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Solving vehicle routing problems with stochastic and correlated travel times and makespan objectives



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ABSTRACT

In this paper, we examine a vehicle routing problem with a makespan objective incorporating both stochastic and correlated travel times, which is usually not considered in routing problems. As an alternative to simulation, we develop an approach based on extreme-value theory to estimate the expected makespan (and standard deviation) and show how this approach can be embedded within an existing routing heuristic. We present results that demonstrate the impact of different correlation patterns and levels of correlation on route planning using real-world motivated instances. Depending on the particular objective, cost savings of up to 13.76% can be obtained by considering correlation.

1. Introduction

Many papers on stochastic vehicle routing problems acknowledge that correlation exists between the travel time on arcs in a network, but assume independence for the sake of simplicity. Many empirical studies, though, have shown that correlation is present at significant levels for many networks. For example, Park and Rilett (1999) analyze data obtained from the Houston freeway system and found considerable correlation between some of the arcs of the freeway, with correlation coefficients as high as 0.75. Travel time correlation was also identified in truck routing data from Sydney, Australia (Figliozzi et al., 2007). He et al. (2002) simulated the Irvine, California corridor network using GPS-based vehicle records and found statistically-significant correlation in travel times, with both positive and negative correlation coefficients. Guo (2006) suggests arc travel times may be correlated due to weather effects and secondary incidents and their impact on the behavior of roadway users, and notes that assuming independent travel times can lead to biased solutions. Eliasson (2007) also found positive and negative correlations using an automatic travel time measurement system in Stockholm. Jenelius (2012) notes that poor weather conditions can affect travel times on several arcs. Nicholson (2015) suggests that negative correlation can occur when a bottleneck in one location results in

increased speeds on the downstream locations. Fan et al. (2005) note that travel time correlation might also be an important aspect to consider in routing emergency vehicles through networks after a natural disaster.

Given how prevalent correlation can be, it is important to understand how it can potentially impact the solutions to vehicle routing problems with stochastic travel times, esp. if correlation is considered within individual and across different routes. Minimizing the combined sum of the mean travel times of the individual routes, for example, will not be affected by correlation. Thus, assuming independence and ignoring correlation for this objective is reasonable. Correlation, though, will affect the distribution of the travel times of the individual routes, and, consequently, will also affect the distribution of the makespan, which is the longest expected travel time of any route in a set of routes. It is straightforward to compute the variance of an individual route while considering correlation, as noted in Rostami et al. (2017). It is not straightforward, though, how to use the distribution information from the individual routes to compute the distribution of the makespan. Bertazzi et al. (2015) note that minimizing makespan is a relevant objective in a number of situations, such as driver workload balance and disaster relief efforts. In terms of workload balance, the makespan represents the completion time of the longest driver route. If one driver has a much longer route than other drivers, this can promote job dissatisfaction and

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impact driver retention, which is a known business challenge (Ulmer et al., 2017). In disaster relief, the makespan reflects the time when all destinations affected by the disaster have been served.

Most of the established techniques that incorporate correlation involve the use of simulation. These are discussed in our literature review in Section 2. In such a simulation, the values of travel times on arcs are sampled based on all of the stochastic information available, including correlation. We know of only one other paper (Rajabi-Bahaabadi et al., 2019) that estimates a distribution through formulas used to combine arc distributions as we do, but the problem and its objective are quite different as well as the process of evaluating the quality of the estimates.

In this paper, we utilize ideas from extreme-value theory to build high quality estimates of the distribution of the makespan for a set of routes. This is challenging because the correlation on the arcs leads to correlation on the routes which impacts the distribution of the makespan. We present evidence of the theoretical quality of our approach for examples, as compared with simulation, for three potential types of underlying travel time distributions and positively and negatively correlated travel times. We then discuss how our approach can be embedded within any vehicle routing heuristic to find the minimum expected makespan or expected makespan plus standard deviation. The addition of the standard deviation to the makespan here adds a measure of reliability to our objective. For example, if two sets of routes had very similar expected makespan measures, this combined objective would indicate a preference for the more reliable one as indicated through the addition of the standard deviation. To demonstrate the impact of including correlation in a routing context, we embed our approach within a simple vehicle routing heuristic and do tests with a variety of different instance types with different levels of correlation and instance sizes that follow real-world traffic patterns. Although we cannot guarantee the optimality of the results, we can observe consistent patterns indicating that correlation has the largest impact on (1) instances with high correlation for arcs near a particular incident and (2) for objectives of makespan plus standard deviation with savings of up to 13.76% compared to the no-correlation solution.

A brief review of related literature is presented in the next section. In Section 3, we discuss how correlation can impact the distribution of the individual routes and the makespan. We describe a step-by-step procedure using extreme-value theory to estimate these distributions. We evaluate the accuracy of these estimations through simulation in Section 4. Our ideas are then incorporated into a vehicle-routing algorithm in Section 5, and computational experiments are presented in Section 6. Finally, we conclude the paper and present suggestions for extensions of this research in Section 7.

2. Literature review

This overview of related routing problems consists of three parts. First, we present a review of the research on min-max objectives for node-routing problems. Then, stochastic routing problems are discussed, focusing on problems with random travel times. Finally, path finding and routing problems that incorporate travel time correlation are reviewed.

2.1. The min-max routing problem

Frederickson et al. (1976) were the first to consider min-max, multiple-route routing problems. They show that the multiple traveling salesman problem (kTSP), the multiple Chinese postman problem (kCPP), and the multiple stacker-crane problem (kSCP) are NP-complete, and they develop heuristics using previously-developed methods from the respective minimum-cost, single-route heuristics. Arkin et al. (2006) also consider several node-routing problems and provide approximation algorithms for each.

Several authors solve the multiple traveling salesman problem using metaheuristics. This research includes the use of tabu search (França et al., 1995), neural networks (Modares et al., 1999), genetic algorithms

(Carter and Ragsdale, 2006; Singh and Baghel, 2009), and ant colony optimization (Liu et al., 2009). An extension to this problem is the location-allocation kTSP, where the location of the depot is found in addition to the allocation of customers to routes (Averbakh and Berman, 1997; Nagamochi and Okada, 2004; Xu et al., 2013).

The vehicle routing problem is a generalization of the multiple traveling salesman problem by including capacities on the vehicles (CVRP), time windows on deliveries to customers (VRPTW), etc. Golden et al. (1997) present an adaptive-memory heuristic for the kTSP and the CVRP with and without the multiple use of vehicles. Modares et al. (1999) suggest the use of neural networks for the CVRP, and Ren (2011) proposes a hybrid genetic algorithm. A branch-and-cut search has been developed by Applegate et al. (2002) to optimally solve a 120-customer, 4-vehicle problem using distributed processing. Campbell et al. (2008) note that the min-max VRP is applicable in the routing of critical supplies for disaster relief; they also formulate a second objective in which the average time to each customer is minimized. Bertazzi et al. (2015) conduct a worst-case analysis comparing the min-max objective and the traditional min-sum objective and find that the length of the longest route when solving the min-sum VRP can be as much as k times the longest route in the min-max problem, where k is the number of vehicles, motivating the need to design efficient heuristics for the min-max problem.

Several variants of the min-max vehicle-routing problem have also been studied. Carlsson et al. (2009), Narasimha et al. (2013), and Wang et al. (2015) consider the min-max VRP with multiple depots, in which a vehicle must start and end at the same depot. Xu et al. (2010) investigate the VRP in which a depot must be selected from a given set, and Valle et al. (2011) consider the VRP in which not all customers need to be visited by a vehicle as long as they are close enough to another customer that is on the route. Yakıcı and Karasakal (2013) extend the problem to account for split deliveries and heterogeneous demand.

2.2. Routing problems with stochastic travel times

Next, we will review the literature related to routing that considers stochastic travel times. Various single-route routing problems have also incorporated stochastic travel times, such as the traveling salesman problem (beginning with Leipälä (1978) and Kao (1978)) and the Chinese postman problem (Tan et al., 2005). A few network flow problems also incorporate stochastic times. For instance, Boyles and Waller (2010) minimize a linear combination of expected flow cost and variance of this cost.

The first multiple-route problem, to our knowledge, that considers stochastic travel times is Cook and Russell (1978). In this paper, the authors use simulation to evaluate the quality of deterministically-generated routes on the VRP with stochastic travel times. Laporte et al. (1992) are the first to incorporate stochastic travel times in a VRP model. They present two versions of the problem: (i) chance-constrained programming and (ii) stochastic programming with recourse. They use a branch-and-cut algorithm to solve problems with up to 20 vehicles and travel times that can take on a value from as many as five discrete states. Lambert et al. (1993) develop a model specific to a particular banking context in which travel time is based on a given probability the route is congested. Kenyon and Morton (2003) consider two models with different objective functions: (i) minimizing the expected time that all vehicles will return to the depot and (ii) maximizing the probability of completing the routes by a given deadline. They develop a branch-and-cut approach to solve the problem when the cardinality of the sample space is small and embed a sampling-based procedure for larger sample spaces or continuous random parameters. The only min-max routing problem identified in the literature that incorporates stochastic travel times is Kenyon and Morton (2003), and they note that for large problems “Monte Carlo sampling may be the only viable option.” While they note that the travel times may be dependent in their models, they do not explicitly evaluate the effect of correlated

travel times.

The vehicle routing problem with time windows (VRPTW) and stochastic travel times has been investigated using genetic algorithms (Ando and Taniguchi, 2006) and tabu search (Russell and Urban (2008); Li et al. (2010); Taş et al. (2013); Zhang et al. (2013)). These models include a penalty in the objective function and/or a constraint limiting the probability of not meeting the time-window constraint. Taş et al. (2014a,b) propose an exact solution approach based on column generation and a branch-and-price method. Lecluyse et al. (2009) also consider the VRPTW, but include the standard deviation of the travel time as part of the objective function. Based on extreme-value theory, Ehmke et al. (2015) approximate chance constraints for the VRPTW and adapt a tabu-search algorithm to ensure a certain service level of arriving within customer time windows. Adulyasak and Jaillet (2015) consider the VRP with deadlines in both the stochastic (known probability distribution) and robust (unknown, but belonging to a family of distributions) cases. Taş et al. (2014a,b) consider time dependency in the modeling of stochastic travel times. Time dependency has also been considered in modeling the dial-a-ride problem with scheduled pickup and drop-off times by Fu (2002), Xiang et al. (2008), and Schilde et al. (2014). Related problems in which the customer service times are stochastic are considered by Sungur et al. (2010), Lei et al. (2012), and Errico et al. (2016).

2.3. Stochastic routing and path-finding problems with correlation

We were able to identify several papers that consider stochastic travel times and correlation mostly for path finding approaches, but recently also for some vehicle routing problems.

A constrained stochastic shortest path problem is considered in Wang et al. (2016). The authors develop a Lagrangian relaxation solution approach to find the least expected travel time path satisfying side constraints. In this case, correlation of arc travel times is represented by a joint probability distribution function. Some authors consider correlation rather indirectly, such as by looking for set of non-dominant paths (e.g., Ji et al. (2011)) instead of using correlation matrices. Many path-finding papers that consider correlation use an objective that combines the mean and standard deviation of the path length. One example is Xing and Zhou (2011), who refer to this problem as the reliable shortest path problem. The authors use sample-based approximation to explicitly consider the arc correlation in path-based travel time computations. They use a Monte Carlo simulation to represent the inherent correlation among arc travel times without explicitly requiring a variance-covariance matrix. Second, Shahabi et al. (2013) also consider correlation in the robust shortest path problem. They reformulate this as a conic quadratic problem and develop an outer approximation algorithm. Third, in Zockaie et al. (2014), the minimum path travel time budget problem (MPTTB) on a Chicago-based network is considered. The travel time budget is defined as a linear combination of mean travel time and standard deviation. Two solution methods are implemented and compared: an outer approximation method based on a cutting-plane algorithm and Monte Carlo simulations. Only spatial correlation between neighboring adjacent arcs is investigated. Fourth, Zhang et al. (2017) also consider a combination of mean and standard deviation as objective function and propose a 4-step Lagrangian relaxation approach to solve the reliable shortest path problem with correlated travel times. Numerical experiments comparing their results with those from the methods in Xing and Zhou (2011) and Shahabi et al. (2013) on three real transportation networks are presented. Last, Zhang and Khani (2019) transform the reliable shortest path problem into a standard shortest path problem and a convex problem and develop an eigendecomposition method to decompose the covariance matrices. The algorithm outperforms many previously presented methods.

