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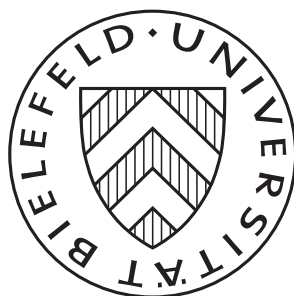
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Berge Equilibria – An Algebraic Approach

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Berge Equilibria – An Algebraic Approach*

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Abstract

Berge equilibrium offers an alternative to Nash equilibrium in game theory, emphasizing cooperative stability rather than individual optimization. Despite recent interest, a systematic study of Berge equilibria in finite normal form games is still lacking, with fundamental questions like existence remaining open. This paper characterizes Berge equilibria through a polynomial system of equations, enabling computational algebra and algebraic geometry methods to analyze them. Algorithms based on Gröbner bases determine the existence and computation of Berge equilibria. Furthermore, we show that the set of games admitting completely mixed Berge equilibria is contained within a determinantal variety, whose dimension we explicitly bound from above.

Keywords: Berge equilibrium, Algebraic Methods in Economics, Gröbner Bases

JEL Subject Classification: C72, C02, C63

1 Introduction

The notion of Berge equilibrium is an interesting alternative to the prevalent Nash equilibrium in game theory. Unlike Nash equilibrium, which is based on the idea that no player can unilaterally improve their own payoff, Berge equilibrium focuses on cooperative stability in non-cooperative games. Berge equilibrium captures a more altruistic or benevolent decision-making process, where each player supports others rather than acting selfishly.

While the notion has found applications in various papers recently (e.g. Ünveren, Donduran, and Barokas (2023), Haller (2024), Graziano, Pesce, and Platino (2024)), a systematic study of Berge equilibria in finite normal form games is lacking so far. In particular, even the basic issue of existence remains open.

In this paper, we characterize Berge equilibria in all finite games by a suitable polynomial system of equations, making it amenable to methods from computational algebra and algebraic geometry.

Using methods from computational algebra, we provide algorithms for generic games that decide whether a Berge equilibrium exists, and, if so, compute all Berge equilibria with the help of Gröbner basis techniques.

Using algebraic geometry, we demonstrate that the set of games capable of admitting completely mixed Berge equilibria forms a subset of a determinantal variety. We compute the dimension of this variety, providing an upper bound on the dimension of games possessing completely mixed Berge equilibria. For

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games with more than two players, completely mixed Berge equilibria exist only in a relatively low-dimensional set of games. For N -player games with two actions per player, less than 50% of games admit such equilibria.

Our paper contributes to the growing body of literature that applies tools from algebraic geometry to game theory. Despite their potential, computational algebra methods are rarely utilized in economics. Recently, Gröbner bases theory has been fruitfully applied in different areas, such as in General Equilibrium Theory for the analysis of the equilibrium conditions in markets (see Kubler and Schmedders (2010)), in Econometrics to address the global identification problem in dynamic stochastic general equilibrium (DSGE) models (see Kocięcki and Kolasa (2023)) and in Game Theory for the computation of all Nash equilibria in games with polynomial payoff functions (see Datta (2010)).

Over the past decade, the concept of Berge equilibrium has gained increasing attention as a plausible alternative to Nash equilibrium for modeling diverse socioeconomic behaviors in strategic interactions (Colman, Corner, Musy, and Tazdait (2011), Larbani and Zhukovskii (2017), Haller (2024)). Several studies have sought to establish sufficient conditions for the existence of pure strategy Berge equilibria (Abalo and Kostreva (2004, 2005); Nessaha, Larbani, and Tazdaitb (2007)). Notably, Colman, Corner, Musy, and Tazdait (2011) show that in N -person games, Berge equilibria form a subset of Nash equilibria that are common to all permutations of the game. Musy, Pottier, and Tazdait (2012) reformulate the existence theorem from Colman, Corner, Musy, and Tazdait (2011) without relying on Nash equilibrium, introducing the concept of best support from co-players and defining Berge equilibria as the intersection of best support correspondences. Further developments include Courtois, Nessah, and Tazdait (2017), which provides existence conditions and computational methods for two refinements of Berge equilibrium: Berge–Vaisman and Berge–Nash equilibrium. From a computational standpoint, Corley and Kwain (2015) proposes an algorithm using the disappointment matrix to compute all pure strategy Berge equilibria in N -person normal-form games. The only known method addressing mixed strategy Berge equilibria is presented in Sawicki, Pykacz, and Bytner (2021), which develops an algorithm for identifying all Berge equilibria—both pure and mixed—in N -person games with two strategies per player. A comprehensive literature review on Berge equilibria in static normal-form games can be found in Larbani and Zhukovskii (2017). Our approach extends this line of research by establishing generalized existence conditions for pure strategy Berge equilibria that encompass and go beyond the results in Corley and Kwain (2015) and Nessaha, Larbani, and Tazdaitb (2007); see Remark 2.7.

The paper is structured as follows. The next section provides the polynomial characterization of Berge equilibria. Section 3 outlines the algebraic techniques employed in our analysis and provides examples that illustrate the application of Gröbner basis methods. Section 4 introduces the concept of (Berge) regular games, while Section 5 presents algorithms for computing all Berge equilibria, when they exist. Within the class of all games, Section 6 characterizes the subset of a determinantal variety that contains all regular games admitting completely mixed Berge equilibria. The last section concludes.

2 Polynomial Characterization of Berge equilibria

We consider a finite normal form game, following the notation of Datta (2010). The set of players is denoted by $I = \{1, \dots, N\}$. Each player i has a finite set of pure strategies $S_i = \{s_{i0}, \dots, s_{id_i}\}$ with $|S_i| = d_i + 1 \geq 2$. $S = \prod_{i \in I} S_i$ is the set of pure strategy profiles with cardinality $D_S := |S| = \prod_{i \in I} (d_i + 1)$. Player i has the payoff function $u_i : S \rightarrow \mathbb{R}$. The game in normal form is written as $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$.

In the following we write $[d_i] = \{0, 1, \dots, d_i\}$ and set $D = \sum_{i \in I} d_i$; in particular, it holds $\sum_{i \in I} |S_i| = \sum_{i \in I} (d_i + 1) = D + N$.

The set Σ_i of mixed strategies of player i is the set of all functions $\sigma_i : S_i \rightarrow [0, 1]$ such that $\sum_{j=0}^{d_i} \sigma_i(s_{ij}) = 1$. We write $\sigma_{ij} = \sigma_i(s_{ij})$. We let $\Sigma = \prod_{i \in I} \Sigma_i$ be the set of strategy profiles. We let $I_{-i} = I \setminus \{i\}$, $\Sigma_{-i} = \prod_{j \in I_{-i}} \Sigma_j$ and we call an element of Σ_{-i} a $(-i)$ -strategy profile. We denote by σ_{-i} the image of $\sigma \in \Sigma$ under the projection π_{-i} onto Σ_{-i} ; further, we let $D_{-i} = \prod_{j=1, j \neq i}^N [d_j]$. Player i 's

expected payoff from a strategy profile $\sigma = (\sigma_1, \dots, \sigma_N) \in \Sigma$ is given by

$$u_i(\sigma) = \sum_{s_1 \in S_1} \dots \sum_{s_N \in S_N} u_i(s_1, \dots, s_N) \sigma_1(s_1) \dots \sigma_N(s_N).$$

We also introduce the notation $\sigma_{-i}^j := \sigma_{1j_1} \dots \sigma_{i-1, j_{i-1}} \sigma_{i+1, j_{i+1}} \dots \sigma_{Nj_N}$ for the probability that the coalition $(-i)$ plays the profile s_{-i}^j under σ_{-i} , and we write

$$u_i^j(\sigma_i, s_{-i}^j) := \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^j) \sigma_i(s_{ij_i}) = \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^j) \sigma_{ij_i} \quad (1)$$

for the expected payoff of player i when the coalition $(-i)$ uses the pure strategies $s_{-i}^j = (s_{1j_1}, \dots, s_{i-1, j_{i-1}}, s_{i+1, j_{i+1}}, \dots, s_{Nj_N})$.

A strategy profile $\sigma^* = (\sigma_1^*, \dots, \sigma_N^*)$ is said to be a *Berge equilibrium* if for each player $i \in I$ it holds

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i^*, \sigma_{-i})$$

for each $(-i)$ -strategy profile $\sigma_{-i} \in \Sigma_{-i}$.

We characterize (mixed and pure) Berge equilibria as the nonnegative solutions to certain polynomial systems (system (2) of Proposition 2.1 or system (7) of Proposition 2.4 below). As players of coalition $-i$ try to maximize player i 's utility, we start with a typical Kuhn–Tucker-like characterization of optimal strategies.

Proposition 2.1 *The strategy profile*

$$\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{R}^{D+N},$$

where

$$\sigma_i := (\sigma_{i0}, \dots, \sigma_{id_i}) \quad \text{for each } i \in I,$$

is a Berge equilibrium if and only if there exists

$$v = (v_1, \dots, v_N) \in \mathbb{R}^N,$$

where $v_i = (v_i^j \mid j \in D_{-i})$, such that (σ, v) satisfies the following system of polynomial equations

$$\begin{cases} v_i^j + u_i^j(\sigma_i, s_{-i}^j) = v_i^0 + u_i^0(\sigma_i, s_{-i}^0) & \text{for each } i \in I \text{ and } j \in D_{-i} \setminus \{0\} \\ \sigma_{-i}^j v_i^j = 0 & \text{for each } i \in I \text{ and } j \in D_{-i} \\ \sum_{j=0}^{d_i} \sigma_{ij} = 1 & \text{for each } i \in I, \end{cases} \quad (2)$$

and $\sigma_{ij} \geq 0$ and $v_i^j \geq 0$ for each $i \in I$ and for each $j \in D_{-i}$.

All proofs can be found in the appendix.

As one can see from the second condition in (2), the variables v_i^j are the complementary slackness variables for the optimization of coalition $-i$ given a possible $-i$ -strategy profile j . In particular, they characterize the support of the possibly (semi-)mixed equilibrium $-i$ -strategy profile σ_i^j .

Preferences that admit an expected utility representation are invariant under positive affine transformations. It is thus no surprise that Berge equilibria are preserved by positive affine transformations of payoff functions. We record this fact in the following remark.

Remark 2.2 Let u_i , $i \in I$, be the payoff functions of a given game $G = (I, (S_i, u_i)_{i \in I})$ and let σ be a Berge equilibrium of G . By Proposition 2.1, there exists $v \in \mathbb{R}^N$ such that system (2) holds, and $\sigma_{ij} \geq 0$ and $v_i^j \geq 0$ for each $i \in I$ and for each $j \in D_{-i}$. We consider a new game $\tilde{G} = (I, (S_i, \tilde{u}_i)_{i \in I})$ defined by some affine transformations \tilde{u}_i of the payoff functions u_i , that is $\tilde{u}_i = \alpha_i u_i + \beta_i$, where $\alpha_i, \beta_i \in \mathbb{R}$,

$\alpha_i > 0$, for each $i \in I$. We prove that \tilde{G} and G have the same Berge equilibria. Let $\mathbf{w}_i = \alpha_i \mathbf{v}_i \geq 0$, for each $i \in I$. It is straightforward to verify that (2) is equivalent to the following system

$$\begin{cases} w_i^j + \alpha_i u_i^j(\boldsymbol{\sigma}_i, \mathbf{s}_{-i}^j) = w_i^0 + \alpha_i u_i^0(\boldsymbol{\sigma}_i, \mathbf{s}_{-i}^0) & \text{for each } i \in I \text{ and } \mathbf{j} \in D_{-i} \setminus \{\mathbf{0}\} \\ \sigma_{-i}^j w_i^j = 0 & \text{for each } i \in I \text{ and } \mathbf{j} \in D_{-i} \\ \sum_{j=0}^{d_i} \sigma_{ij} = 1 & \text{for each } i \in I. \end{cases} \quad (3)$$

In the game \tilde{G} , we consider the expected payoff of player i when the coalition $-i$ uses the pure strategies \mathbf{s}_{-i}^j and the probability of \mathbf{s}_{-i}^j being played under σ_{-i} , that is

$$\tilde{u}_i^j(\boldsymbol{\sigma}_i, \mathbf{s}_{-i}^j) = \sum_{j_i=0}^{d_i} \tilde{u}_i(s_{ij_i}, \mathbf{s}_{-i}^j) \sigma_{ij_i} = \alpha_i u_i^j(\boldsymbol{\sigma}_i, \mathbf{s}_{-i}^j) + \beta_i \sum_{j_i=0}^{d_i} \sigma_{ij_i}, \quad (4)$$

for each $i \in I$ and $\mathbf{j} \in D_{-i}$. Plugging (4) into system (3) we get

$$\begin{cases} w_i^j + \tilde{u}_i^j(\boldsymbol{\sigma}_i, \mathbf{s}_{-i}^j) = w_i^0 + \tilde{u}_i^0(\boldsymbol{\sigma}_i, \mathbf{s}_{-i}^0) & \text{for each } i \in I \text{ and } \mathbf{j} \in D_{-i} \setminus \{\mathbf{0}\} \\ \sigma_{-i}^j w_i^j = 0 & \text{for each } i \in I \text{ and } \mathbf{j} \in D_{-i} \\ \sum_{j=0}^{d_i} \sigma_{ij} = 1 & \text{for each } i \in I. \end{cases}$$

Applying Proposition 2.1, it follows that $\boldsymbol{\sigma}$ is a Berge equilibrium of \tilde{G} .

Observe that according to Proposition 2.1, the computation of Berge equilibria requires solving an overdetermined system with many more equations than free variables if there are more than two players.

Remark 2.3 We count the number of equations and unknowns of system (2). The first line of (2) yields $\sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1) - N$ equations; the second line yields $\sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1)$ equations; finally, the third line leads to N equations. All in all, system (2) is made up of

$$2 \sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1) \quad (5)$$

equations. Regarding the unknowns, the vector $\boldsymbol{\sigma}$ has $\sum_{j=1}^N (d_j + 1) = \sum_{j=1}^N d_j + N$ components, whereas \mathbf{v} is made up of $\sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1)$ elements, so that the number of variables involved in system (2) is

$$\sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1) + \sum_{j=1}^N d_j + N. \quad (6)$$

For $N = 2$, the number of equations of system (2), which is $2(d_1 + d_2 + 1)$ equals the number of its unknowns. For $N > 2$, system (2) is overdetermined, having many more equations than unknowns; indeed,

$$\begin{aligned} 2 \sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1) &= \sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1) + \sum_{i=1}^N \left(\sum_{k=0}^{N-1} \sum_{K \subseteq \{1, \dots, N\} \setminus \{i\}, |K|=k} \prod_{j \in K} d_j \right) \\ &> \sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1) + N + (N-1) \sum_{j=1}^N d_j \\ &> \sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1) + \sum_{j=1}^N d_j + N. \end{aligned}$$

If each player has the same number of pure strategies, that is $d_i = d$ for each $i \in I$, then from (5) and (6) it follows that system (2) is made up by $2N(d+1)^{N-1}$ equations in $N(d+1)((d+1)^{N-2} + 1)$ unknowns. The difference between the number of equations and the number of unknowns is

$$N(d+1)((d+1)^{N-2} - 1) \begin{cases} = 0 & \text{if } N = 2 \\ > 0 & \text{if } N > 2. \end{cases}$$

In the following, we provide alternative conditions to find Berge equilibria simply by solving a polynomial system in the variables σ , thus eliminating the complementary slackness variables.

Proposition 2.4 *The strategy profile*

$$\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{R}^{D+N},$$

where

$$\sigma_i := (\sigma_{i0}, \dots, \sigma_{id_i}) \quad \text{for each } i \in I,$$

is a Berge equilibrium if and only if the following conditions hold:

(i) σ is a positive solution of the following system of polynomial equations

$$\begin{cases} \sigma_{-i}^j p_i^j(\sigma) = 0 & \text{for each } i \in I \text{ and } j \in D_{-i} \\ \sum_{j=0}^{d_i} \sigma_{ij} = 1 & \text{for each } i \in I, \end{cases} \quad (7)$$

where, for each $i \in I$,

$$\begin{aligned} p_i^0(\sigma) &:= \sum_{j \in D_{-i}, j \neq \mathbf{0}} \sigma_{-i}^j (u_i^j(\sigma_i, s_{-i}^j) - u_i^0(\sigma_i, s_{-i}^0)) \\ p_i^j(\sigma) &:= p_i^0(\sigma) + (u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j)), \quad \text{for each } j \in D_{-i} \setminus \{\mathbf{0}\}, \end{aligned} \quad (8)$$

are polynomial functions of degree N ;

(ii) $p_i^j(\sigma) \geq 0$ for each $i \in I$ and $j \in D_{-i}$.

