ECONSTOR
Make Your Publications Visible.

A Service of

ZBW Leibniz-Informationszentrum Wirtschaft
Leibniz Information Centre for Economics

Caporale, Guglielmo Maria; Gil-Alana, Luis Alberiko

Working Paper

A Fractional Integration Model with Autoregressive Processes

CESifo Working Paper, No. 11984

Provided in Cooperation with:
Ifo Institute – Leibniz Institute for Economic Research at the University of Munich

Suggested Citation: Caporale, Guglielmo Maria; Gil-Alana, Luis Alberiko (2025) : A Fractional Integration Model with Autoregressive Processes, CESifo Working Paper, No. 11984, CESifo GmbH, Munich

This Version is available at:
https://hdl.handle.net/10419/324975

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

WWW.ECONSTOR.EU

Mitglied der Leibniz-Gemeinschaft

CESifo WORKING PAPERS

11984
2025

July 2025

A Fractional Integration Model with Autoregressive Processes

Guglielmo Maria Caporale, Luis Alberiko Gil-Alana

CESifo

Impressum:

CESifo Working Papers
ISSN 2364-1428 (electronic version)
Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH
The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute
Poschingerstr. 5, 81679 Munich, Germany
Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email office@cesifo.de
Editor: Clemens Fuest
https://www.ifo.de/en/cesifo/publications/cesifo-working-papers
An electronic version of the paper may be downloaded
· from the SSRN website: www.SSRN.com
· from the RePEc website: www.RePEc.org
· from the CESifo website: https://www.ifo.de/en/cesifo/publications/cesifo-working-papers

A FRACTIONAL INTEGRATION MODEL WITH AUTOREGRESSIVE PROCESSES

Guglielmo Maria Caporale
Brunel University of London, London, UK

Luis A. Gil-Alana
University of Navarra, Faculty of Economics, Pamplona, Spain
and Universidad Francisco de Vitoria, Madrid, Spain

June 2025

Abstract

This note puts forward a new modelling approach that includes both fractional integration and autoregressive processes in a unified framework. The proposed model is very general and includes other more standard approaches such as the AR(F)IMA models. Some Monte Carlo evidence shows that the suggested framework outperforms standard AR(F)IMA specifications in capturing the properties of the series examined.

Keywords: Time series modelling; stationarity; fractional integration; autoregressions

JEL Classification: C22, C50

Corresponding author: Professor Guglielmo Maria Caporale, Department of Economics, Finance and Accounting, Brunel University of London, Uxbridge, UB8 3PH, UK. Email: Guglielmo-Maria.Caporale@brunel.ac.uk; https://orcid.org/0000-0002-0144-4135

Prof. Luis A. Gil-Alana gratefully acknowledges financial support from the project from ‘Ministerio de Ciencia, Innovación y Universidades`Agencia Estatal de Investigación' (AEI) Spain and `Fondo Europeo de Desarrollo Regional' (FEDER), Grant D2023-149516NB-I00 funded by MCIN/AEI/ 10.13039/501100011033. He also acknowledges support from an internal Project of the Universidad Francisco de Vitoria.

1. Introduction

This note proposes a new modelling framework combining fractional integration with autoregressive models to analyse the degree of persistence of time series. Specifically, in contrast to AutoRegressive Fractionally Integrated Moving Average (ARFIMA) models, the proposed model includes both fractional integration and AR processes in the same polynomial. This framework is then compared to the ARFIMA(1,d, 0) one by means of Monte Carlo simulations. The layout of this note is the following: Section 2 describes the model; Section 3 presents the Monte Carlo results; Section 4 concludes and suggests possible extensions of the analysis.