Other path-finding problems with different objectives include Zockaie et al. (2013) and Zockaie et al. (2016). Zockaie et al. (2013) consider a shortest path problem with a desired on time arrival probability

(SPOTAR) between an origin-destination pair in a network with random travel times. To quantify the impact of correlation, a two-stage algorithm based on Monte Carlo simulation is presented. They investigate only spatial correlation between neighboring adjacent arcs as in Zockaie et al. (2014). One of the conclusions is that correlation has a significant impact on path choice. Zockaie et al. (2016) extends the Monte Carlo simulation approach, developed in Zockaie et al. (2013) and Zockaie et al. (2014) for SPOTAR and MTTBP, for stochastic time varying networks with spatial and temporal travel time correlations. For temporal correlation, consecutive time intervals are assumed to be correlated. For spatial correlation, instead of considering a correlation only between two neighboring arcs, the authors consider it between three neighboring arcs. We allow for any arcs in the network to be correlated, but we do not explicitly consider different time periods.

Yang and Zhou (2017) focus on a priori reliable paths. The authors develop two two-stage nonlinear models with on-time arrival probability and percentile travel time objective functions and reformulate them as equivalent mixed-integer models. To allow time-dependent arc travel times to be spatially and temporally correlated, a sample-based representation is adopted. A Lagrangian decomposition method is proposed to solve the reformulated problems.

There are a few recent vehicle routing papers that include correlation. Guo et al. (2019) describe and implement a method to generate a valid correlation matrix for spatially and temporally correlated travel times. Given the high dimensionality, their approach provides an intuitive and logical method to create a valid correlation matrix. They use a heuristic test to identify whether there are enough simulation scenarios for their numerical results to be valid. Jiang and Mahmassani (2014) consider a time-dependent vehicle routing problem with time windows in congested urban environments. They use simulation to retrieve information on the dynamics of traffic within the road network. Their solution approach is based on a two-phase set-partitioning-based heuristic. It is applied to two real-world networks, Fort Worth and Chicago, and time-dependent travel times are obtained under various incidents. Numerical results show that different events may significantly affect the quality of the solution. For instance, weather incidents such as heavy snow fall increase the total time of the route plan by more than 45%, and not considering the incidents leads to multiple violations of customer time windows. We use their observations about disruptions in one of our tested correlation patterns.

Rostami et al. (2017) investigate a CVRP with correlated stochastic travel times. The objective function is to maximize the travel time reliability, which is represented as a linear combination of travel time and its variance. They model the problem as a binary quadratic program and propose two alternative set partitioning reformulations. They solve their problem for two different types of correlation: correlation only between adjacent arcs and correlation between all arcs. To obtain the variance part of their the objective function, a covariance matrix is randomly generated based on mean travel times between arcs. Computational results based on well-known Solomon datasets show that travel time variance could be reduced up to 70% for some instances. We mimic their use of Solomon datasets, but use different correlation matrices and a makespan objective.

Rajabi-Bahaabadi et al. (2019) is a recent paper that addresses a variant of a vehicle routing problem with soft time windows and correlated travel times. The objective is to minimize the total cost, which consists of travel cost between arcs and penalties for early/late arrival. The authors consider a stochastic model with random early and late arrival times since arc travel times are random variables. Empirical travel time data is taken, and different distributions are fitted to this data. It turns out that the shifted log-normal distribution provides a reasonable fit. The authors combine the assumption of shifted log-normal distribution for individual arcs with the use of the Fenton-Wilkinson method to estimate the distribution of arrival times. An evaluation of accuracy is performed with only one instance of a Solomon dataset. A hybrid two level solution method is proposed where a tabu search is incorporated

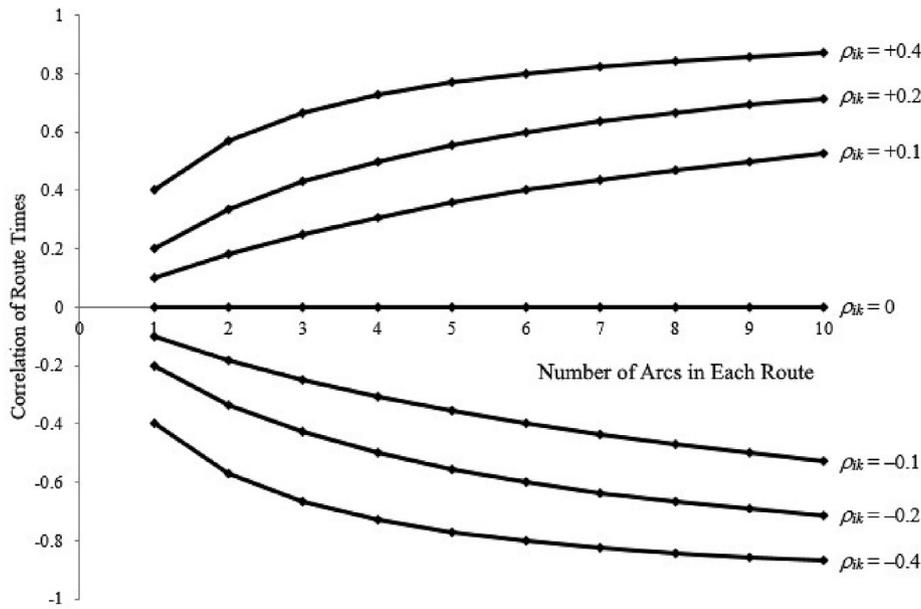


Fig. 1. Correlation of route times for identically-distributed arc travel times.

into a max-min ant system to minimize total costs. Numerical results show that solutions based on ignoring correlation yield higher total costs.

3. Modeling the distribution of the makespan

It is well known in certain settings how correlation affects the variance of a combination of variables. For example, the benefits of risk pooling (or portfolio diversification) decrease as the correlation between the variables becomes more positive. However, with a makespan objective, it is not as clear what the impact of correlation on vehicle routing solutions will be. In this section, we first describe the statistical background needed to evaluate our makespan objective in a vehicle-routing environment, and then present an algorithm for applying these ideas to an existing set of routes.

3.1. Statistical modeling

We begin with a pair of routes R_1 and R_2 each consisting of a set of arcs Π_1 and Π_2 , respectively. Suppose the travel time on each arc $i \in \Pi_p$ follows an arbitrary distribution with a mean of μ_i and a variance of σ_i^2 where arc $i \in \Pi_p, p \in \{1, 2\}$. These times are not required to be independent nor identically distributed (non-IID). The correlation of the time to travel arc i and arc k is ρ_{ik} ($i, k \in \Pi_p$); hence the covariance is $\sigma_{ik} = \rho_{ik}\sigma_i\sigma_k$, where $\sigma_{ii} = \sigma_i^2$ and $\rho_{ii} = 1$.

3.1.1. Distribution of individual routes

A route consists of traveling from the depot, over a set of arcs, and back to the depot. The travel time for the entire route will have a mean of $E(R_p) = \sum_{i \in \Pi_p} \mu_i$ and a variance of $Var(R_p) = \sum_{i \in \Pi_p} \sum_{k \in \Pi_p} \sigma_{ik}$, where $\sigma_{ii} = \sigma_i^2, \rho_{ii} = 1$, and $p \in \{1, 2\}$. Larson (1982) notes that “any linear combination of any normal random variables, independent or not, has a normal probability law,” so the route travel time will be normally distributed for normally-distributed arc travel times. Otherwise, the Central Limit Theorem has been shown to be valid for non-iid variables only in specific situations. Thus, we cannot be assured of normality, particularly for small numbers of arcs on a route.

3.1.2. Correlation between routes

Since the travel times between arcs are correlated, the distributions of the routes will also be correlated. We next analyze how to calculate the

correlation between two routes. The correlation of two linear combinations of variables can be determined as follows (see, e.g., Johnson and Wichern (2007)). Let X_1 and X_2 be two random vectors of size n_1 and n_2 , respectively (hence, we index i and k from 1 to n_1 for X_1 as well as j and l from $n_1 + 1$ to $n_1 + n_2$ for X_2), with variance/covariance matrices of V_{11} and V_{22} , respectively. The covariances between pairs of variables from different sets (cross-correlation) are $Cov(X_1, X_2) = V_{12}$. Then, we define the linear combination of variables as $\tilde{X}_1 = a'X_1$ and $\tilde{X}_2 = b'X_2$. The correlation between these two linear combinations of variables can then be expressed as:

$$Corr(\tilde{X}_1, \tilde{X}_2) = \frac{Cov(\tilde{X}_1, \tilde{X}_2)}{\sqrt{Var(\tilde{X}_1)}\sqrt{Var(\tilde{X}_2)}} = \frac{a'V_{12}b}{\sqrt{a'V_{11}a}\sqrt{b'V_{22}b}} \quad (1)$$

Next, every route can be represented as a linear combination. That is, the sum of the arc travel times for a given route is a simple sum of variables. Hence, the coefficient vectors a and b will be vectors of ones:

$$a'V_{11}a = \sum_{i=1}^{n_1} \sum_{k=1}^{n_1} \sigma_{ik} \quad (2)$$

$$b'V_{22}b = \sum_{j=n_1+1}^{n_1+n_2} \sum_{l=n_1+1}^{n_1+n_2} \sigma_{jl} \quad (3)$$

$$a'V_{12}b = \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_1+n_2} \sigma_{ij} \quad (4)$$

Thus, the correlation between two routes is as follows:

$$Corr(R_1, R_2) = \frac{\sum_{i \in \Pi_1} \sum_{j \in \Pi_2} \sigma_{ij}}{\sqrt{\sum_{i \in \Pi_1} \sum_{k \in \Pi_1} \sigma_{ik}} \sqrt{\sum_{j \in \Pi_2} \sum_{l \in \Pi_2} \sigma_{jl}}} \quad (5)$$

$$= \frac{\sum_{i \in \Pi_1} \sum_{j \in \Pi_2} \rho_{ij} \sigma_i \sigma_j}{\sqrt{\sum_{i \in \Pi_1} \sum_{k \in \Pi_1} \rho_{ik} \sigma_i \sigma_k} \sqrt{\sum_{j \in \Pi_2} \sum_{l \in \Pi_2} \rho_{jl} \sigma_j \sigma_l}}$$

3.1.3. Computing the makespan of a pair of routes

Next, we turn our attention to determining the effect of correlated route times on the expected makespan.

