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# **A FRACTIONAL INTEGRATION MODEL AND TESTING PROCEDURE WITH ROOTS WITHIN THE UNIT CIRCLE**

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## **Abstract**

This paper puts forward a general statistical model in the time domain based on the concept of fractional integration. More specifically, in the proposed framework instead of imposing that the roots are strictly on the unit circle, we also allow them to be within the unit circle. This approach enables us to specify the time series in terms of its infinite past, with a rate of dependence between the observations much smaller than that produced by the classic  $I(d)$  representations. Both Monte Carlo experiments and empirical applications to climatological and financial data show that the proposed approach performs well.

**Keywords:** fractional integration; unit roots; testing procedure

**JEL codes:** C22

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## 1. Introduction

This paper puts forward a general statistical model in the time domain based on the concept of fractional integration. More specifically, in the proposed framework instead of imposing that the roots are strictly on the unit circle, we also allow them to be within the unit circle. The reason is that a fractional root can be any real number, and therefore may or may not lie on the unit circle, which contains only complex numbers with a magnitude (or modulus) of 1. This approach enables us to specify the time series in terms of its infinite past, with a rate of dependence between the observations much smaller than that produced by the classic  $I(d)$  representations. In particular, we consider processes of the following form:

$$(1 - \alpha L)^d x_t = u_t, \quad t = 1, 2, \dots \quad (1)$$

with  $|\alpha| < 1$  and  $d > 0$ . For simplicity, we assume that  $u_t$  is iid. Clearly, if  $\alpha = 1$  in (1),  $x_t$  is  $I(d)$  and exhibits long memory since  $d > 0$ . However, if  $|\alpha| < 1$ , the process is no longer characterised by long memory, although  $x_t$  will still be a function of its infinite past as long as  $d$  is a real value.

The remainder of the paper is organised as follows: Section 2 describes the time series model; Section 3 presents a procedure for simultaneously testing the two parameters,  $d$  and  $\alpha$ , in (1); Section 4 reports on various Monte Carlo experiments; Section 5 discusses some empirical applications to show the usefulness of the proposed approach; Section 6 offers some concluding remarks. Additional technical details are provided in the Appendices.

## 2. The Statistical Model

The starting point is the model given by (1). As already mentioned, if  $\alpha = 1$ , the polynomial in (1) is exactly on the unit circle, and  $x_t$  is said to be integrated of order  $d$ , denoted by  $x_t \sim I(d)$ .

Note that given that  $d > 0$ , the polynomial  $(1-L)^d$  can be expressed in terms of its Binomial expansion, such that, for all real  $d$ ,

$$(1-L)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j L^j = 1 - dL + \frac{d(d-1)}{2!} L^2 - \frac{d(d-1)(d-2)}{3!} L^3 + \dots$$

These processes (of the form  $(1-L)^d x_t = u_t$ ) were introduced by Granger (1980), Granger and Joyeux (1980) and Hosking (1981), and they have been widely employed in the time series literature (Diebold and Rudebusch, 1989; Gil-Alana and Robinson, 1997; etc). A useful survey of  $I(d)$  statistical models can be found in Baillie (1996) and reviews of the relevant theoretical and applied issues can be found in Gil-Alana and Hualde (2016) and, more recently, in Hualde and Nielsen (2016).

For the purpose of the present paper, we define long memory as a process whose spectral density function,  $f(\lambda)$ , tends to infinity as  $\lambda$  approximates 0, i.e.,

$$\lim_{\lambda \rightarrow 0^+} f(\lambda) = \infty. \quad (2)$$

In the case of  $\alpha = 1$ , the spectral density function of  $x_t$  is:

$$f(\lambda) = \frac{\sigma^2}{2\pi} |1 - e^{i\lambda}|^{-2d} = \frac{\sigma^2}{2\pi} (2 - 2\cos \lambda)^{-d} = \frac{\sigma^2}{2\pi} \left(2 \sin^2 \frac{\lambda}{2}\right)^{-d}, \quad (3)$$

where  $\sigma^2 = \text{Var}(u_t)$ , implying that at  $\lambda = 0$ , the spectral density function is unbounded for  $d > 0$ . However, if  $|\alpha| < 1$ , then

$$f(\lambda) = \frac{\sigma^2}{2\pi} |1 - \alpha e^{i\lambda}|^{-2d} = \frac{\sigma^2}{2\pi} (1 + \alpha^2 - 2\alpha \cos \lambda)^{-d}, \quad (4)$$

which is bounded for all  $\alpha \neq 1$ , and thus, in this case ( $|\alpha| < 1$ ), the process is stationary for all real values  $d$ . If  $d$  is an integer value,  $x_t$  will be a function of a finite number of lag values of  $x_t$ , that is,  $x_t$  is an  $AR(d)$  process in this case. However, if  $d$  is a real number, one can still use the Binomial expansion, such that

$$(1 - \alpha L)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j (\alpha L)^j = 1 - d \alpha L + \frac{d(d-1)}{2!} (\alpha L)^2 - \frac{d(d-1)(d-2)}{3!} (\alpha L)^3 + \dots, \quad (5)$$

and  $x_t$  in (1) will be a function of its infinite past.

Note that, using (5), (1) can be written as:

$$x_t = \sum_{j=1}^{\infty} a_j x_{t-j} + \varepsilon_t, \quad t = 1, 2, \dots, \quad (6)$$

with

$$a_j = (-1)^{j+1} \frac{d(d-1)\dots(d-j+1)!}{j!} \alpha^j, \quad j \geq 1,$$

which is an AR( $\infty$ ) process. Similarly, (1) can also be expressed as

$$x_t = (1 - \alpha L)^{-d} \varepsilon_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}, \quad t = 1, 2, \dots, \quad (7)$$

and using again the standard Binomial expansion:

$$(1 - \alpha z)^{-d} = 1 + \sum_{j=1}^{\infty} \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} (\alpha z)^j,$$

where  $\Gamma(x)$  stands for the Gamma function, it follows that

$$b_j = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \alpha^j, \quad j \geq 1.$$

Thus,  $x_t$  also admits a MA( $\infty$ ) representation.

Figure 1 displays the region of  $(\alpha, d)$ -values for the interval  $(0, 2)$ . It can be seen that the process considered is stationary for all  $\alpha < 1$ . Also, if  $\alpha = 1$  and  $d < 0.5$ ,  $x_t$  is stationary, while it is not for the remaining values of these parameters. Note that if  $\alpha > 1$  the process is explosive. Concerning the stationary region, as  $d$  moves from 0 towards 1, and  $\alpha$  from 0 to higher values, the dependence between the observations becomes higher.

**Figure 1: Stationary / nonstationary ( $\alpha, d$ ) region in Equation (1)**

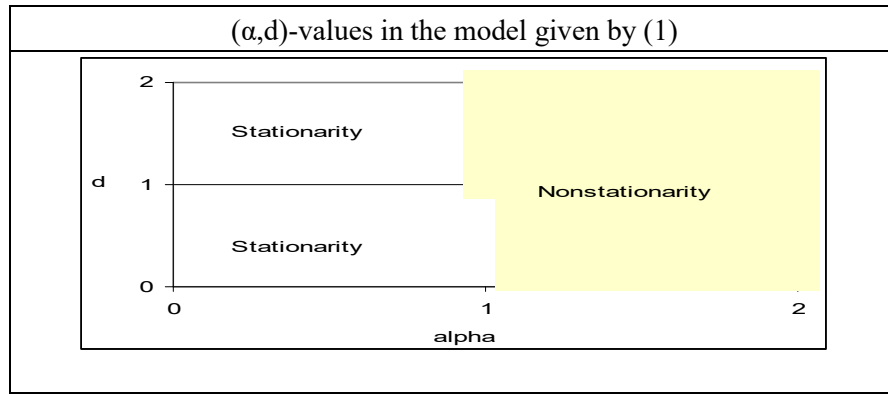


Figure 2 displays plots of simple realizations of processes of the form given by (1) for different values of  $\alpha$  and  $d$ . The plots on the left-hand-side of the figure correspond to the case of long memory ( $\alpha = 1$ ), while those on the right-hand-side to short memory, with  $\alpha = 0.80$ . In all cases we take  $d = 0.35, 0.75$  and  $1$ . In particular, the upper-left plot corresponds to the case with  $d = 1$  and  $\alpha = 0.35$ , namely to an  $I(0.35)$  process, which exhibits long memory despite being stationary; by contrast, in the two plots below ( $I(0.75)$  and  $I(1)$ ), the processes are clearly nonstationary. Finally, stationarity is found in all cases when  $\alpha = 0.80$ .

