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Imperfect Credibility *versus* No Credibility of Optimal Monetary Policy*

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Abstract

A minimal central bank credibility, with a non-zero probability of not reneging his commitment ("quasi-commitment"), is a necessary condition for anchoring inflation expectations and stabilizing inflation dynamics. By contrast, a complete lack of credibility, with the certainty that the policy maker will renege his commitment ("optimal discretion"), leads to the local instability of inflation dynamics. In the textbook example of the new-Keynesian Phillips curve, the response of the policy instrument to inflation gaps for optimal policy under quasi-commitment has an opposite sign than in optimal discretion, which explains this bifurcation.

JEL classification numbers: C61, C62, E31, E52, E58.

Keywords: Ramsey optimal policy, Imperfect commitment, Discretion, New-Keynesian Phillips curve, Imperfect credibility.

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Titre en français: Crédibilité imparfaite ou absence de crédibilité de la politique monétaire optimale.

Résumé:

Une crédibilité minimale de la banque centrale, avec une probabilité non-nulle de ne pas revenir sur son engagement, est une condition nécessaire pour ancrer les anticipations d'inflation et garantir le retour de l'inflation vers sa cible de long terme. En revanche, l'absence complète de crédibilité, avec la certitude que la banque centrale reviendra sur son engagement ("discretion optimale"), implique une bifurcation de la dynamique de l'inflation vers des trajectoires déflationnistes ou hyper-inflationnistes. Dans l'exemple de la courbe de Phillips des nouveaux Keynésiens, la politique optimale à très faible crédibilité impose une réponse de l'instrument de politique monétaire aux écarts de l'inflation de signe opposé à la politique de discrétion optimale, ce qui explique cette bifurcation.

Mots-clés: Politique optimale à la Ramsey, discrétion, crédibilité imparfaite, engagement limité, Courbe de Phillips des nouveaux Keynésiens.

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1 INTRODUCTION

A key determinant of the efficiency of stabilization policy is the degree of the credibility of policy makers, measured by their probability of not reneging their commitment or the probability of a change of the head of the central bank. This paper shows that the equilibrium in a model of optimal policy under quasi-commitment (Schaumburg and Tambalotti [2007], Fujiwara, Kam and Sunakawa [2019]) is completely different from the discretion equilibrium with no commitment and more relevant for policy makers, even if the probability of non-reneging tends to zero (near-zero credibility). This latter discretion equilibrium, however, is presented as a relevant benchmark equilibrium in numerous papers since Clarida, Gali and Gertler [1999], for example in Gali [2015].

In the discretion equilibrium, each period-specific policy maker does a static optimization ignoring expectations, whereas the monetary policy transmission mechanism includes dynamics related to private sector expectations. Phillips, already in 1954, warned about the dramatic errors of static optimization when the underlying transmission mechanism is dynamic:

The time path of income, production and employment during the process of adjustment is not revealed. It is quite possible that certain types of policy may give rise to undesired fluctuations, or even cause a previously stable system to become unstable, although the final equilibrium position as shown by a static analysis appears to be quite satisfactory (Phillips [1954], p. 290).

In discretion, the policy maker's rule, which is optimal for a static model, leads to a sub-optimal positive feedback mechanism once expectations are taken into account. It results in a locally unstable equilibrium in the space of inflation and of the cost-push shock. Therefore, discretion equilibrium requires with an infinite precision the knowledge of the parameters of the monetary policy transmission in order to force an exact correlation between inflation and the cost-push shock. In practice, this perfect knowledge never occurs, so that the probability to shift to inflation or deflation spirals is equal to one in the discretion equilibrium.