Remark 2.5 We note that in the first block of (7) some of the equations are redundant. In particular, for each $i \in I$, recalling (8) and exploiting relation (26) (see Appendix A) it holds

$$\begin{aligned} \sum_{j \in D_{-i}} \sigma_{-i}^j p_i^j(\sigma) &= \sum_{j \in D_{-i}} \sigma_{-i}^j \left(p_i^0(\sigma) + (u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j)) \right) \\ &= \sum_{j \in D_{-i}} \sigma_{-i}^j p_i^0(\sigma) + \sum_{j \in D_{-i}} \sigma_{-i}^j (u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j)) \\ &= p_i^0(\sigma) - p_i^0(\sigma) = 0. \end{aligned} \quad (9)$$

Consequently, from (7) and (9), we conclude that σ is a Berge equilibrium of the game if and only if Proposition 2.4 holds with system (7) replaced by

$$\begin{cases} \sigma_{-i}^j p_i^j(\sigma) = 0 & \text{for each } i \in I \text{ and } j \in D_{-i} \setminus \{\mathbf{0}\} \\ \sum_{j=0}^{d_i} \sigma_{ij} = 1 & \text{for each } i \in I. \end{cases} \quad (10)$$

We conclude this section by characterizing Berge equilibria in pure and completely mixed strategies.

Corollary 2.6 *A pure strategy profile $s = (s_{1l_1}, \dots, s_{Nl_N}) \in S$ is a Berge equilibrium if and only if*

$$\min_{j \in D_{-i} \setminus \{\mathbf{t}^{-i}\}} \left(u_i(s) - u_i(s_{il_i}, s_{-i}^j) \right) \geq 0 \quad \text{for each } i \in I. \quad (11)$$

Remark 2.7 Corollary 2.6 recovers the main result of Corley and Kwain (2015), Theorem 3, where the authors provide a condition for finding pure strategies Berge equilibria using the so-called disappointment matrix. It is also equivalent to the conditions for the existence of pure Berge equilibria provided in Nessaha, Larbani, and Tazdaitb (2007), Lemma 2.1 and Proposition 2.1.

Let us now come to the case of completely mixed Berge equilibria that has not been studied in detail in the literature. As usual, we call a profile σ *completely mixed* if every player i plays each strategy j with strictly positive probability, i.e. for all $i \in I, j \in [d_i]$ we have $\sigma_{ij} > 0$. Let $i \in \{1, \dots, N\}$. We order the elements j_l of the set D_{-i} assuming that the first element is $j_1 = \mathbf{0}$.

Proposition 2.8 The strategy profile $\sigma = ((\sigma_{10}, \dots, \sigma_{1d_1}), \dots, (\sigma_{N0}, \dots, \sigma_{Nd_N})) \in \mathbb{R}^{D+N}$ is a completely mixed Berge equilibrium if and only if $\sigma_i = (\sigma_{i0}, \dots, \sigma_{id_i})$ is a strictly positive solution of the linear system

$$A_i \sigma_i = b_i \quad (12)$$

for each $i \in I$, where $A_i \in \mathbb{R}^{|D_{-i}| \times (d_i+1)}$, with

$$A_i(l, m) := \begin{cases} 1 & \text{if } l = 1 \\ u_i(s_{i(m-1)}, s_{-i}^{j_l}) - u_i(s_{i(m-1)}, s_{-i}^{\mathbf{0}}) & \text{if } 2 \leq l \leq |D_{-i}| \end{cases}, \quad (13)$$

for each $l = 1, \dots, |D_{-i}|$ and $m = 1, \dots, d_i + 1$, and

$$b_i := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{|D_{-i}|}. \quad (14)$$

In mixed Nash equilibrium, the other players make player i indifferent between all her pure strategies. In Berge equilibrium, player i mixes in such a way that all coalitions $-i$ are indifferent among all pure $(-i)$ -strategy profiles they might consider in order to make him better off. This is the meaning of equation (12).

3 Gröbner basis techniques for existence and computation of Berge equilibria

The main object of study of Algebraic Geometry and Commutative Algebra are algebraic varieties, which are geometric entities defined as the solution sets of polynomial equations. The origins of the two disciplines are traced back to the 19th and the (early) 20th century. Throughout the 20th century the ties between the two areas fastly grew up also driven by the emergence of important computational aspects, primarily due to Buchberger's contributions on algorithms for manipulating systems of polynomial equations. In particular, in 1965 the concept of Gröbner bases was introduced by Buchberger (2006), who also developed an algorithm to compute Gröbner bases of given ideals, which allowed for the simplification of polynomial systems. Then, in the late 20th century, with the advent of computers, software packages (like CoCoA (1988), Macaulay2 by Grayson and Stillman (1993) and Singular by Decker, Greuel, Pfister, and Schönemann (2024)) were developed, making Gröbner bases computations doable to all researchers and highly relevant for various disciplines (such as Robotics, Cryptography, Combinatorics, Statistics, Mathematical Optimization), thus establishing themselves as a fundamental tool to bridge theory and application across diverse fields. Recently, in economics, Gröbner bases theory has been fruitfully applied in different areas, such as in General Equilibrium Theory for the analysis of the equilibrium conditions in markets (see Kubler and Schmedders (2010)), in Econometrics to address the global identification problem in dynamic stochastic general equilibrium (DSGE) models (see Kocięcki and Kolasa (2023)) and in Game Theory for the computation of all Nash equilibria in games with polynomial payoff functions (see Datta (2010)).

Despite the wide range of applications, the main and best known task of Gröbner basis technique is to provide a simplified representation of a set of polynomial equations, which enables to compute its solutions. In the rest of this section we provide a brief introduction to Gröbner basis and the main algebraic/geometric objects involved when studying systems of polynomial equations. For a more detailed introduction to the topic we refer the interested reader to Cox, Little, and O'Shea (2005, 2015) and Kreuzer and Robbiano (2000). The definitions and results presented in this section are drawn from these reference textbooks.

Formally, the set of all the solutions of a given system of polynomial equations $f_1(x_1, \dots, x_n) = \dots = f_s(x_1, \dots, x_n) = 0$ in n variables with coefficients in a field \mathbb{K} is the *affine algebraic variety* $\mathcal{V}(f_1, \dots, f_s) \subseteq \mathbb{K}^n$ defined by

$$\mathcal{V}(f_1, \dots, f_s) := \{(a_1, \dots, a_n) \in \mathbb{K}^n \mid f_i(a_1, \dots, a_n) = 0 \text{ for all } 1 \leq i \leq s\} \subseteq \mathbb{K}^n,$$

which is the main geometrical object of Computational Algebraic Geometry. The set

$$\mathcal{I}(f_1, \dots, f_s) := \{p_1 f_1 + \dots + p_s f_s \mid p_i \in \mathbb{K}[x_1, \dots, x_n] \text{ for } i = 1, \dots, s\} \subseteq \mathbb{K}[x_1, \dots, x_n]$$

is the *ideal generated by* f_1, \dots, f_s , one of the most important algebraic structures in polynomial rings.

The definition of Gröbner bases of ideals is constructed upon monomial orders, which are defined as follows.

Definition 3.1 A monomial order on $\mathbb{K}[x_1, \dots, x_n]$ is any relation $>$ on the set of monomials x^α in $\mathbb{K}[x_1, \dots, x_n]$ (or equivalently on the exponent vectors $\alpha \in \mathbb{Z}_{\geq 0}^n$) satisfying:

- (i) $>$ is a total (linear) ordering relation;
- (ii) $>$ is compatible with multiplication in $\mathbb{K}[x_1, \dots, x_n]$, in the sense that if $x^\alpha > x^\beta$ and x^γ is any monomial, then $x^\alpha x^\gamma = x^{\alpha+\gamma} > x^{\beta+\gamma} = x^\beta x^\gamma$;
- (iii) $>$ is a well-ordering, that is, every non-empty collection of monomials has a smallest element under $>$.

Note that in the univariate case ($n = 1$) the only monomial order in the polynomial ring $\mathbb{K}[x]$ is the degree order given by $\dots > x^{m+1} > x^m > \dots > x^2 > x > 1$, whereas polynomial rings in several variables ($n > 1$) admit many different monomial orders. In this case, a choice on the ordering of x_1, \dots, x_n is firstly required (usually the variables are ordered as follows: $x_1 > x_2 > \dots > x_n$). Then, based on the previous choice, there are still many possibilities to define a monomial order; for instance, the lexicographic order, the graded lexicographic order, the graded reverse lexicographic order. Monomial orders allow to introduce the leading term of a polynomial (see Definition 3.2) which is needed to define Gröbner bases (see Definition 3.3).

Definition 3.2 Let $>$ be a monomial order on $\mathbb{K}[x_1, \dots, x_n]$ and let $f = \sum_{\alpha} c_{\alpha} x^{\alpha}$ be a polynomial in $\mathbb{K}[x_1, \dots, x_n]$. The leading term of f (with respect to $>$), denoted by $LT_{>}(f)$ or simply by $LT(f)$ if no confusion arises, is the product $c_{\alpha} x^{\alpha}$ where x^{α} is the largest monomial appearing in f in the ordering $>$.

Definition 3.3 Fix a monomial order $>$ on $\mathbb{K}[x_1, \dots, x_n]$ and let $\mathcal{I} \subset \mathbb{K}[x_1, \dots, x_n]$ be an ideal. A Gröbner basis for \mathcal{I} (with respect to $>$) is a finite collection of polynomials $\mathcal{G} = \{g_1, \dots, g_t\} \subset \mathcal{I}$ with the property that for every nonzero $f \in \mathcal{I}$, its leading term $LT(f)$ is divisible by $LT(g_i)$ for some $i \in \{1, \dots, t\}$.

Gröbner bases are characterized by many different properties, some of them being used to provide alternative equivalent definitions. We recall here two fundamental facts: a Gröbner basis $\mathcal{G} = \{g_1, \dots, g_t\}$ of \mathcal{I} is a special kind of generating set of the ideal, that is $\mathcal{I}(g_1, \dots, g_t) = \mathcal{I}$, and Gröbner bases exist for all ideals in $\mathbb{K}[x_1, \dots, x_n]$. Further, Gröbner basis computation can provide a systematic method for eliminating variables from systems of polynomial equations, as it can be used as a multivariate non-linear generalization of Gaussian elimination for linear systems (see Lazard (1983)). The basic strategy of elimination theory is provided by the following two main theorems: the Elimination Theorem, which is based on the lexicographic monomial order, and the Extension Theorem. Note that, for the reader's convenience, the Extension Theorem is stated only for the case of eliminating the first variable x_1 , but it can be applied as well when eliminating any number of variables.

Theorem 3.4 (The Elimination Theorem) Let $\mathcal{I} \subseteq \mathbb{K}[x_1, \dots, x_n]$ be an ideal and let \mathcal{G} be a Gröbner basis of \mathcal{I} w.r.t. the monomial order $>_{\text{Lex}}$ (with $x_1 > x_2 > \dots > x_n$). Then, for every $0 \leq l \leq n$, the set

$$\mathcal{G}_l = \mathcal{G} \cap \mathbb{K}[x_{l+1}, \dots, x_n]$$

is a Gröbner basis of the l -th elimination ideal $\mathcal{I}_l = \mathcal{I} \cap \mathbb{K}[x_{l+1}, \dots, x_n] \subseteq \mathbb{K}[x_{l+1}, \dots, x_n]$.

Theorem 3.5 (The Extension Theorem) Let $\mathcal{I} = \mathcal{I}(f_1, \dots, f_s) \subseteq \mathbb{K}[x_1, \dots, x_n]$ and let $\mathcal{I}_1 = \mathcal{I} \cap \mathbb{K}[x_2, \dots, x_n] \subseteq \mathbb{K}[x_2, \dots, x_n]$. For each $1 \leq i \leq s$ write f_i in the form

$$f_i = c_i(x_2, \dots, x_n)x_1^{N_i} + \text{terms in which } x_1 \text{ has degree } < N_i,$$

where $N_i \geq 0$ and $c_i(x_2, \dots, x_n) \in \mathbb{K}[x_2, \dots, x_n]$ is nonzero. Suppose that we have a partial solution $(a_2, \dots, a_n) \in \mathcal{V}(\mathcal{I}_1)$. If $(a_2, \dots, a_n) \notin \mathcal{V}(\mathcal{I}(c_1, \dots, c_s))$ and \mathbb{K} is algebraically closed, then there exists $a_1 \in \mathbb{K}$ such that $(a_1, a_2, \dots, a_n) \in \mathcal{V}(\mathcal{I})$.

If the ideal \mathcal{I} is *zero-dimensional*, that is if the associated system of polynomial equations has a finite number of solutions (note that again this can be tested by looking at the leading terms of a Gröbner basis of \mathcal{I}), the previous theorems provide a procedure to compute $\mathcal{V}(\mathcal{I})$. Indeed, let \mathcal{G} be a Gröbner basis of \mathcal{I} w.r.t. the monomial order $>_{\text{Lex}}$ (with $x_1 > x_2 > \dots > x_n$); the last coordinate of a solution of the associated polynomial system is a root a_n of the greatest common divisor of the polynomials \mathcal{G}_{n-1} , that is the polynomials of the basis that depend only on the last variable. If the assumptions of the Extension Theorem are satisfied, the substitution $x_n = a_n$ can be performed in \mathcal{G} , and the second-last coordinate of such solution is a root of the greatest common divisor of the polynomials in $\mathcal{G}_{n-2}|_{\{x_n=a_n\}}$. Then, the iterations continue till the first variable is considered, so that all elements of $\mathcal{V}(\mathcal{I})$ are computed. This approach describes an effective and quite simple solving process which is outlined in Algorithm `SolutionSet` in Section 5 below.

4 Regular games

The algorithms that we develop below focus on the generic class of games that possess at most finitely many locally isolated Berge equilibria. In this section, we specify the notion of regularity that will be used throughout, adapting the notion that van Damme (1991) developed for Nash equilibria to the case of Berge equilibria. The reader who is not interested in these technical details is invited to jump immediately to Section 5 on algorithms.

In the setting of Section 2, let $\mathcal{G}(I, (S_i)_{i \in I})$ be the set of all games with pure strategies spaces S_1, \dots, S_N . Imposing a fixed ordering on the payoff functions, there is one-to-one correspondence between the space of all the games $\mathcal{G}(I, (S_i)_{i \in I})$ and \mathbb{R}^{NDs} . In particular, each game $\bar{\Gamma}$ in $\mathcal{G}(I, (S_i)_{i \in I})$ is represented by its *payoff vector*

$$\bar{\mathbf{u}} = (\bar{u}_1, \dots, \bar{u}_N) \in \mathbb{R}^{NDs}.$$

Thus, the game $\bar{\Gamma} \in \mathcal{G}(I, (S_i)_{i \in I})$ can be simply denoted by $\bar{\mathbf{u}}$, and its (possibly empty) set of Berge equilibria by $B(\bar{\mathbf{u}})$.

Inspired by van Damme (1991), we introduce the notions of *isolated* and *strongly stable* Berge equilibria. A Berge equilibrium is said to be *isolated* if there is no other Berge equilibrium of the game lying arbitrarily close to it. A Berge equilibrium is said to be *strongly stable* if it undergoes continuous and unique changes under small perturbations in the original payoff functions.

Definition 4.1 Let $\bar{\Gamma} \in \mathcal{G}(I, (S_i)_{i \in I})$ be an N -person normal form game defined by $\bar{\mathbf{u}} = (\bar{u}_1, \dots, \bar{u}_N) \in \mathbb{R}^{NDs}$. Let $\bar{\sigma}$ be a Berge equilibrium of $\bar{\Gamma}$. Then:

- (i) $\bar{\sigma}$ is *isolated* if there exists a neighborhood $U_{\bar{\sigma}}$ of $\bar{\sigma}$ such that $U_{\bar{\sigma}} \cap B(\bar{\mathbf{u}}) = \{\bar{\sigma}\}$;
- (ii) $\bar{\sigma}$ is *strongly stable* if there exist neighborhoods $U_{\bar{\mathbf{u}}}$ of $\bar{\mathbf{u}}$ and $U_{\bar{\sigma}}$ of $\bar{\sigma}$ such that:

$$(a) |B(\mathbf{u}) \cap U_{\bar{\sigma}}| = 1 \text{ for each } \mathbf{u} \in U_{\bar{\mathbf{u}}};$$

(b) the mapping $b : U_{\bar{\mathbf{u}}} \rightarrow U_{\bar{\boldsymbol{\sigma}}}$ defined by $b(\mathbf{u}) = B(\mathbf{u}) \cap U_{\bar{\boldsymbol{\sigma}}}$ is continuous.

