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Decadence and Military Overstretch: Modeling the Dynamics of National Decline

Yong Jin Kim*

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Summary

The paper presents a theoretical model and its simulation to explore the process of the fall of nations by focusing on moral and cultural decline, as well as military overstretch. In the model, a representative agent divides his time among military spending, material production, the consumption of scientific ideas and that of religious ideas. Both religious and scientific capital accumulate through learning-by-doing mechanism of the consumption of religious ideas and that of scientific ideas, respectively, at the national level. ‘Religious’ means ‘decadent’ or ‘nonproductive emotional’. Scientific capital boosts income, while religious capital does not. And wealthier nations experience increased decadence and military overstretch, which in turn accelerates moral and cultural decline, increasing the growth rate of religious capital and reducing that of scientific capital and income. The pseudo saddle path with a higher substitutability between religious and scientific idea consumption leads to a steady state like equilibrium. But it endogenously forks into one of two extreme potential outcomes, a religion (decadence) dominated path with slow income growth or a science dominated path with rapid growth, based on the initial conditions and policies. Affluent nations with a high level of decadence or of military overstretch are more prone to rapid decline after peaking, forking into a religion dominated path with slow income growth. The paper suggests that effective policy interventions, such as reducing decadence and military overstretch, increasing the consumption of scientific ideas, and maintaining balanced military spending, could help guide nations toward a science dominated path with sustained growth. These policies could prevent nations from veering into a religion dominated path with slow income growth.

Key Words: the rise and fall of nations, decadence, overstretch, saddle path equilibrium, learning by doing.

JEL Classification Number: H56, N10, O43, D91

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I. Introduction

Hard times create strong men.
Strong men create good times.
Good times create weak men.
And weak men create hard times.¹

Since recent Ukraine-Russian and Hamas-Israelite wars in addition to the US-China hegemony war increased the doubt whether the US dominant global hegemony as well as its power and willingness is waning, it has become one of the very interesting research topics how nations rise and fall.² In fact, the cyclical nature of the rise and fall of nations has been a focal point of historical inquiry for centuries. While quite a few scholars have proposed various frameworks to understand the factors that drive these cycles, decadence and military overstretch are often cited as critical factors especially in the decline phase.³

Considering these skyrocketed geopolitical risks, understanding the dynamics that lead to the decline of nations and civilizations is crucial for predicting their future trends and formulating government policies to avoid potential downfalls. In this context, after surveying various theories mainly through the lens of these concepts, we will propose a theoretical model of decadence and military overstretch in the declining process of nations, and its simulation to verify whether it can emulate the main features of the historical process. Decadence, defined as a state of moral or cultural (civic virtue) decline and characterized by excessive indulgence in pleasure and luxury, in addition to military overstretch beyond economic sustainability, plays a crucial role in undermining the foundational values and structures that sustain societies.

In the model economy, one representative agent allocates the endowed one unit of time, among military spending, material production, time consumption of scientific ideas and that of religious ideas. And the substitutability between the two idea consumptions is greater than one. Religious (resp., scientific) capital accumulates through the aggregate experiences on the national level of consuming religious (resp., scientific) ideas, through learning-by-doing. ‘Religious’ means ‘decadent’ or ‘nonproductive emotional’. Scientific capital increases income, while religious capital does not.

The positive feedback mechanism of habit formation embedded in the model proceeds as follows. The higher ratio of religious capital (resp., scientific capital) to the other capital (first step), the relatively cheaper religious ideas (resp., the relatively cheaper scientific ideas and higher income) agents enjoy (second step), raising religious idea consumption (resp., scientific idea consumption) relatively more through substitution (third step), thus increasing religious capital (resp. scientific capital) higher than scientific capital (resp., religious capital) raising the capital ratio due to the learning-by-doing

¹ This aphorism is cited from “Those Who Remain” written by G. Michael Hopf in 2016.

² The concept is illustrated by recent works such as Turchin [2023], Hanson [2023], Strauss and Howe [1997], Dalio [2021], and Cline [2014], among others.

³ “Many historians have postulated reasons for the collapse of the Western Roman Empire. Their conclusions usually belong in two broad schools: (1) external factors, such as military threats and barbarian invasions or (2) internal factors, such as a decline in “civic virtue” and military and economic capability. Most historians believe that the fall was due to a combination of both internal and external factors but come down more heavily on one or the other as the most important cause of the fall. Modern scholarship has introduced additional factors such as climate change, epidemic diseases, and environmental degradation as important reasons for the decline. Some historians have postulated that the Roman Empire did not fall at all, but that the “decline” was instead a gradual, albeit often violent, transformation into the societies of the Middle Ages.” (“Historiography of the fall of the Western Roman Empire,” Wikipedia)

mechanism (fourth step). This process suppresses (resp., enhances) income growth. Now, the process will repeat going to the first step of this positive feedback loop, reinforcing its effect.

The model also assumes that as nations become richer, their decadence and military overstretch increase, and that decadence increases the growth rate of religious capital while military overstretch decreases the growth rate of scientific capital and income. Thus, as nations become richer, the ratio of the growth rate of religious capital to that of scientific capital increases. Thus, this process continues going to the first step of the above positive feedback loop. Thus, as nations become rich, decadent and complacent, they can be trapped into the above positive feedback loop declining their economies deeper and deeper.

Based on the model, the paper characterizes its dynamics and executes simulations at and after reaching the pseudo steady state of the pseudo saddle path equilibrium, to explore the declining process of nations after their peaks.⁴ The major findings are as follows. First, there exists a unique pseudo saddle path equilibrium which leads to a pseudo steady state and afterwards forks into one of the two extreme paths, a scientific idea dominating path with rapid income growth or a religious idea dominating path with very slow growth. Second, nations having experienced affluence with a high level of decadence and military overstretch are more likely to fork into a religious idea dominating path with very slow growth without government policies. Last, effective policies, targeting to decrease the level of decadence and military overstretch and to increase the consumption of scientific ideas and military spending to an appropriate level, can lead nations into a favorable saddle path equilibrium forking into a scientific idea dominating path with rapid income growth.

The paper proceeds as follows. Section II summarizes various theories about the rise and fall of nations. It also explores five specific theories in detail on the mechanism of decadence and military overstretch in the falling process of nations. Section III presents a theoretical model of decadence and military overstretch, and its simulation to explore the mechanism of leading nations to decline endogenously after peaking, and to help formulate appropriate policies. And Section IV concludes.

II. Survey of Various Theories on the Rise and Fall of Nations

This section reviews theories about the rise and fall of nations, focusing on cultural aspects like decadence, lack of social cohesion⁵, and military overstretch, which later helps construct the theoretical model in Section III.

⁴ The pseudo saddle path equilibrium means: Even though it looks like a usual saddle path equilibrium with its steady state, after reaching its steady state, it will endogenously fork into one of these two extreme paths, depending very sensitively on its initial conditions and policies. We also name 'pseudo steady state' instead of 'steady state', because the steady state like equilibrium endogenously violates the steady state conditions right after reaching this equilibrium. We will explain these concepts in detail later.

⁵ For a more detailed classification of these theories along several different dimensions, refer to Table 1. This section and the next are written with the help of ChatGPT4o. The following quote from Ferguson [2013] gives an overview. "The reasons for decline are fourfold: a failure of institutions, economics, politics, and society. These failures are often rooted in complacency after periods of prolonged prosperity." (Introduction, p. 14 Ferguson [Penguin Press, 2013]).

2.1. Categorizing Theories along the Factors Causing Nations Lose Wars.⁶

The decline of a nation usually becomes visible and is defined by its defeat of major wars. In this context, this overview categorizes the theories in the dimension of factors affecting the possibility of losing wars.

1. Loss of Income and Wealth

The decline of nations is often triggered by defeat in war, which can be attributed to the erosion of wealth and income. Scholars like Kennedy [1987] argue that nations tend to overreach, stretching their resources thin, while others like Tainter [1988] and Cline [2014] emphasize the role of societal complexity. As a nation becomes more complex, maintaining its infrastructure and systems becomes increasingly difficult, leading to inefficiency and collapse. Periods of sustained peace and prosperity can create asset bubbles, and when these bubbles burst, the resulting financial collapse not only wipes out wealth but also fractures society, as noted by Dalio [2020].

2. Decadence and Loss of Asabiyyah

Dalio [2020] also argues that during these peaceful times, societies become increasingly decadent, indulging in luxury and pleasure at the expense of long-term stability. This decadence weakens social cohesion-what Khaldun [1377] called "asabiyyah"-and leads to reduced investment in technology, further diminishing a nation's competitiveness. Other historians and thinkers like Glubb [1976] and Turchin [2010] besides Dalio [2020] have similarly observed that when societies fall into decadence, they lose the drive that once made them strong and innovative.

3. Underestimation of Black Swan Events

Furthermore, there is often a tendency to underestimate the possibility of catastrophic "black swan" events, such as the outbreak of war. Hanson [2001] has pointed out that the destruction caused by war is frequently much greater than anticipated. This underestimation is compounded by generational differences in the perception of war risk. Khaldun [1377] observed that societies tend to lose their martial vigor over generations, while more recent theorists like Turchin [2010], Strauss and Howe [1997], and Dalio [2020] argue that these cycles of generational change matter in shaping attitudes toward conflict.

4. Erosion of Trust in Public Goods and Defense

Social cohesion also erodes as internal conflicts escalate, leading to declining trust in the state's ability to provide essential public goods, including national defense. Fog [2023] suggested that when societies are divided, people become less willing to contribute to the defense effort, leading to a moral hazard problem. This often results in a shift toward more authoritarian governance structures. The use of mercenaries, as discussed by Gibbon [1776] and Hanson [2001], is a symptom of this decline, as states become less able to rely on their own citizens for defense. At the same time, the growing wealth gap between labor and capital leads to further social tensions. Turchin [2010] and Goldstone [1991] identified "elite overproduction"-where too many elites compete for limited

⁶ In the domain of economic growth theory, quite a few papers attempt to verify whether one country's economic growth is affected by its cultural aspect, proxied mainly by 'social capital' or 'trust' constructed using survey data covering after WWII. The growth mechanism of social capital or trust operates mainly through increasing trade, cooperation, saving, innovation, positive attitude towards entrepreneurship and others. These papers include, for example, Knack and Keefer [1997], Guiso et al. [2006, 2009], Sapienza et al. [2006], Alesina and Giuliano [2013], Gorodnichenko and Roland [2017], and others. With their data set covering only periods after WWII, these papers focus mainly on how cultural aspects (proxied by trust or social capital) affect economic growth, not what causes the rise and fall of nations, since the rise and fall cycle lasts much longer than their data coverage (following Glubb [1976] it is about 250 years.). One of interesting results of these growth-oriented papers is that individualism is more efficient in terms of economic growth, innovation and productivity than collectivism, as Alesina and Giuliano [2013] and Gorodnichenko and Roland [2017] show. In contrast, many papers that focus on the rise and fall of nations insist that extreme individualism, bringing forth decadence or lack of social cohesion at their peak of prosperity, is one of the main causes leading to their decline and calling for higher level of civic virtue or social solidarity to avoid the decline especially in the face of intensive warfare. In addition, Fog [2017] and Turchin [2022, 2023] show that nations as well as their religions tend to become more authoritarian and hierarchical, when they are facing a higher intensity of warfare or external collective threat.

resources-and demographic pressures as key drivers of societal unrest.

5. Fiscal Mismanagement and Military Overstretch

Another critical factor in national decline is the mismanagement of state finances. Goldstone [1991] observed that elites often waste public resources on personal luxuries, weakening the state's ability to respond to crises. Meanwhile, Kennedy [1987] pointed to excessive military spending as another strain on state finances, which can hasten the decline of even the most powerful nations.

6. External Shocks

Natural disasters and other external shocks-such as earthquakes, famines, droughts, pandemics, and large-scale population movements-can greatly influence a nation's income and wealth.

2.2. Categorizing Theories along the Factors Causing Nations Collapse

Various theories provide perspectives on what causes civilizations to collapse, from moral decay to military overreach. Below are the main categories.

Moral and Ethical Decline⁷: Gibbon [1776–1788] attributes the fall of the Roman Empire to moral decay, political corruption, and military weakness. Glubb [1976] identifies stages in an empire's life cycle, with moral decline being a key factor in the final stage.

Social Cohesion and Fragmentation: Khaldun's [1377] asabiyyah theory emphasizes that social cohesion is vital for the rise of states. As societies become wealthy, cohesion weakens, leading to eventual collapse. Peter Turchin's [2003, 2010] cliodynamics uses demographic pressures and elite competition to explain cycles of growth and decline.

Economic and Military Factors: Kennedy [1987] argues that great powers fall when their military ambitions outstrip economic capabilities. Ray Dalio's Big Cycle theory [2020] examines how economic imbalances, excessive debt accumulation and the resulting bubble burst, and shifts in global power dynamics lead to rise, peak, and collapse.

Institutional Effectiveness: Tainter [1988] suggests that increasing societal complexity leads to diminishing returns, causing collapse. Ward-Perkins [2005] argues that the Roman Empire's fall was abrupt and catastrophic due to its high level of societal complexity.

2.3. Several Detailed Mechanisms of the Rise and Fall of Nations

This subsection narrows down to five critical theories, quantitatively analyzed, related to moral and ethical decline,

⁷ The study by Heckman et al. (2006), using extensive micro data, investigates the impact of non-cognitive skills, such as attitudes and behaviors, on individuals' future achievements, emphasizing their significance alongside health and cognitive skills in determining educational and labor market outcomes. The study highlights that traits such as motivation, persistence, and self-control are crucial in shaping economic outcomes, often proving to be as important as cognitive abilities in predicting success across various domains. Non-cognitive skills play a significant role in education, as attitudes like self-discipline and a positive outlook towards learning greatly influence educational attainment. Drawing an analogy from development theory at the macro level which emphasizes three factors such as geography, institution, and culture, the noncognitive skill of individuals at the micro level corresponds to culture at the macro level among the three. Similarly, health and cognitive skills at the micro level correspond to tangible aspects like geography and institution at the macro level of development theory. In short, this paper highlights the importance of cultural factors-such as civic virtue and social cohesion-for a nation's success at the societal level, much like Heckman et al. (2006) emphasizes the role of qualities like motivation, persistence, and self-control in determining individual success.

lack of social cohesion, and economic and military overstretch.

2.3.1. Ibn Khaldun's Asabiyyah

Ibn Khaldun's concept of asabiyyah (social cohesion), developed in the 14th century (1377), explains how strong group solidarity allows for the rise of dynasties and states. In time, prosperity weakens asabiyyah, causing internal divisions and weakening the state's ability to resist external threats. Eventually, this erosion leads to a state's decline and collapse.

In the 14th century, Arab historian and philosopher Khaldun introduced a concept that would become central to his understanding of the rise and fall of states: Asabiyyah. This term, which can be translated as "social cohesion" or "group solidarity," refers to unity, loyalty, and mutual support within a group, whether based on kinship, tribal affiliation, or shared purpose.

Khaldun's theory posits that the initial stages of a dynasty or empire are marked by strong Asabiyyah. A group bonded by this powerful social cohesion can unite to conquer territories and establish new ruling entities. This unity provides the energy and determination needed to overcome challenges and achieve collective goals.

However, as the state prospers and becomes more established, the sense of Asabiyyah begins to erode. The initial group that rose to power, hardened by shared experiences and struggles, gives way to later generations who grow accustomed to luxury, comfort, and individual interests. This shift leads to internal divisions and weakens the state's collective strength.

Table 2-1 captures the cyclical process outlined by Khaldun, highlighting the role of Asabiyyah at each stage.

2.3.2. Peter Turchin's Cliodynamics

Peter Turchin's theory of cliodynamics provides a mathematical and historical framework for understanding the rise and fall of civilizations through socio-political dynamics. In his works from 2003 and 2010, Turchin focuses on the interplay of demographic pressures and elite competition as key drivers of these cyclical patterns. Central to his theory is the idea that societies are not static entities but are subject to recurring cycles of growth, stability, and collapse. By using historical data and mathematical models, Turchin illustrates that periods of societal expansion are often followed by stagnation and decline, driven by internal pressures rather than external threats. Table 2-2 captures this cyclical process, highlighting the role of demographic pressure, resource scarcity and elite overproduction at each stage.

One of the key factors in Turchin's model is demographic pressure, which refers to the strain that population growth places on a society's resources. As the population grows, so too does its demand for land, food, and economic opportunities. In time, this leads to resource scarcity, pushing different segments of society into conflict. For example, landowners might fight for control over shrinking agricultural space, while common citizens compete for jobs in an increasingly constrained labor market. This scarcity is a critical precursor to social instability, as the rising tension between resource demand and availability fuels discontent, protest, and potentially revolt.

Elite competition is another central aspect of Turchin's cliodynamics. As societies grow and prosper, they tend to produce a surplus of elites—people who hold power, influence, or wealth. However, the opportunities for these elites to maintain their status do not grow at the same rate. In time, the competition among elites for political and economic dominance intensifies, leading to factionalism, internal conflict, and, eventually, a breakdown in the cohesion necessary for stable governance. According to Turchin, this overproduction of elites exacerbates the pressure on the societal structure, as different factions vie for control, often leading to violent conflict.

According to Turchin, these internal pressures—resource scarcity and elite competition—push societies toward periods of instability and eventual collapse. He points to examples such as the fall of the Roman Empire and the

decline of medieval European kingdoms to illustrate how these cycles have recurred throughout history. By understanding these dynamics, Turchin suggests that it may be possible to foresee or mitigate societal collapse, though reversing these cycles is often difficult due to the complexity of socio-political dynamics.

2.3.3. Agner Fog's Regality Theory

Agner Fog's regality theory [2017] posits that societies adapt their political structures in response to external threats. Societies facing threats become more hierarchical and authoritarian (regal), while peaceful societies adopt more egalitarian structures (kungic).⁸ This adaptability plays a key role in the rise and fall of empires, driven by feedback loops of growth, expansion, and eventual collapse due to overpopulation and inequality.

Here is a related quote from Fog [2017]. "Combining regality theory with cultural selection theory and historical dynamics, we may summarize the main factors leading to the rise and fall of empires. The growth of an empire can be explained by a positive feedback process involving war, growing regality, increasing food production, population growth, improved weapons technology, military strength, and expanding territory. The decline may involve the following factors: loss of regality when the empire has reached the limits to its growth, overpopulation, poverty, and famine, a growing elite, growing inequality, and conflicts between elite and commoners as well as between different factions within the elite, collapse of the economy and weakening of the state. (p.71, Fog [2017])

Table 2-4 shows that the dynamics of empire rise and fall result in cycles of growth and decline, with empires potentially experiencing several such cycles before complete disintegration, as the table explains. The interplay between external threats and internal structures is central to understanding the rise and fall of empires.

2.3.4. Paul Kennedy's Rise and Fall of Great Powers

Kennedy's "The Rise and Fall of the Great Powers" [1987] focuses on the balance between economic strength and military commitments. Great powers decline when their military ambitions exceed their economic capacity, a concept known as "imperial overstretch." Nations expand their influence through economic growth, but excessive military spending drains resources, leading to decline.

Kennedy's work provides a comprehensive analysis of the economic and military dynamics that influence the trajectories of great nations. By highlighting the importance of maintaining a balance between economic strength and military commitments (guns and butter) and warning against the dangers of imperial overstretch, Kennedy's theory offers valuable insights into the rise and fall of historical empires and contemporary global powers. Kennedy outlines the typical stages of the rise and fall of great powers in Table 2-5. His work underscores the need for prudent economic and military policies to sustain national power and influence in an ever-competitive international system.

2.3.5. Dalio's Big Cycle Theory

Dalio's "big cycle" theory examines the rise and decline of dominant powers through economic cycles, political stability, and educational systems. He argues that robust education, economic productivity, and political stability mark the ascent of an empire. In time, however, as wealth and power concentrate, social and economic inequalities increase, leading to internal conflicts and eventual decline (Dalio 2020). Dalio identifies declining education and rising debt as early indicators of an empire's downfall. While his framework offers valuable insights into the economic underpinnings of societal cycles, it has been criticized for its deterministic outlook and insufficient consideration of cultural and ideological factors.

⁸ Table 2-3 compares characteristics of Regal (Warlike) to those of Kungic (Peaceful) Societies.

Dalio's study over the past 500 years indicates that education-comprising skills, knowledge, and cultural values such as determination, civility, and work ethics-initiates the rise of other factors contributing to empires' growth. A decline in education often foreshadows an empire's fall. He explains a cyclical pattern, the "big cycle," involving the emergence of a new dominant power and monetary system, leading to a fresh world order. This cycle occurs universally and timelessly.

Dalio's Big Cycle theory offers a comprehensive framework for understanding the long-term dynamics of economic and political change as Table 2-6 shows.

2.4. Survey of Literature Contending the Existence of Long Complacent Peak Periods followed by the Rapid Decay of Nations

The concept that empires experience a prolonged period of prosperity followed by a rapid decline is a well-documented theme in historical literature. Here are some renowned works and theories that support this idea:

Gibbon's seminal work, *The History of the Decline and Fall of the Roman Empire*, describes how the Roman Empire experienced a long period of prosperity and complacency before its decline. Gibbon attributes the fall to a combination of internal decay and external pressures. He famously notes that Rome's decline was not abrupt but the result of gradual decay following its peak in power and prosperity.

Kennedy's *The Rise and Fall of the Great Powers* examines how economic and military factors contribute to the rise and decline of major powers from 1500 to 2000. Kennedy argues that great powers often reach a "complacent" peak where they overextend themselves militarily and economically, leading to a decline. This military overstretch is typically a result of prolonged success and prosperity, which breeds complacency and neglect of emerging challenges.

There are several recent studies arguing the slow decay of Rome, rather than an abrupt one, as follows:

Cameron [1993] provides a detailed analysis of the Roman Empire's transition from a period of prosperity to decline, focusing on the gradual changes in political, economic, and social structures. And she emphasizes the slow decay due to internal weaknesses and external pressures rather than a sudden collapse.

Here is a quote from Peter Heather saying, "The empire's fall was the culmination of a process that had been centuries in the making," which reflects the idea that the collapse of the Western Roman Empire was not the result of a single event or a short-term crisis but rather the outcome of a long, complex, and multifaceted process that evolved over several centuries. "Heather argues that the Empire's downfall was not an abrupt fall from grace, but a gradual erosion exacerbated by relentless barbarian invasions, economic instability, and political fragmentation." (BooKey, Book Review).

Rostovtzeff [1926] provides a comprehensive analysis of the gradual economic and social changes that contributed to the decline of the Roman Empire, and he emphasizes the slow decay of the economic foundations and social structures over time. "In his view, an alliance of the rural proletariat with the military in the third century A.D. destroyed the beneficent rule of an urban bourgeoisie." (p.18, Bowersock [2024]).

Luttwak [1976] explores the strategic and military aspects of Rome's gradual decline, highlighting the interplay between military overstretch and internal decay. "Defense-in-depth is the term used by American political analyst Luttwak (born 1942) to describe his theory of the defensive strategy employed by the Late Roman army in the third and fourth centuries AD. Luttwak's *Grand Strategy of the Roman Empire* (1976) launched the thesis that in the third and early fourth centuries, the Imperial Roman army's defense strategy mutated from "forward defense" (or "preclusive defense") during the Principate era (30 BC-AD 284) to "defense-in-depth" in the fourth century. "Forward-" or "preclusive" defense aimed to neutralize external threats before they breached the Roman borders: the barbarian regions neighboring the borders were envisaged as the theatres of operations. In contrast, "defense-in-depth" would not attempt to prevent incursions into Roman territory but aimed to neutralize them on Roman soil-in effect turning border provinces into combat zones." (Wikipedia).

Southern [2001] emphasizes the prolonged nature of Rome's decline, highlighting how multi-factors such as political, social, military, and economic ones interacted in to weaken the empire.

In contrast, Ward-Perkins, in his book *The Fall of Rome and the End of Civilization*, argues that the decline of the Roman Empire was abrupt and catastrophic rather than gradual and transformative. He challenges the revisionist view that emphasizes the continuity and adaptation of Roman institutions, instead presenting evidence that the collapse of Rome led to a significant breakdown in material culture and economic systems.

Ward-Perkins also stresses the cultural and intellectual decline that accompanied the fall of Rome. He argues that the abrupt loss of Roman literacy, technological skills, and educational systems had a profound impact on the subsequent medieval world. The disappearance of Roman urban centers and the reduction in the quality of life illustrate the destructive effects of the empire's collapse.

Ward-Perkins highlights how the invasions by barbarian groups such as the Goths and Vandals resulted in the destruction of the sophisticated Roman economy, leading to a marked decline in living standards, urban life, and infrastructure. He uses archaeological data to demonstrate a rapid decrease in the production and distribution of goods, which indicates a severe disruption of trade networks and economic stability.

These works except Ward-Perkins illustrate the common pattern of empires reaching a peak of prosperity for rather long periods of time, often leading to complacency and eventual decline. This cyclical pattern emphasizes the importance of adaptability and resilience in maintaining long-term success.

III. A Model of Decadence and Military Overstretch

Before presenting and solving the model formally, we will provide its bird's-eye view story.

3.1. A Brief Story Line of the Model

The model describes a dynamic economy in which individuals allocate their endowed unit of time among three activities: spending time in a library $h(t)$, spending time in a temple $s(t)$, and producing consumption goods $u(t)$. The size of the library $H(t)$ and the size of the temple $R(t)$ influence how much time people spend in each institution. Specifically, the larger an institution, the more time people tend to allocate to it, provided a certain condition is met—namely, $\varepsilon > 1$, which implies a higher degree of substitutability between $h(t)$ and $s(t)$. This condition is assumed in the paper. In essence, people are drawn to larger and more influential institutions because they offer better services. The more time individuals spend in a growing institution, the faster that institution expands.

As income increases, we assume that both decadence and military overstretch also rise very slowly. Decadence is conceptualized as a decline in productivity driven by overindulgence in pleasure and luxury, while military overstretch refers to productivity losses associated with territorial expansion. Increased decadence accelerates the growth of the temple, whereas military overstretch slows the growth of the library. Since income is positively linked to the size of the library, but not to the size of the temple, a larger library is associated with higher income levels.

Public spending on national defense introduces another important dynamic. Although allocating more time and resources to defense may initially reduce current income by decreasing time spent on productive work, it enhances the growth rate of the library by providing better protection against external threats from neighboring countries. This, in turn, promotes higher future income growth.

The model features a positive feedback loop: as an institution becomes larger, individuals spend more time there, which accelerates its growth and further increases its size, perpetuating this cycle. This is referred to as a 'habit formation' mechanism or a 'positive feedback mechanism'.⁹

The system can reach a balanced steady state-referred to as a "pseudo steady state" or a "pseudo saddle path equilibrium"-in which the time allocated to the library and the temple remains constant, and both institutions grow at the same rate. At this equilibrium, the relative size of the library to that of the temple stabilizes. However, the model predicts that this balance is inherently unsteady. As income continues to grow, decadence and military overstretch also increase slowly and persistently, eventually leading to a 'bad equilibrium' where the temple expands disproportionately compared to the library at an accelerating pace, resulting in a rapid decline of income.⁹

To prevent this adverse outcome-where the dominance of the temple suppresses income growth-the model recommends policies that boost the size or growth rate of the library relative to the temple. Such policies should aim to curb the growth of decadence by discouraging excessive time spent at the temple, promoting greater engagement with the library, moderating the cultural tendency toward military overstretch, and appropriately increasing defense spending to safeguard and accelerate the library's growth.

3.2. A Formal Setup of the Model

**Hell is full of good meanings and wishes,
and heaven is full of good works.**

The model is a modified extension of Kim and Lee [2015]. All identical agents over a unit measure in this model economy are endowed with one unit of time. And m units out of the endowed time are allocated to national security as obligatory military service imposed by government.

Facing the exogenously given aggregate level of religious capital and that of scientific capital, economic agents allocate the remaining $1 - m$ units of time (net of military spending) across three different activities to maximize their utilities. They spend time on consuming scientific and religious ideas, in addition to working for material production and consumption to satisfy their basic biological needs. More specifically, they spend a certain fraction of time ($h(t)$) adopting and consuming scientific ideas from the society's scientific capital ($H(t)$), and another fraction ($s(t)$) adopting and consuming religious ideas from the society's religious capital ($R(t)$). Here 'religious' means 'decadent' or 'nonproductive emotional'. Finally, the remaining time ($u(t)$) is supplied for final goods production ($Y(t)$) which is identical to material consumption ($c(t)$) without saving or investment in a closed economy.

A nation's scientific and religious capital accumulate through aggregate learning-by-doing, based on the Uzawa-Rosen's formula. Economic agents regard military spending m and the paths of these aggregate religious and scientific capital $\{H(t), R(t); t = t_0, \infty\}$ as given exogenously.¹⁰ Given the military spending and these exogenous capital paths, the decentralized competitive market problem can

⁹ <Figure 1> intuitively explains a mechanism of 'a protracted peak followed by a rapid decline'.

