized bibliography for portfolio optimization. Ehrgott et al. [1, pp. 20–30], Bokrantz and Fredriksson [28, pp. 684–691], and Schöbel and Zhou-Kangas [29, pp. 786–792] still exploit 2-objective-model illustrations.

### 2.4. Appraising portfolio selection

#### 2.4.1. The disadvantage

Many researchers criticize portfolio selection in the following aspects:

- Jobson and Korkie [48], Michaud [49] emphasize the weak practicability.
- Black and Litterman [50], Jagannathan and Ma [51], DeMiguel et al. [52], and DeMiguel et al. [53] highlight the formulation insufficiency.
- Best and Grauer [5], Chopra and Ziemba [54] discover that portfolio selection is sensitive to the parameters.
- DeMiguel et al. [55], Ledoit and Wolf [56] concentrate on the parameter estimation.
- Rockafellar and Uryasev [57] prove some weakness of variance and propose CVaR. Kara et al. [58] dissect robust CVaR models under parallelepiped uncertainty and enhance stability of portfolio allocation. Kalayci et al. [59] formulate mutual relevance of finance and investor sentiment by stochastic differential equations and employ multivariate-adaptive-regression-splines models. Schöttle et al. [60] associate Black–Litterman models with Bayes models and extend both groups of models for robust portfolio optimization.
- More generally, Savku and Weber [61] focus on models of delayed Markov regime-switching jump diffusions, conduct the sufficient and necessary conditions, and prove the unique existence of the adjoint equations. Alp and Korn [62] affix jump diffusions to continuous-time portfolio selection and display implications for stock-price drifts. Korn and Müller [63] instigate worst-case portfolio models and enrich indifference frontiers.

#### 2.4.2. The advantage

Nevertheless, portfolio selection enjoys some advantage as follows:

- Portfolio selection is unanimously viewed as the birth of modern finance (as proclaimed by Rubinstein [3, p. 1041]). Therefore, portfolio selection is universally described in classic finance textbooks (e.g., those of Ross et al. [64], Back [65], Campbell [66], Brealey et al. [67], and Bodie et al. [68]).
- Under quadratic utility function or independent multivariate normal distributions for individual stocks (as outlined by Huang and Litzenberger [69, p. 61]), portfolio selection is consistent with the principle of maximizing expected utility (as initiated by Von Neumann and Morgenstern [70]). The principle is exceedingly fundamental in economics and finance.

<sup>4</sup> The analysis also works for (5) with fixed  $t_3$ .



- Portfolio selection is computationally readily solved by parametric-quadratic programming or just quadratic programming. Specifically, Markowitz [71] and Markowitz and Todd [72, p. 176] prove that efficient frontiers are typically composed of piece-wise hyperbolic segments. Markowitz and Todd [72, pp. 301–338] fashion a parametric-quadratic-programming algorithm and the software. Qi [73] maneuvers the software and reports intricate information by the piecewise-segment structure. Hirschberger et al. [74] also commence their parametric-quadratic-programming algorithm and software. Efficient frontiers can also be calculated by quadratic programming which is a common function of Microsoft Excel or Matlab. Steuer et al. [75] document comparative issues in large-scale portfolio optimization.
- Portfolio selection provides a natural background for the capital asset pricing model (CAPM) of Sharpe [42]. Although CAPM does not comprehensively withstand empirical tests, CAPM is widely included in finance textbooks due to its insight. Moreover, testing CAPM implicitly requires independent and identical distributions. The requirement is infrequently met in the empirical tests. Furthermore, Roll [76] even questions the testability.

### 3. Performing robust optimization for (5)

We perform the robust optimization first, because (5) operates for general  $\mu_3$ . Specifically for (5), we compute the mean-parameterized nondominated path indirectly (i.e., computing the path with  $t_3$ ) and further compute the path directly in  $(z_1, z_2, z_3)$  space (i.e., dropping  $t_3$ ). We will belabor the scenario of  $\mu_3$  for counter-COVID in the next section.

We inherit and revise the following assumptions from Qi et al. [13]:

**Assumption 1.**  $\Sigma$  is invertible and thus positive definite.

**Assumption 2.** Vectors  $\mu_2$ ,  $\mu_3$ ,  $\mathbf{A}_3$ , and  $\mathbf{1}$  are linearly independent.

#### 3.1. Computing mean-parameterized nondominated paths indirectly with $t_3$

We utilize a weighted-sum method to (5) and obtain as follows:

$$\min \mathbf{x}^T \Sigma \mathbf{x} - \lambda_2 \mathbf{x}^T \mu_2 - \lambda_3 \mathbf{x}^T (\mu_3 + t_3 \mathbf{A}_3), \lambda_2 \geq 0, \lambda_3 \geq 0 \quad (13)$$

subject to  $\mathbf{x}^T \mathbf{1} = 1$ .

In finance,  $\lambda_3$  can be interpreted as investors' risk-tolerance parameter to expected counter-COVID. Moreover, with  $-\mathbf{x}^T \Sigma \mathbf{x} + \lambda_2 \mathbf{x}^T \mu_2 + \lambda_3 \mathbf{x}^T (\mu_3 + t_3 \mathbf{A}_3)$  as the expected utility, (5) maximizes the expected utility. We apply a Lagrange-multiplier method to (13) as follows:

$$L(\mathbf{x}, \ell) = \mathbf{x}^T \Sigma \mathbf{x} - \lambda_2 \mathbf{x}^T \mu_2 - \lambda_3 \mathbf{x}^T (\mu_3 + t_3 \mathbf{A}_3) + \ell (\mathbf{x}^T \mathbf{1} - 1),$$

where  $\ell$  is a Lagrange-multiplier scalar. Because  $\Sigma$  is positive definite by Assumption 1,  $\mathbf{x}$  is the optimal solution of (13) if and only if the following two partial-derivative equations hold:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{x}} &= 2 \Sigma \mathbf{x} - \lambda_2 \mu_2 - \lambda_3 (\mu_3 + t_3 \mathbf{A}_3) + \ell \mathbf{1} = \mathbf{0} \\ \frac{\partial L}{\partial \ell} &= \mathbf{x}^T \mathbf{1} - 1 = 0. \end{aligned}$$

We premultiply the first partial-derivative equation by  $\Sigma^{-1}$  and obtain the following  $\mathbf{x}$ :

$$\mathbf{x} = \frac{1}{2} (\lambda_2 \Sigma^{-1} \mu_2 + \lambda_3 \Sigma^{-1} \mu_3 + \lambda_3 t_3 \Sigma^{-1} \mathbf{A}_3 - \ell \Sigma^{-1} \mathbf{1}). \quad (14)$$

We next substitute (14) into the second partial-derivative equation, utilize (11)–(12), and obtain as follows:

$$\begin{aligned} \mathbf{1}^T \frac{1}{2} (\lambda_2 \Sigma^{-1} \mu_2 + \lambda_3 \Sigma^{-1} \mu_3 + \lambda_3 t_3 \Sigma^{-1} \mathbf{A}_3 - \ell \Sigma^{-1} \mathbf{1}) &= 1 \\ \lambda_2 c + \lambda_3 e + \lambda_3 t_3 \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3 - \ell f &= 2 \\ \ell &= \frac{1}{f} (\lambda_2 c + \lambda_3 e + \lambda_3 t_3 \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3 - 2). \end{aligned}$$

We then substitute  $\ell$  back to (14) and obtain as follows:

$$\begin{aligned} \mathbf{x}(\lambda_2, \lambda_3, t_3) &= \frac{1}{2} (\lambda_2 \Sigma^{-1} \mu_2 + \lambda_3 \Sigma^{-1} \mu_3 + \lambda_3 t_3 \Sigma^{-1} \mathbf{A}_3 \\ &\quad - \frac{1}{f} (\lambda_2 c + \lambda_3 e + \lambda_3 t_3 \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3 - 2) \Sigma^{-1} \mathbf{1}) \\ &= \frac{1}{2} \lambda_2 (\Sigma^{-1} \mu_2 - \frac{c}{f} \Sigma^{-1} \mathbf{1}) + \frac{1}{2} \lambda_3 (\Sigma^{-1} \mu_3 - \frac{e}{f} \Sigma^{-1} \mathbf{1}) + \frac{1}{f} \Sigma^{-1} \mathbf{1} \\ &\quad + \frac{1}{2} \lambda_3 t_3 (\Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \Sigma^{-1} \mathbf{1}). \end{aligned}$$

We use symbol  $\mathbf{x}(\lambda_2, \lambda_3, t_3)$  instead of  $\mathbf{x}$  in order to specify that  $\mathbf{x}(\lambda_2, \lambda_3, t_3)$  depends on  $\lambda_2$ ,  $\lambda_3$ , and  $t_3$ . To clearly see the structure of  $\mathbf{x}(\lambda_2, \lambda_3, t_3)$ , we introduce symbols  $\mathbf{h}_0$  and  $\mathbf{h}_3$  as follows:

$$\begin{aligned} \mathbf{h}_0 &\equiv \frac{1}{2} \lambda_2 (\Sigma^{-1} \mu_2 - \frac{c}{f} \Sigma^{-1} \mathbf{1}) + \frac{1}{2} \lambda_3 (\Sigma^{-1} \mu_3 - \frac{e}{f} \Sigma^{-1} \mathbf{1}) + \frac{1}{f} \Sigma^{-1} \mathbf{1} \\ &= \Sigma^{-1} (\frac{1}{2} \lambda_2 (\mu_2 - \frac{c}{f} \mathbf{1}) + \frac{1}{2} \lambda_3 (\mu_3 - \frac{e}{f} \mathbf{1}) + \frac{1}{f} \mathbf{1}) \end{aligned} \quad (15)$$

$$\begin{aligned} \mathbf{h}_3 &\equiv \frac{1}{2} \lambda_3 (\Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \Sigma^{-1} \mathbf{1}) \\ &= \Sigma^{-1} \frac{1}{2} \lambda_3 (\mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \mathbf{1}). \end{aligned} \quad (16)$$

$\mathbf{h}_0$  is the optimal solution of (13) with  $t_3 = 0$  and the optimal solution of the weighted-sum method of (4). We then reexpress  $\mathbf{x}(\lambda_2, \lambda_3, t_3)$  as follows:

$$\mathbf{x}(\lambda_2, \lambda_3, t_3) = \mathbf{h}_0 + t_3 \mathbf{h}_3. \quad (17)$$

