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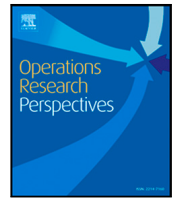
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Nonconvex multicommodity near equilibrium models: Energy markets perspective

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ABSTRACT

In this paper we explore the application of the minimum total opportunity cost (MTOC) model of Fuller and Celebi (2017) to multicommodity market planning models containing binary variables and price sensitive demands, with or without substitution among commodities. We present a greatly simplified derivation of the MTOC approximation of Fuller and Celebi (2017), here called the near equilibrium (NE) model, a mixed integer program with nonlinearities only in the objective function. For some models, the NE solution achieves the MTOC solution exactly, as in an example. We provide a simple example of capacity expansion in gas and electricity markets that are linked through substitution in demand and in the possibility of using gas to produce electricity. In several cases, we compare the NE solution to the social welfare (SW) maximization solution calculated by a sequential optimization algorithm. In one case, the sequential optimization algorithm fails to converge, due to the binary variables. For the other cases, the NE model has smaller producer opportunity costs – in particular, in most cases, smaller make whole payments that bring negative producer profits up to zero – at some sacrifice of social welfare. We suggest that the NE model could be useful to government regulators as a supplementary tool along with SW models, as the NE solution usually reduces subsidies needed for make whole payments, and sometimes benefits consumers compared to the SW solution.

1. Introduction

When a market model has nonconvexities such as binary variables, the prices from the model's solution do not necessarily induce producers to produce the quantities in the model's solution — e.g., some producers might have a negative profit if the model's solution were followed. Because this is a practical difficulty encountered in short-term models used to run electricity markets, there has been much research in recent decades into various methods to correct or reduce this problem in short-term market models for a single commodity, electricity, with or without price sensitivity in demand — see, e.g., [1,2], and [3] for discussions of these methods. The present paper explores the application of one of these methods to multicommodity market models having price sensitive demand functions with substitutability among commodities. In an illustration, consumers satisfy their energy needs using gas and electricity, in a long-term model of investment in new capacities for gas and electricity production.

Our method also applies to the special cases of single commodity models or multicommodity models with no substitution. For example, it applies to electricity market models that contain nonconvexities due to

binary variables for long term decisions on building new capacities [4], and it applies to short term models with binary variables for starting or shutting down the operating generators [1]. The commonly used social welfare maximization model does not always provide price signals that induce the profit maximizing electricity producers to carry out the market regulator's or market operator's desired actions, because the regulator's/operator's desired actions can produce a smaller profit than producers could theoretically have if they could choose their actions based only on the prices. Our method provides a way to minimize the discrepancy between producers' theoretical profits and the profits that actually come from following the regulator's or operator's desired actions, i.e., to minimize producers' *opportunity costs*.¹ Minimization of opportunity costs (compared to a social welfare maximizing solution) usually tends to reduce *make whole* payments to producers that bring negative profits up to zero.

Social welfare (SW) maximization is a widely used one-shot optimization approach to model single commodity markets. When the producers' feasible sets and cost functions are convex, the SW solution

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¹ Opportunity cost normally means a decision maker's loss of value of the next best choice, when the best choice is made. Here, a market regulator *forces* a private actor to lose the extra profit that, in theory, could have been earned by just responding optimally to the prices announced by the regulator.

is an equilibrium: producers and consumers have no incentive to deviate from the solution, given the prices produced by the model. The social welfare objective is the integral of the inverse demand function (representing consumers' value) minus producers' costs. Extending the social welfare model to a multicommodity setting requires a multicommodity version of a consumer value function, but such a function does not exist if the multicommodity inverse demand function is asymmetric in its cross-commodity effects, i.e., consumers' behaviour to substitute one commodity for another [5]. There are two types of methods to formulate and solve such asymmetric, multicommodity, *convex* market models: sequential optimization methods, the earliest of which was the Project Independence Evaluation System (PIES) method [5,6]; and the mixed complementarity problem (MCP) approach [5]. Both methods can also be applied to models with symmetric inverse demand.

In this paper, we explore and compare the application of the SW model (as the PIES extension) and a proposed variant of the MCP approach [1] when the multicommodity market model contains nonconvexities due to binary variables. We show that the SW PIES extension has the same producer incentive problems as in the single commodity electricity models, and in addition, the presence of binary variables can cause nonconvergence of the PIES algorithm. The MCP variant reduces the producer incentive problems, and since it is a one-shot optimization method, there is no question of convergence of an algorithm.

The MCP variant, due to Gabriel et al. [7], is here called the MC model, as it minimizes the sum of complementarities subject to mixed integer constraints containing continuous and binary variables; details are given in Appendix. In [1], the minimum total opportunity cost (MTOC) model is proposed as a generalization of the minimum uplift (MU) model [2,8,9], which is a method to reduce the above mentioned problems with electricity market models. It is shown in [1] that the MC model provides an upper bound on the MTOC optimal value, and for some models, the MC model gives an exact solution of the MTOC model. While the MU model keeps the physical quantity results of the social welfare model but adjusts prices, introducing possible inconsistencies between demand quantities from the model and actual demands which respond to the adjusted prices, both the MTOC and MC models produce demands that are consistent with the model's prices. It was noted in [1] that this consistency makes the MC model a candidate for a multicommodity near-equilibrium model; but only a simple mathematical illustration was given, in an online appendix, without any attempt to interpret the meaning of such a model or its possible utility for analysts.

In this paper, we present a new, simplified derivation of the MC model as a relaxation of the MTOC model, for the case of linear production costs and constraints. As the derivation avoids any reliance on minimization of complementarities, we give this model a different name — the approximate near equilibrium model, or NE for short.² The MTOC and NE models here are slightly different from the MTOC and MC models of Fuller and Celebi [1], to accommodate some features of an illustrative model in Section 3.

The main contributions of this paper are as follows.

1. A general NE model is proposed as a solution concept for multicommodity market models having both binary and continuous variables. A simplified derivation from the MTOC model is presented, along with a simplified proof that the NE model gives an upper bound on the optimal value of the MTOC model.
2. A simple illustrative model of investment in new capacity for the production of two commodities, natural gas and electricity, is presented. It has the property that the NE solution is an exact solution of the MTOC model. An SW model is also defined to produce solutions to compare with the MTOC solution.
3. Several data sets are used for comparisons of SW and NE solutions, and in one case showing the possibility of a non-convergent PIES algorithm. For cases when the PIES algorithm converges, differences between SW and NE solutions are discussed, focusing on differences in make whole payments, social welfare, total opportunity cost, producers' profit and consumers' welfare. Some cases have *concave* variable costs, as piecewise linear approximations, presenting a second source of nonconvexity in addition to the binary variables.
4. It is proposed that the NE solution of the illustrative model can be interpreted as an alternative possible recommendation to government bodies that oversee and approve major new capacity investments that may have impacts on related industries such as gas and electricity supplies and demands. In short, the NE model can be another tool for market analysts, in addition to social welfare oriented models.

The remainder of this paper is organized as follows. Section 2 presents a brief background on the general forms of the MTOC and NE models, describes the SW model and PIES algorithm, and describes measures to compare SW and NE solutions. Section 3 presents the NE and SW versions of a simple two-commodity model for investment in new capacities. Section 4 illustrates the simple two-commodity model with several data sets and compares the SW and NE solutions. Finally, Section 5 summarizes the paper and highlights future directions of this research, including some interesting possibilities in recent research by other authors. An Appendix outlines the derivation of the NE model by the minimum complementarity procedure of Fuller and Celebi [1], in order to prove that a zero optimal value of the NE model implies that the NE solution is also a solution to the SW model.

2. General forms of the SW and NE models

In this section, we outline the derivation of a near equilibrium (NE) model which is general enough to be applicable as an alternative to many multicommodity models of energy markets which contain some binary variables. The NE model has been applied to short-term market operation models by Fuller and Celebi [1], but in this paper, we explore its application to long-term multicommodity planning models, where demand substitution can be a significant consideration, with a simple model of investment in new gas and electricity capacities. Examples of large scale models which could benefit from the NE approach include the MESSAGE [10] and TIMES [11] energy modelling systems, which allow for the use of binary variables to represent lumpy investments or other features requiring discrete variables. One of the first such models represented energy supplies and demands in the city of Geneva [12], using the TIMES precursor MARKAL. For recent reviews of energy planning models, with or without discrete variables, see [13,14]. There are two main points of interest in the use of the NE model for long-term multicommodity models with binary variables: (1) the NE model can be solved as a one-shot optimization, with readily available optimization software and no need to code a specialized algorithm, and (2) the NE solutions, which are often somewhat different from solutions based on optimization of social welfare, may be of interest to government regulators and policy makers, for reasons that we discuss in Section 4.