In order to introduce *regularity*, we recall that each Berge equilibrium is defined as a positive solution of system (10) such that $p_i^j(\boldsymbol{\sigma}) \geq 0$ for each $i \in I$ and $j \in D_{-i}$ (see Proposition 2.4 and Remark 2.5). Henceforth, we find it convenient to explicitly denote the dependence of each p_i^j on the payoff functions u_i , that is we will write $p_i^j(\boldsymbol{\sigma}, u_i)$. We let $F = (F_1, \dots, F_N) : \mathbb{R}^{D+N} \times \mathbb{R}^{ND_S} \rightarrow \mathbb{R}^{\sum_{i \in I} |D_{-i}|}$ be the mapping defined by system (10), where $F_i : \mathbb{R}^{D+N} \times \mathbb{R}^{D_S} \rightarrow \mathbb{R}^{|D_{-i}|}$ has the following components

$$\begin{aligned} F_{i,0}(\boldsymbol{\sigma}, u_i) &:= \sum_{j=0}^d \sigma_{ij} - 1 \\ F_{i,j}(\boldsymbol{\sigma}, u_i) &:= \sigma_{-i}^j p_i^j(\boldsymbol{\sigma}, u_i) \quad \text{for each } j \in D_{-i} \setminus \{0\}, \end{aligned} \quad (15)$$

for each $i \in I$. Let $\nabla F(\boldsymbol{\sigma}, \mathbf{u})$ and $\nabla_{\boldsymbol{\sigma}} F(\boldsymbol{\sigma}, \mathbf{u})$ be the Jacobian matrices of F w.r.t. all the variables $(\boldsymbol{\sigma}, \mathbf{u})$ and w.r.t. $\boldsymbol{\sigma}$ respectively. A game $\bar{\Gamma}$ defined by the payoff function $\bar{\mathbf{u}}$ is *regular* if for any (real) solution $\bar{\boldsymbol{\sigma}}$ of the polynomial system (10) the Jacobian matrix $\nabla_{\boldsymbol{\sigma}} F(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$ has full rank and the rank of $\nabla F(\boldsymbol{\sigma}, \mathbf{u})$ is constant and equal to $\text{rank}(\nabla_{\boldsymbol{\sigma}} F(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}))$ in a neighborhood of $(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$. The formalization of this concept is provided in Definition 4.3.

Remark 4.2 We observe that the rank of $\nabla_{\boldsymbol{\sigma}} F(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$ is at most $D + N$. Indeed, recalling that $D = \sum_{i=1}^N d_i$ and $\sum_{i=1}^N |D_{-i}| = \sum_{i=1}^N \prod_{j=1, j \neq i}^N (d_j + 1)$, it immediately follows that in the case $N = 2$ it holds

$$\sum_{i=1}^N |D_{-i}| = D + N,$$

whereas for $N \geq 3$ we have

$$\sum_{i=1}^N |D_{-i}| > \sum_{i=1}^{N-1} (d_{i+1} + 1) + (d_1 + 1) = \sum_{i=1}^N (d_i + 1) = D + N.$$

Definition 4.3 Let $\bar{\Gamma} \in \mathcal{G}(I, (S_i)_{i \in I})$ be an N -person normal form game defined by $\bar{\mathbf{u}} = (\bar{u}_1, \dots, \bar{u}_N) \in \mathbb{R}^{ND_S}$ and $V_{\mathbb{R}}(\bar{\mathbf{u}}) \subseteq \mathbb{R}^{D+N}$ be the (real) zero set of the polynomial system $F(\boldsymbol{\sigma}, \bar{\mathbf{u}}) = 0$, where F is the mapping defined in (15). The game $\bar{\Gamma}$ is (Berge) *regular* if for each $\bar{\boldsymbol{\sigma}} \in V_{\mathbb{R}}(\bar{\mathbf{u}})$ the following conditions are satisfied:

- (i) $\text{rank}(\nabla_{\boldsymbol{\sigma}} F(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})) = D + N$;
- (ii) there exists a neighborhood W of $(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$ such that

$$\text{rank}(\nabla F(\boldsymbol{\sigma}, \mathbf{u})) = D + N, \quad \forall (\boldsymbol{\sigma}, \mathbf{u}) \in W.$$

Remark 4.4 In 2-players games, (Berge) regularity simply corresponds to the nonsingularity of the Jacobian matrix $\nabla_{\boldsymbol{\sigma}} F(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$ for each $\bar{\boldsymbol{\sigma}} \in V_{\mathbb{R}}(\bar{\mathbf{u}})$ (see Remark 4.2 and Definition 4.3).

In regular games, Berge equilibria are strongly stable isolated points.

Theorem 4.5 In (Berge) regular games,

- (i) every Berge equilibrium is strongly stable,
- (ii) every strongly stable equilibrium is isolated. In particular, all Berge equilibria are isolated.

Finally, regularity has a remarkable implication on Berge equilibria: regular games admit at most one completely mixed Berge equilibrium, regardless of the number of players or strategies. This stands in stark contrast to the rapid growth of completely mixed Nash equilibria, as discussed in McKelvey and McLennan (1997).

Proposition 4.6 (Berge) regular games admit at most one completely mixed Berge equilibrium.

5 Algorithms for Computing Berge Equilibria

Having transformed the problem of finding Berge equilibria into an equivalent algebraic polynomial system (see Proposition 2.1) or an equivalent semi-algebraic system (see Proposition 2.4), we can use techniques such as Gröbner bases and elimination theory (see Section 3) to simplify and solve these systems. The process provides a structured way to compute equilibria, offering insights beyond traditional equilibrium analysis. Building on the results from Section 2 and the theory recalled in Section 3, this section introduces algorithms for computing Berge equilibria. Specifically, Algorithm **AllBerge** is designed to compute all Berge equilibria, while Algorithms **PureBerge** and **CompletelyMixedBerge** provide pseudocode for computing pure and completely mixed Berge equilibria, respectively. From now on, we assume that the games under consideration are regular (see Section 4). The algorithms presented in this paper have been implemented using the CoCoA computer algebra system CoCoA (1988). The full source code is publicly available in the GitHub repository: <https://github.com/mltorrente/BergeEquilibria.git>.

5.1 Pseudocode for computing all Berge equilibria

In Proposition 2.4 (and Remark 2.5) all Berge equilibria are characterized as the nonnegative solutions to the polynomial system (10) satisfying some positivity conditions (see Proposition 2.4-item(ii)). Based on such result, we introduce Algorithm **AllBerge** that, starting from any regular game with N -players, sets of pure strategies S_1, \dots, S_N and payoff functions u_1, \dots, u_N , performs the following steps. At first, it constructs the polynomial set F associated to the system (10) and the ideal $\mathcal{I}(F)$. Then, the solution set \mathcal{S} of the system $F = 0$ is computed by applying Algorithm **SolutionSet** (described below) to $\mathcal{I}(F)$. To this aim we recall that, by the regularity assumption, all real zeros of $F = 0$ are locally isolated points (see Corollary ??). Finally, Algorithm **AllBerge** collects in the variable BE all the solutions $z \in \mathcal{S}$ which are both positive and satisfy the positivity condition $p_i^j(z) \geq 0$ for each $i \in I$ and $j \in D_{-i}$, and return it. From Proposition 2.4, the returned set BE contains all Berge equilibria of the input game.

Algorithm 1 - AllBerge: Computation of All Berge Equilibria

Require: Regular game defined by:

set of players $I = \{1, \dots, N\}$

sets of pure strategies $S_1 = \{s_{10}, \dots, s_{1d_1}\}, \dots, S_N = \{s_{N0}, \dots, s_{Nd_N}\}$

payoff functions u_1, \dots, u_N

Ensure: Berge Equilibria BE

```

1: BE =  $\emptyset$  {Initialize the set of Berge Equilibria}
2:  $\sigma_i = (\sigma_{i0}, \dots, \sigma_{id_1})$  for each  $i \in I$ 
3:  $\sigma = (\sigma_1, \dots, \sigma_N)$  {Construct the set of variables}
4:  $[d_i] = \{0, 1, \dots, d_i\}$  for each  $i \in I$ 
5:  $D_{-i} = \prod_{k=1, k \neq i}^N [d_k]$  for each  $i \in I$ 
6: for all  $i \in I$  and  $j \in D_{-i}$  do
7:    $u_i^j(\sigma_i, s_{-i}^j) = \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^j) \sigma_{ij_i}$  {Compute all  $u_i^j(\sigma_i, s_{-i}^j)$ }
8: end for
9: for all  $i \in I$  do
10:   $p_i^0(\sigma) = \sum_{j \in D_{-i}, j \neq 0} \sigma_{-i}^j (u_i^j(\sigma_i, s_{-i}^j) - u_i^0(\sigma_i, s_{-i}^0))$  {Compute all  $p_i^j(\sigma)$ }
11:  for all  $j \in D_{-i} \setminus \{0\}$  do
12:     $p_i^j(\sigma) = p_i^0(\sigma) + (u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j))$ 
13:  end for
14: end for
15:  $F = \emptyset$ 
16: for all  $i \in I$  do
17:  for all  $j \in D_{-i} \setminus \{0\}$  do
18:    append  $\sigma_{-i}^j p_i^j(\sigma)$  to  $F$  {Compute the polynomial set  $F$ }
19:  end for
20:  append  $\sum_{i=0}^{d_i} \sigma_{ij} - 1$  to  $F$ 

```

```

21: end for
22: let  $\mathcal{S}$  be the output of Algorithm SolutionSet applied to  $\sqrt{\mathcal{I}(F)}$   $\{S = \mathcal{V}_{\mathbb{R}}(\sqrt{\mathcal{I}(F)})\}$ 
23: for all  $z \in \mathcal{S}$  do
24:   if  $z \geq 0$  and  $p_i^j(z) \geq 0$  for each  $i \in I$  and  $j \in D_{-i}$  then
25:     append  $z$  to BE
26:   end if
27: end for
28: if  $\text{BE} = \emptyset$  then
29:   print "Berge equilibria do not exist"
30: else
31:   print "The game admits Berge equilibria"
32: end if
33: return BE  $\{\text{Return the set of All Berge Equilibria}\}$ 

```

Algorithm **SolutionSet** is designed to compute all the solutions of a given zero-dimensional ideal. It is based on the Elimination Theorem and the Extension Theorem (recalled in Section 3, see Theorem 3.4 and 3.5). Starting from a given zero-dimensional ideal \mathcal{I} of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$, Algorithm **SolutionSet** first of all computes the Gröbner basis \mathcal{G} of \mathcal{I} w.r.t. the lexicographic monomial order ($>_{\text{Lex}}$) with $x_1 > x_2 > \dots > x_n$. Then, it performs a stepwise computation based on the number of the variables. The main **for** loop is indexed by the integer variable l that ranges between $n - 1$ (starting value) to 0 (final value). In particular, at the first iteration $l = n - 1$, the partial solution set is $\mathcal{S}_{n-1} = \emptyset$ and $\mathcal{G}_{n-1} = \mathcal{G} \cap \mathbb{K}[x_n]$ is a Gröbner basis of the $(n - 1)$ th elimination ideal \mathcal{I}_{n-1} . The set \mathcal{S}_{n-1} is then updated with all the common roots of the polynomials belonging to \mathcal{G}_{n-1} , that are computed by simply finding all the common zeros of the GCD of the elements of \mathcal{G}_{n-1} . At the l th iteration, with $0 \leq l < n - 1$, the partial solution set is $\mathcal{S}_l = \emptyset$, $\mathcal{G}_l = \mathcal{G} \cap \mathbb{K}[x_{l+1}, \dots, x_n]$ is a Gröbner basis of \mathcal{I}_l and \mathcal{L}_l contains the leading coefficients polynomials of \mathcal{G}_l , that is the nonzero polynomials $c(x_{l+2}, \dots, x_n) \in \mathbb{K}[x_{l+2}, \dots, x_n]$ such that each $f \in \mathcal{G}_l$ rewrites in the form

$$f = c(x_{l+2}, \dots, x_n)x_{l+1}^d + \text{terms in which } x_{l+1} \text{ has degree } < d,$$

where $d \geq 0$. Then, an inner loop applies. The **for** condition considers at each step one partial solution $s \in \mathcal{S}_{l+1}$ at a time: if not all elements of \mathcal{L}_l simultaneously vanish at s , then the univariate polynomial set \mathcal{H} is constructed with the evaluations of all elements of \mathcal{G}_l at $(x_{l+1}, \dots, x_n) = s$, that is $\mathcal{H} = \mathcal{G}_l|_{\{(x_{l+1}, \dots, x_n) = s\}}$. The set of all the common roots of the GCD of \mathcal{H} is denoted by \mathcal{T} and each $t \in \mathcal{T}$ is used to "extend" the partial solution s , that is the pair $(t, s) \in \mathbb{K}^{n-l}$ is appended to \mathcal{S}_l . The outer iterations continue till $l = 0$ when the first variable is considered. Finally, \mathcal{S}_0 is assigned to the returned variable \mathcal{S} which, by Theorem 3.5 and 3.4, contains all elements of $\mathcal{V}(\mathcal{I})$.

Algorithm 2 - SolutionSet: Computation of the solution set of a zero-dimensional ideal

Require: $\mathcal{I} \subset \mathbb{K}[x_1, \dots, x_n]$ zero-dimensional ideal

Ensure: solution set $\mathcal{S} = \mathcal{V}(\mathcal{I})$

```

1: compute the reduced Gröbner basis  $\mathcal{G}$  of  $\mathcal{I}$  w.r.t. Lex with  $x_n < \dots < x_1$ 
2: for  $l = n - 1, \dots, 0$  do
3:    $\mathcal{S}_l = \emptyset$   $\{\text{Initialize } \mathcal{S}_l = \emptyset\}$ 
4:    $\mathcal{G}_l = \mathcal{G} \cap \mathbb{K}[x_{l+1}, \dots, x_n]$   $\{\text{Compute a Gröbner basis of } l\text{th elimination ideal}\}$ 
5:   if  $l = n - 1$  then
6:      $\mathcal{S}_{n-1} = \text{RootsOf}(\text{GCD}(\mathcal{G}_{n-1}))$   $\{\mathcal{S}_{n-1} \text{ contains the common roots of } \mathcal{G}_{n-1}\}$ 
7:   else
8:      $\mathcal{L}_l = \{\text{LCP}(f, x_{l+1}) \mid f \in \mathcal{G}_l\}$   $\{\text{Compute the Leading Coefficients Polynomials of } \mathcal{G}_l\}$ 
9:     for all  $s \in \mathcal{S}_{l+1}$  do
10:      if  $\text{subst}(c, [x_{l+2}, \dots, x_n], s) \neq 0$  for some  $c \in \mathcal{L}_l$  then
11:         $\mathcal{H} = \{\text{subst}(f, [x_{l+2}, \dots, x_n], s) \mid f \in \mathcal{G}_l\}$   $\{\mathcal{H} \text{ contains univariate polys in } x_{l+1}\}$ 
12:         $\mathcal{T} = \text{RootsOf}(\text{GCD}(\mathcal{H}))$   $\{\mathcal{T} \text{ contains the common roots of } \mathcal{H}\}$ 
13:        for all  $t \in \mathcal{T}$  do

```

```

14:         append  $(t, s)$  to  $\mathcal{S}_l$                                  $\{\mathcal{S}_l$  contains the extensions of the partial solutions $\}$ 
15:     end for
16: end if
17: end for
18: end if
19: end for
20:  $\mathcal{S} = \mathcal{S}_0$ 
21: return  $\mathcal{S}$                                                      $\{\text{Return the solution set } \mathcal{S}\}$ 

```

Example 5.1 We consider the game *Battle of the Sexes* illustrated by Table 1.