Moreover, we substitute (17) into (5) as follows:

$$\begin{aligned} z_1(\lambda_2, \lambda_3, t_3) &= (\mathbf{h}_0 + t_3 \mathbf{h}_3)^T \Sigma (\mathbf{h}_0 + t_3 \mathbf{h}_3) \\ &= \mathbf{h}_0^T \Sigma \mathbf{h}_0 + 2(\mathbf{h}_0^T \Sigma \mathbf{h}_3) t_3 + (\mathbf{h}_3^T \Sigma \mathbf{h}_3) t_3^2 \end{aligned} \quad (18)$$

$$\begin{aligned} z_2(\lambda_2, \lambda_3, t_3) &= (\mathbf{h}_0 + t_3 \mathbf{h}_3)^T \mu_2 \\ &= \mathbf{h}_0^T \mu_2 + (\mathbf{h}_3^T \mu_2) t_3 \end{aligned} \quad (19)$$

$$\begin{aligned} z_3(\lambda_2, \lambda_3, t_3) &= (\mathbf{h}_0 + t_3 \mathbf{h}_3)^T (\mu_3 + t_3 \mathbf{A}_3) \\ &= \mathbf{h}_0^T \mu_3 + (\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3) t_3 + (\mathbf{h}_3^T \mathbf{A}_3) t_3^2. \end{aligned} \quad (20)$$

We summarize the computation above into the following theorem and corollary:

**Theorem 2.** For given  $\lambda_2$ ,  $\lambda_3$ , and  $t_3$ , (18)–(20) describe one nondominated objective vector of (5) with the given  $t_3$ . For given  $t_3$  and with  $\lambda_2$  varying and with  $\lambda_3$  varying, (18)–(20) describe the nondominated set of (5) with the given  $t_3$  by (13). For given  $\lambda_2$  and  $\lambda_3$  and with  $t_3$  varying, (18)–(20) describe a trail of one nondominated objective vector of (5) with the varying of  $t_3$ . The trail stands as the solution of (5) and describes the nondominated objective vector's sensitivity with respect to  $t_3$ .

**Proof.** The results rest on the relationship between (6) and (7) in Theorem 1. Specifically, Qi et al. [13] prove that the efficient set of (5) equals the weakly efficient set of (5) for fixed  $t_3$ . Therefore, for the fixed  $t_3$  and with  $\lambda_2$  varying and with  $\lambda_3$  varying, (18)–(20) describe the nondominated set of (5).  $\square$

The trail is called mean-parameterized nondominated path.

**Corollary 1.** (18)–(20) are all continuous with respect to  $t_3$ .

**Proof.** (18)–(20) are all elementary functions with respect to  $t_3$  and thus continuous.  $\square$

Therefore, minute  $t_3$  results in relatively small change of  $(z_1, z_2, z_3)$  of (5).

#### 3.2. Computing mean-parameterized nondominated paths directly in $(z_1, z_2, z_3)$ space

In this subsection, we compute coefficients of (18)–(20), accordingly transform (18)–(20), drop  $t_3$  during the transformation, and directly

compute mean-parameterized nondominated paths for case  $\eta_{21} \neq 0$  and case  $\eta_{21} = 0$ . We evidently stage the steps in subsections.

### 3.2.1. Computing coefficients for $t_3^2$ of (18)–(20)

We discover a relationship between the coefficients for  $t_3^2$  of (18) and (20) (i.e.,  $\mathbf{h}_3^T \Sigma \mathbf{h}_3$  and  $\mathbf{h}_3^T \mathbf{A}_3$ ) in the following lemma:

**Lemma 1.** For the coefficients for  $t_3^2$  of (18) and (20), the following relationship holds

$$\mathbf{h}_3^T \Sigma \mathbf{h}_3 = \frac{\lambda_3}{2} \mathbf{h}_3^T \mathbf{A}_3.$$

**Proof.** By utilizing (11) and (15)–(16), we compute the coefficients as follows: (i.e.,  $\mathbf{h}_3^T \Sigma \mathbf{h}_3$  and  $\mathbf{h}_3^T \mathbf{A}_3$ ) as follows:

$$\begin{aligned} \mathbf{h}_3^T \Sigma \mathbf{h}_3 &= \left( \Sigma^{-1} \frac{1}{2} \lambda_3 (\mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \mathbf{1}) \right)^T \Sigma \left( \Sigma^{-1} \frac{1}{2} \lambda_3 (\mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \mathbf{1}) \right) \\ &= \frac{1}{4} (\mathbf{A}_3^T \Sigma^{-1} \mathbf{A}_3) \lambda_3^2 - \frac{1}{2} \frac{(\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3)^2}{f} \lambda_3^2 + \frac{1}{4} \frac{(\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3)^2}{f^2} (\mathbf{1}^T \Sigma^{-1} \mathbf{1}) \lambda_3^2 \\ &= \frac{1}{4} (\mathbf{A}_3^T \Sigma^{-1} \mathbf{A}_3) \lambda_3^2 - \frac{1}{4} \frac{(\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3)^2}{f} \lambda_3^2 \\ &= \frac{1}{4} \left( \mathbf{A}_3^T \Sigma^{-1} \mathbf{A}_3 - \frac{(\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3)^2}{f} \right) \lambda_3^2 \\ \mathbf{h}_3^T \mathbf{A}_3 &= \left( \Sigma^{-1} \frac{1}{2} \lambda_3 (\mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \mathbf{1}) \right)^T \mathbf{A}_3 \\ &= \frac{1}{2} \left( \mathbf{A}_3^T \Sigma^{-1} \mathbf{A}_3 - \frac{(\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3)^2}{f} \right) \lambda_3. \end{aligned}$$

We multiply the second expression by  $\frac{\lambda_3}{2}$  and then obtain the result of this lemma.  $\square$

### 3.2.2. Computing coefficients for $t_3$ of (18)–(20)

In order to check the possible relationship  $2\mathbf{h}_0^T \Sigma \mathbf{h}_3 = \frac{\lambda_3}{2} (\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3)$  on the basis of Lemma 1, we further compute the coefficient (i.e.,  $\mathbf{h}_0^T \Sigma \mathbf{h}_3$ ) for  $t_3$  of (18) and the coefficient (i.e.,  $\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3$ ) for  $t_3$  of (20) in the following lemmas:

**Lemma 2.** For the coefficient for  $t_3$  of (18), the following relationship holds

$$\mathbf{h}_0^T \Sigma \mathbf{h}_3 = \frac{1}{4} \lambda_2 \lambda_3 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 + \frac{1}{4} \lambda_3^2 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3.$$

**Proof.** By utilizing (11) and (15)–(16), we compute the coefficient as follows:

$$\begin{aligned} \mathbf{h}_0^T \Sigma \mathbf{h}_3 &= \left( \Sigma^{-1} \left( \frac{1}{2} \lambda_2 (\mu_2 - \frac{c}{f} \mathbf{1}) + \frac{1}{2} \lambda_3 (\mu_3 - \frac{e}{f} \mathbf{1}) + \frac{1}{f} \mathbf{1} \right) \right)^T \\ &\quad \Sigma \left( \Sigma^{-1} \frac{1}{2} \lambda_3 (\mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \mathbf{1}) \right) \\ &= \frac{1}{4} \lambda_2 \lambda_3 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{4f} \lambda_2 \lambda_3 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{1} \\ &\quad + \frac{1}{4} \lambda_3^2 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{4f} \lambda_3^2 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{1} \\ &\quad + \frac{1}{2f} \lambda_3 \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{2f^2} \lambda_3 \mathbf{1}^T \Sigma^{-1} \mathbf{1} \\ &= \frac{1}{4} \lambda_2 \lambda_3 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{4f} \lambda_2 \lambda_3 (\cancel{\mathbf{1}} - \frac{\cancel{c}}{f}) \\ &\quad + \frac{1}{4} \lambda_3^2 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{4f} \lambda_3^2 (\cancel{\mathbf{1}} - \frac{\cancel{e}}{f}) \\ &\quad + \frac{\cancel{1}}{2f} \lambda_3 \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{2f} \lambda_3 \cancel{\mathbf{1}} \\ &= \frac{1}{4} \lambda_2 \lambda_3 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 + \frac{1}{4} \lambda_3^2 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3. \quad \square \end{aligned}$$

**Lemma 3.** For the coefficient for  $t_3$  of (20), the following relationship holds

$$\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3 = \lambda_3 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 + \frac{1}{2} \lambda_2 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 + \frac{1}{f} \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3.$$

**Proof.** By utilizing (11) and (15)–(16), we compute the coefficient as follows:

$$\begin{aligned} \mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3 &= \left( \Sigma^{-1} \frac{1}{2} \lambda_3 (\mathbf{A}_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{f} \mathbf{1}) \right)^T \mu_3 \\ &\quad + \left( \Sigma^{-1} \left( \frac{1}{2} \lambda_2 (\mu_2 - \frac{c}{f} \mathbf{1}) + \frac{1}{2} \lambda_3 (\mu_3 - \frac{e}{f} \mathbf{1}) + \frac{1}{f} \mathbf{1} \right) \right)^T \mathbf{A}_3 \\ &= \frac{1}{2} \lambda_3 \mathbf{A}_3^T \Sigma^{-1} \mu_3 - \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{A}_3}{2f} \lambda_3 \mathbf{1}^T \Sigma^{-1} \mu_3 \\ &\quad + \frac{1}{2} \lambda_2 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 + \frac{1}{2} \lambda_3 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 \\ &\quad + \frac{1}{f} \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3 \\ &= \lambda_3 (\mu_3 - \frac{e}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 + \frac{1}{2} \lambda_2 (\mu_2 - \frac{c}{f} \mathbf{1})^T \Sigma^{-1} \mathbf{A}_3 \\ &\quad + \frac{1}{f} \mathbf{1}^T \Sigma^{-1} \mathbf{A}_3. \quad \square \end{aligned}$$

We thus negate the possible relationship  $2\mathbf{h}_0^T \Sigma \mathbf{h}_3 = \frac{\lambda_3}{2} (\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3)$  in the following lemma:

**Lemma 4.** For the coefficients for  $t_3$  of (18) and (20),  $2\mathbf{h}_0^T \Sigma \mathbf{h}_3$  and  $\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3$  do not satisfy  $2\mathbf{h}_0^T \Sigma \mathbf{h}_3 = \frac{\lambda_3}{2} (\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3)$ ; that is,

$$2\mathbf{h}_0^T \Sigma \mathbf{h}_3 - \frac{\lambda_3}{2} (\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3) \neq 0.$$