The exact distribution of the maximum of two correlated, non-IID, Gaussian random variables, $Z = \max\{R_1 \sim N(\mu_{R_1}, \sigma_{R_1}^2), R_2 \sim N(\mu_{R_2}, \sigma_{R_2}^2)\}$

Algorithm 1 Calculate expected makespan and standard deviation

for each route $R_p \in \mathcal{R}$ **do**
 Compute $E(R_p)$ and $Var(R_p)$, i.e. mean and variance of completion time (for details, see Section 3.2.1)
end for
for each pair of routes $(R_p, R_q) \in \mathcal{R}$ **do**
 Compute $Corr(R_p, R_q)$, i.e. correlation coefficients of completion times (for details, see Section 3.2.1)
end for
Randomly pick a pair M of routes $R_p, R_q \in \mathcal{R}$
Compute μ_M and σ_M^2 , i.e. mean and variance of makespan of merged route M (for details, see Section 3.2.2)
Delete routes R_p, R_q from \mathcal{R}
for all routes $R_r \in \mathcal{R}, p \neq r, q \neq r$ **do**
 Compute correlation coefficients $Corr(R_M, R_r)$ (for details, see Section 3.2.2)
end for
while $|\mathcal{R}| > 0$ **do**
 Randomly pick any route $R_p \in \mathcal{R}$
 Compute μ_M and σ_M^2 , i.e. mean and variance of makespan of M (for details, see Section 3.2.3) based on merging route p with M
 Delete route R_p from \mathcal{R}
 for all routes $R_r \in \mathcal{R}, p \neq r$ **do**
 Compute correlation coefficients $Corr(R_M, R_r)$ (for details, see Section 3.2.3)
 end for
end while
return μ_M and σ_M^2 contain the final distribution information and represent the mean and variance of the route plan's makespan (for details, see Section 3.2.3)

is known (Nadarajah and Kotz, 2008). The mean μ_Z and the variance σ_Z^2 are:

$$\mu_Z = \mu_{R_1} \Phi[\alpha] + \mu_{R_2} \Phi[-\alpha] + \sigma_{\Delta R} \varphi[\alpha] \quad (6)$$

$$\sigma_Z^2 = (\mu_{R_1}^2 + \sigma_{R_1}^2) \Phi[\alpha] + (\mu_{R_2}^2 + \sigma_{R_2}^2) \Phi[-\alpha] + (\mu_{R_1} + \mu_{R_2}) \sigma_{\Delta R} \varphi[\alpha] - \mu_Z^2 \quad (7)$$

where:

$$\sigma_{\Delta R} = \sqrt{\sigma_{R_1}^2 + \sigma_{R_2}^2 - 2\sigma_{R_1}\sigma_{R_2}Corr(R_1, R_2)} \quad (8)$$

$$\alpha = \frac{(\mu_{R_1} - \mu_{R_2})}{\sigma_{\Delta R}}, \quad (9)$$

and where φ and Φ are the density and cumulative distribution functions for the standard normal distribution, respectively, and the correlation between the two routes, $Corr(R_1, R_2)$, is obtained from (5).

3.1.4. Observations from statistical modeling

To illustrate the effect of arc travel time correlation on route travel time correlation, suppose we have two routes R_1, R_2 with the same number of arcs (i.e., $|\Pi_1| = |\Pi_2| = n$), and that the variances of the travel times for all arcs are equal ($\sigma_i^2 = \sigma_j^2 \forall i, j \in \Pi_p, p \in \{1, 2\}$). Further assume that the magnitude, but not necessarily the sign, of the correlation coefficients between all arcs $i, j \in \Pi_p, p \in \{1, 2\}$ are the same and are equal to ρ_{ij} . Equation (5) then reduces to:

$$Corr(R_1, R_2) = \frac{n\rho_{ij}}{1 + (n-1)\rho_{ik}} \quad (10)$$

Fig. 1 illustrates the correlation of route times when the correlations of arc times are positive within a route (ρ_{ij}) and either positive ($\rho_{ik} = \rho_{ij}$) or negative ($\rho_{ik} = -\rho_{ij}$) between routes. If $n = 1$, the route correlation is the arc correlation. However, the correlation of route times approaches one as the number of arcs in the routes increases. And the effect is significant even for the size of problems typically encountered in routing problems; for an arc correlation of $\rho_{ij} = 0.20$, the route correlation equals 0.60 for $n = 6$ arcs. So even weakly correlated arc travel times can result in strongly correlated route times that we must deal with in designing routes. When the variances and correlations are not the same for all arcs, we can take advantage of these relationships in designing routes by appropriately assigning arcs to routes to influence the route correlations to our advantage.

To isolate the effect of correlated route times on the makespan, suppose the two routes have equal means and variances, $\mu_{R_1} = \mu_{R_2} = \mu_R$ and $\sigma_{R_1}^2 = \sigma_{R_2}^2 = \sigma_R^2$. Then, from Equation (8):

$$\sigma_{\Delta R} = \sqrt{2\sigma_R^2(1 - Corr(R_1, R_2))}. \quad (11)$$

For equal means, $\alpha = 0$, and for a normal distribution, $\Phi[0] = 0.5$ and $\varphi[0] = \frac{1}{\sqrt{2\pi}}$, so the expected value of the maximum of the two routes will be:

$$\mu_Z = \mu_R + \sigma_{\Delta R} \sqrt{\frac{1 - Corr(R_1, R_2)}{\pi}}. \quad (12)$$

If the routes are perfectly positively correlated (i.e. $Corr(R_1, R_2) = 1$), the expected value of the maximum will simply be the mean of each route, since the two routes will always have the same time. However, as the correlation decreases, μ_Z increases.

3.2. Estimation of makespan objectives

We next present pseudocode to describe the procedure for estimating the distribution of the makespan considering correlation between arcs and routes. The different steps of the pseudocode point to the relevant subsections for more detailed equations. The general idea is to compute the mean and variance for individual routes first, then combine the created route information recursively applying extreme-value theory, following the example of authors like Clark (1961). The resulting estimates should be able to be used in a variety of arc- and node-routing algorithms by simply replacing the objective-function computation (either minimize expected makespan or makespan plus standard deviation) with the one used here. Our pseudocode is presented in Algorithm 1.

3.2.1. Initialization of route information

For the initialization of route information, let us consider a set of routes $\mathcal{R} = R_1, \dots, R_v$ where v is the number of vehicles. We let Π_p be the set of arcs in route p (a given assignment of arcs to routes). Suppose the travel time on each arc follows an arbitrary distribution with a mean of μ_i and a variance of σ_i^2 as well as a correlation of the time to travel arc i and arc k of ρ_{ik} ($i, k \in \Pi_p$); hence the covariance is $\sigma_{ik} = \rho_{ik}\sigma_i\sigma_k$, where $\sigma_{ii} = \sigma_i^2$ and $\rho_{ii} = 1$. Following Section 3.1.1, we calculate the mean and variance of the time to complete each route. For each route R_p in \mathcal{R} :

$$E(R_p) = \sum_{i \in \Pi_p} \mu_i \quad (13)$$

$$Var(R_p) = \sum_{i \in \Pi_p} \sum_{k \in \Pi_p} \sigma_{ik} = \sum_{i \in \Pi_p} \sum_{k \in \Pi_p} \rho_{ik} \sigma_i \sigma_k \quad (14)$$

Following Section 3.1.2, we calculate the correlation coefficient of the time to complete each pair of routes (using indices i and k for route p with $|\Pi_p|$ arcs assigned to it, and using indices j and l for route q with $|\Pi_q|$ arcs). Then, for each pair of routes R_p and R_q in \mathcal{R} :

$$Corr(R_p, R_q) = \frac{\sum_{i \in \Pi_p} \sum_{j \in \Pi_q} \rho_{ij} \sigma_i \sigma_j}{\sqrt{\sum_{i \in \Pi_p} \sum_{k \in \Pi_p} \rho_{ik} \sigma_i \sigma_k} \sqrt{\sum_{j \in \Pi_q} \sum_{l \in \Pi_q} \rho_{jl} \sigma_j \sigma_l}} \quad (15)$$

At this point, we have the distribution of the time to complete each route. To simplify subsequent expressions, let $\mu_{R_p} = E(R_p)$ be the expected (mean) time to complete route R_p , and $\sigma_{R_p}^2 = Var(R_p)$ be the variance of the time to complete route p .

3.2.2. Combination of route information with extreme-value theory

Let us now begin creating the makespan distribution merging the distribution information (mean and variance) of the completion times for the first pair of routes. As noted in Algorithm 1, we can begin by merging any pair of routes. Likewise, as we iterate through the routes, we can choose any remaining route to be the next route to be merged. For the sake of simplicity of presentation, we will assume the first two routes that are merged are R_1 and R_2 . Then, the routes are merged in order of R_3, \dots, R_v .

We compute the mean and the variance of the maximum time to complete the first two routes (as shown in Section 3.1.3). Let $M_2 = \max\{R_1, R_2\}$, and μ_{M_2} and ν_{M_2} represent the mean and the second raw moment for this pair of routes, respectively:

$$\mu_{M_2} = \mu_{R_1} \Phi[\alpha] + \mu_{R_2} \Phi[-\alpha] + \sigma_{\Delta R} \varphi[\alpha] \quad (16)$$

$$\nu_{M_2} = (\mu_{R_1}^2 + \sigma_{R_1}^2) \Phi[\alpha] + (\mu_{R_2}^2 + \sigma_{R_2}^2) \Phi[-\alpha] + (\mu_{R_1} + \mu_{R_2}) \sigma_{\Delta R} \varphi[\alpha] \quad (17)$$

where

$$\sigma_{\Delta R} = \sqrt{\sigma_{R_1}^2 + \sigma_{R_2}^2 - 2\sigma_{R_1}\sigma_{R_2}Corr(R_1, R_2)} \quad (18)$$

$$\alpha = \frac{(\mu_{R_1} - \mu_{R_2})}{\sigma_{\Delta R}}, \quad (19)$$

and φ and Φ are the density and cumulative distribution functions for the distributions used (either normal, shifted gamma or shifted exponential). Then, the mean and variance of the maximum of the two route times will be μ_{M_2} and $\sigma_{M_2}^2 = \nu_{M_2} - \mu_{M_2}^2$.

Now, we recompute correlation coefficients from M_2 and the remaining routes $p = 3, 4, \dots, v$:

$$Corr(M_2, R_p) = \frac{\sigma_{R_1}Corr(R_1, R_p)\Phi[\alpha] + \sigma_{R_2}Corr(R_2, R_p)\Phi[-\alpha]}{\sigma_{M_2}} \quad (20)$$

3.2.3. Recursive combination of route information

Next, we recursively merge our combined routes with a new route and look for their maximum distribution. We repeat until there are no more routes left to merge. We begin with M_2 and a mean and variance of a third route (for simplicity of presentation, R_3) and call their maximum distribution M_3 ; that is, $M_3 = \max\{R_1, R_2, R_3\} = \max\{M_2, R_3\}$. We compute the mean and variance of M_3 based on Equations (21) and (22) with $q = 3$. Then, we calculate the correlation coefficient of M_3 and each of remaining routes following the correlation formula presented in Equation (25) for $q = 3$ and $p = 4, \dots, v$. We repeat the above steps until all routes R_4, \dots, R_v have been merged and the maximum distribution $M_v = \max\{M_{v-1}, R_v\}$, has been created. For $q = v$, we compute the mean and variance based on Equations (21) and (22), but do not need to compute any remaining correlation coefficients using Equation (25).

$$\mu_{M_q} = \mu_{M_{q-1}} \Phi[\alpha] + \mu_{R_q} \Phi[-\alpha] + \sigma_{\Delta R} \varphi[\alpha] \quad (21)$$

$$\nu_{M_q} = (\mu_{M_{q-1}}^2 + \sigma_{M_{q-1}}^2) \Phi[\alpha] + (\mu_{R_q}^2 + \sigma_{R_q}^2) \Phi[-\alpha] + (\mu_{M_{q-1}} + \mu_{R_q}) \sigma_{\Delta R} \varphi[\alpha] \quad (22)$$

where

$$\sigma_{\Delta R} = \sqrt{\sigma_{M_{q-1}}^2 + \sigma_{R_q}^2 - 2\sigma_{M_{q-1}}\sigma_{R_q}Corr(M_{q-1}, R_q)} \quad (23)$$

$$\alpha = \frac{(\mu_{M_{q-1}} - \mu_{R_q})}{\sigma_{\Delta R}}, \quad (24)$$

and φ and Φ are the density and cumulative distribution functions for the distributions used (either normal, shifted gamma or shifted exponential). Then, the mean and variance of the maximum of the two route times will be μ_{M_q} and $\sigma_{M_q}^2 = \nu_{M_q} - \mu_{M_q}^2$.