¹⁰ The assumption that individuals do not consider investing in capitals regarding them exogenously given is to capture not only agents' myopic behavior in the competitive market, but also the fact that intellectual property right is not protected in the long-term frame of this model.

be described by the central planner's one period problem of ¹¹

$$\begin{aligned}
& \max_{\{c(t), s(t), h(t)\}_{t=t_0}^{\infty}} \int_{t_0}^{\infty} e^{-\rho t} U(c(t), R(t)s(t), H(t)h(t)) dt \\
\text{s. t. } & c(t) = A(t)u(t)H(t)^{\kappa} = A(t)(1 - m - s(t) - h(t))H(t)^{\kappa} \\
& \dot{R}(t) = \xi s(t)R(t)^{12} \\
& \dot{H}(t) = \eta h(t)H(t) \\
& u(t) + s(t) + h(t) = 1 - m, \text{ and} \\
& U(c(t), R(t)s(t), H(t)h(t)) = [(c(t))^{\frac{\sigma-1}{\sigma}} + \beta\{((R(t)s(t))^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(H(t)h(t))^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}\}^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}},
\end{aligned}$$

where ρ represents the instantaneous time discount rate of the preference, ξ the accumulation efficiency of religious capital, η that of scientific capital¹³, $H(t)$ the society's scientific capital, $R(t)$ the society's religious capital, $h(t)$ the fraction of time to adopt and consume scientific (intellectual) ideas from the society's scientific capital, $s(t)$ the fraction of time to adopt and consume religious (divine) ideas from the society's religious capital, m the national military spending, $u(t)$ the remaining time supplied for the final goods production, $A(t)$ the production efficiency assumed as constant A , $c(t)$ material consumption, $\sigma > 1$ the elasticity of substitution between material consumption and abstract (religious and scientific) idea consumption, ε that between religious and scientific idea consumption, $Y(t)$ the final goods production, and we assume that $c(t) = Y(t) = (1 - m - s(t) - h(t))K(t) = (1 - m - s(t) - h(t))H(t)^{\kappa}$, where $K(t)$ represents the production capital, assuming that scientific capital $H(t)$ will be transformed into production capital through the relationship of $K(t) = H(t)^{\kappa}$ with $0 < \kappa < 1$.¹⁴

This section provides only the main results in the form of lemmas and propositions whose proofs are in the appendices.

3.3. Solving the Model

Firstly, we will solve a simple model without military spending nor the effect of decadence and military overstretch. Later, we will solve an extended model by sequentially adding military spending initially and next the accumulation efficiencies of religious and scientific capital described as functions of m , $H(t)$ or $R(t)$. However, the FOCs of the extended model are identical to those of the simple model without military spending nor decadence or military overstretch. In the case of the model with military spending, it is due to the changing of variables, while with decadence and military overstretch, it is because agents treat the capital stocks as exogenously given, irrespective of their functional forms of accumulation efficiencies, when they maximize their utilities.

3.3.1. The First Order Conditions

¹¹ We can assume that their preference changes from $U(c(t))$ to $U(c(t), R(t)s(t), H(t)h(t))$ when $c(t)$ surpasses \bar{c} , according to Maslow [1943, 1950]. Thus, after $c(t)$ surpasses a certain level of consumption \bar{c} , they maximize the problem with the preference of $U(c(t), R(t)s(t), H(t)h(t)) = [(c(t))^{\frac{\sigma-1}{\sigma}} + \beta\{((R(t)s(t))^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(H(t)h(t))^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}\}^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$ instead of $U(c(t))$, appreciating the consumption of abstract ideas. And we set $\gamma = \varphi = 1$ later to simplify the problem.

¹² This equation implies an aggregate learning-by-doing on a national level. This concept can be described more exactly by $\dot{R}(t) = \xi \left(\int_0^1 s(t; i) di \right) R(t)$, where $s(t; i)$ is an i -th agent's time consumption of religious ideas equal for all agents $i \in [0, 1]$. Thus, since $\int_0^1 s(t; i) di$ represents the aggregate experiences of consumption at t on a national level, using $s(t; i) = s(t)$, we will have $\dot{R}(t) = \xi \left(\int_0^1 s(t) di \right) R(t) = \xi s(t)R(t)$. The same logic applies to $\dot{H}(t) = \eta h(t)H(t)$.

¹³ We assume later in the paper that these accumulation efficiency parameters (ξ and η) can be affected by scientific capital $H(t)$ and religious capital $R(t)$.

¹⁴ This assumption is necessary to focus only on the dynamics between scientific and religious capital accumulation.

The Hamiltonian \mathbf{H} is represented by

$$\begin{aligned}\mathbf{H} &= [(A((1-m-s-h)H^\kappa))^{\frac{\sigma-1}{\sigma}} + \beta\{((Rs)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(Hh)^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}\}^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} \rightarrow \\ &= (1-m)[\{A(1-s'-h')H^\kappa\}^{\frac{\sigma-1}{\sigma}} + \beta\{((Rs')^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(Hh')^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}\}^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}\end{aligned}$$

where $x' = \frac{x}{1-m}$ for $x \in \{c, v, s, h, \mathbf{H}, \omega\}$. The maximization problem of the model given above becomes equal to that of the simplified model without military spending through changing variables by $x' = \frac{x}{1-m}$. We omit time subscripts when not invoking confusion.

$$(1) \quad \mathbf{H}' = \frac{\mathbf{H}}{1-m} = [\{A(1-s'-h')H^\kappa\}^{\frac{\sigma-1}{\sigma}} + \beta\{((Rs')^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(Hh')^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}\}^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$$

$$(2) \quad \frac{\partial H'}{\partial s'} = 0: v'^{\frac{1}{\sigma}} c'^{\frac{-1}{\sigma}} A H^\kappa - \beta v'^{\frac{1}{\sigma}} \omega'^{\frac{-1}{\sigma}} \omega'^{\frac{1}{\varepsilon}} (Rs')^{\frac{-1}{\varepsilon}} R = 0,$$

where $c = Y = A(1-m-s-h)H^\kappa \rightarrow c' = \frac{c}{1-m} = A(1-s'-h')H^\kappa$, $v' = [c'^{\frac{\sigma-1}{\sigma}} + \beta\{((Rs')^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(Hh')^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}\}^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$, and $\omega' = ((Rs')^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(Hh')^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}$.

$$(3) \quad \frac{\partial H'}{\partial h'} = 0: v'^{\frac{1}{\sigma}} c'^{\frac{-1}{\sigma}} A H^\kappa - \gamma \beta v'^{\frac{1}{\sigma}} \omega'^{\frac{-1}{\sigma}} \omega'^{\frac{1}{\varepsilon}} (Hh')^{\frac{-1}{\varepsilon}} H = 0$$

$$(4) \quad \dot{R}(t) = \xi s(t) R(t) \Rightarrow \dot{R} = \xi s(t) = \xi'' s(t)', \text{ where } \xi'' = (1-m)\xi$$

$$(5) \quad \dot{H}(t) = \eta h(t) H(t) \Rightarrow \dot{H} = \eta h(t) = \eta'' h'(t), \text{ where } \eta'' = (1-m)\eta$$

where “ \wedge ” on a variable represents its ‘growth rate’, while $x' = \frac{x}{1-m}$ for $x \in \{c, v, s, h, \mathbf{H}, \omega\}$ and $x'' = (1-m)x$ for $x \in \{\xi, \eta\}$. In the simplified model, the capital accumulation efficiencies are modified by a multiplicative factor $1-m$, whereas time consumptions of scientific and religious ideas are modified by a factor $\frac{1}{1-m}$.

Now, we assume the accumulation efficiency of religious capital as $\xi = \xi(H(t), R(t))$. It is because we assume that as income ($H(t)$) grows, raising the extent and quality of network system, the accumulation efficiency of religious capital increases. This will facilitate the exchange of nonproductive emotional ideas and make agents more decadent.¹⁵ In contrast, with a given income level ($H(t)$), as $R(t)$ increases, the accumulation efficiency of religious capital decreases, because the increased traffic of nonproductive emotional ideas causes traffic congestion in the network system. Here the religious capital is the society’s pool of nonproductive ideas such as decadent, emotional, religious, artistic and others, not affecting material production productivity.

In addition, we assume the accumulation efficiency of scientific capital as $\eta = \eta(H(t), m)$. It is because it increases in the income share of military spending (m) relative to that of its neighboring enemies,¹⁶ and decreases in the extent of its conquered land and of its border lines with the neighboring

¹⁵ This is to capture the concept ‘decadence’ that is one of the major factors contributing to the fall of empires, which is emphasized by quite a few historians as Section II surveys.

¹⁶ ‘ m ’ representing the level of military spending as an income share also indirectly captures the cultural level of ‘civic militarism’. The civic militarism in addition to civic virtue, courage, discipline, training, individual creativity, and adaptability,

countries (proxied by $H(t)$), causing the inefficiency of military overstretch.¹⁷ This military overstretch mechanism is described by $\eta = \eta(H(t), m)$, which decreases in scientific capital ($H(t)$) and increases in m . Note that when maximizing their virtually one period maximization problem, agents take into consideration the aggregate levels of religious and scientific capital as exogenously given, not the specific functional forms of their accumulation efficiencies. The specific functional forms of the accumulation efficiencies of capitals will be explained in detail later.

The model considering a military spending of m units of time can be solved using the same FOCs of the model without it with the changing of variables in (1) to (5), as stated before.

3.3.2. Differential Equation System of the Model

This subsection presents the differential equation system to solve the model and to characterize the dynamic behavior of its solutions, based on the FOCs. For this, some algebraic manipulations are required.

From (2) and (3), we derive

$$(6) \quad \frac{h}{s} = \gamma^\varepsilon \left(\frac{H}{R}\right)^{\varepsilon-1} \Rightarrow \hat{h} = \hat{s} + (\varepsilon - 1)(\hat{H} - \hat{R}).$$

This equation represents the agent's optimizing behavior in terms of the elasticity of substitution (ε) between scientific and religious ideas, and of their relative price between $1/H$ (price for a piece of scientific idea) and $1/R$ (price for a piece of religious idea) measured in unit of time. The equation shows that the higher the elasticity of substitution ($\varepsilon > 1$) and the lower the relative price, the higher the ratio of time consumption between these two ideas. For simplicity, we assume $\gamma = 1$ without sacrificing the main implications of the model in the below.

After deriving $\omega = Rs(1 + \frac{h}{s})^{\frac{\varepsilon}{\varepsilon-1}}$ using (6), plugging this into (3) gives

$$(7) \quad (1 - m - s - h) = (1 + \frac{h}{s})^{\frac{\varepsilon-\sigma}{\varepsilon-1}} \left(\frac{h}{s}\right)^{\frac{-(\varepsilon-\sigma)}{\varepsilon-1}} H^{-(1-\kappa)(\sigma-1)} h$$

Taking log on both sides of (7) and differentiating this with respect to t solves for \hat{h} as

$$(8) \quad \hat{h} = (1 - m - s - h) \frac{1}{1-m-s} \left\{ \frac{\varepsilon-\sigma}{\varepsilon-1} \frac{s}{s+h} (\hat{h} - \hat{s}) + (1 - \kappa)(\sigma - 1) \hat{H} \right\} - \frac{s}{1-m-s} \hat{s}.$$

Thus, with (4), (5), (6) and (8), we derive

$$(9) \quad \begin{aligned} \dot{h} = & \frac{s}{1-m} \left((1 - m - s - h) \frac{\varepsilon-\sigma}{\varepsilon-1} \frac{h}{s+h} + h \right) (\varepsilon - 1) (\eta(H(t), m) h - \xi(H(t), R(t)) s) \\ & + \frac{(1 - m - s - h) h}{1 - m} (1 - \kappa)(\sigma - 1) \eta(H(t), m) h \end{aligned}$$

as emphasized in Hanson [2001], represents the aspect of the western culture that have empowered the West to historically have dominated other areas in military power. Specifically, Hanson sets the stage by explaining that the Western way of war is distinctive and has led to the West's military superiority. This superiority is not based on genetic or geographic factors but on cultural and institutional differences. He asserts that Western culture promotes values such as individualism, civic militarism, capitalism, rationalism, and freedom, which collectively contribute to military success. His main argument is that cultural and moral decay leads to the loss of civic militarism and subsequently the decline of civilizations.

¹⁷ This is to capture the concept 'military overstretch' of Kennedy [1987].

Similarly, using (6) and (9), we derive \dot{s} as

$$(10) \quad \begin{aligned} \dot{s} &= s \left(\hat{h} - (\varepsilon - 1)(\hat{H} - \hat{R}) \right) \\ &= \frac{s}{1-m} \left((1-m-s-h) \frac{\varepsilon-\sigma}{\varepsilon-1} \frac{s}{s+h} - (1-m-s) \right) (\varepsilon-1) (\eta(H(t), m)h - \xi(H(t), R(t))s) \\ &\quad + \frac{(1-m-s-h)s}{1-m} (1-\kappa)(\sigma-1)\eta(H(t), m)h \end{aligned}$$

3.3.3. Characterization of the Pseudo Steady State and Stability Conditions

Using the equations derived above, we characterize the dynamic behavior of the model around a pseudo-steady state. In this state, both $h(t)$ and $s(t)$ are constant, and the growth rate of scientific capital and that of religious capital are equal.

(9) and (10) imply that the steady state of $\dot{h} = \dot{s} = 0$ should have both $1-m-s-h = 0$ and $\hat{H} - \hat{R} = 0$, neglecting the trivial case of $h = s = 0$. The lemma below summarizes the steady state conditions.

Lemma 1: (Steady State Conditions): At the steady state of $\dot{h} = \dot{s} = 0$, neglecting the trivial case of $h = s = 0$, the following two conditions should hold.

$$(11) \quad s + h = 1 - m.$$

$$(12) \quad \hat{H} = \hat{R}.$$

The output growth of this economy is described by

$$(13) \quad \hat{Y} = (1 - m - s - h) + \kappa \hat{H} = \kappa \hat{H} - \frac{s}{1-m-s-h} \hat{s} - \frac{h}{1-m-s-h} \hat{h}$$

In the steady state, assuming capital accumulation efficiencies stay constant, the growth rate of scientific and of religious capital are equal, ensuring that the economy remains stable in the long run, unless disrupted by external forces such as changes in military spending or cultural shifts.

The differential equation system of (4), (5), (9), (10), and (13) solves for the time paths of H , R , h , s , and Y , given the initial conditions of military spending (m) and government policies in addition to capital stocks ($R(0)$, $H(0)$) (or two initial idea consumptions ($s(0)$, $h(0)$)).

From (9) and (10), we can study the dynamics of $s + h$ through the relationship of

$$(14) \quad \begin{aligned} \dot{s} + \dot{h} &= \frac{s(1-m-s-h)}{1-m} \left(\frac{\varepsilon-\sigma}{\varepsilon-1} - 1 \right) (\varepsilon-1) (\eta(H(t), m)h - \xi(H(t), R(t))s) \\ &\quad + \frac{(s+h)(1-m-s-h)}{1-m} (1-\kappa)(\sigma-1)\eta(H(t), m)h \\ &= \frac{(1-m-s-h)}{1-m} \{s(1-\sigma)(\varepsilon-1)(\eta(H(t), m)h - \xi(H(t), R(t))s) + (s \\ &\quad + h)(1-\kappa)(\sigma-1)\eta(H(t), m)h\} \\ &= \frac{(1-m-s-h)}{1-m} \{s(\sigma-1)(\varepsilon-1)(\xi(H(t), R(t))s) + ((s+h)(1-\kappa) - s(\varepsilon-1))(\sigma-1)\eta(H(t), m)h\} \end{aligned}$$

Assuming that $\{s(\sigma-1)(\varepsilon-1)(\xi(H(t), R(t))s) + ((s+h)(1-\kappa) - s(\varepsilon-1))(\sigma-1)\eta(H(t), m)h\} > 0$, which holds true if $\varepsilon > 1$ and $\kappa + \varepsilon \leq 2$ with $\sigma > 1$, or if $0 < \varepsilon < 1$ and $\kappa + \varepsilon > 2$ with $0 < \sigma < 1$, (or when being sufficiently close to the steady state with $\xi s \cong \eta h$, if κ

$\kappa < 1$ with $\sigma > 1$, or if $\kappa > 1$ with $0 < \sigma < 1$), we can easily infer from (14) that the unique solution for $\dot{s} + \dot{h} = 0$ at the steady state should satisfy¹⁸

$$(11) \quad s + h = 1 - m,$$

and that $s + h$ increases to $1 - m$ in t until reaching the steady state.¹⁹

(14) also says that, around the steady state, if $\sigma > 1$ and $\kappa < 1$, $\dot{s} + \dot{h}$ will be positive. It is because: (a) $1 - m - s - h > 0$ is positive and close to zero, (b) $\eta(H(t), m)h - \xi(H(t), R(t))s$ is very small valued, and (c) $\frac{s+h}{s}(1 - \kappa)(\sigma - 1)\eta(H(t), m)h$ is positive.

And if the ratio of the capital growth rates, $\eta(H(t), m)h/\xi(H(t), R(t))s$, increases, raising the capital ratio $\frac{H(t)}{R(t)}$, h/s will increase due to (6). And this will lead to a further increase in the ratio of the capital growth rates $(\eta(H(t), m)/\xi(H(t), R(t))) (h/s)$, if the induced positive effect on h/s dominates the negative effect of the ratio of the capital accumulation efficiencies $\frac{\eta(H(t), m)}{\xi(H(t), R(t))}$ caused by the increased $H(t)/R(t)$ and $H(t)$. This will further increase the ratio of the capital growth rates $\eta(H(t), m)h/\xi(H(t), R(t))s$, repeating this positive feedback loop again. We will elaborate more on this positive feedback mechanism later.

(6) gives another steady state characterization of s and h , stating the two capital growth rates are equal, identical to (12), as

$$(15) \quad \widehat{h(t)} - \widehat{s(t)} = (\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)}) = 0 \rightarrow \widehat{H(t)} = \widehat{R(t)} \\ \rightarrow \eta(H(t), m)h(t) = \xi(H(t), R(t))s(t), \text{ if } \varepsilon \neq 1.$$

Before exploring further implications, we will introduce the concept ‘pseudo steady state’, because (9) and (10) do not hold any more right after reaching the steady state, since $H(t)$ and $R(t)$ increase continuously in t due to (4) and (5).

Definition 1: (Pseudo Steady State) ‘Pseudo steady state’ is defined as the state in which $\dot{h} = \dot{s} = 0$, (11) and (12) hold, which however will be violated right after, because the RHSs of (9) and (10) are affected by continuously increasing state variables of $H(t)$ and $R(t)$.

¹⁸ From (14), we can easily infer that there can exist another trivial solution of $s(t) = h(t) = 0$ satisfying the relationship of $\dot{s} + \dot{h} = 0$. In the paper, we neglect this trivial case. And if (14) equals zero exactly with $s + h = 1 - m$, then goods consumption c will be zero, pushing its marginal utility to infinity. Thus, this steady state satisfying $\dot{h} = \dot{s} = 0$, with (14) equal to zero, is the convergent point which the equilibrium path approaches as t goes to infinity. This means that the steady state at a specific finite time does not exist. Thus, we assume that when the equilibrium path arrives at a small neighborhood of the steady state, agents regard it having reached the steady state satisfying (11) and (12). More formally, we assume that agents cannot perceive a small number less than a certain threshold level of biological perception magnitude (or computational accuracy limit), regarding it to be zero, due to their limited cognitive (or computational) ability. Thus, during the path stays in a small neighborhood of the steady state, agents regard the equilibrium path to have reached the steady state, in which it holds $\dot{h} = \dot{s} = 0$, since the magnitudes of $s + h - 1 - m$ and $\eta(H(t), m)h - \xi(H(t), R(t))s$ are too small to perceive, regarding them to be zero. For further exposition, refer to Appendix B11. (According to ChatGPT4o: **How many digits can computers calculate?** 1. Basic Accuracy (Single Precision): Computers can handle numbers with about 7 to 8 digits of accuracy. 2. Higher Accuracy (Double Precision): For more precise calculations, computers use a system that handles numbers with about 15 to 17 digits of accuracy. 3. Super High Accuracy (Special Cases): If even more accuracy is needed (like for very complex simulations or cryptography), some systems can handle numbers with 20 or more digits of accuracy.)

¹⁹ Refer to Appendix B10, for a further analysis of the differential equation of (14).

The implication of this definition goes as follows. We can easily infer that the steady state condition of both $\xi(H(t^*), R(t^*))s(t^*) = \eta(H(t^*), m)h(t^*)$ and $h(t^*) = s(t^*) = 0$ will soon be violated right after reaching the steady state.²⁰ It is because $H(t)$ and $R(t)$ increase continuously in t with both $h \neq 0$ and $s \neq 0$, changing the steady state values of $\eta^* = \eta(H(t^*), m)$ and $\xi^* = \xi(H(t^*), R(t^*))$, thus violating the relationships of (9) $h(t^*) = 0$, (10) $s(t^*) = 0$, and (12) $\eta(H(t^*), m) = \xi(H(t^*), R(t^*))$, right after achieving these. However, because $H(t)$ and $R(t)$ stay almost constant since $h(t) \cong s(t) \cong 0$, around t^* , this pseudo steady state will look like the usual steady state for a while.²¹ Thus, we call this steady state as ‘pseudo’ steady state because even though the ‘seeming’ steady state may look like a usual steady state for a while, it does not last long.²²

(11) and (12) solve the unique pseudo steady state as

$$(16) \quad s^* = (1 - m) \frac{\eta(H(t^*), m)}{\xi(H(t^*), R(t^*)) + \eta(H(t^*), m)}, \text{ and } h^* = (1 - m) \frac{\xi(H(t^*), R(t^*))}{\xi(H(t^*), R(t^*)) + \eta(H(t^*), m)}.$$

Thus, we have the following lemma.

Lemma 2: (i) If $\varepsilon > 1$ and $\kappa + \varepsilon \leq 2$ with $\sigma > 1$, or if $0 < \varepsilon < 1$ and $\kappa + \varepsilon > 2$ with $0 < \sigma < 1$, (or when being close to the steady state with $\xi s \cong \eta h$, if $\kappa < 1$ with $\sigma > 1$, or if $\kappa > 1$ with $0 < \sigma < 1$), $s + h$ continuously increases to $1 - m$. (ii) And it results in the unique pseudo steady state satisfying $\dot{s} = \dot{h} = 0$ with $s^* = (1 - m) \frac{\eta(H^*, m)}{\xi(H^*, R^*) + \eta(H^*, m)}$ and $h^* = (1 - m) \frac{\xi(H^*, m)}{\xi(H^*, R^*) + \eta(H^*, m)}$, with exogenously given m , where $H^* = H(t^*)$ and $R^* = R(t^*)$. (iii) After reaching the pseudo steady state, the relationship of $s + h = 1 - m$ will hold for all $t \geq t^*$.

With a given $H(t^*)$ and $R(t^*)$, (9) and (10) solve for local stability conditions around the pseudo steady state of $s^* = (1 - m) \frac{\eta(H^*, m)}{\xi(H^*, R^*) + \eta(H^*, m)}$ and $h^* = (1 - m) \frac{\xi(H^*, m)}{\xi(H^*, R^*) + \eta(H^*, m)}$ as Appendix B1 shows.

Definition 2: (Pseudo Local Stable Equilibrium) This means the equilibrium leading to the pseudo steady state satisfying the pseudo local stability conditions stated in Proposition 1 below.

Definition 3: (Pseudo Local Saddle Path Equilibrium) This means the equilibrium leading to the pseudo steady state satisfying the pseudo local saddle path equilibrium conditions stated in Proposition 1 below.

Proposition 1: (Pseudo Local Stable and Pseudo Local Saddle Path Equilibrium Conditions)

If $\kappa < 1$, $\sigma > 1$ and $0 < \varepsilon < 1$, or if $\kappa > 1$, $0 < \sigma < 1$ and $0 < \varepsilon < 1$, the unique pseudo steady state equilibrium with given $H(t^*)$ and $R(t^*)$ is pseudo locally stable, while if $\kappa < 1$, $\sigma > 1$ and $\varepsilon > 1$, or if $\kappa > 1$, $0 < \sigma < 1$ and $\varepsilon > 1$, that is a unique pseudo local saddle path equilibrium with

²⁰ With the assumption of the ‘limited cognitive (or computational) ability’, we can say “We can easily see that when the equilibrium path reaches a small neighborhood of the steady state, agents think that it has reached the steady state with $\dot{h} = \dot{s} = 0$, (11) and (12), and that it will soon deviate from this convergent path right after leaving it”. The pseudo steady state of the pseudo saddle path equilibrium can be more formally defined as “if the capital growth rate difference of $\eta(H(t), m)h - \xi(H(t), R(t))s$ reaches a small neighborhood of zero, then both \dot{s} and \dot{h} will be zero on the pseudo saddle path equilibrium.” For their convergence speed, refer to Appendix B9.

²¹ (17) gives the pseudo steady state ratio of $H(t^*)/R(t^*)$ as stated later.

²² Later, we assume this condition under the name of ‘Assumption One’ to emulate the historical phenomena. The concept ‘pseudo steady state’ is introduced to describe long, steady, complacent, peaceful, rich and decadent periods of empires, which are endogenously followed by abruptly declining periods.

given $H(t^*)$ and $R(t^*)$.

(Proof) In Appendix B1.

Proposition 1 implies that if the elasticity of substitution between religious and scientific idea consumption is large enough, with $\varepsilon > 1$, the economy can collapse depending on its initial conditions of $(R(0), H(0))$, or those of $(s(0), h(0))$, since it is on the pseudo saddle path. The paper mainly focuses on the pseudo saddle path equilibrium with $\varepsilon > 1$, because its main objective is modeling the phenomena that nations collapse endogenously, after their long periods of complacent affluence.

In this context, the main story line of the dynamics described by the model consists of two components²³: (a) pseudo steady state of a saddle path equilibrium (describing unstable long and complacent periods of empires' peak times), and (b) endogenous mechanism breaking this seemingly steady state into a declining equilibrium path (describing rapid and total decline after their long and complacent periods).

3.4. Dynamics with the Mechanism of Decadence and Military Overstretch

By modeling the mechanism of decadence and military overstretch as described before, this subsection explores how the severity of decadence and military overstretch effects can be mitigated to avoid an endogenous decline by relevant policies, such as increasing military spending, decreasing decadence contagion, increasing time consumption of scientific ideas, or containing military overstretch.

To incorporate the effect of military spending on a country's security into the model, we describe its effect on the accumulation efficiency of scientific capital by employing the contest-winning probability model as²⁴:

$$\eta_0 (1 + d_1 \left(\frac{m}{c_m + m} - 0.5 \right)),$$

where m represents the military spending in time unit (or in the fraction of GDP), c_m its main enemy's military spending, η_0 the net accumulation efficiency of scientific capital without military overstretch effect, and d_1 an exogenously given parameter representing a cultural aspect. This expression implies that the effect on the accumulation efficiency of scientific capital is zero when its military spending equals that of the enemy.

Based on the above model, the military overstretch mechanism is described by

$$\eta'(t) \equiv \eta_0 (1 + d_1 \left(\frac{mH(t)}{c_1 H^e(t)^{c_2} + mH(t)} - 0.5 \right)).$$

We assume that the total military spending of a country's rivals along its border increases in their income $H^e(t)$. This is described by $c_1 H^e(t)^{c_2}$ with $c_1 > 0$ and $c_2 > 1$. Here $c_1 H^e(t)^{c_2}$ represents the sum of military spendings of other neighboring enemy countries along the border of the overextended conquered land which increases in their income $H^e(t)$.²⁵ c_1 and c_2 measure the level of military overstretch representing a country's military policy or culture. To simplify the analysis, we assume $H^e(t) = H(t)$ as

²³ For transitional dynamics of various paths of the model, refer to phase diagrams for both the stable and saddle path equilibrium presented in Appendix A.6.

²⁴ Refer to Tullock (1980) for various specifications of the contest winning probability model.

²⁵ This functional form is to describe the 'military overstretch' of Kennedy (1987). Refer to Appendix C for further discussion on it.

$$\eta'(t) \equiv \eta(H(t), m) = \eta_0 (1 + d_1 \left(\frac{mH(t)}{c_1 H(t)^{c_2} + mH(t)} - 0.5 \right)) = \eta_0 (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)).$$

In other words, if military investments are identical for all countries given by $mH(t)$, then the above specification implies that the number of neighboring countries equals $(c_1/m)H(t)^{c_2-1}$, increasing in $H(t)$. Thus, if the income growth rate of the enemies becomes higher (resp., lower) than that of this country, then c_2 increases (resp., decreases), raising (resp., lowering) military overstretch. And we can easily infer that a decrease in military overstretch (i.e., a decrease in c_1 or c_2) increases $\eta(H(t), m)$, $H(t)$ and $H(t)/R(t)$ just like the effect of an increase in m . In the simulation, we assume that government can affect c_1 and c_2 in addition to η_0 by controlling the network system of spillover and transfer of scientific ideas.

To incorporate the decadence effect into the model, we assume:

$$\dot{R}(t) = \xi(H(t), R(t))s(t)R(t),$$

where $\xi'(t) \equiv \xi(H(t), R(t)) \equiv \xi_0 (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) = \xi_0 (1 + c_4 \left(\frac{H(t)}{c_3 R(t) + H(t)} \right)^\theta)$, with $c_3, c_4, \theta > 0$.²⁶

Here ξ_0 is the net accumulation efficiency of religious capital without decadence effect. This equation says that the accumulation efficiency of religious capital increases as $H(t)/R(t)$ grows. The reason is that the accumulation efficiency of religious capital depends positively on the level of viral networks, such as trade, transportation, and other information network systems, which increase in national income and wealth, proxied by $H(t)$. When a nation becomes more affluent, increasing $H(t)/R(t)$, its increased network flow of emotional, nonproductive ideas leads to more decadent behavior, lowering work ethic, civic virtue, social solidarity, and military spirit (civic militarism), and widening the gap between wealth, values, and political orientations.

Conversely, as $R(t)$ increases with a fixed $H(t)$, the accumulation efficiency of religious capital decreases due to increased congestion in the network traffic of emotional, nonproductive ideas, caused by the increase of $R(t)$, while the network system's capacity, proxied by $H(t)$, remains fixed. In the simulation, we assume that government can control ξ_0 by managing the channels of contagion and the transfer of emotional ideas and memes. And c_3, c_4 and θ measure the level of decadence as representing a country's cultural aspect.