2. The Modelling Framework

Fractional integration is a generalization of the unit root models where the degree of differencing required to make a series stationary or I(0) can be any real value including fractional ones. More precisely, a process $\{x_t, t = 0, \pm 1, \ldots\}$ is said to be integrated of order d (denoted as I(d)) if it can be represented as:

$$(1 - L)^d x_t = u_t, \quad t = 0, \pm 1, \ldots, \tag{1}$$

where u_t is assumed to be I(0) and defined for the purposes of the present note as a covariance-stationary process with a spectral density function that is positive and finite at the zero frequency. Thus, u_t in (1) can be an AR(1) process of the form:

$$(1 - \rho L)u_t = \varepsilon_t, \quad t = 0, \pm 1, \ldots, \tag{2}$$

with ε_t being a white noise process. In this context, x_t can be said to follow an ARFIMA(1, d, 0) process.[1]

[1] The concept of fractional integration was originally introduced by Granger (1980, 1981) and Granger and Joyeux (1980) and it has been widely employed in the literature for modelling macroeconomic and financial time series data (Baillie, 1996; Gil-Alana and Robinson, 1997; etc.)

In this note we propose a new modelling approach based on the following specification:

$$(1 - \rho L)^d x_t = \varepsilon_t, \quad t = 0, \pm 1, ..., \tag{3}$$

with -1 < ρ < 1, and d being a real number. Then, since $|\rho L| < 1$, one can use a Binomial expansion such that, for all real d,

$$(1 - \rho L)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j (\rho L)^j = 1 - d\rho L + \frac{d(d-1)}{2}\rho^2 L^2 - ...,$$

which implies

$$(1 - \rho L)^d x_t = x_t - \rho d x_{t-1} + \frac{d(d-1)}{2}\rho^2 x_{t-2} - \ldots .$$

If d = 0 in (3), $x_t = \varepsilon_t$, and the series is a white noise process; if d = 1, x_t becomes a stationary AR(1) process, and with d = 2, the process becomes a stationary AR(2) with two identical roots equal to ρ, and therefore (3) can be expressed as

$$(1 - \rho L)^2 x_t = (1 - 2\rho L + \rho^2 L^2) x_t = \varepsilon_t, \tag{4}$$

and the two roots of the inverted polynomial are:

$$z_i = \frac{2\rho \pm \sqrt{4\rho^2 - 4\rho^2}}{2} = \rho,$$

which is smaller than 1 in absolute value by construction.

3. Simulation Results

We carry out Monte Carlo simulations to compare the performance of the proposed model to that of the standard ARFIMA(p, 1, 0) specification given by (1) and (2). For this purpose, we start by generating Gaussian series using the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Wetterling (1986) and Equation (3) for different values of ρ and d. Figures 1, 2, 3 display the results for a sample size T = 100, 500, 1000

respectively, whilst (d, ρ) = (0.25, 0.25); (0.25, 0.75); (0.75, 0.25) and (0.75, 0.75) in all cases. Specifically, the left-hand side plots correspond to the model

$$(1 - \rho L)(1 - L)^{d} u_t = \varepsilon_t, \quad t = 0, \pm 1, ..., \tag{5}$$

while the right-hand side ones are those for the new model given by equation (3). It can be seen that, when d = ρ = 0.25, the series appear to be stationary, but the ARFIMA model exhibits higher dependence. When increasing the value of ρ to 0.75, dependence increases, and is again more pronounced in the case of the ARFIMA specification. When d = 0.75 the ARFIMA models are non-stationary whilst the proposed ones are still stationary but exhibit a high degree of dependence. Thus, the new modelling framework allows for a richer degree of flexibility in the dynamic specification of the stationary processes. Also note that when allowing ε_t in (3) to be AR(1), the ARFIMA(1, d, 0) model is obtained if one sets ρ = 1.

4. Conclusions

This note puts forward a general fractional integration framework which includes the standard AR(F)IMA models. Monte Carlo simulations indicate that the proposed specification performs better than standard AR(F)IMA ones in capturing the properties of the data. Since the model is shown to be stationary for d = 1 and 2, future research should estimate the two parameters in (3) as well as test different null hypotheses for the AR and the fractional parameters.