Then, we recompute correlation coefficients between M_q and the remaining routes $p = q + 1, \dots, v$:

$$Corr(M_q, R_p) = \frac{\sigma_{M_{q-1}}Corr(M_{q-1}, R_p)\Phi[\alpha] + \sigma_{R_q}Corr(R_q, R_p)\Phi[-\alpha]}{\sigma_{M_q}} \quad (25)$$

Using these formulas, we can create an optimization model to solve for the set of routes that minimizes the makespan or makespan plus standard deviation. Unfortunately, the resulting model is a nonlinear integer program, which are notoriously difficult to solve exactly. Due to the recursion, the recombination of routes and the recalculation of correlation coefficients require additional efforts that are not necessary for deterministic routing problems. However, it is possible to include this recursive approximation in efficient local search procedures. Before we

Table 1
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Expected Makespan – Normal Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	0.0203	0.0141	0.0170	0.0196	0.0265
	0.6	0.0165	0.0191	0.0208	0.0210	0.0193
	0.4	0.0101	0.0147	0.0354	0.0173	0.0264
	0.2	0.0475	0.0209	0.0176	0.0405	0.0232
	0.0	0.0188	0.0410	0.0339	0.0515	0.0528
Correlation Matrix 2	0.2	0.0398	0.0419	0.0348	0.0374	0.0233
	0.4	0.0405	0.0278	0.0421	0.0241	0.0251
	0.6	0.0534	0.0117	0.0213	0.0256	0.0183
	0.8	0.0578	0.0096	0.0144	0.0290	0.0244
Correlation Matrix 3	0.2	0.0830	0.1175	0.1039	0.1871	0.2557
	0.4	0.1518	0.2281	0.2799	0.5126	0.7044
	0.6	0.3168	0.3668	0.5043	0.9237	1.1397
	0.8	0.5220	0.4859	0.7769	1.3307	1.5874

present such a heuristic in Section 5, we will analyze the quality of the recursive approximation approach in terms of statistical accuracy.

4. Evaluating the accuracy of the estimates

To evaluate the accuracy of the estimates for more than two routes and/or for non-normal route times, in this section, we compare the results of our recursive approximation with results from simulation. For our simulation experiment, we consider a theoretical setting with given instances with various numbers of routes, routes with varying numbers of arcs as well as various travel time distributions. We compare the accuracy of our estimates for completely positively correlated travel times as well as systematic mixtures of positively and negatively correlated travel times. Evaluation in a routing context and more realistic instances following urban traffic patterns will be demonstrated in Sections 5 and 6.

Depending on the true distribution of the arc travel times, creating makespan distributions with extreme-value theory can be inaccurate. For normally-distributed arc travel times, the route times will also be normally distributed, and Equations (16) and (17) will provide an accurate estimate. For non-normally distributed arc travel times, even for those situations in which the Central Limit Theorem may be valid, the number of arcs in a route may not be sufficient to assure convergence to the normal distribution. Furthermore, even if all of the route times were approximately normally distributed, the distribution of the maximum time, which we need to evaluate our makespan objective with our step-by-step approach, will not be normal. In fact, it will be skewed. As a consequence, the route times will neither be identical nor independent due to the correlated arc times as illustrated in the previous section, but since we treat them as normally distributed, the resulting estimates may be inaccurate.

The distributions of the arc travel times, including the mean and coefficient of variation, are randomly generated in the manner of Ehmke et al. (2015). The distance between each pair of customers is the Euclidean distance between randomly-generated coordinates in two-dimensional space with a uniform distribution between 0 and 40. The mean travel time is set equal to this distance, while the coefficient of variation is randomly generated with a uniform distribution between 0.1 and 0.3. Three travel time distributions are evaluated – a normal distribution, a shifted gamma distribution, and a shifted exponential distribution – with parameters set such that the appropriate mean and coefficient of variation are obtained with a skewness equal to 0 (normal), 1 (shifted gamma), or 2 (shifted exponential). We generate five instances for each test scenario.

In addition to a benchmark scenario with uncorrelated travel times (an identity correlation matrix), three types of correlation matrices are constructed. For the first scenario, the travel times for each pair of arcs

Table 2
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Expected Makespan – Shifted Gamma Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	0.1610	0.1766	0.1548	0.1455	0.1064
	0.6	0.1957	0.1841	0.1361	0.0641	0.1287
	0.4	0.2050	0.1544	0.1110	0.0481	0.1158
	0.2	0.2382	0.1204	0.0588	0.1264	0.0960
	0.0	0.2560	0.0833	0.0562	0.2448	0.0999
Correlation Matrix 2	0.2	0.2314	0.0797	0.1135	0.1873	0.0682
	0.4	0.2147	0.0667	0.1193	0.1656	0.0775
	0.6	0.1912	0.0456	0.0750	0.1207	0.0731
	0.8	0.1834	0.0418	0.0633	0.1021	0.0290
Correlation Matrix 3	0.2	0.1560	0.1540	0.1065	0.1149	0.2239
	0.4	0.2016	0.2517	0.2724	0.4790	0.6340
	0.6	0.3420	0.3655	0.6102	1.0313	1.1576
	0.8	0.7078	0.5305	1.0445	1.5948	1.6693

Table 3
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Expected Makespan – Shifted Exponential Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	0.3335	0.3009	0.1845	0.2567	0.1444
	0.6	0.3524	0.3007	0.1988	0.1753	0.1553
	0.4	0.3612	0.2722	0.1856	0.1431	0.1396
	0.2	0.3494	0.2231	0.1677	0.1963	0.1384
	0.0	0.4004	0.1713	0.1994	0.3968	0.1310
Correlation Matrix 2	0.2	0.4331	0.1573	0.2478	0.3627	0.1604
	0.4	0.4402	0.1278	0.2849	0.3035	0.1407
	0.6	0.3925	0.1079	0.2587	0.2485	0.0918
	0.8	0.3274	0.0913	0.2081	0.2612	0.0699
Correlation Matrix 3	0.2	0.6288	0.6507	0.6340	0.9166	0.8230
	0.4	0.4578	0.5062	0.7393	0.7834	0.9837
	0.6	0.9347	0.8008	1.3129	1.5489	1.7297
	0.8	1.5093	1.1784	1.9794	2.4415	2.5177

are assumed to be positively equicorrelated; four values representing increasing interrelationships are used ($\rho_{ik} = +0.2, +0.4, +0.6,$ and $+0.8,$ for $i \neq k$). The other two scenarios incorporate negative correlations; however, due to the need for a positive semidefinite correlation matrix, some of the correlation coefficients are necessarily positive. The second scenario is such that the travel times are positively and negatively correlated within a route and with arcs in all other routes. The third scenario is such that the travel times are positively correlated within a route and positively correlated with all arcs in half of the other routes, negatively correlated with arcs in the other half of the routes. Tables 19–21 of the electronic appendix illustrate the three types of matrices for a 3-route/4-arcs per route instance with a correlation coefficient of 0.2.

To evaluate its accuracy, the expected makespan estimated by our recursive approximation approach is compared to the results of a simulation using @Risk with 10,000 simulation runs. Tables 1–3 provide the mean absolute percent error (MAPE) of the five instances for each test scenario. Results are provided for sets of 3, 5, and 10 routes; for 2, 4, and 8 arcs per route; as well as for normal, shifted gamma, and shifted exponential travel time distributions.

As illustrated in Tables 1–3, our recursive approach is remarkably accurate in estimating the expected makespan. For normally-distributed travel times (Table 1), all of the MAPEs for Correlation Matrices 1 and 2 are less than one-tenth of one percent, with no apparent effect of the number of routes or the number of arcs per route. Not surprisingly, the accuracy is slightly less for the gamma and exponential distributions (Tables 2 and 3), although there is still less than one-half of one percent

Table 4
Mean Percent Deviation of the Expected Makespan from the Uncorrelated Makespan.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	-4.07	-1.46	-2.40	-2.57	-1.44
	0.6	-2.90	-1.11	-1.77	-1.85	-1.05
	0.4	-1.85	-0.74	-1.16	-1.20	-0.68
	0.2	-0.90	-0.37	-0.57	-0.58	-0.33
	0.0	0.00	0.00	0.00	0.00	0.00
Correlation Matrix 2	0.2	-0.77	-0.37	-0.48	-0.50	-0.32
	0.4	-1.59	-0.73	-0.96	-1.02	-0.66
	0.6	-2.45	-1.07	-1.43	-1.57	-1.00
	0.8	-3.38	-1.35	-1.85	-2.17	-1.35
Correlation Matrix 3	0.2	+0.85	+1.02	+1.82	+2.08	+2.26
	0.4	+1.61	+2.06	+3.42	+3.80	+4.22
	0.6	+2.30	+3.10	+4.88	+5.33	+6.01
	0.8	+2.94	+4.12	+6.25	+6.75	+7.65

error for Correlation Matrices 1 and 2. For these distributions, increasing the number of arcs per route (e.g., going from 2 to 4 to 8 arcs per route with 5 routes) improves the accuracy of our approximation, likely due to the fact that the route times approach normality as the number of arcs grows larger.

The results from Correlation Matrix 3 are somewhat different. Although the recursive approximation is still reasonably accurate – MAPEs within 1.6, 1.7, and 2.6 percent for the normal, shifted gamma, and shifted exponential travel-time distributions, respectively – the errors are consistently higher. While it is not apparent as to why the esti-

mates are worse in this situation, it is interesting to note that Correlation Matrices 1 and 2 provide expected makespans that are smaller than the uncorrelated case ($\rho_{ik} = 0$), while Correlation Matrix 3 results in larger makespans (as shown in Table 4). Correlation Matrix 1 has fairly high, positively-correlated route times, which decreases the expected makespan (see Fig. 2). Correlation Matrix 2 has positively- and negatively-correlated arcs within a route, resulting in a lower variance, and also results in route times that have low correlations. Correlation Matrix 3, on the other hand, results in relatively highly-correlated route times, some of which are negative; therefore, increasing the expected makespan.

Table 5
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Standard Deviation – Normal Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	9.63	2.59	4.00	7.25	4.29
	0.6	14.11	3.63	6.48	10.22	5.78
	0.4	17.95	4.88	8.92	13.15	7.08
	0.2	21.95	7.87	12.34	15.92	8.51
	0.0	26.70	12.86	18.25	22.02	14.32
Correlation Matrix 2	0.2	26.95	13.89	19.61	23.13	16.94
	0.4	28.96	19.41	23.40	27.65	25.06
	0.6	32.74	29.65	31.81	35.25	37.00
	0.8	40.46	46.60	45.14	45.62	51.50
Correlation Matrix 3	0.2	30.35	16.34	25.77	30.42	21.47
	0.4	30.50	18.45	28.49	33.03	25.10
	0.6	32.45	22.02	32.85	37.08	30.11
	0.8	34.28	25.42	36.45	40.90	34.35