**(Insert Figures 2 – 4 about here)**

Figure 3 displays the first 50 sample autocorrelation values of the realizations in Figure 2. It can be seen that in the long memory cases these decay at a hyperbolic rate, and in the short memory case the rate of decay slows down as  $d$  increases. Figure 4 displays the periodograms, with higher peaks corresponding to lower frequencies in the long memory cases.

**(Insert Figure 5 about here)**

Figure 5 displays the impulse response functions obtained from the coefficients of the infinite MA representation in (7). As expected, mean reversion occurs in all cases except when  $\alpha = d = 1$  (the random walk model). These values correspond to the coefficients  $b_j$  in the



MA( $\infty$ ) representation of  $x_t$  in (7). This figure shows that when increasing  $\alpha$  and  $d$  the impulse responses have a longer time horizon before disappearing completely.

### 3. The Testing Procedure

In this section we develop a score test of the null hypothesis:

$$H_o : (\alpha, d)' = (\alpha_o, d_o)', \quad (8)$$

in the model given by (1) for any real values  $d_o > 0$  and  $|\alpha_o| < 1$ , under the assumption that  $u_t$  in (1) is a sequence of zero-mean uncorrelated random variables with unknown variance  $\sigma^2$ .

Specifically, the score statistic for  $H_o$  in (8) is constructed as follows. Consider a column vector of parameters  $\eta = (\alpha, d, \sigma^2)'$  and let  $L(\eta)$  be an objective function, such as the negative of the log-likelihood, which has a minimum under  $H_o$  at  $\hat{\eta} = (\alpha_o, d_o, \hat{\sigma}^2)'$ .

Following Rao (1973), a score statistic is:

$$\frac{\partial L(\eta)}{\partial \eta'} \left( E_o \left[ \frac{\partial L(\eta)}{\partial \eta} \frac{\partial L(\eta)}{\partial \eta'} \right] \right)^{-1} \frac{\partial L(\eta)}{\partial \eta} \Big|_{\substack{\alpha = \alpha_o \\ d = d_o \\ \sigma^2 = \hat{\sigma}^2}}, \quad (9)$$

where the expectation is taken under  $H_o$  (8) prior to substitution of  $\hat{\sigma}^2$ . In standard problems such as ours, the null limit distribution of the test will be unaffected if the inverted matrix is replaced by alternatives such as a sample average or the Hessian. However, testing the null of long memory, i.e.,  $\alpha = 1$ , leads to a non-standard null limit distribution.

With Gaussian  $u_t$ , the negative of the log-likelihood with  $\eta = (\alpha, d, \sigma^2)'$  is:

$$L(\eta) = \frac{T}{2} \log 2\pi\sigma^2 + \frac{1}{2\sigma^2} \sum_{t=1}^T [\rho(L; \alpha, d) x_t]^2,$$

where  $\rho(L; \alpha, d) = (1 - \alpha L)^d$ . In Appendix A, it is shown that (9) takes the form:

$$\hat{S} = \frac{T}{\hat{\sigma}^4} [\hat{a}_1 \hat{a}_2] \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}, \quad (10)$$

where  $T$  is the sample size, and

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^T \hat{u}_t^2; \quad \hat{u}_t = (1 - \alpha_o L)^{d_o} x_t;$$

$$\hat{a}_1 = - \sum_{j=1}^{T-1} \psi_j^{(\alpha)} C_{\hat{u}}(j); \quad \hat{a}_2 = - \sum_{j=1}^{T-1} \psi_j^{(d)} C_{\hat{u}}(j);$$

where  $C_{\hat{u}}(j) = \frac{1}{T-j} \sum_{t=1}^{T-j} \hat{u}_t \hat{u}_{t+j}$ , and  $\psi_j^{(\alpha)}$  and  $\psi_j^{(d)}$  are obtained by expanding:

$$\frac{\partial \log \rho(L; \alpha_o, d_o)}{\partial \alpha} = \frac{-d_o L}{(1 - \alpha_o L)} = \sum_{j=1}^{\infty} \psi_j^{(\alpha)} L^j, \quad (11)$$

and

$$\frac{\partial \log \rho(L; \alpha_o, d_o)}{\partial d} = \log(1 - \alpha_o L) = \sum_{j=1}^{\infty} \psi_j^{(d)} L^j, \quad (12)$$

Then, clearly

$$\psi_j^{(\alpha)} = -d_o \alpha_o^{j-1} \quad \text{and} \quad \psi_j^{(d)} = \frac{-1}{j} \alpha_o^j$$

Finally:

$$\hat{a}_{11} = \sum_{j=1}^{T-1} \left(1 - \frac{j}{T}\right) \left(\psi_j^{(\alpha)}\right)^2; \quad \hat{a}_{12} = \sum_{j=1}^{T-1} \left(1 - \frac{j}{T}\right) \psi_j^{(\alpha)} \psi_j^{(d)}; \quad (13)$$

$$\hat{a}_{21} = \sum_{j=1}^{T-1} \left(1 - \frac{j}{T}\right) \psi_j^{(d)} \psi_j^{(\alpha)}; \quad \hat{a}_{22} = \sum_{j=1}^{T-1} \left(1 - \frac{j}{T}\right) \left(\psi_j^{(d)}\right)^2.$$

Asymptotically, the above expressions can be approximated by:

$$\begin{aligned} \tilde{a}_{11} &= \sum_{j=1}^{\infty} \left(\frac{-1}{j} \alpha^j\right)^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} \alpha^{2j} \\ \tilde{a}_{12} &= \tilde{a}_{21} = \sum_{j=1}^{\infty} \left(\frac{-1}{j} \alpha^j\right) \left(-d \alpha^{j-1}\right) = d \sum_{j=1}^{\infty} \frac{1}{j} \alpha^{2j-1} \\ \tilde{a}_{22} &= \sum_{j=1}^{\infty} \left(-d \alpha^{j-1}\right)^2 = d^2 \sum_{j=1}^{\infty} \alpha^{2j-2}, \end{aligned} \quad (14)$$

and their limits for different values of  $\alpha$  are given in Table 1. Routine calculations lead to the frequency domain version of the test, which is given by:

$$\tilde{S} = \frac{T}{\tilde{\sigma}^4} [\tilde{a}_1 \tilde{a}_2] \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix}, \quad (15)$$

where

$$\tilde{a}_i = \frac{-2\pi}{T} \sum_k^* \psi_i(\lambda_k) I_{\hat{u}}(\lambda_k),$$

where  $I_{\hat{u}}(\lambda_k)$  is the periodogram of  $\hat{u}_t$ :

$$I_{\hat{u}}(\lambda_k) = \frac{1}{2\pi T} \left| \sum_{t=1}^T \hat{u}_t e^{it\lambda_k} \right|^2,$$

and  $\tilde{a}_{ij}$ ,  $i, j = 1, 2$  can be approximated by:

$$\tilde{a}_{ij} = \frac{2}{T} \sum_{k=1}^* \psi_i(\lambda_k) \psi_j(\lambda_k),$$

where

$$\psi_1(\lambda_k) = \log(1 + \alpha^2 - 2\alpha \cos \lambda_k) \quad (16)$$

and

$$\psi_2(\lambda_k) = \frac{d(\alpha - \cos \lambda_k)}{1 + \alpha^2 - 2\alpha \cos \lambda_k}, \quad (17)$$

with  $\lambda_k = 2\pi k/T$ .