**Proof.** On the basis of Lemmas 2–3, we negate the possible relationship.  $\square$

### 3.2.3. Transforming (18)–(20) and dropping $t_3$

In order to drop  $t_3$  and clearly present (18)–(20), we introduce the following symbols by Lemma 1:

$$z_1(\lambda_2, \lambda_3, t_3) = \eta_{10} + \eta_{11} t_3 + \frac{\lambda_3}{2} \eta_{32} t_3^2 \quad (21)$$

$$z_2(\lambda_2, \lambda_3, t_3) = \eta_{20} + \eta_{21} t_3 \quad (22)$$

$$z_3(\lambda_2, \lambda_3, t_3) = \eta_{30} + \eta_{31} t_3 + \eta_{32} t_3^2 \quad (23)$$

$$\eta_{10} \equiv \mathbf{h}_0^T \Sigma \mathbf{h}_0 \quad \eta_{11} \equiv 2(\mathbf{h}_0^T \Sigma \mathbf{h}_3) \quad (24)$$

$$\eta_{20} \equiv \mathbf{h}_0^T \mu_2 \quad \eta_{21} \equiv \mathbf{h}_3^T \mu_2 \quad (25)$$

$$\eta_{30} \equiv \mathbf{h}_0^T \mu_3 \quad \eta_{31} \equiv \mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3 \quad \eta_{32} \equiv \mathbf{h}_3^T \mathbf{A}_3. \quad (26)$$

We next utilize Lemma 1, compute (21)–(23) in order to cancel the  $t_3^2$  term, and obtain as follows:

$$\begin{aligned} z_1(\lambda_2, \lambda_3, t_3) - \frac{\lambda_3}{2} z_3(\lambda_2, \lambda_3, t_3) &= \eta_{10} + \eta_{11} t_3 + \frac{\lambda_3}{2} \cancel{\eta_{32} t_3^2} - \frac{\lambda_3}{2} \eta_{30} \\ &\quad - \frac{\lambda_3}{2} \eta_{31} t_3 - \frac{\lambda_3}{2} \cancel{\eta_{32} t_3^2} \\ &= (\eta_{10} - \frac{\lambda_3}{2} \eta_{30}) + (\eta_{11} - \frac{\lambda_3}{2} \eta_{31}) t_3. \end{aligned} \quad (27)$$

Moreover, by Lemma 4, (24), and (26), we know that the coefficient for  $t_3$  of (27) is not zero as follows:

$$\eta_{11} - \frac{\lambda_3}{2} \eta_{31} = 2(\mathbf{h}_0^T \Sigma \mathbf{h}_3) - \frac{\lambda_3}{2} (\mathbf{h}_3^T \mu_3 + \mathbf{h}_0^T \mathbf{A}_3) \neq 0. \quad (28)$$

(22) and (27) are linear functions of  $t_3$ ; (23) is a quadratic function of  $t_3$ . We solve  $t_3$  from one linear function and substitute  $t_3$  into the other functions. Technically, we compute a consequential coefficient  $\eta_{21}$  for

(22) by (11), (16) and (25) as follows:

$$\begin{aligned}\eta_{21} &= \mathbf{h}_3^T \boldsymbol{\mu}_2 = (\boldsymbol{\Sigma}^{-1} \frac{1}{2} \lambda_3 (\mathbf{A}_3 - \frac{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{A}_3}{f} \mathbf{1}))^T \boldsymbol{\mu}_2 \\ &= \frac{1}{2} \lambda_3 (\mathbf{A}_3^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_2 - \frac{c}{f} \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{A}_3) = \frac{1}{2} \lambda_3 \mathbf{A}_3^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_2 - \frac{c}{f} \mathbf{1}).\end{aligned}\quad (29)$$

For (29), we know  $\boldsymbol{\mu}_2 - \frac{c}{f} \mathbf{1} \neq \mathbf{0}$  because  $\boldsymbol{\mu}_2$  and  $\mathbf{1}$  are independent by Assumption 2. We thus know  $\boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_2 - \frac{c}{f} \mathbf{1}) \neq \mathbf{0}$  because  $\boldsymbol{\Sigma}$  is invertible by Assumption 1. However,  $\mathbf{A}_3^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_2 - \frac{c}{f} \mathbf{1})$  can be 0. Moreover,  $\lambda_3$  can be 0. Therefore for (29), we have two exhaustive cases:  $\eta_{21} \neq 0$  and  $\eta_{21} = 0$ . We then directly compute mean-parameterized nondominated paths for the two cases. For directness and simplicity reasons, we notate  $z_1(\lambda_2, \lambda_3, t_3)$  as  $z_1$ ,  $z_2(\lambda_2, \lambda_3, t_3)$  as  $z_2$ , and  $z_3(\lambda_2, \lambda_3, t_3)$  as  $z_3$  during the computation.

### 3.2.4. Case $\eta_{21} \neq 0$

**Theorem 3.** For  $\eta_{21} \neq 0$ , the mean-parameterized nondominated path of (5) is a curve in  $(z_1, z_2, z_3)$  space as the intersection of (30) and (31) as follows:

$$2\eta_{21}z_1 - (2\eta_{11} - \lambda_3\eta_{31})z_2 - \eta_{21}\lambda_3z_3 = \eta_{21}(2\eta_{10} - \lambda_3\eta_{30}) - (2\eta_{11} - \lambda_3\eta_{31})\eta_{20} \quad (30)$$

$$\eta_{21}^2z_3 = \eta_{32}z_2^2 + (\eta_{21}\eta_{31} - 2\eta_{20}\eta_{32})z_2 + (\eta_{21}^2\eta_{30} - \eta_{20}\eta_{21}\eta_{31} + \eta_{20}^2\eta_{32}). \quad (31)$$

**Proof.** By (22), we compute  $t_3$  as follows:

$$t_3 = \frac{z_2 - \eta_{20}}{\eta_{21}}. \quad (32)$$

We next substitute (32) into (27) as follows:

$$z_1 - \frac{\lambda_3}{2}z_3 = (\eta_{10} - \frac{\lambda_3}{2}\eta_{30}) + (\eta_{11} - \frac{\lambda_3}{2}\eta_{31})\frac{z_2 - \eta_{20}}{\eta_{21}}.$$

We reorganize the expression above and obtain (30). By (15)–(16), (24), (25), and (26), all the coefficients of (30) do not depend on  $t_3$ . Moreover, with one  $z_1$  term,  $z_2$  term, and  $z_3$  term, (30) is a plane in  $(z_1, z_2, z_3)$  space.

We also substitute (32) into (23) as follows:

$$z_3 = \eta_{30} + \eta_{31}\left(\frac{z_2 - \eta_{20}}{\eta_{21}}\right) + \eta_{32}\left(\frac{z_2 - \eta_{20}}{\eta_{21}}\right)^2.$$

We reorganize the expression above and obtain (31). By (15)–(16), (24), (25), and (26), all the coefficients of (31) do not depend on  $t_3$ . Moreover, with one  $z_2$  term,  $z_2^2$  term, and  $z_3$  term, (31) is parabola in  $(z_2, z_3)$  subspace and thus a degenerate paraboloid in  $(z_1, z_2, z_3)$  space. Here, “degenerate” means that the coefficients of both the  $z_1$  term and the  $z_1^2$  term are 0.  $\square$

Geometrically, (30) is a plane in  $(z_1, z_2, z_3)$  space; (31) is a degenerate paraboloid in  $(z_1, z_2, z_3)$  space or a parabola in  $(z_2, z_3)$  subspace. We have already depicted a mean-parameterized nondominated path (i.e., (30) and (31)) in Fig. 1.

### 3.2.5. Case $\eta_{21} = 0$

**Theorem 4.** For  $\eta_{21} = 0$ , the mean-parameterized nondominated path of (5) is a curve in  $(z_1, z_2, z_3)$  space as the intersection of (33) and (34) as follows:

$$z_2 = \eta_{20} \quad (33)$$

$$\begin{aligned}&4\eta_{32}z_1^2 + \lambda_3^2\eta_{32}z_3^2 - 4\lambda_3\eta_{32}z_1z_3 \\ &+ (4\eta_{11}\eta_{31} - 2\lambda_3\eta_{31}^2 + 4\lambda_3\eta_{30}\eta_{32} - 8\eta_{10}\eta_{32})z_1 \\ &+ (-4\eta_{11}^2 + 2\lambda_3\eta_{11}\eta_{31} - 2\lambda_3^2\eta_{30}\eta_{32} + 4\lambda_3\eta_{10}\eta_{32})z_3 \\ &+ 4\eta_{11}^2\eta_{30} - 2\lambda_3\eta_{11}\eta_{30}\eta_{31} - 4\eta_{10}\eta_{11}\eta_{31} + 2\lambda_3\eta_{10}\eta_{31}^2 \\ &+ \lambda_3^2\eta_{30}^2\eta_{32} + 4\eta_{10}^2\eta_{32} - 4\lambda_3\eta_{10}\eta_{30}\eta_{32} = 0.\end{aligned}\quad (34)$$

**Proof.** With  $\eta_{21} = 0$ , (22) becomes simpler as follows:

$$z_2 = \eta_{20} + 0t_3 = \eta_{20}.$$

(33) is a plane which passes through  $(0, \eta_{20}, 0)$  and is perpendicular to  $z_2$  axis in  $(z_1, z_2, z_3)$  space. By (28), we solve  $t_3$  from (27) as follows:

$$\begin{aligned}z_1 - \frac{\lambda_3}{2}z_3 &= (\eta_{10} - \frac{\lambda_3}{2}\eta_{30}) + (\eta_{11} - \frac{\lambda_3}{2}\eta_{31})t_3 \\ t_3 &= \frac{1}{2\eta_{11} - \lambda_3\eta_{31}}(2z_1 - \lambda_3z_3 + (\lambda_3\eta_{30} - 2\eta_{10})).\end{aligned}$$