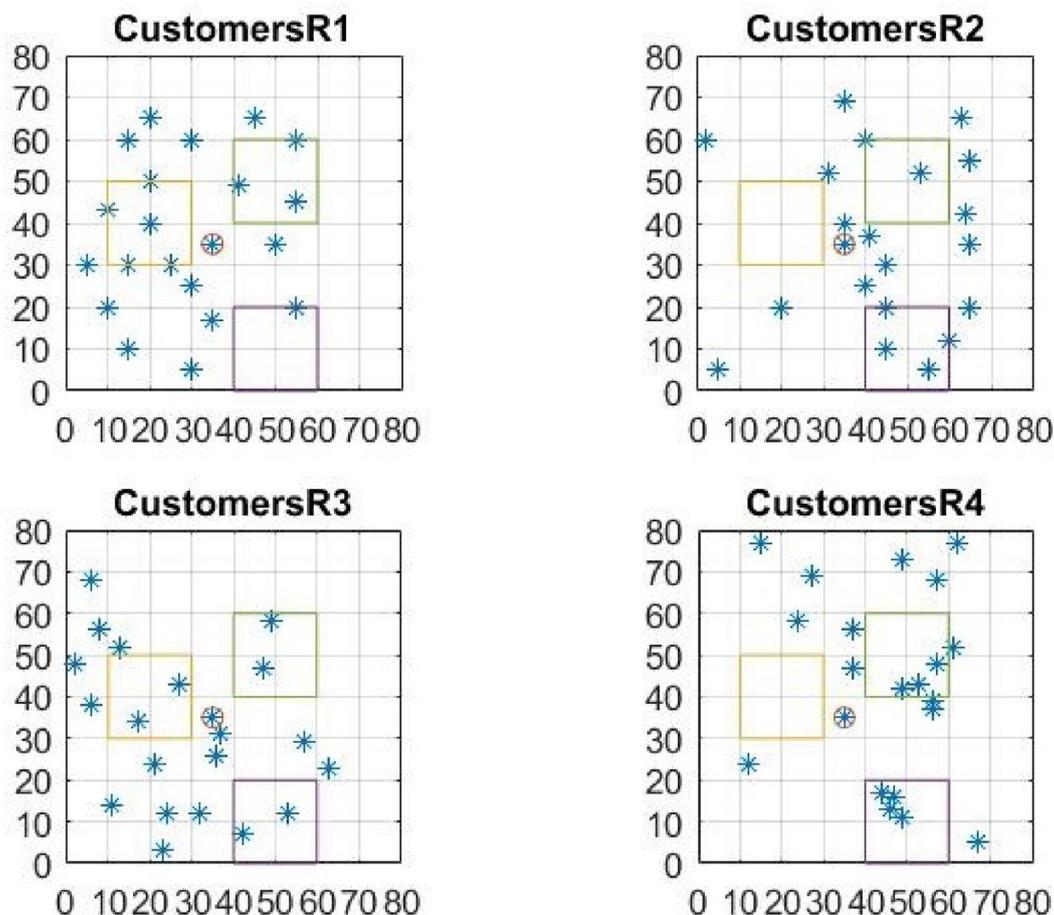


Fig. 2. Setup for INC correlation pattern.

Table 6
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Standard Deviation – Shifted Gamma Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	5.83	1.91	2.86	4.59	2.63
	0.6	8.90	2.97	4.45	6.50	3.50
	0.4	11.13	4.07	6.17	8.41	4.58
	0.2	13.47	5.24	7.83	10.71	5.29
	0.0	16.58	7.50	11.01	14.39	8.87
Correlation Matrix 2	0.2	16.56	7.76	11.08	13.84	9.43
	0.4	16.66	9.19	12.60	15.66	12.27
	0.6	18.34	13.94	15.81	19.08	18.76
	0.8	23.58	25.72	25.26	26.88	30.59
	0.2	18.72	9.51	15.54	20.35	13.36
Correlation Matrix 3	0.4	20.58	11.32	19.65	25.04	17.31
	0.6	22.23	13.67	23.53	28.85	21.09
	0.8	23.69	15.63	26.60	31.72	24.12

Table 7
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Standard Deviation – Shifted Exponential Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	9.63	2.59	4.00	7.25	4.29
	0.6	14.11	3.63	6.48	10.22	5.78
	0.4	17.95	4.88	8.92	13.15	7.08
	0.2	21.95	7.87	12.34	15.92	8.51
	0.0	26.70	12.86	18.25	22.02	14.32
Correlation Matrix 2	0.2	26.95	13.89	19.61	23.13	16.94
	0.4	28.96	19.41	23.40	27.65	25.06
	0.6	32.74	29.65	31.81	35.25	37.00
	0.8	40.46	46.60	45.14	45.62	51.50
	0.2	30.35	16.34	25.77	30.42	21.47
Correlation Matrix 3	0.4	30.50	18.45	28.49	33.03	25.10
	0.6	32.45	22.02	32.85	37.08	30.11
	0.8	34.28	25.42	36.45	40.90	34.35

Table 8
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Expected Makespan plus Standard Deviation – Normal Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	0.0241	0.0224	0.0355	0.0307	0.0670
	0.6	0.0810	0.0419	0.0343	0.0901	0.0526
	0.4	0.1239	0.0482	0.0509	0.1329	0.0621
	0.2	0.2044	0.0552	0.0663	0.1894	0.0757
	0.0	0.2470	0.0972	0.1546	0.2472	0.0911
Correlation Matrix 2	0.2	0.1956	0.0850	0.1409	0.2387	0.0686
	0.4	0.1820	0.0672	0.0672	0.1842	0.0929
	0.6	0.2093	0.0337	0.0629	0.1095	0.0798
	0.8	0.1929	0.0117	0.0435	0.0477	0.0408
	0.2	0.2869	0.1847	0.3799	0.4551	0.3220
Correlation Matrix 3	0.4	0.4031	0.2613	0.4129	0.4780	0.3899
	0.6	0.4827	0.3575	0.4257	0.4804	0.5633
	0.8	0.6166	0.5201	0.3731	0.5105	0.8310

Again, this highlights the need for an appropriate routing algorithm that can take advantage of correlated travel times.

While the estimated expected makespan is quite accurate in general, the same cannot be said of the standard deviation (Tables 5–7). The estimate of standard deviation of the makespan is only relatively accurate

Table 9
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Expected Makespan plus Standard Deviation – Shifted Gamma Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	0.9252	0.2195	0.3312	0.5669	0.3434
	0.6	1.3587	0.3429	0.5441	0.8493	0.4537
	0.4	1.6199	0.4600	0.7479	1.0353	0.5277
	0.2	1.8680	0.5424	0.8609	1.2410	0.5206
	0.0	2.0779	0.6489	1.0257	1.3847	0.5975
Correlation Matrix 2	0.2	2.0088	0.6616	1.0119	1.2273	0.6116
	0.4	1.8753	0.7114	1.0696	1.3000	0.7757
	0.6	1.8653	0.9726	1.2008	1.4506	1.1342
	0.8	2.0964	1.5811	1.7455	1.8173	1.7384
	0.2	2.3525	0.9742	1.5080	1.7772	0.9149
Correlation Matrix 3	0.4	2.4059	1.2194	1.9053	2.0491	1.1265
	0.6	2.5010	1.4827	2.2599	2.2236	1.3023
	0.8	2.4273	1.6897	2.4830	2.3279	1.4936

Table 10
Mean Absolute Percent Deviation of the Recursive Approximation vs. Simulated Expected Makespan plus Standard Deviation – Shifted Exponential Distribution.

	ρ_{ik}	5 routes/2 arcs ea.	3 routes/4 arcs ea.	5 routes/4 arcs ea.	10 routes/4 arcs ea.	5 routes/8 arcs ea.
Correlation Matrix 1	0.8	1.5588	0.3085	0.5311	0.9059	0.5998
	0.6	2.2931	0.4176	0.8448	1.3469	0.7432
	0.4	2.7512	0.5392	1.1494	1.7126	0.8267
	0.2	3.1825	0.8281	1.4687	1.9562	0.8605
	0.0	3.6130	1.1555	1.8766	2.3290	1.0586
Correlation Matrix 2	0.2	3.5489	1.2247	1.9359	2.3337	1.2421
	0.4	3.6662	1.6979	2.2456	2.6489	1.8506
	0.6	3.8829	2.5114	2.9628	3.2389	2.7830
	0.8	4.4062	3.8882	4.0853	4.0865	4.0980
	0.2	3.9053	1.4024	2.6043	2.6593	1.5607
Correlation Matrix 3	0.4	3.8939	1.7617	2.8401	3.0395	1.7323
	0.6	3.9030	2.1995	3.2908	3.2957	2.1317
	0.8	3.8615	2.6201	3.6430	3.5390	2.5300

in the case of the Correlation Matrix 1. For other correlation matrices, across all considered distributions, the estimate becomes less accurate. However, since the standard deviation is added to the expected makespan, the objective will still reflect a measure of reliability in the makespan which is the goal of this second objective.

For the objective that combines the mean as well as the variability, we evaluate the accuracy of the expected value plus one standard deviation of the makespan. As seen in Table 8, the accuracy of estimates for normally-distributed travel times is within one percent of the simulated values for all instances. In fact, the values for some instances are better than the deviation of the expected value alone, as an underestimated standard deviation somewhat offsets an overestimated mean. Tables 9 and 10 provide the accuracy for the shifted gamma and shifted exponential distributions. In these situations, the estimate of the makespan variance is considerably worse than that realized with the normal distribution. Thus, the mean absolute percent deviation of the expected value plus one standard deviation of the makespan is as high as 2.5% for the shifted gamma and as high as 4.5% for the shifted exponential.

5. Embedding estimates within a vehicle routing heuristic

In Section 3, we presented an algorithm that estimates the distribution of makespan based on extreme-value theory. In the following, we embed our algorithm within a vehicle-routing heuristic to find the assignment of customers to routes that minimize the makespan (with and without the addition of standard deviation). To this end, we use a well-

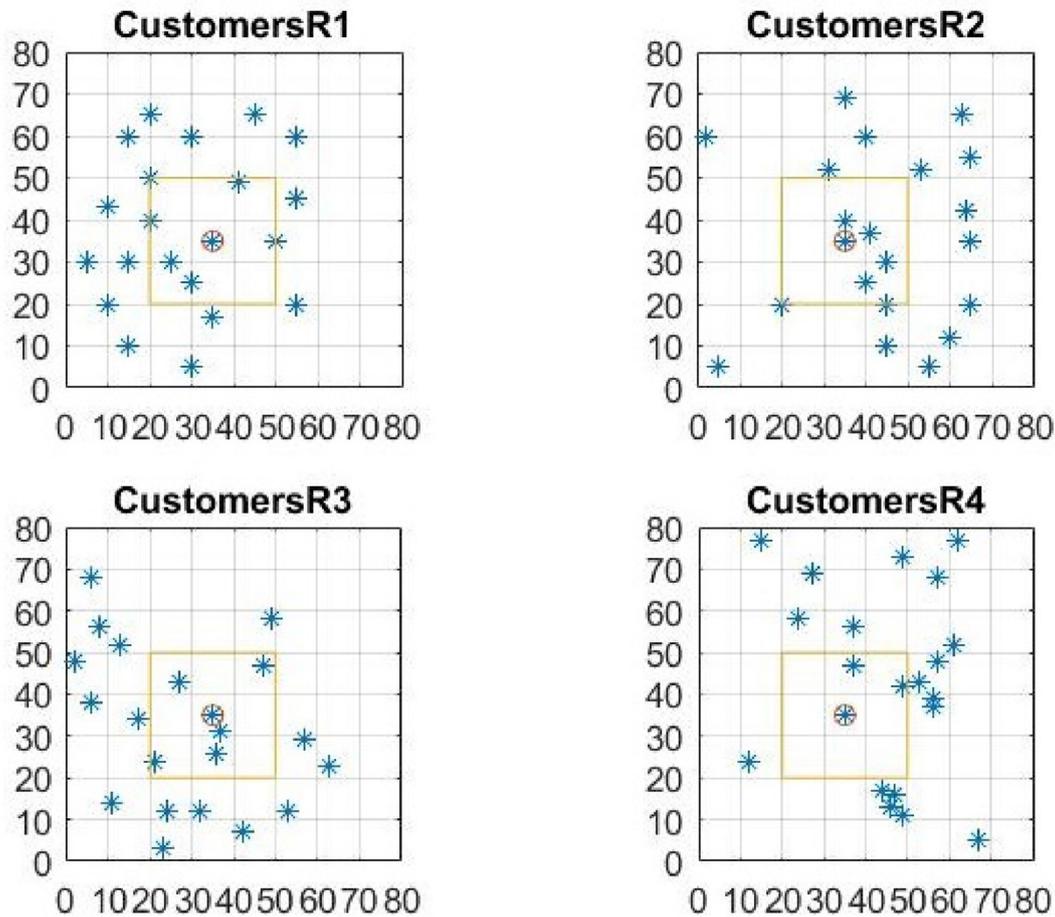


Fig. 3. Setup for I/O correlation pattern.

known construction heuristic (GRASP, see Resende and Ribeiro (2003) for details) and two adapted variants of standard local search neighborhoods for improvement (1-shift operator and Exchange operator, see, for instance, Aarts et al. (2003) for details). Our estimates could similarly be embedded within other, more complex heuristics. The algorithm is as follows.