When changing the policy from commitment or quasi-commitment to discretion the qualitative properties of the dynamic system change dramatically. The probability of reneging commitment serves as a bifurcation parameter, the bifurcation occurs when going from probability zero to any probability, even *infinitesimal* small, larger than zero. With quasi-commitment, the private sector's expectations of inflation are taken into account in the equilibrium, even with an extremely low probability of not reneging commitment. The policy maker's negative feedback rule is such that the policy instrument responds with an opposite sign to inflation as in discretion equilibrium. This ensures the locally stable dynamics of inflation. The inflation auto-correlation (or growth factor) parameter shifts from above one (discretion) to below one (quasi-commitment). Shifting from discretion equilibrium to quasi-commitment corresponds to a saddle-node bifurcation of inflation dynamics.

However, the existing literature did not mention the bifurcation when shifting from quasi-commitment to discretion equilibrium. Schaumburg and Tambalotti's ([2007], p.304) statement that "*quasi-commitment converges to full commitment for [the probability of reneging commitment tends to zero]*" is valid. But their second statement that "*it also converges to discretion when [the probability of reneging commitment tends to one]*" is not valid, as demonstrated in this paper. Schaumburg and Tambalotti's ([2007], figure

4, p.318) numerical examples suggest that their second statement is likely to be false. There is a gigantic gap between the initial jumps of inflation which is nearly the *double* for discretion (4.9) as compared to optimal policy under quasi-commitment (2.5) for the lowest numerical value of the probability of not reneging commitment that they have chosen. Because of this large gap, Schaumburg and Tambalotti [2007] conclude that "*most of gains of commitment accrue at relatively low levels of credibility*". We show that most of the large gains of commitment accrue at extremely low levels of credibility. Finally, the impulse response functions, welfare losses and initial anchors (or jumps) of inflation are much larger with discretion than with near-zero credibility.

Section 2 presents Ramsey optimal policy under imperfect commitment and section 3 discretionary policy. Section 4 demonstrates the existence of a bifurcation and calculates policy rule parameters comparing near-zero credibility *versus* discretion. Section 5 compares initial anchors of inflation on the cost-push shock, impulse response functions and welfare for near-zero credibility *versus* discretion. The last section concludes.

2 RAMSEY OPTIMAL POLICY UNDER QUASI-COMMITMENT

Following Schaumburg and Tambalotti [2007], we assume that the mandate to minimize the loss function is delegated to a sequence of policy makers (indexed by $j, k, \dots$) with a commitment of random duration. The length of their tenure depends on a sequence of exogenous independently and identically distributed Bernoulli signals $\{\eta_t\}_{t \geq 0}$ with $E_t[\eta_t]_{t \geq 0} = 1 - q$, with $0 < q \leq 1$. The case $q = 0$ of discretionary policy is treated separately in the next section. If $\eta_t = 1$, a new policy maker takes office at the beginning of time t and is not committed to the policy of his/her predecessor. Otherwise, the incumbent stays on. A higher probability of not reneging commitment q can be interpreted as a higher credibility. This leads to use a "credibility adjusted" discount factor βq in the policy maker's optimal behavior. A policy maker with little credibility does not give a large weight on future welfare losses.

At the start of his tenure, policy maker j solves the following problem, where subscript k corresponds to the new policy maker. Welfare is maximized subject to the new-Keynesian Phillips curve with slope κ and subject to the auto-correlation ρ of a cost-push shock u_t and the constant variance σ_u^2 of its identically and independently normally distributed disturbances $\eta_{u,t}$:

$$V^{jj}(u_0) = E_0 \sum_{t=0}^{t=+\infty} (\beta q)^t \left[-\frac{1}{2} \left(\pi_t^2 + \frac{\kappa}{\varepsilon} x_t^2 \right) + \beta (1 - q) V^{jk}(u_t) \right] \quad (1)$$

s.t. $\pi_t = \kappa x_t + \beta q E_t \pi_{t+1} + \beta (1 - q) E_t \pi_{t+1}^k + u_t$ (Lagrange multiplier γ_{t+1})

$u_t = \rho u_{t-1} + \eta_{u,t}, \forall t \in \mathbb{N}, u_0 \text{ given}, 0 < \rho < 1, \varepsilon > 1, 0 < \beta < 1,$

where x_t represents the welfare-relevant output gap, i.e. the deviation between (log) output and its efficient level, π_t the rate of inflation between periods $t - 1$ and t , β the discount factor, and E_t the expectation operator. The utility the central bank obtains if next period's objectives change is denoted V^{jk} . Inflation expectations are an average between two terms in the new-Keynesian Phillips curve. The first term, with weight q is the inflation that would prevail under the current regime upon which there is commitment.