	I	II
I	(2,1)	(0,0)
II	(0,0)	(1,2)

Table 1: 2-player game Battle of the Sexes

This is a regular 2-player game in which each player has the two strategies I and II, hence $N = 2$, $d_1 = d_2 = 1$, $S_1 = \{s_{10}, s_{11}\} = \{I, II\}$ and $S_2 = \{s_{20}, s_{21}\} = \{I, II\}$. We apply Algorithm **AllBerge** and follow its main steps. Algorithm **AllBerge** initializes the variables $\sigma = ((\sigma_{10}, \sigma_{11}), (\sigma_{20}, \sigma_{21}))$ and constructs the polynomial set F (see (10)), which is made up of 4 polynomials in 4 variables (see also Proposition 2.4 and Remark 2.5):

$$F = \{-(2\sigma_{10} - \sigma_{11})\sigma_{20}\sigma_{21}, -\sigma_{10}\sigma_{11}(\sigma_{20} - 2\sigma_{21}), \sigma_{10} + \sigma_{11} - 1, \sigma_{20} + \sigma_{21} - 1\}. \quad (16)$$

Then, Algorithm **SolutionSet** is applied to the (radical) ideal $\mathcal{I}(F)$ to solve the polynomial system $F = 0$ via the method of variables' elimination (see Elimination Theorem and Extension Theorem). The reduced Gröbner basis \mathcal{G} , w.r.t. the lexicographical ordering on the monomials of $\mathbb{K}[\sigma]$ with $\sigma_{21} < \sigma_{20} < \sigma_{11} < \sigma_{10}$, of $\mathcal{I}(F)$ is computed

$$\mathcal{G} = \begin{cases} g_1 = \sigma_{20} + \sigma_{21} - 1 \\ g_2 = \sigma_{10} + \sigma_{11} - 1 \\ g_3 = \frac{1}{3}\sigma_{21}(\sigma_{21} - 1)(3\sigma_{11} - 2) \\ g_4 = (\sigma_{11} + \sigma_{21} - 1)(\sigma_{11} - \sigma_{21}) \\ g_5 = \frac{1}{3}\sigma_{21}(\sigma_{21} - 1)(3\sigma_{21} - 1). \end{cases}$$

The procedure of variables' elimination starts. The set $\mathcal{G}_{21} = \mathcal{G} \cap \mathbb{K}[\sigma_{21}] = \{g_5\}$ is a Gröbner basis of the last elimination ideal $\mathcal{I}_{21} = \mathcal{I} \cap \mathbb{K}[\sigma_{21}]$. The unique solutions of the equation $g_5 = 0$ are

$$\sigma_{21} = 0, \quad \sigma_{21} = \frac{1}{3}, \quad \sigma_{21} = 1,$$

which are used to extend to all complete solutions of the polynomial system. At the end, Algorithm **SolutionSet** returns the following 5 solutions:

$$\begin{array}{lll} \xi_1 = (0, 1, 1, 0) & \xi_2 = (1, 0, 1, 0) & \xi_3 = (0, 1, 0, 1) \\ \xi_4 = (1, 0, 0, 1) & \xi_5 = (1/3, 2/3, 2/3, 1/3) & \end{array}$$

Control is then passed back to Algorithm **AllBerge** which checks whether the condition of Proposition 2.4 item (ii) is verified on the solutions. While ξ_2 , ξ_3 and ξ_5 satisfy the condition, ξ_1 and ξ_4 fail to meet it. Algorithm **AllBerge** stops and returns two pure Berge equilibria $\sigma_{10} = \sigma_{20} = 1$ and $\sigma_{10} = \sigma_{20} = 0$ and one completely mixed Berge equilibrium $((\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}))$. Note that this is one of the rare cases where Berge and Nash equilibria coincide as we shall see below.

Example 5.2 Our techniques allow to determine whether Berge equilibria exist. Consider the following regular three player game (compare Sturmfels (2002), Section 6.2), with $d_1 = d_2 = d_3 = 1$ and where player 1 chooses the row, player 2 the column, and player 3 the matrix.

$$C_1 : \begin{pmatrix} (0, 12, 11) & (11, 6, 3) \\ (6, 10, 0) & (6, 8, 2) \end{pmatrix} \quad C_2 : \begin{pmatrix} (6, 7, 11) & (1, 8, 3) \\ (4, 12, 14) & (8, 1, 7) \end{pmatrix}.$$

We apply Algorithm **AllBerge** to this game. Algorithm **AllBerge** initializes the variables $\sigma = ((\sigma_{10}, \sigma_{11}), (\sigma_{20}, \sigma_{21}), (\sigma_{30}, \sigma_{31}))$ and constructs the polynomial set F (see (10)), which is made up of 12 polynomials in 6 variables (see also Proposition 2.4 and Remark 2.5):

$$F = \left\{ \begin{array}{l} \sigma_{31}\sigma_{20}(6\sigma_{10}\sigma_{20}\sigma_{31} + 11\sigma_{10}\sigma_{21}\sigma_{30} + \sigma_{10}\sigma_{21}\sigma_{31} - 2\sigma_{11}\sigma_{20}\sigma_{31} + 2\sigma_{11}\sigma_{21}\sigma_{31} - 6\sigma_{10} + 2\sigma_{11}), \\ \sigma_{30}\sigma_{21}(6\sigma_{10}\sigma_{20}\sigma_{31} + 11\sigma_{10}\sigma_{21}\sigma_{30} + \sigma_{10}\sigma_{21}\sigma_{31} - 2\sigma_{11}\sigma_{20}\sigma_{31} + 2\sigma_{11}\sigma_{21}\sigma_{31} - 11\sigma_{10}), \\ \sigma_{31}\sigma_{21}(6\sigma_{10}\sigma_{20}\sigma_{31} + 11\sigma_{10}\sigma_{21}\sigma_{30} + \sigma_{10}\sigma_{21}\sigma_{31} - \sigma_{10} - 2\sigma_{11}\sigma_{20}\sigma_{31} + 2\sigma_{11}\sigma_{21}\sigma_{31} - 2\sigma_{11}), \\ \sigma_{31}\sigma_{10}(-5\sigma_{10}\sigma_{20}\sigma_{31} + 2\sigma_{10}\sigma_{21}\sigma_{31} - 2\sigma_{11}\sigma_{20}\sigma_{30} + 2\sigma_{11}\sigma_{21}\sigma_{30} - 5\sigma_{11}\sigma_{21}\sigma_{31} + 5\sigma_{20} - 2\sigma_{21}), \\ \sigma_{30}\sigma_{11}(-5\sigma_{10}\sigma_{20}\sigma_{31} + 2\sigma_{10}\sigma_{21}\sigma_{31} - 2\sigma_{11}\sigma_{20}\sigma_{30} + 2\sigma_{11}\sigma_{21}\sigma_{30} - 5\sigma_{11}\sigma_{21}\sigma_{31} + 2\sigma_{20} - 2\sigma_{21}), \\ \sigma_{31}\sigma_{11}(-5\sigma_{10}\sigma_{20}\sigma_{31} + 2\sigma_{10}\sigma_{21}\sigma_{31} - 2\sigma_{11}\sigma_{20}\sigma_{30} + 2\sigma_{11}\sigma_{21}\sigma_{30} - 5\sigma_{11}\sigma_{21}\sigma_{31} + 5\sigma_{21}), \\ \sigma_{21}\sigma_{10}(-8\sigma_{10}\sigma_{21}\sigma_{30} - 8\sigma_{10}\sigma_{21}\sigma_{31} - 11\sigma_{11}\sigma_{20}\sigma_{30} + 3\sigma_{11}\sigma_{20}\sigma_{31} - 9\sigma_{11}\sigma_{21}\sigma_{30} - 4\sigma_{11}\sigma_{21}\sigma_{31} + 8\sigma_{30} + 8\sigma_{31}), \\ \sigma_{20}\sigma_{11}(-8\sigma_{10}\sigma_{21}\sigma_{30} - 8\sigma_{10}\sigma_{21}\sigma_{31} - 11\sigma_{11}\sigma_{20}\sigma_{30} + 3\sigma_{11}\sigma_{20}\sigma_{31} - 9\sigma_{11}\sigma_{21}\sigma_{30} - 4\sigma_{11}\sigma_{21}\sigma_{31} + 11\sigma_{30} - 3\sigma_{31}), \\ \sigma_{21}\sigma_{11}(-8\sigma_{10}\sigma_{21}\sigma_{30} - 8\sigma_{10}\sigma_{21}\sigma_{31} - 11\sigma_{11}\sigma_{20}\sigma_{30} + 3\sigma_{11}\sigma_{20}\sigma_{31} - 9\sigma_{11}\sigma_{21}\sigma_{30} - 4\sigma_{11}\sigma_{21}\sigma_{31} + 9\sigma_{30} + 4\sigma_{31}), \\ \sigma_{10} + \sigma_{11} - 1, \\ \sigma_{20} + \sigma_{21} - 1, \\ \sigma_{30} + \sigma_{31} - 1 \end{array} \right\}$$

Algorithm **SolutionSet** is then applied to the (radical) ideal $\mathcal{I}(F)$ to solve the polynomial system $F = 0$. The reduced Gröbner basis \mathcal{G} , w.r.t. the lexicographical ordering on the monomials of $K[\sigma]$ with $\sigma_{31} < \sigma_{30} < \sigma_{21} < \sigma_{20} < \sigma_{11} < \sigma_{10}$, of $\mathcal{I}(F)$ is computed:

$$\mathcal{G} = \left\{ \begin{array}{l} g_1 = \sigma_{10} + \sigma_{11} - 1 \\ g_2 = \sigma_{11}(\sigma_{11} - 1) \\ g_3 = \sigma_{20} + \sigma_{21} - 1 \\ g_4 = \sigma_{21}(\sigma_{21} - 1) \\ g_5 = \sigma_{30} + \sigma_{31} - 1 \\ g_6 = \sigma_{31}(\sigma_{31} - 1) \end{array} \right.$$

The procedure of variables' elimination starts. The set $\mathcal{G}_{31} = \mathcal{G} \cap \mathbb{K}[\sigma_{31}] = \{g_6\}$ is a Gröbner basis of the last elimination ideal $\mathcal{I}_{31} = \mathcal{I} \cap \mathbb{K}[\sigma_{31}]$. The unique solutions of the equation $g_6 = 0$ are 0 and 1, which are used to extend to all complete solutions of the polynomial system. At the end, Algorithm **SolutionSet** returns the following 8 solutions

$$\begin{array}{ll} \xi_1 = (0, 1, 0, 1, 0, 1) & \xi_5 = (1, 0, 0, 1, 0, 1) \\ \xi_2 = (0, 1, 0, 1, 1, 0) & \xi_6 = (1, 0, 0, 1, 1, 0) \\ \xi_3 = (0, 1, 1, 0, 0, 1) & \xi_7 = (1, 0, 1, 0, 0, 1) \\ \xi_4 = (0, 1, 1, 0, 1, 0) & \xi_8 = (1, 0, 1, 0, 1, 0). \end{array}$$

Control is then passed back to Algorithm **AllBerge** which checks whether the condition of Proposition 2.4 item (ii) is verified. Since none of them meet it, Algorithm **AllBerge** terminates with the conclusion that the game has no Berge equilibria.

5.2 Pseudocode for computing pure Berge equilibria

Based on Corollary 2.6, we introduce Algorithm **PureBerge** that, starting from any game with N -players, sets of pure strategies S_1, \dots, S_N and payoff functions u_1, \dots, u_N , finds all pure strategies Berge equilibria as follows. Similarly to Corley and Kwain (2016) and Nessaha, Larbani, and Tazdaitb (2007), we consider a transformation of all the player's payoffs. For each pure strategy profile $s \in S$ and each player $i \in I$

the difference between the actual payoff $u_i(s)$ and the best payoff that the player i could get by choosing among all $-i$ coalitions is computed. If such values are positive for each $i \in I$, then the pure strategy s , which is a Berge equilibrium by Corollary 2.6, is appended to the variable PBE, that is returned by Algorithm **PureBerge** once all pure strategies have been analyzed.

Algorithm 3 - PureBerge: Computation of Pure Berge Equilibria

Require: Game defined by:

- set of players $I = \{1, \dots, N\}$
- sets of pure strategies $S_1 = \{s_{10}, \dots, s_{1d_1}\}, \dots, S_N = \{s_{N0}, \dots, s_{Nd_N}\}$
- payoff functions u_1, \dots, u_N

Ensure: Pure Berge Equilibria PBE

- 1: $PBE = \emptyset$ {Initialize the set of Pure Berge Equilibria}
 - 2: $[d_i] = \{0, 1, \dots, d_i\}$ for each $i \in I$
 - 3: **for all** $\mathbf{l} = (l_1, \dots, l_N) \in \prod_{i=1}^N [d_i]$ **do**
 - 4: $s = (s_{l_1}, \dots, s_{l_N}) \in S$
 - 5: $\text{zero}(N) = (0 \mid i \in I)$
 - 6: $z = \text{zero}(N)$
 - 7: **for** $i = 1, \dots, N$ **do**
 - 8: $z_i = \min_{j \in D_{-i} \setminus \{i\}} (u_i(s) - u_i(s_{il_i}, s_{-i}^j))$
 - 9: **end for**
 - 10: **if** $z \geq \text{zero}(N)$ componentwise **then**
 - 11: append s to PBE
 - 12: **end if**
 - 13: **end for**
 - 14: **return** PBE {Return the set of all Pure Berge Equilibria}
-

5.3 Pseudocode for computing Completely Mixed Berge equilibria

The computation of all completely mixed Berge equilibria relies upon Proposition 2.8 and is implemented in Algorithm **CompletelyMixedBerge**, whose input is any N -player regular game with sets of pure strategies S_1, \dots, S_N and payoff functions u_1, \dots, u_N . The algorithm's main **for** loop is indexed by the integer i , ranging from 1 to N , representing the considered player. At each iteration, D_{-i} is the number of $-i$ coalitions (as in our general setting), whereas $A_i \in \mathbb{R}^{|D_{-i}| \times (d_i+1)}$ and $b_i \in \mathbb{R}^{|D_{-i}| \times 1}$ are defined as in (13) and (14) respectively. The linear system $A_i \sigma_i = b_i$ is solved and the (componentwise) strictly positive solution is denoted by S_i . By Proposition 2.8, in any completely mixed Berge equilibrium, player i plays a strategy of the set S_i . At the end of the **for** loop, if all S_i are not empty, (S_1, \dots, S_N) is the unique completely mixed Berge equilibrium of the game (see also Proposition 4.6), which is returned by the algorithm.

Algorithm 4 - CompletelyMixedBerge: Computation of Completely Mixed Berge Equilibria

Require: Regular game defined by:

- set of players $I = \{1, \dots, N\}$
- sets of pure strategies $S_1 = \{s_{10}, \dots, s_{1d_1}\}, \dots, S_N = \{s_{N0}, \dots, s_{Nd_N}\}$
- payoff functions u_1, \dots, u_N

Ensure: Completely Mixed Berge Equilibrium CMBE

- 1: $CMBE = \emptyset$ {Initialize the set of Completely Mixed Berge Equilibria}
- 2: $\sigma_i = (\sigma_{i0}, \dots, \sigma_{id_i})$ for each $i \in I$ {Construct the set of variables}
- 3: $[d_i] = \{0, 1, \dots, d_i\}$ for each $i \in I$
- 4: $D_{-i} = \prod_{k=1, k \neq i}^N [d_k]$ for each $i \in I$
- 5: **for** $i = 1, \dots, N$ **do**
- 6: construct $A_i \in \mathbb{R}^{|D_{-i}| \times (d_i+1)}$ as defined in (13)
- 7: construct $b_i \in \mathbb{R}^{|D_{-i}|}$ as defined in (14)
- 8: let S_i be the (componentwise) strictly positive solution of $A_i \sigma_i = b_i$


```

9: end for
10: if  $S_i = \emptyset$  for some  $i \in I$  then
11:   print "The game does not admit completely Berge equilibria"
12: else
13:   CMBE =  $\{(S_1, \dots, S_N)\}$ 
14: end if
15: return CMBE {Return the Completely Mixed Berge Equilibrium}

```

Example 5.3 Consider the following regular three player game with $d_1 = d_2 = d_3 = 1$ and where player 1 chooses the row, player 2 the column, and player 3 the matrix.

$$C_1 : \begin{pmatrix} (0, 2, 7) & (10, 8, 3) \\ (10, 8, 3) & (0, 2, 7) \end{pmatrix} \quad C_2 : \begin{pmatrix} (10, 8, 3) & (0, 2, 7) \\ (0, 2, 7) & (10, 8, 3) \end{pmatrix}.$$

Intuitively, in a completely mixed Berge equilibrium, each player has to make the other players indifferent in their concern about her payoff. We apply Algorithm `CompletelyMixedBerge` to this game. Algorithm `CompletelyMixedBerge` constructs the three linear systems:

$$A_1 \sigma_1 = b_1, \quad A_2 \sigma_2 = b_2 \quad \text{and} \quad A_3 \sigma_3 = b_3,$$

where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 10 & -10 \\ 10 & -10 \\ 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 1 \\ 6 & -6 \\ 6 & -6 \\ 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 1 \\ -4 & 4 \\ -4 & 4 \\ 0 & 0 \end{pmatrix}$$

and $b_1 = b_2 = b_3 = (1, 0, 0, 0)^t$. Then, for each $i = 1, \dots, 3$, the set S_i of (componentwise) strictly positive solutions of $A_i \sigma_i = b_i$ is computed:

$$S_1 = S_2 = S_3 = \left\{ \left(\frac{1}{2}, \frac{1}{2} \right) \right\}.$$

The strategy $\left(\left(\frac{1}{2}, \frac{1}{2} \right), \left(\frac{1}{2}, \frac{1}{2} \right), \left(\frac{1}{2}, \frac{1}{2} \right) \right)$ is returned as the unique completely mixed Berge equilibrium of the game.