**(Insert Table 1 about here)**

### Theorem

Let us consider the model given by (1), where  $u_t$  is a stationary random variable satisfying

$$E(u_t | B_{t-1}) = 0, \quad (T1)$$

and

$$E(u_t^2 | B_{t-1}) = \sigma^2 \text{ almost surely,} \quad (T2)$$

where  $0 < \sigma^2 < \infty$ , and  $B_t$  is the  $\sigma$ -field of events generated by  $u_s$ ,  $s \leq t$ .

Also, the functions  $\psi$  in (16) and (17) must satisfy some technical restrictions to justify approximating integrals by sums. These restrictions correspond to those in Class H in Robinson (1994) (page 1433). Then, under  $H_0$  defined as in (8),

$$\hat{S} \rightarrow_d \chi_2^2$$

Note that (T1) and (T2) imposes a martingale difference assumption on the white noise  $u_t$  which is substantially weaker than the Gaussianity assumed for developing the test statistic, since it requires only a second moment condition, which is clearly a minimal requirement. Finite sample critical values based on Monte Carlo simulation results are reported in Table 2.

**(Insert Table 2 about here)**

Note that the model and the testing procedure described above can be extended to include deterministic terms, with  $x_t$  in (1) denoting the errors in the following multiple regression model (instead of being observable):

$$y_t = \beta' z_t + x_t, \quad t = 1, 2, \dots \quad (18)$$

where  $y_t$  and the  $(k \times 1)$  vector  $z_t$  are observable and  $\beta$  is a  $(k \times 1)$  unknown vector. We assume that the elements of  $z_t$  are non-stochastic, such as polynomials in  $t$ , to include, for instance, the case of a linear time trend, if  $z_t = (1, t)'$ . Note that under  $H_0$  (6), the model becomes:

$$(1 - \alpha_o L)^{d_o} y_t = \beta' w_t + u_t, \quad t = 1, 2, \dots$$

where  $w_t = (1 - \alpha_o L)^{d_o} z_t$ . Using now the OLS estimate of  $\beta$ :

$$\hat{\beta} = \left( \sum_{t=1}^T w_t w_t' \right)^{-1} \sum_{t=1}^T w_t (1 - L)^{d_o} y_t,$$

the residuals are:

$$\hat{u}_t = (1 - L)^{d_o} y_t - \hat{\beta}' w_t, \quad t = 1, 2, \dots \quad (19)$$

and the functional form of the test statistic takes the same form as before, replacing the values of  $\hat{u}_t$  with those given by (19).

#### 4. A Monte Carlo Simulation Study

This section examines the finite sample behaviour of the test described in Section 3 by means of Monte Carlo simulations. The computations were carried out using Fortran, and the codes of the programs are available from the author upon request.

**(Insert Tables 3 – 5 about here)**

Tables 3 – 5 report the rejection frequencies of the test statistic  $\hat{S}$  in (8), testing  $H_0$  (6) for  $\alpha_o$  and  $d_o$  values equal to 0.25, 0.50, 0.75 and 1.00, in a model given by (1) with  $(\alpha, d) = (0.25, 0.50), (0.50, 0.50), (0.75, 0.50)$  and  $(1.00, 0.50)$ . The nominal size is 5% in all cases and  $T = 100$  (Table 3), 500 (Table 4) and 1000 (Table 5). Starting with  $T = 100$ , we see that the size is slightly above its nominal value (0.0524) and the rejection frequencies are very low in some cases; for example, if  $\alpha = 0.25$  and  $d = 0.50$ , the rejection probability at  $(\alpha_o, d_o) = (0.50, 0.25)$  is smaller than 10% implying that the procedure has some problems in identifying the true parameters. When increasing the sample size (in Tables 4 and 5) the size of the test converges to its nominal value (0.0489 with  $T = 500$  and 0.0495 with  $T = 1000$ ), and the rejection frequencies are now higher in all cases.

## 5. Empirical Applications

This section follows the proposed approach to examine long-range dependence in two sets of series, namely (i) water levels in the Nile River dataset, Northern Hemisphere temperatures and weight measurements of a 1kg check standard weight performed at the National Institute of Standards and Technology (NIST, formerly, NBS) in the US; (ii) stock returns in five Latin American countries. These series have been chosen for illustrative purposes, all of them having been analysed in previous studies using either autoregressive or fractional processes.

We analyse first the Nile River data collected by Toussoun (1925) starting in 622 CE and ending in 1284 CE, specifically the yearly minimal water levels of the river, measured at the Roda Gauge near Cairo. This dataset can also be found in Beran (1994, pp.237-239).<sup>1</sup>

**(Insert Figure 6 about here)**

We perform the procedure described in Section 3, testing first  $H_0$  (6) in the model given by (1) for  $(\alpha_0, d_0)$ -values:  $\alpha_0 = 0.01, 0.02, \dots, 1$ ;  $d_0 = 0.01, 0.02, \dots, 2$ . From all these combinations there emerges only a single case when  $H_0$  (6) cannot be rejected at the 5% significance level – this corresponds to  $\alpha_0 = 1$  and  $d_0 = 0.72$ , which is consistent with the long-range dependence observed in the series. Next, we also consider two specifications including respectively an intercept only, and an intercept as well as a linear time trend. These two sets of results are very similar, and thus we only report in Figure 6 those based on the model with an intercept only. It can be seen that the non-rejection values of  $d$  range between 0.35 and 0.50 and occur when  $\alpha_0$  is in the interval  $[0.95, 1]$ . In the case of no regressors,  $d$  is higher than 0.5, which implies non-stationarity, while in the cases of an intercept and/or a linear time trend  $d$  appears to be smaller than 0.5 and thus implies stationarity. We would argue that the specifications with deterministic terms are preferable, since the model without regressors assumes a zero mean for the series, whose values are

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<sup>1</sup> Toussoun (1925) collected data up until 1921, but 622-1284 is the longest period without gaps.

instead all positive, Moreover, the former produce results which are consistent with those of Beran (1994), who examined the same series using ARFIMA models and concluded that the fractional differencing parameter  $d$  was around 0.40 with a 95% confidence interval of (0.34, 0.46).

**(Insert Figure 7 about here)**

The second series examined is monthly temperatures in the Northern Hemisphere over the years 1854-1989, which are obtained from the database of the Climate Research Unit of the University of East Anglia, Norwich, England (Jones and Briffa, 1992). More precisely, this series reports the temperature (degrees C) difference (anomaly) from the monthly average over the period 1950-1979, and its plot (Beran, 1994, p.30) suggests an upward trend that may reflect global warming during the last 100 years. It can be seen that in the case of the model without regressors the null is rejected for all values of  $\alpha_0$  and  $d_0$ . When including a linear time trend there is a wide range of non-rejection values, which are displayed in Figure 7. These are between 0.32 and 0.57 for  $d_0$  and between 0.8 and 1 for  $\alpha_0$ . The lowest statistic corresponds to  $\alpha_0 = 0.97$  and  $d_0 = 0.40$ , which implies short memory but with an AR coefficient very close to 1. Beran (1994) used a Whittle estimator for  $d$  in an ARFIMA(0,  $d$ , 0) model with a linear time trend and estimated the value of  $d$  to be 0.37, and the trend coefficient 0.00032.

The third series reports high precision weight measurements of a 1kg check standard weight performed at the National Institute of Standards and Technology (NIST, formerly, NBS), Gaithersburg, USA (see Pollak, Croarkin and Hagwood, 1993; Graf, Hample and Tacier, 1984; Graf, 1983). The measurements were taken between June 24, 1963 and October 17, 1975, using the same weight machine. The differences (in micrograms) from 1kg were recorded. One difficulty with this dataset is that the dates at which measurements were taken are not equidistant. This makes the analysis of short-term correlation problematic. However, the overall long-range dependence structure is likely to be largely unaffected. In the case of

this series, when including no regressors, all non-rejection values occur for  $\alpha_0 = 1$ , (i.e., long memory), with values of  $d_0$  ranging between 0.93 and 1.13. However, if an intercept is also included in the model, the region of non-rejection values is wider (see Figure 8) with  $\alpha_0$  equal to or slightly smaller than 1, and  $d_0$  ranging between 0.02 and 0.23. The lowest statistic occurs at  $\alpha_0 = 0.98$  and  $d_0 = 0.11$ , and setting  $\alpha_0 = 1$ ,  $d_0 = 0.13$ . By applying standard semi-parametric long-memory methods, Beran (1994, p.99) found in this case estimates of  $d$  ranging between 0.02 and 0.28, and when using other, more robust techniques (p.140) an estimate of about 0.10.