We also present the corresponding social welfare (SW) models, for the case of a single social welfare objective (when inverse demand is symmetric), and for the case of a sequence of approximate social welfare objectives (when inverse demand is asymmetric, using the PIES algorithm). This is followed by a discussion of methods to compare the SW and NE solutions.

Most of the symbols for this section are as follows. All vectors are column vectors; in the models, ^T indicates the transpose of a vector or matrix.

² Readers familiar with noncooperative game theory should take care not to read NE as Nash Equilibrium.

x_f, y_f : vectors of continuous and binary variables for firm f
 c_f, d_f : vectors of cost coefficients of x_f, y_f for firm f
 E_f, D_f : matrices multiplying x_f, y_f in constraints of firm f
 b_f : vector of right side parameters of constraints of firm f
 α_f, β_f : vectors of dual variables for binary-relaxed model, firm f
 q : vector of demand quantity variables for all commodities
 q^{fix} : vector of parameters for fixed source of supply for q
 p : vector of prices of commodities in q vector
 $q(p)$: vector of demand functions, i.e., demands as functions of prices
 $p(q)$: vector of inverse demand functions, i.e., prices as functions of demands
 A_f, F_f : matrices such that $A_f x_f + F_f y_f =$ net supplies from f towards q
 $B(q)$: benefit to consumers of having amounts q (for symmetric $p(q)$)

In the definition of A_f, F_f , *net supplies* means amounts of the commodities produced by firm f minus amounts of the commodities purchased by firm f as inputs to production — e.g., natural gas purchased by a firm producing electricity with natural gas turbines.

2.1. NE model derivation

The model below differs from that of Fuller and Celebi [1] by (1) including binary variables in the expressions for supply from producers, and (2) including fixed sources of supply; both of the new features are present in the illustration of Sections 3 and 4, necessitating a modified presentation of the model derivations. The NE model below is otherwise very similar to the minimum complementarity (MC) model in [1], but here we present a greatly simplified derivation — for the case of linear cost and constraint functions — that also clearly shows how improvements could be made in future research. The NE model is here derived directly from the MTOC model as a relaxation. In contrast, the MC model was derived by Fuller and Celebi [1] from an MCP variant and is then shown to give an upper bound on the optimal MTOC value (see the Appendix for details). In addition to the inclusion of binary variables in supply expressions and fixed sources of supply, the differences between the NE model here and the MC model of Fuller and Celebi [1] are: whereas Fuller and Celebi [1] assume a convex variable cost function $c_f(x_f)$ and vector-valued convex functions $E_f(x_f)$ contributing to the constraints of firm f , here we assume linearity, i.e., $c_f^T x_f$ and $E_f x_f$; furthermore, Fuller and Celebi [1] use a vector-valued demand function $q(p)$ but here we use the inverse demand functions $p(q)$, to facilitate the definition of the SW models.

The first step in defining the MTOC and NE models is to specify the individual firm model. We assume that the firm is a price-taker, i.e., it treats the market price vector p as a parameter, beyond its control, and, if allowed, it would choose its continuous and binary variables in order to maximize its profit:

Price-Taker problem (PT):

$$\max_{x_f, y_f} (A_f^T p - c_f)^T x_f + (F_f^T p - d_f)^T y_f \quad (1a)$$

$$\text{s.t. } E_f x_f + D_f y_f \leq b_f \quad (1b)$$

$$x_f \geq 0, y_f \in \{0, 1\}^{\dim(y_f)} \quad (1c)$$

In [1], the minimum total opportunity cost (MTOC) model is defined, with price and variable values that produce the smallest possible total opportunity cost, V^{MTOC} . With $\hat{x}_f(p), \hat{y}_f(p)$ denoting optimal solutions to the price-taker problems PT, which depend on the price vector p , MTOC is defined as follows:

MTOC:

$$V^{MTOC} =$$

$$\min_{x_f, y_f, p} \sum_f \left\{ (A_f^T p - c_f)^T \hat{x}_f(p) + (F_f^T p - d_f)^T \hat{y}_f(p) \right\} - \sum_f \left\{ (A_f^T p - c_f)^T x_f + (F_f^T p - d_f)^T y_f \right\} \quad (2a)$$

$$\text{s.t. } b_f - E_f x_f - D_f y_f \geq 0 \quad \forall f \quad (2b)$$

$$q(p) - \sum_f (A_f x_f + F_f y_f) = q^{fix} \quad (2c)$$

$$x_f \geq 0, y_f \in \{0, 1\}^{\dim(y_f)} \quad \forall f \quad (2d)$$

The MTOC model above is not in a convenient form for the use of software such as GAMS, because the functional forms of $\hat{x}_f(p)$ and $\hat{y}_f(p)$ cannot be specified in an efficient way, in general, because PT is a mixed integer model. However, we show below that we can derive a mixed integer nonlinear program (MINLP) approximation to MTOC, denoted NE, whose optimal solution gives an upper bound $V^{NE} \geq V^{MTOC}$. Furthermore, for the example in Sections 3 and 4, $V^{NE} = V^{MTOC}$.

Consider any linear program (LP) approximation of PT that relaxes the binary variables to be between 0 and 1, and that produces an optimal solution $\tilde{x}_f(p), \tilde{y}_f(p)$ with a value that is an upper bound on PT. Examples include: the straightforward replacement of y_f by a vector of real variables bounded below by 0 and above by 1, but otherwise keeping the form of (1a)–(1c); an equivalent, sharper reformulation of PT with extra constraints and possibly extra variables (see, e.g., [15]); and a convex hull approximation to PT. Clearly, replacing $\hat{x}_f(p), \hat{y}_f(p)$ in (2a) with $\tilde{x}_f(p), \tilde{y}_f(p)$ produces an approximation to MTOC whose optimal solution is an upper bound on V^{MTOC} . The last step to produce an MINLP approximation to MTOC relies on the dual of the LP approximation to PT.

For the remainder of the paper, we consider only the PT approximation which is the straightforward replacement of y_f by a vector of real variables between 0 and 1, leaving the exploration of other approximations to future research. The binary-relaxed price-taker problem in primal form is (dual variable vectors are in parentheses):

$$\max_{x_f, y_f} (A_f^T p - c_f)^T x_f + (F_f^T p - d_f)^T y_f \quad (3a)$$

$$\text{s.t. } E_f x_f + D_f y_f \leq b_f \quad (\alpha_f) \quad (3b)$$

$$y_f \leq 1 \quad (\beta_f) \quad (3c)$$

$$x_f, y_f \geq 0 \quad (3d)$$

The binary-relaxed price-taker problem in dual form is the following LP:

$$\min_{\alpha_f, \beta_f} b_f^T \alpha_f + 1^T \beta_f \quad (4a)$$

$$\text{s.t. } c_f - A_f^T p + E_f^T \alpha_f \geq 0 \quad (4b)$$

$$d_f - F_f^T p + D_f^T \alpha_f + \beta_f \geq 0 \quad (4c)$$

$$\alpha_f, \beta_f \geq 0 \quad (4d)$$

The MINLP approximation to MTOC, NE, is created by replacing the MTOC objective terms $(A_f^T p - c_f)^T \hat{x}_f(p) + (F_f^T p - d_f)^T \hat{y}_f(p)$ by the dual PT objective $b_f^T \alpha_f + 1^T \beta_f$, and including the dual PT constraints:

$$\min_{\alpha_f, \beta_f, x_f, y_f, p} \sum_f \left\{ b_f^T \alpha_f + 1^T \beta_f \right\} - \sum_f \left\{ (A_f^T p - c_f)^T x_f + (F_f^T p - d_f)^T y_f \right\} \quad (5)$$

s.t. (2b), (2d), (4b), (4c), (4d) $\forall f$, and (2c),

which has nonlinearities only in the objective terms $-\sum_f p^T A_f x_f - \sum_f p^T F_f y_f$. Using the constraint (2c) of MTOC, we choose to convert these nonlinear terms to the equivalent term $-(q(p) - q^{fix})^T p$, and to change variables from the p vector to the q vector, producing the term

$-p^T(q)(q - q^{fix})$, which, under some conditions on the inverse demand function, can be convex in q . The near equilibrium MINLP follows.