The second term with weight $1 - q$ is the inflation that would be implemented under the alternative regime, which is taken as given by the current central bank. The key change with respect to the model without the possibility of a change of monetary policy is that the narrow range of values for the discount factor around 0.99 for quarterly data (4% discount rate) is much wider for the "credibility weighted discount factor" of the policy maker: $\beta q \in]0, 0.99]$, with limit numerical value $q = 10^{-7} > 0$ in this paper.

The slope of the new-Keynesian Phillips is a decreasing function of the household's elasticity of substitution between each differentiated intermediate goods ($\varepsilon > 1$):

$$\lim_{\varepsilon \rightarrow +\infty} \kappa = 0 < \kappa = \left(\sigma + \frac{\varphi + \alpha_L}{1 - \alpha_L} \right) \frac{(1 - \theta)(1 - \beta\theta)}{\theta} \frac{(1 - \alpha_L)}{(1 - \alpha_L + \alpha_L \varepsilon)} < \kappa_{\max} = \lim_{\varepsilon \rightarrow 1^+} \kappa \quad (2)$$

with $\varepsilon > 1$, $0 < \beta, \alpha_L, \theta < 1$, $\sigma > 0$, $\varphi > 0$.

$$\kappa_{\max} = \lim_{\varepsilon \rightarrow 1^+} \kappa = \left(\sigma + \frac{\varphi + \alpha_L}{1 - \alpha_L} \right) \frac{(1 - \theta)(1 - \beta\theta)}{\theta} (1 - \alpha_L).$$

Gali's ([2015], chapter 3) calibration of structural parameters is as follows: The representative household's discount factor $\beta = 0.99$ for a logarithmic utility of consumption $\sigma = 1$ and a unitary Frisch elasticity of labor supply $\varphi = 1$. The household's elasticity of substitution between each differentiated intermediate goods $\varepsilon = 6$. The production function is $Y = A_t L^{1 - \alpha_L}$ where Y is output, L is labor, A_t represents the level of technology. The measure of decreasing returns to scale of labor is $0 < \alpha_L = 1/3 < 1$. The proportion of firms who do not reset their price each period $0 < \theta = 2/3 < 1$ which corresponds to an average price duration of three quarters. For these parameters, the maximal value of the slope of the new-Keynesian Phillips curve when varying the elasticity of substitution between intermediate goods, $\kappa_{\max} = 0.34$ is obtained when the elasticity of substitution tends to one. The auto-correlation of the cost-push shock is $\rho = 0.8$.

If the policy maker is maximizing welfare (Gali [2015]), the cost of changing the policy instrument x_t is scaled by $\frac{\kappa}{\varepsilon}$ which is a decreasing function of the household's elasticity of substitution between each differentiated intermediate goods ($\varepsilon > 1$):

$$0 < \frac{\kappa}{\varepsilon} = \left(\sigma + \frac{\varphi + \alpha_L}{1 - \alpha_L} \right) \frac{(1 - \theta)(1 - \beta\theta)}{\theta} \frac{(1 - \alpha_L)}{(1 - \alpha_L + \alpha_L \varepsilon)} \frac{1}{\varepsilon} < \kappa(\varepsilon) < \kappa_{\max}.$$

With Gali's [2015] parameters, the relative weight of the variance of the policy instrument (output gap) is a very low proportion ($\frac{\kappa}{\varepsilon} = 2.125\%$) of the weight on the variance of the policy target (inflation).