Finally, we analyse daily data on stock market returns in Venezuela, Argentina, Brazil, Mexico and Colombia, which have been obtained from the International Monetary Fund database. The starting date is 4 January 1988 for Argentina and Mexico; 2 January 1990 for Venezuela; 2 January 1992 for Colombia, and 4 July 1994 for Brazil. The end date is 22 January 2002 in all cases.

**(Insert Table 6 about here)**

It can be seen in Table 6 that for Mexico  $\alpha$  is exactly equal to 1, which implies long memory with an order of integration equal to 0.13. Short memory is found in all other cases. In particular, for Colombia,  $\alpha = 0.87$  and  $d = 0.33$ ; for Argentina,  $\alpha = 0.57$  and  $d = 0.21$ ; for Venezuela and Brazil  $d = 1.92$  but  $\alpha$  is then very close to 0 (0.09 for Venezuela and 0.06 for Colombia). Table 7 reports the impulse response for each series; these are significant in all cases in the first period, up to the third period in Mexico, and up to the fifth in Colombia.

## **6. Conclusions**

This paper proposes a general statistical model in the time domain (with a corresponding testing procedure) which is based on the concept of fractional integration and allows the roots



of the process to be not only on but also within the unit circle. This is because a fractional root can be any real number, and therefore may or may not lie on the unit circle. The evidence based on both Monte Carlo simulations and some empirical applications shows that the proposed framework performs well in finite samples and captures accurately the stochastic behaviour of various series ranging from climatological to financial ones. Future work could extend the proposed model to allow for cyclical structures, where the singularity or pole in the spectrum occurs at one or more frequencies away from zero.

## Appendix A: Derivation of the score statistic $\hat{S}$

The negative of the log-likelihood under (1) and Gaussianity of  $u_t$  is given by:

$$L(\alpha, d, \sigma^2) = \frac{T}{2} \log 2\pi\sigma^2 + \frac{1}{2\sigma^2} \sum_{t=1}^T [\rho(L; \alpha, d) x_t]^2,$$

for any admissible  $\alpha, d$  and  $\sigma^2$ . Then,

$$\frac{\partial L(\alpha, d, \sigma^2)}{\partial \alpha} = \frac{1}{\sigma^2} \sum_{t=1}^T \left[ \frac{\partial \rho(L; \alpha, d)}{\partial \alpha} x_t \right] \rho(L; \alpha, d) x_t,$$

so that

$$\frac{\partial L(\alpha_o, d_o, \sigma^2)}{\partial \alpha} = \frac{1}{\sigma^2} \sum_{t=1}^T \left[ \frac{\partial \log \rho(L; \alpha_o, d_o)}{\partial \alpha} u_t \right] u_t,$$

where  $u_t = (1 - \alpha_o L)^{d_o} x_t$ . Using now the expansion in (11), we obtain that

$$\frac{\partial L(\alpha_o, d_o, \sigma^2)}{\partial \alpha} = \frac{1}{\sigma^2} \sum_{t=1}^T \left[ \sum_{j=1}^{\infty} \psi_j^{(\alpha)} u_{t-j} \right] u_t = \frac{1}{\sigma^2} \sum_{j=1}^{T-1} \psi_j^{(\alpha)} \sum_{t=1}^{T-j} u_t u_{t+j} \approx \frac{T}{\sigma^2} \sum_{j=1}^{T-1} \psi_j^{(\alpha)} C_u(j),$$

where  $C_u(j) = \frac{1}{T-j}$ .

Similarly,

$$\frac{\partial L(\alpha_o, d_o, \sigma^2)}{\partial d} = \frac{1}{\sigma^2} \sum_{t=1}^T \left[ \frac{\partial \log \rho(L; \alpha_o, d_o)}{\partial d} u_t \right] u_t,$$

and using now (12),

$$\frac{\partial L(\alpha_o, d_o, \sigma^2)}{\partial d} = \frac{1}{\sigma^2} \sum_{t=1}^T \left[ \sum_{j=1}^{\infty} \psi_j^{(d)} u_{t-j} \right] u_t = \frac{1}{\sigma^2} \sum_{j=1}^{T-1} \psi_j^{(d)} \sum_{t=1}^{T-j} u_t u_{t+j} \approx \frac{T}{\sigma^2} \sum_{j=1}^{T-1} \psi_j^{(d)} C_u(j).$$

Finally,  $\frac{\partial L(\alpha_o, d_o, \sigma^2)}{\partial \sigma^2} = \frac{-1}{2\sigma^4} \sum_{t=1}^T (u_t^2 - \sigma^2)$ , and this last expression vanishes for

$\sigma^2 = \hat{\sigma}^2$ . It also follows that the expectation in (9) is:

$$T \begin{pmatrix} \hat{a} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix},$$

where  $\hat{a}$  is a (2x2) matrix formed by the elements  $\hat{a}_{ij}$  in (13), which can be asymptotically approximated (for  $|\alpha| < 1$ ) by  $\tilde{a}$  in (14). Then, using (9), we deduce  $\hat{S}$ .

## Appendix B: Proof of the theorem

Note that the test statistic is based on the Lagrange Multiplier (LM) principle and thus everything is evaluated under the null hypothesis  $H_0$  (7), using  $\alpha_0$  and  $d_0$  in the equations and with the residuals being  $I(0)$ .

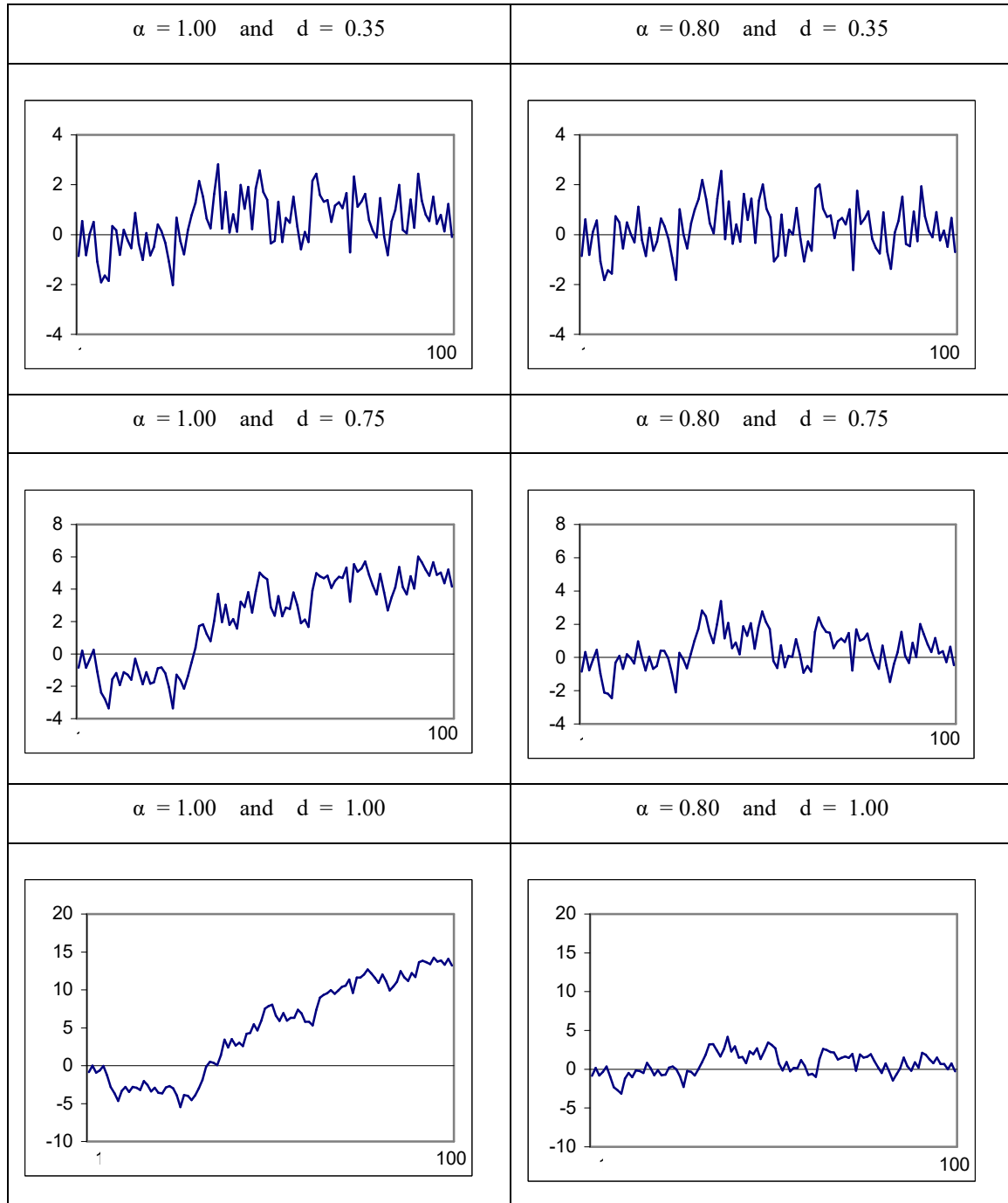
Calling  $\tilde{a}^t = [\tilde{a}_1; \tilde{a}_2]$ ,  $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & 0 \\ 0 & \tilde{a}_{22} \end{pmatrix}$ , and following conditions (T1) and (T2) along with the class H of functions for  $\Psi(\lambda_k)^t = [\psi_1(\lambda_k); \psi_1(\lambda_k)]$  as defined in Robinson (1994, page 1433), the proof is a straightforward extension of that of Theorem 1 in that paper using Lemmas 1, 2 and 3 and an application of a martingale difference Central Limit Theorem as in Brown (1971). Specifically, the proof that  $\widehat{\sigma^2} \rightarrow \sigma^2$  follows from the fact that  $u_t^2 - \sigma^2$  are stationary martingale differences.