NE:

$$V^{NE} = \min_{x_f, y_f, q, \alpha_f, \beta_f} \sum_f \left\{ \begin{array}{l} b_f^T \alpha_f + 1^T \beta_f \\ + c_f^T x_f + d_f^T y_f \end{array} \right\} - p^T(q)(q - q^{fix}) \quad (6a)$$

$$\text{s.t. } b_f - E_f x_f - D_f y_f \geq 0 \quad \forall f \quad (6b)$$

$$c_f - A_f^T p(q) + E_f^T \alpha_f \geq 0 \quad \forall f \quad (6c)$$

$$d_f - F_f^T p(q) + D_f^T \alpha_f + \beta_f \geq 0 \quad \forall f \quad (6d)$$

$$q - \sum_f (A_f x_f + F_f y_f) = q^{fix} \quad (6e)$$

$$\alpha_f \geq 0, \beta_f \geq 0, x_f \geq 0, y_f \in \{0, 1\}^{\dim(y_f)} \quad \forall f \quad (6f)$$

When $F_f^T = 0$ and $q^{fix} = 0$, the model NE is equivalent to the model MC of Fuller and Celebi [1], for the case of linear costs and constraints: the constraints (6b)–(6f) are identical to the constraints of MC; and (6a) differs from the MC objective only by the use of $p(q)$ instead of $q(p)$, and by the substitution of $-p^T(q)q$ for $-\sum_f p^T A_f x_f$. Our contributions here are in the inclusion of the cases $F_f^T \neq 0$ and $q^{fix} \neq 0$, and in the simplification of the derivation of NE and of the proof of Theorem 1 below.

Given a solution to NE, denoted by NE , the price $p^{NE} = p(q^{NE})$, and the profit Π_f^{NE} of firm f is

$$\Pi_f^{NE} = (A_f^T p^{NE} - c_f)^T x_f^{NE} + (F_f^T p - d_f)^T y_f^{NE}. \quad (7)$$

If a market operator gives instructions x_f, y_f that lead to firm f having a negative profit, then a make whole payment, MW_f to firm f would bring the firm's profit up to zero:

$$MW_f^{NE} = \begin{cases} -\Pi_f^{NE}, & \text{if } \Pi_f^{NE} < 0 \\ 0, & \text{if } \Pi_f^{NE} \geq 0 \end{cases}. \quad (8)$$

In general, the solution of the price-taker model PT for firm f , (1a)–(1c), using the price p^{NE} , will be different from the NE solution; the PT profit $\Pi_f^{PT}(p^{NE})$ at optimal $\hat{x}_f(p^{NE}), \hat{y}_f(p^{NE})$, is

$$\Pi_f^{PT}(p^{NE}) = (A_f^T p^{NE} - c_f)^T \hat{x}_f(p^{NE}) + (F_f^T p - d_f)^T \hat{y}_f(p^{NE}). \quad (9)$$

Since x_f^{NE}, y_f^{NE} is feasible in PT, $\Pi_f^{PT}(p^{NE}) \geq \Pi_f^{NE}$, and the opportunity cost for firm f is defined as

$$OC_f^{NE} = \Pi_f^{PT}(p^{NE}) - \Pi_f^{NE}. \quad (10)$$

A strictly positive opportunity cost could be due to negative profit Π_f^{NE} but a zero price-taker profit $\Pi_f^{PT}(p^{NE})$, or it could instead be due to the price-taker profit being strictly greater than a positive Π_f^{NE} .

The total opportunity cost

$$TOC^{NE} = \sum_f OC_f^{NE} \quad (11)$$

is therefore nonnegative, and can be thought of as a measure of disequilibrium: the larger the total opportunity cost, the greater is the incentive for one or more firms to deviate from the NE solution.

The value of TOC^{NE} cannot be less than the minimum total opportunity cost V^{MTOC} ; therefore

$$V^{MTOC} \leq TOC^{NE}. \quad (12)$$

Furthermore, for the price vector $p = p^{NE}$, it is clear from the structure of the model NE that $\alpha_f^{NE}, \beta_f^{NE}$ are optimal in (4a)–(4d), which implies, by LP duality, that any optimal primal solution $\tilde{x}_f(p^{NE}), \tilde{y}_f(p^{NE})$ to the binary-relaxed price-taker problem (3a)–(3d) satisfies

$$\sum_f (A_f^T p^{NE} - c_f)^T \tilde{x}_f(p^{NE}) + (F_f^T p - d_f)^T \tilde{y}_f(p^{NE}) = \sum_f b_f^T \alpha_f^{NE} + 1^T \beta_f^{NE}.$$

Therefore, using the form of the NE objective in (5), together with the fact that the optimal value of the relaxed PT problem is an upper bound on the optimal value of the exact PT problem, it follows that

$$\begin{aligned} V^{NE} &= \sum_f ((A_f^T p^{NE} - c_f)^T \tilde{x}_f(p^{NE}) + (F_f^T p - d_f)^T \tilde{y}_f(p^{NE})) \\ &\quad - \sum_f ((A_f^T p^{NE} - c_f)^T x_f^{NE} + (F_f^T p - d_f)^T y_f^{NE}) \\ &\geq \sum_f ((A_f^T p^{NE} - c_f)^T \hat{x}_f(p^{NE}) + (F_f^T p - d_f)^T \hat{y}_f(p^{NE})) \\ &\quad - \sum_f ((A_f^T p^{NE} - c_f)^T x_f^{NE} + (F_f^T p - d_f)^T y_f^{NE}) \\ &= TOC^{NE}, \text{ by (7), (9), (10) and (11).} \end{aligned}$$

Note that, if the binary-relaxed PT problem always has optimal solutions with binary values in the y_f vector, for all firms f , and for any price vector p , then $V^{NE} = V^{MTOC}$.

The above observations prove the following theorem.³

Theorem 1. $V^{MTOC} \leq TOC^{NE} \leq V^{NE}$; and if, for any price vector p , the binary-relaxed PT model has a binary solution, then $V^{MTOC} = TOC^{NE} = V^{NE}$.

Remark 1. For models such as the large scale energy sector models developed in the MESSAGE [10] or TIMES [11] frameworks, the NE model might only be guaranteed to find a solution which gives an upper bound V^{NE} on the minimum total opportunity cost. However, we note in Section 3 that many energy planning and capacity expansion models share a key feature that is used in the proof that the binary-relaxed PT model of Section 3 has a binary optimal solution, suggesting that this might be a property of other models, too.

Remark 2. If $p(q)$ is a linear vector valued function, then the only nonlinear term in NE is $-p^T(q)q$, which is quadratic and, under some assumptions, it can be convex, as is the case in the illustrative model of Sections 3 and 4. This allows the use of mixed integer quadratic convex programming solvers such as CPLEX.

Remark 3. The vector q and the vector valued function $p(q)$ can be composed not only of demands for different commodities, but for the general form of NE, these vectors can distinguish demands in different geographical locations and in different time periods. See, e.g., [1] for an example of a two-period unit commitment model with an electricity transmission network.

Remark 4. The objective function (6a) of NE contains the terms that would appear in a monopoly model of a single firm, i.e., to minimize the sum of all firms' costs, minus all revenue from sales to consumers, subject to (6b), (6e) and (6f). However, it is important to note that the other terms in the objective (6a), together with the other constraints (6c) and (6d) force the NE model to have the characteristics of a model of price-taking firms, because price-taker behaviour was assumed in deriving NE from the model of each firm (1a)–(1c). In a computational illustration of Section 4, we show that the monopoly solution is very different from the NE solution.

2.2. Social Welfare (SW) model

If the inverse demand function is symmetric (i.e., $\partial p_j / \partial q_k = \partial p_k / \partial q_j$ for all pairs of commodities j, k), then a consumer value, or benefit function, $B(q)$, exists such that

$$\partial B(q) / \partial q_j = p_j(q) \quad \forall j, \quad (13)$$

³ For the case $F_f = 0, q^{fix} = 0$, this result is also in [1], Theorem 1 and Corollary 6, but with a more complicated proof.

and the social welfare maximization objective is defined as consumer benefit minus producers' cost of producing q :

$$SW = B(q) - \sum_f \{c_f^T x_f + d_f^T y_f\}, \quad (14)$$

with constraints (6b), (6e), nonnegativity for x_f and the binary requirements for y_f . One particular type of symmetric $p(q)$ is to have each good's price depend only on its own quantity, i.e., $\partial p_j / \partial q_k = 0$ for all $j \neq k$.