Step 1. Create Input Data. We create the correlation matrix and the covariance matrix for a given set of depot, customers and their locations to prepare the data input required for the algorithm presented in Section 3.2.

Step 2. Initialize Routes. For the creation of routes, we assume that the number of routes is given, and we aim at creating routes for our makespan objective (with or without standard deviation) for the given number of routes. We initialize the routes by assigning the farthest, second-farthest, etc. customers to the routes until every route has one initial customer. Since we start with empty routes that contain only a depot, the farthest customer is the one with the highest mean travel time to the depot, the second-farthest customer has the second-highest mean travel time, and so on.

Step 3. Insert Customers. We insert the remaining customers following the idea of GRASP. In each insertion step, we evaluate the consequences of insertion for our complex makespan objective for all remaining customers, routes and insertion positions in these routes. Out of the three “best” customers and insertion points, we randomly choose one for insertion. We continue until all customers have been assigned. We need to evaluate the current solution with the Algorithm 1 of Section 3.2 whenever we check if an insertion to a particular insertion point would

be beneficial or not. In the end, all customers have been inserted, and a first feasible solution has been created. Due to the random nature of our GRASP approach, we repeat the construction of the initial route plan five times and improve each of the five created route plans as follows.

Step 4. Improve Routes (1-shift). The created initial solution is subject to improvement with standard local search neighborhood operators. First, we apply the idea of 1-shift, which investigates whether it is beneficial to shift a customer from its current position to all other possible insertion points within the same or the other routes. We identify the most beneficial shift with regard to our objective, keep the improved solution, and continue with 1-shift operations as long as we can find improvements.

Step 5. Improve Routes (Exchange). We use the well-known exchange operator to systematically switch two customers between different routes. We compute the consequences that each switch would have on our makespan objective (with or without standard deviation) and perform the first exchange that leads to an improvement. If we found an improvement, we start the exchange operator again. We continue the exchange operator until no improvement of the objective function is possible any more. Then, we activate 1-shift again (Step 4), and both operators run alternately until we cannot find improved solutions any more.

Step 6. Solution selection. Once all five runs are completed, we select a route plan that yield smallest objective function value as a best one.

Since insertion of a customer or the application of a local search move can impact the expected travel times of the individual routes and, consequently, the makespan objective of the full solution, we need to

Table 11
Summary results for minimizing makespan (normal distribution, 20 customers).

#veh	objective	base	I/O 50	I/O 75	I/O 90	INC 50	INC 75	INC 90
stdv = 0.25 * mean								
3	best sol makespan	125.49	124.66	124.19	123.87	126.49	126.84	127.33
	zero corr sol makespan	125.49	124.78	124.42	124.19	126.58	127.06	127.33
	% different to best sol	0.00%	-0.11%	-0.19%	-0.28%	-0.09%	-0.19%	0.00%
5	best sol makespan	100.52	98.81	97.77	97.05	101.67	102.02	102.20
	zero corr sol makespan	100.52	98.98	98.16	97.65	102.01	102.68	103.06
	% different to best sol	0.00%	-0.15%	-0.37%	-0.59%	-0.34%	-0.65%	-0.85%
stdv = 0.5 * mean								
3	best sol makespan	134.35	132.32	131.12	130.34	136.41	137.06	137.44
	zero corr sol makespan	134.35	132.70	131.82	131.26	136.82	137.89	138.49
	% different to best sol	0.00%	-0.30%	-0.55%	-0.73%	-0.31%	-0.62%	-0.78%
5	best sol makespan	110.60	106.99	104.78	102.95	112.82	113.53	113.92
	zero corr sol makespan	110.60	107.23	105.34	104.12	113.65	115.00	115.77
	% different to best sol	0.00%	-0.21%	-0.51%	-1.07%	-0.74%	-1.28%	-1.60%
stdv = 0.75 * mean								
3	best sol makespan	143.62	139.94	138.17	136.30	146.27	147.25	147.68
	zero corr sol makespan	143.62	141.02	139.61	138.72	150.14	152.90	154.45
	% different to best sol	0.00%	-0.73%	-1.01%	-1.69%	-2.44%	-3.53%	-4.20%
5	best sol makespan	121.03	115.39	111.94	109.32	124.73	126.00	126.22
	zero corr sol makespan	121.03	115.64	112.57	110.58	126.54	128.93	130.29
	% different to best sol	0.00%	-0.22%	-0.56%	-1.12%	-1.45%	-2.32%	-3.13%

evaluate the full solution by recomputing the correlation matrix whenever we perform an insertion or move. Compared to deterministic routing, where it is sufficient to analyze the values of the affected arcs instead of a full solution evaluation, this leads to a significant increase in run times.

6. Experimental design and results

In this section, we introduce the instances and present the computational results of experiments with our stochastic vehicle routing algorithm considering stochastic and correlated travel times for different makespan objectives.

6.1. Instances

We analyze the impact of correlation in optimizing routes based on well-known benchmark instances with correlation patterns motivated by the literature on correlation. In particular, we use the dataset R101 of the Solomon datasets. We use the first customer as depot location. We form four sets of instances based on the next 20 customers, i.e., depot location and customers 2, 3, ..., 21 form the instance R101-1 (short: R1), 22, 23, ..., 41 form R101-2 (short: R2), etc. We mainly focused on fairly small instances so that we could understand and visualize the impact of the correlation on vehicle routing. In addition, we created four 40-customer datasets based on 20 customer datasets by considering different combinations of 20 customer datasets. Namely, we used the following combinations: R1+R3, R1+R4, R2+R3 and R2+R4. To all instances, we apply the following correlation patterns:

- Correlation pattern type *incidents (INC)*: Guo (2006) suggests arc travel times may be correlated due to weather effects and secondary incidents. Zockaie et al. (2016) consider different incidents to produce travel time variations in each correlation pattern. In addition, our motivation comes from the real life observation that traffic incidents may often create local traffic jams that are not observed on a regular day-to-day basis. Hence, for the incident correlation pattern, we define three locations of “incidents” randomly in the Euclidean plane. We assume that there is positive correlation between all arcs within 5 unit blocks of the incident locations. If arcs start or finish in the same block, they are positively correlated. This is illustrated in Fig. 2.

- Correlation pattern type *inner/outer (I/O)*: Nicholson (2015) suggests that negative correlation can occur when a bottleneck in one location results in increased speeds on the downstream locations. This motivates us to consider an inner/outer correlation pattern: if during a given time period a large amount of traffic is going into the same direction, the reverse direction is more likely to be less loaded. Hence, for our correlation pattern, we designate a block in the middle of the investigated area (coordinates 20–50, 20–50) to be the “downtown”. All arcs into the downtown block are positively correlated and all arcs out of the downtown block are positively correlated. The two groups are negatively correlated (i.e. if they are going into different directions). This is illustrated in Fig. 3.

We conduct experiments for the above instances and correlation patterns for four different correlation coefficients, namely zero correlation, 0.5, 0.75 and 0.9 (for negative correlation the negative of this). We also consider different coefficient of variation levels of travel times: 25% of mean, 50% of mean, 75% of mean. We consider three different distributions of travel times: normal distribution, shifted gamma distribution and shifted exponential distribution. The number of vehicles is fixed and is either 3 or 5 (just 5 for 40 customer datasets). Finally, we investigate two different parameterizations of our objective function (makespan with/without standard deviation). For each result, we report the objective function value and the individual components, i.e. the makespan and standard deviation, and compare it to the zero-correlation solutions.

We implemented our algorithms in Java 8 on a laptop with Win10 OS, Intel i7-7700HQ processor and 8 GB of RAM. The computation time for a 20-customer, three-vehicle makespan objective function instance takes a minimum of 2.6 s, maximum of 10.9 s, and an average of 5.5 s. Five-vehicle instances take a minimum of 3.6 s, maximum of 10.6 s, and an average of 6.3 s. For the 40 customer instances, the minimum runtime is 214.9 s, the maximum is 425.8 s, and an average is 308.7 s. As we discuss in Section 7, there are many opportunities to reduce this runtime as part of our future work.

6.2. Results

We first present summary tables and then discuss illustrative examples of routes created with our stochastic routing algorithm.

Table 12
Summary results for minimizing makespan plus standard deviation (normal distribution, 20 customers).

#veh	objective	base	I/O 50	I/O 75	I/O 90	INC 50	INC 75	INC 90
stdv = 0.25 * mean								
3	best sol makespan	127.67	126.44	125.89	125.49	126.49	126.84	127.04
	best sol stdv	9.08	8.15	7.49	7.06	11.65	12.38	12.79
	best sol ms + stdv	136.75	134.59	133.38	132.55	138.14	139.22	139.83
	zero corr sol makespan	127.67	126.91	126.50	126.24	128.78	129.24	129.49
	zero corr sol stdv	9.08	8.01	7.81	7.52	12.17	13.42	14.12
	zero corr sol ms + stdv	136.75	134.93	134.32	133.77	140.95	142.66	143.60
	% different to best sol	0.00%	-0.24%	-0.71%	-0.94%	-1.98%	-2.35%	-2.54%
5	best sol makespan	102.39	100.81	99.93	99.33	103.62	103.92	104.11
	best sol stdv	8.06	6.88	6.15	5.37	8.91	9.31	9.53
	best sol ms + stdv	110.45	107.68	106.08	104.70	112.53	113.23	113.63
	zero corr sol makespan	102.39	100.87	100.12	99.68	103.93	104.62	105.02
	zero corr sol stdv	8.06	6.87	6.19	5.72	9.32	9.93	10.29
	zero corr sol ms + stdv	110.45	107.74	106.31	105.40	113.25	114.56	115.31
	% different to best sol	0.00%	-0.05%	-0.20%	-0.64%	-0.65%	-1.18%	-1.47%
stdv = 0.5 * mean								
3	best sol makespan	134.55	132.66	131.28	131.63	136.57	137.23	137.62
	best sol stdv	17.57	15.50	14.41	13.54	21.33	22.83	23.70
	best sol ms + stdv	152.12	148.16	145.69	145.18	157.90	160.06	161.32
	zero corr sol makespan	134.55	132.89	132.01	131.46	138.19	139.70	140.55
	zero corr sol stdv	17.57	16.10	15.31	14.80	23.49	25.93	27.28
	zero corr sol ms + stdv	152.12	149.00	147.32	146.27	161.67	165.63	167.83
	% different to best sol	0.00%	-0.58%	-1.12%	-0.68%	-2.20%	-3.17%	-3.66%
5	best sol makespan	110.70	106.92	105.02	103.20	112.84	113.67	114.01
	best sol stdv	14.62	12.17	10.57	9.28	16.54	17.22	17.76
	best sol ms + stdv	125.32	119.10	115.59	112.48	129.38	130.89	131.77
	zero corr sol makespan	110.70	107.29	105.38	104.15	113.97	115.44	116.28
	zero corr sol stdv	14.62	12.42	11.20	10.44	17.45	18.88	19.72
	zero corr sol ms + stdv	125.32	119.71	116.58	114.59	131.42	134.32	136.00
	% different to best sol	0.00%	-0.52%	-0.81%	-1.79%	-1.54%	-2.50%	-3.03%
stdv = 0.75 * mean								
3	best sol makespan	143.29	140.26	138.31	137.51	146.28	147.54	149.02
	best sol stdv	25.59	22.67	20.90	18.98	31.38	33.34	35.36
	best sol ms + stdv	168.88	162.93	159.21	156.49	177.66	180.88	184.38
	zero corr sol makespan	143.29	140.34	138.72	137.69	149.21	151.70	153.09
	zero corr sol stdv	25.59	22.96	21.52	20.59	34.37	37.98	39.99
	zero corr sol ms + stdv	168.88	163.29	160.24	158.29	183.58	189.68	193.08
	% different to best sol	0.00%	-0.20%	-0.59%	-1.03%	-3.04%	-4.38%	-4.12%
5	best sol makespan	121.19	115.47	112.33	110.28	124.68	125.70	126.27
	best sol stdv	21.43	18.07	15.36	12.55	24.44	25.53	26.18
	best sol ms + stdv	142.61	133.54	127.69	122.83	149.11	151.23	152.45
	zero corr sol makespan	121.19	116.03	113.11	111.20	126.31	128.61	129.92
	zero corr sol stdv	21.43	18.12	16.26	15.09	25.92	28.13	29.41
	zero corr sol ms + stdv	142.61	134.15	129.37	126.29	152.23	156.74	159.33
	% different to best sol	0.00%	-0.49%	-1.32%	-2.75%	-2.01%	-3.40%	-4.15%

Table 13
Summary results for minimizing makespan (normal distribution, 40 customers).