Differentiating the Lagrangian with respect to the policy instrument (output gap x_t) and to the policy target (inflation π_t) yields the first order conditions for $t = 1, 2, \dots$:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \pi_t} = 0 : \pi_t + \gamma_{t+1} - \gamma_t = 0 \\ \frac{\partial \mathcal{L}}{\partial x_t} = 0 : \frac{\kappa}{\varepsilon} x_t - \kappa \gamma_{t+1} = 0 \end{cases} \Rightarrow \begin{cases} x_t = x_{t-1} - \varepsilon \pi_t \\ x_t = \varepsilon \gamma_{t+1} = \varepsilon (\gamma_t - \pi_t) \end{cases}$$

The central bank's Euler equation ($\frac{\partial \mathcal{L}}{\partial \pi_t} = 0$) links recursively the future or current value of central bank's policy instrument x_t to its current or past value x_{t-1} , because of the central bank's relative cost of changing her policy instrument is strictly positive $\alpha_x = \frac{\kappa}{\varepsilon} > 0$. This non-stationary Euler equation adds an unstable eigenvalue in the central bank's Hamiltonian system including three laws of motion of one forward-looking variable

(inflation π_t) and of two predetermined variables (u_t, x_t) or (u_t, γ_t) .

The transversality condition $\gamma_0 = 0$ minimizes the loss function with respect to inflation at the initial date:

$$\gamma_0 = 0 \Rightarrow x_{-1} = -\varepsilon\gamma_0 = 0 \text{ so that } \pi_0 = -\frac{1}{\varepsilon}x_0 \text{ or } x_0 = -\varepsilon\pi_0 .$$

It predetermines the policy instrument which allows to anchor the forward-looking policy target (inflation). The inflation Euler equation corresponding to period 0 is not an effective constraint for the central bank choosing its optimal plan in period 0. The former commitment to the value of the policy instrument of the previous period x_{-1} is not an effective constraint. The policy instrument is predetermined at the value zero $x_{-1} = 0$ at the period preceding the commitment. Combining the two first order conditions to eliminate the Lagrange multipliers yields the optimal initial anchor of forward inflation π_0 on the predetermined policy instrument x_0 .

Chatelain and Ralf's [2019] algorithm for Ramsey optimal policy with forcing variables seeks a stable subspace of dimension two in a system of three equations including the first order Euler condition on the policy instrument (or on the Lagrange multiplier on inflation). The representation of the optimal policy rule depends on current private sectors variables:

$$x_t = F_\pi \pi_t + F_u u_t. \quad (3)$$

This representation of the optimal policy rule is simpler than other observationally equivalent alternatives proposed for example by Gali [2015] or by Schaumburg and Tambalotti [2007], where the policy instrument depends on its lagged value instead of inflation. Chatelain and Ralf [2019] provide the details of the solution. The characteristic polynomial of the Hamiltonian system is:

$$\lambda^2 - \left(1 + \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q}\right) \lambda + \frac{1}{\beta q} = 0.$$

Its solution inside the unit circle is denoted "inflation eigenvalue" $\lambda(q, \varepsilon)$:

$$\lambda(q, \varepsilon) = \frac{1}{2} \left(\left(1 + \frac{1}{\beta q} + \frac{\varepsilon \kappa}{\beta q}\right) - \sqrt{\left(1 + \frac{1}{\beta q} + \frac{\varepsilon \kappa}{\beta q}\right)^2 - \frac{4}{\beta q}} \right) = \frac{1}{\beta q} - \frac{\kappa}{\beta q} F_\pi.$$

The optimal policy rule parameters are:

$$F_\pi = \left(\frac{\lambda(q, \varepsilon)}{1 - \lambda(q, \varepsilon)} \right) \varepsilon > 0 \text{ and } F_u = \frac{-1}{1 - \beta q \rho \lambda(q, \varepsilon)} F_\pi.$$