6 Games Admitting Completely Mixed Berge Equilibria

As discussed in Remark 2.3, the reader may already anticipate that Berge equilibria do not generally exist, since their characterization involves solving an overdetermined system of equations. This intuition is confirmed in the analysis that follows. We proceed to identify and characterize the class of games that can admit completely mixed Berge equilibria. Our results show that, for games with more than two players, this class forms a subset of significantly lower dimension within the space of all games. Specifically, we describe the determinantal variety that encompasses all such games, providing a precise geometric understanding of the structural constraints underlying the existence of completely mixed Berge equilibria.

For simplicity, we consider the case of games with N players, each having the same number $d+1 \geq 2$ of pure strategies. We write $[d] = \{0, 1, \dots, d\}$. The finite set of pure strategies of the i th player is $S_i = \{s_{i0}, \dots, s_{id}\}$ and the set of strategy profiles is $S = \prod_{i \in I} S_i$ with cardinality $D_S := |S| = (d+1)^N$. In this setting $D = Nd$, $D_{-i} = [d]^{N-1}$ and $|D_{-i}| = (d+1)^{N-1}$, for each $i \in I$. For each player i , we consider $S_{-i} = \prod_{j \in I, j \neq i} S_j$, that is the set of pure strategies of the $(-i)$ -coalition. Using the notation of Section 4, we recall that each game in $\mathcal{G}(I, (S_i)_{i \in I})$ is identified by its payoff vector $\bar{\mathbf{u}} \in \mathbb{R}^{N(d+1)^N}$ and that $\mathbb{R}^{N(d+1)^N}$ represents the space of all games in $\mathcal{G}(I, (S_i)_{i \in I})$.

Let $i \in I$. Recalling the notation of Proposition 2.8, we order the elements \mathbf{j}_l of the set D_{-i} assuming that the first element is $\mathbf{j}_1 = \mathbf{0}$. It is convenient to reorganize the payoff functions of the the game as follows. For each $i \in I$, we let

$$\xi^i := (\xi_1^i, \dots, \xi_{d+1}^i) \in \mathbb{R}^{(d+1)^N - (d+1)},$$

where

$$\xi_m^i := (\xi_{2,m}^i, \dots, \xi_{(d+1)^{N-1},m}^i) \in \mathbb{R}^{(d+1)^{N-1}-1} \quad \forall m \in \{1, \dots, d+1\}$$

and

$$\xi_{l,m}^i := u_i(s_{i(m-1)}, s_{-i}^{j_l}) - u_i(s_{i(m-1)}, s_{-i}^0) \in \mathbb{R} \quad \forall l \in \{2, \dots, (d+1)^{N-1}\}, \forall m \in \{1, \dots, d+1\}.$$

The space of coordinates of the i th player is denoted by $\mathcal{Z}_i := \mathbb{R}^{(d+1)^N - (d+1)}$ and the space of payoff functions of all players is $\mathcal{Z} := \prod_{i=1}^N \mathcal{Z}_i$. In such a space, we aim to characterize the payoff functions such that the associated game admits a completely mixed Berge equilibrium. It is trivial to note that, having imposed a fixed ordering on D_{-i} , each game is uniquely represented as the point

$$\xi := (\xi^1, \dots, \xi^N) \in \mathcal{Z}. \quad (17)$$

Further, we introduce the set of variables for player i :

$$\mathbf{z}^i := (\mathbf{z}_1^i, \dots, \mathbf{z}_{d+1}^i),$$

where

$$\mathbf{z}_m^i := (z_{2,m}^i, \dots, z_{(d+1)^{N-1},m}^i) \quad \forall m \in \{1, \dots, d+1\},$$

and

$$z_{lm}^i := u_i(s_{i(m-1)}, s_{-i}^{j_l}) - u_i(s_{i(m-1)}, s_{-i}^0) \quad \forall l \in \{2, \dots, (d+1)^{N-1}\}, \forall m \in \{1, \dots, d+1\}. \quad (18)$$

Let $P^i := \mathbb{R}[\mathbf{z}^i]$ be the polynomial ring in the variables \mathbf{z}^i with real coefficients. The following result holds.

Theorem 6.1 (The class of games with completely mixed Berge equilibria)

(i) *The affine variety*

$$V := \prod_{i=1}^N V_i \subseteq \mathcal{Z},$$

where

$$V_i := V(I_i) \subseteq \mathcal{Z}_i$$

and $I_i := \langle G_i \rangle$ is the ideal generated by the polynomials set

$$G_i := \{f_{\mathbf{l}}^i \mid \mathbf{l} = (l_2, \dots, l_{d+2}) \text{ and } 2 \leq l_2 < \dots < l_{d+2} \leq (d+1)^{N-1}\} \quad (19)$$

with

$$f_{\mathbf{l}}^i := \det \begin{pmatrix} z_{l_2 1}^i & \dots & z_{l_2(d+1)}^i \\ \vdots & & \vdots \\ z_{l_{d+2} 1}^i & \dots & z_{l_{d+2}(d+1)}^i \end{pmatrix} \in P^i, \quad (20)$$

contains the set of all games that admit completely mixed Berge equilibria.

(ii) *The payoff functions in $\mathcal{Z} \setminus V$ give rise to games that do not have completely mixed Berge equilibria.*

(iii) *The dimension of the set of games that admit completely mixed Berge equilibria is at most $Nd(d+1)^{N-1}$.*

(iv) For each $i \in I$ we consider the affine variety

$$W_i := V(J_i) \subseteq \mathcal{Z}_i \quad (21)$$

where

$$J_i := \langle \{g_l^i \mid l = (l_2, \dots, l_{d+1}) \text{ and } 2 \leq l_2 < \dots < l_{d+1} \leq (d+1)^{N-1}\} \rangle$$

and

$$g_l^i := \det \begin{pmatrix} 1 & \dots & 1 \\ z_{l_2 1}^i & \dots & z_{l_2(d+1)}^i \\ \vdots & & \vdots \\ z_{l_{d+1} 1}^i & \dots & z_{l_{d+1}(d+1)}^i \end{pmatrix} \in P^i. \quad (22)$$

It holds $W_i \subseteq V_i$ and, letting

$$\mathcal{T}_i := V_i \setminus W_i, \quad (23)$$

it follows that $V_i = W_i \cup \mathcal{T}_i$. Further, the set

$$\mathcal{T} := \prod_{i=1}^N \mathcal{T}_i \subseteq V$$

contains the set of games that admit at most one completely mixed Berge equilibrium.

(v) The payoff functions in $V \setminus \mathcal{T}$ give rise to games that have none or infinitely many completely mixed Berge equilibria. Further, if the game is regular then it has no completely mixed Berge equilibria.

(vi) In the above setting, we consider a game represented by $\xi = (\xi^1, \dots, \xi^N) \in \mathcal{Z}$ (see (17)) and assume that $\xi \in \mathcal{T}$. For each $i \in I$, let $\mathbf{l}^i = (l_2^i, \dots, l_{d+1}^i)$ be such that $2 \leq l_2^i < \dots < l_{d+1}^i \leq (d+1)^{N-1}$ and $g_{\mathbf{l}^i}^i(\xi^i) \neq 0$. Let

$$\sigma_{it} = \frac{(-1)^t h_{(\mathbf{l}^i)-t}(\xi^i)}{g_{\mathbf{l}^i}^i(\xi^i)} > 0, \quad \text{for each } t \in \{0, \dots, d\}, \text{ for each } i \in I, \quad (24)$$

where

$$h_{(\mathbf{l}^i)-t} := \det \begin{pmatrix} z_{l_2 1}^i & \dots & z_{l_2 t}^i & z_{l_2(t+2)}^i & \dots & z_{l_2(d+1)}^i \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{l_{d+1} 1}^i & \dots & z_{l_{d+1} t}^i & z_{l_{d+1}(t+2)}^i & \dots & z_{l_{d+1}(d+1)}^i \end{pmatrix}.$$

If

$$\sigma_{it} > 0 \quad \text{for each } t \in \{0, \dots, d\}, \text{ for each } i \in I, \quad (25)$$

then

$$\sigma = (\sigma_{10}, \dots, \sigma_{1d}, \dots, \sigma_{N0}, \dots, \sigma_{Nd})$$

is the unique completely Berge equilibrium of the game. Otherwise, the game admits no completely mixed Berge equilibria.

Theorem 6.1 provides a characterization of games according to their location in the space of payoff functions of all players. Recalling that each game is identified by the point $\xi = (\xi^1, \dots, \xi^N) \in \mathcal{Z}$, the main results of Theorem 6.1 can be summarized as follows: if $\xi \in \mathcal{Z} \setminus V$ then the game has no completely mixed Berge equilibria; if $\xi \in \mathcal{T}$ then the game admits at most one completely mixed Berge equilibrium (and conditions for its existence are provided by (25), which can be used to define a semi-algebraic subset of \mathcal{T}); finally, if $\xi \in V \setminus \mathcal{T}$ then the game may admit none or infinitely many completely mixed Berge

equilibria. In this last case, assuming that the game is regular, we can conclude that no completely mixed Berge equilibrium exists.

The result shows that the set of games admitting completely mixed Berge equilibria is a null set of substantially lower dimension within the class of all games. Relative to the dimension of the space of all games, we have a ratio of

$$\frac{Nd(d+1)^{N-1}}{N[(d+1)^N - (d+1)]} \approx \frac{d}{d+1}$$

for large N .

7 Conclusion

This paper presents a novel algebraic framework for analyzing Berge equilibria in finite normal-form games. By characterizing Berge equilibria as solutions to systems of polynomial equations, we bring powerful tools from computational algebra and algebraic geometry – particularly Gröbner basis techniques – into the study of strategic interactions. Our approach provides a systematic method for determining the existence of Berge equilibria and computing them explicitly, both in pure and mixed strategies.

In addition to the theoretical formulation, we have developed algorithms capable of identifying all Berge equilibria in regular games, including completely mixed equilibria. We also show that the set of games admitting such equilibria lies within a determinantal variety, and we provide an explicit computation of its dimension. This insight clarifies the structural rarity of Berge equilibria, especially in multi-player games, and contributes to a deeper understanding of their geometric properties.

Beyond these technical achievements, our work lays the foundation for a richer algebraic treatment of alternative solution concepts in game theory. One promising avenue for future research is to explore the relationship between Berge equilibria and strong Nash equilibria. Both concepts involve stability against deviations by coalitions rather than individuals, yet they rest on fundamentally different principles—benevolence in the case of Berge, and mutual self-interest in the case of strong Nash. Investigating formal connections or identifying conditions under which the two coincide could deepen our understanding of strategic cooperation and coalition-proof stability.

Moreover, the algebraic framework developed here may be extended to dynamic games, stochastic settings, or games with continuous strategy spaces, potentially linking to equilibrium refinements in broader economic and strategic contexts. In sum, this paper not only contributes a new computational and structural understanding of Berge equilibria but also opens several directions for advancing the interplay between game theory and computational algebra.

A Proofs

A.1 Proof of Proposition 2.1

Proof. We rewrite the expected payoff to player i as follows:

$$\begin{aligned}
u_i(\sigma_i, \sigma_{-i}) &= \sum_{s_1 \in S_1} \dots \sum_{s_N \in S_N} u_i(s_1, \dots, s_N) \sigma_1(s_1) \dots \sigma_N(s_N) \\
&= \sum_{s_{-i} \in S_{-i}} \sum_{s_i \in S_i} u_i(s_i, s_{-i}) \sigma_1(s_1) \dots \sigma_N(s_N) \\
&= \sum_{s_{-i} \in S_{-i}} \sigma_1(s_1) \dots \sigma_{i-1}(s_{i-1}) \sigma_{i+1}(s_{i+1}) \dots \sigma_N(s_N) \sum_{s_i \in S_i} u_i(s_i, s_{-i}) \sigma_i(s_i) \\
&= \sum_{s_{-i} \in S_{-i}} \sigma_1(s_1) \dots \sigma_{i-1}(s_{i-1}) \sigma_{i+1}(s_{i+1}) \dots \sigma_N(s_N) \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}) \sigma_i(s_{ij_i}) \\
&= \sum_{j=(j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_N) \in D_{-i}} \sigma_1(s_{1j_1}) \dots \sigma_{i-1}(s_{i-1, j_{i-1}}) \sigma_{i+1}(s_{i+1, j_{i+1}}) \dots \sigma_N(s_{Nj_N}) \cdot \\
&\quad \cdot \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^j) \sigma_i(s_{ij_i}) \\
&= \sum_{j=(j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_N) \in D_{-i}} \sigma_1 s_{1j_1} \dots \sigma_{i-1, j_{i-1}} \sigma_{i+1, j_{i+1}} \dots \sigma_N s_{Nj_N} \cdot \\
&\quad \cdot \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^j) \sigma_i(s_{ij_i}) \\
&= \sum_{j=(j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_N) \in D_{-i}} \sigma_1 s_{1j_1} \dots \sigma_{i-1, j_{i-1}} \sigma_{i+1, j_{i+1}} \dots \sigma_N s_{Nj_N} u_i^j(\sigma_i, s_{-i}^j)
\end{aligned}$$

Note that for each $j \in D_{-i}$ the function $u_i^j(\sigma_i, s_{-i}^j)$ is a polynomial with real coefficients of degree one in the variables $\sigma_{i0}, \dots, \sigma_{id_i}$.

Assume that σ is a Berge equilibrium. In particular, it holds $\sigma_{ij} \geq 0$ and $\sum_{j=0}^{d_i} \sigma_{ij} = 1$ for each $i \in I$ and for each $j = 0, \dots, d_i$. For each $i \in I$ and for each $j \in D_{-i}$ let $v_i^j := u_i(\sigma) - u_i^j(\sigma_i, s_{-i}^j)$; then $v_i^j \geq 0$. The quantities $v_i^j + u_i^j(\sigma_i, s_{-i}^j)$ are all equal to $u_i(\sigma)$ for each $j \in D_{-i}$ and so they are all equal to each other. Recall that $\sigma_{-i}^j = \sigma_1 s_{1j_1} \dots \sigma_{i-1, j_{i-1}} \sigma_{i+1, j_{i+1}} \dots \sigma_N s_{Nj_N}$; we consider the quantity $\sigma_{-i}^j v_i^j$

$$\begin{aligned}
\sigma_{-i}^j v_i^j &= \sigma_{-i}^j (u_i(\sigma) - u_i^j(\sigma_i, s_{-i}^j)) - u_i(\sigma) + \sum_{k \in D_{-i}} \sigma_{-i}^k u_i^k(\sigma_i, s_{-i}^k) \\
&= \left(\sum_{k \in D_{-i}, k \neq j} \sigma_{-i}^k u_i^k(\sigma_i, s_{-i}^k) \right) - (1 - \sigma_{-i}^j) u_i(\sigma) \\
&= \left(\sum_{k \in D_{-i}, k \neq j} \sigma_{-i}^k u_i^k(\sigma_i, s_{-i}^k) \right) - \\
&\quad - \left(\sum_{k_1=0}^{d_1} \sigma_1 s_{1k_1} \dots \sum_{k_{i-1}=0}^{d_{i-1}} \sigma_{i-1, k_{i-1}} \sum_{k_{i+1}=0}^{d_{i+1}} \sigma_{i+1, k_{i+1}} \dots \sum_{k_N=0}^{d_N} \sigma_N s_{Nk_N} - \sigma_{-i}^j \right) u_i(\sigma) \\
&= \sum_{k \in D_{-i}, k \neq j} \sigma_{-i}^k (u_i^k(\sigma_i, s_{-i}^k) - u_i(\sigma)) \leq 0.
\end{aligned}$$

Since $\sigma_{-i}^j v_i^j \geq 0$ then it follows that $\sigma_{-i}^j v_i^j = 0$ for each $i \in I$ and for each $j \in D_{-i}$.