References

Baillie, R.T., 1996. Long memory processes and fractional integration in econometrics. Journal of Econometrics 73, 5-59.

Gil-Alana, L.A. and Robinson, P.M. (1997) Testing of unit roots and other nonstationary hypotheses in macroeconomic time series. Journal of Econometrics, 80, 241–268.

Granger, C.W.J. (1980) Long memory relationships and the aggregation of dynamic models. Journal of Econometrics, 14, 227–238.

Granger, C.W.J. (1981) Some properties of time series data and their use in econometric model specification. Journal of Econometrics, 16, 121–130.

Granger, C.W.J. and Joyeux, R. (1980) An introduction to long memory time series and fractionally differencing. Journal of Time Series Analysis, 1, 15–29.

Press, W.H., Flannery, B.P., Teukolsky, S.A. and Wetterling, W.T. (1986), Numerical Recipes: The Art of Scientific Computing, Cambridge University Press, Cambridge.

Figure 1: Time series realizations (T = 100)

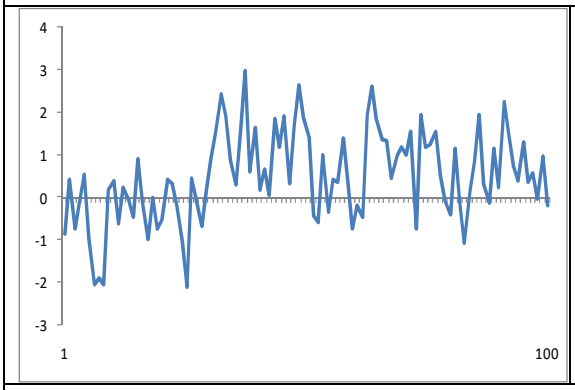
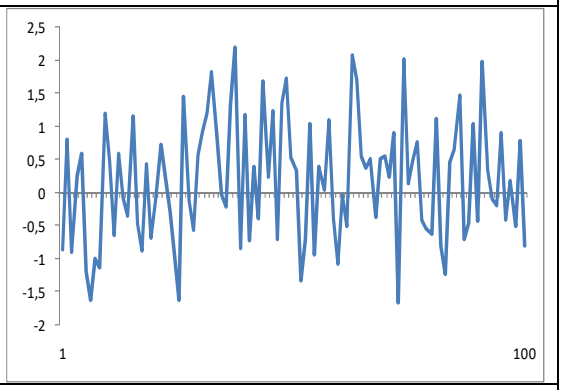
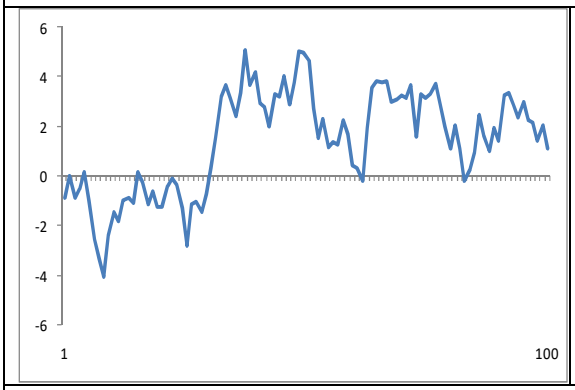
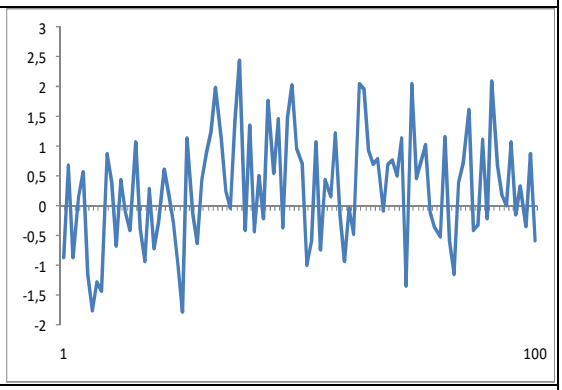
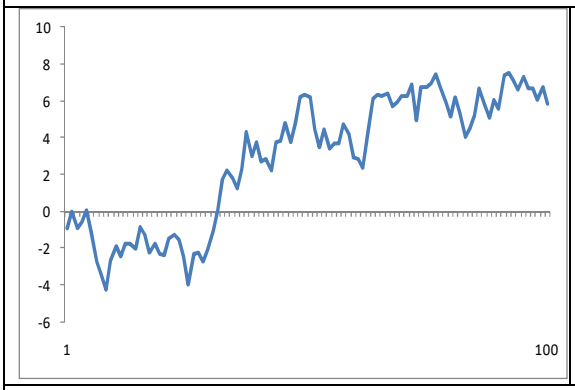
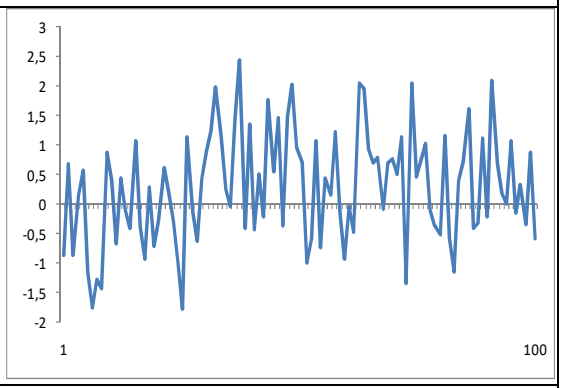
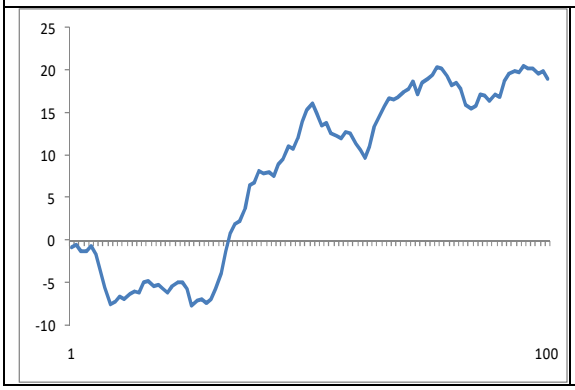
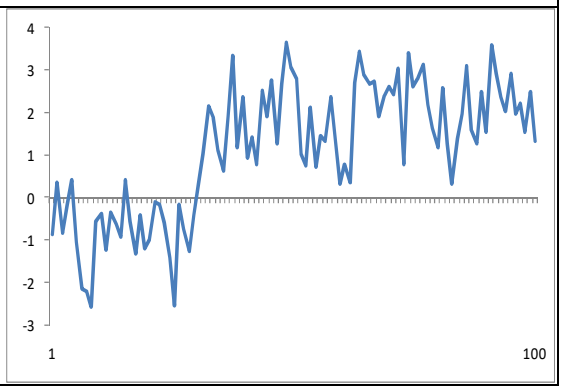
ARFIMA (1, d, 0)	The new modelling framework
d = 0.25 and rho = 0.25	
	
d = 0.25 and rho = 0.75	
	
d = 0.75 and rho = 0.25	
	
d = 0.75 and rho = 0.75	
	

Figure 2: Time series realizations (T = 500)