We substitute  $t_3$  into (23) and obtain as follows:

$$\begin{aligned}z_3 &= \eta_{30} + \eta_{31}\frac{1}{2\eta_{11} - \lambda_3\eta_{31}}(2z_1 - \lambda_3z_3 + (\lambda_3\eta_{30} - 2\eta_{10})) \\ &+ \eta_{32}\frac{1}{(2\eta_{11} - \lambda_3\eta_{31})^2}(2z_1 - \lambda_3z_3 + (\lambda_3\eta_{30} - 2\eta_{10}))^2 \\ (4\eta_{11}^2 - 4\lambda_3\eta_{11}\eta_{31} + \lambda_3^2\eta_{31}^2)z_3 &= (4\eta_{11}^2 - 4\lambda_3\eta_{11}\eta_{31} + \lambda_3^2\eta_{31}^2)\eta_{30} \\ &+ (4\eta_{11}\eta_{31} - 2\lambda_3\eta_{31}^2)z_1 + (\lambda_3^2\eta_{31}^2 - 2\lambda_3\eta_{11}\eta_{31})z_3 \\ &+ (2\eta_{11}\eta_{31} - \lambda_3\eta_{31}^2)(\lambda_3\eta_{30} - 2\eta_{10}) \\ &+ \eta_{32}(4z_1^2 + \lambda_3^2z_3^2 + (\lambda_3\eta_{30} - 2\eta_{10})^2 - 4\lambda_3z_1z_3 \\ &+ 4(\lambda_3\eta_{30} - 2\eta_{10})z_1 - 2(\lambda_3\eta_{30} - 2\eta_{10})\lambda_3z_3) \\ 4\eta_{32}z_1^2 + \lambda_3^2\eta_{32}z_3^2 - 4\lambda_3\eta_{32}z_1z_3 &= \\ &+ (4\eta_{11}\eta_{31} - 2\lambda_3\eta_{31}^2 + 4\lambda_3\eta_{30}\eta_{32} - 8\eta_{10}\eta_{32})z_1 \\ &+ (-4\eta_{11}^2 + 4\lambda_3\eta_{11}\eta_{31} - \lambda_3^2\eta_{31}^2 + \lambda_3^2\eta_{31}^2 - 2\lambda_3\eta_{11}\eta_{31} \\ &- 2\lambda_3^2\eta_{30}\eta_{32} + 4\lambda_3\eta_{10}\eta_{32})z_3 \\ &+ 4\eta_{11}^2\eta_{30} - 4\lambda_3\eta_{11}\eta_{30}\eta_{31} + \lambda_3^2\eta_{30}^2\eta_{32} \\ &+ 2\lambda_3\eta_{11}\eta_{30}\eta_{31} - \lambda_3^2\eta_{30}^2\eta_{31} - 4\eta_{10}\eta_{11}\eta_{31} + 2\lambda_3\eta_{10}\eta_{31}^2 \\ &+ \lambda_3^2\eta_{30}^2\eta_{32} + 4\eta_{10}^2\eta_{32} - 4\lambda_3\eta_{10}\eta_{30}\eta_{32} = 0 \\ 4\eta_{32}z_1^2 + \lambda_3^2\eta_{32}z_3^2 - 4\lambda_3\eta_{32}z_1z_3 &= \\ &+ (4\eta_{11}\eta_{31} - 2\lambda_3\eta_{31}^2 + 4\lambda_3\eta_{30}\eta_{32} - 8\eta_{10}\eta_{32})z_1 \\ &+ (-4\eta_{11}^2 + 2\lambda_3\eta_{11}\eta_{31} - 2\lambda_3^2\eta_{30}\eta_{32} + 4\lambda_3\eta_{10}\eta_{32})z_3 \\ &+ 4\eta_{11}^2\eta_{30} - 2\lambda_3\eta_{11}\eta_{30}\eta_{31} - 4\eta_{10}\eta_{11}\eta_{31} + 2\lambda_3\eta_{10}\eta_{31}^2 \\ &+ \lambda_3^2\eta_{30}^2\eta_{32} + 4\eta_{10}^2\eta_{32} - 4\lambda_3\eta_{10}\eta_{30}\eta_{32} = 0.\end{aligned}$$

By (15)–(16), (24), (25), and (26), all the coefficients of (34) do not depend on  $t_3$ . Moreover, with one  $z_1^2$  term,  $z_3^2$  term,  $z_1z_3$  term,  $z_1$  term, and  $z_3$  term, (34) is a quadratic curve in  $(z_1, z_3)$  subspace. Therefore, (34) is a degenerate quadratic surface in  $(z_1, z_2, z_3)$  space. Here, “degenerate” means that the coefficients of the  $z_2^2$  term, the cross terms for  $z_2$ , and the  $z_2$  term are all 0.  $\square$

Geometrically, (33) is a plane in  $(z_1, z_2, z_3)$  space; (34) is a degenerate quadratic surface in  $(z_1, z_2, z_3)$  space.

### 3.3. Generalization

The models (4)–(5) and analysis do not rely on special properties of COVID. Therefore, investors can harness the models and analysis for general 3-objective robust optimization (e.g., for other disaster management).

### 3.4. Normative methodology vs. positive methodology

Markowitz [71], Markowitz and Todd [72] inspect a general portfolio-selection model (1) and orchestrate parametric-quadratic programming to solve (1). Sharpe [42, p. 426] and Markowitz [77, p. 99] call this general-model methodology as normative methodology.

In contrast, on the basis of (8), Sharpe [42, pp. 433–434] assumes investor homogeneity and originates his profound CAPM (capital asset pricing model). Although (8) and investor homogeneity are too simplistic, Sharpe [42, p. 434] justifies his methodology by arguing for CAPM’s insights. Sharpe [42, p. 425] and Markowitz [77, p. 99] call



this methodology of simplistic assumptions but profound insights as positive methodology.

We principally pursue positive methodology by extending (8) but analytically instigating robust optimization.

In the future, we will impose practical constraints in the form of inequality to (4)–(5), but we will lose the analytical solutions. Hirschberger et al. [78], Jayasekara et al. [79] propose parametric-quadratic programming to solve a general multiple-objective portfolio-selection model (3). We will try to exploit the numerical solutions (rather than the analytical solutions).

#### 4. Traditional capital allocation by a risk-free asset and the extension for robust optimization of multiple-objective portfolio selection

Investments textbooks (e.g., that of Bodie et al. [68, p. 159]) emphasize the importance of interest rates and prescribe capital allocation to both a risk-free asset  $r_f$  and a risky asset. In this section, we extend the capital allocation for multiple-objective portfolio selection (4). In addition, we contemplate robust optimization for both  $z_3 = \mathbf{x}^T(\mu_3 + t_3\mathbf{A}_3)$  of (5) and  $r_f$ .

##### 4.1. Traditional capital allocation

Bodie et al. [68, p. 277] assumes a line which passes through  $r_f$  and is tangent to an efficient frontier at  $\mathbf{z}_t$  (as depicted in Panel A of Fig. 2). Bodie et al. [68, pp. 168–170] exhibit the process of building portfolios of  $r_f$  and  $\mathbf{z}_t$ , call the process as capital allocation, and call the portfolios as capital allocation line.

Bodie et al. [68, pp.161 & 188–190] delineate the fundamental principle of maximizing expected utility and assume the following expected-utility model:

$$\max E(u(r)) = z_2 - Az_1, \quad (35)$$

where  $r$  is portfolio return,  $z_1$  and  $z_2$  are calculated in (1), and  $A$  stands for investors' risk-aversion coefficient. Bodie et al. [68, p. 175] deliberate the tangency between an indifference curve  $z_2 - Az_1 = \alpha$  of (35) and the capital allocation line of  $r_f$  and  $\mathbf{z}_t$  (as depicted in Panel B of Fig. 2). Bodie et al. [68, p. 175] locate the tangent portfolio  $\mathbf{z}_o$  as the optimal portfolio of (35). That is,  $\mathbf{z}_o$  achieves the optimal balance between  $r_f$  and  $\mathbf{z}_t$ .

##### 4.2. Extending capital allocation for multiple-objective portfolio selection (4)

We have already depicted the nondominated set of (4) as a surface with label  $t_3 = 0$  in Panel B of Fig. 1. Please note that all the horizontal axes of Fig. 2 are standard deviation (instead of variance). We redraw the nondominated set in Panel C of Fig. 2. By following Bodie et al. [68, p. 277], we assume a plane which passes through  $r_f$  and is tangent to the surface at  $\mathbf{z}_t$ . We depict the plane as yellow in Panel C of Fig. 2.

(4) extends (1) in  $\max z_3 = \mathbf{x}^T\mu_3$ .  $z_3 = \mathbf{x}^T\mu_3$  is a linear function of  $\mathbf{x}$ . Hence, we can readily extend the traditional capital allocation (as presented by Bodie et al. [68, pp. 168–170]) into multiple-objective capital allocation. The capital allocation line in Panel B of Fig. 2 extends into a capital allocation plane in Panel C of Fig. 2.

For (4), we extend the utility function  $u(r)$  into  $u(r, c)$  where  $c$  is portfolio counter-COVID. We also implement the fundamental principle of maximizing expected utility, extend (35), and assume the following expected-utility model:

$$\max E(u(r, c)) = z_2 - Az_1 + Bz_3, \quad (36)$$

where  $B$  stands for investors' risk-aversion coefficient;  $z_3$  is calculated in (4).

By following Bodie et al. [68, p. 175], we assume the tangency between an indifference surface  $z_2 - Az_1 = \alpha_{0,0}$  of (36) and the capital

allocation plane of  $r_f$  and  $\mathbf{z}_t$ . We depict the tangency in Panel D of Fig. 2. By following Bodie et al. [68, p. 175], we locate the tangent portfolio  $\mathbf{z}_o$  as the optimal portfolio of (36). That is,  $\mathbf{z}_o$  achieves the optimal balance between  $r_f$  and  $\mathbf{z}_t$ .

We follow standard finance principles and envision the extension. However, the specific mathematical accomplishment is nontrivial. For example, we need the analytical expression of the nondominated set of (4) in order to compute the tangency in Panel C of Fig. 2. The analytical expression of the tangency will be convoluted, because the nondominated set is a portion of the paraboloid (10) and the paraboloid's six coefficients are all complex. For instance, Qi et al. [13, p. 169] report the following coefficients:

$$\begin{aligned} \mathbf{d}_2^T \Sigma \mathbf{d}_2 &= \frac{1}{|\mathbf{C}|^2} (ad^2f^2 - 2ade^2f + ae^4 - b^2df^2 + b^2e^2f \\ &\quad + 2bcdef - 2bce^3 - c^2d^2f + c^2e^2d) \\ \mathbf{x}_0^T \Sigma \mathbf{x}_0 &= \frac{1}{|\mathbf{C}|^2} (a^2d^2f - a^2de^2 - 2ab^2df + ab^2e^2 \\ &\quad + 2abcde - ac^2d^2 - 2b^3ce + b^4f + b^2c^2d), \end{aligned}$$

where  $a$  to  $f$  and  $\mathbf{C}$  are introduced in (11).