The initial value of the policy instrument x_0 is anchored on the initial value of the cost-push shock u_0 because of the feedback policy rule. This implies an optimal initial anchor of inflation π_0 on the initial value of the cost-push shock u_0 :

$$\left. \begin{array}{l} x_0 = F_\pi \pi_0 + F_u u_0 \\ \pi_0 = -\frac{1}{\varepsilon} x_0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x_0 = \varepsilon \frac{-F_u}{F_\pi + \varepsilon} u_0 = -\varepsilon \frac{\lambda(q, \varepsilon)}{1 - \beta q \rho \lambda(q, \varepsilon)} u_0 \\ \pi_0 = -\frac{F_u}{F_\pi + \varepsilon} u_0 = \frac{\lambda(q, \varepsilon)}{1 - \beta q \rho \lambda(q, \varepsilon)} u_0. \end{array} \right.$$

The dynamical system in the space of target variables is locally stable with two eigen-

values inside the unit circle: $0 < \rho < 1$ and $0 < \lambda(q, \varepsilon) < 1$:

$$\begin{pmatrix} E_{t-1}\pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} \lambda(q, \varepsilon) & -\frac{1}{\beta q} - \frac{\kappa}{\beta q} F_u \\ 0 & \rho \end{pmatrix}^t \begin{pmatrix} \frac{\lambda(q, \varepsilon)}{1 - \beta q \rho \lambda(q, \varepsilon)} \\ 1 \end{pmatrix} u_0.$$

The positive correlation ($\kappa > 0$) between current inflation and current output gap implies a negative correlation ($-\frac{\kappa}{\beta q}$) between future inflation and current output gap. The positive sign of the policy rule parameter ($F_\pi(q, \varepsilon) > 0$) satisfies this necessary condition in order to lean against inflation spirals:

$$0 < \lambda(q, \varepsilon) = \frac{1}{\beta q} - \frac{\kappa}{\beta q} F_\pi(q, \varepsilon) < 1 < \frac{1}{\beta q} \Rightarrow -\frac{\kappa}{\beta q} F_\pi(q, \varepsilon) < 0.$$

The sign of the correlation of expected inflation with the policy instrument determines the sign of the response of the policy instrument to the policy target in the optimal policy rule. By contrast, in the accelerationist Phillips curve, Clarida, Gali and Gertler [2001] mention that there is a positive sign of the correlation between expected inflation and current output gap ($-\frac{\kappa}{\beta q}$). If $\kappa < 0$, because negative feedback requires $-\frac{\kappa}{\beta q} F_\pi < 0$, the sign of the optimal policy rule is negative ($F_\pi < 0$).

3 DISCRETION

There is a long history of different definitions of "discretion" for stabilization policy (Chatelain and Ralf [2020a]). Clarida, Gali and Gertler [1999] and Gali ([2015], chapter 5) define "discretion" (or discretion equilibrium) as the case where policy makers re-optimize with certainty each period.

Proposition 1 *When policy makers re-optimize each period ($q = 0$), they do static optimization each period even if the private sector's transmission mechanism is dynamic.*

Proof. If $q = 0$, the policy maker's discounted loss function boils down to a *static utility* ($(\beta \cdot 0)^0 = 1$, $(\beta \cdot 0)^t = 0$, $t = 1, 2, \dots$). The policy maker only values the current period, as he knows he will be replaced next period, whatever the duration of the next period:

$$V^{jj}(u_0) = -E_0 \sum_{t=0}^{t=+\infty} (\beta q)^t \left[\frac{1}{2} \left(\pi_t^2 + \frac{\kappa}{\varepsilon} x_t^2 \right) + \beta (1 - q) V^{jk}(u_t) \right] = -\frac{1}{2} \left(\pi_0^2 + \frac{\kappa}{\varepsilon} x_0^2 \right).$$