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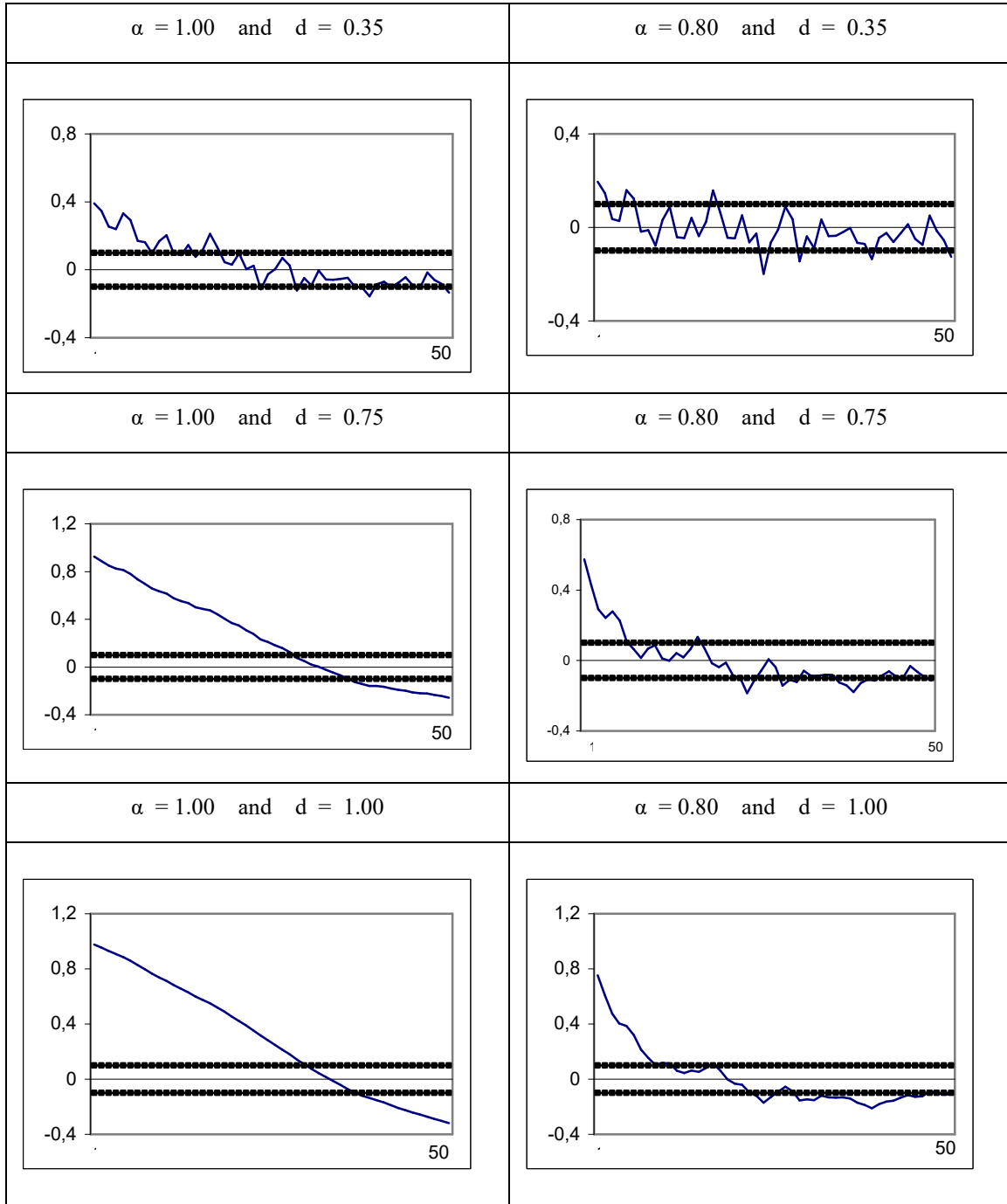
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**Figure 2 : Simple realizations of models of the form  $(1 - \alpha L)^d x_t = \varepsilon_t$**



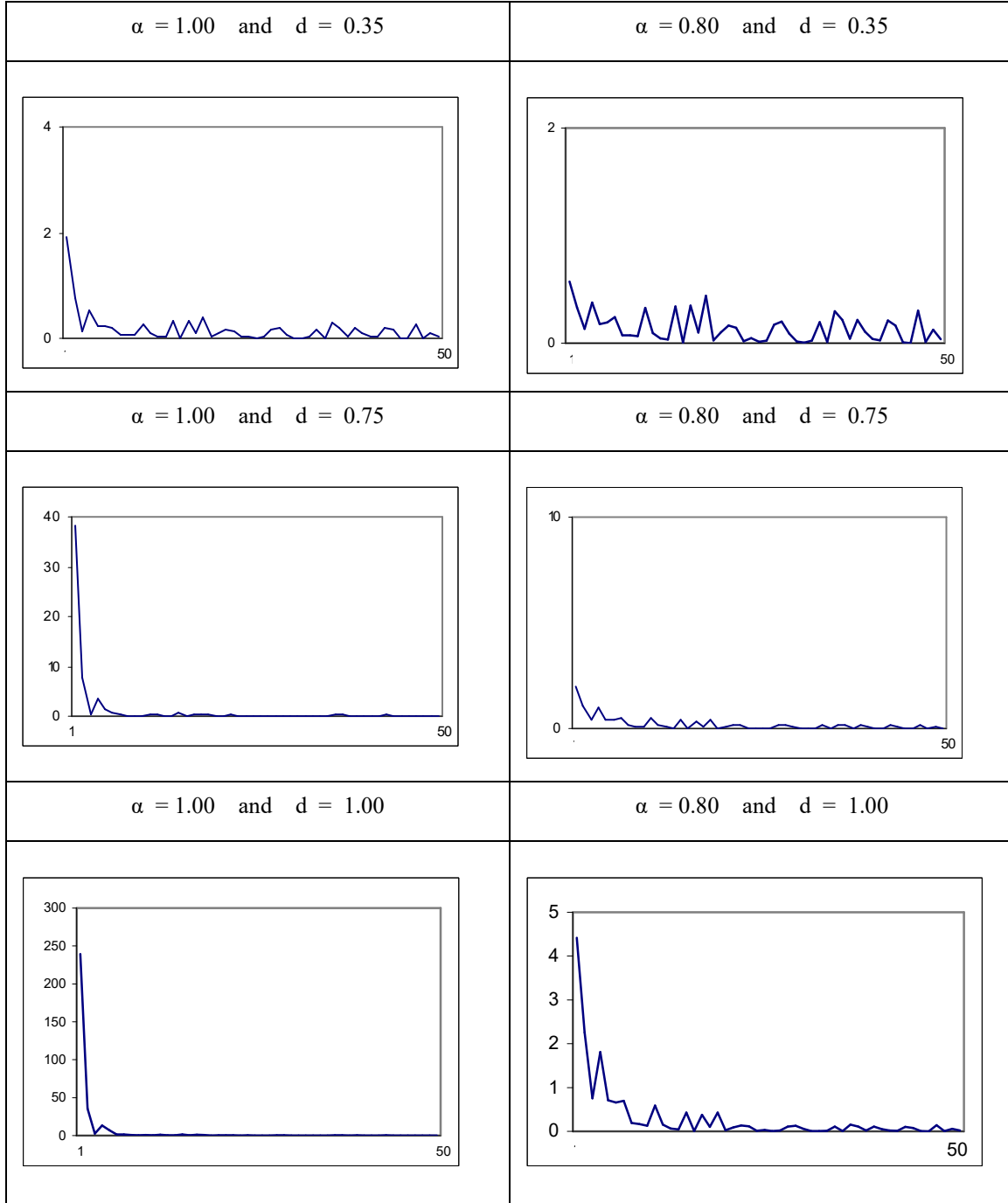
The plots are sample realizations of the model in Equation (1) for a range of values for  $\alpha$  and  $d$ .