To solve the model, we derived a differential equation system consisting of five equations as

$$(9) \quad \hat{s} = \frac{1}{1-m} \left\{ \left[(1-m-s) - (1-m) \frac{h}{s+h} \varepsilon + (1-m-s-h) \frac{h}{s+h} \sigma \right] \eta(H(t), m) h(t) + \left[-(1-m-s) + (1-m) \frac{h}{s+h} \varepsilon + (1-m-s-h) \frac{s}{s+h} \sigma \right] \xi(H(t), R(t)) s(t) \right\}$$

$$(10) \quad \hat{h} = \frac{1}{1-m} \left\{ \left[-s + (1-m) \frac{s}{s+h} \varepsilon + (1-m-s-h) \frac{h}{s+h} \sigma \right] \eta(H(t), m) h(t) + [s - (1-m) \frac{s}{s+h} \varepsilon + (1-m-s-h) \frac{s}{s+h} \sigma] \xi(H(t), R(t)) s(t) \right\}$$

$$(13) \quad \hat{Y} = (1-m-s-h) + \kappa \hat{H} = \kappa \hat{H} - \frac{s}{1-m-s-h} \hat{s} - \frac{h}{1-m-s-h} \hat{h}$$

$$(4) \quad \hat{R}(t) = \xi(H(t), R(t))s(t)$$

$$(5) \quad \hat{H}(t) = \eta(H(t), m)h(t)$$

²⁶ This functional form follows a logistic growth model. The typical model is the differential equation $x'(t) = x(t) \left(1 - \frac{x(t)}{K} \right)$. This equation will be solved as $x(t) = \frac{KCe^t}{1+Ce^t}$ which describes $x(t)$ growing in an elongated S-shape in t .

(9) and (10) imply that the steady state enforces the optimal time allocation of $h(t)$, $s(t)$, and $u(t)$, to be constant, which causes $H(t)$ and $R(t)$ to grow at the rate of $\eta(H(t), m)h(t)$ and $\xi(H(t), R(t))s(t)$ respectively in t . However, as stated before, such constant time allocations for idea consumption at the steady state cannot hold as time increases. It is because the continuous growth of $H(t)$ and $R(t)$ in t affecting the capital accumulation efficiencies, violates the steady state conditions described by $\dot{h} = \dot{s} = 0$, (11), and (12), as mentioned before.

The differential equation system describes complicated nonlinear dynamics. For example, as income (proxied by scientific capital) grows, the decadence effect on religious capital increases, raising its growth rate due to its increased accumulation efficiency, which in turn lowers the growth rate of scientific capital through the substitution effect. However, the decadence effect will be restrained around the steady state, both because it increases in $\frac{H(t)}{R(t)}$ and because both scientific and religious capital grow at the same rate at the steady state, not affecting $\frac{H(t)}{R(t)}$. In contrast, as income grows in t , increasing the border line, the military overstretch effect on scientific capital is intensified, lowering its growth rate due to the decreased accumulation efficiency of scientific capital, which in turn raises the growth rate of religious capital through the substitution effect. These processes facilitate the violation of the steady state conditions specified in Proposition 1.

From (B17), (B19), and (15), we have

$$\begin{aligned}
(17) \quad & (1 + d_1 \left(\frac{mH(t)}{c_1 H(t)^{c_2} + mH(t)} - 0.5 \right)) \eta_0 [A^{\sigma-1} H(t)^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1 - m) \\
& = (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 (\frac{R(t)}{H(t)})^{\varepsilon-1} [A^{\sigma-1} H(t)^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1 - m) \\
& \rightarrow (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)) \eta_0 = (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 (\frac{R(t)}{H(t)})^{\varepsilon-1} \\
& \rightarrow (1 - 0.5d_1) \eta_0 \cong (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 (\frac{R(t)}{H(t)})^{\varepsilon-1} \text{ with } H(t) \text{ becoming very large.}
\end{aligned}$$

where $h = [A^{\sigma-1} H(t)^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1 - m)$.

(17) implies that $H(t)/R(t)$ decreases continuously for $t > t^*$ with $\varepsilon > 1$, violating the steady state necessary conditions, if the continuously increasing capital stocks affect accumulation efficiencies ($\eta'(t)$ and $\xi'(t)$).

(11), (B17) and (B19) give another relationship between $H(t)$ and $R(t)$ at the pseudo steady state as

$$\begin{aligned}
(18) \quad & s + h = [A^{\sigma-1} H(t)^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1 - m) + \\
& (\frac{R(t)}{H(t)})^{\varepsilon-1} [A^{\sigma-1} H(t)^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1 - m) \\
& = 1 - m \rightarrow \\
& [A^{\sigma-1} H(t)^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1 + (\frac{R(t)}{H(t)})^{\varepsilon-1}) = 1
\end{aligned}$$

However, because in the pseudo steady state with its timing (and also $H(t)$) very much extended, $H(t)^{(\sigma-1)(\kappa-1)}$ becomes very small close to zero with $\sigma > 1$ and $\kappa < 1$, (18) reduces to a tautology of $[(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1}(1 + (\frac{R(t)}{H(t)})^{\varepsilon-1}) = 1$. Thus, in the pseudo steady state with its timing (and also $H(t)$) very much extended, only (17) gives the pseudo steady state ratio of $H(t^*)/R(t^*)$.

Because $\widehat{h(t)} = -\frac{s}{h}\widehat{s(t)}$ due to (6) and $s(t) + h(t) = 1 - m$ after reaching the steady state, we have

$$(19) \quad \begin{aligned} \widehat{h(t)} \left(1 + \frac{h(t)}{s(t)}\right) &= (\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)}) \\ &\rightarrow \widehat{h(t)} = \frac{s(t)}{1-m}(\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)}). \end{aligned}$$

Because after reaching the steady state $h(t)$ moves in the opposite direction to that of $s(t)$ due to $s(t) + h(t) = 1 - m$, (19) gives with $\varepsilon > 1$,

$$(20) \quad \begin{aligned} \widehat{H(t)} > \widehat{R(t)} &\leftrightarrow \widehat{h(t)} > 0 \text{ and } \widehat{s(t)} < 0 \\ \text{and } \widehat{H(t)} < \widehat{R(t)} &\leftrightarrow \widehat{h(t)} < 0 \text{ and } \widehat{s(t)} > 0, \text{ for } t > t^*. \end{aligned}$$

We call the strategy of $\{\widehat{h(t)} > 0, \widehat{s(t)} < 0\}$ for $t > t^*$ as the science strategy, while that of $\{\widehat{h(t)} < 0, \widehat{s(t)} > 0\}$ for $t > t^*$ the religion strategy in the below. (20) implies habit formation mechanism. That is, if $\widehat{H(t)} > \widehat{R(t)}$ (resp., $\widehat{H(t)} < \widehat{R(t)}$), then $\widehat{h(t)} > 0$ and $\widehat{s(t)} < 0$ (resp., $\widehat{h(t)} < 0$ and $\widehat{s(t)} > 0$), for $t > t^*$. This further increases the gap between the two capital growth rates as $\widehat{H(t)} \gg \widehat{R(t)}$ (resp., $\widehat{H(t)} \ll \widehat{R(t)}$) due to (4) and (5) and if Assumption One in the below holds. This process will repeat continuously in this positive feedback loop.

Using the relationship of (6) $z(t) \equiv \frac{h(t)}{s(t)} = (\frac{H(t)}{R(t)})^{\varepsilon-1}$, (17) will be changed into

$$(21) \quad \begin{aligned} \widehat{H(t)} &= (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)) \eta_0 h(t) \\ &= \widehat{R(t)} = (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 s(t) = (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 (\frac{H(t)}{R(t)})^{1-\varepsilon} h(t). \end{aligned}$$

From (6) and (21), setting $G(z(t)) \equiv \frac{H(t)}{R(t)} = z(t)^{\frac{1}{\varepsilon-1}}$ with $z(t) = \frac{h(t)}{s(t)} = \frac{h(t)}{1-m-h(t)}$, we will have

$$(22) \quad \begin{aligned} LHS &\equiv \widehat{H(t)}/h(t) = (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)) \eta_0 \\ &= RHS \equiv \frac{\widehat{R(t)}}{h(t)} = \frac{\left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right) \xi_0 s(t)}{h(t)} \\ &= (1 + c_4 \left(\frac{z(t)^{\frac{1}{\varepsilon-1}}}{c_3 + z(t)^{\frac{1}{\varepsilon-1}}} \right)^\theta) \xi_0 \frac{1}{z(t)} - \left(\frac{1}{z(t)} + c_4 \left(\frac{z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + z(t)^{\frac{1}{\varepsilon-1}}} \right)^\theta \right) \xi_0 \end{aligned}$$

We should note that after reaching the pseudo steady state for $t > t^*$, (17) will not hold, but (11) will. However, at $t = t^*$, (17) and (11) hold. To explore the dynamics of capitals and of the optimal

strategies after reaching the pseudo steady state, Appendix B2 derives <Graph 1> mainly based on (22). Its x-axis represents the current ratio between the two capital growth rates (i.e., virtually the time consumption ratio between scientific and religious ideas), while y-axis does virtually the direction of the incremental movement (i.e., the time derivative) of this ratio next period, under certain conditions we assume.

The government policy effect of a change in the net capital accumulating efficiencies, in military overstretch policy parameters, or in military spending on the dynamics of $s(t), h(t), H(t)$ and $R(t)$ can be explored easily by tracing the induced changes both in the two roots of $w_i(H(t), R(t), m)$ and in $w(t) = \frac{\widehat{H(t)} - \widehat{R(t)}}{\widehat{H(t)}} = 1 - \frac{\widehat{R(t)}}{\widehat{H(t)}}$ in <Graph 1>.²⁷ It is because the y-axis virtually represents the time derivative of the x-axis under certain conditions we assume.

Note also that changing net accumulation efficiencies (η_0, ξ_0) will change only $w(t) = \frac{\widehat{H(t)} - \widehat{R(t)}}{\widehat{H(t)}} =$

$$1 - \frac{\widehat{R(t)}}{\widehat{H(t)}} = 1 - \frac{(1+c_4) \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \xi_0 s(t)}{(1+d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right) \eta_0 h(t)} \quad \text{not the two roots, while } c_1, c_2 \text{ or military spending will}$$

change both $w(t)$ and the roots of $w_i(H(t), R(t), m)$ on the x-axis, in <Graph 1>. But we can safely regard the two roots remain constant when executing experiments of m , since the changes in these roots caused by a change in military spending are very small and negligible.

Since $B > 0$, $A < 0$ and C is close to zero and negative as assumed above, we have two positive real roots. It is because $B^2 - 4AC > 0$ holds true as long as $c_2 - 1$ (or c_1 or d_1) are positive and close to zero, making C close to zero.

(B13)

$$w_1(H(t), R(t), m) = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = \frac{-\frac{B}{A} - \sqrt{\left(\frac{B}{A}\right)^2 - \frac{4C}{A}}}{2} > 0 \quad , \quad \text{and} \quad w_2(H(t), R(t), m) = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{-\frac{B}{A} + \sqrt{\left(\frac{B}{A}\right)^2 - \frac{4C}{A}}}{2} > 0 \quad , \quad \text{where} \quad A = - \left\{ \frac{\left(\frac{H(t)}{R(t)}\right)^{\varepsilon-1} (\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right)}{1 + \left(\frac{H(t)}{R(t)}\right)^{\varepsilon-1}} \right\} < 0, \quad B = \left(\varepsilon - 1 \right) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) > 0, \quad C = -(c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} < 0.$$

To explore further implications in a simple setup, we assume Assumption One below.²⁸ This assumption not only simplifies the dynamic analysis of the model, but also helps describe the slow decay of nations after a prolonged affluence of their peak.

²⁷ Since a change in c_1 , c_2 or in military spending negligibly affects these solutions of the two roots, we have only to consider the induced change of $w(t) = 1 - \widehat{R(t)}/\widehat{H(t)}$ caused by it, to figure out the dynamics of time spent in ideas, of capital stocks and of income, with the assumption that $c_2 - 1$ (or c_1 or d_1) is positive and close to zero. For example, a decrease in c_1 or c_2 or an increase in m (resp., an increase in η_0 , or a decrease in ξ_0) increases $w(t) = 1 - \widehat{R(t)}/\widehat{H(t)}$ in <Graph 1> with $w_1(H(t), R(t), m)$ getting close to zero (resp., with $w_1(H(t), R(t), m)$ not changing).

²⁸ This assumption is necessary in our model setup to account for the well-established empirical fact that empires experience a prolonged period of prosperity followed by a rapid decline, as Subsection 2.3 surveys the related literature. For example, as

Assumption One: (Limiting the Size of Parameters) θ (or c_3 or c_4) and $c_2 - 1$, (or c_1 or d_1) are positive and small enough close to zero, with $0 < \theta < \varepsilon - 1$, such that (23) holds, for any change in exogenously given initial parameters (x) including policy parameters of m , c_1 , $c_2 - 1$, η_0 and ξ_0 , in the region $w(t) \in [w_1(H(t), R(t), m) + \delta, \alpha]$ with any very small positive constant δ and any $\alpha \in (w_1(H(t), R(t), m) + \delta, 1)$.²⁹

$$(23) \quad \text{sign} \left(\frac{d\widehat{H(t;x)}}{dt} = \frac{d(\eta(H(t;x), m; x)h(t;x))}{dt} = h(t; x) \frac{d\eta(H(t;x), m; x)}{dt} + \eta(H(t; x), m; x) \frac{dh(t;x)}{dt} \right) = \text{sign} \left(\eta(H(t; x), m; x) \frac{dh(t;x)}{dt} \right),$$

$$\text{and } \text{sign} \left(\frac{d\widehat{R(t;x)}}{dt} = \frac{d(\xi(H(t;x), R(t;x); x)s(t;x))}{dt} = s(t; x) \frac{d\xi(H(t;x), R(t;x); x)}{dt} + \xi(H(t; x), R(t; x); x) \frac{ds(t;x)}{dt} \right) = \text{sign} \left(\xi(H(t; x), R(t; x); x) \frac{ds(t;x)}{dt} \right).$$

Assumption One implies that the growth rate of scientific capital (resp., religious capital) is affected more by the change in the time consumption of scientific ideas (resp., of religious ideas) than by the change in the capital accumulation efficiency in the region $w(t) \in [w_1(H(t), R(t), m) + \delta, \alpha]$, at any time, with appropriate parameter values. That is, with appropriate parameter values, the direction of an endogenous change in $h(t)$ (resp., $s(t)$) will determine the direction of the resulting change of the growth rate of scientific capital $H(t)$ (resp., religious capital $R(t)$), dominating the indirect effect of the induced change in its capital accumulating efficiency on the growth rate of $H(t)$ (resp., $R(t)$).

Note also that the direction of the change in $s(t)$ is opposite to that in $h(t)$ due to the relationship of $h(t) + s(t) = 1 - m$ after reaching the pseudo steady state, for all $t > t^*$.

Lemma 3: (i) No matter how small valued θ (or c_4 or c_3) and $c_2 - 1$ (c_1 or d_1) are, with $0 < \theta < \varepsilon - 1$, there exists a small $\delta > 0$ around the pseudo steady state $w(t^*) = 0$ such that $\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt} < 0$ for any $w(t) \in [0, \delta]$, violating (23). (ii) But, in the region $w(t) \in [w_1(H(t), R(t), m) + \delta, \alpha]$ with any very small positive constant δ and any $\alpha \in (w_1(H(t), R(t), m) + \delta, 1)$ in <Graph 1>³⁰, we can restrict the parameter values of θ (or c_3 or c_4)

noted before, Cameron [1993] emphasizes that the Roman Empire's decline was a protracted and complex process. She argues that the decline should be understood in terms of a long-term transformation rather than a sudden fall. This perspective challenges the traditional narrative of a catastrophic collapse, instead highlighting the adaptive changes in political structures, economic systems, and cultural practices over time. Cameron stresses the importance of understanding the continuous shifts and adaptations that allowed the empire to endure despite mounting challenges.

²⁹ If some of these parameters are changed at t' , then new model economies are virtually created at t' with the changed initial conditions including capital stocks $H(t')$ and $R(t')$, induced by the changes in those parameters, showing new dynamics of $h(t)$, $s(t)$, $H(t)$, and $R(t)$. In this context, (23) says that across a reasonable set of different models with these changed initial conditions of parameters and capitals, the direction of the change in the growth rate of religious or scientific capital at any time is equal to that in its respective time consumption of ideas, in the region $w(t) \in [w_1(H(t), R(t), m) + \delta, \alpha]$.

³⁰ Note that we restrict the upper limit of the region by α by $w(t) \in [w_1(H(t), R(t), m) + \delta, \alpha]$. It is because: If $w(t)$ goes to one (i.e., $\widehat{R(t)}/\widehat{H(t)}$ goes to zero) without the restriction, with $h(t)$ and $s(t)$ converging to $1 - m$ and zero respectively, $H(t)$ diverges to infinity much faster than $R(t)$. In this case, the indirect effects might dominate the direct effects, if the decrease in $\eta(H(t), m)$ dominates the increase in $h(t)$ over time, and if the increase in $\xi(H(t), R(t))$ dominates the decrease in $s(t)$. This may not happen in our model, because the changes in accumulation efficiencies caused

and $c_2 - 1$ (or c_1 or d_1) such that (23) holds. (iii) In addition, if $w(t) \in (-\infty, 0]$, a decadence equilibrium ($\widehat{h(t)} < 0$ and $\widehat{s(t)} > 0$) with $\widehat{H(t)} - \widehat{R(t)} < 0$, persists for all $t > t^*$. (iv) Lastly, if $w(t) \in [0, w_1(H(t), R(t), m)]$, $w(t)$ will continuously decrease to be negative, eventually resulting in a decadence equilibrium ($\widehat{h(t)} < 0$ and $\widehat{s(t)} > 0$) with $\widehat{H(t)} - \widehat{R(t)} < 0$, which persists afterwards.

(Proof) In Appendix B3.

From <Graph 1> we can easily infer that (23) does not hold in the region of $(0, w_1(H(t), R(t), m))$. It is because the graph shows that $\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt} < 0$ for any $w(t) \in (0, w_1(H(t), R(t), m))$, contrasting the fact that we should have $\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt} > 0$ if $\widehat{H(t)} > \widehat{R(t)}$ ($\widehat{h(t)} > 0$ and $\widehat{s(t)} < 0$), which holds true for any $w(t) \in (0, w_1(H(t), R(t), m))$, and if (23) holds. At $w(t) = 0$, if (23) holds, $\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt}$ should be zero, not negative, because $\widehat{H(t)} = \widehat{R(t)} \leftrightarrow \widehat{h(t)} = 0$ and $\widehat{s(t)} = 0$ from (6) and (11).

For the purpose of this paper, we assume that the parameter values of θ (or c_4 or c_3) and $c_2 - 1$ (or c_1 or d_1) satisfy (ii) of Lemma 3 with (23) holding for the region $w(t) \in [w_1(H(t), R(t), m) + \delta, \alpha]$ with any very small positive constant δ and any $\alpha \in (w_1(H(t), R(t), m) + \delta, 1)$, resulting in the science equilibrium ($\widehat{h(t)} > 0$ and $\widehat{s(t)} < 0$) for all $t > t^*$ in this region.

(iii) of Lemma 3 theoretically helps explain why nations decline endogenously in the model with decadence and military overstretch, after reaching the pseudo steady state at $w(t^*) = 0$, without any government interventions. Its simulation results of Appendix A.1 demonstrate this, comparing the dynamics of the model with decadence and military overstretch to that without them.

Lemma 4: Assume θ (or c_4 or c_3) and $c_2 - 1$ (or c_1 or d_1) are positive and very close to zero, with $0 < \theta < \varepsilon - 1$. If $\frac{H(t)}{R(t)}$ and $H(t)$ increase to infinity, $w_1(H(t), R(t), m)$ will strictly decrease to zero, and $w_2(H(t), R(t), m)$ to $1 + \frac{s(t)}{h(t)}$ and finally to 1.

(Proof) In Appendix B4.

Lemma 5: Assume θ (or c_4 or c_3) and $c_2 - 1$ (or c_1 or d_1) are positive and close to zero. Then we will have $sign((\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)})) = sign(\widehat{h(t)} - \widehat{s(t)}) = sign(\frac{1}{\widehat{H(t)}} \frac{d}{dt} \widehat{H(t)} - \frac{1}{\widehat{R(t)}} \frac{d}{dt} \widehat{R(t)}) = sign(\frac{d}{dt} \frac{\widehat{H(t)}}{\widehat{R(t)}}) = sign(\frac{d}{dt} (\widehat{H(t)} - \widehat{R(t)}) / \widehat{H(t)}) = sign(\frac{d}{dt} (\widehat{H(t)} - \widehat{R(t)}))$. The last equality holds after reaching the pseudo steady state.

by the diverging $H(t)$ are limited under the current specification. It is because $\eta_0 (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right))$ decreases to $\eta_0 (1 - 0.5 d_1)$, while $\xi_0 (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta)$ increases to $\xi_0 (1 + c_4)$, as $H(t)$ and $H(t)/R(t)$ increase to infinity. And note that d_1 and c_4 are very small valued. Anyway, these dynamics are not the subject of this paper.

(Proof) In Appendix B5.

Note that if $\varepsilon > 1$, $\text{sign}((\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)})) = \text{sign}(\widehat{H(t)} - \widehat{R(t)})$, while if $0 < \varepsilon < 1$, $\text{sign}((\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)})) = \text{sign}(-(\widehat{H(t)} - \widehat{R(t)}))$.

Proposition 2: (Persistent Decadence Equilibrium)

Assume θ (or c_4 or c_3) and $c_2 - 1$ (or d_1 or c_1) are positive and close to zero, with $0 < \theta < \varepsilon - 1$. We will have a persistent decadence equilibrium with $\widehat{h(t)} < 0$ and $\widehat{s(t)} > 0$ for all $t > t^*$ (i.e., after reaching the pseudo steady state) without any external interventions (government policies, mob psychology or cultural shifts), right after reaching the pseudo steady state.

(Proof) By (iii) of Lemma 3.

Thus, we can see from <Graph 1> that the equilibrium afterwards without external intervention (government policies, mob psychology or cultural shifts) will make agents to take the decadence (religion) strategy ($\widehat{h(t)} < 0$ and $\widehat{s(t)} > 0$) for all $t > t^*$ right after reaching the pseudo steady state, because $H(t)$ and $R(t)$ increase continuously in t violating the steady state condition of $\widehat{H(t)} - \widehat{R(t)} = 0$.

To alter the naturally ensuing decadence equilibrium into the science equilibrium right after reaching the pseudo steady state, government should increase the net accumulation efficiency of scientific capital (η_0), decrease that of religious capital (ξ_0), or increase military spending (m), as proved in the below.³¹ In other words, in order to shift from the steady-state path to a high-growth trajectory dominated by scientific capital, the path must cross a certain threshold (δ) into the high-growth region, analogous to moving from 0 to z in <Graph 1>.

Proposition 3: (Persistent Science Equilibrium)

Assume θ (or c_4 or c_3) and $c_2 - 1$ (or d_1 or c_1) are positive and close to zero, with $0 < \theta < \varepsilon - 1$. After reaching the pseudo steady state of the pseudo saddle path, government policies of increasing η_0 or m , or decreasing ξ_0 , c_1 , or $c_2 - 1$ (through taxes, subsidies, mob psychology or cultural shifts) to increase $\frac{H(t)}{R(t)}$, satisfying the condition below at least just once, can achieve a persistent science equilibrium by inducing agents to adopt the science strategy for all $t > t^*$.

$$w(t) = 1 - \widehat{R(t)}/\widehat{H(t)} = (1 - \frac{\xi(H(t), R(t)) s(H(t), R(t))}{\eta(H(t), m) h(H(t), R(t))})$$

³¹ These policies will make societies more regal in the context of the regality theory of Fog [2017, 2023]. Fog introduces and elaborates on the concept of regality theory to explain why some cultures are more inclined towards war and authoritarianism (regal society), while others lean towards peace and tolerance (kungic society). If perceived levels of conflict and danger are high (resp., low), societies are more likely to adapt their social structures to be more regal (resp., kungic), according to Fog.

$$= 1 - (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 s(H(t), R(t)) / (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)) \eta_0 h(H(t), R(t)) \geq w_1(H(t), R(t), m) + \delta \text{ with any small positive constant } \delta.$$

(Proof) In Appendix B6.

Note that ξ_0 and η_0 do not affect $w_1(H(t), R(t), m)$ while m , c_1 , and $c_2 - 1$ do very little, and that $w_1(H(t), R(t), m)$ decreases in $H(t)/R(t)$. If these once-and-for-all government policies are strong enough to raise $w(t)$ above $w_1(H(t), R(t), m) + \delta$ by increasing $\frac{H(t)}{R(t)}$, $w(t)$ will move towards $w_2(H(t), R(t), m)$ converging to one over time. It is because if $H(t)$ increases faster than $R(t)$ to infinity due to the science strategy, then $w_2(H(t), R(t), m)$ converge to $1 + \frac{s(t)}{h(t)} > 1$ and finally to one, since the science strategy equilibrium will make $s(t)$ converge to 0, and $h(t)$ to $1-m$, as Lemma 4 states.

Note also that we need an additional term δ , with $\delta > \varepsilon$. It is because the path corresponding to an element in a neighborhood $[w_1(H(t), R(t), m), w_1(H(t), R(t), m) + \varepsilon)$ belongs to a religious capital dominated low growth path. And ε goes to zero as decadence and military overstretch effects decrease to zero.

A nation at risk of stagnation can potentially reverse course by reigniting innovation - analogous to historical "renaissances" or reforms that renewed growth. For example, late Qing China's Self-Strengthening Movement, Japan's Meiji Restoration or South Korea's New Village Movement were attempts (with mixed success) to swap decadent stagnation for scientific-industrial progress.

Proposition 4: (Existence of Two Kinds of Self-fulfilling Equilibria)

Assume that θ (or c_4 or c_3) and $c_2 - 1$ (or d_1 or c_1) are positive and close to zero and that the equilibrium is at the pseudo steady state of the pseudo saddle path. If all the other agents take the science strategy increasing $h(t)$ and decreasing $s(t)$ and thus increasing $\widehat{H(t)}/\widehat{R(t)}$, satisfying the condition below, any agent will also take the same strategy.

$$w(t) = 1 - \widehat{R(t)}/\widehat{H(t)} = \left(1 - \frac{\xi(H(t), R(t)) s(H(t), R(t))}{\eta(H(t), m) h(H(t), R(t))} \right) \geq w_1(H(t), R(t), m) + \delta, \text{ with any small positive constant } \delta.$$

In contrast, any agent will also take the decadence (religion) strategy, if all the other agents take this strategy.

(Proof) In Appendix B7.

And this proposition leads to the following one.

Proposition 5: (Existence of Two Populist Equilibria)

At the pseudo steady state of the pseudo saddle path equilibrium, populist policies will have a strong permanent impact to achieve the decadence equilibrium of increasing $s(t)$ and decreasing $h(t)$ (resp., the science equilibrium of increasing $h(t)$ and decreasing $s(t)$), providing positive incentives enough for agents to take the decadence strategy (resp., the science strategy) at least once, through maneuvering

mob psychology, taxes or subsidies, thus lowering $w(t) = 1 - \frac{\widehat{R(t)}}{\widehat{H(t)}} = \left(1 - \frac{\xi(H(t), R(t)) s(H(t), R(t))}{\eta(H(t), m) h(H(t), R(t))}\right)$ to be negative, for all $t > t^*$ (resp., thus raising $w(t)$, satisfying $w(t) \geq w_1(H(t), R(t), m) + \delta$, with any small positive constant δ , for all $t > t^*$).

3.5. Calibration and Simulation Results

This section, along with Appendix A, provides various simulation results experimented with different parameter values. The simulation primarily focuses on whether the dynamic paths of pseudo saddle path equilibria of the theoretical model can emulate stylized properties of historical dynamics of national decline.³² Simulation results of the model consistent with these stylized properties are as follows:

First, the model shows that the processes of the fall of empires can be endogenous, as surveyed in Section II. In Appendix A, especially A.1, A.3, and A.4, all simulation results of pseudo saddle path equilibria with decadence and military overstretch exhibit the endogenous nature of national decline following a prolonged period of prosperity. In contrast, nations remain in a steady-state equilibrium forever without an endogenous decline in the model without decadence and military overstretch, as Appendix A.1 shows.

Second, the driving forces of empire decline can be either decadence or military overstretch, nurtured by a long period of affluent complacency, as surveyed in Section II. The theoretical and empirical models are constructed based on this understanding.

Third, the peaking period of an empire tends to be rather extended, as discussed in Subsection 2.3. Simulations in Appendix A based on the pseudo saddle path equilibrium with decadence and military overstretch, assuming Assumption One, demonstrate that this prolonged peak can indeed occur. A.1, A.3, A.4, and A5 show this.

Fourth, empires often experience a slow, unnoticeable moral and spiritual decline over a long period of peak prosperity, followed by a collapse. All simulations of the pseudo saddle path equilibrium with decadence and military overstretch in Appendix A (A.1, A.3, A.4, and A5) show endogenous, rapid decline after prolonged periods of peak prosperity.

Fifth, the model suggests that effective policies to steer nations toward a favorable saddle path equilibrium—one where scientific idea dominates and rapid income growth follows—include restricting the growth rate of religious capital, reducing decadence and military overstretch, increasing time consumption of scientific ideas and scientific capital, and increasing military spending to an appropriate level. The simulations of A.5 in Appendix A show that the effects of these policies are as predicted, and very sensitive to their changes since the economy is located on the saddle path equilibrium.

Sixth, simulations in Appendix A show that the pseudo saddle path equilibrium is highly sensitive to changes in exogenous parameters representing cultural aspects (with c_1 and c_2 representing military overstretch and c_3 and c_4 decadence) and policy variables (such as military overstretch policy

³² Refer to Appendix A for various simulation results with different sets of exogenous parameter values representing one country's various cultural aspects (c_1 and c_2 representing military overstretch, and c_3 and c_4 decadence) and various government policies (η_0 , ξ_0 , m , c_1 and c_2). Table 3 lists the calibrated parameter values for simulation and explains why and how these parameters are calibrated as such.

(c_1, c_2) , military spending (m), and capital accumulation efficiencies (η_0 and ξ_0)). Whether the equilibrium forks into a science- or religion dominated equilibrium depends very sensitively on variations in these factors.

Seventh, the self-fulfilling equilibrium of the model explains why even tiny small differences in policies or cultural factors can lead to vastly different outcomes, leading to either a rapid decline or sustained growth. Thus, predicting the exact timing of an empire's collapse is very difficult.