The policy maker's infinite horizon transversality condition is always satisfied as he does not survive the current period and as his discount factor is zero after his single period of life. Because the policy maker does not value the future, he does not take into account private sector expectations in the new-Keynesian Phillips curve. He only considers a static Phillips curve with an exogenous intercept:

$$\pi_0 = \kappa x_0 + \beta 0 E_0 \pi_1 + \beta (1 - 0) E_0 \pi_1^k + u_0 = \kappa x_0 + u_0 + \beta E_0 \pi_1^{k=1}, u_0 + \beta E_0 \pi_1^{k+1=1} \text{ given.}$$

The superscript for the policy maker index k is now identical to the period index, as each period corresponds to a new policy maker. ■

As expected inflation is not taken into account by successive policy makers, one order (or one dimension) of the dynamics of inflation is removed in discretion. By contrast, for

a non-zero probability of not reneging commitment, ($q > 0$), the policy maker takes into account private sector expectations. This causes the bifurcation of the dynamic system between discretion ($q = 0$) versus quasi-commitment ($q \in]0, 1]$), see section 3.

Proposition 2 *The policy makers' static optimizations of an otherwise dynamic transmission mechanism implies locally unstable dynamics of the dynamic transmission mechanism.*

Proof. On an iso-utility curve, the derivative of the loss function is equal to zero:

$$dL = \pi_0 d\pi_0 + \frac{\kappa}{\varepsilon} x_0 dx_0 = 0 \Rightarrow \frac{\partial \pi_0}{\partial x_0} = -\frac{\kappa}{\varepsilon} \frac{x_0}{\pi_0}.$$

The policy maker's first order condition is such that the iso-utility ellipse is tangent to the slope of the static Phillips curve with a given intercept:

$$\frac{\partial \pi_0}{\partial x_0} = \kappa = -\frac{\kappa}{\varepsilon} \frac{x_0}{\pi_0} \Rightarrow x_0 = -\varepsilon \pi_0.$$

This static optimization implies a proportional policy rule with the exact negative correlation of the policy instrument (output gap) with the policy target (inflation). The proportional parameter is the opposite of the household's elasticity of substitution between goods. Any increase of current inflation is instantaneously related to a decrease of current output. This static optimal program is repeatedly solved each period by each new policy maker of period t :

$$x_t = -\varepsilon \pi_t \text{ for } t = 0, 1, 2, \dots \text{ with } \varepsilon > 1. \quad (4)$$

Substituting this policy rule in the new-Keynesian Phillips curve, one has this recursive unstable dynamics for inflation, denoted the discretionary inflation eigenvalue $\lambda(0, \varepsilon)$ for $q = 0$:

$$E_t \pi_{t+1}^{t+1} = \left(\frac{1 + \kappa \varepsilon}{\beta} \right) \pi_t - \frac{1}{\beta} u_t \text{ with } 1 < \frac{1}{\beta} < \lambda(0, \varepsilon) = \frac{1 + \kappa \varepsilon}{\beta}.$$

The policy maker lives only the current period. He sets a zero weight on future inflation. For the policy maker, it does not matter what happens as long as it is not in his instantaneous lifetime. *The policy maker does not care in his loss function that his static optimal policy leads to sub-optimal unstable dynamics in the future.* The economy dynamic system is locally unstable with the inflation dynamics eigenvalue outside the unit circle ($\lambda(0, \varepsilon) > \frac{1+\kappa}{\beta} > 1$) and the cost-push shock autoregressive root inside the unit circle ($0 < \rho < 1$):

$$\begin{pmatrix} \lambda(0, \varepsilon) = \frac{1}{\beta} + \frac{\kappa}{\beta} \varepsilon & -\frac{1}{\beta} \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} E_t \pi_{t+1}^{t+1} \\ u_{t+1} \end{pmatrix}. \quad (5)$$

■

The optimal policy for a static Phillips curve assuming a positive correlation between current inflation and current output gap ($\kappa > 0$) but excluding inflation expectations, leads to an optimal solution where the output gap responds negatively to inflation ($F_\pi = -\varepsilon < 0$). This example shows that the optimal solution of static optimization can be different from the solution of dynamic optimization (Phillips [1954]).