**Figure 3: Correlograms of the realizations in Figure 2**



The large sample standard error under the null hypothesis of no autocorrelation is  $1/\sqrt{T}$

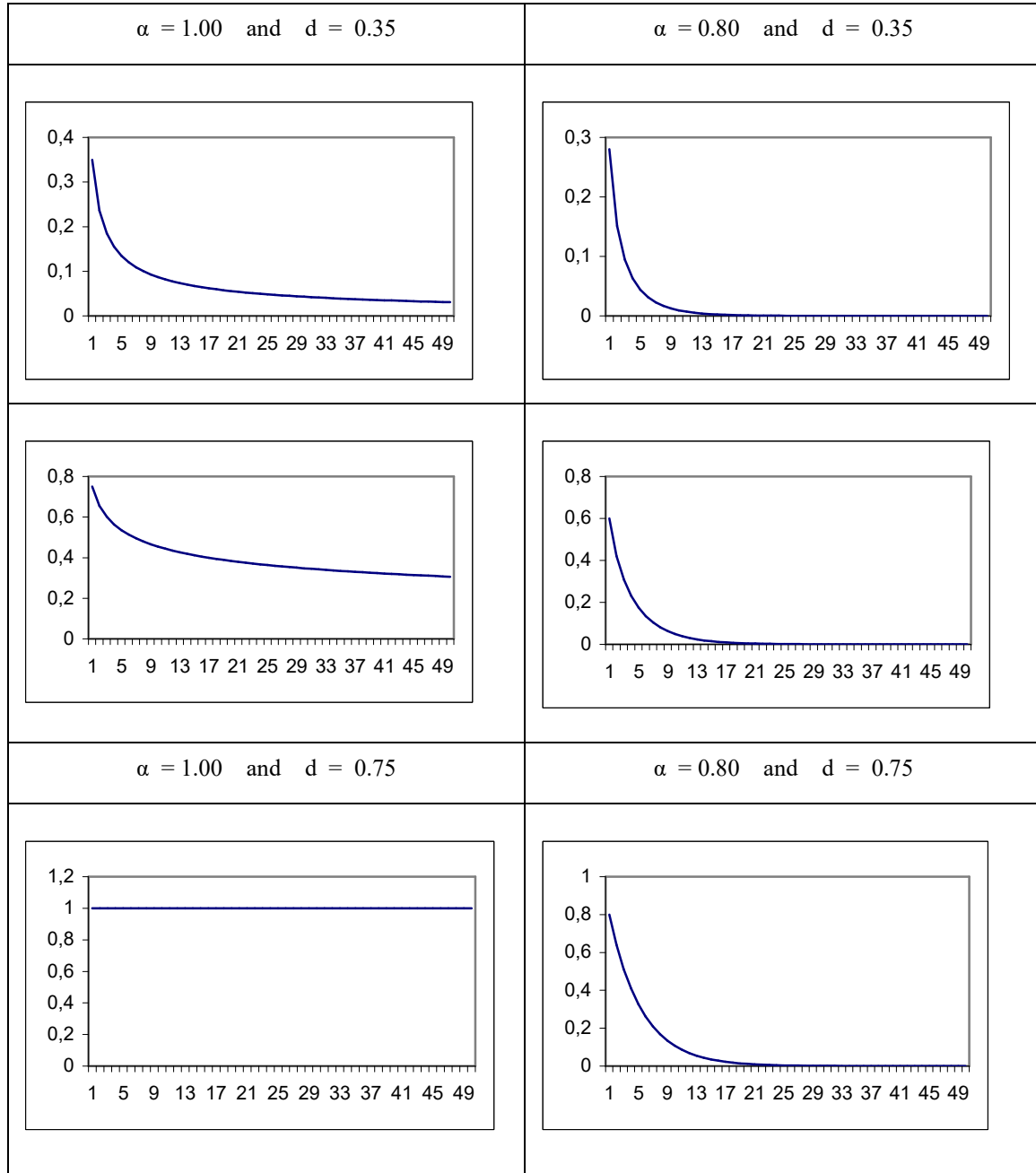
**Figure 4: Periodograms of the realizations in Figure 2**



The periodograms were computed on the basis of the discrete Fourier frequencies  $\lambda_j = 2\pi j/T$ .