If instead, the inverse demand function is asymmetric, then a consumer benefit function cannot be defined (see, e.g., [5]). However, it is possible to extend the social welfare maximization model by defining an approximate consumer benefit function to repeatedly maximize approximate social welfare, updating and improving the benefit function for each maximization, and converging to an equilibrium, under certain conditions. The model and algorithm that we use in the illustration of Section 4 is the first such model and algorithm, known by the acronym PIES (for Project Independence Evaluation System, created at the US Department of Energy in the 1970s).⁴ This variation on the equilibrium model of social welfare maximization first defines a *diagonalization* of the inverse demand vector valued function: the inverse demand for the price of the j th commodity q_j , given the demands of the other commodities \tilde{q}_{-j} , is $\hat{p}_j(q_j; \tilde{q}_{-j}) = p_j(\tilde{q}_1, \dots, \tilde{q}_{j-1}, q_j, \tilde{q}_{j+1}, \dots)$. The approximate benefit to consumers of having the amount q_j is $\hat{B}_j(q_j; \tilde{q}_{-j})$, the integral of $\hat{p}_j(q_j; \tilde{q}_{-j})$, and $\sum_j \hat{B}_j(q_j; \tilde{q}_{-j})$ takes the place of $B(q)$ in (14) for an approximate social welfare maximization model. If q_j is a vector of demands distinguished by location and period as well as commodity, then $\hat{B}_j(q_j; \tilde{q}_{-j})$ is the sum over locations and periods of all such integrals. A solution to the PIES model is defined as a solution (x^*, y^*, q^*) to the approximate social welfare maximization model, having the property that $q^* = \tilde{q}$:

PIES model: Find (x^*, y^*, q^*) that solves

$$\max_{x_f, y_f, q} \sum_j \hat{B}_j(q_j; \tilde{q}_{-j}) - \sum_f \{c_f^T x_f + d_f^T y_f\} \quad (15a)$$

$$\text{s.t. (6b), (6e)} \quad (15b)$$

$$x_f \geq 0, y_f \in \{0, 1\}^{\dim(y_f)} \quad \forall f \quad (15c)$$

with $\tilde{q} = q^*$.

The PIES algorithm starts by solving (15a)–(15c) with a first guess of demands \tilde{q} ; the first solution q^* replaces \tilde{q} , and (15a)–(15c) is solved again; the updating of \tilde{q} and solving is repeated until q^* is acceptably close to \tilde{q} . See [6] for conditions on $p(q)$ that ensure convergence, when the firm models are convex (in particular, no binary variables): essentially, the cross- q effects $\partial p_j / \partial q_k$, $k \neq j$, should be small compared with $\partial p_j / \partial q_j$. In Section 4, we illustrate that when the firm models contain binary variables, the PIES algorithm can sometimes fail to converge, even if $p(q)$ satisfies the convergence conditions for convex firm models.

The PIES model can be viewed as a generalization of social welfare maximization because, although it was designed for an asymmetric $p(q)$, the PIES model can also be defined when $p(q)$ is symmetric: at a PIES solution, $\hat{B}_j(q_j^*; \tilde{q}_{-j}) = \hat{B}_j(q_j^*; q_{-j}^*) = \hat{B}_j(q^*)$, so at q^* , $\sum \hat{B}_j(q^*)$ is the total consumer benefit, having the property (13). If symmetry is due to $\partial p_j / \partial q_k = 0$ for all $j \neq k$, then, in addition, the PIES algorithm solves in one optimization step. Therefore, for the remainder of the paper, when we refer to an SW model, we have in mind the more general PIES model.

We can calculate profits, make whole payments and opportunity costs for an SW solution, Π_f^{SW} , $\Pi_f^{PT}(p^{SW})$, MW_f^{SW} , OC_f^{SW} , TOC_f^{SW} in the same manner as with a solution of the NE model (7)–(11),

⁴ This describes the PIES-q variant [5,6], which updates guesses of cross-commodity effects in the inverse demand function. The PIES-p variant updates cross-price effects in the demand function.

but using the price and variable values from the SW solution, $p^{SW} = p(q^{SW})$.

In the appendix, we show that if the NE model has a zero optimal value, $V^{NE} = 0$, then the solution to the NE model also solves the SW model. This suggests the intuition that if the binary nature is expected to be significant but not overwhelming – i.e., the total opportunity cost is small – then the two solutions might be somewhat similar; we see this in some illustrations of Section 4.

2.3. Measures for comparison of SW and NE solutions

If an analyst has found solutions to both the NE and SW models, then comparisons of the two solutions can give insights into decisions, as we illustrate in Section 4. In addition to detailed comparisons of particular activities of all firms, some macro-level comparisons can be useful, e.g., market prices, firms' profits, total make whole payments (which would need to come from extra lines on customers' bills, or from general taxation by government), firms' opportunity costs (indicating firms' degree of dissatisfaction with quantity decisions, given the prices), and if possible, comparisons of consumers' surplus (consumer benefit minus money paid by consumers) and social welfare. We show below that although it is impossible to formulate a consumer benefit function when the multicommodity inverse demand function is asymmetric, it is possible to estimate an approximate difference in consumer benefit between the two solutions, and with this difference, approximate differences in consumers' surplus and social welfare can be estimated.

In microeconomic theory, an inverse demand function is derived from a model of the consumer who maximizes utility in the choices of amounts of several commodities available at given prices, subject to a constraint on the consumer's total income. Harberger [16] showed that, in this multicommodity setting, it is possible to estimate an approximation of the difference in consumer benefit (in monetary units) between the commodity choices corresponding to two different sets of commodity prices. The result, which relies on an assumption of a small change in marginal utility of income across the two sets of prices, is easy to describe: for each commodity separately, plot the two price-quantity pairs on a graph, draw a straight line between them, and calculate the area of the trapezoid from the quantity axis to the straight line, giving the difference in consumer benefit for this one commodity.

Following this simple result, given the price and demand vectors of the solutions of the two models, (p^{SW}, q^{SW}) and (p^{NE}, q^{NE}) we estimate the difference in consumer benefit for commodity j (in the SW minus NE sense) as $\frac{1}{2}(p_j^{SW} + p_j^{NE})(q_j^{SW} - q_j^{NE})$. The difference in consumers' surplus for commodity j subtracts the difference in consumers' payments (also in the SW minus NE sense)⁵:

$$\Delta CS_j = \frac{1}{2}(p_j^{SW} + p_j^{NE})(q_j^{SW} - q_j^{NE}) - (p_j^{SW} q_j^{SW} - p_j^{NE} q_j^{NE}) \quad (16)$$

and the total estimated difference ΔCS , across all commodities j is found by summing (16) over j .

The difference in social welfare for commodity j starts from the difference in consumer benefit, and subtracts the difference in costs of firms that produce commodity j ⁶ (indicated by $f(j)$):

$$\Delta SW_j = \frac{1}{2}(p_j^{SW} + p_j^{NE})(q_j^{SW} - q_j^{NE}) - \sum_{f(j)} (c_f^T x_f^{SW} + d_f^T y_f^{SW} - c_f^T x_f^{NE} - d_f^T y_f^{NE}) \quad (17)$$

and the total estimated difference in social welfare ΔSW is calculated by summing (17) over all commodities.

⁵ If consumers are required to pay any extra amounts – e.g., make whole payments to compensate producers who would otherwise have negative profits – then ΔCS_j must be further adjusted by subtracting the difference in such payments between the two solutions, as we illustrate in Section 4.

⁶ If a firm produces more than one commodity, then it might be impossible to separate some costs by commodity. For this type of model, only the total difference in social welfare, across all commodities, is possible to calculate.

3. NE and SW models with two commodities

In this section, we describe simple NE and SW two-commodity models that are used in the next section to illustrate the possible behaviours of the models and the SW algorithm, and to suggest uses in policy analysis. We do not claim that these models are sufficiently realistic to support real world decision making. In order to make the discussion easier to follow, we name the two commodities gas and electricity. Each firm produces only one commodity, but the models allow for the possibility that a firm might purchase some of the other commodity as an input to production, e.g., producing electricity from gas turbines. The models have only one time period and one geographical location, which means that each model has two market clearing constraints (one for gas, the other for electricity). The single period is a year that is far enough in the future that demands at that time will be significantly affected by prices, i.e., inverse demand for a commodity is a function of demand for that commodity and for the other commodity. The models' decision variables are the sizes of new gas and electricity production capacities, but a given amount of capacity is proportional to the amount of the commodity produced over the year — thus, annual production is taken as the measure of capacity.