Last, since the model's capacity to emulate historical patterns of national rise and fall is demonstrated, it can be used as a useful tool for understanding the endogenous processes behind these dynamics and for formulating appropriate policies.

IV. Conclusion

The rise and fall of nations are complex processes influenced by a multitude of factors, as our surveyed theories suggest. Theories such as asabiyyah, cliodynamics, regality theory, the big cycle, and the Empire Cycle provide valuable insights into these dynamics. Many of them show that decadence and military overstretch play critical roles in the decline of nations, driven by internal affluence and complacency, as well as by external policies or psychological factors.

After reviewing these theories, we developed a model and its simulation aimed at predicting and potentially mitigating future national collapse by formulating appropriate policies. Main results we learned from the analysis of the model and its simulation exercises are as follows:

First, there exists a unique pseudo saddle path equilibrium. This equilibrium resembles a usual saddle path with a steady state, but eventually, it will fork into a religion dominated path with low economic growth without government intervention. It is also important to note that the pseudo steady state resembles a typical steady state for a considerable period until its rapid collapse.

Second, which equilibrium the economy will fork into (between a religion- or a science dominated equilibrium) depends very sensitively on initial conditions representing a country's cultural aspects of decadence and military overstretch, government policies, or even mob psychology. It is due to habit formation mechanism in the accumulation of religious and scientific capital.

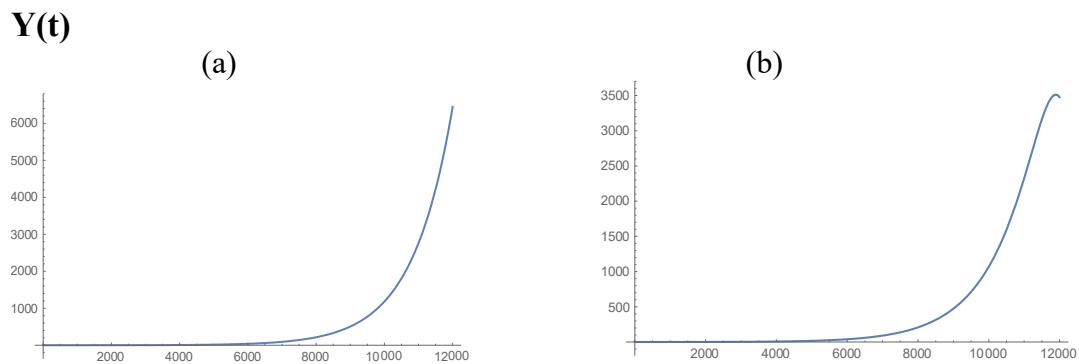
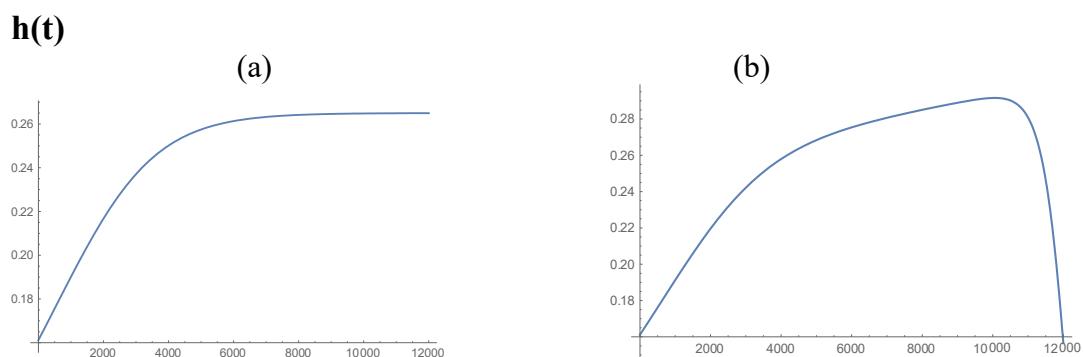
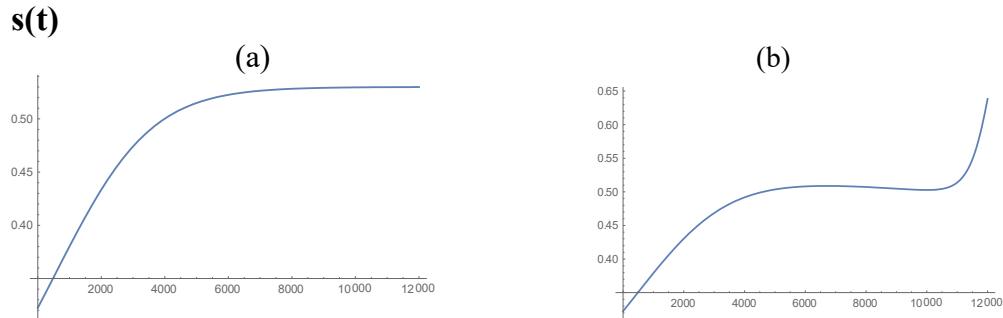
Last, to avoid forking into a religion dominated equilibrium (which would represent national collapse), government should implement policies that prevent citizens from falling into excessive decadence, and that incentivize them to consume more scientific ideas and to increase military spending to an appropriate level.

For future research, we will refine the model by calibrating its parameter values more realistically and applying it to contemporary societies to quantitatively describe and predict the challenges and opportunities they face. In other words, a well-designed and quantified theoretical model, combined with well-measured proxies for decadence and military overstretch, will allow us to diagnose and forecast the direction where these countries are heading, rise or fall, and prescribe the necessary policies to prevent their collapse.

Appendix A: Simulation Results of the Model

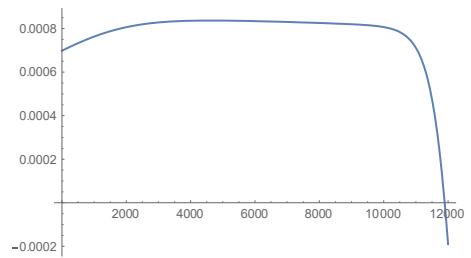
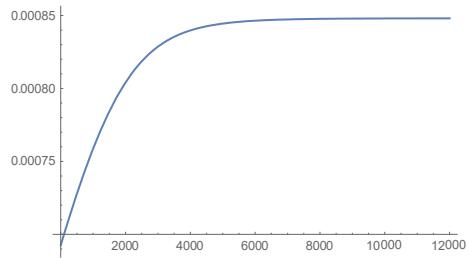
A.1. Saddle Path Equilibrium without Decadence or Military Overstretch, and That with Them

- (a) Saddle path with epsilon 1.5 ($h(0) = 0.1611613722665686$, $s(0) = 0.3223227445331372$) w/o decadence or military overstretch
- (b) Saddle path with epsilon 1.5 ($h(0) = 0.1611613722665686$, $s(0) = 0.3223227545331372$) w/ decadence and military overstretch



Growth Rate of $Y(t)$

- (a)
- (b)

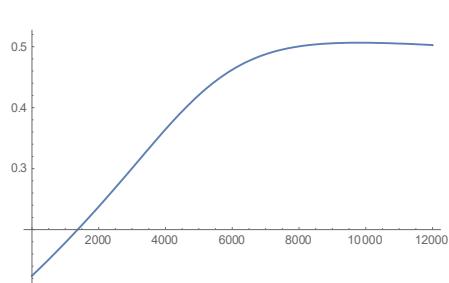


A.2. Stable Path Equilibrium with Epsilon 0.8 with Decadence and Military Overstretch, Changing Initial Capital Ratios

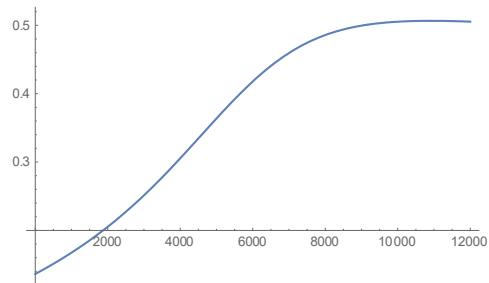
- (a) $R/H=4, h(0)=0.163996, s(0)=0.124285$
- (b) $R/H=1/4, h(0)=0.103257, s(0)=0.136248$

s(t)

(a)

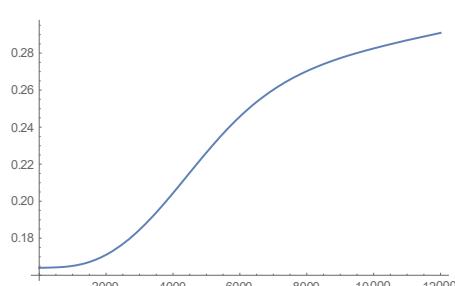


(b)

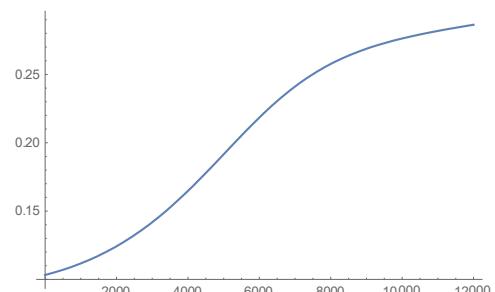


h(t)

(a)



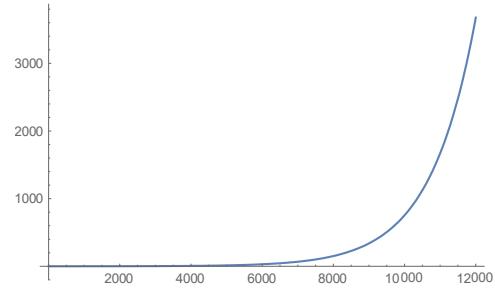
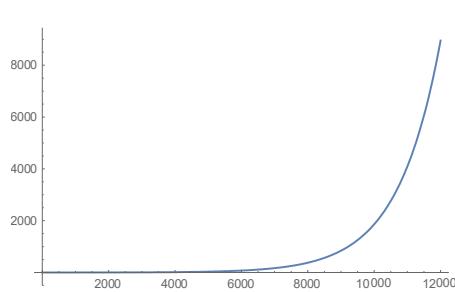
(b)



Y(t)

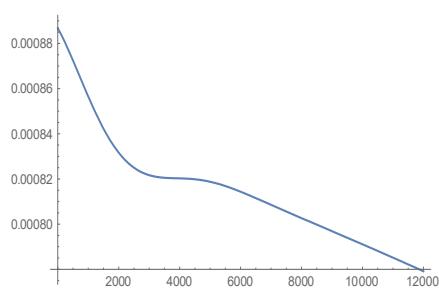
(a)

(b)

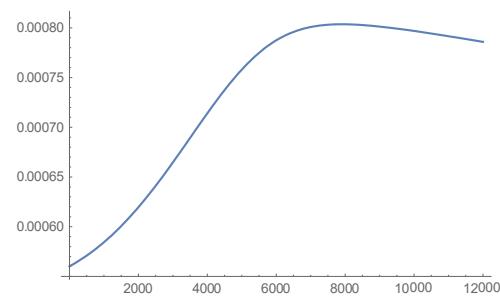


Growth Rate of $Y(t)$

(a)



(b)

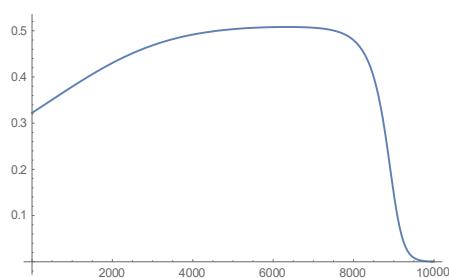


A.3. Saddle Path Equilibrium with Decadence and Military Overstretch, Changing Initial Capital Ratios

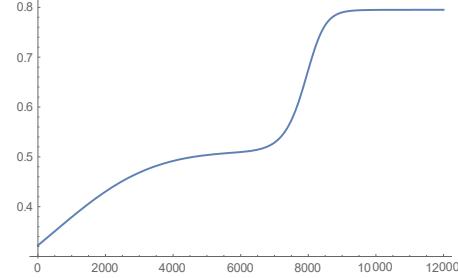
- (a) $R(t)/H(t)=3.9999999$
- (b) $R(t)/H(t)=4.0000001$
- (c) $R(t)/H(t)=4$ (default case)

$s(t)$

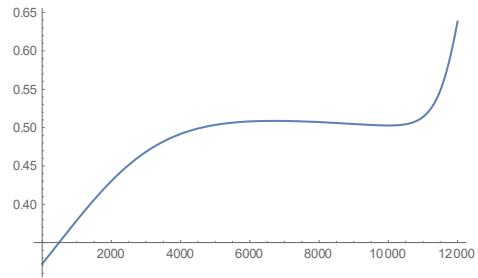
(a)



(b)

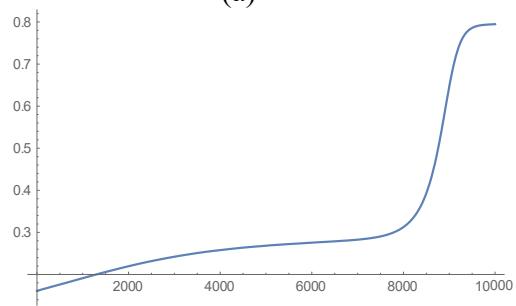


(c)

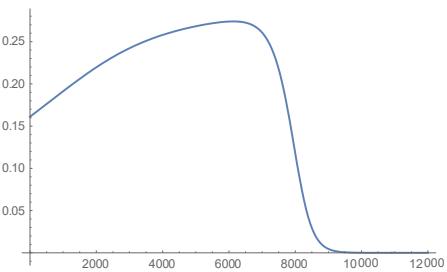


$h(t)$

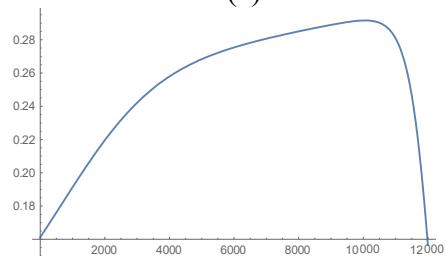
(a)



(b)

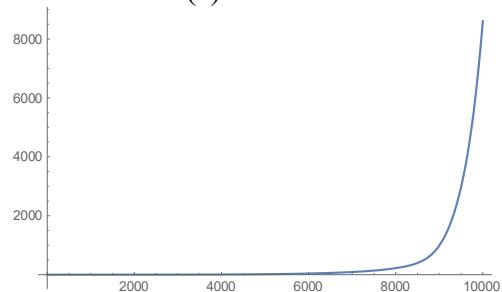


(c)

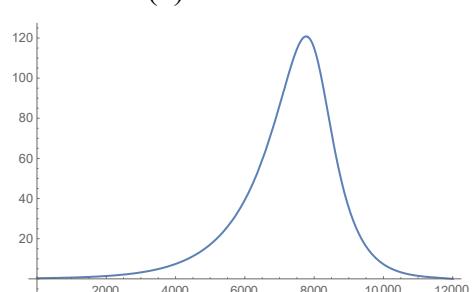


$Y(t)$

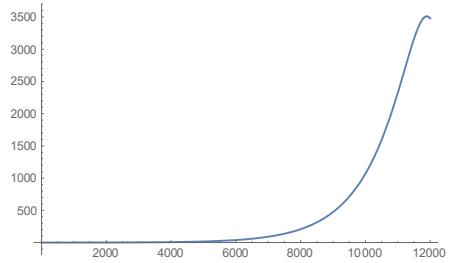
(a)



(b)

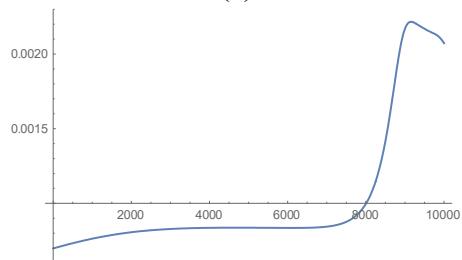


(c)

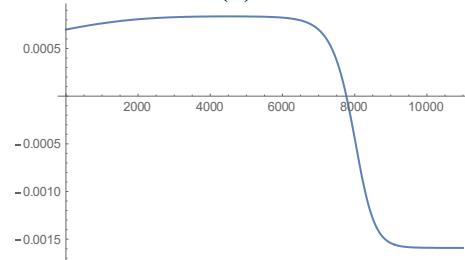


Growth Rate of $Y(t)$

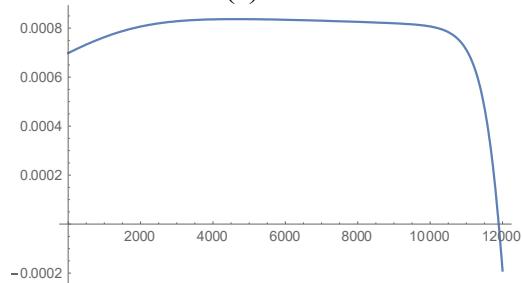
(a)



(b)



(c)



A.4. Saddle Path Equilibrium with Decadence and Military Overstretch, Changing Military Spendings

$h(0)=0.1611613722667713$, $s(0)=0.3223227445335426$

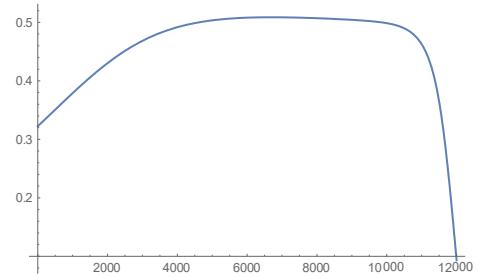
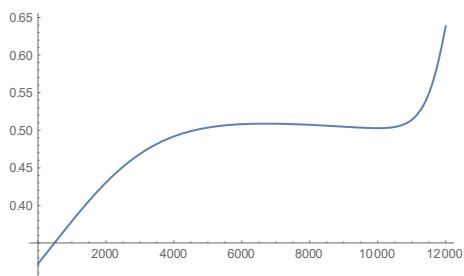
(a) $m=0.204961611998$ (default case)

(b) $m=0.204961611999$

s(t)

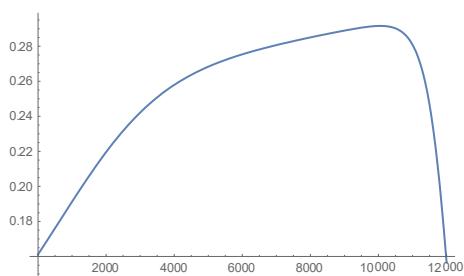
(a)

(b)

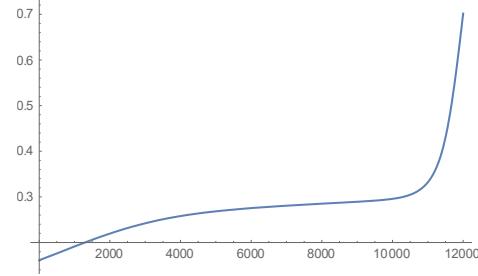


h(t)

(a)

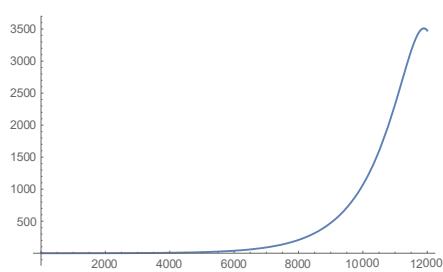


(b)

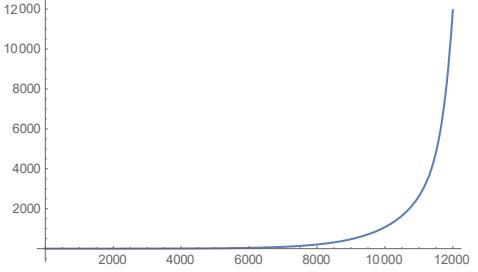


Y(t)

(a)



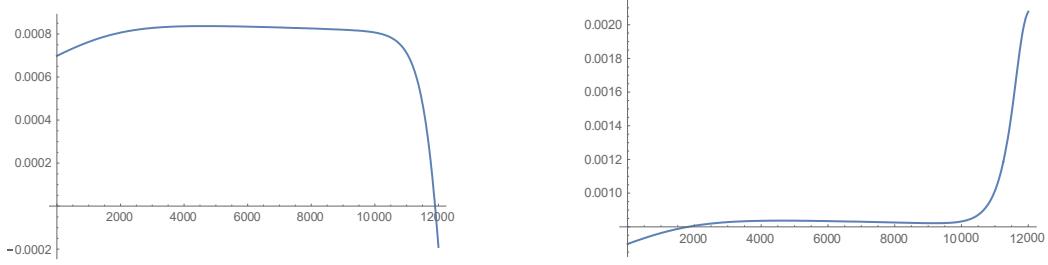
(b)



Growth Rate of Y(t)

(a)

(b)

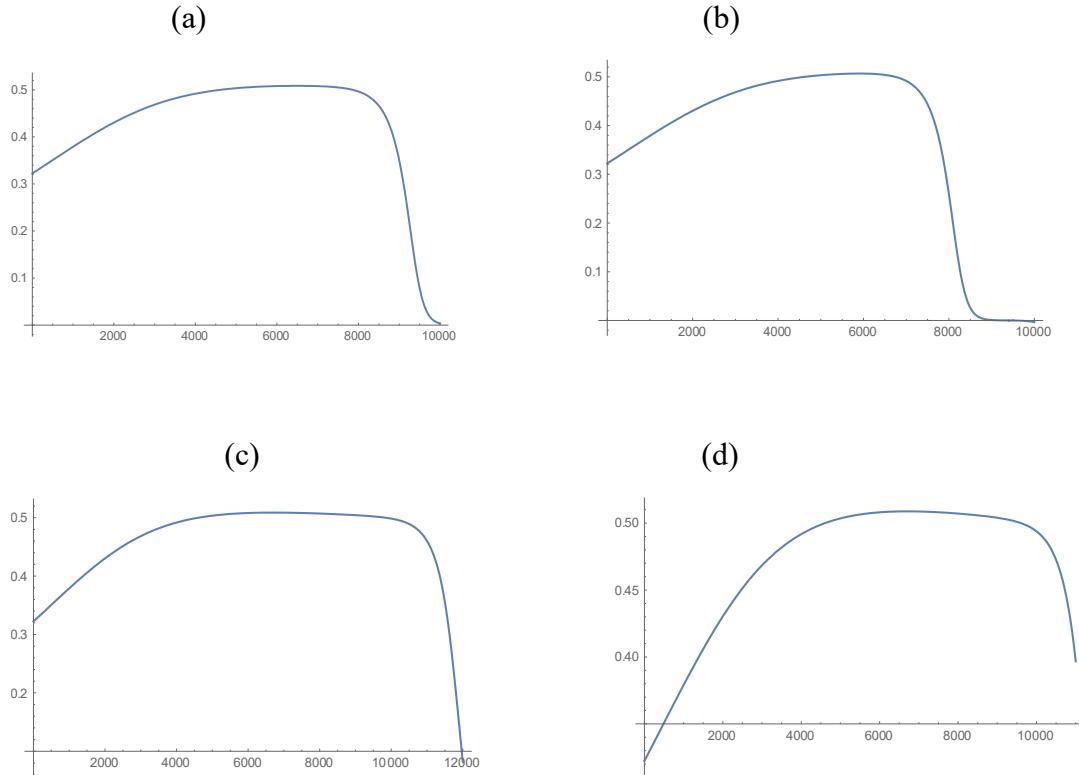


A.5. Saddle Path Equilibrium with Decadence and Military Overstretch, Changing Policies to Avoid Collapse (Capital Accumulation Efficiencies (ξ_0 , η_0) and Military Overstretch (c_1, c_2))

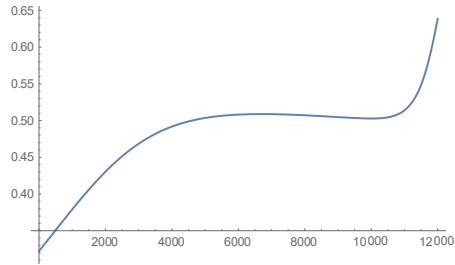
- (a) Epsilon=1.5, m=0.204961611998, $\xi_0=0.01$, $\eta_0=0.02000000001$, $c_1 = 0.1$, $c_2 = 1.02$
- (b) Epsilon=1.5, m=0.204961611998, $\xi_0=0.0099999999$, $\eta_0=0.02$, $c_1 = 0.1$, $c_2 = 1.02$
- (c) Epsilon=1.5, m=0.204961611998, $\xi_0=0.01$, $\eta_0=0.02$, $c_1 = 0.099999999999$, $c_2 = 1.02$
- (d) Epsilon=1.5, m=0.204961611998, $\xi_0=0.01$, $\eta_0=0.02$, $c_1 = 0.1$, $c_2 = 1.01999999999$
- (e) Epsilon=1.5, m=0.204961611998, $\xi_0=0.01$, $\eta_0=0.02$, $c_1 = 0.1$, $c_2 = 1.02$ (default case)

ξ_0 is the net accumulation efficiency of religious capital and η_0 that of scientific capital.

$s(t)$

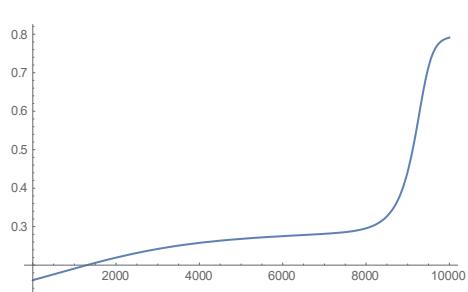


(e)

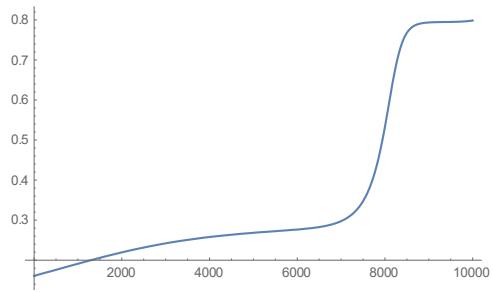


h(t)

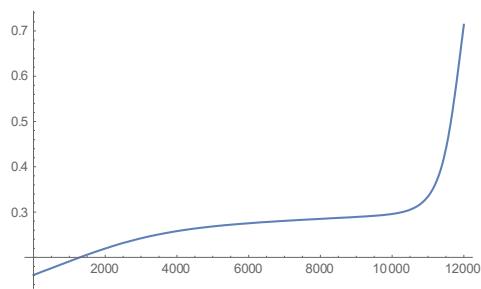
(a)



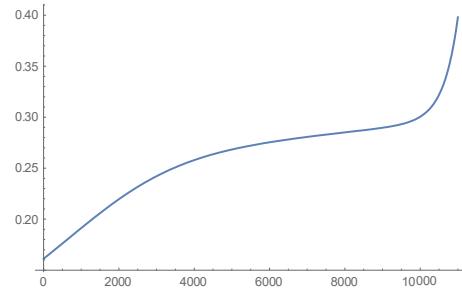
(b)



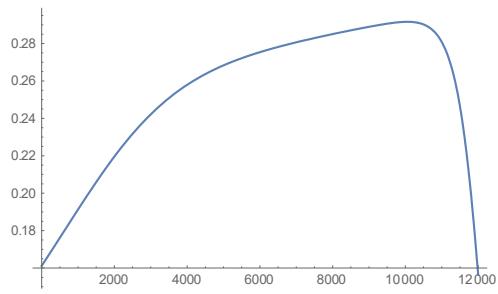
(c)



(d)

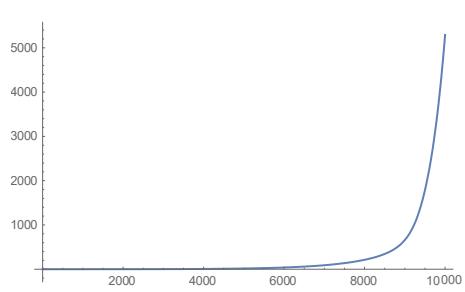


(e)

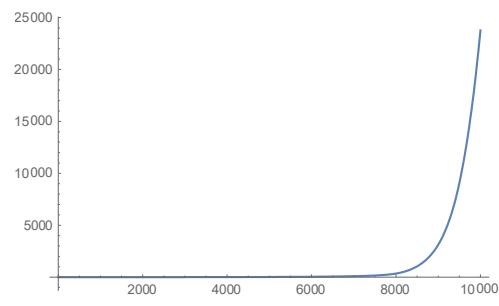


Y(t)

(a)

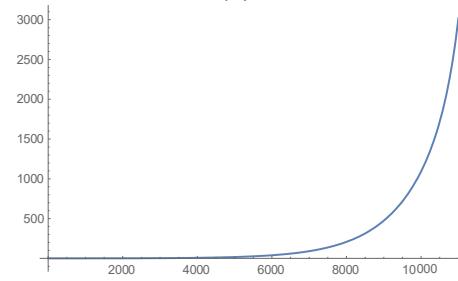
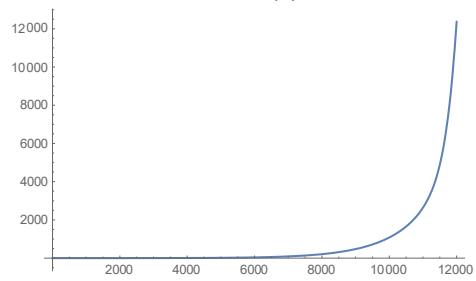


(b)