Viceversa: suppose that σ_{ij} and v_i^j are nonnegative and satisfy (2). Let σ be the strategy profile defined by $\sigma(s_{ij}) = \sigma_{ij}$. Fix a player i and consider, for each $k \neq i$, mixed strategies σ'_k . Recall that $\sum_{j_k=0}^{d_k} \sigma_{kj_k} = \sum_{j_k=0}^{d_k} \sigma'_{kj_k} = 1$ for each k ; then

$$\begin{aligned} 1 &= \prod_{k \neq i} \left(\sum_{j_k=0}^{d_k} \sigma_{kj_k} \right) = \sum_{j \in D_{-i}} \sigma_{1j_1} \cdots \sigma_{i-1,j_{i-1}} \cdot \sigma_{i+1,j_{i+1}} \cdots \sigma_{Nj_N} = \sum_{j \in D_{-i}} \sigma_{-i}^j \quad (26) \\ 1 &= \prod_{k \neq i} \left(\sum_{j_k=0}^{d_k} \sigma'_{kj_k} \right) = \sum_{j \in D_{-i}} \sigma'_{1j_1} \cdots \sigma'_{i-1,j_{i-1}} \cdot \sigma'_{i+1,j_{i+1}} \cdots \sigma'_{Nj_N} = \sum_{j \in D_{-i}} (\sigma')_{-i}^j. \end{aligned}$$

Then

$$\sum_{j \in D_{-i}} \left((\sigma')_{-i}^j - \sigma_{-i}^j \right) = 0,$$

hence there exists some j such that $(\sigma')_{-i}^j - \sigma_{-i}^j \geq 0$. W.l.o.g. we suppose $(\sigma')_{-i}^0 - \sigma_{-i}^0 \geq 0$. We consider

$$\begin{aligned} u_i(\sigma) - u_i(\sigma_i, \sigma'_{-i}) &= \sum_{j \in D_{-i}} (\sigma_{-i}^j - (\sigma')_{-i}^j) u_i^j(\sigma_i, s_{-i}^j) \\ &= \sum_{j \in D_{-i}} (\sigma_{-i}^j - (\sigma')_{-i}^j) u_i^j(\sigma_i, s_{-i}^j) - u_i^0(\sigma_i, s_{-i}^0) \sum_{j \in D_{-i}} (\sigma_{-i}^j - (\sigma')_{-i}^j) \\ &= \sum_{j \in D_{-i}} (\sigma_{-i}^j - (\sigma')_{-i}^j) \left(u_i^j(\sigma_i, s_{-i}^j) - u_i^0(\sigma_i, s_{-i}^0) \right) \\ &= \sum_{j \in D_{-i}, j \neq 0} (\sigma_{-i}^j - (\sigma')_{-i}^j) \left(v_i^0 - v_i^j \right) \\ &= \sum_{j \in D_{-i}, j \neq 0} (\sigma_{-i}^j - (\sigma')_{-i}^j) v_i^0 - \sum_{j \in D_{-i}, j \neq 0} \sigma_{-i}^j v_i^j + \sum_{j \in D_{-i}, j \neq 0} (\sigma')_{-i}^j v_i^j \\ &= ((\sigma')_{-i}^0 - (\sigma)_{-i}^0) v_i^0 + \sum_{j \in D_{-i}, j \neq 0} (\sigma')_{-i}^j v_i^j \geq 0, \end{aligned}$$

therefore σ is a Berge equilibrium.

A.2 Proof of Proposition 2.4

Proof. From System (2) it holds

$$v_i^j = v_i^0 + \left(u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j) \right) \text{ for each } i \in I \text{ and } j \in D_{-i} \setminus \{0\}, \quad (27)$$

where $u_i^j(\sigma_i, s_{-i}^j)$, defined in (1), is a linear function in the variables $\sigma_{i0}, \dots, \sigma_{id_i}$, for each $i \in I$ and $j \in D_{-i}$. Further, again from system (2), we have $\sigma_{-i}^j v_i^j = 0$ for each $i \in I$ and $j \in D_{-i}$. Consequently, for each $i \in I$, it follows that

$$0 = \sum_{j \in D_{-i}} \sigma_{-i}^j v_i^j. \quad (28)$$

Using (27) in (28) we get

$$\begin{aligned}
0 &= \sum_{j \in D_{-i}} \sigma_{-i}^j v_i^j \\
&= \sigma_{-i}^0 v_i^0 + \sum_{j \in D_{-i}, j \neq 0} \sigma_{-i}^j \left[v_i^0 + \left(u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j) \right) \right] \\
&= \left(\sum_{j \in D_{-i}} \sigma_{-i}^j \right) v_i^0 + \sum_{j \in D_{-i}, j \neq 0} \sigma_{-i}^j \left(u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j) \right)
\end{aligned} \tag{29}$$

Since

$$\sum_{j \in D_{-i}} \sigma_{-i}^j = \prod_{k=1, k \neq i}^N \left(\sum_{j=1}^{d_k} \sigma_{kj} \right) = 1,$$

then (29) becomes

$$0 = v_i^0 + \sum_{j \in D_{-i}, j \neq 0} \sigma_{-i}^j (u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j)),$$

from which, using (8) and (27), it follows

$$\begin{aligned}
v_i^0 &= p_i^0(\sigma) & \text{for each } i \in I \\
v_i^j &= p_i^j(\sigma) & \text{for each } i \in I \text{ and } j \in D_{-i} \setminus \{0\}.
\end{aligned}$$

Consequently, each v_i^j is a polynomial function of degree N of σ , and solving system (2) reduces to solve (7). Thus, we conclude that σ is a Berge equilibrium if and only if conditions (i) and (ii) simultaneously hold.

A.3 Proof of Corollary 2.6

Proof. Let $\mathbf{l} = (l_1, \dots, l_N)$; the pure strategy $s = (s_{1l_1}, \dots, s_{Nl_N}) \in S$ is represented by the strategy profile

$$\xi(\mathbf{l}) = (\xi(\mathbf{l})_{10}, \dots, \xi(\mathbf{l})_{1d_1}, \dots, \xi(\mathbf{l})_{N0}, \dots, \xi(\mathbf{l})_{Nd_N}) \in \mathbb{R}^{D+N}$$

such that

$$\xi(\mathbf{l})_{ij} = \begin{cases} 1 & \text{if } j = l_i \\ 0 & \text{if } j \neq l_i \end{cases} \quad \text{for each } i \in I.$$

Since

$$\xi(\mathbf{l})_{-i}^j = \begin{cases} 0 & \text{if } j \neq l^{-i} \\ 1 & \text{if } j = l^{-i} \end{cases} \quad \text{for each } i \in I \text{ and } j \in D_{-i}, \tag{30}$$

it follows that

$$\xi(\mathbf{l})_{-i}^j p_i^j(\xi(\mathbf{l})) = 0 \quad \text{for each } i \in I \text{ and } j \in D_{-i} \setminus \{l^{-i}\} \tag{31}$$

and

$$\xi(\mathbf{l})_{-i}^{l^{-i}} p_i^{l^{-i}}(\xi(\mathbf{l})) = p_i^{l^{-i}}(\xi(\mathbf{l})). \tag{32}$$

If $l^{-i} = 0$, from (8) and (30) it follows

$$p_i^{l^{-i}}(\xi(\mathbf{l})) = p_i^0(\xi(\mathbf{l})) = \sum_{j \in D_{-i}, j \neq 0} \sigma_{-i}^j \left(u_i^j(\xi(\mathbf{l})_i, s_{-i}^j) - u_i^0(\xi(\mathbf{l})_i, s_{-i}^0) \right) = 0; \tag{33}$$

otherwise, if $\mathbf{l}^{-i} \neq \mathbf{0}$, using again (8) and (30) we have

$$\begin{aligned}
p_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})) &= p_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})) + \left(u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) - u_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{l}^{-i}}) \right) \\
&= \sum_{\mathbf{j} \in D_{-i}, \mathbf{j} \neq \mathbf{0}} \boldsymbol{\xi}(\mathbf{l})_{-i}^{\mathbf{j}} \left(u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) - u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) \right) \\
&\quad + \left(u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) - u_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{l}^{-i}}) \right) \\
&= \sum_{\mathbf{j} \in D_{-i}, \mathbf{j} \neq \mathbf{0}, \mathbf{j} \neq \mathbf{l}^{-i}} \boldsymbol{\xi}(\mathbf{l})_{-i}^{\mathbf{j}} \left(u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) - u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) \right) \\
&\quad + (1 - \boldsymbol{\xi}(\mathbf{l})_{-i}^{\mathbf{l}^{-i}}) \left(u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) - u_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{l}^{-i}}) \right) \\
&= 0.
\end{aligned} \tag{34}$$

By combining (32), (33) and (34) it follows that

$$\boldsymbol{\xi}(\mathbf{l})_{-i}^{\mathbf{l}^{-i}} p_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})) = 0. \tag{35}$$

Finally, from (31) and (35), it follows that $\boldsymbol{\xi}(\mathbf{l})$ is a positive solution of the polynomial system (7), so that condition (i) of Proposition 2.4 is satisfied.

Regarding condition (ii) of Proposition 2.4, we first observe that $p_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})) = 0$ for each $i \in I$ (see (33) and (34)). We evaluate $p_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l}))$ for each $i \in I$ and $\mathbf{j} \in D_{-i} \setminus \{\mathbf{l}^{-i}\}$. If $\mathbf{l}^{-i} = \mathbf{0}$ then

$$\begin{aligned}
p_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})) &= p_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})) + \left(u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) - u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) \right) \\
&= u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) - u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) \\
&= \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^{\mathbf{0}}) \boldsymbol{\xi}(\mathbf{l})_{ij_i} - \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^{\mathbf{j}}) \boldsymbol{\xi}(\mathbf{l})_{ij_i} \\
&= u_i(s_{il_i}, s_{-i}^{\mathbf{0}}) - u_i(s_{il_i}, s_{-i}^{\mathbf{j}}) \\
&= u_i(s) - u_i(s_{il_i}, s_{-i}^{\mathbf{j}}) \quad \text{for each } i \in I \text{ and } \mathbf{j} \in D_{-i} \setminus \{\mathbf{l}^{-i}\}
\end{aligned} \tag{36}$$

If $\mathbf{l}^{-i} \neq \mathbf{0}$ then

$$\begin{aligned}
p_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})) &= \sum_{\mathbf{j} \in D_{-i}, \mathbf{j} \neq \mathbf{0}} \boldsymbol{\xi}(\mathbf{l})_{-i}^{\mathbf{j}} \left(u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) - u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) \right) \\
&= u_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{l}^{-i}}) - u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) \\
&= \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^{\mathbf{l}^{-i}}) \boldsymbol{\xi}(\mathbf{l})_{ij_i} - \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^{\mathbf{0}}) \boldsymbol{\xi}(\mathbf{l})_{ij_i} \\
&= u_i(s_{il_i}, s_{-i}^{\mathbf{l}^{-i}}) - u_i(s_{il_i}, s_{-i}^{\mathbf{0}}) \\
&= u_i(s) - u_i(s_{il_i}, s_{-i}^{\mathbf{0}})
\end{aligned}$$

and

$$\begin{aligned}
p_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})) &= p_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})) + \left(u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) - u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) \right) \\
&= u_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{l}^{-i}}) - u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) + u_i^{\mathbf{0}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{0}}) - u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) \\
&= u_i^{\mathbf{l}^{-i}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{l}^{-i}}) - u_i^{\mathbf{j}}(\boldsymbol{\xi}(\mathbf{l})_i, s_{-i}^{\mathbf{j}}) \\
&= \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^{\mathbf{l}^{-i}}) \boldsymbol{\xi}(\mathbf{l})_{ij_i} - \sum_{j_i=0}^{d_i} u_i(s_{ij_i}, s_{-i}^{\mathbf{j}}) \boldsymbol{\xi}(\mathbf{l})_{ij_i} \\
&= u_i(s_{il_i}, s_{-i}^{\mathbf{l}^{-i}}) - u_i(s_{il_i}, s_{-i}^{\mathbf{j}}) \\
&= u_i(s) - u_i(s_{il_i}, s_{-i}^{\mathbf{j}}), \quad \text{for each } i \in I \text{ and } \mathbf{j} \in D_{-i} \setminus \{\mathbf{l}^{-i}\}
\end{aligned} \tag{37}$$

As a consequence of (36) and (37), and since $p_i^{l-i}(\xi(l)) = u_i(s) - u_i(s_{il_i}, s_{-i}^{l-i}) = 0$ condition (ii) of Proposition 2.4 is satisfied if and only if condition (11) holds. Finally, applying Proposition 2.4 the result is proved.

A.4 Proof of Proposition 2.8

Proof. The strategy profile σ is completely mixed if and only if $\sigma_{-i}^j > 0$ for each $i \in I$ and $j \in D_{-i}$. Further, by applying Proposition 2.4, a necessary and sufficient condition for σ to be a completely mixed Berge equilibrium is that it is a solution of the following system

$$\begin{cases} p_i^j(\sigma) = 0 & \text{for each } i \in I \text{ and } j \in D_{-i} \\ \sum_{j=0}^{d_i} \sigma_{ij} = 1 & \text{for each } i \in I \end{cases} \quad (38)$$

By using (8), system (38) becomes

$$\begin{cases} u_i^j(\sigma_i, s_{-i}^j) - u_i^0(\sigma_i, s_{-i}^0) = 0 & \text{for each } i \in I \text{ and } j \in D_{-i} \setminus \{0\} \\ \sum_{j=0}^{d_i} \sigma_{ij} = 1 & \text{for each } i \in I, \end{cases} \quad (39)$$

and, recalling the definition of u_i^j (see (1)), system (39) is equivalent to the linear systems:

$$\begin{cases} \sum_{k=0}^{d_i} \sigma_{ik} = 1 \\ \sum_{k=0}^{d_i} \left(u_i(s_{ik}, s_{-i}^j) - u_i(s_{ik}, s_{-i}^0) \right) \sigma_{ik} = 0 & \text{for each } j \in D_{-i} \setminus \{0\}, \end{cases} \quad (40)$$

for each $i \in I$. System (40) is made up by $|D_{-i}|$ equations in $d_i + 1$ unknowns and, by exploiting the ordering on elements of D_{-i} , it can immediately be rewritten in matrix form as formula (12).

A.5 Proof of Theorem 4.5

Before providing the proof of Theorem 4.5, we state the following result, which is a generalization of the classical Implicit Function Theorem (see for instance Rudin (1964), Theorem 9.28) to the overdetermined case.