A firm's costs include fixed costs, regardless of the size of the new capacity (e.g., land acquisition costs) and variable costs that vary with the size of the new capacity. The fixed cost in the model is the portion of the actual fixed cost that is allocated to the year in question, by an engineering-economic calculation. Similarly, part of the cost of installing the new capacity is allocated to the year in question and added to the annual cost of operation, to give the variable cost in the model. Each firm's variable cost is described by a piecewise linear function composed of several connected line segments; as detailed below, this allows for the representation of concave or convex variable costs, while ensuring that the NE model solution is a solution to the minimum total opportunity cost model MTOC.

Each inverse demand function is linear in the two demand commodities. As in [1], a complementarity submodel of demand ensures that, e.g., if the gas demand is zero, then the gas price can exceed the value of the inverse demand function.

The symbols are defined as follows. Note that every variable and parameter is a real or binary number, not a vector. The symbols have been chosen to correspond, as far as possible, to the vector symbols of the general NE model (6a)–(6f) — e.g. the vector x_f in the general model corresponds to a vector whose components are the individual real variables $x_{f,i}$, below.

- G : set of all firms f that produce gas
- E : set of all firms f that produce electricity
- $x_{f,i}$: variable for capacity *above the minimum* for firm f , on segment i
- $y_{f,i}$: binary variable equal to 1 if segment i is chosen by f , 0 otherwise
- $\alpha_{f,i}, \beta_{f,i}, \phi_f$: dual variables of relaxed PT model (appearing in NE)
- d_f : fixed cost for firm f , regardless of capacity size
- $d_{f,i}^s$: variable cost of capacity equal to minimum on segment i , for f
- $c_{f,i}^s$: variable cost per unit of $x_{f,i}$
- $m_{f,i}^s$: minimum capacity on segment i for firm f
- $M_{f,i}^s$: maximum capacity on segment i for firm f
- r_f : amount of the *other* commodity purchased by f , per unit of f capacity
- g^{ex}, e^{ex} : parameters for existing capacities of gas and electricity
- q^g, q^e : variables for demands for gas and electricity
- p^g, p^e : variables for prices for gas and electricity
- a^g, b^{gg}, b^{ge} : parameters of inverse demand function for gas, all ≥ 0

- a^e, b^{ee}, b^{eg} : parameters of inverse demand function for electricity, all ≥ 0
- u^g, u^e : binary variables used to enforce complementarity in demand submodel
- M : large positive parameter used for complementarity in demand submodel

Note that the total capacity, if segment i is selected, is $m_{f,i}^s + x_{f,i}^s$, and that the upper bound on $x_{f,i}^s$ is $(M_{f,i}^s - m_{f,i}^s)$. Furthermore, if no segment is selected for firm f , then the capacity of firm f is zero.

In the following model of the reaction to prices by firm f (a particular form of the general price-taker model (1a)–(1c)), the symbol p is for the price of the commodity produced by f , and p' is for the price of the other commodity which could be purchased by f as an input — e.g., if f produces electricity, then $p = p^e$ and $p' = p^g$.

$$\max_{x_{f,i}, y_{f,i}} \sum_i \{(p - c_{f,i}^s - r_f p')x_{f,i} + (m_{f,i}^s p - d_f - d_{f,i}^s - r_f m_{f,i}^s p')y_{f,i}\} \quad (18a)$$

$$\text{s.t. } x_{f,i} - (M_{f,i}^s - m_{f,i}^s)y_{f,i} \leq 0 \quad \forall i \quad [\alpha_{f,i}], \quad (18b)$$

$$\sum_i y_{f,i} - 1 \leq 0 \quad [\phi_f], \quad (18c)$$

$$x_{f,i} \geq 0, \quad y_{f,i} \in \{0, 1\} \quad \forall i \quad (18d)$$

The firm model (18a)–(18d) has the property that, for any prices p, p' , the binary-relaxed model (a linear program) has an optimal solution which satisfies the binary restrictions on $y_{f,i}$. The proof is simple: for each segment i , evaluate the objective at $x_{f,i} = M_{f,i}^s - m_{f,i}^s$, $y_{f,i} = 1$ and choose the segment having the largest such value, i.e., the largest profit V_i^* ; then an optimal solution is either (a) set $y_{f,i^*} = 1$, $x_{f,i^*} = M_{f,i^*}^s - m_{f,i^*}^s$ and all other variables to zero if $V_{i^*} \geq 0$, or (b) set $y_{f,i} = 0$, $x_{f,i} = 0$ for all i if $V_{i^*} < 0$. Therefore, the following NE model finds an optimal solution to the MTOC model, according to Theorem 1.

A key feature of the model (18a)–(18d) is the constraint (18c) which requires that at most one of the firm's binary variables can equal 1. This is crucial in the above proof that the binary-relaxed model has a binary optimal solution. This feature often appears in energy planning and capacity expansion models; see, e.g., [11] (section 10 on the lumpy investment extension, where a new capacity variable equals one of several preset sizes or zero), and Pirnia et al. [4] (where any new facility can be built in at most one future year). Therefore, depending on the rest of a firm model's structure, it may be possible to prove that the binary-relaxed firm model has a binary optimal solution, and thus that the NE model provides an optimal solution to the MTOC model.

In the model below, the constraints (19g)–(19l) enforce complementarity relations on demands, prices and inverse demand expressions: they require that a commodity's price be given by a linear inverse demand function of demands, except when demand is zero, when the price may exceed the linear inverse demand function. For example, if the binary variable $u^g = 0$, then (19g) and (19h) imply that $p^g = a^g - b^{gg}q^g - b^{ge}q^e$; but if $u^g = 1$ then constraint (19h) has no effect, (19g) says that $p^g \geq a^g - b^{gg}q^g - b^{ge}q^e$, and (19k) together with nonnegativity of q^g require that $q^g = 0$.

In the following NE model, the binary variables are $y_{f,i}, u^g, u^e$ and the continuous variables are $x_{f,i}, q^g, q^e, p^g, p^e, \alpha_{f,i}, \beta_{f,i}, \phi_f$; for the sake of brevity, this set of variables is called ζ .

$$\min_{\zeta} \sum_{f \in G} (c_{f,i}^s x_{f,i} + (d_f + d_{f,i}^s)y_{f,i} + \beta_{f,i} + \phi_f) \quad (19a)$$

$$-(q^g - g^{ex})(a^g - b^{gg}q^g - b^{ge}q^e) - (q^e - e^{ex})(a^e - b^{ee}q^e - b^{eg}q^g) \quad (19b)$$

$$\text{s.t. } x_{f,i} - (M_{f,i}^s - m_{f,i}^s)y_{f,i} \leq 0 \quad \forall f, i, \quad (19c)$$

$$\sum_i y_{f,i} - 1 \leq 0 \quad \forall f, \quad (19d)$$

$$\sum_{f \in G} m_{f,i}^s y_{f,i} + x_{f,i} + g^{ex} - \sum_{f \in E} r_f (m_{f,i}^s y_{f,i} + x_{f,i}) - q^g = 0, \quad (19e)$$

$$\sum_{f \in E} m_{f,i}^s y_{f,i} + x_{f,i} + e^{ex} - \sum_{f \in G} r_f (m_{f,i}^s y_{f,i} + x_{f,i}) - q^e = 0, \quad (19f)$$

$$a^g - b^{gg} q^g - b^{ge} q^e - p^g \leq 0, \quad (19g)$$

$$a^g - b^{gg} q^g - b^{ge} q^e - p^g + M u^g \geq 0, \quad (19h)$$

$$a^e - b^{ee} q^e - b^{eg} q^g - p^e \leq 0, \quad (19i)$$

$$a^e - b^{ee} q^e - b^{eg} q^g - p^e + M u^e \geq 0, \quad (19j)$$

$$q^g - M(1 - u^g) \leq 0, \quad (19k)$$

$$q^e - M(1 - u^e) \leq 0, \quad (19l)$$

$$c_{f,i}^g + r_f p^g - p^g + \alpha_{f,i} \geq 0 \quad \forall f \in G, \forall i, \quad (19m)$$

$$c_{f,i}^g + r_f p^g - p^e + \alpha_{f,i} \geq 0 \quad \forall f \in E, \forall i, \quad (19n)$$