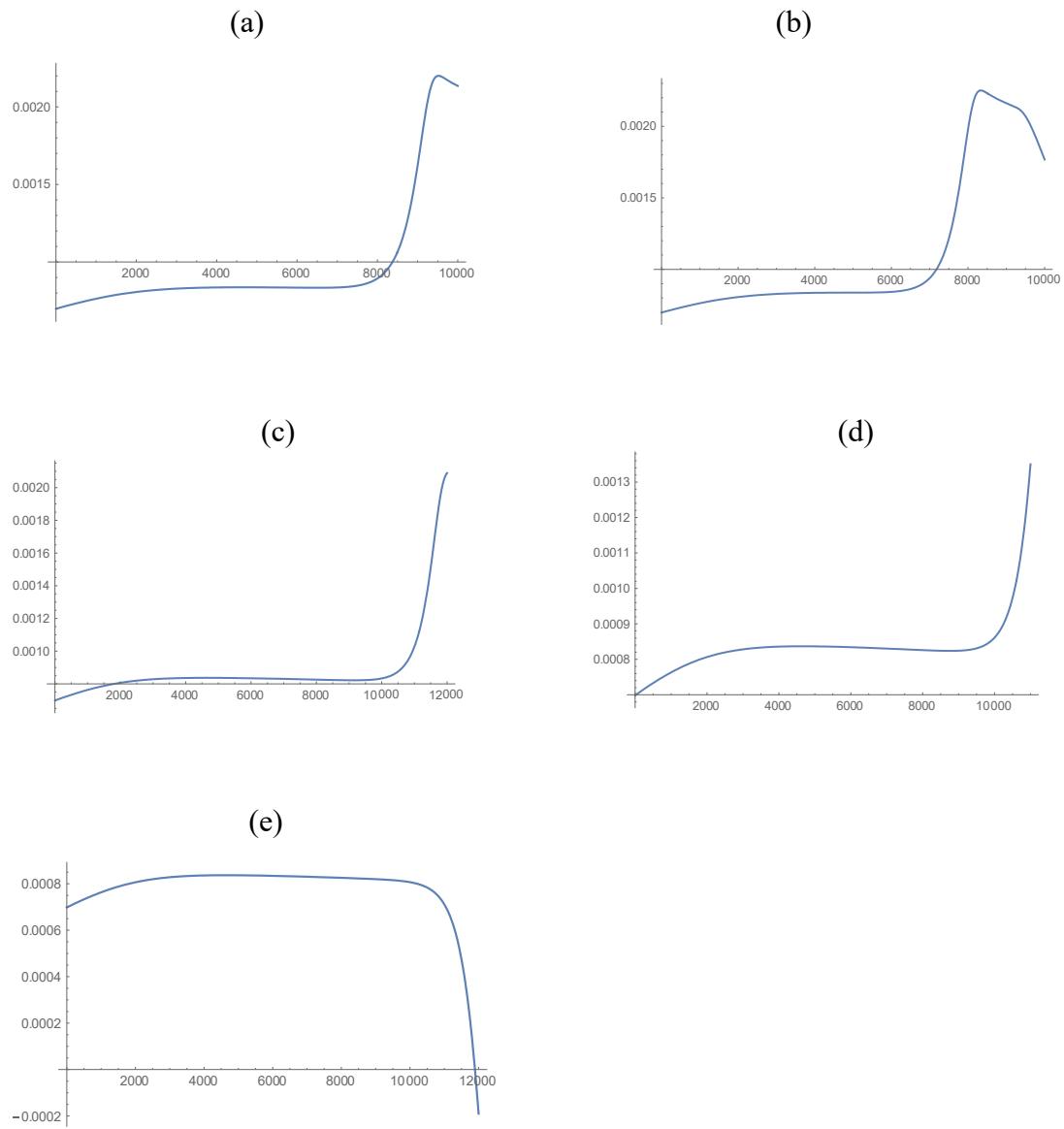
(c)

(d)



(e)

Growth Rate of Y(t)

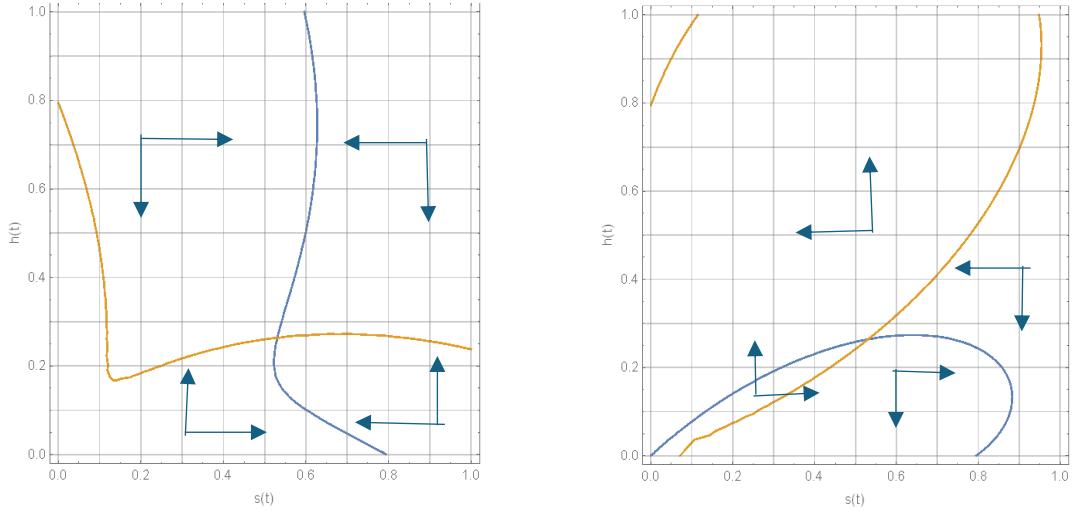


A.6. Phase Diagrams for the Stable and Saddle Path Equilibrium.

- (a) Stable Equilibrium with Epsilon=0.8
- (b) Saddle Path Equilibrium with Epsilon=1.5 (default case)

(a) Epsilon=0.8

(b) Epsilon=1.5



Appendix B1: Proof of Proposition 1

Given $H(t^*)$ and $R(t^*)$, (9) and (10) solve for local stability conditions around the pseudo steady state of $s^* = (1 - m) \frac{\eta(H^*, m)}{\xi(H^*, R^*) + \eta(H^*, m)}$ and $h^* = (1 - m) \frac{\xi(H^*, m)}{\xi(H^*, R^*) + \eta(H^*, m)}$ as follows.

First, we define:

$$f(s, h) \equiv (1 - m)\dot{s}$$

$$= s^2 \left((1 - m - s - h) \frac{\varepsilon - \sigma}{\varepsilon - 1} \frac{1}{s + h} - \frac{1 - m - s}{s} \right) (\varepsilon - 1) (\eta(H(t), m)h - \xi(H(t), R(t))s) + (1 - m - s - h)s(1 - \kappa)(\sigma - 1)\eta(H(t), m)h$$

$$g(s, h) \equiv (1 - m)\dot{h} = sh \left((1 - m - s - h) \frac{\varepsilon - \sigma}{\varepsilon - 1} \frac{1}{s + h} + 1 \right) (\varepsilon - 1) (\eta(H(t), m)h - \xi(H(t), R(t))s) + (1 - m - s - h)h(1 - \kappa)(\sigma - 1)\eta(H(t), m)h$$

The partial derivatives, $a_{11} = \frac{\partial f}{\partial s}$, $a_{12} = \frac{\partial f}{\partial h}$, $a_{21} = \frac{\partial g}{\partial s}$, and $a_{22} = \frac{\partial g}{\partial h}$, evaluated at (s^*, h^*) are given by³³:

³³ Before executing algebra further, we will calculate $\frac{\partial H(t)}{\partial h(t)}$ and $\frac{\partial R(t)}{\partial s(t)}$ first.

From the relationship that $H(t; h(t)) \cong H(t - \Delta t; h(t - \Delta t))(1 + \eta(H(t - \Delta t; h(t - \Delta t), m))h(t - \Delta t)\Delta t)$, we derive

$$\frac{\partial}{\partial h(t)} H(t; h(t)) = \lim_{\Delta h(t) \rightarrow 0} \frac{H(t - \Delta t; h(t - \Delta t))(1 + \eta(H(t - \Delta t; h(t - \Delta t), m))h(t - \Delta t) + \Delta h(t)\Delta t) - H(t - \Delta t; h(t - \Delta t))(1 + \eta(H(t - \Delta t; h(t - \Delta t), m))h(t - \Delta t)\Delta t)}{\Delta h(t)} =$$

$$\lim_{\Delta h(t) \rightarrow 0} \frac{H(t - \Delta t; h(t - \Delta t))\eta(H(t - \Delta t; h(t - \Delta t), m))\Delta h(t)}{\Delta h(t)} = \lim_{\Delta h(t) \rightarrow 0} H(t - \Delta t; h(t - \Delta t))(\eta(H(t - \Delta t; h(t - \Delta t), m)))\Delta t = 0.$$

This is the ratio of the instantaneous change of $H(t)$ with respect to $h(t)$, as $\Delta h(t) \rightarrow 0$ (also meaning $\Delta t \rightarrow 0$). It is because we are experimenting perturbations around t^* with $\Delta h(t)$ and with the timing interval of Δt .

Similarly, because $R(t; s(t)) \cong R(t - \Delta t; s(t - \Delta t))(1 + \xi(H(t), R(t - \Delta t; s(t - \Delta t)))s(t - \Delta t)\Delta t)$, we also have

$$\frac{\partial}{\partial s(t)} R(t; s(t)) =$$

$$\lim_{\Delta s(t) \rightarrow 0} \frac{R(t - \Delta t; s(t - \Delta t))(1 + \xi(H(t), R(t - \Delta t; s(t - \Delta t)))s(t - \Delta t) + \Delta s(t)\Delta t) - R(t - \Delta t; s(t - \Delta t))(1 + \xi(H(t), R(t - \Delta t; s(t - \Delta t)))s(t - \Delta t)\Delta t)}{\Delta s(t)} =$$

$$\lim_{\Delta s(t) \rightarrow 0} \frac{R(t - \Delta t; s(t - \Delta t))\xi(H(t), R(t - \Delta t; s(t - \Delta t)))\Delta s(t)}{\Delta s(t)} = \lim_{\Delta s(t) \rightarrow 0} R(t - \Delta t; s(t - \Delta t))(\xi(H(t), R(t - \Delta t; s(t - \Delta t))))\Delta t = 0.$$

Thus, neglecting the derivative of each capital stock with respect to its time consumption of ideas, we can easily infer that the following derivations of the extended model are identical to those of the simple model as in the below.

$$(B1) \quad a_{11} = \frac{\partial f}{\partial s} = f_1(s, h) = sh(\varepsilon - 1)\xi + sB = C\xi + sB$$

$$(B2) \quad a_{22} = \frac{\partial g}{\partial h} = g_2(s, h) = sh(\varepsilon - 1)\eta + hB = C\eta + hB$$

$$(B3) \quad a_{12} = \frac{\partial f}{\partial h} = f_2(s, h) = -sh(\varepsilon - 1)\eta + sB = -C\eta + sB$$

$$(B4) \quad a_{21} = \frac{\partial g}{\partial s} = g_1(s, h) = -sh(\varepsilon - 1)\xi + hB = -C\xi + hB,$$

where $B = -(1 - \kappa)(\sigma - 1)\eta(H(t), m)h$, $C = s(\varepsilon - 1)(1 - m - s) = sh(\varepsilon - 1)$.

To prove the proposition, we need the following algebra.

$$(B5) \quad \det \equiv a_{11}a_{22} - a_{12}a_{21} = BC(\eta + \xi)(1 - m) = -(1 - \kappa)(\sigma - 1)\eta(H(t), m)h(\varepsilon - 1)sh(\eta + \xi)(1 - m)$$

$$(B6) \quad \text{tr}(A) = sh(\eta + \xi)(\varepsilon - 1) - (1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h$$

$$(B7) \quad (\text{tr}(A))^2 = (a_{11} + a_{22})^2 = (sh(\eta + \xi)(\varepsilon - 1))^2 + ((1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h)^2 - 2sh(\eta + \xi)(\varepsilon - 1)(1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h,$$

With (B7), we can easily derive

$$(B8) \quad \begin{aligned} (\text{tr}(A))^2 - 4\det &= (sh(\eta + \xi)(\varepsilon - 1))^2 + ((1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h)^2 + \\ &- 2sh(\eta + \xi)(\varepsilon - 1)(1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h + 4(1 - \kappa)(\sigma - 1)\eta(H(t), m)h(\eta + \xi)(\varepsilon - 1)sh(1 - m) \\ &= (sh(\eta + \xi)(\varepsilon - 1))^2 + ((1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h)^2 + 2(1 - \kappa)(\sigma - 1)\eta(H(t), m)h(\eta + \xi)(\varepsilon - 1)sh(1 - m) \\ &= (sh(\eta + \xi)(\varepsilon - 1) + (1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h)^2 > 0 \end{aligned}$$

Using (B1)-(B8) and Taylor-expanding (9) and (10) around the unique pseudo steady state (s^*, h^*) provide the local stability and the local saddle path conditions as follows:

The stability conditions for having a pair of negative eigenvalues are: (i) $\text{tr}(A) \equiv a_{11} + a_{22} < 0$ (the sum of eigenvalues are negative), (ii) the determinant is positive (both eigenvalues have the same sign), and (iii) $(\text{tr}(A))^2 - 4 \cdot \det > 0$ (eigenvalues are real numbers).

(B5) implies that if $\kappa < 1$, $\sigma > 1$ and $0 < \varepsilon < 1$, or if $\kappa > 1$, $0 < \sigma < 1$ and $0 < \varepsilon < 1$, we will have a positive determinant, proving (ii). On the other hand, if $\kappa < 1$, $\sigma > 1$ and $\varepsilon > 1$, or if $\kappa > 1$, $0 < \sigma < 1$ and $\varepsilon > 1$, we will have a negative determinant.

$$(B8) \quad \begin{aligned} (\text{tr}(A))^2 - 4\det &= (sh(\eta + \xi)(\varepsilon - 1))^2 + ((1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h)^2 + \\ &- 2sh(\eta + \xi)(\varepsilon - 1)(1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h + 4(1 - \kappa)(\sigma - 1)\eta(H(t), m)h(\eta + \xi)(\varepsilon - 1)sh(1 - m) \\ &= (sh(\eta + \xi)(\varepsilon - 1) + (1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h)^2 > 0, \text{ which holds true for any parameter values, proving (iii).} \end{aligned}$$

We can also easily derive that

$\text{tr}(A) = sh(\eta + \xi)(\varepsilon - 1) - (1 - m)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h < 0$, if $\kappa < 1$, $\sigma > 1$ and $0 < \varepsilon < 1$, or if $\kappa > 1$, $0 < \sigma < 1$ and $0 < \varepsilon < 1$, giving negative $\text{tr}(A)$, proving (i). Thus, if $\kappa < 1$, $\sigma > 1$ and $0 < \varepsilon < 1$, or if $\kappa > 1$, $0 < \sigma < 1$ and $0 < \varepsilon < 1$, the first part of the proposition 'the pseudo local stability' is proved.

We also prove by (B5) that if $\kappa < 1$, $\sigma > 1$ and $\varepsilon > 1$, or if $\kappa > 1$, $0 < \sigma < 1$ and $\varepsilon > 1$, $\det < 0$, satisfying the condition for real number conditions of $(\text{tr}(A))^2 - 4\det > 0$. Then, one eigenvalue is positive and the other one is negative, leading to a pseudo saddle path equilibrium. Hence, the second part of the proposition, 'the existence of a unique pseudo local saddle path,' is proved.

Appendix B2: Characterizing the Dynamics of Capital Stocks and Optimal Strategies after Reaching the Pseudo Steady State

To explore these dynamics, we take derivatives on both sides of (22) as follows:

$$(B9) \quad \begin{aligned} \frac{dLHS}{dt} &= \frac{d\left(\frac{H(t)}{h(t)}\right)}{dH(t)} \frac{dH(t)}{dt} = \frac{d}{dH(t)} \left(\eta_0 \left(1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right) \right) \right) \frac{dH(t)}{dt} = \frac{d}{dH(t)} \left(\eta_0 d_1 \frac{m}{c_1 H(t)^{c_2-1} + m} \right) \frac{dH(t)}{dt} \\ &= -\eta_0 d_1 \frac{(c_2 - 1)mc_1 H(t)^{c_2-2}}{(c_1 H(t)^{c_2-1} + m)^2} \frac{dH(t)}{dt} = -\eta_0 d_1 \frac{m}{(c_1 H(t)^{c_2-1} + m)} \frac{(c_2 - 1)c_1 H(t)^{c_2-1}}{(c_1 H(t)^{c_2-1} + m)} \frac{dH(t)}{H(t)dt} \\ &= -(c_2 - 1) \left[\eta_0 \left(1 - 0.5d_1 + d_1 \frac{m}{c_1 H(t)^{c_2-1} + m} \right) \right]^2 - \eta_0 (1 - 0.5d_1) \eta_0 \left(1 - 0.5d_1 + d_1 \frac{m}{c_1 H(t)^{c_2-1} + m} \right) \frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} h(t) \end{aligned}$$

with $c_2 > 1$.

Similarly, for the right-hand side of (22), we have:

$$\begin{aligned}
(B10) \quad \frac{dRHS}{dt} &= \frac{d(\frac{\bar{R}(t)}{h(t)})}{dz(t)} \frac{dz(t)}{dt} = \frac{d}{dz(t)} \{ \xi_0 (1/z(t) + c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta) \} \frac{dz(t)}{dt} = \xi_0 \left\{ \left(\frac{\theta - (\varepsilon-1)}{\varepsilon-1} - \right. \right. \\
&\quad \left. \left. \theta \frac{\frac{1}{\varepsilon-1}(\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta - 1/z(t) \right) \frac{1}{z(t)} \frac{dz(t)}{dt} \right\} \\
&= \frac{\theta - (\varepsilon-1)}{\varepsilon-1} \xi_0 \left(\left(1 - \frac{(\varepsilon-1)}{\theta - (\varepsilon-1)} \theta \frac{1}{\varepsilon-1} \frac{(\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta \right) \frac{1}{z(t)} \frac{dz(t)}{dt} - \xi_0 z(t)^{-1} \frac{1}{z(t)} \frac{dz(t)}{dt} \\
&= \frac{\theta - (\varepsilon-1)}{\varepsilon-1} \xi_0 \left(\left(1 - \frac{\theta}{\theta - (\varepsilon-1)} \frac{(\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta \right) \frac{dln(z(t))}{dt} - \xi_0 z(t)^{-1} \frac{dln(z(t))}{dt} \\
&= - \frac{\varepsilon-1-\theta}{\varepsilon-1} \xi_0 \left(1 + \frac{\theta}{(\varepsilon-1-\theta)} \frac{(\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) - \xi_0 z(t)^{-1} (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) \\
&= - \frac{\varepsilon-1-\theta}{\varepsilon-1} \xi_0 \left(\left(1 + \frac{\theta}{\varepsilon-1-\theta} \left(1 - \frac{c_3}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) \right) c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta \right) (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) - \xi_0 z(t)^{-1} (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) \\
&= - \frac{\varepsilon-1-\theta}{\varepsilon-1} \xi_0 \left(\frac{\varepsilon-1}{\varepsilon-1-\theta} - \frac{\theta}{\varepsilon-1-\theta} \frac{c_3}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) - \xi_0 z(t)^{-1} (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) \\
&= - \xi_0 \left(1 - \frac{\theta}{\varepsilon-1} \frac{c_3}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) - \xi_0 z(t)^{-1} (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) \\
&= - \xi_0 \{ (1/z(t)) + c_4 \left(\frac{\gamma^{\frac{-\varepsilon}{\varepsilon-1} z(t)^{\frac{1-\frac{1}{\theta}(\varepsilon-1)}{\varepsilon-1}}}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} }{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right)^\theta \} - 1/z(t) \} (\varepsilon-1) \left(1 - \frac{\theta}{\varepsilon-1} \frac{c_3}{c_3 + (\gamma^{-\varepsilon} z(t))^{\frac{1}{\varepsilon-1}}} \right) (\widehat{H(t)} - \widehat{R(t)}) - \xi_0 z(t)^{-1} (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) \\
&= - \left[- \frac{\widehat{R(t)}}{h(t)} + \xi_0 \left(\frac{1}{z(t)} \right) \right] \left((\varepsilon-1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \right) (\widehat{H(t)} - \widehat{R(t)}) - \xi_0 z(t)^{-1} (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) \\
&= - \frac{\widehat{R(t)}}{h(t)} \left((\varepsilon-1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \right) (\widehat{H(t)} - \widehat{R(t)}) - \xi_0 z(t)^{-1} \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} (\widehat{H(t)} - \widehat{R(t)})
\end{aligned}$$

with $\frac{dz(t)}{dt} > 0 (\leftrightarrow h'(t) > 0)$, $\frac{dz(t)}{dt} < 0 (\leftrightarrow h'(t) < 0)$, $0 < \theta < 1$ and $\varepsilon > 1 + \theta$.

Now, from (6), we can easily prove that $\frac{dln(z(t))}{dt} = \widehat{h(t)} - \widehat{s(t)} = (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)})$.

By combining equations (B9) and (B10), we can derive a differential equation describing the dynamics of the difference of capital growth rates as

(B11)

$$\frac{d(\widehat{H(t)}/h(t))}{dt} - \frac{d(\widehat{R(t)}/h(t))}{dt} = \frac{1}{h(t)} \left(\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt} \right) - \frac{\widehat{H(t)} - \widehat{R(t)}}{h(t)^2} \frac{dh(t)}{dt} \rightarrow \left(\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt} \right) = \left(\frac{d(\widehat{H(t)}/h(t))}{dt} - \frac{d(\widehat{R(t)}/h(t))}{dt} \right) h(t) + (\widehat{H(t)} - \widehat{R(t)}) \widehat{h(t)} = \left(\frac{d(\widehat{H(t)}/h(t))}{dt} - \frac{d(\widehat{R(t)}/h(t))}{dt} \right) h(t) + \frac{s(t)}{1-m} (\varepsilon - 1) (\widehat{H(t)} - \widehat{R(t)}) (\widehat{H(t)} - \widehat{R(t)}).$$

Using (6), (B9), (B10) and (B11), we derive

(B12)

$$\begin{aligned} \frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt} &= -(c_2 - 1) \left\{ \eta_0 \left(1 - 0.5d_1 + d_1 \frac{m}{c_1 H(t)^{c_2-1} + m} \right) h(t) \right\}^2 \frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} \\ &\quad + (c_2 - 1) \eta_0 (1 - 0.5d_1) \left\{ \eta_0 \left(1 - 0.5d_1 + d_1 \frac{m}{c_1 H(t)^{c_2-1} + m} \right) h(t) \right\} \frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} h(t) \\ &\quad + \left((\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \right) \widehat{R(t)} (\widehat{H(t)} - \widehat{R(t)}) \\ &\quad + \xi_0 s(t) \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} (\widehat{H(t)} - \widehat{R(t)}) + \frac{1}{1+z(t)} (\varepsilon - 1) (\widehat{H(t)} - \widehat{R(t)}) (\widehat{H(t)} - \widehat{R(t)}) \\ &= -(c_2 - 1) \left(\frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} \right) \widehat{H(t)} \left\{ \widehat{H(t)} - \eta_0 (1 - 0.5d_1) h(t) \right\} + \left((\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \right) (\widehat{R(t)}) (\widehat{H(t)} - \widehat{R(t)}) + \xi_0 s(t) \\ &\quad \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} (\widehat{H(t)} - \widehat{R(t)}) + \frac{1}{1+z(t)} (\varepsilon - 1) (\widehat{H(t)} - \widehat{R(t)})^2 \\ &= \{\widehat{H(t)}\}^2 \left\{ -(c_2 - 1) \left(\frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} \right) \left(1 - \frac{1 - 0.5d_1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} \right) \right. \\ &\quad \left. + \left((\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) \right) Z(t) (1 - Z(t)) + \frac{1}{1+z(t)} (\varepsilon - 1) (1 - Z(t))^2 \right\} \\ &= \{\widehat{H(t)}\}^2 \left\{ -(c_2 - 1) \left(\frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} \right) \left(1 - \frac{1 - 0.5d_1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} \right) + \left((\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) \right) (1 - Z(t)) - \right. \\ &\quad \left. \left((\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) \right) (1 - Z(t)) (1 - Z(t)) + \frac{1}{1 + (\frac{H(t)}{R(t)})^{\varepsilon-1}} (\varepsilon - 1) (1 - Z(t))^2 \right\} \\ &= \{\widehat{H(t)}\}^2 \left\{ \{-(c_2 - 1) \left(\frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} \right) \left(1 - \frac{1 - 0.5d_1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} \right) \right. \\ &\quad \left. + \left((\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) \right) (1 - Z(t)) \right\} (1 - Z(t)) \\ &\quad - \left\{ \frac{(\frac{H(t)}{R(t)})^{\varepsilon-1}}{1 + (\frac{H(t)}{R(t)})^{\varepsilon-1}} (\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) \right\} (1 - Z(t))^2 \} \end{aligned}$$

where $Z(t) \equiv \frac{\widehat{R(t)}}{\widehat{H(t)}}$ and $w(t) = 1 - Z(t)$ with $(\varepsilon - 1 - \theta) > 0$.

Now, we can graphically describe the above differential equation by <Graph 1> below, after solving two roots from the RHS of (B12) after setting it to zero. We can derive the two positive roots $w_1(H(t), R(t), m) > 0$ and $w_2(H(t), R(t), m) > 0$ from this equation as

(B13)

$$w_1(H(t), R(t), m) = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = \frac{-\frac{B}{A} - \sqrt{\frac{(B/A)^2 - 4C}{A}}}{2} > 0, \text{ and } w_2(H(t), R(t), m) = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{-\frac{B}{A} + \sqrt{\frac{(B/A)^2 - 4C}{A}}}{2} > 0,$$

where we assume that $A = -\left\{ \frac{\left(\frac{H(t)}{R(t)}\right)^{\varepsilon-1}}{1 + \left(\frac{H(t)}{R(t)}\right)^{\varepsilon-1}}(\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) \right\} < 0$, $B = \left((\varepsilon - 1) - \theta \frac{c_3}{c_3 + \frac{H(t)}{R(t)}} \left(1 - \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right)^{-1} \right) \right) > 0$, $C = -(c_2 - 1) \left(\frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} \right) \left(1 - \frac{1 - 0.5d_1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} \right) = -(c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} < 0$

For the assumed signs of A, B , and C above to hold true, the exogenous parameter values should be small enough.

We can easily infer the effect of a change in capital accumulating efficiencies or in military spending on capital growth rates through identifying the induced changes in the two roots and in $w(t) = 1 - \widehat{R(t)}/\widehat{H(t)}$ in <Graph 1>.³⁴ It is because y-axis actually represents the time derivative of x-axis representing the difference of two capital growth rates in <Graph 1>. We can also find out what will happen to the time paths of capital accumulations, of time spent in consuming ideas, and of income if accumulating efficiencies or military spending change exogenously. Note that changing accumulation efficiencies or military spending will change both $w(t) = 1 - \widehat{R(t)}/\widehat{H(t)}$ and the solutions of $w_i(H(t), R(t), m)$ in <Graph 1>.

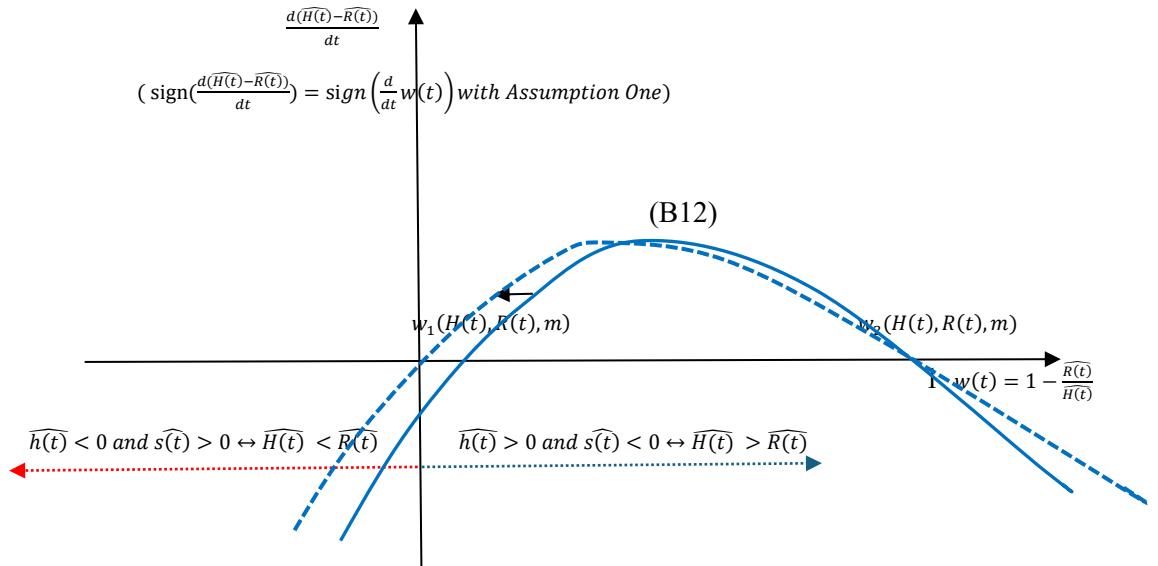
Since $B > 0$, $A < 0$, and $C < 0$ close to zero and negative as assumed above, we have two positive real roots. It is because as long as $B^2 - 4AC > 0$, which holds true as long as $c_2 - 1$ (or c_1 or d_1) are positive and close to zero, forcing C close to zero.

As $H(t)$ increases to infinity, with $c_2 - 1 > 0$ (or if $c_2 - 1$ (or c_1 or d_1) is positive and close to zero), the term $C = -(c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} < 0$ approaches zero, shifting the real line curve to the dotted line curve in <Graph 1>, resulting in $w_1(H(t), R(t), m) \cong 0$, and if $\frac{H(t)}{R(t)}$ as well as $H(t)$ approaches infinity, $w_2(H(t), R(t), m)$ converge to $1 + \frac{s(t)}{h(t)} > 1$ and finally to one.