#veh	objective	base	I/O 50	I/O 75	I/O 90	INC 50	INC 75	INC 90
stdv = 0.25 * mean								
5	best solution	117.55	116.12	116.29	115.04	118.69	119.44	119.26
	zero correlation	117.55	116.79	116.40	116.16	119.85	120.95	121.55
	% different to best sol	0.00%	-0.56%	-0.11%	-0.94%	-0.93%	-1.23%	-1.88%
stdv = 0.5 * mean								
5	best solution	127.05	124.54	124.30	122.23	131.34	131.94	132.93
	zero correlation	127.05	125.54	124.75	124.26	133.15	135.73	137.17
	% different to best sol	0.00%	-0.83%	-0.35%	-1.67%	-1.46%	-2.90%	-3.18%
stdv = 0.75 * mean								
5	best solution	136.76	133.71	131.51	131.01	143.14	144.01	145.70
	zero correlation	136.76	134.39	133.14	132.37	144.53	148.31	149.86
	% different to best sol	0.00%	-0.50%	-1.21%	-1.01%	-0.98%	-3.01%	-2.84%

6.2.1. Summary tables

Summary results are given in Tables 11–14 for the experiments involving normal distributions. Detailed results for these are in the appendix in Tables 22–31. Note that the relative patterns in the results are quite stable across the datasets, i.e. including correlation information consistently improves the quality of the results, indicating the heuristic is performing well.

For Table 11, we will first define the columns: #veh refers to the number of vehicles used, objective refers to the objective of the routing problem solved, base refers to the objective function for the no correlation case, I/O 50, I/O 75, I/O 90 refer to the corresponding inner-outer correlation matrix with 0.5/-0.5, 0.75/-0.75, 0.9/-0.9 correlation values, respectively, and INC 50, INC 75, INC 90 refer to the corresponding incidence correlation matrix with 0.5/-0.5, 0.75/-0.75, 0.9/-0.9

Table 14
Summary results for minimizing makespan plus standard deviation (normal distribution, 40 customers).

#veh	objective	base	I/O 50	I/O 75	I/O 90	INC 50	INC 75	INC 90
stdv = 0.25 mean								
5	best sol makespan	116.33	115.70	115.20	115.07	118.21	119.51	119.95
	best sol stdv	6.96	6.15	6.08	5.92	8.58	8.81	9.36
	best sol ms + stdv	123.30	121.84	121.28	120.99	126.78	128.32	129.32
	0 corr sol makespan	116.33	115.83	115.57	115.42	119.76	121.16	121.93
	0 corr sol stdv	6.96	6.50	6.24	6.08	9.77	10.90	11.52
	0 corr sol ms + stdv	123.30	122.33	121.81	121.49	129.53	130.04	133.45
	% different to best sol	0.00%	-0.40%	-0.44%	-0.41%	-2.17%	-1.34%	-3.20%
	stdv = 0.5 mean							
5	best sol makespan	126.83	125.03	124.34	123.70	130.88	132.56	133.52
	best sol stdv	12.85	11.74	10.74	9.63	16.27	17.54	18.54
	best sol ms + stdv	139.68	136.78	135.08	133.34	147.16	150.10	152.06
	0 corr sol makespan	126.83	125.20	124.34	123.81	132.82	135.43	136.90
	0 corr sol stdv	12.85	11.99	11.54	11.26	17.91	20.13	21.36
	0 corr sol ms + stdv	139.68	137.19	135.88	135.07	150.73	155.56	158.26
	% different to best sol	0.00%	-0.30%	-0.59%	-1.30%	-2.43%	-3.64%	-4.08%
	stdv = 0.75 mean							
5	best sol makespan	138.20	133.77	133.04	132.51	144.31	144.78	147.55
	best sol stdv	18.88	17.17	15.56	13.56	22.93	25.76	27.05
	best sol ms + stdv	157.08	150.94	148.60	146.07	167.25	170.53	174.60
	0 corr sol makespan	138.20	136.07	134.84	134.37	148.45	152.61	154.89
	0 corr sol stdv	18.88	17.39	17.00	17.00	27.61	31.13	33.07
	0 corr sol ms + stdv	157.08	153.75	152.22	151.38	176.07	183.74	187.96
	% different to best sol	0.00%	-1.87%	-2.44%	-3.63%	-5.27%	-7.75%	-7.65%

correlation values, respectively. In terms of the rows, **stdv = 0.* mean** is where * is the ratio of the standard deviation of travel time to the mean for each arc (i.e. coefficient of variation), **best sol makespan** is the expected makespan of the best solution obtained for the given correlation matrix, **zero corr sol makespan** is the expected makespan of the zero correlation solution evaluated using the given correlation matrix, and **% different to best solution** is the percentage difference between the best solution for expected makespan and the zero correlation solution evaluated with the given correlation matrix (i.e. the savings in expected makespan from including correlation in our routing algorithm).

From Table 11 that only considers expected makespan, a few patterns are apparent. Increasing the correlation across each correlation pattern leads to increased importance in modeling the correlation in the routing. For example, for 5 vehicles and stdv = 0.25 mean, we see savings of 0.34%, 0.65%, and 0.85% for INC 50, INC 75, and INC 90. This is not surprising since more correlation should yield larger impact on solutions. We also see that a larger coefficient of variation yields larger impacts from modeling correlation. For example, for I/O 50 and 3 vehicles, we see the savings go from 0.11% to 0.30% to 0.73% as the standard deviation increases. We will look at an example with a coefficient of variation of 0.75 in the next section to understand the changes in the routes creating the savings. In general, there is no strong relationship between the number of vehicles and the savings from modeling correlation. We see that with 20 customers most experiments yield an average savings of around 1% or less except with the highest coefficient of variation and the INC correlation patterns. Here, we see values ranging as high as 4.20% average improvement from modeling correlation.

Next, we will consider Table 12 that optimizes for expected makespan plus standard deviation. The columns are the same as for Table 11, but there are additional rows. These include **best sol stdv** which is the standard deviation of the best solution obtained with the given correlation matrix, **best sol ms + stdv** the sum of the makespan and standard deviation of the best solution obtained for the given correlation matrix, **zero corr sol stdv** the standard deviation of the zero correlation solution evaluated using the given correlation matrix, **zero corr sol ms + stdv** makespan plus standard deviation of the zero correlation solution evaluated using the given correlation matrix, and **% difference to best solution** now represents percentage difference between the zero correlation solution evaluated with the given correlation matrix and the best solution for expected makespan and standard deviation.

Here, we observe generally larger average savings from considering

Table 15
Summary results for minimizing makespan (shifted gamma distribution, 20 customers).

# veh	objective	base	INC 50	INC 75	INC 90
stdv = 0.25 * mean					
3	best solution	120.53	121.50	122.06	122.32
	zero correlation	120.53	121.57	122.06	122.32
	% different to best sol	0.00%	-0.06%	0.00%	0.00%
5	best solution	100.85	102.00	102.33	102.52
	zero correlation	100.85	102.27	102.85	103.29
	% different to best sol	0.00%	-0.27%	-0.51%	-0.75%
stdv = 0.5 * mean					
3	best solution	132.85	134.79	135.45	135.84
	zero correlation	132.85	135.62	136.92	138.00
	% different to best sol	0.00%	-0.62%	-1.07%	-1.56%
5	best solution	111.85	114.30	115.00	115.53
	zero correlation	111.85	115.50	116.93	118.02
	% different to best sol	0.00%	-1.04%	-1.65%	-2.11%
stdv = 0.75 * mean					
3	best solution	143.45	146.58	147.58	148.01
	zero correlation	143.45	149.47	152.13	153.59
	% different to best sol	0.00%	-1.93%	-3.00%	-3.63%
5	best solution	122.58	124.17	125.24	125.89
	zero correlation	122.58	127.27	128.97	130.40
	% different to best sol	0.00%	-2.43%	-2.89%	-3.46%

correlation. This is particularly apparent for the INC correlation pattern as we create average savings as high as 2.54% even with a coefficient of variation of only 0.25. We note that the highest average savings across the table is 4.38%, though, which is only slightly higher than 4.20% in Table 11, but many more values in the tables have savings more than 1%. We again see savings increase when considering correlation, but not so clearly with increasing coefficient of variation. This seems to be a change created by the new objective. The dramatic increase in magnitude of savings when incorporating correlation with the INC correlation patterns is quite interesting and is one of the patterns we try to understand better in the next section.

Tables 13 and 14 are similar to Tables 11 and 12, but they represent results for 40-customers datasets to see the impact of instance size. We did these experiments only with the 5 vehicle instances. Not surprisingly, similar patterns can be observed. For instance, increasing the correlation

Table 16
Summary results for minimizing makespan plus standard deviation (shifted gamma distribution, 20 customers).

#veh	objective	base	INC 50	INC 75	INC 90
stdv = 0.25 * mean					
3	best sol makespan	126.30	126.19	126.89	127.67
	best sol stdv	11.08	11.76	12.76	13.10
	best sol ms + stdv	137.37	137.95	139.65	140.77
	0 corr sol makespan	126.30	128.20	128.88	129.61
	0 corr sol stdv	11.08	11.76	14.04	15.37
	0 corr sol ms + stdv	137.37	139.96	142.92	144.99
	% different to best sol	0.00%	-1.46%	-2.34%	-3.00%
5	best sol makespan	102.39	103.22	104.40	105.39
	best sol stdv	8.78	10.54	10.64	11.19
	best sol ms + stdv	111.16	113.77	115.04	116.58
	0 corr sol makespan	102.39	103.14	104.36	105.11
	0 corr sol stdv	8.78	10.68	11.47	12.93
	0 corr sol ms + stdv	111.16	113.81	115.82	118.04
	% different to best sol	0.00%	-0.04%	-0.68%	-1.25%
stdv = 0.5 * mean					
3	best sol makespan	135.09	136.99	137.63	138.33
	best sol stdv	17.53	18.35	19.70	20.86
	best sol ms + stdv	152.62	155.34	157.33	159.19
	0 corr sol makespan	135.09	137.73	139.07	139.97
	0 corr sol stdv	17.53	21.76	23.46	25.52
	0 corr sol ms + stdv	152.62	159.50	162.54	165.48
	% different to best sol	0.00%	-2.68%	-3.31%	-3.96%
5	best sol makespan	108.71	110.54	111.70	113.62
	best sol stdv	16.74	17.94	19.14	19.49
	best sol ms + stdv	125.45	128.48	130.84	133.12
	0 corr sol makespan	108.71	111.34	113.78	115.68
	0 corr sol stdv	16.74	19.24	20.28	21.79
	0 corr sol ms + stdv	125.45	130.58	134.06	137.47
	% different to best sol	0.00%	-1.63%	-2.46%	-3.27%
stdv = 0.75 * mean					
3	best sol makespan	143.59	146.14	147.96	149.73
	best sol stdv	27.39	32.72	34.63	35.56
	best sol ms + stdv	170.98	178.85	182.58	185.29
	0 corr sol makespan	143.59	148.42	151.05	152.62
	0 corr sol stdv	27.39	35.57	38.61	42.16
	0 corr sol ms + stdv	170.98	183.99	189.66	194.78
	% different to best sol	0.00%	-2.87%	-3.88%	-5.12%
5	best sol makespan	116.60	120.91	122.55	123.56
	best sol stdv	23.81	25.39	26.51	27.61
	best sol ms + stdv	140.42	146.30	149.06	151.17
	0 corr sol makespan	116.60	123.53	125.56	127.33
	0 corr sol stdv	23.81	26.96	29.61	31.48
	0 corr sol ms + stdv	140.42	150.49	155.17	158.81
	% different to best sol	0.00%	-2.86%	-4.10%	-5.05%

leads to increased importance of modeling the correlation in the routing (see, for example, Table 13 results for INC 50, INC 75 and INC 90). In addition, a combined objective function of makespan and standard deviation in many cases yields larger savings from considering correlation than a makespan objective function. One can observe this by comparing INC results in Tables 13 and 14 for the same standard deviation. Results for 40 customers exhibit some slightly larger changes than 20 customers for some experiments, but there does not seem to be a clear increase in value of incorporating correlation for larger customer instances.