$$d_f + d_{f,i}^g + r_f m_{f,i}^g p^g - m_{f,i}^g p^e - (M_{f,i}^g - m_{f,i}^g) \alpha_{f,i} + \beta_{f,i} + \phi_f \geq 0 \quad \forall f \in G, \forall i, \quad (19o)$$

$$d_f + d_{f,i}^g + r_f m_{f,i}^g p^g - m_{f,i}^g p^e - (M_{f,i}^g - m_{f,i}^g) \alpha_{f,i} + \beta_{f,i} + \phi_f \geq 0 \quad \forall f \in E, \forall i, \quad (19p)$$

$$x_{f,i}, q^g, q^e, \alpha_{f,i}, \beta_{f,i}, \phi_f \geq 0, \quad (19q)$$

$$y_{f,i}, u^g, u^e \in \{0, 1\} \quad (19r)$$

The approximate social welfare maximization at each step of the PIES algorithm (15a)–(15c) is as follows:

$$\max_{x_{f,i}, y_{f,i}, q^g, q^e} (a^g - \frac{1}{2} b^{gg} q^g - b^{ge} q^e) q^g + (a^e - \frac{1}{2} b^{ee} q^e - b^{eg} q^g) q^e \quad (20a)$$

$$- \sum_{f,i} (c_{f,i}^g x_{f,i} + (d_f + d_{f,i}^g) y_{f,i}) \quad (20b)$$

$$\text{s.t. } x_{f,i} - (M_{f,i}^g - m_{f,i}^g) y_{f,i} \leq 0 \quad \forall f, i \quad (20c)$$

$$\sum_i y_{f,i} - 1 \leq 0 \quad \forall f \quad (20d)$$

$$\sum_{f \in G, i} m_{f,i}^g y_{f,i} + x_{f,i} + g^{ex} - \sum_{f \in E, i} r_f (m_{f,i}^g y_{f,i} + x_{f,i}) - q^g = 0 \quad (20e)$$

$$\sum_{f \in E, i} m_{f,i}^g y_{f,i} + x_{f,i} + e^{ex} - \sum_{f \in G, i} r_f (m_{f,i}^g y_{f,i} + x_{f,i}) - q^e = 0 \quad (20f)$$

$$q^g, q^e \geq 0, \quad x_{f,i} \geq 0, \quad y_{f,i} \in \{0, 1\} \quad \forall f, i \quad (20g)$$

where \bar{q}^g, \bar{q}^e are the demand quantities from the solution of the previous iteration, or from the first guess at the first iteration.

4. Computational results

In the illustrations below, we assume that there are two gas firms and two electricity firms that can provide new capacity, and that the electricity firm number 2 needs to purchase gas as an input (its $r_f > 0$), while electricity firm number 1 produces electricity without gas (its $r_f = 0$), and neither gas firm uses electricity ($r_f = 0$). Data for the piece-wise linear variable cost functions are derived from smooth linear-quadratic functions. We start with lower and upper bounds k_f, K_f on capacity z_f , and a variable cost function $V_f(z_f) = \gamma_f z_f + \frac{1}{2} \delta_f z_f^2$. Then we define a number N of straight line segments, to approximate the variable cost function, on the interval $[k_f, K_f]$, such that each segment's end points lie on the graph of $V_f(z_f)$. For segment $i, i = 1, \dots, N$, the minimum production is $m_{f,i}^g = k_f + (i-1)(K_f - k_f)/N$, the maximum production is $M_{f,i}^g = m_{f,i}^g + (K_f - k_f)/N$, the slope of the segment is $c_{f,i}^g = (V_f(M_{f,i}^g) - V_f(m_{f,i}^g))/(M_{f,i}^g - m_{f,i}^g)$, and the value of the variable cost at the lower endpoint is $d_{f,i}^g = V_f(m_{f,i}^g)$.

We first discuss four cases for which PIES converges, labelled cases A, B, C and D, followed by a discussion of a case where PIES fails to converge. The data for the base case, Case A, are in Table 1, where all $\delta_f < 0$, so that $V_f(z_f)$ is concave, indicating economies of scale, another source of nonconvexity in addition to the binary variables. Data for the three other PIES-convergent cases are described in Table 2 in terms of differences from Table 1. Note that for all four cases, the diagonally

Table 1

Data for the base-case: Case A.

Data for firms f									
	Gas		Electric			Gas		Electric	
	1	2	1	2		1	2	1	2
$K_f =$	400	310	250	200	$\gamma_f =$	13	15	50	3
$k_f =$	250	80	200	60	$\delta_f =$	-.02	-.01	-.003	-.001
$d_f =$	2600	1000	1100	250	$N =$	5	5	5	5
Other supply data									
	$g^{ex} =$		50	$e^{ex} =$		20	$r_{e2} =$		1.5
Inverse demand data									
	$a^g =$		40	$b^{gg} =$.06	$b^{ge} =$.002
	$a^e =$		90	$b^{eg} =$.003	$b^{ee} =$.086

Table 2

Data for Cases B, C, D as changes from Case A.

Case B	$\delta_f^B = \delta_f^A $, for convex $V_f(z_f)$
Case C	$\delta_f = 0, N = 1, a^g = 45, a^e = 100$
Case D	$\delta_f^D = \delta_f^A , N = 1, a^g = 45, a^e = 100, g^{ex} = 210, e^{ex} = 220$

Table 3

Some aggregate results for the base-case A.

	SW		NE	
	Gas	Electric	Gas	Electric
Price p^g, p^e	12.1	49.3	16.7	52.8
Demand q^g, q^e	449.2	457.6	374.7	420.0
Profit	1450		4330	
Opportunity cost	3940		858	
Make whole payment	3940		858	
Profit + Make Whole	5390		5188	
ΔCS			3392	
$\Delta CS - \text{Make Whole}$			310	
$\Delta \text{ Social Welfare}$			215	
$\Delta \text{ Profit}$			-2880	
$\Delta (\text{Profit} + \text{Make Whole})$			202	

(Δ indicates a difference in the SW-NE sense.)

Table 4

Firm-specific results for base-case A.

	SW				NE			
	Gas		Elec.		Gas		Elec.	
Firm	1	2	1	2	1	2	1	2
Capacity	400	299.2	237.6	200	400	224.7	200	200
Profit	-1348	-1409	-1183	5390	470	-370	-489	4718
Opportunity cost	1348	1409	1183	0	0	370	489	0

dominant values of the b^{**} parameters of inverse demand ensure two properties: (i) the PIES convergence criteria for convex models are satisfied; and (ii) the term $-p(q)q$ in the objective (6a) is convex.

All models were formulated in GAMS and solved with CPLEX. Because the models are very small, the computation times were only a few seconds.

4.1. Detailed results of base case A

The aggregate results for Case A are displayed in Table 3, and some details of individual firms are in Table 4. Prices for gas and electricity are higher in the NE solution than in the SW solution; NE demand quantities are lower in the NE solution. The higher NE prices result in smaller opportunity costs for the producers and therefore lower make whole payments by the regulators compared to the SW solution. The NE solution has the larger total profit across all firms, before the make whole adjustment, but after the make whole payments are added to profits, the SW solution has the larger total profit (\$202 higher). The opportunity costs for both models are entirely due to firms preferring

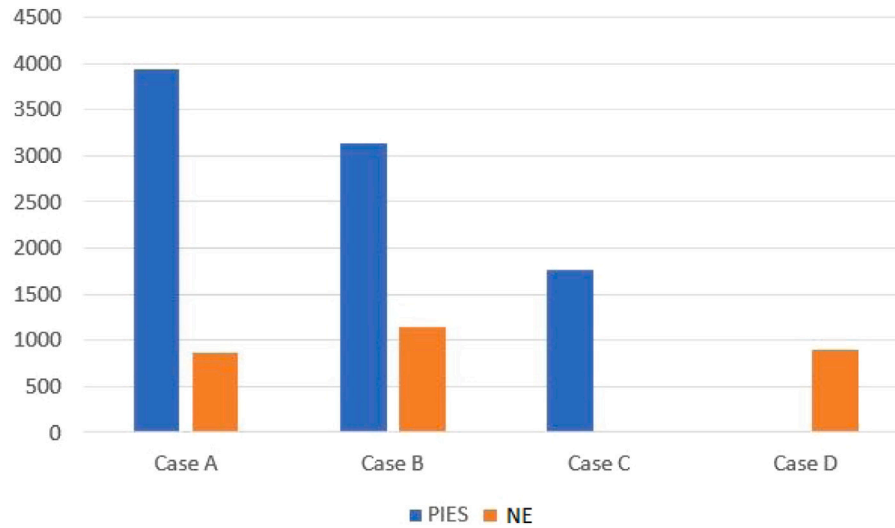


Fig. 1. make whole payments from different scenarios.

zero profit (from building no new facilities) to negative profit; but note that in some other cases, some opportunity costs arise due to positive profits being less than the optimal response to prices.