³⁴ Since changes in military spending negligibly affect these two roots, we have only to consider the change in $w(t) = 1 - \widehat{R(t)}/\widehat{H(t)}$, assuming the values of these two roots constant, to figure out the dynamic paths of time spent on ideas and of capital stocks, with the assumption that $c_2 - 1$ (or c_1 or d_1) is positive and close to zero. For example, an increase in m , η_0 , or c_3 or a decrease in ξ_0 increases $w(t) = 1 - \widehat{R(t)}/\widehat{H(t)}$ in <Graph 1> with $w_1(H(t), R(t), m) \cong 0$.

$$(20) \quad \widehat{H(t)} > \widehat{R(t)} \leftrightarrow \widehat{h(t)} > 0 \text{ and } \widehat{s(t)} < 0$$

and $\widehat{H(t)} < \widehat{R(t)} \leftrightarrow \widehat{h(t)} < 0 \text{ and } \widehat{s(t)} > 0$



<Graph 1: Dynamics of Capital Growth Rates after Reaching the Pseudo Steady State>

Appendix B3: Proof of Lemma 3

(23) defines the indirect effect of $\frac{d\widehat{H}(t)}{dt}$ (resp., $\frac{d\widehat{R}(t)}{dt}$) as $h(t) \frac{d\eta(H(t),m)}{dt}$ (resp., $s(t) \frac{d\xi(H(t),R(t))}{dt}$) and the direct effect as $\eta(H(t),m) \frac{dh(t)}{dt}$ (resp., $\xi(H(t),R(t)) \frac{ds(t)}{dt}$).

For $\frac{d\widehat{H}(t)}{dt}$, we will prove (i) of Lemma 3 that there exists a small δ' such that, for $w(t) \in [0, \delta']$ around the pseudo steady state $w(t^*) = 0$, the indirect effect is greater than the direct effect as $\left| h(t) \frac{d\eta(H(t),m)}{dt} \right| > \left| \eta(H(t),m) \frac{dh(t)}{dt} \right|$. In the region where $\widehat{H}(t) > \widehat{R}(t)$, as t increases, the directions of these two effects, direct and indirect, differ, so we must verify the net effect on the capital growth rate. In contrast, the net effect is obvious in the region where $\widehat{H}(t) < \widehat{R}(t)$, because the directions, direct and indirect, coincide for $\frac{d\widehat{H}(t)}{dt}$. After dividing both sides of this inequality by $h(t)$, we can easily infer that $\left| \frac{d\eta(H(t),m)}{dt} \right| > \left| \eta(H(t),m) \frac{dh(t)}{dt} \frac{1}{h(t)} \right|$ holds if $\left| \frac{d\eta(H(t),m)}{dt} \right| > (1 + 0.5d_1)\eta_0 \left| \frac{dh(t)}{dt} \frac{1}{h(t)} \right|$. It is because $(1 + 0.5d_1)\eta_0 > \eta(H(t),m)$. This holds true because, as δ goes to zero, $\max_{t \in [t^*, t^* + \delta]} \left| \frac{dh(t)}{dt} \frac{1}{h(t)} \right|$ (and thus $(1 + 0.5d_1)\eta_0 \left| \frac{dh(t)}{dt} \frac{1}{h(t)} \right|$) converges to zero, since $\frac{dh(t)}{dt} \frac{1}{h(t)} = 0$ at t^* and this function is continuous in t . Here, we assumed that $t(\delta') = \delta > 0$, meaning the time at which $w(t) = \delta'$. Thus, we can adjust δ (or δ') to set the upper limit $(1 + 0.5d_1)\eta_0 \left| \frac{dh(t)}{dt} \frac{1}{h(t)} \right|$ of the direct effect $\eta(H(t),m) \left| \frac{dh(t)}{dt} \frac{1}{h(t)} \right|$ to any small magnitude we want. Thus, we can finish proving (i) of Lemma 3, by additionally proving that the indirect effect is greater than some positive constant for any $t \in [t^*, t^* + \delta]$ in the region of $\widehat{H}(t) > \widehat{R}(t)$. We can prove this as follows. $\left| \frac{d\eta(H(t),m)}{dt} \right|$ for any $t \in [t^*, t^* + \delta]$ is greater than a fixed positive constant because $\frac{d_1 mc_1 (c_2 - 1) H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 (1 - 0.5d_1)\eta_0 h(t^*) < \min_{t \in [t^*, t^* + \delta]} \left| \frac{d\eta(H(t),m)}{dt} \right| \leq \left| \frac{d\eta(H(t),m)}{dt} \right|$ for any $t \in [t^*, t^* + \delta]$. It is because $\frac{d\eta(H(t),m)}{dt} = \frac{d_1 mc_1 (c_2 - 1) H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 \eta(H(t),m) h(t)$, as we can see in the below, because $H(t)$ and $h(t)$ increases in t in the region of $\widehat{H}(t) > \widehat{R}(t)$, and because $(1 - 0.5d_1)\eta_0 < \eta(H(t),m)$. And also if $c_1 < \frac{c_m + m}{H(t^* + \delta)^{c_2 - 1}}$, which is satisfied with a sufficiently small c_1 , then $\frac{H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2}$ increases in t ($\because \frac{d}{dt} \ln \left(\frac{H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \right) > 0$, for all $t \in [t^*, t^* + \delta]$). Setting $x = H(t)^{c_2 - 1}$ with $\frac{d}{dt} x > 0$, $\frac{d}{dx} \ln \left(\frac{x}{(c_1 x + m)^2} \right) = \frac{1}{x} - 2c_1 \frac{1}{c_1 x + m} > 0$, for all $t \in [t^*, t^* + \delta] \leftrightarrow x < \frac{c_1 x + m}{2c_1}$, for all $t \in [t^*, t^* + \delta] \leftrightarrow c_1 x < m$, for all $t \in [t^*, t^* + \delta] \leftrightarrow c_1 < \frac{m}{x}$, for all $t \in [t^*, t^* + \delta] \leftrightarrow c_1 < \frac{m}{H(t^* + \delta)^{c_2 - 1}}$ ($\because H(t)$ increases in t and $c_2 > 1$)).

Thus, (i) of Lemma 3 will be proved by decreasing δ sufficiently such that $\eta(H(t),m) \frac{dh(t)}{dt} \frac{1}{h(t)} < (1 + 0.5d_1)\eta_0 \left| \frac{dh(t)}{dt} \frac{1}{h(t)} \right| < \frac{d_1 mc_1 (c_2 - 1) H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 (1 - 0.5d_1)\eta_0 h(t^*) < \left| \frac{d\eta(H(t),m)}{dt} \right|$.

To prove (ii) of Lemma 3, stating that ‘for the region $[w_1(H(t),R(t),m) + \delta, \alpha]$ with $w_1(H(t),R(t),m) + \delta < \alpha < 1$ and very small δ , we can set parameters of $c_2 - 1$ (c_1 or d_1) such that Assumption One (Equation (23)) holds’, we will verify the following two statements: Before proving them, we first define what $t(\alpha)$ denotes. $t(\alpha)$ means the time t at which $w(t) = \frac{\widehat{H}(t) - R(t)}{\widehat{H}(t)} = 1 - \frac{R(t)}{\widehat{H}(t)}$ equals α . $t(\alpha)$ strictly increases in α , because $w(t) = 1 - \widehat{R}(t)/\widehat{H}(t) = 1 - \xi(H(t),R(t))s(t)/\eta(H(t),m)h(t)$ strictly increases in t and in $h(t)$, when direct effect dominating indirect one in the region $[w_1(H(t),R(t),m) + \delta, \alpha]$ with $w_1(H(t),R(t),m) + \delta < \alpha < 1$, and also because (B17) and (B19) imply that each capital ratio corresponds to each $h(t)$ and thus $s(t) = 1 - m - h(t)$, since $h(t)$ strictly increases in $H(t)/R(t)$.

First, we will verify that $\max_{w \in [w_1(H(t),R(t),m) + \delta, \alpha]} \left| \frac{d\eta(H(t),m)}{dt} \right|$ converges to zero, as any one of the parameters ($c_2 - 1$, c_1 or d_1) approaches zero.

$$\begin{aligned} & \max_{w \in [w_1(H(t),R(t),m) + \delta, \alpha]} \left| \frac{d\eta(H(t),m)}{dt} \right| \\ &= \max_{w \in [w_1(H(t),R(t),m) + \delta, \alpha]} \left| \frac{d_1 mc_1 (c_2 - 1) H(t)^{c_2 - 2}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 \frac{dH(t)}{dt} \right| \end{aligned}$$

$$\begin{aligned}
&= \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \frac{d_1 m c_1 (c_2 - 1) H(t)^{c_2 - 2}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 \eta(H(t), m) h(t) H(t) \right| \\
&= \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \frac{d_1 m c_1 (c_2 - 1) H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 \eta(H(t), m) h(t) \right| \\
&\because \frac{dH(t)}{dt} = \eta(H(t), m) h(t) H(t). \\
&< \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| (1 - m) \frac{d_1 m c_1 (c_2 - 1) H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 \eta(H(t), m) \right| \\
&\because h(t) < 1 - m \\
&< \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| (1 - m) \frac{d_1 m c_1 (c_2 - 1) H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 \eta_0 (1 + 0.5d_1) \eta_0 \right| \\
&\because \eta(H(t), m) < (1 + 0.5d_1) \eta_0 \\
&= (1 - m) d_1 m (c_2 - 1) (1 + 0.5d_1) \eta_0^2 \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \frac{c_1 H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \eta_0 \right| \\
&\leq (1 - m) d_1 m (c_2 - 1) (1 + 0.5d_1) \eta_0^2 \frac{c_1 H(t(\alpha))^{c_2 - 1}}{(c_1 H(t(\alpha))^{c_2 - 1} + m)^2} \\
&\because \text{assuming } \frac{m}{c_1 H(t)^{c_2 - 1}} > 1, \text{ then } \frac{c_1 H(t)^{c_2 - 1}}{(c_1 H(t)^{c_2 - 1} + m)^2} \text{ increases in } c_1 H(t)^{c_2 - 1} \text{ and in } t \quad (\because \frac{d}{dx} \ln \left(\frac{x}{(x+m)^2} \right) > 0 \text{ with } x = c_1 H(t)^{c_2 - 1}, \text{ for all } t \leftarrow \text{Setting } x = c_1 H(t)^{c_2 - 1} \text{ with } \frac{d}{dt} x > 0, \frac{d}{dx} \ln \left(\frac{x}{(x+m)^2} \right) = \frac{1}{x} - 2 \frac{1}{x+m} > 0, \text{ for all } t \leftrightarrow x < \frac{x+m}{2}, \text{ for all } t \leftrightarrow x < m, \text{ for all } t).
\end{aligned}$$

Thus, the first statement is proved.

The second statement we will prove is: $|\eta(H(t), m) \frac{dh(t)}{dt} \frac{1}{h(t)}|$ for any $w(t) \in [w_1(H(t), R(t), m) + \delta, \alpha]$ is greater than some positive constant, irrespective of any parameter values of $(c_2 - 1, c_1 \text{ or } d_1)$.

$$\begin{aligned}
&|\eta(H(t), m) \frac{dh(t)}{dt} \frac{1}{h(t)}| \text{ for all } w(t) \in (w_1(H(t), R(t), m) + \delta, \alpha] \\
&\geq \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} |\eta(H(t), m) \frac{dh(t)}{dt} \frac{1}{h(t)}| \\
&= \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \eta(H(t), m) \frac{s(t)}{1-m} (\varepsilon - 1) (\widehat{H(t)} - \widehat{R(t)}) \right| \\
&\because \text{from (19)} \quad \widehat{h(t)} = \frac{s(t)}{1-m} (\varepsilon - 1) (\widehat{H(t)} - \widehat{R(t)}) \\
&> \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \eta(H(t), m) \frac{s(t)}{1-m} (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta) \widehat{H(t)} \right| \\
&\because w(t) = (\widehat{H(t)} - \widehat{R(t)}) / \widehat{H(t)}. \\
&= \frac{1}{1-m} (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta) \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} |\eta(H(t), m) \eta(H(t), m) h(t) s(t)| \\
&\because \widehat{H(t)} = \eta(H(t), m) h(t).
\end{aligned}$$

$$> \frac{((1 - 0.5d_1)\eta_0)^2}{1 - m} (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta) \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} |h(t)s(t)|$$

$$\because \eta(H(t), m) > (1 - 0.5d_1)\eta_0.$$

$$> \frac{((1 - 0.5d_1)\eta_0)^2}{1 - m} (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta) h(t(w_1)) s(t(\alpha))$$

$$> \frac{((1 - 0.5d_1)\eta_0)^2}{1 - m} (\varepsilon - 1)(w_1(H(t), R(t), m) + \delta)h(t^*)s(t(\alpha))$$

$\because h(t)$ increases while $s(t)$ decreases in t in the region $w(t) = 1 - \frac{\widehat{R(t)}}{H(t)} \in [w_1(H(t), R(t) + \delta, m), \alpha]$.

$$\cong ((1 - 0.5d_1)\eta_0)^2 \frac{h(t^*)}{1 - m} \{ (c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} + (\varepsilon - 1)\delta \} s(t(\alpha))$$

(\because It is because the convergence speed of $w_1(H(t), R(t), m)$ equals $\frac{-C(t)}{B(t)}$, which is shown as follows. The limit here means taking limit with respect to the parameters ($c_2 - 1$, c_1 and d_1) going to zero with $H(t)/R(t) = H(t^*)/R(t^*)$. And note that $A(t)$ and $B(t)$ defined in (B13) are not affected with these parameters. Let $x(t) = D(t) - \sqrt{D(t)^2 - E(t)} = w_1(H(t), R(t), m)$ and $y(t) = D(t) + \sqrt{D(t)^2 - E(t)} = w_2(H(t), R(t), m)$, where $D(t) = -B(t)/(2A(t)) > 0$, $C(t) = -(c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} < 0$, and $E(t) = C(t)/A(t)$ with $A(t), B(t)$ and $C(t)$ defined in (B13).

$\lim_{\text{parameters} \rightarrow 0} \{D(t)\} = -\frac{0.5B(t)}{A(t)} = D(t)$, and $\lim_{\text{parameters} \rightarrow 0} \{E(t)\} = \lim_{\text{parameters} \rightarrow 0} \{C(t)\}/A(t) = 0$. Then, since $x(t)y(t) = E(t) = C(t)/A(t)$, for an arbitrary very small ω , $\lim_{\text{parameters} \rightarrow \omega} \{C(t)/A(t)\} = \lim_{\text{parameters} \rightarrow \omega} \{C(t)\}/A(t) = \lim_{\text{parameters} \rightarrow \omega} \{E(t)\} = \lim_{\text{parameters} \rightarrow \omega} \{x(t)y(t)\} = \lim_{\text{parameters} \rightarrow \omega} \{y(t)\} \lim_{\text{parameters} \rightarrow \omega} \{x(t)\} \cong 2D(t) \lim_{\text{parameters} \rightarrow \omega} \{x(t)\} = -B(t)/A(t) \lim_{\text{parameters} \rightarrow \omega} \{w_1(H(t), R(t), m)\}$. Thus, if the parameters are small enough, $w_1(H(t), R(t), m) \cong \frac{-C(t)}{B(t)} \cong (c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} \frac{1}{\varepsilon - 1}$ where $B(t) \cong (\varepsilon - 1)$.)

To finish proving the latter part of (ii) of Lemma 3, we should additionally prove that

$$(1 - m)d_1 m(c_2 - 1)(1 + 0.5d_1)\eta_0^2 \frac{\frac{c_1 H(t(\alpha))^{c_2-1}}{(c_1 H(t(\alpha))^{c_2-1} + m)^2}}{\cong ((1 - 0.5d_1)\eta_0)^2 \frac{h(t^*)}{1 - m} \{ (c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} + (\varepsilon - 1)\delta \} s(t(\alpha))}$$

, as $c_2 - 1$, c_1 or d_1 decreases enough. It is easy to prove this because as any one of these parameters goes to zero, LHS of this inequality approaches zero while its RHS does not due to a positive constant δ .

For $\frac{d\widehat{R(t)}}{dt}$, for the proof of (i) of Lemma 3, we will prove that there exists a small δ such that, for any $t \in [t^*, t^* + \delta]$, around the pseudo steady state time t^* , the direct effect is greater than the indirect effect, as $|s(t) \frac{d\xi(H(t), R(t))}{dt}| < |\xi(H(t), R(t)) \frac{ds(t)}{dt}|$. After dividing both sides of this inequality by $s(t)$, the inequality of $\left| \frac{d\xi(H(t), R(t))}{dt} \right| < \xi(H(t), R(t)) \left| \frac{ds(t)}{dt} \frac{1}{s(t)} \right|$ holds if $\left| \frac{d\log \xi(H(t), R(t))}{d\log(\frac{H(t)}{R(t)})} (\widehat{H(t)} - \widehat{R(t)}) \right| < \left| \frac{ds(t)}{dt} \frac{1}{s(t)} \right| = \left| \frac{1-m-s}{1-m} (\varepsilon - 1) (\widehat{H(t)} - \widehat{R(t)}) \right| \leftrightarrow \left| \frac{d\log \xi(H(t), R(t))}{d\log(\frac{H(t)}{R(t)})} \right| < \left| \frac{1-m-s}{1-m} (\varepsilon - 1) \right|$. It is because $\frac{d\xi(H(t), R(t))}{dt} = \frac{d\xi(H(t), R(t))}{d\log(\frac{H(t)}{R(t)})} \frac{d\log(\frac{H(t)}{R(t)})}{dt} = \frac{d\xi(H(t), R(t))}{d\log(\frac{H(t)}{R(t)})} (\widehat{H(t)} - \widehat{R(t)}) = \xi(H(t), R(t)) \frac{d\log \xi(H(t), R(t))}{d\log(\frac{H(t)}{R(t)})} (\widehat{H(t)} - \widehat{R(t)})$. It is also because $\frac{d\log \xi(H(t), R(t))}{d\log(\frac{H(t)}{R(t)})} = \frac{c_4}{1 + c_4 (\frac{x}{c_3+x})^\theta} (\frac{x}{c_3+x})^{\theta-1} \theta \frac{c_3}{(c_3+x)^2} x$ where $x = \frac{H(t)}{R(t)}$. Thus, since $\frac{H(t)}{R(t)}$ and $s(t)$ is bounded for all $t \in [t^*, t^* + \delta]$, as θ , c_4 or c_3 approaches zero, $\left| \frac{d\log \xi(H(t), R(t))}{d\log(\frac{H(t)}{R(t)})} \right| = \left| \frac{c_4}{1 + c_4 (\frac{x}{c_3+x})^\theta} (\frac{x}{c_3+x})^{\theta-1} \theta \frac{c_3}{(c_3+x)^2} x \right| < \left| \frac{1-m-s}{1-m} (\varepsilon - 1) \right|$ for any $t \in [t^*, t^* + \delta]$ will hold. Also, with the fact that as δ approaches zero, s converges to s^* and x converges to $H(t^*)/R(t^*)$, we prove that there exists a small δ such that, for any $t \in [t^*, t^* + \delta]$, the indirect effect is dominated by the direct effect as $|s(t) \frac{d\xi(H(t), R(t))}{dt}| < |\xi(H(t), R(t)) \frac{ds(t)}{dt}|$ when θ , c_4 or c_3 approaches zero. However, considering both $|\xi(H(t), R(t)) \frac{ds(t)}{dt}| < \xi_0(1 + c_4) \left| \frac{ds(t)}{dt} \right|$ and the fact that $\frac{ds(t^*)}{dt} = 0$ as δ approaches zero, $|\xi(H(t), R(t)) \frac{ds(t)}{dt}|$ will converge to zero. Thus, in the region of $t \in [t^*, t^* + \delta]$, we can easily infer that the net effect of $\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt}$ will be negative with a sufficiently small δ by considering both cases of $\frac{d\widehat{H(t)}}{dt}$ and $\frac{d\widehat{R(t)}}{dt}$, proving (i) of Lemma 3.

And for $\frac{d\widehat{R}(t)}{dt}$, since the directions of the two effects are different for both regions of $\widehat{H}(t) > \widehat{R}(t)$ and $\widehat{H}(t) < \widehat{R}(t)$, thus to prove (ii) of Lemma 3 we should take into consideration both of these two regions.

To prove the latter part of (ii) of the Lemma 3, that ‘for the region $w \in [w_1(H(t), R(t), m) + \delta, \alpha]$, with $w_1(H(t), R(t), m) + \delta < \alpha < 1$, we can set the parameters $(\theta, c_4$ or $c_3)$, such that Assumption One (Equation (23)) holds’, we have only to verify the following two statements. First, we will verify that $\max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \frac{d\xi(H(t), R(t))}{dt} \right|$ converges to zero, as any of the parameters $(\theta, c_4$ or $c_3)$ approaches zero.

$$\begin{aligned}
& \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \frac{d\xi(H(t), R(t))}{dt} \right| \\
&= \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \frac{d\xi(H(t), R(t))}{d(\frac{H(t)}{R(t)})} \frac{d(\frac{H(t)}{R(t)})}{d(\ln(\frac{H(t)}{R(t)}))} \frac{d(\ln(\frac{H(t)}{R(t)}))}{dt} \right| \\
&= \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \xi_0 \theta c_4 \left(1 - \frac{c_3}{c_3 + H(t)/R(t)}\right)^{\theta-1} c_3 \left(\frac{1}{c_3 + H(t)/R(t)}\right)^2 \frac{H(t)}{R(t)} ((\widehat{H}(t) - \widehat{R}(t)) \right| \\
&= \xi_0 \theta c_4 c_3 \max_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \left(1 - \frac{c_3}{c_3 + H(t)/R(t)}\right)^{\theta-1} \left(\frac{1}{c_3 + H(t)/R(t)}\right)^2 \frac{H(t)}{R(t)} ((\widehat{H}(t) - \widehat{R}(t)) \right| \\
&< \xi_0 \theta c_4 c_3 \max_{w \in [w_1(H(t), R(t), m), \alpha]} \left| \left(1 - \frac{c_3}{c_3 + H(t)/R(t)}\right)^{\theta-1} \frac{1}{c_3 + H(t)/R(t)} ((\widehat{H}(t) - \widehat{R}(t)) \right| \\
&< \xi_0 \theta c_4 c_3 \left(1 - \frac{c_3}{c_3 + \frac{H(t^*)}{R(t^*)}} \right)^{\theta-1} \frac{1}{c_3 + \frac{H(t^*)}{R(t^*)}} \eta_0 (1 + 0.5d_1)(1 - m)
\end{aligned}$$

Thus, the first statement is proved. The second statement we will prove is: $\left| \xi(H(t), R(t)) \frac{ds(t)}{dt} \frac{1}{s(t)} \right|$ for any $w \in [w_1(H(t), R(t), m) + \delta, \alpha]$ is greater than some positive constant, irrespective of the parameters $(\theta, c_4$ or $c_3)$.

$$\begin{aligned}
& \left| \xi(H(t), R(t)) \frac{ds(t)}{dt} \frac{1}{s(t)} \right| \\
&\geq \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \left| \xi(H(t), R(t)) \frac{ds(t)}{dt} \frac{1}{s(t)} \right| \\
&> \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \xi_0 \left| \frac{ds(t)}{dt} \frac{1}{s(t)} \right| \\
&\because \xi(H(t), R(t)) > \xi_0. \\
&= \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \xi_0 \left| \frac{h(t)}{1-m} (\varepsilon - 1) (\widehat{H}(t) - \widehat{R}(t)) \right| \\
&\because \text{from (19)} \quad \widehat{s(t)} = -\frac{h(t)}{1-m} (\varepsilon - 1) (\widehat{H}(t) - \widehat{R}(t)) \\
&> \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} \xi_0 \left| \frac{h(t)}{1-m} (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta) \widehat{H}(t) \right| \\
&\because w_1(H(t), R(t), m) + \delta \leq w(t) = (\widehat{H}(t) - \widehat{R}(t)) / \widehat{H}(t). \\
&= \frac{1}{1-m} \xi_0 (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta) \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} |\eta(H(t), m) h(t)^2| \\
&\because \widehat{H}(t) = \eta(H(t), m) h(t). \\
&> \frac{\xi_0 (1 - 0.5d_1) \eta_0}{1 - m} (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta) \min_{w \in [w_1(H(t), R(t), m) + \delta, \alpha]} |h(t)^2| \\
&\because \eta(H(t), m) > (1 - 0.5d_1) \eta_0.
\end{aligned}$$

$$> \frac{\xi_0(1-0.5d_1)\eta_0}{1-m} h(t^*)^2 (\varepsilon - 1) (w_1(H(t), R(t), m) + \delta)$$

$\because h(t)$ increases in t in the region $w(t) = 1 - \frac{\widehat{R}(t)}{\widehat{H}(t)} \in [w_1(H(t), R(t), m) + \delta, \alpha]$.

$$\simeq \frac{\xi_0(1-0.5d_1)\eta_0}{1-m} h(t^*)^2 \left\{ (c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} + (\varepsilon - 1) \delta \right\}$$

\because Refer to the corresponding case of $\frac{d\widehat{H}(t)}{dt}$.

Thus, the second statement will be proved by additionally verifying that when the parameters (θ , c_4 or c_3) approach zero, $\xi_0(1-0.5d_1) \frac{h(t^*)}{1-m} \eta_0 \{ (c_2 - 1) \left(\frac{d_1 c_1 H(t)^{c_2-1} m}{(c_1 H(t)^{c_2-1} + m)^2} \right) \frac{1}{1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} + (\varepsilon - 1) \delta \} h(t^*)$ remain greater than some positive constant, because C and δ will not be affected by these parameters by (B13).

And in the region of $w \in [w_1(H(t), R(t), m) + \delta, \alpha]$, we can also easily infer that the net effect of $\frac{d(\widehat{H}(t) - \widehat{R}(t))}{dt}$ will be positive by considering both cases of $\frac{d\widehat{H}(t)}{dt}$ and $\frac{d\widehat{R}(t)}{dt}$.

We will prove the statement of (iii) of Lemma 3, “If $w(t) \in (-\infty, 0]$, decadence equilibrium, $\widehat{H}(t) < \widehat{R}(t)$ ($\widehat{h}(t) < 0$ and $\widehat{s}(t) > 0$) with $\frac{d(\widehat{H}(t) - \widehat{R}(t))}{dt} < 0$, persists for all t ”. It is because: In the region $w(t) \in (-\infty, 0)$, for $\frac{d\widehat{H}(t)}{dt}$, the directions of both the direct and the indirect effect coincide to be negative, while for $\frac{d\widehat{R}(t)}{dt}$ they do not with the positive direct effect and the negative indirect effect. However, even if $\frac{d}{dt}w(t) = \frac{d((\widehat{H}(t) - \widehat{R}(t))/\widehat{H}(t))}{dt} > 0$, meaning $w(t)$ increasing in t , with the indirect effect of $\frac{d\widehat{R}(t)}{dt}$ negative enough, $w(t)$ cannot increase above zero. It is because as $w(t)$ approaches zero, as previously proved, the indirect effect converges to zero, which makes $\frac{d(\widehat{H}(t) - \widehat{R}(t))}{dt}$ negative, implying $\frac{d}{dt}w(t) = \frac{d((\widehat{H}(t) - \widehat{R}(t))/\widehat{H}(t))}{dt} < 0$, forcing $w(t)$ to be negative again. It is because at $w(t) = 0$ (i.e., $\frac{\widehat{H}(t)}{\widehat{R}(t)} = 1$), $\frac{d(\widehat{H}(t) - \widehat{R}(t))}{dt} < 0$ ($\leftrightarrow \frac{d\widehat{H}(t)}{dt} < \frac{d\widehat{R}(t)}{dt}$) implies $\frac{d}{dt}(\frac{\widehat{H}(t)}{\widehat{R}(t)}) < 0$ ($\leftrightarrow \frac{d(\frac{\widehat{H}(t) - \widehat{R}(t)}{\widehat{H}(t)})}{dt} = \frac{d}{dt}w(t) < 0$). Thus, we proved the statement “If $w(t) \in (-\infty, 0]$, decadence equilibrium, $(\widehat{h}(t) < 0$ and $\widehat{s}(t) > 0$) with $\widehat{H}(t) < \widehat{R}(t)$ persists for all t ”.

Finally, we will prove the last statement of (iv) of Lemma 3, “And if $w(t) \in [0, w_1(H(t), R(t), m)]$, $w(t)$ will continuously decrease to be negative, eventually resulting in decadence equilibrium $(\widehat{h}(t) < 0$ and $\widehat{s}(t) > 0$) with $\widehat{H}(t) - \widehat{R}(t) < 0$ persisting afterwards”. It is because: In the region $w(t) \in [0, w_1(H(t), R(t), m)]$, where $\widehat{H}(t) \geq \widehat{R}(t)$ ($\widehat{h}(t) \geq 0$ and $\widehat{s}(t) \leq 0$), with both $\frac{d(\widehat{H}(t) - \widehat{R}(t))}{dt} \leq 0$ ($\leftrightarrow \frac{d\widehat{H}(t)}{dt} \leq \frac{d\widehat{R}(t)}{dt}$) and $\frac{\widehat{H}(t)}{\widehat{R}(t)} > 1$, or with both $\frac{d(\widehat{H}(t) - \widehat{R}(t))}{dt} < 0$ ($\leftrightarrow \frac{d\widehat{H}(t)}{dt} < \frac{d\widehat{R}(t)}{dt}$) and $\frac{\widehat{H}(t)}{\widehat{R}(t)} \geq 1$, it is easy to infer that $w(t)$ will continuously decrease to zero. It is because: Assuming $\frac{\widehat{H}(t)}{\widehat{R}(t)} \geq z > 0$ if $\frac{d\widehat{H}(t)}{dt} < z \frac{d\widehat{R}(t)}{dt}$, or assuming $\frac{\widehat{H}(t)}{\widehat{R}(t)} > z > 0$ if $\frac{d\widehat{H}(t)}{dt} \leq z \frac{d\widehat{R}(t)}{dt}$, then $\frac{\widehat{H}(t)}{\widehat{R}(t)}$ will decrease in t . It is because the ratio will be $\frac{\widehat{H}(t+\Delta t)}{\widehat{R}(t+\Delta t)} = \frac{\widehat{H}(t) + \frac{d\widehat{H}(t)}{dt} \Delta t}{\widehat{R}(t) + \frac{d\widehat{R}(t)}{dt} \Delta t} < \frac{\widehat{H}(t)}{\widehat{R}(t)}$ at $t + \Delta t$. Here $z = 1$.