##### 4.3. Further extension for robust optimization of multiple-objective portfolio selection by $\mu_3 + t_3\mathbf{A}_3$ of (5) and risk-free rate $r_f + t_f$

We extend (5) in this subsection by appending the risk-free asset to (5) and investigating uncertainty (i.e.,  $r_f + t_f$ ) of risk-free rate.

###### 4.3.1. First robust optimization by (5)

First, we still deploy (5) and have illuminated the analysis in Section 3.

###### 4.3.2. Recognizing uncertainty for the risk-free rate and setting $r_f + t_f$

During COVID, the Federal Reserve substantially revises monetary policies and adjusts the risk-free rate from 1.75% on January 30, 2020, to 0.25% on March 16, 2020, to 1.75% on June 16, 2022.<sup>5</sup> We recognize the uncertainty for the risk-free rate and set  $r_f + t_f$  for the risk-free rate (instead of  $r_f$  alone), where  $t_f$  is a (scalar) parameter for the uncertain amount. Therefore, we resolve robust optimization for both  $\mu_3 + t_3\mathbf{A}_3$  of (5) and  $r_f + t_f$ .

###### 4.3.3. Second robust optimization by risk-free rate $r_f + t_f$

We experiment with four cases (i.e.,  $t_3 = 0$  or  $t_3 = 0.05$  and  $t_f = 0$  or  $t_f = 0.01$ ) in order to obtain intuition. We start from a relative simple case of  $t_3 = 0$  and  $t_f = 0$ . For this case, we have already depicted the nondominated set of (5) with  $t_3 = 0$ ,  $r_f$ , the tangent portfolio  $\mathbf{z}_t$ , and the capital allocation plane in Panel C of Fig. 2. We also have depicted the tangent portfolio  $\mathbf{z}_o$  between an indifference surface  $z_2 - Az_1 + Bz_3 = \alpha_{0,0}$  of (36) and the capital allocation plane of  $r_f$  and  $\mathbf{z}_t$  in Panel D of Fig. 2. We duplicate the capital allocation plane, indifference surface  $z_2 - Az_1 + Bz_3 = \alpha_{0,0}$ , and tangent portfolio  $\mathbf{z}_o$  in Panel E of Fig. 2.  $\mathbf{z}_o$  achieves the optimal balance between  $r_f$  and  $\mathbf{z}_t$ . We attach label  $t_3 = 0$  and  $t_f = 0$ .

For the case of  $t_3 = 0$  and  $t_f = 0.01$ , we depict a plane which passes through  $r_f + 0.01$  and is tangent to the nondominated set of (5) with  $t_3 = 0$  in Panel E of Fig. 2. The plane is the capital allocation plane of  $r_f + 0.01$  and the tangent portfolio  $\mathbf{z}_t$ . We also depict the tangent portfolio  $\mathbf{z}_o$  between an indifference surface  $z_2 - Az_1 + Bz_3 = \alpha_{0,0.01}$  of (36) and the capital allocation plane.  $\mathbf{z}_o$  achieves the optimal balance between  $r_f + 0.01$  and  $\mathbf{z}_t$ . We attach label  $t_3 = 0$  and  $t_f = 0.01$ .

For the case of  $t_3 = 0.05$  and  $t_f = 0.01$ , we depict a plane which passes through  $r_f + 0.01$  and is tangent to the nondominated set of (5) with  $t_3 = 0.05$  in Panel E of Fig. 2. The plane is the capital allocation

<sup>5</sup> Data source: the Federal Reserve, <https://www.federalreserve.gov/>, July 2, 2022.

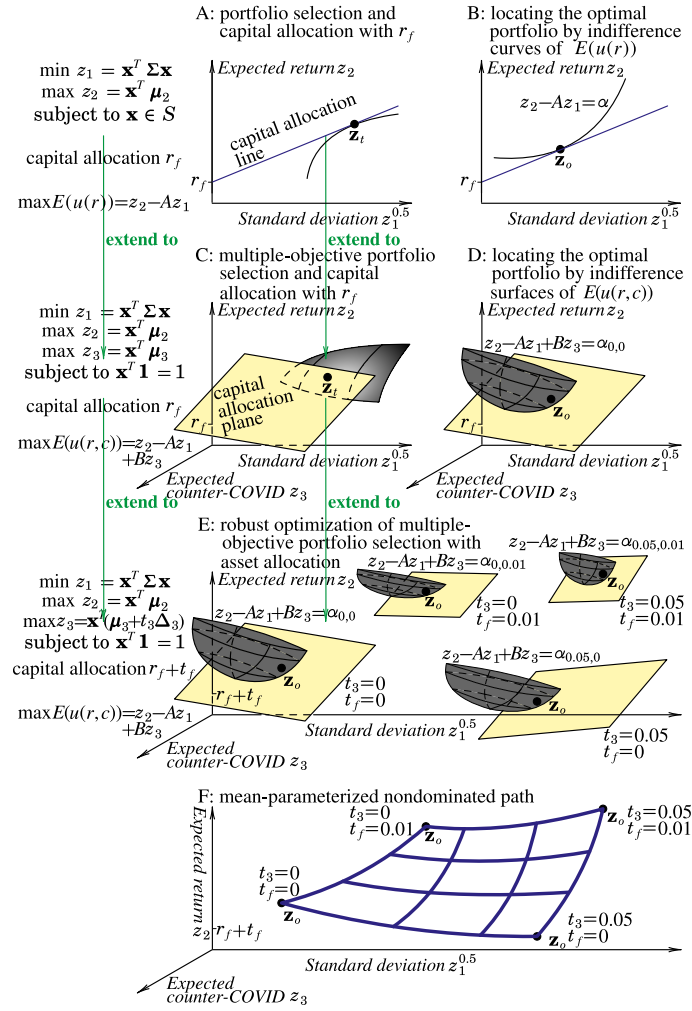


Fig. 2. Traditional capital allocation, the extension for robust optimization of multiple-objective portfolio selection, and the mean-parameterized nondominated path.

plane of  $r_f + 0.01$  and the tangent portfolio  $\mathbf{z}_t$ . We also depict the tangent portfolio  $\mathbf{z}_o$  between an indifference surface  $z_2 - Az_1 + Bz_3 = \alpha_{0,0.05,0.01}$  of (36) and the capital allocation plane.  $\mathbf{z}_o$  achieves the optimal balance between  $r_f + 0.01$  and  $\mathbf{z}_t$ . We attach label  $t_3 = 0.05$  and  $t_f = 0.01$ .

For the case of  $t_3 = 0.05$  and  $t_f = 0$ , we depict a plane which passes through  $r_f$  and is tangent to the nondominated set of (5) with  $t_3 = 0.05$  in Panel E of Fig. 2. The plane is the capital allocation plane of  $r_f$  and the tangent portfolio  $\mathbf{z}_t$ . We also depict the tangent portfolio  $\mathbf{z}_o$  between an indifference surface  $z_2 - Az_1 + Bz_3 = \alpha_{0,0.05,0}$  of (36) and the capital allocation plane.  $\mathbf{z}_o$  achieves the optimal balance between  $r_f$  and  $\mathbf{z}_t$ . We attach label  $t_3 = 0.05$  and  $t_f = 0$ .

We then connect the  $\mathbf{z}_o$  of the four cases and of other cases. The connection forms a surface as depicted in Panel F of Fig. 2. The surface is the mean-parameterized nondominated path of robust optimization of multiple-objective portfolio selection by  $\boldsymbol{\mu}_3 + t_3 \Delta_3$  of (5) and risk-free rate  $r_f + t_f$ .

## 5. Originating a counter-COVID measure by stakeholder theory

We propose a counter-COVID measure, justify it by stakeholder theory, report the rating of the 30 component stocks of Dow Jones Industrial Average, and discuss the implications.

### 5.1. Originating a counter-COVID measure

Under the sub-prime mortgage crisis, Yiannaki [80] appraises firms' risk management through categories responsibility, finance, supervision and resource development, technological networking advance, and

innovation and radical change. Morgan Stanley Capital International, Inc. (MSCI) measures ESG (environmental, social, and governance) by setting categories and setting binary variables under the categories.<sup>6</sup> Therefore, We originate a counter-COVID measure for stocks as follows:

- By stakeholder theory, we set categories managers, fund providers, products, employees, media, and society. We then set 25 binary variables under the categories. We depict the categories and variables in Fig. 3.
- We search firm performance through firm websites, industry associations, government agencies, etc.
- For financial variables (e.g., return on equity), if a firm's performance is better than that of industry median, we assign the variable as 1. Otherwise, we assign the variable as 0. We obtain the firm data and industry median through East Money Corporation.<sup>7</sup>
- For other variables, we directly rate the variables as 1 with supportive proofs or as 0 without supportive proofs.
- We sum the 25 variables as the counter-COVID measure.

We list the variables and brief variable descriptions in Table 1. For example in the second row, variable management ability measures emergency response mechanism for COVID.

<sup>6</sup> Data source: database MSCI ESG KLD of Wharton Research Data Service, <https://wrds-www.wharton.upenn.edu>, December 21, 2020.

<sup>7</sup> <http://global.eastmoney.com>, May 27, 2021.

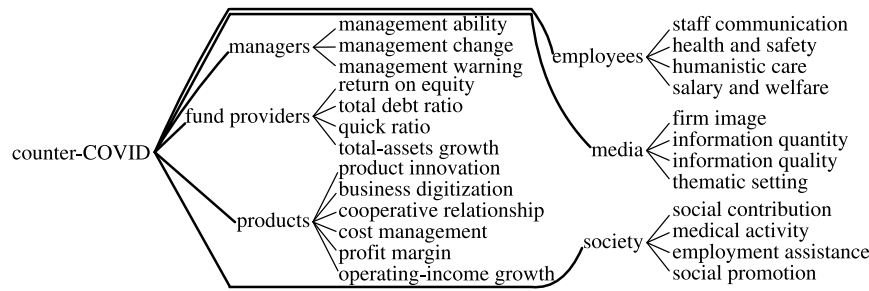


Fig. 3. The categories and variables for the counter-COVID measure.