On the consumer side, the results indicate a higher consumer surplus for the SW model than the NE model (\$3392 higher), before accounting for the make whole payments. However, this difference in net consumer surplus is reduced to \$310 if the difference in make whole payment is subtracted from the difference in consumer surplus, as it is assumed that the consumers pay for the make whole payments.

Overall, in this case we observe that the difference in social welfare favours the SW model, which is not surprising because PIES is oriented to maximizing social welfare, albeit in an iterative fashion. After accounting for the transfer of the make whole payments from consumers to producers, both consumers and producers are better off with the SW solution than the NE solution. However, the much larger make whole payment in the SW solution could be a consideration in the choice by government regulators between the two solutions — e.g., a large subsidy at the time of construction might be criticized as corporate welfare, while a much smaller subsidy could be more easily defended.

With the data in Table 1, a monopoly model was constructed, as discussed in Remark 4 of Section 2.1. As suggested in Remark 4, the monopoly solution was very different from the NE solution: the monopoly prices were much higher, at 30.56 and 70.63 for gas and electric; monopoly demands were much lower, at 150 and 220 for gas and electric; social welfare was worse for the monopoly solution than for the NE solution, by 2489, much more than the SW–NE social welfare difference in Table 3; and the consumers' surplus difference (in the NE-monopoly sense), adjusted for make whole payments, was 8504, which is much larger than the SW–NE difference of 310 in Table 3.

4.2. Comparison of SW and NE for Cases A, B, C and D

We further study the performance of the proposed NE model in comparison to the SW model to explore its advantages and disadvantages for various policy implications. Cases B, C and D are variations of the base case A - the changes from Case A are shown in Table 2. The comparisons are based on the results shown in Fig. 1 for make whole payments, and Fig. 2, in which we compare the differences between SW and NE (in the SW minus NE sense) with respect to:

- consumers surplus (CS)
- consumers surplus minus make whole payments (CS-MW)
- social welfare (SW)
- total profit

- total profit plus make whole payments (Profit+MW).

As shown in Fig. 1, in cases A, B and C, the amount of make whole payment in the SW model is much higher than in the NE model (in case C, make whole payments are zero in the NE solution). In these three cases, the prices are lower in the SW solution than in the NE solution, causing more negative profits and thus larger make whole payments. This is understandable because the SW solution ignores fixed costs in the determination of prices. If government regulators are particularly sensitive to publicity about large make whole payments – e.g., if the make whole payments come as subsidies from government – then the regulators might have a strong preference for the NE solution in cases A, B and especially C.

However, the comparison does not always go this way, as shown in case D, where the SW solution has zero make whole payments but the NE solution has a positive amount. This peculiarity arises because the prices are higher in the SW solution than in the NE solution.

Next we turn to the SW–NE welfare differences in Fig. 2. In cases A, B and C, the consumers surplus criterion favours the SW solution by a large amount before accounting for differences in make whole payments, but by a smaller amount after the make whole adjustment (much smaller for cases A and B). However, the large negative consumers surplus difference in case D indicates a strong preference for the NE solution by this criterion, although the preference is reduced a little when the make whole payments in the NE solution are considered.

The total profit difference, before the make whole adjustment, suggests a preference by producers for the NE solution in cases A, B and C, but instead a strong preference for the SW solution in case D. When profit is adjusted to account for make whole payments, the SW solution is preferred in all four cases, on this producer oriented criterion.

The SW–NE total social welfare difference favours SW in all four cases, but it is conceivable that government regulators could nevertheless prefer the NE solution because of the large make whole payments in cases A, B and C, or because the case D NE solution is much better for consumers in spite of the need for some make whole payments.

4.3. PIES fails to converge

The only data differences in this case, compared with case A in Table 1, are in the two cross-commodity demand function parameters b^{ge} , b^{eg} , which are increased by a factor of 10 to $b^{ge} = .02$, $b^{eg} = .03$, thus enhancing the effect in the PIES algorithm of the previous iteration's other commodity on the inverse demand. Note that the PIES convergence criteria, and convexity of $-p(q)q$ are still satisfied.

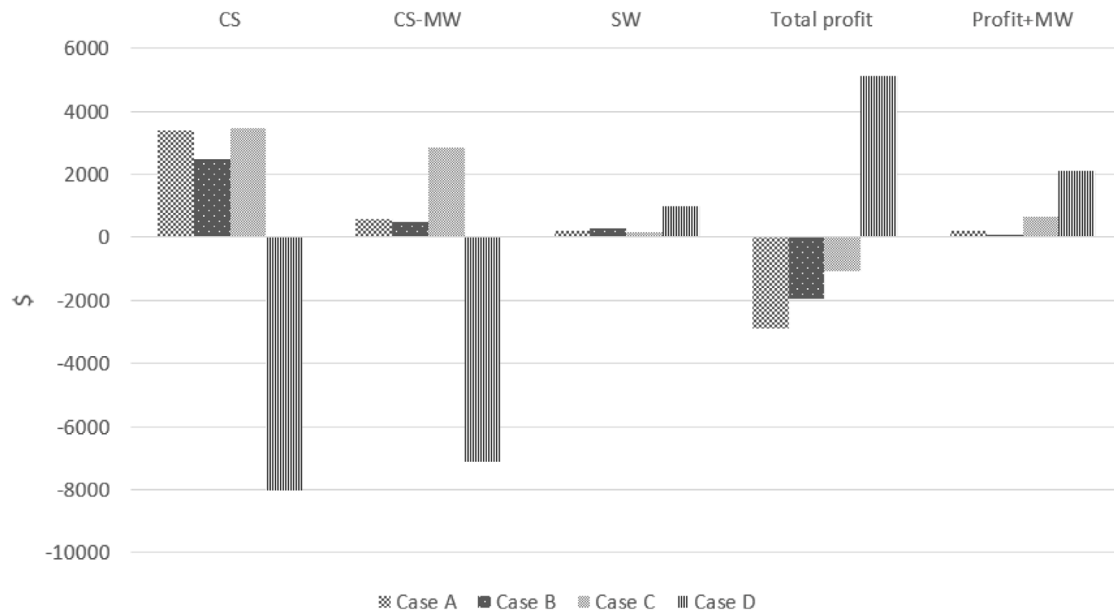


Fig. 2. Differences between SW and NE results (SW-NE) for CS, CS-MW, SW, Total profit, and Profit+MW.

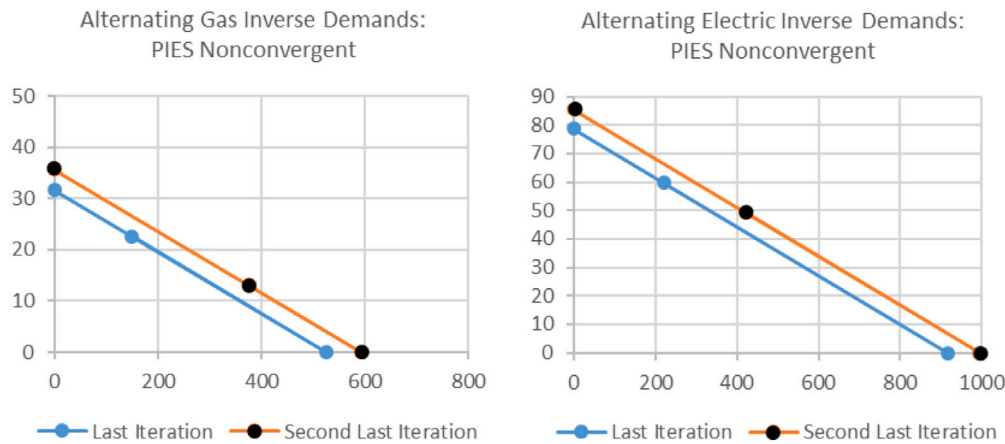


Fig. 3. Two alternating positions of partial inverse demands.