Appendix B4: Proof of Lemma 4

If $H(t)$ increases sufficiently in t with $c_2 - 1 > 0$ (or if $c_2 - 1$ (or c_1 or d_1) is positive and close to zero), the term $C = -(c_2 - 1) \left(\frac{c_1 H(t)^{c_2-1}}{c_1 H(t)^{c_2-1} + m} \right) \left(1 - \frac{1-0.5d_1}{1+d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)} \right)$ converges to zero, shifting the real line curve to the dotted line curve in <Graph 1>, resulting in $w_1(H(t), R(t), m)$ decreasing close to 0. And because A decreases to $-\frac{h(t)}{h(t)+s(t)} (\varepsilon - 1)$ while B increases to $\varepsilon - 1$, as $\frac{H(t)}{R(t)}$ and $H(t)$ go to infinity, $w_2(H(t), R(t), m)$ converges to $1 + \frac{s(t)}{h(t)}$ and finally reaches one, since

$s(t)$ converges to 0 while $h(t)$ to $1-m$. In this case, noting that $w_1(H(t), R(t), m)$ decreases in $\frac{H(t)}{R(t)}$, it will be easily satisfied with an appropriate increase in $h(t)$ and $H(t)$ that

$$w(t) = 1 - \frac{\xi(H(t), R(t))s(H(t), R(t))}{\eta(H(t), m)h(H(t), R(t))}$$

$$= 1 - (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 s(H(t), R(t)) / (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right) \eta_0 h(H(t), R(t))) \geq w_1(H(t), R(t), m) + \delta.$$

Appendix B5: Proof of Lemma 5

Since (6) holds for all t , differentiating both sides of (6) with respect to t gives

$$(6) \quad \frac{h(t)}{s(t)} = \gamma^\varepsilon \left(\frac{H(t)}{R(t)} \right)^{\varepsilon-1} \Rightarrow \widehat{h(t)} = \widehat{s(t)} + (\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)}) \rightarrow (\widehat{h(t)} - \widehat{s(t)}) = (\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)}).$$

From (6) and (B10), we have:

$$\frac{d}{dt} (\ln(\widehat{H(t)}) - \ln(\widehat{R(t)})) = (\widehat{\eta(t)'} - \widehat{\xi(t)'}) + (\widehat{h(t)} - \widehat{s(t)}) = (\widehat{\eta(t)'} - \widehat{\xi(t)'}) + (\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)})$$

Note that the sign of $\widehat{\eta(t)'}$ (resp., $\widehat{\xi(t)'}$) is dominated by the sign of $\widehat{h(t)}$ (resp., $\widehat{s(t)}$) in this equation, because θ (or c_4) and $c_2 - 1$ (or c_1 or d_1) are positive and close to zero, ensuring that (23) holds due to Assumption One.

$$(B14) \quad \text{sign} \left(\frac{d\widehat{H(t)}}{dt} = \frac{d(\eta(H(t), m)h(t))}{dt} = \frac{h(t)d\eta(H(t), m)}{dt} + \frac{\eta(H(t), m)dh(t)}{dt} \right) = \text{sign} \left(\frac{\eta(H(t), m)dh(t)}{dt} \right),$$

$$\text{and } \text{sign} \left(\frac{d\widehat{R(t)}}{dt} = \frac{d(\xi(H(t), R(t))s(t))}{dt} = \frac{s(t)d\xi(H(t), R(t))}{dt} + \frac{\xi(H(t), R(t))ds(t)}{dt} \right) = \text{sign} \left(\frac{\xi(H(t), R(t))ds(t)}{dt} \right).$$

Thus, we prove the first and the second equalities of $\text{sign}((\varepsilon - 1)(\widehat{H(t)} - \widehat{R(t)})) = \text{sign}(\widehat{h(t)} - \widehat{s(t)}) = \text{sign}(\frac{\widehat{H(t)}}{\widehat{H(t)}} - \frac{\widehat{R(t)}}{\widehat{R(t)}})$.

Finally since $\frac{d}{dt} \frac{\widehat{H(t)}}{\widehat{R(t)}} = \frac{\widehat{H(t)}\widehat{R(t)} - \widehat{R(t)}\widehat{H(t)}}{\widehat{R(t)}^2} = \frac{\widehat{R(t)}\widehat{H(t)}}{\widehat{R(t)}^2} \left(\frac{\widehat{H(t)}}{\widehat{H(t)}} - \frac{\widehat{R(t)}}{\widehat{R(t)}} \right)$, with $\widehat{H(t)}, \widehat{R(t)} > 0$, we can prove the third equality.

We can prove the fourth equality, using the third equality and deriving

$$\text{sign} \left(\frac{d}{dt} \left(\frac{\widehat{H(t)} - \widehat{R(t)}}{\widehat{H(t)}} \right) \right) = \text{sign} \left(-\frac{d}{dt} \left(\frac{\widehat{R(t)}}{\widehat{H(t)}} \right) \right) = \text{sign} \left(\frac{\widehat{H(t)}}{\widehat{H(t)}} - \frac{\widehat{R(t)}}{\widehat{R(t)}} \right) = \text{sign} \left(\frac{d}{dt} \frac{\widehat{H(t)}}{\widehat{R(t)}} \right).$$

For the last equality, considering that $h(t) + s(t) = 1 - m$ after reaching the pseudo steady state (meaning $\text{sign} \left(\frac{dh(t)}{dt} \right) = \text{sign} \left(-\frac{ds(t)}{dt} \right)$), we can easily derive $\text{sign} \left(\frac{dh(t)}{dt} \right) = \text{sign} \left(\frac{dh(t)}{h(t)dt} \right) = \text{sign} \left(-\frac{ds(t)}{dt} \right) = \text{sign} \left(-\frac{ds(t)}{s(t)dt} \right) = \text{sign} \left(\frac{d\widehat{H(t)}}{dt} \right) = \text{sign} \left(-\frac{d\widehat{R(t)}}{dt} \right) = \text{sign} \left(-\frac{d\widehat{R(t)}}{R(t)dt} \right)$. Thus, we can easily prove that $\text{sign} \left(\frac{\widehat{H(t)}}{\widehat{H(t)}} - \frac{\widehat{R(t)}}{\widehat{R(t)}} \right) = \text{sign}(\widehat{H(t)} - \widehat{R(t)})$, since $\widehat{H(t)}$ has the opposite sign of $\widehat{R(t)}$ and $\widehat{H(t)}, \widehat{R(t)} > 0$.

Appendix B6: Proof of Proposition 3

Since $w_1(H(t), R(t), m)$ decreases in $\frac{H(t)}{R(t)}$, we can make $w(t) = (\widehat{H(t)} - \widehat{R(t)})/\widehat{H(t)}$ greater than or equal to $w_1(H(t), R(t), m) + \delta$ by sufficiently increasing η_0 (or by increasing m) or by decreasing ξ_0 . This shift prompts agents to adopt the science strategy, leading to increased consumption of scientific ideas and decreased consumption of religious ideas.

By executing a one-time government policy intervention to sufficiently increase η_0 or m , or to sufficiently decrease ξ_0 , c_1 or $c_2 - 1$, the condition $w(t) \geq w_1(H(t), R(t), m) + \delta$ will be satisfied. It is because these policies increase $\frac{H(t)}{R(t)}$ and thus $w(t)$, and because $w_1(H(t), R(t), m)$ decreases in $\frac{H(t)}{R(t)}$. This ensures that $\frac{d(\widehat{H(t)} - \widehat{R(t)})}{dt} > 0$ ($\leftrightarrow \frac{d}{dt}w(t) = \frac{d}{dt}(\widehat{H(t)} - \widehat{R(t)})/\widehat{H(t)} > 0$), causing $w(t)$ to continuously increase to be greater than $w_1(H(t), R(t), m) + \delta$ close to zero afterwards, ensuring that agents stick to the science strategy. ($\widehat{H(t)} - \widehat{R(t)} > 0 \rightarrow \widehat{h(t)} - \widehat{s(t)} > 0 \rightarrow \widehat{h(t)} > 0$ and $\widehat{s(t)} < 0$ (due to $h(t) + s(t) = 1 - m \rightarrow \widehat{H(t)} - \widehat{R(t)} \gg 0$ (due to Assumption One) \rightarrow go to the first step of this positive feedback loop).

Appendix B7: Proof of Proposition 4

Assume that θ (or c_4) and $c_2 - 1$ (or d_1 or c_1) are positive and close to zero. Based on Lemma 5, if all the other agents take the science strategy (increasing $h(t)$ and decreasing $s(t)$ possibly due to mob psychology) with the decision of $\{\widehat{h(t)} > 0 \text{ and } \widehat{s(t)} < 0\}$, then it follows that $\widehat{H(t)} > \widehat{R(t)}$ and $\frac{d}{dt}\frac{\widehat{H(t)}}{\widehat{R(t)}} = \frac{d}{dt}w(t) > 0$, resulting in $w(t) = (\widehat{H(t)} - \widehat{R(t)})/\widehat{H(t)} \geq w_1(H(t), R(t), m) + \delta$ for all $t > t^*$, as we can see from <Graph 1>. Note here that $w_1(H(t), R(t), m)$ decreases in $\frac{H(t)}{R(t)}$. Thus, any agent will take the same strategy. In contrast, if all the other agents take the decadence (religion) strategy (to increase $s(t)$ and to decrease $h(t)$ possibly due to mob psychology) of $\{\widehat{h(t)} < 0, \widehat{s(t)} > 0\}$, then it follows that $\widehat{H(t)} < \widehat{R(t)}$ and $\frac{d}{dt}\frac{\widehat{H(t)}}{\widehat{R(t)}} < 0$, for all $t > t^*$, resulting in $w(t) < 0$ for all $t > t^*$ as we can see from <Graph 1>. Thus, any agent will take the same strategy. Thus, the proposition is proved: once a certain strategy (either a science or a decadence strategy) is adopted by the majority of agents, the dynamics reinforce the initial majority choice, inducing the entire population to join this strategy based on majority influences.

Appendix B8: Transforming an Initial Condition of $\{H(0), R(0)\}$ into that of $\{h(0), s(0)\}$, and Providing the Closed Form Solutions for $h(t)$ and $s(t)$ in $H(t)$ and $R(t)$.

From (3), we have

$$\frac{\partial H'}{\partial h'} = 0: v'^{\frac{1}{\sigma}} c'^{\frac{-1}{\sigma}} A H^\kappa = \gamma \beta v'^{\frac{1}{\sigma}} \omega'^{\frac{-1}{\sigma}} \omega'^{\frac{1}{\varepsilon}} (H h')^{\frac{-1}{\varepsilon}} H$$

which is simplified to

$$c'^{\frac{-1}{\sigma}} A H^\kappa = \gamma \beta \omega'^{\frac{-1}{\sigma}} \omega'^{\frac{1}{\varepsilon}} (H h')^{\frac{-1}{\varepsilon}} H.$$

This gives,

$$\begin{aligned} (A(1 - s' - h') H^{\kappa-1})^{\frac{-1}{\sigma}} A H^{\kappa-1} &= \gamma \beta \left(\left\{ \left(\frac{R}{H} s' \right)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma (h')^{\frac{\varepsilon-1}{\varepsilon}} \right\}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{\frac{-1}{\sigma} + \frac{1}{\varepsilon}} (h')^{\frac{-1}{\varepsilon}} \\ &\rightarrow (A(1 - m - s - h))^{\frac{-1}{\sigma}} A H^{\frac{(\sigma-1)(\kappa-1)}{\sigma}} = \gamma \beta \left(\left\{ \left(\frac{R}{H} s \right)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma (h)^{\frac{\varepsilon-1}{\varepsilon}} \right\}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{\frac{-1}{\sigma} + \frac{1}{\varepsilon}} (h)^{\frac{-1}{\varepsilon}} \end{aligned}$$

Setting $A'^{\frac{\sigma-1}{\sigma}} = A^{\frac{\sigma-1}{\sigma}} H^{\frac{(\sigma-1)(\kappa-1)}{\sigma}}$ and $\gamma = \beta = 1$ to simplify the problem, we have:

$$\begin{aligned} (B15) \quad (1 - m - s - h)^{\frac{-1}{\sigma}} A'^{\frac{\sigma-1}{\sigma}} H^{\frac{(\sigma-1)(\kappa-1)}{\sigma}} &= \left[\left\{ \left(\frac{R}{H} s \right)^{\frac{\varepsilon-1}{\varepsilon}} + (h)^{\frac{\varepsilon-1}{\varepsilon}} \right\}^{\frac{\varepsilon}{\varepsilon-1}} \right]^{\frac{-1}{\sigma} + \frac{1}{\varepsilon}} (h)^{\frac{-1}{\varepsilon}} \rightarrow \\ ((1 - m - h)/h - s/h)^{\frac{-1}{\sigma}} A'^{\frac{\sigma-1}{\sigma}} &= \left[\left\{ \left(\frac{R}{H} s/h \right)^{\frac{\varepsilon-1}{\varepsilon}} + 1 \right\}^{\frac{\varepsilon}{\varepsilon-1}} \right]^{\frac{-1}{\sigma} + \frac{1}{\varepsilon}} \rightarrow \\ ((1 - m - h)/h - s/h)^{\frac{-1}{\sigma}} A'^{\frac{\sigma-1}{\sigma}} &= \left[\left\{ \left(\frac{R}{H} s/h \right)^{\frac{\varepsilon-1}{\varepsilon}} + 1 \right\}^{\frac{\varepsilon}{\varepsilon-1}} \right]^{\frac{-1}{\sigma} + \frac{1}{\varepsilon}} \rightarrow \end{aligned}$$

$$(6) \quad \begin{aligned} ((1-m-h)/h - s/h) A'^{1-\sigma} &= \{(\frac{R}{H} s/h)^{\frac{\varepsilon-1}{\varepsilon}} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} \rightarrow \\ \frac{h'}{s'} &= \gamma^\varepsilon (\frac{R}{H})^{\varepsilon-1} \rightarrow \frac{h}{s} = \gamma^\varepsilon (\frac{R}{H})^{\varepsilon-1} \rightarrow \frac{s}{h} = \gamma^{-\varepsilon} (\frac{R}{H})^{\varepsilon-1} \end{aligned}$$

Substituting (6) into (B15), this gives a solution for $h(t)$ in terms of $R(t)$ and $H(t)$:

(B16)

$$\begin{aligned} ((1-m-h)/h - s/h) A'^{1-\sigma} &= \{(\frac{R}{H} s/h)^{\frac{\varepsilon-1}{\varepsilon}} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} \rightarrow \\ ((1-m-h)/h - (\frac{R}{H})^{\varepsilon-1}) A'^{1-\sigma} &= \{(\frac{R}{H} (\frac{R}{H})^{\varepsilon-1})^{\frac{\varepsilon-1}{\varepsilon}} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} \rightarrow \\ (1-m-h)/h - (\frac{R}{H})^{\varepsilon-1} &= A'^{\sigma-1} \{(\frac{R}{H} (\frac{R}{H})^{\varepsilon-1})^{\frac{\varepsilon-1}{\varepsilon}} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} \rightarrow \\ (1-m-h)/h &= A'^{\sigma-1} \{(\frac{R}{H})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R}{H})^{\varepsilon-1} \rightarrow \\ \frac{1-m}{h} - 1 &= A'^{\sigma-1} \{(\frac{R}{H})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R}{H})^{\varepsilon-1} \rightarrow \\ (1-m)/h &= A'^{\sigma-1} \{(\frac{R}{H})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R}{H})^{\varepsilon-1+1} \rightarrow \\ (B17) \quad h &= [A'^{\sigma-1} H^{(\sigma-1)(\kappa-1)} \{(\frac{R}{H})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R}{H})^{\varepsilon-1} + 1]^{-1} (1-m) \rightarrow \\ h(1 + (\frac{R}{H})^{\varepsilon-1} + A'^{\sigma-1} H^{(\sigma-1)(\kappa-1)} \{(\frac{R}{H})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}}) &= (1-m) \end{aligned}$$

From (6) ($s = (\frac{R}{H})^{\varepsilon-1} h$) and (B17), we have

$$(B18) \quad u = (1-m-s-h) = A'^{\sigma-1} H^{(\sigma-1)(\kappa-1)} \{(\frac{R}{H})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} h$$

Finally, (6) and (B17) provide the closed form solution for $s(t)$ in terms of $H(t)$ and $R(t)$ as

$$(B19) \quad s = (\frac{R}{H})^{\varepsilon-1} [A'^{\sigma-1} H^{(\sigma-1)(\kappa-1)} \{(\frac{R}{H})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R}{H})^{\varepsilon-1} + 1]^{-1} (1-m)$$

Given the state variables of $\{H(t), R(t)\}$ at t , the closed form solution for $\{h(t), s(t)\}$ is given by (B17) and (B19).

From (B17), (B19), and (21), we have:

$$\begin{aligned} (B20) \quad \widehat{h(t)} &= \widehat{s(t)} + (\varepsilon-1)(\widehat{H(t)} - \widehat{R(t)}) = 0 \Rightarrow \eta'(t)h(t) = \xi'(t)s(t) \rightarrow \\ (1 + d_1 \left(\frac{mH(t)}{c_1 H(t)^{c_2} + mH(t)} - 0.5 \right)) \eta_0 h(t) &= \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right) \xi_0 s(t) \rightarrow \\ (1 + d_1 \left(\frac{mH(t)}{c_1 H(t)^{c_2} + mH(t)} - 0.5 \right)) \eta_0 [A'^{\sigma-1} H^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1-m) &= (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 (\frac{R(t)}{H(t)})^{\varepsilon-1} [A'^{\sigma-1} H^{(\sigma-1)(\kappa-1)} \{(\frac{R(t)}{H(t)})^{\varepsilon-1} + 1\}^{\frac{\varepsilon-\sigma}{\varepsilon-1}} + (\frac{R(t)}{H(t)})^{\varepsilon-1} + 1]^{-1} (1-m) \rightarrow \\ (1 + d_1 \left(\frac{mH(t)}{c_1 H(t)^{c_2} + mH(t)} - 0.5 \right)) \eta_0 &= (1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta) \xi_0 (\frac{R(t)}{H(t)})^{\varepsilon-1} \end{aligned}$$

where $\eta'(t) = (1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5 \right)) \eta_0$, and $\xi'(t) = \left(1 + c_4 \left(\frac{\frac{H(t)}{R(t)}}{c_3 + \frac{H(t)}{R(t)}} \right)^\theta \right) \xi_0$

This maps the initial condition of $\{H(0), R(0)\}$ into that of $\{h(0), s(0)\}$, and provides the closed-form solutions for $h(t)$ and $s(t)$, in terms of $R(t)$ and $H(t)$, completing the proof of Proposition 4.

Appendix B9: Convergence Speed towards the Pseudo Steady State before Reaching It

Define ‘convergence speed’ as follows: If $x(t)$ converges to x^* and satisfies $|x(t) - x^*| < (resp., >) b \exp(-at)$ for all $t > t^*$ for some t^* and $b > 0$, then we will say $x(t)$ converges to x^* at the speed higher (resp., lower) than a . And if $x(t)$ converges to x^* and satisfies both $|x(t) - x^*| < b \exp(-at)$ for all $t > t^*$ for some t^* and $b > 0$ and $|x(t) - x^*| > c \exp(-at)$ for all $t > t^{**}$ for some t^{**} and $c > 0$ then we will say $x(t)$ converges to x^* at the speed equal to a . In the below, second order error terms are approximated out.

We derive the convergence speeds for \dot{s} , \dot{h} , and $\dot{s} + \dot{h}$ when $t \leq t^*$ as follows.

From (14) we derive the following relationship.

$$(14) \quad \dot{s} + \dot{h}$$

$$\begin{aligned} &= \frac{(1-m-s-h)}{1-m} \{s(\sigma-1)(\varepsilon-1)(\xi(H(t), R(t))s - \eta(H(t), m)h) + (s+h)(1-\kappa)(\sigma-1)\eta(H(t), m)h\} \\ &\cong \frac{(1-m-s-h)}{1-m} (1-m)(1-\kappa)(\sigma-1)\eta(H(t), m)h < (1-m-s-h)(1-\kappa)(\sigma-1)\eta_0(1+0.5d_1)(1-m) \\ &\because 2 > \varepsilon + \kappa \leftrightarrow (1-\kappa) > (\varepsilon-1) \text{ and assuming } \frac{(1-m-s-h)}{1-m} (\xi(H(t), R(t))s - \eta(H(t), m)h) \cong 0, \quad \eta(H(t), m) \ll 1 + 0.5d_1 \text{ and } s+h < 1-m. \\ &= (1-m-s-h)(1-m)(1-\kappa)(\sigma-1)\eta_0(1+0.5d_1) \\ &= (1-m-s-h)\Delta < \infty \end{aligned}$$

where $\Delta = (1-m)(1-\kappa)(\sigma-1)\eta_0(1+0.5d_1)$.

Thus, (B25) helps prove that if $1-m-(s(t)+h(t)) < (1-m) \frac{1}{1+K \exp(t(1-\kappa)(\sigma-1)\eta_0(1-0.5d_1)h(0))} < \delta \equiv \frac{\varepsilon}{\Delta} > 0$, then we will have $0 < \dot{s} + \dot{h} < \infty$ with $K = \frac{s(0)+h(0)}{1-m-(s(0)+h(0))}$.

Thus, we can say $1-m-(s+h)$ converges to zero at the speed higher (resp., lower) than $(1-\kappa)(\sigma-1)\eta_0(1-0.5d_1)h(0)$ (resp., $(1-\kappa)(\sigma-1)\eta_0(1+0.5d_1)(1-m)$).

In addition, we can also prove the following inequalities from (9) and (10), when it approximately holds that $s+h = 1-m$ around the pseudo steady state with $t \leq t^*$.

$$\begin{aligned} (B21) \quad 0 < \dot{h} &= \frac{s}{1-m} \left((1-m-s-h) \frac{\varepsilon-\sigma-h}{\varepsilon-1 s+h} + h \right) (\varepsilon-1)(\eta(H(t), m)h - \xi(H(t), R(t))s) \\ &\quad + \frac{(1-m-s-h)h}{1-m} (1-\kappa)(\sigma-1)\eta(H(t), m)h \\ &\cong \frac{s+h}{1-m} \dot{h} + \frac{(1-m-s-h)h}{1-m} (1-\kappa)(\sigma-1)\eta(H(t), m)h \rightarrow \\ &\because \text{Neglecting the second order error terms with } \frac{(1-m-s-h)}{1-m} (\xi(H(t), R(t))s - \eta(H(t), m)h) \cong 0, (\varepsilon-1)(\hat{H} - \hat{R}) = \hat{h} - \hat{s} \cong \frac{s+h}{sh} \dot{h} \text{ due to } s+h \cong 1-m. \\ &\cong \frac{1-m}{1-m-s-h} \frac{(1-m-s-h)h}{1-m} (1-\kappa)(\sigma-1)\eta(H(t), m)h = h^2(1-\kappa)(\sigma-1)\eta(H(t), m) \end{aligned}$$

With $t \leq t^*$, $\frac{d}{dt} \ln(h(t)) = (1-\kappa)(\sigma-1)\eta(H(t), m)h(t) \rightarrow h(t) = h^* \exp\{-(1-\kappa)(\sigma-1) \left(\int_{t^*}^{t^*} \eta(H(s), m)h(s) ds \right)\}$.

Thus, the convergence speed of \dot{h} is lower than $(1 - \kappa)(\sigma - 1)\eta_0(1 + 0.5d_1)(1 - m)$ and higher than $(1 - \kappa)(\sigma - 1)\eta_0(1 - 0.5d_1)h(0)$, for $t \leq t^*$.

Because around the pseudo steady state, $\dot{h} \cong -\dot{s}$ and $(\varepsilon - 1)(\hat{H} - \hat{R}) = \hat{h} - \hat{s} \cong \frac{s+h}{sh}h$ due to $s + h \cong 1 - m$, we can infer that the convergence speeds of \dot{h} , \dot{s} , $1 - m - s - h$ and $(\hat{H} - \hat{R})$ are equal around the pseudo steady state.

Similarly, we can solve for \dot{s} around the steady state as follows.

$$0 < \dot{s} = \frac{s}{1 - m} \left((1 - m - s - h) \frac{\varepsilon - \sigma}{\varepsilon - 1} \frac{s}{s + h} - (1 - m - s) \right) (\varepsilon - 1) (\eta(H(t), m)h - \xi(H(t), R(t))s)$$

$$+ \frac{(1 - m - s - h)s}{1 - m} (1 - \kappa)(\sigma - 1)\eta(H(t), m)h$$

$$\cong \frac{s + h}{1 - m} \dot{s} + \frac{(1 - m - s - h)s}{1 - m} (1 - \kappa)(\sigma - 1)\eta(H(t), m)h \rightarrow$$

\because Neglecting the second order error terms with $\frac{(1-m-s-h)}{1-m}(\xi(H(t), R(t))s - \eta(H(t), m)h) \cong 0$, $(\varepsilon - 1)(\hat{H} - \hat{R}) = \hat{h} - \hat{s} \cong -\frac{s+h}{sh}\dot{s}$ due to $s + h \cong 1 - m$.

$$= \frac{1 - m}{1 - m - s - h} \frac{(1 - m - s - h)s}{1 - m} (1 - \kappa)(\sigma - 1)\eta(H(t), m)h = s(1 - \kappa)(\sigma - 1)\eta(H(t), m)h$$

$$\frac{d}{dt} \ln(s(t)) = (1 - \kappa)(\sigma - 1)\eta(H(t), m)h(t) \rightarrow s(t) = s^* \exp\{-(1 - \kappa)(\sigma - 1) \left(\int_t^{t^*} \eta(H(z), m)h(z) dz \right)\}, \text{ when } t \leq t^*.$$

Thus, the convergence speed of \dot{s} is lower than $(1 - \kappa)(\sigma - 1)\eta_0(1 + 0.5d_1)(1 - m)$ and higher than $(1 - \kappa)(\sigma - 1)\eta_0(1 - 0.5d_1)h(0)$, when $t \leq t^*$.

Because around the pseudo steady state $\dot{h} \cong -\dot{s}$ and $(\varepsilon - 1)(\hat{H} - \hat{R}) = \hat{h} - \hat{s} \cong \frac{s+h}{sh}\dot{h}$ due to $s + h \cong 1 - m$, we can infer that the convergence speeds of \dot{h} , \dot{s} , $1 - m - s - h$ and $(\hat{H} - \hat{R})$ are equal around the pseudo steady state when $t \leq t^*$.

When $t > t^*$ (i.e., after reaching the steady state), $s(t)$ and $h(t)$ will optimize the maximization problem by satisfying the only first order condition (6) with the constraint of $s(t) + h(t) = 1 - m$.

We can also prove that the convergence speed of $w_1(H(t), R(t), m)$ equals $C(t)$ in (B13) as follows when $t > t^*$.

Let $x(t) = D(t) - \sqrt{D(t)^2 - E(t)} = w_1(H(t), R(t), m)$ and $y(t) = D(t) + \sqrt{D(t)^2 - E(t)} = w_2(H(t), R(t), m)$, with $D(t) > 0$, $\lim_{t \rightarrow \infty} \{D(t)\} = \bar{D}$, and $\lim_{t \rightarrow \infty} \{E(t)\} = 0$. Then, $\lim_{t \rightarrow \infty} \{E(t)\} = \lim_{t \rightarrow \infty} \{C(t)/A(t)\} = \lim_{t \rightarrow \infty} \{x(t)y(t)\} = \lim_{t \rightarrow \infty} \{y(t)\} \lim_{t \rightarrow \infty} \{x(t)\} = 2\bar{D} \lim_{t \rightarrow \infty} \{x(t)\} = 2\bar{D} \lim_{t \rightarrow \infty} \{w_1(H(t), R(t), m)\}$. Thus, the convergence speed of $w_1(H(t), R(t), m)$ equals $E(t)$ and $C(t)$ where $D(t) = -0.5B(t)/A(t)$, and $E(t) = C(t)/A(t)$ in (B13). The convergence speed of $C(t)$ is lower than $(c_2 - 1)\eta_0(1 + 0.5d_1)(1 - m)$ and higher than $(c_2 - 1)\eta_0(1 - 0.5d_1)h(0)$.

Appendix B10: Solving the Differential Equation of (14)

$$(14) \quad \dot{s} + \dot{h}$$

$$= \frac{(1-m-s-h)}{1-m} \{s(\sigma - 1)(\varepsilon - 1)(\xi(H(t), R(t))s - \eta(H(t), m)h) + (s + h)(1 - \kappa)(\sigma - 1)\eta(H(t), m)h\}$$

$$\rightarrow \dot{x} \cong (1 - Ax)Bx = Bx - ABx^2, \text{ where } x = (s + h)/(1 - m), A = 1, \text{ and } B = (1 - \kappa)(\sigma - 1)\eta(H(t), m)h,$$

assuming that $\frac{(1-m-s-h)}{1-m}\xi(H(t), R(t))s - \eta(H(t), m)h \approx 0$, and that $\eta(H(t), m)h \cong \eta(H^*(t), m)h^*$ ('*' meaning 'at the steady state') does not change in time. Second order error terms are approximated out in the below.