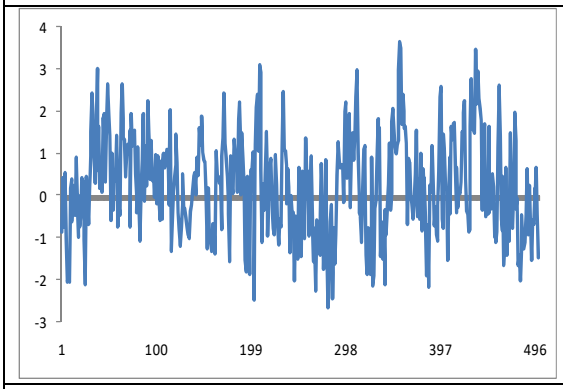
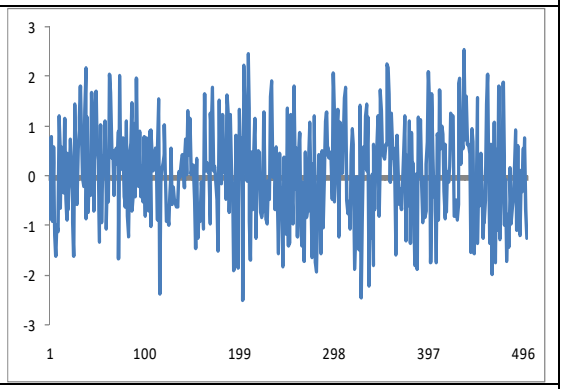
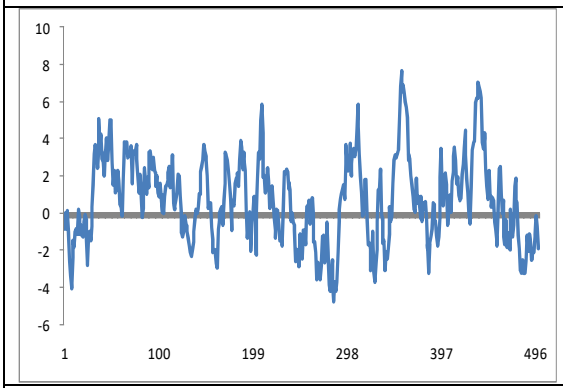
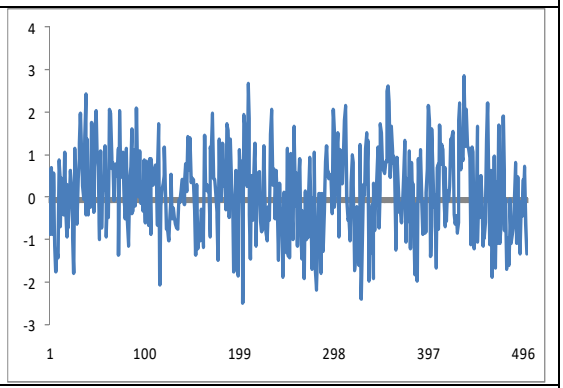
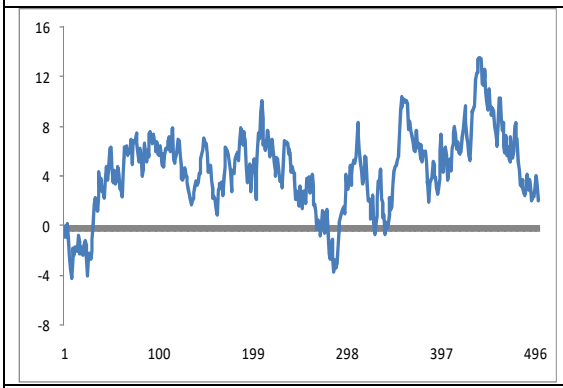
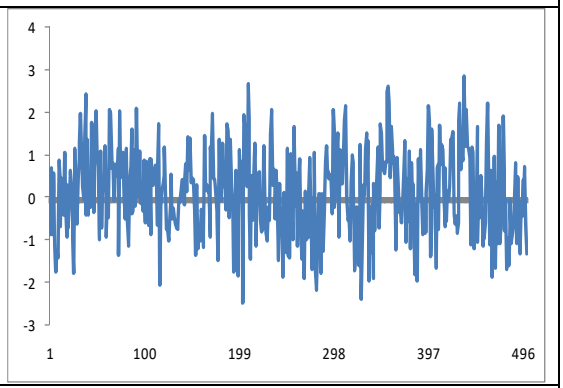
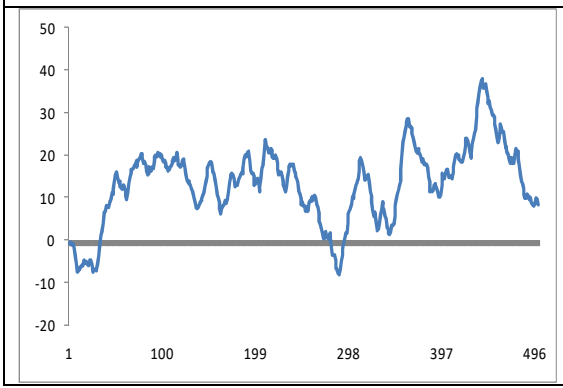
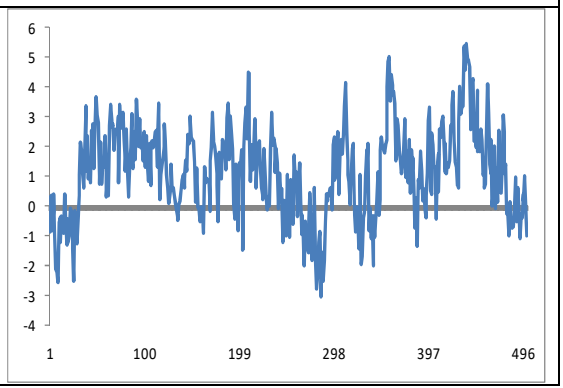