**Table 1**  
The variables and variable descriptions for the counter-COVID measure.

Variables	Variable descriptions
management ability	emergency response mechanism for COVID
management change	management adjustment for COVID
management warning	warning COVID risk in reports
return on equity	$\frac{\text{net income}}{\text{equity}}$
total debt ratio	$\frac{\text{total liabilities}}{\text{total assets}}$
quick ratio	$\frac{\text{current assets} - \text{inventory}}{\text{current liabilities}}$
total-assets growth	$\frac{\text{total assets of this year} - \text{total assets of last year}}{\text{total assets of last year}}$
product innovation	new product development for COVID
business digitization	online marketing, business support, and customer service
cooperative relationship	good relationships with suppliers and customers
cost management	$\frac{\text{operating cost}}{\text{operating income}}$
profit margin	$\frac{\text{net income}}{\text{sales}}$
operating-income growth	$\frac{\text{operating income of this year} - \text{operating income of last year}}{\text{operating income of last year}}$
staff communication	remote working and training for employees
health and safety	practice to reduce health and safety incidents
humanistic care	good employee relations
salary and welfare	employees' basic salary and related benefits
firm image	positive evaluation for corporate image
information quantity	sufficient information disclosure
information quality	fast and effective information disclosure
thematic setting	topics or websites specifically for COVID news
social contribution	provide social contributions such as donations
medical activity	medical equipment donation or institution cooperation
employment assistance	assistance to unemployed persons for COVID
social promotion	publicize knowledge and precautions of COVID prevention

## 5.2. Justifying the counter-COVID measure

Freeman and Reed [81, p. 89] and Freeman et al. [82] define stakeholder as “those groups without whose support the organization would cease to exist. The list of stakeholders originally included shareholders, employees, customers, suppliers, lenders, and society”. For crisis management, Ulmer [83] and Coombs [84] emphasize strong communication between corporations and stakeholders. Sheth [85] suggests the purpose of business as to nurture its stakeholders. Verma and Gustafsson [86] contend that COVID-19 has pushed businesses to rapidly operate in newer and more resilient ways. Therefore, we set categories managers, fund providers (i.e., shareholders and lenders), products, employees, media, and society.

Specifically, for category managers, James et al. [87] contend that managers play important roles in crisis management. Seetharaman [88] and Cortez and Johnston [89] argue that firms should develop capabilities to counter COVID. Therefore, we set variables management ability, management change, management warning.

For fund providers, Jensen [90] argues that lenders are concerned about loan safety. Amankwah-Amoah et al. [91] warn the danger of excessive corporate debts. Mirza et al. [92] report that COVID threatens the liquidity and solvency of business in Europe. Therefore, we set variables return on equity, total debt ratio, quick ratio, and total-assets growth.

For category products, Vial [93] prices the importance of digitization. Chesbrough [94] suggests that innovation plays important

roles during COVID crises. Seetharaman [88] suggests that COVID actually presents opportunities to firms for digitization or online service. Crick and Crick [95] propose collaborative business-to-business marketing strategies for COVID. In practice, MSCI combines customers and suppliers of stakeholder of Freeman and Reed [81, p. 89] into products. Therefore, we set variables product innovation, business digitization, cooperative relationship, cost management, profit margin, and operating-income growth.

For category employees, Mazzei et al. [96] emphasize the importance of employee relationships and internal communication during corporate crises. Cortez and Johnston [89] suggest that firms reduce distractions and alleviate family tension. Therefore, we set variables staff communication, health and safety, humanistic care, and salary and welfare.

For category media, Veil et al. [97] review the research for social media in crisis communication. Claeys and Cauberghe [98] stress the importance of crisis response and crisis timing strategies. Cortez and Johnston [89] report that using social media during COVID crises can be very effective. Therefore, we set variables firm image, information quantity, information quality, and thematic setting.

For category society, Sheth [85] stresses that business and society are closely interconnected. He and Harris [99] examine the relationship between COVID and corporate social responsibility. Therefore, we set variables social contribution, medical activity, employment assistance and social promotion.

**Table 2**

Rating counter-COVID for the 30 component stocks of Dow Jones Industrial Average of the first quarter of 2020.

Variables	AAPL	AMGN	AXP	BA	CAT	CRM	CSCO	CVX	DIS	DOW	GS	HD	HON	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PG	TRV	UNH	V	VZ	WBA	WMT
management ability	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
management change	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	0	1	1	1	1
management warning	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
return on equity	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1
total debt ratio	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
quick ratio	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	0	1	0	0	0
total-assets growth	0	0	0	1	0	1	0	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1	1	0	0	1	0	0	1	0
product innovation	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
business digitization	1	1	1	0	1	1	1	0	0	1	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
cooperative relationship	1	1	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1
cost management	1	1	0	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	0	1
profit margin	1	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
operating-income growth	1	1	0	0	0	1	0	1	1	0	1	1	0	0	1	0	0	1	1	1	1	1	1	0	0	1	0	0	0	0
staff communication	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1
health and safety	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
humanistic care	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	1	1
salary and welfare	1	1	1	0	1	1	0	1	0	1	1	1	1	0	0	1	1	0	1	1	0	1	0	1	0	1	0	1	0	1
firm image	1	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
information quantity	0	1	1	1	0	1	0	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	1	0	1	1	1	1
information quality	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
thematic setting	0	1	1	1	1	1	0	1	0	1	1	1	1	1	0	1	1	0	1	1	1	1	0	0	1	1	1	1	1	1
social contribution	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
medical activity	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1
employment assistance	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	0	1	0	1	0	0	0	1
social promotion	1	1	0	1	0	1	0	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	0	1	1	1
sum, counter-COVID	20	22	18	17	16	24	14	20	8	19	21	21	19	19	19	21	18	20	22	22	22	24	16	17	18	21	16	21	19	22

### 5.3. Rating the 30 component stocks of Dow Jones Industrial Average

Hence, we rate the 30 component stocks of Dow Jones Industrial Average for the first quarter and second quarter of 2020 and list the rating in Tables 2–3. We have deposited the 30 stocks' rating in Mendeley Data.<sup>8</sup> For example for Apple (AAPL) for the first quarter, the rating of management ability is 1 in the second row of Table 2; the counter-COVID is 20 in the last row.

Moreover, we compute the sample mean vector of the 30 stocks' counter-COVID of the two quarters, assume the vector as  $\mu_3$ , and report  $\mu_3$  in Table 4. For example for AAPL, we compute  $(20 + 17) \div 2 = 18.5$ .

### 5.4. Enrichment and implications of the measure

The counter-COVID measure is exploratory and thus unpolished. Many measurement details need to be further explored. For example, Greco et al. [100] systematically review the weighting, aggregating, and robustness of composite indicators. Greco et al. [101] further initiate robust non-compensatory approaches for composite indicators by PROMETHEE methods. We will deploy the methodologies.

Then, multiple-objective optimization, multiple-criteria analysis, risk management, and other operations-research methodologies can be utilized on the basis of the measure. For example, Drnovšek et al. [102] propose a multi-criteria risk evaluation model to identify COVID patients. Then, counter-COVID can be included in the multi-criteria risk evaluation model. Nikolopoulos et al. [103] utilize machine learning to predict the growth rate of COVID. Then by machine learning and big data, counter-COVID database can be established to support government decision-making.

## 6. Illustrations by Dow Jones Industrial Average and its component stocks

In this section, we illustrate our computations for Section 3. We first select four component stocks and present full computations. In the next subsection, we will report the more meaningful and stronger result of the 30 component stocks but skip some 30-vector computations which are too long to be reported. We have deposited the 30-stock computation in Mendeley Data.<sup>9</sup>

### 6.1. Full computations for 4 component stocks

#### 6.1.1. Sampling

We select four component stocks of Dow Jones Industrial Average as follows: Apple (AAPL), Amgen (AMGN), American Express (AXP), and Boeing (BA). We collect the stocks' monthly returns and monthly prices from January 1, 2020 to June 30, 2020.<sup>10</sup>

#### 6.1.2. Estimating portfolio-selection parameters

We follow [104, p. 87], compute the returns' sample covariance matrix and sample mean vector, and adjust them into quarterly data by following [68, pp.121 & 135]. We next assume them as  $\Sigma$  and  $\mu_2$  and report them as follows:

$$\Sigma = \begin{bmatrix} 0.0318 & 0.0148 & 0.0298 & 0.0470 \\ 0.0148 & 0.0253 & 0.0073 & 0.0007 \\ 0.0298 & 0.0073 & 0.0373 & 0.0524 \\ 0.0470 & 0.0007 & 0.0524 & 0.1372 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 0.1287 \\ 0.0079 \\ -0.1065 \\ -0.1874 \end{bmatrix} \quad \mu_3 = \begin{bmatrix} 18.5 \\ 22 \\ 18 \\ 17 \end{bmatrix}.$$

We retrieve  $\mu_3$  for the four stocks from Table 4 and report  $\mu_3$  above.

#### 6.1.3. Computing a price-weighted stock-market index as a benchmark

As a benchmark, we compute a price-weighted stock-market index (as described by Elton et al. [104, p. 21]) by the average monthly prices (75.4132, 219.3351, 99.9822, and 201.8013 with the sum 596.5318) as follows:

$$\mathbf{x}_i = \begin{bmatrix} 75.4132 \\ 596.5318 \\ 219.3351 \\ 596.5318 \\ 99.9822 \\ 596.5318 \\ 201.8013 \\ 596.5318 \end{bmatrix} = \begin{bmatrix} 0.1264 \\ 0.3677 \\ 0.1676 \\ 0.3383 \end{bmatrix} \quad \mathbf{z}_i = \begin{bmatrix} 0.0344 \\ -0.0621 \\ 19.1957 \end{bmatrix}.$$

We substitute  $\mathbf{x}_i$  into (4), compute  $\mathbf{z}_i$ , and report  $\mathbf{z}_i$  above.