In addition to the results for normally distributed travel times presented in Tables 11–14, we conducted similar experiments for shifted gamma (Tables 15 and 16) and shifted exponential distributions (Tables 17 and 18) for INC correlation pattern cases to see the impact of different distributions. Detailed results for these can be found in the appendix (Tables 32 – 37 for shifted gamma distribution and Tables 38 – 43 for shifted exponential distribution). For both of these distributions, the average percentage gain for the makespan only objective function remain relatively similar to the normal distribution case (Tables 15 and 17), as well as for the makespan plus standard deviation objective for the shifted gamma, except for the case when $stdv = 0.75 * mean$ (Table 16). However, for the shifted exponential distribution, the combined

Table 17
Summary results for minimizing makespan (shifted exponential distribution, 20 customers).

# veh	Objective	base	INC 50	INC 75	INC 90
stdv = 0.25 * mean					
3	best solution	134.15	134.97	135.59	136.14
	zero correlation	134.15	135.25	135.79	136.15
	% different to best sol	0.00%	-0.21%	-0.14%	0.00%
	stdv = 0.5 * mean				
5	best solution	108.27	109.52	109.90	110.09
	zero correlation	108.27	109.70	110.55	110.95
	% different to best sol	0.00%	-0.16%	-0.59%	-0.78%
	stdv = 0.75 * mean				
3	best solution	141.71	143.88	144.56	144.95
	zero correlation	141.71	144.06	145.27	145.82
	% different to best sol	0.00%	-0.12%	-0.49%	-0.60%
	5	best solution	116.65	119.01	119.76
zero correlation		116.65	119.26	120.88	121.46
% different to best sol		0.00%	-0.22%	-0.92%	-1.07%
stdv = 0.75 * mean					
3	best solution	151.14	153.98	155.01	155.48
	zero correlation	151.14	156.97	159.79	162.43
	% different to best sol	0.00%	-1.91%	-2.99%	-4.28%
	5	best solution	128.09	131.98	133.34
zero correlation		128.09	133.43	136.66	138.01
% different to best sol		0.00%	-1.09%	-2.43%	-3.22%

makespan plus standard deviation objective function improvements are generally higher (Table 18) despite the lack of the accuracy (see Section 4). The lack of the accuracy for the zero correlation solution evaluated with the any other correlation matrix (INC 50, INC 75, INC 90) may have contributed to the objective function improvement.

6.2.2. Illustrative examples of routes

To understand how correlation is causing these changes in solution values, we next examine a series of illustrative examples. We will look at examples of how the different objectives and correlation patterns change the design of the routes. We will use instances with 3 vehicles, a coefficient of variation of 75% and normally distributed travel times.

We start with the makespan objective and see how an incident correlation pattern changes the resulting routes. For R1 with a maximum positive correlation of 75%, the solution found for the incident correlation instance has a 2.65% improvement over using the routes from the no correlation version. A comparison of the routes found without correlation and with incident correlation is in Fig. 4.

The two route plans are somewhat different: for instance, it can be observed that one of the customers at (42,48), who was assigned to route 1 with no correlation, has been reassigned to route 2 because of the fact that arcs coming to and leaving from that customer are positively correlated since they are starting and finishing in one of the incident locations.

For R2 with a maximum positive correlation of 90%, the solution found for the incident correlation instance has a 6.51% improvement over using the routes from the no correlation version. The routes found are in Fig. 5.

It is interesting to see that the solution with correlation here shows a crossing in the solution. The crossing is beneficial because the arcs are in the incident zone and become positively correlated as a result.

We can see the differences that inner/outer correlation can create for instance R3 (see Fig. 6). For R3 with a maximum positive correlation of 90%, the solution found for the incident correlation instance has a 2.92% improvement over using the routes from the no correlation version.

For the inner/outer case, the customer that is the closest one to the depot gets reassigned to the different route due to its location inside of the downtown box.

With standard deviation in the objective, we sometimes see similar changes as with only makespan, but at other times the routes change more. For R4, with a maximum positive correlation of 75%, the solution

Table 18
Summary results for minimizing makespan plus standard deviation. (Shifted exponential distribution, 20 customers).

#veh	Objective	base	INC 50	INC 75	INC 90
stdv = 0.25 * mean					
3	best sol makespan	131.64	130.41	130.78	130.98
	best sol stdv	9.37	12.02	12.76	13.18
	best sol ms + stdv	141.01	142.44	143.54	144.17
	0 corr sol makespan	131.64	133.05	133.51	134.02
	0 corr sol stdv	9.37	13.01	14.52	15.13
	0 corr sol ms + stdv	141.01	146.06	148.03	149.15
	% different to best sol	0.00%	-2.55%	-3.13%	-3.46%
5	best sol makespan	107.83	109.11	109.43	109.63
	best sol stdv	8.50	9.38	9.80	10.02
	best sol ms + stdv	116.34	118.50	119.23	119.65
	0 corr sol makespan	107.83	109.65	110.37	110.85
	0 corr sol stdv	8.50	10.11	11.04	11.96
	0 corr sol ms + stdv	116.34	119.76	121.41	122.81
	% different to best sol	0.00%	-1.07%	-1.83%	-2.64%
stdv = 0.5 * mean					
3	best sol makespan	138.74	140.80	141.48	141.88
	best sol stdv	18.13	20.81	21.99	23.05
	best sol ms + stdv	156.86	161.61	163.47	164.93
	0 corr sol makespan	138.74	142.50	144.06	144.94
	0 corr sol stdv	18.13	22.75	25.76	27.20
	0 corr sol ms + stdv	156.86	165.25	169.83	172.14
	% different to best sol	0.00%	-2.25%	-3.89%	-4.37%
5	best sol makespan	116.59	118.83	119.71	120.07
	best sol stdv	15.41	15.73	16.36	17.59
	best sol ms + stdv	132.00	134.56	136.06	137.65
	0 corr sol makespan	116.59	120.03	121.58	122.46
	0 corr sol stdv	15.41	17.39	19.10	20.96
	0 corr sol ms + stdv	132.00	137.42	140.68	143.42
	% different to best sol	0.00%	-2.13%	-3.39%	-4.19%
stdv = 0.75 * mean					
3	best sol makespan	147.74	150.81	152.10	152.45
	best sol stdv	26.40	31.57	33.61	35.35
	best sol ms + stdv	174.14	182.37	185.71	187.81
	0 corr sol makespan	147.74	153.87	156.44	157.88
	0 corr sol stdv	26.40	34.45	38.72	41.36
	0 corr sol ms + stdv	174.14	188.32	195.16	199.24
	% different to best sol	0.00%	-3.26%	-5.09%	-6.09%
5	best sol makespan	127.63	131.29	132.37	132.97
	best sol stdv	22.59	25.73	26.87	27.55
	best sol ms + stdv	150.22	157.02	159.25	160.52
	0 corr sol makespan	127.63	133.02	135.44	136.82
	0 corr sol stdv	22.59	27.36	30.65	33.46
	0 corr sol ms + stdv	150.22	160.38	166.09	170.28
	% different to best sol	0.00%	-2.14%	-4.30%	-6.08%

found for the incident correlation instance has a 5.73% improvement over using the routes from the no correlation version with just the makespan and 11.35% with makespan and standard deviation. The routes found with no correlation and the combined objective and the routes found with inner/outer correlation are in Fig. 7.

For this set of routes, the effect of the incident locations is the most visible. One can notice that all the routes look different in the base case and with the incident correlation. If we look at the top right incident location, we can see that route 1 changed such that the many customers present in route 1 in the base case are reassigned to a different route. We can also notice that those relocated customers were not in the incident location. This fact would suggest that correlation is important in this case and is the most influential. Here the moved customers are “close” to the customers in route 1 in the base case (which we typically want in a routing solution), but the impact of correlation is strong enough to put them on route 3 in the makespan plus standard deviation solution.

7. Summary and Future Work

Using ideas from extreme-value theory, we developed a method to incorporate travel time correlation in computing the distribution of the makespan for a vehicle routing problem with stochastic travel times. We verified the quality of our approach and integrated it into a heuristic for the creation of routes for a fleet of vehicles. Using real world motivated correlation patterns, we explored the effect of the different correlation levels on the value of the objective function along with the set of resulting routes. We observed that correlation can play a significant role even for small instances with 20 customers and fleets of 3/5 vehicles. Taking correlation into account can result in an 13.76% decrease in the objective function value for these relatively small cases. These savings result from changes in the routes to take advantage of the impact of correlation that are counter-intuitive to typical route structures.

Future research directions include algorithmic and implementation improvements to decrease the computation time for larger datasets. Currently, we use routing algorithms “as is” and apply our full set of formulas to evaluate the new objection function after every change in the routes. This is required since a small change in one of the routes affects the correlation coefficients of pairs of routes and hence the expected makespan computation. Hence, in future work, following runtime reduction techniques as presented in Campbell and Thomas (2009), our idea is to develop smart ways to store portions of calculations from the current best solution, so the full solution would not need to be recomputed. We anticipate such changes could make it possible to solve real-world sized datasets. In our future work, we would also like to consider time-dependent correlation patterns to reflect periods of high traffic.

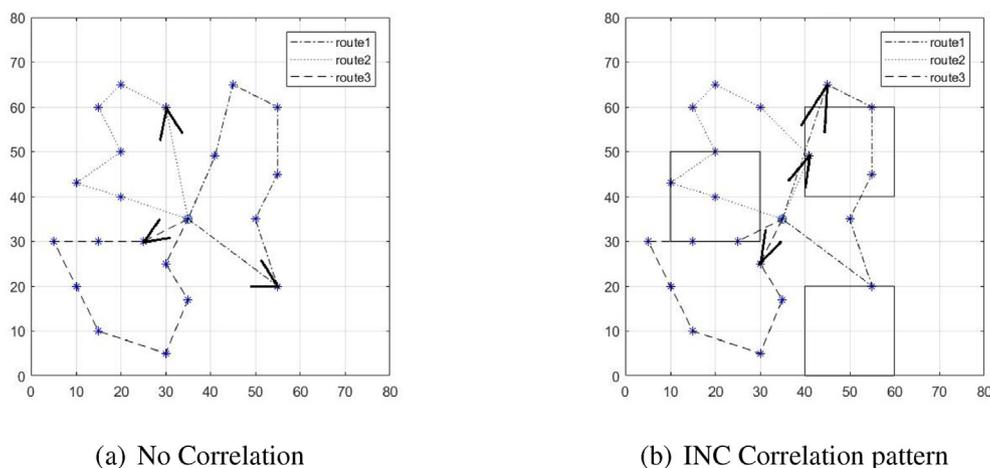


Fig. 4. Routes for R1, makespan objective.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Appendix A. Supplementary data

Supplementary data to this article can be found online at <https://doi.org/10.1016/j.ejtl.2021.100029>.

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