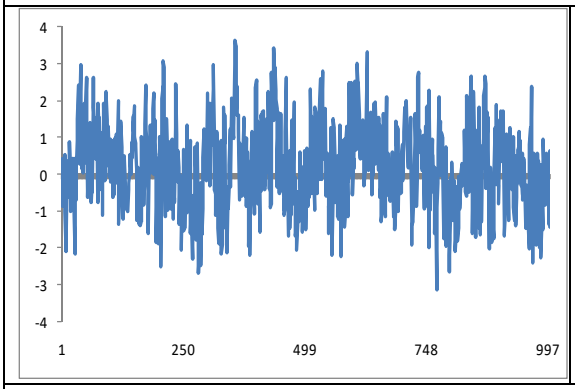
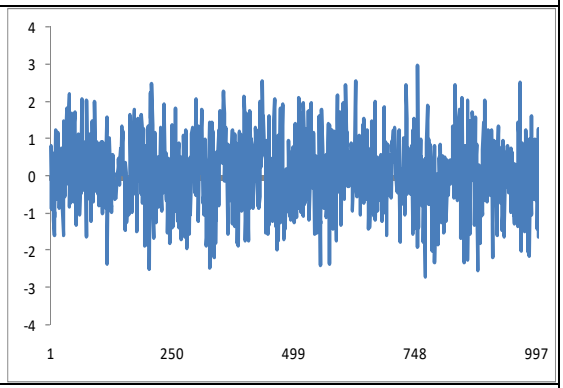
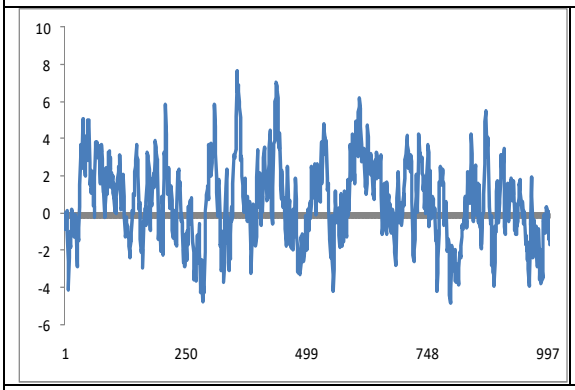
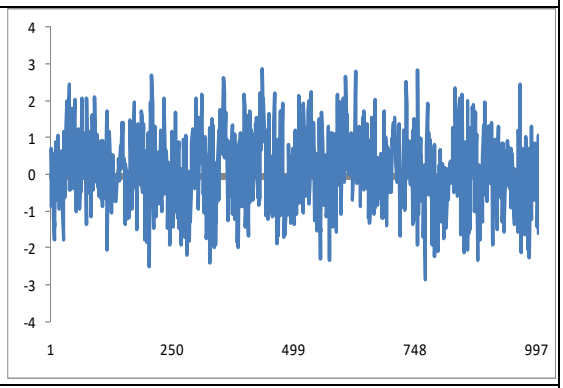
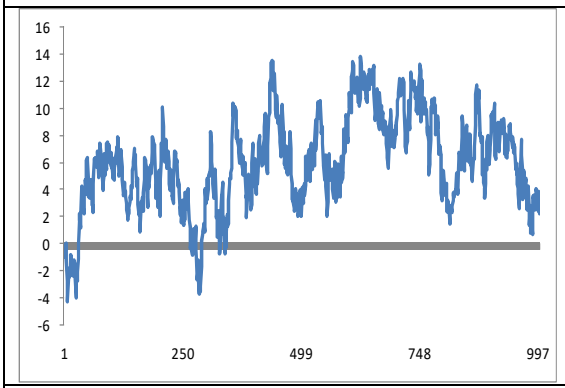
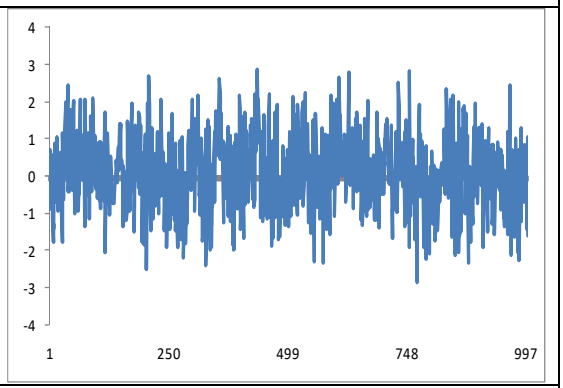
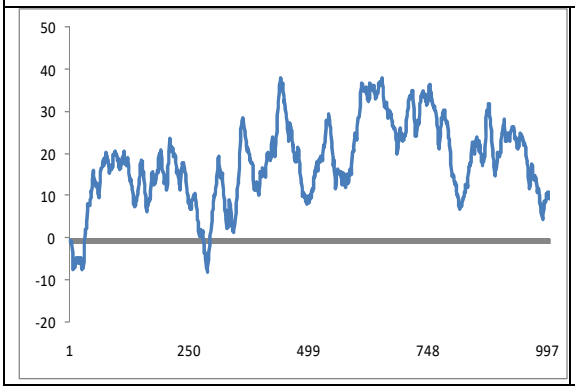
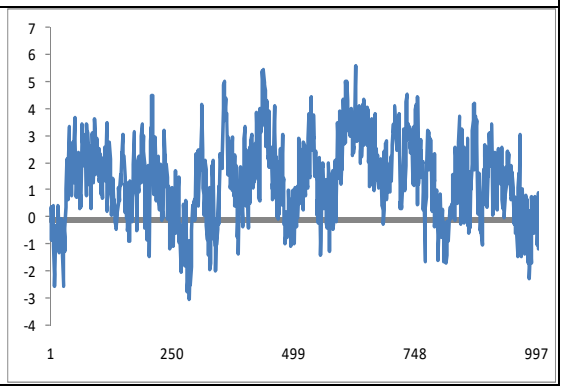
ARFIMA (1, d, 0)	The new modelling framework
d = 0.25 and rho = 0.25	
	
d = 0.25 and rho = 0.75	
	
d = 0.75 and rho = 0.25	
	
d = 0.75 and rho = 0.75	
	

Figure 3: Time series realizations (T = 1000)

ARFIMA (1, d, 0)	The new modelling framework
d = 0.25 and rho = 0.25	
	
d = 0.25 and rho = 0.75	
	
d = 0.75 and rho = 0.25	
	
d = 0.75 and rho = 0.75