#### 6.1.4. Dominating the stock-market index $\mathbf{z}_i$ as the benchmark

We compute (10) as follows:

$$z_1 = 0.1097z_2^2 + 0.0024z_3^2 + 0.0165z_2z_3 - 0.3157z_2 - 0.0977z_3 + 1.0183.$$

Qi et al. [13] prove the expression above as a paraboloid. Therefore with  $z_{i1} = 0.0344$ , we obtain the intersection of plane  $z_1 = 0.0344$  and (10) as an ellipse as follows:

$$0.1097z_2^2 + 0.0024z_3^2 + 0.0165z_2z_3 - 0.3157z_2 - 0.0977z_3 + 1.0183 = 0.0344.$$

<sup>8</sup> <http://dx.doi.org/10.17632/g3dn7mzvhhb.1>, June 13, 2021.

<sup>9</sup> <http://dx.doi.org/10.17632/g3dn7mzvhhb.1>, June 13, 2021.

<sup>10</sup> Data source: <https://finance.yahoo.com/>, September 25, 2020.

**Table 3**

Rating counter-COVID for the 30 component stocks of Dow Jones Industrial Average of the second quarter of 2020.

Variables	AAPL	AMGN	AXP	BA	CAT	CRM	CSCO	CVX	DIS	DOW	GS	HD	HON	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PG	TRV	UNH	V	VZ	WBA	WMT
management ability	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
management change	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	0	1	1	1	1
management warning	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
return on equity	1	1	1	1	1	1	1	1	0	0	1	0	1	1	1	1	0	1	0	1	1	1	1	0	1	1	1	1	0	1
total debt ratio	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
quick ratio	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	1	1	0	1	0	0	1	0
total-assets growth	0	1	0	1	0	1	0	0	0	0	1	1	1	0	1	0	1	1	1	1	1	0	0	1	0	0	1	1	1	0
product innovation	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
business digitization	1	0	1	0	1	1	1	0	0	1	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
cooperative relationship	1	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1
cost management	1	1	0	0	1	1	1	0	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	0	1
profit margin	1	1	1	0	1	1	1	1	0	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	1
operating-income growth	1	1	0	0	0	1	1	0	1	0	1	1	0	0	1	0	1	0	1	1	1	0	1	1	1	0	1	0	0	0
staff communication	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
health and safety	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
humanistic care	0	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1
salary and welfare	1	1	1	0	1	1	0	1	0	1	1	1	1	0	1	1	1	0	1	1	0	1	0	1	0	1	0	1	0	1
firm image	1	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
information quantity	0	1	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1
information quality	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
thematic setting	0	1	1	1	1	1	0	1	0	1	1	1	1	1	0	1	1	1	0	1	1	1	0	0	1	1	1	1	1	1
social contribution	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
medical activity	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
employment assistance	1	1	0	0	1	1	0	0	0	0	1	0	0	0	0	0	1	0	1	0	1	1	0	1	0	1	0	0	0	1
social promotion	1	1	0	1	0	1	0	1	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	0	1	1	1
sum, counter-COVID	17	22	18	17	16	24	15	18	7	17	21	22	19	18	20	20	19	20	21	22	20	23	16	18	18	21	17	22	17	21

**Table 4**Computing the sample mean of the counter-COVID for the 30 stocks of the first and second quarters of 2020 and assuming the mean as  $\mu_3$ .

	AAPL	AMGN	AXP	BA	CAT	CRM	CSCO	CVX	DIS	DOW	GS	HD	HON	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PG	TRV	UNH	V	VZ	WBA	WMT
$\mu_3$	18.5	22.0	18.0	17.0	16.0	24.0	14.5	19.0	7.5	18.0	21.0	21.5	19.0	18.5	19.5	20.5	18.5	20.0	21.5	22.0	21.0	23.5	16.0	17.5	18.0	21.0	16.5	21.5	18.0	21.5

Qi [105] configures portfolios on the ellipse to dominate the stock-market index  $\mathbf{z}_i$  in sample. We follow the configuration and choose the following portfolios which dominate the index  $\mathbf{z}_i$ :

$$\begin{aligned}\mathbf{z}_{d1} &= (0.0344, -0.0621, 23.2712) & \mathbf{z}_{d2} &= (0.0344, 0.0652, 22.5972) \\ \mathbf{z}_{d3} &= (0.0344, 0.1924, 21.6188) & \mathbf{z}_{d4} &= (0.0344, 0.3197, 19.3739).\end{aligned}$$

We follow Qi [106, pp. 142–144] and compute  $(\lambda_2, \lambda_3)$  for  $\mathbf{z}_{d1}$ ,  $\mathbf{z}_{d2}$ ,  $\mathbf{z}_{d3}$ , and  $\mathbf{z}_{d4}$  as follows:

$$\begin{aligned}(0.0547, 0.0125) & \text{ for } \mathbf{z}_{d1} & (0.0715, 0.0114) & \text{ for } \mathbf{z}_{d2} \\ (0.0833, 0.0088) & \text{ for } \mathbf{z}_{d3} & (0.0741, 0.0001) & \text{ for } \mathbf{z}_{d4}.\end{aligned}\quad (37)$$

#### 6.1.5. Computing the mean-parameterized nondominated path indirectly

For the three stocks, we expect that the counter-COVID of Apple and of Boeing will gradually improve, that the counter-COVID of Amgen will gradually worsen, and that the counter-COVID of American Express will remain the same. Therefore, we take  $\Delta_3$  as follows:

$$\Delta_3 = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T.$$

We compute (15)–(17) as follows:

$$\begin{aligned}\mathbf{h}_0 &= \lambda_2 \begin{bmatrix} 29.5325 \\ -10.1638 \\ -14.7361 \\ -4.6326 \end{bmatrix} + \lambda_3 \begin{bmatrix} -126.7724 \\ 92.8664 \\ 8.4089 \\ 25.4971 \end{bmatrix} + \begin{bmatrix} -0.4890 \\ 0.8000 \\ 0.6423 \\ 0.0467 \end{bmatrix} \\ \mathbf{h}_3 &= \lambda_3 \begin{bmatrix} 357.3616 \\ -152.7205 \\ -160.7231 \\ -43.9180 \end{bmatrix}\end{aligned}$$

$$\mathbf{x}(\lambda_2, \lambda_3, t_3) = \mathbf{h}_0 + t_3 \mathbf{h}_3.$$

We compute (18)–(20) as follows:

$$\begin{aligned}z_1 &= 116.5410\lambda_3^2 t_3^2 - (194.1417\lambda_3^2 - 0.0000\lambda_3 - 35.0637\lambda_2\lambda_3)t_3 \\ &\quad + 3.0787\lambda_2^2 + 141.2911\lambda_3^2 - 21.2563\lambda_2\lambda_3 + 0.0000\lambda_2 - 0.0000\lambda_3 + 0.0177 \\ z_2 &= 35.0637\lambda_3 t_3 + 6.1575\lambda_2 - 21.2563\lambda_3 - 0.1338 \\ z_3 &= 233.0821\lambda_3 t_3^2 + (35.0637\lambda_2 - 388.2834\lambda_3 - 1.2423)t_3 \\ &\quad - 21.2563\lambda_2 + 282.5823\lambda_3 + 20.9087.\end{aligned}$$

#### 6.1.6. Computing the mean-parameterized nondominated path directly

We compute  $\eta_{21} = 35.0637 \neq 0$  by (22)–(25). We compute (30) as follows:

$$\begin{aligned}& 70.1277\lambda_3 z_1 - (388.2834\lambda_3^2 + 2.4845\lambda_3 - 0.0000\lambda_2\lambda_3)z_2 - 35.0637\lambda_3^2 z_3 \\ &= 215.9039\lambda_2^2 \lambda_3 - 745.3237\lambda_2 \lambda_3^2 + 0.0000\lambda_2 \lambda_3 \\ &\quad + 0.0000\lambda_3^3 - 733.1361\lambda_3^2 + 1.2386\lambda_3.\end{aligned}$$

We compute (31) as follows:

$$\begin{aligned}& 1229.4618\lambda_3^2 z_3 - 233.0821\lambda_3 z_2^2 \\ & - (18.7972\lambda_3 - 3705.7270\lambda_3^2 - 1640.9347\lambda_2\lambda_3)z_2 \\ &= 13426.5199\lambda_2^2 \lambda_3 - 164001.2002\lambda_2 \lambda_3^2 - 4997.9223\lambda_2 \lambda_3 + 663352.8012\lambda_3^3 \\ &\quad + 43941.0340\lambda_3^2 + 102.2374\lambda_3.\end{aligned}$$

#### 6.1.7. Checking the domination with $t_3 = 0.05$

We specifically scrutinize the sensitivity for  $\mathbf{z}_{d1}$ ,  $\mathbf{z}_{d2}$ ,  $\mathbf{z}_{d3}$ , and  $\mathbf{z}_{d4}$  with  $t_3 = 0.05$ . Please note that  $\mathbf{z}_i$ ,  $\mathbf{z}_{d1}$ ,  $\mathbf{z}_{d2}$ ,  $\mathbf{z}_{d3}$ , and  $\mathbf{z}_{d4}$  all change with  $t_3 = 0.05$ . We compute the new values by (18)–(20) and (37) and check the domination in Table 5. On the basis of Corollary 1, the change of  $\mathbf{z}_{d1}$ ,  $\mathbf{z}_{d2}$ ,  $\mathbf{z}_{d3}$ , and  $\mathbf{z}_{d4}$  is mild;  $\mathbf{z}_{d1}$  and  $\mathbf{z}_{d4}$  still dominate  $\mathbf{z}_i$ .

### 6.2. Computation results for Dow Jones Industrial Average and the 30 component stocks

We similarly compute for the 30 component stocks of Dow Jones Industrial Average. One technicality is that the sample covariance matrix is singular, so we utilize the shrinkage-estimate method of Ledoit and Wolf [56] with shrinkage-estimate coefficient 0.5. By the instruction of Elton et al. [104, p. 21],  $\mathbf{z}_i$  is Dow Jones Industrial Average. In Table 6, We report  $\mathbf{z}_{d1}$ ,  $\mathbf{z}_{d2}$ ,  $\mathbf{z}_{d3}$ , and  $\mathbf{z}_{d4}$  which dominate  $\mathbf{z}_i$  in sample.