Utilizing the solution for a differential equation in (B23) below, we can solve for $x = s + h$ as

(B24) $s(t) + h(t) = (1 - m) \frac{\exp(at)K}{1 + \exp(at)K} = (1 - m) \frac{\exp(Bt)K}{1 + \exp(Bt)K} = (1 - m) \frac{K \exp(t(1-\kappa)(\sigma-1)\eta(H^*(t),m)h^*)}{1 + K \exp(t(1-\kappa)(\sigma-1)\eta(H^*(t),m)h^*)}$ (representing a logistic growth model) where $a = B$, $b = B$, and $s(0) + h(0) = (1 - m) \frac{K}{1 + K} \rightarrow K = \frac{s(0) + h(0)}{1 - m - (s(0) + h(0))}$.

$$(B25) 1 - m - (s(t) + h(t)) = 1 - m - (1 - m) \frac{\exp(Bt)K}{1 + \exp(Bt)K} = \frac{1 - m}{1 + K \exp(t(1-\kappa)(\sigma-1)\eta(H^*(t),m)h^*)}.$$

Thus, considering Appendix B9, $1 - m - (s + h)$ converges to zero at the speed higher (resp., lower) than $(1 - \kappa)(\sigma - 1)\eta_0(1 - 0.5d_1)h(0)$ (resp., $(1 - \kappa)(\sigma - 1)\eta_0(1 + 0.5d_1)(1 - m)$).

$$(B23) \frac{dx(t)}{dt} = ax(t) - bx(t)^2 \rightarrow \frac{1}{x(a-bx)} dx = dt \rightarrow \frac{1}{a} \left(\frac{1}{x} + \frac{b}{(a-bx)} \right) dx = \frac{1}{a} \left(\frac{b}{bx} + \frac{b}{(a-bx)} \right) dx = dt \rightarrow \frac{1}{a} \left(\frac{b}{y} + \frac{b}{(a-y)} \right) dx = dt \rightarrow \frac{1}{a} \left(\frac{b}{y} + \frac{b}{(a-y)} \right) \frac{1}{b} dy = dt, \text{ where } y = bx \rightarrow \frac{b}{a} \left(\frac{1}{y} + \frac{1}{(a-y)} \right) \frac{1}{b} dy = dt \rightarrow \left(\frac{1}{y} + \frac{1}{(a-y)} \right) dy = adt \rightarrow \ln \left(\frac{y}{a-y} \right) = at + C \rightarrow \frac{y}{a-y} = \exp(at)K \rightarrow y = \frac{a \exp(at) K}{1 + \exp(at) K} \rightarrow x(t) = \frac{a}{b} \frac{\exp(at) K}{1 + \exp(at) K}$$

Appendix B11: Cognitive Ability Limitation and Pseudo Steady State

As mentioned before, we assume agents are limited in cognitive or computational ability such that they cannot perceive the difference in capital growth rates when that is smaller than a certain threshold magnitude, which guarantees the existence of the pseudo steady state in a finite time horizon. That is, if $|\eta(H(t), m)h(t^0) - \xi(H(t), R(t))s(t^0)| \leq \nabla$ with a fixed small threshold level ∇ , agents think that two capital growth rates are equal for $t \in \{t^0, t^0 + \Delta t\}$. Here t^0 is the finite timing at which the equilibrium path firstly reaches this small neighborhood around the pseudo steady state where this inequality holds, and $t^0 + \Delta t$ at which it leaves this small neighborhood.

During the interval of $t \in \{t^0, t^0 + \Delta t\}$ between reaching the neighborhood at t^0 and leaving it at $t^0 + \Delta t$, while $s(t) = s(t^0)$ and $h(t) = h(t^0)$ remain constant, $\eta(H(t), m)h(t^0)$ decreases as $H(t)$ increases. But $\xi(H(t), R(t))s(t^0)$ does not change much because it depends on the ratio of the two capital stocks $(H(t)/R(t))$ whose growth rates are equal at the pseudo steady state. More specifically, because $\eta(H(t), m)$ decreases as time goes on due to the continuous increase in $H(t)$ with fixed $s(t) = s(t^0)$ and $h(t) = h(t^0)$ in this small neighborhood, there exists a threshold time $t^0 + \Delta t$ of leaving this neighborhood, satisfying the following equality.

$$(B26) |\eta(H(t^0 + \Delta t), m)h(t^0) - \xi(H(t^0 + \Delta t), R(t^0 + \Delta t))s(t^0)| = \left| \left(1 + d_1 \left(\frac{m}{c_1 H(t^0 + \Delta t)^{c_2-1} + m} - 0.5 \right) \right) \eta_0 h(t^0) - \left(1 + c_4 \left(\frac{\frac{H(t^0 + \Delta t)}{R(t^0 + \Delta t)}}{c_3 + \frac{H(t^0 + \Delta t)}{R(t^0 + \Delta t)}} \right)^\theta \right) \xi_0 h(t^0) \right| = \nabla.$$

Assuming Δt is small, we can calculate Δt satisfying the condition of leaving the small neighborhood as follows.

With (B26),

$$\begin{aligned} & \left(\frac{d}{dt} |\eta(H(t^0 + \Delta t), m)h(t^0) - \xi(H(t^0 + \Delta t), R(t^0 + \Delta t))s(t^0)| \right) \Delta t \\ &= \left| \frac{d_1 m (c_2 - 1) c_1 H(t^0)^{c_2-1}}{(c_1 H(t^0)^{c_2-1} + m)^2} \eta(H(t^0), m) h(t^0)^2 - \frac{d \xi(H(t^0), R(t^0))}{d(\frac{H(t)}{R(t)})} \frac{H(t^0)}{R(t^0)} (\bar{H}(t^0) - \bar{R}(t^0)) s(t^0) \right| \Delta t \\ &= \left| \frac{d_1 m (c_2 - 1) c_1 H(t^0)^{c_2-1}}{(c_1 H(t^0)^{c_2-1} + m)^2} \eta_0 (H(t^0), m) h(t^0)^2 \right| \Delta t = 2\nabla \end{aligned}$$

$$\therefore \widehat{H(t^o)} - \widehat{R(t^o)} = 0$$

∴ During Δt , the path should trespass the distance of 2∇ before leaving the small neighborhood after having crossed it with its radius ∇ .

Thus,

$$(B27) \quad \Delta t = 2\nabla / \left| \frac{d_1 m (c_2 - 1) c_1 H(t^o)^{c_2 - 1}}{(c_1 H(t^o)^{c_2 - 1} + m)^2} \eta_0(H(t^o), m) h(t^o)^2 \right|$$

Thus, after $t^o + \Delta t$, agents start to perceive the decrease in $H(t)/R(t)$, resulting in that in $h(t)/s(t)$ according to (6), triggering the cumulative declining process of both $\frac{H(t)}{R(t)}$ and $h(t)/s(t)$. It is because during the path has stayed in the small neighborhood of the perceived steady state for Δt , the growth rate of science capital has decreased by 2∇ while that of religion capital does not have changed, satisfying (B27).

The rest of the analysis in the paper remains intact with the existence of the pseudo steady state in a finite time horizon assuming the cognitive (computational) ability limitation.

Appendix C: Paul Kennedy's Military Overstretch Model:

As an empire increases its wealth by extending its border through conquering neighboring countries, it must also increase its expenditure not only to conquer them, but also to defend its increased border and manage its increased conquered territory. Let's assume that the conquered area is a circle with a radius r . Then as r increases by Δr , the area conquered increases by $2\pi r \Delta r$, while the border line by $2\pi \Delta r$. And we further assume that the marginal productivity of a unit land with a distance r from the center is described by $F'(r)$ with $F' > 0$ and $F'' < 0$, for example with a production function $F(k) = k^\alpha$ with $1 > \alpha > 0$. The farther away from the center a land is located, the less is its productivity.

Assume that the military mobilization cost to conquer neighboring countries is included in the administration cost c_a per unit of the conquered land, increasing in r due to the information and coordination cost of transportation and communication increasing in distance.³⁵ And assume also that the defense cost per unit of border line is a constant c_d representing the successful defense cost per one mile.

The following optimization problem solves for the optimal size of an empire r^* as

$$\max_{r^*} \left\{ \int_0^{r^*} 2\pi r (F'(r) - c_a(r)) dr - 2\pi r^* c_d \right\}$$

with its FOC of

$$2\pi r (F'(r) - c_a(r)) - 2\pi c_d = 0$$

$$\rightarrow 2\pi F'(r) \text{ (Marginal Benefit)} = 2\pi \left(c_a(r) + \frac{c_d}{r} \right) \text{ (Marginal Cost)}.$$

$$\rightarrow F'(r) = \left(c_a(r) + \frac{c_d}{r} \right).$$

This FOC implies the following.

³⁵ This is the cost of complexity emphasized by Tainter as in Section II. The macro dynamics of a nation's fall caused by complexity is described and explained utilizing big push and self-fulfilling prophecy mechanism in Kim and Lee [2023]. Here is a quote from the summary of Heather. “As the Roman Empire basked in its zenith, the seed of its eventual downfall had already begun to germinate internally. One of the primary internal challenges was economic in nature. The empire's vast expanse necessitated a complex and costly administrative structure, which in turn strained the fiscal capabilities of the state. Over time, the revenues from conquests dwindled, and the burden of sustaining a sprawling military and bureaucracy grew heavier. The imposition of heavy taxes to meet these demands often led to widespread disenchantment among the populace, unsettling the once sturdy foundation of the empire.”

First, assuming c_d is much bigger than c_a , when r is relatively small, there exists an economy of scale, because $\frac{c_d}{r} + c_a(r)$ decreases at a higher speed than $F(r)$, as r increases.

Second, however, if $F(k) = k^\alpha$ with $1 > \alpha > 0$, then obviously there exists the effect of military overstretch. And as an empire extends its territory (as r increases) continuously, there exists a threshold level of r^* (the size of a nation) above which the marginal cost dominates the marginal benefit, triggering the military overstretch effect. It is because $c_a(r)$ increases while $F'(r)$ decreases, as r increases.

Third, as an empire extends its territory excessively, its demand for military manpower exceeds its available manpower among its own citizens, resulting in employing more of mercenaries or of noncitizens as its military force, increasing its coordination and management cost, lowering civic militarism.

Fourth, as it extends, the probability of facing rising superpowers from neighboring or conquered regions rapidly increases as the following calculation shows. Assume the probability of facing a rising superpower, which cannot be conquered by any means, per square mile internally and externally is p_{is} and p_{es} respectively. Then as the empire extends, its probability of collapse, $(1 - (1 - p_{is})^{\pi r^2} (1 - p_{es})^{2\pi r})$, rapidly increases towards one. As r increases, this probability rapidly goes to one, because the term $(1 - p_{is})^{\pi r^2} (1 - p_{es})^{2\pi r}$, representing the probability of facing no superpower both internally and externally, goes to zero rapidly.

Last, usually an empire cannot build up its military power to the level of perfect defense along the whole border, rather does to a satisfying level such that a significant revolt or assault occurring in one place can be subdued with the help of other places' residing military power mobilized to this place on time. This means if several revolts happen simultaneously in several places internally or along the border, or if a consecutive sequence of significant revolts are happening there, the empire hardly manages them to its survival. This happens more probably, as it extends its territory more.

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Table 1: Comparison of Various Theories

Theorist	Length of the Cycle	Periods Covered	Research Methodology	Key Driving Factor	Phases of the Cycle	Distinguishing Feature	Relevant Historical Facts
Ibn Khaldun (1377)	Cyclical, not precisely defined	Islamic and medieval periods	Historical and sociological analysis	Social cohesion (Asabiyah)	Tribal solidarity, state formation, prosperity, decline	Focus on social cohesion eroding through prosperity	Rise and fall of Islamic dynasties
Edward Gibbon (1776)	Not specified	Roman Empire	Historical documentation	Moral and civic decay	Rise and fall	Focus on internal moral decay and Christianity's role	Fall of the Roman Empire
Max Weber (1905)	Not cyclical	Early modern Europe	Sociological and economic analysis	Religious values (Protestantism)	Development of capitalism	Links religious values to economic behavior	Rise of capitalism in Western Europe
Oswald Spengler (1918)	1000 years (for civilizations)	Western civilization	Philosophical and historical analysis	Cultural and spiritual decline	Birth, growth, maturity, decline	Cyclical theory of cultural life cycles	Predicted Western civilization's decline
Arnold Toynbee (1934)	Cyclical, not strictly defined	Various civilizations	Comparative historical analysis	Creative responses to challenges	Genesis, growth, breakdown, disintegration	Focus on responses to existential challenges	Multiple examples of civilization al rise and fall
John Glubb (1976)	About 250 years	Various empires	Historical analysis	Moral decay	Pioneers, conquest, commerce, affluence, intellect, decadence, decline	Emphasizes moral and ethical decay in later stages	Roman, Arab, British empires
Paul Kennedy (1987)	Not cyclical	Post-medieval to modern era	Economic and military analysis	Economic and military overstretch	Rise, expansion, overreach, decline	Focus on economic resources and military power	Spanish, French, British, and American empires
Joseph Tainter (1988)	Not cyclical	Various ancient societies	Economic and administrative analysis	Costs of maintaining social complexity	Growth, diminishing returns, collapse	Focus on diminishing returns on social complexity	Collapse of Roman and Mayan civilizations

Jack Goldstone (1991)	Not cyclical	Early modern period	Demographic and historical analysis	Demographic pressures	Growth, resource strain, instability	Focuses on population growth and its impact on stability	English Civil War, French Revolution
Strauss and Howe (1997)	80-100 years	Western civilization	Historical and sociological analysis	Generational dynamics	High, Awakening, Unraveling, Crisis	Focus on cyclical generational patterns	Four generational turnings
Agner Fog (1999)	Not cyclical	Various historical periods	Evolutionary psychology and sociocultural analysis	Perceived external threats	Regal (hierarchical) vs. kungic (egalitarian) shifts	Adapts to external threats through hierarchical structures	Warlike vs. peaceful societies
Victor Davis Hanson (2001)	Not cyclical	Western civilization	Military and cultural history	Military and cultural practices	Military dominance, decline	Focuses on Western military traditions and culture	Key Western battles throughout history
Jared Diamond (2005)	Not cyclical	Various ancient and modern societies	Interdisciplinary, environmental analysis	Environmental and societal choices	Sustainability, collapse	Emphasizes environmental degradation and resource management	Collapse of Easter Island, Norse Greenland, Mayan civilization
Brian Ward-Perkins (2005)	Not cyclical	Roman Empire	Archaeological evidence	Barbarian invasions, economic disruption	Sudden collapse	Focuses on material consequences of the collapse	Fall of the Roman Empire, start of Dark Ages
Eric Cline (2014)	Not cyclical	Late Bronze Age	Archaeological and historical analysis	Environmental stress, systemic vulnerabilities	Stability, interconnected collapse	Focuses on interconnected civilizations' vulnerability	Collapse of Mycenaean, Hittite, and Egyptian civilizations
Peter Turchin (2016)	Cyclical (multi-century)	Various civilizations	Mathematical and historical modeling	Population growth, resource strain, elite competition	Stability, elite overproduction, collapse	Uses quantitative models to predict societal dynamics	American history, global rise and fall of empires
Ray Dalio (2020)	Varies (usually multi-century)	Last 500 years	Economic and financial analysis	Debt cycles, leadership, education	Rise, peak, decline	Focus on financial cycles and leadership	Dutch, British, American empires

Table 2-1: Ibn Khaldun's Cycle of Asabiyyah

Stage	Asabiyyah	Explanation
Formation	Strong Asabiyyah	A group with strong Asabiyyah challenges and overthrows a declining state, marked by high levels of solidarity and collective effort.
Consolidation	Stabilized Asabiyyah	The new ruling group stabilizes its power, builds institutions, and establishes governance structures. Asabiyyah remains strong, but the focus shifts from conquest to administration.
Expansion and Prosperity	Weakened Asabiyyah	The state reaches its peak, experiencing economic growth, cultural flourishing, and territorial expansion. However, Asabiyyah begins to weaken as wealth and luxury lead to complacency and internal divisions.
Decline	Deteriorating Asabiyyah	As the ruling elite becomes more decadent and less cohesive, Asabiyyah deteriorates. The state becomes vulnerable to internal strife and external threats, leading to its eventual decline and fall.

Table 2-2: Turchin's Cycle of Cliodynamics

Stage	Explanation
Expansion	During this phase, society experiences population growth and economic prosperity. Resources are plentiful, and social cohesion is high, as there are ample opportunities for elites and common citizens alike.
Resource Strain	As the population continues to grow, the demand for resources such as land, food, and jobs increases. Over time, the availability of these resources becomes limited, leading to competition and social unrest.
Elite Overproduction	The number of elites in society grows, but the opportunities for maintaining or increasing power diminish. This leads to intense competition among elites, factionalism, and the destabilization of governance structures.
Instability and Collapse	As resource scarcity and elite competition escalate, social cohesion breaks down. Internal conflict and civil unrest become widespread, leading to the collapse of the societal structure. Eventually, the population declines, and the cycle may reset.

Table 2-3: Agner Fog's Regal and Kungic Societies

Characteristic	Regal (Warlike) Societies	Kungic (Peaceful) Societies
Political System	Hierarchical with a strong leader	Flat and egalitarian
Group Identity	Strong sense of national or tribal identity	High individualism
Discipline and Punishment	Strict discipline, harsh punishment for deviants	Lax discipline, high tolerance for deviants
Xenophobia	High levels of xenophobia	Tolerance of foreigners
Worldview	The world is seen as full of dangers and enemies	The world is seen as peaceful and safe
Purpose of Individuals	Belief that individuals exist for the benefit of society	Belief that society exists for the benefit of individuals
Religion	Strict religious discipline	Religion has little or no disciplining power
Sexual Morals	Strict sexual morals, high birthrate	High sexual freedom, low birthrate
Parental Investment	Low parental investment, short childhood	High parental investment, long childhood and education
Suicide Rates	Low suicide rate (except for culturally prescribed reasons)	High rate of anomic suicide

Characteristic	Regal (Warlike) Societies	Kungic (Peaceful) Societies
Art and Music	Perfectionist, highly embellished, follows specific schemes	Expresses individual fantasy, appreciation of individuality

Table 2-4: Agner Fog's Cycle

Stage	Keyword	Explanation
Empire Rise	War and Regality	Intergroup conflict increases regality, fostering a preference for strong leadership and coordinated societal efforts.
	Population Growth	As societies become more organized and hierarchical, population growth is supported by increased resource allocation and improved living conditions.
	Technological Advancements	Military and technological innovations enhance an empire's capacity for expansion and control over new territories.
	Territorial Expansion	Successful military campaigns and efficient resource management facilitate territorial growth, reinforcing regality and social cohesion.
Empire Decline	Loss of Regality	As empires reach the limits of their expansion, the cohesive social forces that once drove growth begin to weaken, resulting in a loss of regality.
	Overpopulation and Economic Strain	Overpopulation strains resources, leading to increased prices for land and food, while the value of labor decreases, creating economic hardship.
	Elite Expansion and Inequality	A growing elite exploits economic conditions, extracting surplus from the impoverished populace and exacerbating inequality. This leads to increased competition within the elite and between the elite and commoners.
	Economic Collapse and Political Instability	Heavy taxation and debt, coupled with rebellion and internal conflict, deplete state resources. The government struggles to maintain control, leading to economic breakdown and loss of territorial integrity.

Table 2-5: Paul Kennedy's Stages of the Rise and Fall of Great Powers

Stage	Explanation
Ascent	Nations rise to power through economic growth, technological innovation, and effective governance. They expand their influence through trade, diplomacy, and military strength.
Peak	At their peak, great powers have significant economic and military capabilities, enjoying global influence and dominance.
Overreach	As these nations pursue ambitious foreign policies and military expansions, they begin to overstretch their resources.
Decline	Economic strain from military overstretch leads to reduced military effectiveness, loss of global influence, and eventual decline.

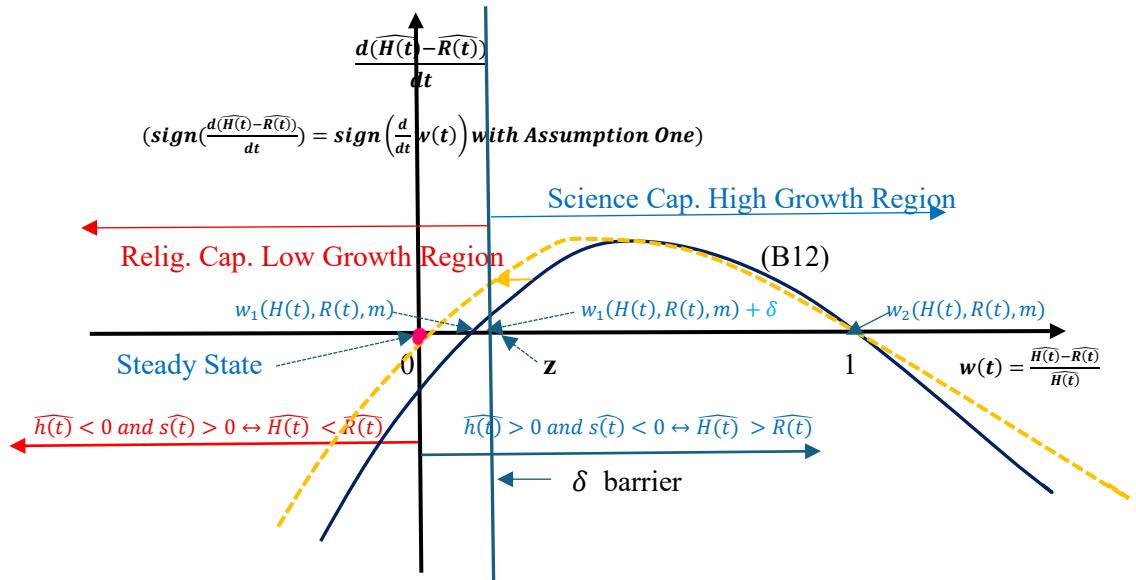
Table 2-6: Stages of Ray Dalio's Big Cycle with Their Key Characteristics

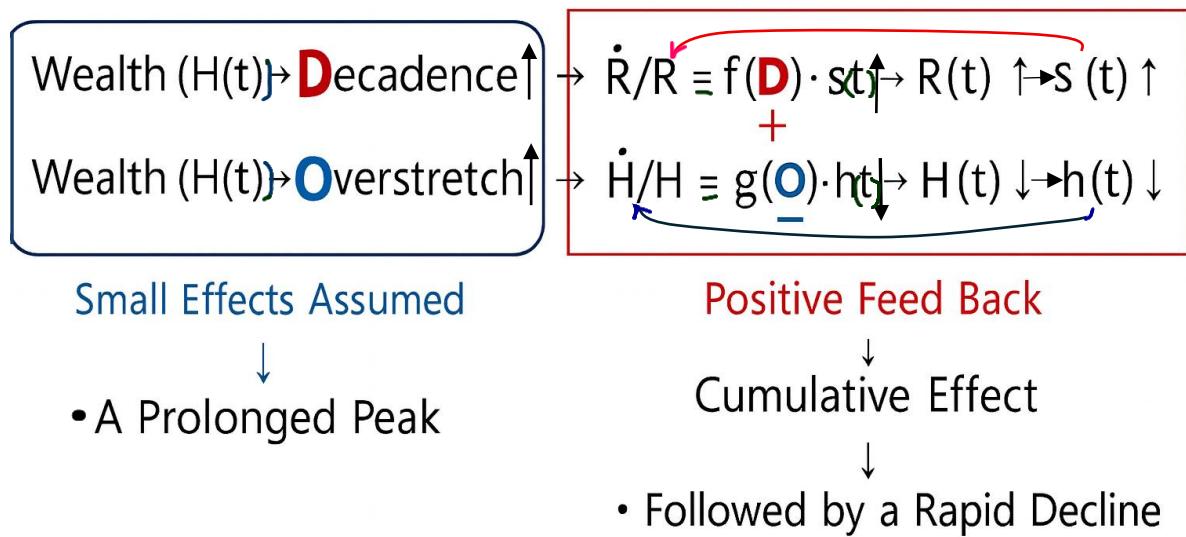
Stage	Keyword	Explanation
The Rise	Leadership and Governance	Strong leadership and governance foster stability, innovation, and economic growth.
	Investment in Education	Heavy investment in education, infrastructure, and technology boosts productivity and competitiveness.
	Capital Formation	Increased savings and investment support economic expansion.

Stage	Keyword	Explanation
The Top	Social Cohesion	High levels of social cohesion and shared values promote collective efforts toward national goals.
	Economic Prosperity	High levels of income, employment, and wealth mark the economy's peak.
	Global Influence	The nation achieves significant global influence, often becoming a dominant economic and military force.
	Cultural Flourishing	Wealth and stability drive cultural, scientific, and intellectual flourishing.
The Decline	Inequality and Complacency	Rising inequality and complacency begin to undermine growth.
	Excessive Debt	Accumulation of excessive debt burdens the economy. Governments and individuals borrow beyond sustainable limits.
	Economic Stagnation	Growth slows, productivity declines, and economic stagnation sets in.
	Social and Political Strife	Increasing inequality leads to political instability and internal conflicts.
The Fall	Loss of Global Influence	The nation loses its competitive edge and global influence as emerging powers rise.
	Economic Crisis	The decline culminates in financial instability, high unemployment, and widespread distress.
	Political Upheaval	Political institutions may collapse or undergo major transformations, leading to revolutions, regime changes, or civil wars.
	Loss of Cohesion	Social cohesion deteriorates as divisions and conflicts intensify.
	Opportunity for Renewal	The crisis phase offers a chance for renewal and restructuring, potentially starting a new growth cycle.

Table 3: Calibration of Parameter Values

Parameter	Explanation	Calibrated Value	Comment
Utility Function	$[(c(t))^{\frac{\sigma-1}{\sigma}} + \beta\{((R(t)s(t))^{\frac{\varepsilon-1}{\varepsilon}} + \gamma(H(t)h(t))^{\frac{\sigma-1}{\sigma}})^{\frac{\varepsilon}{\varepsilon-1}}\}^{\frac{\sigma}{\sigma-1}}$		Elasticity of substitution assumed b/n material ($c(t)$) and idea consumption ($R(t)s(t)$ and $H(t)h(t)$) and also b/n science ($H(t)h(t)$) and religious ideas consumption ($R(t)s(t)$).
ε	Elasticity of substitution b/n ideas	1.5	Substitutability assumed to have corner solutions to emulate the fall of nations.
σ	Elasticity of substitution b/n material and idea consumption.	1.2	Substitutability assumed b/n material and idea consumption, to have a saddle path with $\varepsilon > 1$, $\kappa < 1$ and $\sigma > 1$.
β	Distribution parameter for ideas b/n consumption and ideas	1	Equal weight between material and idea consumption.
γ	Distribution parameter for scientific ideas against religious idea.	1	Equal weight b/n two kinds of idea consumptions
$A = A(0)$	Initial technology level	1	Arbitrary and assumed as constant.
κ	Transformation parameter from scientific capital to production capital as $c(t) = Y(t) = (1 - m - s(t) - h(t))H(t)^\kappa$.	0.3	On a saddle path with $\varepsilon > 1$ in addition to $\kappa < 1$ and $\sigma > 1$ or to $\kappa > 1$ and $0 < \sigma < 1$. And $\varepsilon > 1$ and $\kappa + \varepsilon \leq 2$ with $\sigma > 1$ is a sufficient condition for the total time of idea consumption to increase in t.
$s(0)$	Initial time consumption of religious idea	Endog.	(6) $\frac{s(0)}{h(0)} = \left(\frac{R(0)}{H(0)}\right)^{\varepsilon-1} = \left(\frac{R(0)}{H(0)}\right)^{0.5} = 2$ since we assume $\frac{R(0)}{H(0)} = 4$ and $\varepsilon = 1.5$ with $H(0) = 1$.
$h(0)$	Initial time consumption of scientific idea	Endog.	
$R(0)/H(0)$	Ratio of religion to scientific capital as an exogenously given state variable. Since (6) $\frac{h(t)}{s(t)} = \left(\frac{H(t)}{R(t)}\right)^{\varepsilon-1}$, we are assuming $\frac{s(0)}{h(0)} = \left(\frac{R(0)}{H(0)}\right)^{0.5} = (4)^{0.5} = 2$.	4	Assuming $\left(\frac{R(t)}{H(t)}\right)^{0.5} \cong \left(\frac{R(0)}{H(0)}\right)^{0.5} = 2 \rightarrow \frac{s(t)}{h(t)} \cong 2$ for all t. Since $\frac{\eta'(t)}{\xi'(t)} \cong \frac{\eta_0}{\xi_0} = 2 \rightarrow (15) \eta'(t)h(t) \cong \xi'(t)s(t)$ (two capital growth rates are equal) for all t, justifying the above assumption.
$H(0)$	Initial level of scientific capital	1	Arbitrary
$\eta'(t)$	$(1 + d_1 \left(\frac{m}{c_1 H(t)^{c_2-1} + m} - 0.5\right))\eta_0$		Accumulation efficiency of scientific capital with military overstretch.
η_0	Net accumulation efficiency of scientific capital. Assumed as a policy variable or representing mob psychology.	0.02	The absolute size does not matter here since the measurement unit of time is not defined, while its relative size with respect to the net accumulation efficiency of religious capital is assumed to be twice.
c_1	The level of military overstretch as a policy variable or a cultural aspect.	0.1	Level of military overstretch. Close to zero to make the military overstretch effect of stock variables on capital growth rates to be very small and gradual. Due to Assumption One.
c_2	The level of military overstretch as a policy variable or a cultural aspect. Assumed to be slightly above one.	1.02	Level of military overstretch. Slightly above one to make the military overstretch effect of stock variables on capital growth rates to be very small and gradual. Due to Assumption One.
m	Policy variable of military spending.	0.20496161 1998	Military spending is assumed to be about 20% of GDP. This 12-digit number is specifically calibrated to make the equilibrium on the saddle path equilibrium.
$\xi'(t)$	$(1 + c_4 \left(\frac{H(t)}{R(t)} / (c_3 + \frac{H(t)}{R(t)})\right)^\theta) \xi_0$		Accumulation efficiency of religious capital with decadence.
ξ_0	Net accumulation efficiency of religious capital. Assumed as a policy variable or representing mob psychology.	0.01	The absolute magnitude does not matter here since the measurement of time is not defined, while its relative size wrt net accumulation efficiency of scientific capital is assumed to be half.
c_4	The level of decadence as a cultural aspect.	0.001	Level of decadence. Small valued to make the decadence effect of stock variables on capital growth rates to be small and gradual. Due to Assumption One.
c_3	The level of decadence as a cultural aspect.	5	Arbitrary
θ	The level of decadence as a cultural aspect.	0.0002	Small valued to make the effect of stock variables on capital growth rates to be small and gradual. Due to Assumption One.





A Prolonged Peak Followed by a Rapid Decline

<Figure 1: An Intuitive Mechanism of 'A Prolonged Peak Followed by a Rapid Decline'>