In successive iterations, the PIES solutions alternate between selecting one gas firm and one electric firm, followed by two gas firms and two electric firms. The gas and electric demands alternate between low and high amounts in successive iterations. Fig. 3 shows how this demand alternation causes the position of each commodity's approximate inverse demand (with the other commodity's amount fixed at the previous iteration's value) to shift from one iteration to the next. In Fig. 3, the middle dot of each inverse demand line shows the quantity and price at that iteration of PIES; it can be seen that the demands and prices are very different on successive iterations.

5. Summary and directions for future research

Government regulation and approval of large capacity expansion projects requires careful economic analysis, which can include long term market equilibrium models such as energy planning models that maximize social welfare either as one-shot optimization models, or possibly by sequential optimization in the case of multicommodity models with substitution in demands. However, the large size of such projects naturally leads to models with binary variables, but this causes two problems: prices are not guaranteed to provide correct incentives to producers to choose new capacities that maximize social welfare,

i.e., producers would see opportunity costs; and the sequential optimization algorithm for multicommodity models can fail to converge. The most glaring example of incorrect producer incentives is negative profit, which requires make whole payments to be paid to producers and charged to consumers, if producers are to accept the solution.

In this paper, we propose a general near equilibrium (NE) model, which tends to minimize opportunity costs (and for some models, it exactly minimizes opportunity costs) as a solution concept for single- or multicommodity market models. We do not propose to replace the social welfare oriented analysis; instead, we suggest that the NE model solution could supplement the solution from a social welfare (SW) maximization model.

Using a simple, one period illustrative model of capacity expansion in gas and electricity markets, with substitution in demand, we compare the NE model to the SW model, using the PIES algorithm which maximizes social welfare in an iterative sequence. Solutions for five different data sets confirm, as expected, that the NE solutions have smaller opportunity costs, and the SW solutions have larger social welfare. Of more interest is the failure of the PIES algorithm to converge for one of the data sets; in practice, this would mean that the NE solution is the only solution available. The other four data sets show differences in make whole payments, consumer surplus (before and after adjusting

for make whole payments by consumers), and in profits (before and after make whole payments to producers).

We suggest that government regulators might prefer an NE solution if it leads to smaller make whole payments (i.e., smaller subsidies), or to larger consumer surplus, even at the expense of a small loss in social welfare. Thus, the addition of the NE model to the analyst's toolkit could be useful.

Future research could include the examination of multiperiod NE versions of energy planning models. One important question is whether the NE model exactly minimizes opportunity costs, i.e., whether the price-taker submodel has the desirable property in [Theorem 1](#) (for any price vector p , the binary-relaxed PT model has a binary solution).

For price-taker models that do not have this desirable property, another line of research could be to explore alternative relaxations of the price-taker problem (1a)–(1c) (e.g., convex hull approximations) in the derivation of the NE model. In [\[17\]](#), outer and inner approximations of the convex hulls of players in a Nash game are constructed iteratively; these ideas may be useful for improving the NE approximation. Recent research by Harwood et al. [\[18\]](#) presents an entirely different approach to deal with such PT models: the authors devised a convergent algorithm using a global optimization and semi-infinite approach to solve the MTOC model directly as a bilevel program. The authors tested it on a small example from [\[1\]](#) that lacked the desirable PT property, and found an optimal solution to the MTOC model, different from the MC solution. Future research could examine the performance of the algorithm of Harwood et al. [\[18\]](#), and could possibly consider the NE approximate solution to the MTOC model as an advanced starting point for the algorithm.

CRedit authorship contribution statement

J. David Fuller: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Validation, Visualization, Writing – original draft. **Mehrdad Pirnia:** Data curation, Formal analysis, Funding acquisition, Investigation, Project administration, Resources, Software, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. $V^{NE} = 0$ implies NE solution is SW solution

In this appendix, we show that a zero optimal value for the NE model implies that the NE solution is also a solution to the SW model. The proof uses the original derivation of the NE model based on the minimum complementarity (MC) model employed by Fuller and Celebi [\[1\]](#), so we begin by explaining this.

Starting from the binary-relaxed price-taker problem (3a)–(3d), we derive a mixed complementarity problem (MCP) model of market equilibrium [\[5\]](#), which is composed of the KKT conditions of the binary-relaxed price-taker models, together with a constraint that ensures that net supplies equal demands, and replace the price vector p with the vector-valued inverse demand function $p(q)$:

Binary-Relaxed Mixed Complementarity Problem⁷ (BRMCP):

$$0 \leq b_f - E_f x_f - D_f y_f \perp \alpha_f \geq 0 \quad \forall f \quad (21a)$$

$$0 \leq 1 - y_f \perp \beta_f \geq 0 \quad \forall f \quad (21b)$$

$$0 \leq c_f - A_f^T p(q) + E_f^T \alpha_f \perp x_f \geq 0 \quad \forall f \quad (21c)$$

$$0 \leq d_f - F_f^T p(q) + D_f^T \alpha_f + \beta_f \perp y_f \geq 0 \quad \forall f \quad (21d)$$

$$q^{fix} = q - \sum_f (A_f x_f + F_f y_f) \quad (21e)$$

In each of (21a)–(21d), we refer to the inner product of the vector on the left and on the right as the *complementarity*. Note that the sum of all of the complementarities in BRMCP is equal to zero; but if the y_f are restricted to be binary (as in the next step), then the sum of complementarities could be greater than zero for all values of the variables.

The final step in the derivation of the MC model is to restore the binary restriction on the y_f variables, and formulate a mixed integer nonlinear program which minimizes the sum of the complementarities in BRMCP, subject to all of the constraints of BRMCP (except for $0 \leq 1 - y_f$ and $y_f \geq 0$, which are replaced by $y_f \in \{0, 1\}^{\dim(y_f)}$). The MC objective, to be minimized, is

$$\sum_f (b_f - E_f x_f - D_f y_f)^T \alpha_f + (1 - y_f)^T \beta_f + (c_f - A_f^T p(q) + E_f^T \alpha_f)^T x_f + (d_f - F_f^T p(q) + D_f^T \alpha_f + \beta_f)^T y_f \quad (22)$$

The MC objective is simplified by noting that there are several cancellations. For example, the first term in the summand of (22) includes the term $-x_f^T E_f^T \alpha_f$, but this term is cancelled by the term $+ \alpha_f^T E_f x_f$ from the third term of (22). As in the derivation of the NE model in Section 2.1, we can replace the term $-\sum_f (p^T(q) A_f x_f + p^T(q) F_f y_f)$ by $-p^T(q)(q - q^{fix})$, resulting in an MC model which is precisely the same as the NE model (6a)–(6f) of Section 2.1.

Therefore, if the NE model has a zero optimal value, $V^{NE} = 0$, then the sum of complementarities (22) is zero, which implies that the BRMCP model has a solution with binary values for the y_f variables.

Now consider the KKT conditions of the binary-relaxed SW model. The PIES form (15a)–(15c), with $\tilde{q} = q^*$, is general enough to include both the symmetric and asymmetric inverse demand cases. It is straightforward to check that the KKT conditions of the binary-relaxed PIES model are precisely the BRMCP model (21a)–(21e). Therefore, if the KKT conditions of the binary-relaxed PIES model are sufficient to specify an optimal solution to the binary-relaxed PIES model, then $V^{NE} = 0$ implies that the solution of NE solves the binary-relaxed PIES model with binary values for the y_f variables, i.e., the NE solution solves the PIES model. Because the PIES model includes both the symmetric and asymmetric inverse demand cases, we can say that $V^{NE} = 0$ implies that the solution of NE solves the SW model, provided that the KKT conditions of the binary-relaxed PIES model are sufficient for an optimal solution.

For sufficiency of the KKT conditions, we note that all constraints of the PIES model are linear, satisfying a constraint qualification, and we assume convexity of the approximate consumer benefit functions \hat{B}_j with respect to q_j . Convexity should be an uncontroversial assumption: e.g., if q_j is a single variable in (15a), then the second derivative of $\hat{B}_j(q_j; \tilde{q}_{-j})$ with respect to q_j is negative as it is the rate of change of $p_j(q)$ with respect to q_j .

The discussion above proves the following theorem.

Theorem 2. Assume that $\hat{B}_j(q_j; \tilde{q}_{-j})$ is concave with respect to q_j . If the optimal objective value of NE equals zero, then any solution of NE also solves the PIES model.

⁷ The symbol \perp means (vector on the left)^T(vector on the right) = 0.

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