Theorem A.1 *Let k, m, n be three strictly positive integers such that $k \geq n$. Let F be a C^1 -mapping of an open set $E \subset \mathbb{R}^n \times \mathbb{R}^m$ into \mathbb{R}^k , such that $F(\mathbf{a}, \mathbf{b}) = 0$ for some point $(\mathbf{a}, \mathbf{b}) \in E$. Let $\nabla F(\mathbf{x}, \mathbf{y})$ be the Jacobian of F and $\nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{y})$ be the matrix of first order derivatives of F w.r.t. the variables \mathbf{x} . Assume that*

$$\text{rank}(\nabla_{\mathbf{x}} F(\mathbf{a}, \mathbf{b})) = n \quad (41)$$

and that there exists a neighborhood $W \subseteq E$ of (\mathbf{a}, \mathbf{b}) such that

$$\text{rank}(\nabla F(\mathbf{x}, \mathbf{y})) = n \quad \forall (\mathbf{x}, \mathbf{y}) \in W. \quad (42)$$

Then there exist an open set $\overline{W} \subset \mathbb{R}^n \times \mathbb{R}^m$, with $(\mathbf{a}, \mathbf{b}) \in \overline{W}$, and a neighborhood $\overline{V} \subset \mathbb{R}^m$ of \mathbf{b} , having the following properties:

(i) *to every $\mathbf{y} \in \overline{V}$ corresponds a unique \mathbf{x} such that*

$$(\mathbf{x}, \mathbf{y}) \in \overline{W} \quad \text{and} \quad F(\mathbf{x}, \mathbf{y}) = 0;$$

(ii) *$\mathbf{x} = G(\mathbf{y})$, where G is a C^1 -mapping from \overline{V} into \mathbb{R}^n , $G(\mathbf{b}) = \mathbf{a}$ and*

$$F(G(\mathbf{y}), \mathbf{y}) = 0 \quad \forall \mathbf{y} \in \overline{V}.$$

Proof. From assumption (41) it follows that there exists a neighborhood $W_1 \subseteq E$ of (\mathbf{a}, \mathbf{b}) such that the matrix $\nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{y})$ contains n linearly independent rows. W.l.o.g. we assume that the first n rows of $\nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{y})$ are linearly independent for each $(\mathbf{x}, \mathbf{y}) \in W_1$. Then, by assumption (42), for each $(\mathbf{x}, \mathbf{y}) \in W_2 = W_1 \cap W$, the last $k - n$ rows of $\nabla F(\mathbf{x}, \mathbf{y})$ are linear combination of the first n rows of $\nabla F(\mathbf{x}, \mathbf{y})$. Let $\pi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be the projection map on the first n components and let $f : E \rightarrow \mathbb{R}^n$ be defined by $f = \pi \circ F$. Then, f is a \mathcal{C}^1 -mapping from E to \mathbb{R}^n , $f(\mathbf{a}, \mathbf{b}) = \pi(F(\mathbf{a}, \mathbf{b})) = \pi(0) = 0$ and $\nabla f_{\mathbf{x}}(\mathbf{a}, \mathbf{b})$, being the submatrix of $\nabla_{\mathbf{x}} F(\mathbf{a}, \mathbf{b})$ made up of its first n rows, is invertible. By applying the Implicit Function Theorem to f at the point (\mathbf{a}, \mathbf{b}) it follows that there exist open sets $W_3 \subset \mathbb{R}^n \times \mathbb{R}^m$ and $V \subset \mathbb{R}^m$, with $(\mathbf{a}, \mathbf{b}) \in W_3$ and $\mathbf{b} \in V$, such that to every $\mathbf{y} \in V$ corresponds a unique \mathbf{x} such that $(\mathbf{x}, \mathbf{y}) \in U$ and $F_1(\mathbf{x}, \mathbf{y}) = \dots = F_n(\mathbf{x}, \mathbf{y}) = 0$. Further, $\mathbf{x} = g(\mathbf{y})$, with g is a \mathcal{C}^1 -mapping from V into \mathbb{R}^n , $g(\mathbf{b}) = \mathbf{a}$ and

$$F_1(g(\mathbf{y}), \mathbf{y}) = \dots = F_n(g(\mathbf{y}), \mathbf{y}) = 0 \quad \forall \mathbf{y} \in V. \quad (43)$$

Let $\Gamma : V \rightarrow W_3$ be defined by $\Gamma : \mathbf{y} \mapsto (g(\mathbf{y}), \mathbf{y})$; then, Γ is continuous. Let $\overline{W} = W_2 \cap W_3$; notice that $(\mathbf{a}, \mathbf{b}) \in \overline{W}$. Let \overline{V} be a neighborhood of \mathbf{b} such that $\overline{V} \subseteq \Gamma^{-1}(\overline{W})$ and let $G = g|_{\overline{V}}$. Notice that $\text{Graph}(g|_{\overline{V}}) \subseteq \overline{W}$. We aim to show that $F_{n+1}(G(\mathbf{y}), \mathbf{y}) = \dots = F_k(G(\mathbf{y}), \mathbf{y}) = 0$ for each $\mathbf{y} \in \overline{V}$. Condition (43) yields

$$\nabla F_i(G(\mathbf{y}), \mathbf{y}) \begin{pmatrix} \nabla G(\mathbf{y}) \\ I_m \end{pmatrix} = 0 \quad \forall \mathbf{y} \in \overline{V}, \forall i \in \{1, \dots, n\}. \quad (44)$$

We consider F_{n+1} . By contradiction, we assume that there exists $\overline{\mathbf{y}} \in \overline{V}$ such that $F_{n+1}(G(\overline{\mathbf{y}}), \overline{\mathbf{y}}) \neq 0$. Since \overline{V} is convex, by applying the Mean Value Theorem and recalling that $F_{n+1}(G(\mathbf{b}), \mathbf{b}) = F_{n+1}(\mathbf{a}, \mathbf{b}) = 0$ we have:

$$\begin{aligned} 0 &\neq F_{n+1}(G(\overline{\mathbf{y}}), \overline{\mathbf{y}}) \\ &= F_{n+1}(G(\overline{\mathbf{y}}), \overline{\mathbf{y}}) - F_{n+1}(G(\mathbf{b}), \mathbf{b}) \\ &= \left[\nabla F_{n+1}(G(\mathbf{y}_t), \mathbf{y}_t) \begin{pmatrix} \nabla G(\mathbf{y}_t) \\ I_m \end{pmatrix} \right] (\overline{\mathbf{y}} - \mathbf{b}) \end{aligned} \quad (45)$$

where $\mathbf{y}_t := t\overline{\mathbf{y}} + (1-t)\mathbf{b} \in \overline{V}$ for a suitable $t \in (0, 1)$. Recalling that in W_2 the last $k - n$ rows of $\nabla F(\mathbf{x}, \mathbf{y})$ are a linear combination of its first n rows, we write

$$\nabla F_{n+1}(G(\mathbf{y}_t), \mathbf{y}_t) = \sum_{i=1}^n \lambda_1^i(\mathbf{y}_t) \nabla F_i(G(\mathbf{y}_t), \mathbf{y}_t) \quad (46)$$

for some scalars $\lambda_1^i(\mathbf{y}_t), \dots, \lambda_n^i(\mathbf{y}_t)$. Plugging (44) and (46) into (45) we get $F_{n+1}(G(\overline{\mathbf{y}}), \overline{\mathbf{y}}) = 0$, that is a contradiction. Repeating the same argument for the cases F_{n+2}, \dots, F_k the theorem is proved.

We are now ready to give the proof of Theorem 4.5.

Proof.

- (i) Let $\overline{\Gamma} = (I, (S_i)_{i \in I}, \overline{\mathbf{u}})$ be a regular game with payoff functions $\overline{\mathbf{u}} = (\overline{u}_N, \dots, \overline{u}_N)$. Let $\overline{\sigma}$ be a completely Berge equilibrium of $\overline{\Gamma}$. From Proposition 2.4 and Remark 2.5, it holds $F(\overline{\sigma}, \overline{\mathbf{u}}) = 0$, where F is the \mathcal{C}^1 -mapping defined in (15). Further, since $\overline{\sigma}$ is completely mixed, then $p_i^j(\overline{\sigma}, \overline{\mathbf{u}}) := p_i^j = 0$ for each $i \in I$ and $j \in D_{-i}$, where p_i^j is defined in (8). From Remark 4.2, it follows that $\sum_{i=1}^N |D_{-i}| \geq D + N$. Further, from regularity (see Definition 4.3) it holds

$$\text{rank}(\nabla_{\sigma} F(\overline{\sigma}, \overline{\mathbf{u}})) = D + N$$

and there exists a neighborhood W of $(\overline{\sigma}, \overline{\mathbf{u}})$ such that

$$\text{rank}(\nabla F(\sigma, \mathbf{u})) = D + N \quad \forall (\sigma, \mathbf{u}) \in W.$$

Applying Theorem A.1 to F at the point $(\bar{\sigma}, \bar{u})$, it follows that there exist an open set $\bar{W} \subset \mathbb{R}^{D+N} \times \mathbb{R}^{ND_s}$, with $(\bar{\sigma}, \bar{u}) \in \bar{W}$, a neighborhood $\bar{V} \subset \mathbb{R}^{ND_s}$ of \bar{u} and a C^1 -mapping G from \bar{V} into \mathbb{R}^{D+N} such that to every $u \in \bar{V}$ corresponds a unique σ such that $(\sigma, u) \in \bar{W}$ and $F(\sigma, u) = 0$. Further, $G(\bar{u}) = \bar{\sigma}$ and

$$F(G(u), u) = 0 \quad \forall u \in \bar{V}.$$

W.l.o.g. we may assume \bar{V} small enough so that $G(u) > 0$ for each $u \in \bar{V}$. Hence, $G(u) \in B(u)$ for each $u \in \bar{V}$ (note that for completely mixed strategies condition (ii) of Proposition 2.4 is automatically satisfied). Let $U_{\bar{\sigma}} := \{\sigma \mid (\sigma, u) \in \bar{W}\}$. It is simple to show that $U_{\bar{\sigma}}$ is a neighborhood of $\bar{\sigma}$ containing the unique Berge equilibrium $G(u)$ for each $u \in \bar{V}$, that is

$$|B(u) \cap U_{\bar{\sigma}}| = 1 \quad \forall u \in \bar{V},$$

so that Definition 4.1, (ii), condition (a) is proved. Further, since the mapping $u \mapsto G(u)$ is differentiable, condition (b) is also satisfied, leading to the conclusion that the equilibrium $\bar{\sigma}$ is strongly stable.

Now, we consider the case in which $\bar{\sigma}$ is a semi-mixed Berge equilibrium of Γ , that is a type of mixed strategy where some players randomize their actions while others play a pure strategy. We note that a semi-mixed equilibrium corresponds to a completely mixed equilibrium of a restricted version of the original game, where only a subset of the original pure strategies are available. The proof simply follows by repeating the above discussion in the setting of the restricted game.

(ii) It immediately follows from Definition 4.1, (i) and (ii).

A.6 Proof of Proposition 4.6

Before passing to the proof of Proposition 4.6, we present the following technical lemma which shows that, in the case of completely mixed Berge equilibria, the Jacobian matrix $\nabla_{\sigma} F(\bar{\sigma}, \bar{u})$ can be decomposed into the product of two suitable matrices, each depending only on one of the two inputs $\bar{\sigma}$ and \bar{u} . As a byproduct, the regularity of the completely mixed equilibrium $\bar{\sigma}$ reduces to a simple condition on the rank of a matrix that only depends on the payoffs of the game.

Lemma A.2 *Let $\bar{\Gamma} \in \mathcal{G}(I, (S_i)_{i \in I})$ be an N -person normal form game defined by $\bar{u} = (\bar{u}_1, \dots, \bar{u}_N) \in \mathbb{R}^{ND_s}$ and $\bar{\sigma}$ be a completely mixed Berge equilibrium of $\bar{\Gamma}$. Let*

$$\begin{aligned} A(\bar{u}) &= \begin{pmatrix} A_1(\bar{u}_1) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_2(\bar{u}_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & A_N(\bar{u}_N) \end{pmatrix} \in \mathbb{R}^{(\sum_{i=1}^N |D_{-i}|) \times (D+N)}, \\ \Sigma(\bar{\sigma}) &= \begin{pmatrix} \Sigma_1(\bar{\sigma}) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Sigma_2(\bar{\sigma}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \Sigma_N(\bar{\sigma}) \end{pmatrix} \in \mathbb{R}^{(\sum_{i=1}^N |D_{-i}|) \times (\sum_{i=1}^N |D_{-i}|)}, \\ B(\bar{\sigma}) &= \begin{pmatrix} B_1(\bar{\sigma}) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & B_2(\bar{\sigma}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & B_N(\bar{\sigma}) \end{pmatrix} \in \mathbb{R}^{(\sum_{i=1}^N |D_{-i}|) \times (\sum_{i=1}^N |D_{-i}|)} \end{aligned}$$

where $A_i(\bar{u}_i) \in \mathbb{R}^{|D_{-i}| \times (d_i+1)}$ is defined by (13),

$$\Sigma_i(\bar{\sigma}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \bar{\sigma}_{-i}^{j_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{\sigma}_{-i}^{j_{|D_{-i}|}} \end{pmatrix} \in \mathbb{R}^{|D_{-i}| \times |D_{-i}|}, \quad (47)$$

and

$$B_i(\bar{\sigma}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \bar{\sigma}_{-i}^{j_2} - 1 & \bar{\sigma}_{-i}^{j_3} & \dots & \bar{\sigma}_{-i}^{j_{|D_{-i}|}} \\ 0 & \bar{\sigma}_{-i}^{j_2} & \bar{\sigma}_{-i}^{j_3} - 1 & \dots & \bar{\sigma}_{-i}^{j_{|D_{-i}|}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \bar{\sigma}_{-i}^{j_2} & \bar{\sigma}_{-i}^{j_3} & \dots & \bar{\sigma}_{-i}^{j_{|D_{-i}|}} - 1 \end{pmatrix} \in \mathbb{R}^{|D_{-i}| \times |D_{-i}|}. \quad (48)$$

Then

$$\nabla_{\sigma} F(\bar{\sigma}, \bar{u}) = \Sigma(\bar{\sigma}) B(\bar{\sigma}) A(\bar{u}). \quad (49)$$

and

$$\text{rank}(\nabla_{\sigma} F(\bar{\sigma}, \bar{u})) = \text{rank}(A(\bar{u})).$$

Proof. From system (10), since $\bar{\sigma}_{ij} \neq 0$ for each $i \in I$ and $j \in \{0, \dots, d_i\}$, this is equivalent to $p_i^j(\bar{\sigma}, \bar{u}_i) = 0$ for each $i \in I$ and $\mathbf{j} \in D_{-i}$. Further, Proposition 2.8 yields $A_i(\bar{u}_i) \bar{\sigma}_i = b_i$ and

$$u_i^j(\bar{\sigma}_i, s_{-i}^j) - u_i^0(\bar{\sigma}_i, s_{-i}^0) = 0 \quad \text{for each } i \in I \text{ and } \mathbf{j} \in D_{-i} \setminus \{\mathbf{0}\}, \quad (50)$$

see also (39). Let $i \in I$; from (15) we compute

$$\frac{\partial F_{i0}}{\partial \sigma_{kl}}(\bar{\sigma}, \bar{u}_i) = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i, \end{cases} \quad (51)$$

for each $l \in [d_k]$. Now, we fix $\mathbf{j} \in D_{-i} \setminus \{\mathbf{0}\}$ and let $k \in I$ and $l \in [d_k]$. We consider two cases: $k \neq i$ and $k = i$. If $k \neq i$, using (1) and (8) we first compute

$$\begin{aligned} \frac{\partial p_i^j}{\partial \sigma_{kl}}(\sigma, u_i) &= \frac{\partial p_i^0}{\partial \sigma_{kl}}(\sigma, u_i) + \frac{\partial}{\partial \sigma_{kl}}(u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j)) \\ &= \sum_{\mathbf{m} \in D_{-i} \setminus \{\mathbf{0}\}} \frac{\partial}{\partial \sigma_{kl}} [\sigma_{-i}^{\mathbf{m}}(u_i^{\mathbf{m}}(\sigma_i, s_{-i}^{\mathbf{m}}) - u_i^0(\sigma_i, s_{-i}^0))] + 0 \\ &= \sum_{\mathbf{m} \in D_{-i} \setminus \{\mathbf{0}\}, \sigma_{kl} | \sigma_{-i}^{\mathbf{m}}} \frac{\sigma_{-i}^{\mathbf{m}}}{\sigma_{kl}} (u_i^{\mathbf{m}}(\sigma_i, s_{-i}^{\mathbf{m}}) - u_i^0(\sigma_i, s_{-i}^0)), \end{aligned}$$

from which, using (50), we get

$$\frac{\partial p_i^j}{\partial \sigma_{kl}}(\bar{\sigma}, \bar{u}_i) = 0.$$

Using (15), we compute

$$\frac{\partial F_{i,\mathbf{j}}}{\partial \sigma_{kl}}(\sigma, u_i) = \begin{cases} \frac{\sigma_{-i}^{\mathbf{j}}}{\sigma_{kl}} p_i^{\mathbf{j}}(\sigma, u_i) + \sigma_{-i}^{\mathbf{j}} \frac{\partial p_i^{\mathbf{j}}}{\partial \sigma_{kl}}(\sigma, u_i) & \text{if } \sigma_{kl} | \sigma_{-i}^{\mathbf{j}} \\ \sigma_{-i}^{\mathbf{j}} \frac{\partial p_i^{\mathbf{j}}}{\partial \sigma_{kl}}(\sigma, u_i) & \text{if } \sigma_{kl} \nmid \sigma_{-i}^{\mathbf{j}}, \end{cases} \quad (52)$$

from which, since $p_i^j(\bar{\sigma}, \bar{u}_i) = \frac{\partial p_i^j}{\partial \sigma_{kl}}(\bar{\sigma}, \bar{u}_i) = 0$, we get

$$\frac{\partial F_{i,j}}{\partial \sigma_{kl}}(\bar{\sigma}, \bar{u}_i) = 0. \quad (53)$$