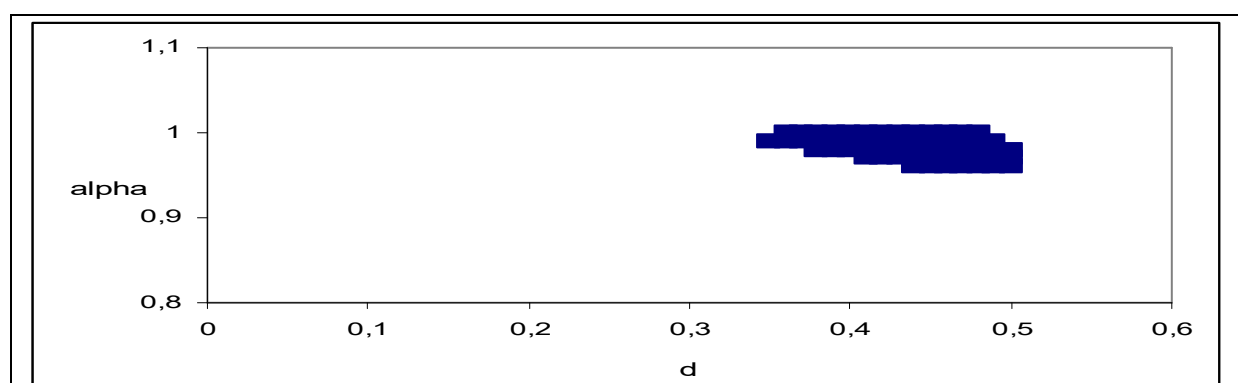


**Figure 5: Impulse response functions of the realizations in Figure 2**



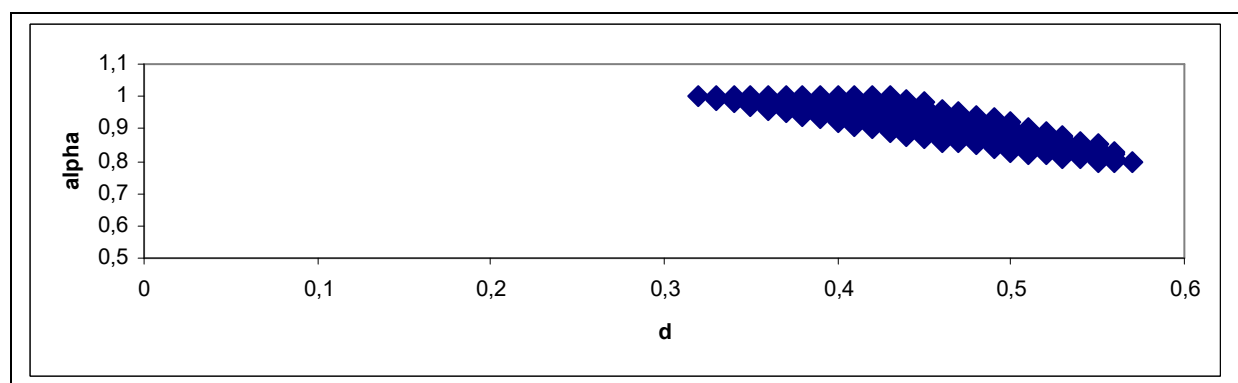
The values indicate the impulse responses based on the coefficients of the  $MA(\infty)$  representation of the series given by Equation (7).

**Figure 6: ( $\alpha_0$ ,  $d_0$ )-values for which  $H_0$  (7) cannot be rejected. Nile river data**



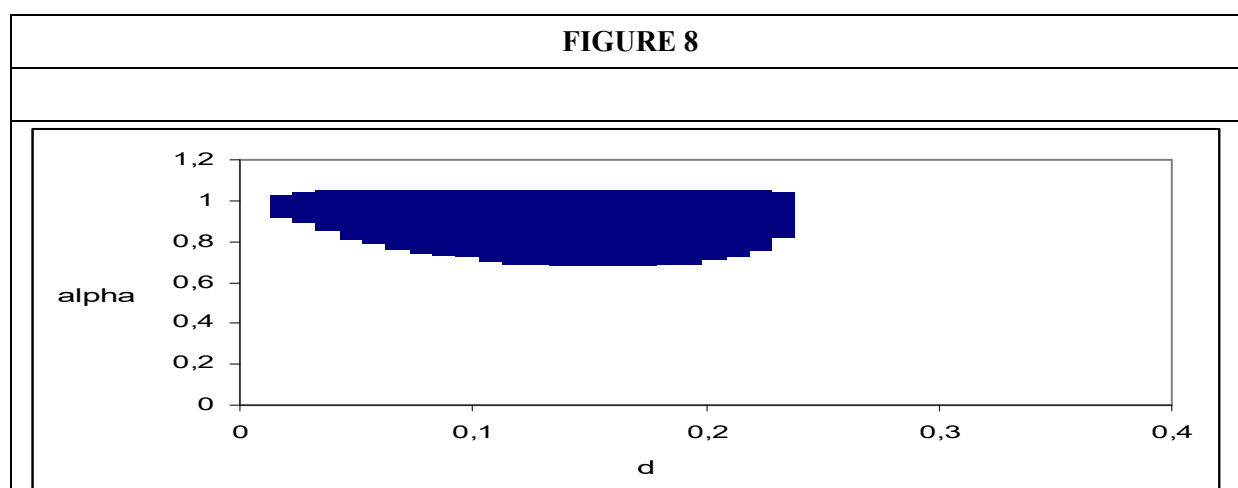
The region in blue corresponds to the values of  $\alpha$  and  $d$  for which the null hypothesis cannot be rejected at the 95% level.

**Figure 7: ( $\alpha_0$ ,  $d_0$ )-values for which  $H_0$  (7) cannot be rejected. Northern Hemisphere data**



The region in blue corresponds to the values of  $\alpha$  and  $d$  for which the null hypothesis cannot be rejected at the 95% level.

**Figure 8: ( $\alpha_0$ ,  $d_0$ )-values for which  $H_0$  (7) cannot be rejected. 1kg standard weight NIST**



The region in blue corresponds to the values of  $\alpha$  and  $d$  for which the null hypothesis cannot be rejected at the 95% level.

**Table 1: Limiting values of the expressions reported in (14)**

$\alpha$	$\tilde{a}_{11}$	$\tilde{a}_{12}$	$\tilde{a}_{13}$
0.10	0.010025	0.100503d	1.010101d
0.20	0.040407	0.204109d	1.041666d
0.30	0.092110	0.314368d	1.098901d
0.40	0.166900	0.435883d	1.190476d
0.50	0.267652	0.575364d	1.333333d
0.60	0.398958	0.743811d	1.562499d
0.70	0.568438	0.961920d	1.960784d
0.80	0.790026	1.277064d	2.777777d
0.90	1.095103	1.845256d	5.263157d
1.00	1.645000	$\infty$	$\infty$

**Table 2: Finite sample critical values of the test statistic for different sample sizes**

$\alpha$	T = 100	T = 300	T = 500	T = 1000	T = 5000
0.1	6.028	5.991	5.908	5.921	5.862
0.2	5.990	6.025	5.925	5.923	5.858
0.3	5.992	6.042	5.901	5.894	5.846
0.4	6.086	6.045	5.921	5.951	5.853
0.5	6.115	6.045	5.918	5.966	5.860
0.6	6.185	6.044	5.999	5.935	5.899
0.7	6.322	6.058	5.988	5.914	5.893
0.8	6.599	6.189	5.990	5.950	5.901
0.9	7.126	6.281	5.962	5.880	6.020

**Table 3: Rejection frequencies of the test statistic in Section 3. T = 300 observations**

d = 0.25 and $\alpha = 0.50$			d = 0.50 and $\alpha = 0.50$		
D	$\alpha$	Rejection prob.	d	$\alpha$	Rejection prob.
0.25	0.25	0.1903	0.25	0.25	0.8856
<b>0.25</b>	<b>0.50</b>	<b>0.0524</b>	0.25	0.50	0.5508
0.25	0.75	0.1181	0.25	0.75	0.1775
0.25	1.00	0.6394	0.25	1.00	0.2395
0.50	0.25	0.0630	0.50	0.25	0.5859
0.50	0.50	0.4349	<b>0.50</b>	<b>0.50</b>	<b>0.0524</b>
0.50	0.75	0.9843	0.50	0.75	0.5000
0.50	1.00	1.0000	0.50	1.00	0.9891
0.75	0.25	0.1407	0.75	0.25	0.2354
0.75	0.50	0.9739	0.75	0.50	0.4349
0.75	0.75	1.0000	0.75	0.75	0.9995
0.75	1.00	1.0000	0.75	1.00	1.0000
1.00	0.25	0.4576	1.00	0.25	0.0914
1.00	0.50	1.0000	1.00	0.50	0.9739
1.00	0.75	1.0000	1.00	0.75	1.0000
1.00	1.00	1.0000	1.00	1.00	1.0000
d = 0.75 and $\alpha = 0.50$			d = 1.00 and $\alpha = 0.50$		
D	$\alpha$	Rejection prob.	d	$\alpha$	Rejection prob.
0.25	0.25	0.9997	0.25	0.25	1.0000
0.25	0.50	0.9852	0.25	0.50	1.0000
0.25	0.75	0.8588	0.25	0.75	0.9993
0.25	1.00	0.3456	0.25	1.00	0.8121
0.50	0.25	0.9883	0.50	0.25	1.0000
0.50	0.50	0.5508	0.50	0.50	0.9852
0.50	0.75	0.1045	0.50	0.75	0.5325
0.50	1.00	0.6143	0.50	1.00	0.1891
0.75	0.25	0.9081	0.75	0.25	0.9998
<b>0.75</b>	<b>0.50</b>	<b>0.0524</b>	0.75	0.50	0.5508
0.75	0.75	0.9053	0.75	0.75	0.2813
0.75	1.00	0.9999	0.75	1.00	0.9723
1.00	0.25	0.6461	1.00	0.25	0.9896
1.00	0.50	0.4349	<b>1.00</b>	<b>0.50</b>	<b>0.0524</b>
1.00	0.75	1.0000	1.00	0.75	0.9960
1.00	1.00	1.0000	1.00	1.00	1.0000

In bold the size of the test statistic. The nominal size is 5% and 10,000 replications were used in each case.