In order to verify the effectiveness of multiple-objective portfolio selection (especially (4)), we choose four out-of-sample periods (as reported in Table 7) and compare  $\mathbf{z}_{d1}$  vs.  $\mathbf{z}_i$  as Dow Jones Industrial Average,  $\mathbf{z}_{d2}$  vs.  $\mathbf{z}_i$ ,  $\mathbf{z}_{d3}$  vs.  $\mathbf{z}_i$ , and  $\mathbf{z}_{d4}$  vs.  $\mathbf{z}_i$ . At this moment, we cherish only the counter-COVID for the first two quarters of the year 2020 and cannot test hypothesis for counter-COVID. However for returns, we then test the following hypothesis for  $\mathbf{z}_{d1}$  vs.  $\mathbf{z}_i$ :

$$H_0 : \text{expected return of } \mathbf{z}_{d1} = \text{expected return of } \mathbf{z}_i \quad (38)$$



**Table 5**New values of  $z_i$ ,  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$  with  $t_3 = 0.05$  and the status of dominating  $z_i$  for the four stocks.

Portfolios	$z_i$	$z_{d1}$	$z_{d2}$	$z_{d3}$	$z_{d4}$
$z_1$	0.0344	0.0341	0.0346	0.0349	0.0344
$z_2$	-0.0621	-0.0402	0.0851	0.2078	0.3199
$z_3$	19.2005	23.0701	22.4467	21.5374	19.4390
dominate $z_i$ ?		yes	no	no	yes

**Table 6** $z_i$  as Dow Jones Industrial Average,  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$  for the 30 stocks in sample.

Portfolios	$z_i$	$z_{d1}$	$z_{d2}$	$z_{d3}$	$z_{d4}$
$z_1$	0.0295	0.0295	0.0295	0.0295	0.0295
$z_2$	-0.0198	-0.0058	0.0010	0.0079	0.0147
$z_3$	19.5125	20.8821	20.8040	20.6196	20.0331

**Table 7**For four out-of-sample periods in 2020, the sample means of the daily returns of  $z_i$  as Dow Jones Industrial Average,  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$ , and the hypothesis result.

Periods	$z_i$	$z_{d1}$	$z_{d2}$	$z_{d3}$	$z_{d4}$
August 25 to August 31	-0.0043	0.0065**	0.0068***	0.0071***	0.0071***
September 16 to October 1	-0.0083	-0.0001*	-0.0000*	0.0000*	0.0000*
November 25 to December 7	-0.0099	0.0001*	0.0001*	0.0001*	0.0001*
December 21 to December 29	-0.0040	0.0003*	0.0005*	0.0006*	0.0006*

Symbols \*, \*\*, and \*\*\* denote 0.10, 0.05, and 0.01 levels of significance, respectively.

 $H_a$  : expected return of  $z_{d1}$  > expected return of  $z_i$ 

We operate paired  $t$ -tests (as described by Anderson et al. [107, pp. 481–484]) for (38) for list the result in Table 7.  $z_i$ ,  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$  are listed in the first row. In the second row, August 25, 2020 to August 31, 2020 is taken an out-of-sample period. For the period, the sample means of the daily returns of  $z_i$  and  $z_{d1}$  are -0.0043 and 0.0065, respectively. The  $p$ -value for (38) is less than 0.05 (as symbolized by \*\*). We reject  $H_0$  and accept  $H_a$  of (38) and conclude that  $z_{d1}$  significantly outperforms Dow Jones Industrial Average. In similar format, the hypothesis result of  $z_{d1}$  vs.  $z_i$ ,  $z_{d2}$  vs.  $z_i$ ,  $z_{d3}$  vs.  $z_i$ , and  $z_{d4}$  vs.  $z_i$  for the four periods are listed.  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$  all significantly outperform Dow Jones Industrial Average in all the four periods.

We compute (18)–(20) as follows:

$$\begin{aligned}
z_1 &= 12.4223\lambda_3^2 t_3^2 - (24.5112\lambda_3^2 + 0.6753\lambda_2\lambda_3 - 0.0000\lambda_3)t_3 \\
&+ 0.0901\lambda_2^2 + 143.6328\lambda_3^2 + 2.9427\lambda_2\lambda_3 - 0.0000\lambda_2 - 0.0000\lambda_3 + 0.0242 \\
z_2 &= -0.6753\lambda_3 t_3 + 0.1803\lambda_2 + 2.9427\lambda_3 - 0.0287 \\
z_3 &= 24.8445\lambda_3 t_3^2 - (0.6753\lambda_2 + 49.0225\lambda_3 - 0.0955)t_3 \\
&+ 2.9427\lambda_2 + 287.2656\lambda_3 + 19.1521.
\end{aligned}$$

We compute (30) as follows:

$$\begin{aligned}
&-1.3506\lambda_3 z_1 - (49.0225\lambda_3^2 - 0.1909\lambda_3 - 0.0000\lambda_2\lambda_3)z_2 + 0.6753\lambda_3^2 z_3 \\
&= 0.0000\lambda_2^2 \lambda_3 + 0.0000\lambda_2 \lambda_3^2 \\
&-0.0022\lambda_2 \lambda_3 - 0.0000\lambda_3^3 + 13.2142\lambda_3^2 - 0.0354\lambda_3.
\end{aligned}$$

We compute (31) as follows:

$$\begin{aligned}
&0.4560\lambda_3^2 z_3 - 24.8445\lambda_3 z_2^2 - (1.3638\lambda_3 - 113.1169\lambda_3^2 - 8.5007\lambda_2\lambda_3)z_2 \\
&= 1.1655\lambda_2^2 \lambda_3 + 68.5145\lambda_2 \lambda_3^2 + 2.0167\lambda_2 \lambda_3 + 917.0025\lambda_3^3 \\
&+ 37.0138\lambda_3^2 - 0.3512\lambda_3.
\end{aligned}$$

We specifically examine the sensitivity for  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$  with  $t_3 = 0.05$ . We compute the new values and check the domination in Table 8.  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$  all still dominate  $z_i$ .

## 7. Conclusions

### 7.1. Future directions

For counter-COVID, we can further consider categories managers, fund providers, products, employees, media, and society. We extend (4) by considering the categories and correspondingly construct the following model:

$$\begin{aligned}
\min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\
\max z_2 &= \mathbf{x}^T \mu_2 \\
\max z_4 &= \mathbf{x}^T \mu_4 \text{ managers} \\
\max z_5 &= \mathbf{x}^T \mu_5 \text{ fund providers} \\
\max z_6 &= \mathbf{x}^T \mu_6 \text{ products} \\
\max z_7 &= \mathbf{x}^T \mu_7 \text{ employees} \\
\max z_8 &= \mathbf{x}^T \mu_8 \text{ media} \\
\max z_9 &= \mathbf{x}^T \mu_9 \text{ society} \\
\text{subject to } \mathbf{A}^T \mathbf{x} &= \mathbf{b},
\end{aligned} \tag{39}$$

where  $\mu_4 \dots \mu_9$  are vectors of stock expected objectives for the categories of counter-COVID (i.e., managers, fund providers, products, employees, media, and society),  $z_4 \dots z_9$  measure the portfolio expected objectives,  $\mathbf{A}$  is a constraint matrix,  $\mathbf{b}$  is the right-hand-side vector, constraint  $\mathbf{A}^T \mathbf{x} = \mathbf{b}$  extends constraint  $\mathbf{x}^T \mathbf{1} = 1$  of (4).

Moreover, because of the uncertainty in measuring the categories, we perform the following robust optimization:

$$\begin{aligned}
\min z_1 &= \mathbf{x}^T \Sigma \mathbf{x} \\
\max z_2 &= \mathbf{x}^T \mu_2 \\
\max z_4 &= \mathbf{x}^T (\mu_4 + t_4 \Delta_4) \\
\max z_5 &= \mathbf{x}^T (\mu_5 + t_5 \Delta_5) \\
\max z_6 &= \mathbf{x}^T (\mu_6 + t_6 \Delta_6) \\
\max z_7 &= \mathbf{x}^T (\mu_7 + t_7 \Delta_7) \\
\max z_8 &= \mathbf{x}^T (\mu_8 + t_8 \Delta_8) \\
\max z_9 &= \mathbf{x}^T (\mu_9 + t_9 \Delta_9) \\
\text{subject to } \mathbf{A}^T \mathbf{x} &= \mathbf{b},
\end{aligned} \tag{40}$$

Table 8

New values of  $z_i$ ,  $z_{d1}$ ,  $z_{d2}$ ,  $z_{d3}$ , and  $z_{d4}$  with  $t_3 = 0.05$  and the status of dominating  $z_i$  for the 30 stocks in sample.

Portfolios	$z_i$	$z_{d1}$	$z_{d2}$	$z_{d3}$	$z_{d4}$
$z_1$	0.0295	0.0294	0.0294	0.0294	0.0295
$z_2$	-0.0198	-0.0060	0.0009	0.0077	0.0147
$z_3$	19.5156	20.8722	20.7942	20.6108	20.0284
dominate $z_i$ ?		yes	yes	yes	yes

where  $t_4 \dots t_9$  are (scalar) parameters for the uncertain amount,  $\Delta_4 \dots \Delta_9$  are fixed vectors for specific directions, and  $t_4 \Delta_4 \dots t_9 \Delta_9$  dictate the uncertainty in measuring  $\mu_4 \dots \mu_9$ .

Qi and Steuer [14] resolve the general model (39). We could enhance the research and resolve (40).

## 7.2. Concluding remarks

For the implications, investors can develop counter-COVID measures and treat them as additional objectives in portfolio selection. Hence, efficient frontiers extend to efficient surfaces. Investors thus enjoy more freedom of choices by balancing and selecting objectives (i.e., variance, expected return, and expected counter-COVID). Tentative empirical results demonstrate that investors can construct portfolios to outperform Dow Jones Industrial Average in sample and out of sample.

Because of the inherent uncertainty in measuring counter-COVID, investors deploy robust optimization (5) and recognize mean-parameterized nondominated paths as the analytical solution of (5) by Theorem 2. Specifically, investors can readily exploit the solutions (e.g., (30) and (31)) by Theorems 3–4. Moreover due to the continuity of mean-parameterized nondominated paths, investors conclude the multiple-objective portfolio selection (4) as relatively robust; that is, minute  $t_3$  results in relatively small change of  $(z_1, z_2, z_3)$  of (5). Technically, investors can pinpoint some portfolios of (4) in order to dominate stock-market indexes and verify the domination by the 30 component stocks of Dow Jones Industrial Average. Furthermore, the domination basically still holds for minute  $t_3$  (e.g.,  $t_3 = 0.05$ ).

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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