Now, we consider $k = i$; using again (15), we have

$$\frac{\partial F_{i,j}}{\partial \sigma_{il}}(\sigma, u_i) = \sigma_{-i}^j \frac{\partial p_i^j}{\partial \sigma_{il}}(\sigma, u_i). \quad (54)$$

Using (1) and (8), we compute

$$\begin{aligned} \frac{\partial p_i^j}{\partial \sigma_{il}}(\sigma, u_i) &= \frac{\partial p_i^0}{\partial \sigma_{il}}(\sigma, u_i) + \frac{\partial}{\partial \sigma_{il}}(u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j)) \\ &= \sum_{\mathbf{m} \in D_{-i} \setminus \{0\}} \sigma_{-i}^{\mathbf{m}} \frac{\partial}{\partial \sigma_{il}}(u_i^{\mathbf{m}}(\sigma_i, s_{-i}^{\mathbf{m}}) - u_i^0(\sigma_i, s_{-i}^0)) + \frac{\partial}{\partial \sigma_{il}}(u_i^0(\sigma_i, s_{-i}^0) - u_i^j(\sigma_i, s_{-i}^j)) \\ &= \sum_{\mathbf{m} \in D_{-i} \setminus \{0\}} \sigma_{-i}^{\mathbf{m}} (u_i(s_{il}, s_{-i}^{\mathbf{m}}) - u_i(s_{il}, s_{-i}^0)) - (u_i(s_{il}, s_{-i}^j) - u_i(s_{il}, s_{-i}^0)). \end{aligned} \quad (55)$$

Coupling (55) with (13), where $A_i = A_i(\bar{u}_i)$, it follows that

$$\frac{\partial p_i^j}{\partial \sigma_{il}}(\sigma, u_i) = \sum_{\mathbf{m} \in D_{-i} \setminus \{0\}} \sigma_{-i}^{\mathbf{m}} A_i(\mathbf{m}, l) - A_i(j, l).$$

By evaluating (53) at $(\bar{\sigma}, \bar{u}_i)$ and using the former expression we obtain

$$\frac{\partial F_{i,j}}{\partial \sigma_{il}}(\bar{\sigma}, \bar{u}_i) = \bar{\sigma}_{-i}^j \sum_{\mathbf{m} \in D_{-i} \setminus \{0\}} \bar{\sigma}_{-i}^{\mathbf{m}} A_i(\mathbf{m}, l) - A_i(j, l). \quad (56)$$

From (51), (53) and (56) it follows that the Jacobian matrix $\nabla_{\sigma} F_i(\bar{\sigma}, \bar{u}_i) \in \mathbb{R}^{|D_{-i}| \times (D+N)}$ of F_i w.r.t. σ is a block matrix with the following structure

$$\nabla_{\sigma} F_i(\bar{\sigma}, \bar{u}_i) = \left(\underbrace{0 \dots 0}_{d_1+1} \mid \dots \mid \underbrace{0 \dots 0}_{d_{i-1}+1} \mid \Sigma_i(\bar{\sigma}) B_i(\bar{\sigma}) A_i(\bar{u}_i) \mid \underbrace{0 \dots 0}_{d_{i+1}+1} \mid \dots \mid \underbrace{0 \dots 0}_{d_N+1} \right), \quad (57)$$

where $\Sigma_i(\bar{\sigma}) \in \mathbb{R}^{|D_{-i}| \times |D_{-i}|}$, $B_i(\bar{\sigma}) \in \mathbb{R}^{|D_{-i}| \times |D_{-i}|}$, and $A_i(\bar{u}_i) \in \mathbb{R}^{|D_{-i}| \times (d_i+1)}$ are defined by (47), (48) and (13), respectively. From (57) and the block structure of the matrices $A(\bar{u})$, $B(\bar{\sigma})$ and $\Sigma(\bar{\sigma})$, formula (49) easily follows.

Since $\bar{\sigma}$ is completely mixed then $\det(\Sigma_i(\bar{\sigma})) \neq 0$, hence $\det(\Sigma(\bar{\sigma})) = \prod_{i \in I} \det(\Sigma_i(\bar{\sigma})) \neq 0$ and $\Sigma(\bar{\sigma})$ is invertible. In order to show the $B(\bar{\sigma})$ has full-rank, we fix $i \in I$ and consider a linear combination of the columns of the matrix $B_i(\bar{\sigma})$ with scalars $\lambda_1, \dots, \lambda_{|D_{-i}|}$. By some computations, it follows that such linear combination is trivial if and only if $\lambda_1 = 0$, $\lambda_2 = \dots = \lambda_{|D_{-i}|}$ and

$$\lambda_2 \left(1 - \sum_{k=2}^{|D_{-i}|} \bar{\sigma}_{-i}^{j_k} \right) = 0.$$

Since $\sum_{k=2}^{|D_{-i}|} \bar{\sigma}_{-i}^{j_k} < 1$, it follows that $\lambda_k = 0$ for each $k \in \{1, \dots, |D_{-i}|\}$, so $B_i(\bar{\sigma})$ is a full-rank matrix. Consequently, the matrix $B(\bar{\sigma})$ is invertible. Then, from (49) and the nonsingularity of $\Sigma(\bar{\sigma})$ and $B(\bar{\sigma})$, it follows that

$$\text{rank}(\nabla_{\sigma} F(\bar{\sigma}, \bar{u})) = \text{rank}(\Sigma(\bar{\sigma}) B(\bar{\sigma}) A(\bar{u})) = \text{rank}(A(\bar{u})),$$

which concludes the proof.

We are now ready to provide the proof of Proposition 4.6.

Proof. Let $\bar{\Gamma} \in \mathcal{G}(I, (S_i)_{i \in I})$ be an N -person normal form (Berge) regular game defined by $\bar{\mathbf{u}} = (\bar{u}_1, \dots, \bar{u}_N) \in \mathbb{R}^{ND_S}$ and let $\bar{\sigma}$ be a completely mixed Berge equilibrium of $\bar{\Gamma}$. The regularity assumption, Lemma A.2 and Remark 4.2 yield

$$\text{rank}(A_i(\bar{u}_i)) = d_i + 1,$$

that is $A_i(\bar{u}_i)$ is invertible, for each $i \in I$. By Proposition 2.8, we conclude that $\bar{\sigma}$ is the unique completely Berge equilibrium of the game.

A.7 Proof of Theorem 6.1

Proof. Using notation (18) and Proposition 2.8, the strategy $\sigma = (\sigma_{10}, \dots, \sigma_{1d}, \dots, \sigma_{N0}, \dots, \sigma_{Nd})$ is a completely mixed Berge equilibrium if and only if $\sigma_i = (\sigma_{i0}, \dots, \sigma_{id})$ is a strictly positive solution of the linear system $A_i \sigma_i = b_i$ for each $i \in I$ (see (12)), where

$$A_i := \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_{21}^i & z_{22}^i & \dots & z_{2(d+1)}^i \\ \vdots & \vdots & & \vdots \\ z_{(d+1)^{N-1}1}^i & z_{(d+1)^{N-1}2}^i & \dots & z_{(d+1)^{N-1}(d+1)}^i \end{pmatrix} \in \mathbb{R}^{(d+1)^{N-1} \times (d+1)}$$

and $b_i \in \mathbb{R}^{(d+1)^{N-1}}$ is given by (14). A necessary condition for the existence of solutions of the linear system $A_i \sigma_i = b_i$ is that the rank of the augmented matrix

$$(A_i | b_i) := \left(\begin{array}{cccc|c} 1 & 1 & \dots & 1 & 1 \\ z_{21}^i & z_{22}^i & \dots & z_{2(d+1)}^i & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ z_{(d+1)^{N-1}1}^i & z_{(d+1)^{N-1}2}^i & \dots & z_{(d+1)^{N-1}(d+1)}^i & 0 \end{array} \right) \in \mathbb{R}^{(d+1)^{N-1} \times (d+2)} \quad (58)$$

is at most $d + 1$. The set of payoff functions such that $\text{rank}((A_i | b_i)) \leq (d + 1)$ is an affine variety of \mathcal{Z}_i defined by the vanishing of all the $(d + 2) \times (d + 2)$ minors of $(A_i | b_i)$. Since the number of columns of $(A_i | b_i)$ is exactly $(d + 2)$, all such minors are given by the determinant of all the submatrices of $(A_i | b_i)$ constructed by selecting $(d + 2)$ rows, that is

$$\det((A_i | b_i)_{lm}) \mid l \in L \text{ and } 1 \leq m \leq d + 2$$

where

$$L := \{l_1, \dots, l_{d+2}\} \subset \{1, \dots, (d + 1)^{N-1}\}$$

such that $1 \leq l_1 < \dots < l_{d+2} \leq (d + 1)^{N-1}$. From (58), it is immediate to observe that if $l_1 > 1$ then the corresponding minor is equal to 0. Hence, we only consider the case $l_1 = 1$ and, by an easy computation, we find that G_i (see (19)) is the set of all nondegenerate minors of $(A_i | b_i)$ of order $(d + 2)$. Consequently, recalling that $I_i = \langle G_i \rangle$, inside the affine variety $V_i = V(I_i) \subseteq \mathcal{Z}_i$ the rank of $(A_i | b_i)$ is at most $(d + 1)$, which is a necessary condition for the linear system (12) to admit solutions. By applying Proposition 2.8 statements (i) and (ii) follow.

Now, we prove item (iii). We consider player $i \in I$. We compute the dimension of the coefficient ring P^i/I_i by applying a classical result on determinantal ideals (see Hashimoto (1994), Corollary 1.4, and Hochster and Eagon (1971)). In our setting, the parameters of Hashimoto (1994), Corollary 1.4, read $m = (d + 1)^{N-1} - 1$ and $n = t = d + 1$, hence:

$$\begin{aligned} \dim(P^i/I_i) &= ((d + 1)^{N-1} - 1)(d + 1) - ((d + 1)^{N-1} - 1 - (d + 1) + 1)((d + 1) - (d + 1) + 1) \\ &= (d + 1)^N - (d + 1) - (d + 1)^{N-1} + (d + 1) \\ &= d(d + 1)^{N-1} \end{aligned}$$

From Bruns and Vetter (2006), Theorem 2.10, it follows that I_i is prime, hence radical. Consequently, $I(V_i) = \sqrt{I_i} = I_i$ and $\dim(V_i) = \dim(P^i/I_i) = d(d+1)^{N-1}$ for each $i \in I$. Finally

$$\dim(V) = \dim\left(\prod_{i=1}^N V_i\right) = Nd(d+1)^{N-1},$$

so item (iii) follows.

We prove item (iv). A necessary condition so that the game admits a unique completely mixed Berge equilibrium is that for each $i \in I$ the linear system $A_i \sigma_i = b_i$ has unique solution. This is equivalent to select inside $V_i \subseteq \mathcal{Z}_i$ the set of payoff functions such that $\text{rank}(A_i) = d+1$. A sufficient condition is that there exists $\mathbf{l} = (l_2, \dots, l_{d+1})$, with $2 \leq l_2 < \dots < l_{d+1} \leq (d+1)^{N-1}$, such that $g_{\mathbf{l}}^i \neq 0$. Recalling the definition of W_i and \mathcal{T}_i (see (21) and (23)), for each payoff function in \mathcal{T}_i it holds that $\text{rank}((A_i | b_i)) = \text{rank}(A_i) = d+1$, hence system $A_i \sigma_i = b_i$ admits unique solution. Consequently, the payoff functions in \mathcal{T} are such that all linear systems $A_i \sigma_i = b_i$, for each $i \in I$, admit unique solution (with no information about their positiveness). By Proposition 2.8, we conclude that the associated games admit at most one completely mixed Berge equilibrium. Now, we show that $W_i \subseteq V_i$. We consider a multi-index $\mathbf{m} = (m_1, \dots, m_{d+1})$ such that $2 \leq m_1 < \dots < m_{d+1} \leq (d+1)^{N-1}$ and let

$$B_i^{\mathbf{m}} := \begin{pmatrix} 1 & \dots & 1 & 1 \\ z_{m_1 1}^i & \dots & z_{m_1(d+1)}^i & z_{m_1 1}^i \\ \vdots & & \vdots & \vdots \\ z_{m_{d+1} 1}^i & \dots & z_{m_{d+1}(d+1)}^i & z_{m_{d+1} 1}^i \end{pmatrix}.$$

By some computations, using (20) and (22), we get:

$$\begin{aligned} 0 &= \det(B_i^{\mathbf{m}}) \\ &= (-1)^{d+3} \det \begin{pmatrix} z_{m_1 1}^1 & \dots & z_{m_1(d+1)}^1 \\ \vdots & & \vdots \\ z_{m_{d+1} 1}^i & \dots & z_{m_{d+1}(d+1)}^i \end{pmatrix} \\ &\quad + \sum_{t=1}^{d+1} (-1)^{t+d+3} z_{m_t 1}^i \det \begin{pmatrix} 1 & \dots & 1 \\ z_{m_1 1}^i & \dots & z_{m_1(d+1)}^i \\ \vdots & & \vdots \\ z_{m_{t-1} 1}^i & \dots & z_{m_{t-1}(d+1)}^i \\ z_{m_{t+1} 1}^i & \dots & z_{m_{t+1}(d+1)}^i \\ \vdots & & \vdots \\ z_{m_{d+1} 1}^i & \dots & z_{m_{d+1}(d+1)}^i \end{pmatrix} \\ &= (-1)^{d+3} \left(f_{\mathbf{m}}^i + \sum_{t=1}^{d+1} (-1)^t z_{m_t 1}^i g_{\mathbf{m}-t}^i \right) \end{aligned} \tag{59}$$

where

$$\mathbf{m}^{-t} := (m_1, \dots, m_{t-1}, m_{t+1}, \dots, m_{d+1}) \quad \text{for each } t \in \{1, \dots, d+1\}.$$

From (59) it follows that $f_{\mathbf{m}}^i = \sum_{t=1}^{d+1} (-1)^{t+1} z_{m_t 1}^i g_{\mathbf{m}-t}^i$, for each $\mathbf{m} = (m_1, \dots, m_{d+1})$ such that $2 \leq m_1 < \dots < m_{d+1} \leq (d+1)^{N-1}$, hence $I_i \subseteq J_i$, which implies that $W_i \subseteq V_i$. In addition, we show that $I_i \subset J_i$. The polynomial set G_i (see (19)) is the reduced Gröbner basis, w.r.t. the standard graded lexicographic monomial ordering, of the ideal I_i of P^i (see Caniglia, Guccione, and Guccione (1990) and Sturmfels (1990)); further, G_i is made up of homogeneous polynomials of degree $d+1$. Consequently, all polynomials of I_i have degree $\geq d+1$. On the other hand, each polynomial $g_{\mathbf{l}}^i \in J_i$ has degree d , and so $g_{\mathbf{l}}^i \notin I_i$. We thus conclude that $I_i \subset J_i$.

We prove item (v). From item (iv) we recall that \mathcal{T}_i is characterized by payoff functions of player i such that $\text{rank}((A_i|b_i)) = \text{rank}(A_i) = d+1$. Consequently, the payoff functions in $V \setminus \mathcal{T}$ are such that for some $i \in I$ the corresponding linear system $A_i \sigma_i = b_i$ either admits no solution or infinitely many solutions. Hence, the associated game may exhibit none or infinitely many completely mixed Berge equilibria. In the special case of regular games, since only isolated Berge equilibria are possible, it immediately follows that no completely mixed Berge equilibria exist.

Finally, we prove item (vi). If $\xi = (\xi^1, \dots, \xi^N) \in \mathcal{T}$, then from item (iv) the linear system $A_i \sigma_i = b_i$ has unique solution. Since $g_i^i(\xi^i) \neq 0$, solution (24) immediately follows. Finally, if condition (25) holds then $\sigma_i = (\sigma_{i0}, \dots, \sigma_{id})$ is the strategy of the i -th player of a completely mixed Berge equilibrium, otherwise no completely mixed Berge equilibrium exists.

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