**Table 4: Rejection frequencies of the test statistic in Section 3. T = 500 observations**

d = 0.25 and $\alpha = 0.50$			d = 0.50 and $\alpha = 0.50$		
D	$\alpha$	Rejection prob..	d	$\alpha$	Rejection prob.
0.25	0.25	0.2776	0.25	0.25	0.9850
<b>0.25</b>	<b>0.50</b>	<b>0.0489</b>	0.25	0.50	0.7701
0.25	0.75	0.2215	0.25	0.75	0.2573
0.25	1.00	0.8999	0.25	1.00	0.2562
0.50	0.25	0.0644	0.50	0.25	0.8042
0.50	0.50	0.6924	<b>0.50</b>	<b>0.50</b>	<b>0.0489</b>
0.50	0.75	1.0000	0.50	0.75	0.8240
0.50	1.00	1.0000	0.50	1.00	1.0000
0.75	0.25	0.2136	0.75	0.25	0.3514
0.75	0.50	0.9992	0.75	0.50	0.6924
0.75	0.75	1.0000	0.75	0.75	1.0000
0.75	1.00	1.0000	0.75	1.00	1.0000
1.00	0.25	0.6918	1.00	0.25	0.1082
1.00	0.50	1.0000	1.00	0.50	0.9992
1.00	0.75	1.0000	1.00	0.75	1.0000
1.00	1.00	1.0000	1.00	1.00	1.0000
d = 0.75 and $\alpha = 0.50$			d = 1.00 and $\alpha = 0.50$		
D	$\alpha$	Rejection prob.	d	$\alpha$	Rejection prob.
0.25	0.25	1.0000	0.25	0.25	1.0000
0.25	0.50	0.9997	0.25	0.50	1.0000
0.25	0.75	0.9772	0.25	0.75	1.0000
0.25	1.00	0.3726	0.25	1.00	0.9413
0.50	0.25	0.9999	0.50	0.25	1.0000
0.50	0.50	0.7701	0.50	0.50	0.9997
0.50	0.75	0.1792	0.50	0.75	0.7660
0.50	1.00	0.9050	0.50	1.00	0.2183
0.75	0.25	0.9890	0.75	0.25	1.0000
<b>0.75</b>	<b>0.50</b>	<b>0.0489</b>	0.75	0.50	0.7701
0.75	0.75	0.9973	0.75	0.75	0.5990
0.75	1.00	1.0000	0.75	1.00	0.9999
1.00	0.25	0.8536	1.00	0.25	0.9999
1.00	0.50	0.6924	<b>1.00</b>	<b>0.50</b>	<b>0.0489</b>
1.00	0.75	1.0000	1.00	0.75	1.0000
1.00	1.00	1.0000	1.00	1.00	1.0000

In bold the size of the test statistic. The nominal size is 5% and 10,000 replications were used in each case.

**Table 5: Rejection frequencies of the test statistic in Section 3. T = 1000 observations**

d = 0.25 and $\alpha = 0.50$			d = 0.50 and $\alpha = 0.50$		
D	$\alpha$	Rejection prob.	d	$\alpha$	Rejection prob.
0.25	0.25	0.4971	0.25	0.25	1.0000
<b>0.25</b>	<b>0.50</b>	<b>0.0495</b>	0.25	0.50	0.9716
0.25	0.75	0.5221	0.25	0.75	0.4638
0.25	1.00	0.9983	0.25	1.00	0.3605
0.50	0.25	0.0768	0.50	0.25	0.9811
0.50	0.50	0.9571	<b>0.50</b>	<b>0.50</b>	<b>0.0495</b>
0.50	0.75	1.0000	0.50	0.75	0.9950
0.50	1.00	1.0000	0.50	1.00	1.0000
0.75	0.25	0.4080	0.75	0.25	0.6061
0.75	0.50	1.0000	0.75	0.50	0.9571
0.75	0.75	1.0000	0.75	0.75	1.0000
0.75	1.00	1.0000	0.75	1.00	1.0000
1.00	0.25	0.9511	1.00	0.25	0.1658
1.00	0.50	1.0000	1.00	0.50	1.0000
1.00	0.75	1.0000	1.00	0.75	1.0000
1.00	1.00	1.0000	1.00	1.00	1.0000
d = 0.75 and $\alpha = 0.50$			d = 1.00 and $\alpha = 0.50$		
D	$\alpha$	Rejection prob.	d	$\alpha$	Rejection prob.
0.25	0.25	1.0000	0.25	0.25	1.0000
0.25	0.50	1.0000	0.25	0.50	1.0000
0.25	0.75	1.0000	0.25	0.75	1.0000
0.25	1.00	0.4635	0.25	1.00	0.9992
0.50	0.25	1.0000	0.50	0.25	1.0000
0.50	0.50	0.9716	0.50	0.50	1.0000
0.50	0.75	0.4031	0.50	0.75	0.9690
0.50	1.00	0.9992	0.50	1.00	0.3959
0.75	0.25	1.0000	0.75	0.25	1.0000
<b>0.75</b>	<b>0.50</b>	<b>0.0495</b>	0.75	0.50	0.9716
0.75	0.75	1.0000	0.75	0.75	0.9571
0.75	1.00	1.0000	0.75	1.00	1.0000
1.00	0.25	0.9900	1.00	0.25	1.0000
1.00	0.50	0.9571	<b>1.00</b>	<b>0.50</b>	<b>0.0495</b>
1.00	0.75	1.0000	1.00	0.75	1.0000
1.00	1.00	1.0000	1.00	1.00	1.0000

In bold the size of the test statistic. The nominal size is 5% and 10,000 replications were used in each case.

**Table 6: Estimates of  $\alpha$  and d for each country**

Country	$\alpha$	d	Test Statistic
Argentina	0.57	0.21	0.0054
Mexico	1.00	0.13	1.0599
Venezuela	0.09	1.92	0.1002
Brazil	0.06	1.92	0.0983
Colombia	0.87	0.33	0.0234

**Table 7: Impulse response functions of the selected models for each country**

	Mexico	Colombia	Argentina	Venezuela	Brazil
1	<b>0.13000000</b>	<b>0.28710002</b>	<b>0.11969999</b>	<b>0.17280000</b>	<b>0.11520000</b>
2	<b>0.07345000</b>	<b>0.16610171</b>	0.04127854	0.02270592	0.01009152
3	<b>0.05214950</b>	<b>0.11223494</b>	0.01733286	0.00267022	0.00079118
4	0.04080698	<b>0.08128896</b>	0.00792848	0.00029559	5.8389E-05
5	0.03370657	<b>0.06124473</b>	0.00380520	3.1498E-05	4.1479E-06
6	0.02881912	0.04733299	0.00188338	3.2695E-06	2.8700E-07
7	0.02523731	0.03723821	0.00095237	3.3290E-07	1.9500E-08
8	0.02249275	0.02968398	0.00048925	3.3400E-08	1.3000E-09
9	0.02031845	0.02390253	0.00025439	3.3000E-09	1.0000E-10
10	0.01855075	0.01940192	0.00013355	3.0000E-10	0
11	0.01708355	0.01585154	7.0655E-05	0	0
12	0.01584500	0.01302085	3.7622E-05	0	0
13	0.01478460	0.01074431	2.0142E-05	0	0
14	0.01386584	0.00890020	1.0833E-05	0	0
15	0.01306162	0.00739731	5.8495E-06	0	0
16	0.01235140	0.00616617	3.1696E-06	0	0
17	0.01171930	0.00515314	1.7227E-06	0	0
18	0.01115286	0.00431636	9.3880E-07	0	0
19	0.01064218	0.00362281	5.1290E-07	0	0
20	0.01017924	0.00304626	2.8080E-07	0	0

The impulse responses are obtained on the basis of the coefficients of the  $MA(\infty)$  representation of the series given by Equation (7). Significant values (at the 95% level) appear in bold.