The policy makers' sub-optimal static optimizations of a dynamic transmission mechanism implies locally unstable dynamics in the space of policy targets. Starting with

Hypothesis H_1 : Policy makers re-optimize every period, $q = 0$,

building epicycles on epicycles, four additional restrictions on private sector's behavior have to be imposed in order to restrict inflation dynamics to the stable subspace of dimension one within the unstable space of dimension two. These four assumptions are not explicitly spelled out in Clarida, Gali and Gertler [1999] and in Gali [2015]:

Hypothesis H_2 : The private sector expectation of inflation $E_t \pi_1^{k+1=1}$ does not depend on current and past values of inflation.

Hypothesis H_3 : The private sector does not select paths with inflation or deflation spirals.

Hypothesis H_4 : The policy instrument x_t is a non-predetermined variable without a given initial condition x_0 .

Hypothesis H_5 : Policy makers and the private sector measure with infinite precision the initial and the future values of variables and the structural parameters of the transmission mechanism.

Proposition 3 *Under assumptions H_i ($i = 1, \dots, 5$), Blanchard and Kahn's [1980] determinacy condition forces a unique solution with an exact correlation of inflation and output gap with the cost-push shock.*

Proof. Blanchard and Kahn's [1980] solution is given by the unique slope of the eigenvectors of the given stable eigenvalue $0 < \rho < 1$ of the predetermined cost-push shock:

$$\begin{pmatrix} \frac{1}{\beta} + \frac{\kappa}{\beta}\varepsilon & -\frac{1}{\beta} \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \rho \begin{pmatrix} \pi_t \\ u_t \end{pmatrix} \Rightarrow \left(\frac{1}{\beta} + \frac{\kappa}{\beta}\varepsilon - \rho \right) \pi_t = \frac{1}{\beta} u_t.$$

There is an exact positive correlation between inflation and the cost-push shock:

$$\pi_t = \left(\frac{1}{1 - \beta\rho + \kappa\varepsilon} \right) u_t. \quad (6)$$

Combining this equation with the policy rule leads to an exact negative correlation between output gap x_t and the cost-push shock u_t :

$$x_t = -\varepsilon \left(\frac{1}{1 - \beta\rho + \kappa\varepsilon} \right) u_t. \quad (7)$$

■

The dynamics of inflation is locally unstable in the space (π_t, u_t) of dimension two. However, inflation dynamics is constrained to vary in a stable subspace of dimension one, which is lower than the dimension of the policy target and forcing variables dynamics. For example, if u_t is exogenous imported inflation, inflation π_t would be exactly correlated with imported inflation, without any relation to other drivers of inflation within the country.

For practitioners of monetary policy, the assumption H_5 of perfect measurement stretches credulity to its limit, because of the measurement issues for inflation, for the output gap, for the cost-push shock, for the slope $\kappa(\beta, \varepsilon, \alpha_L, \theta, \sigma, \varphi)$ of the new-Keynesian Phillips curve (Mavroeidis *et al.* [2015]) and for the correlation of expected inflation with current inflation $(1/\beta)$. Because of these imperfect measurements, in the real world, the

probability to select unstable paths with inflation and deflation spirals is equal to one with the discretion policy rule.

4 BIFURCATION

4.1 Inflation Eigenvalue

Going from monetary policy with quasi commitment to discretionary policy results in a fundamental change of the properties of the dynamical system.

Proposition 4 *For any value of the elasticity of substitution between goods $\varepsilon > 1$:*

(i) *There is a saddle-node bifurcation on the inflation eigenvalue when shifting from quasi-commitment ($q \in]0, 1]$ with stable eigenvalue $0 < \lambda(q, \varepsilon) < 1$) to discretion ($q = 0$, with unstable eigenvalue $\lambda(0, \varepsilon) > 1$):*

$$0 < \lambda_{\min} < \lambda(q, \varepsilon) < \frac{1}{1 + \kappa_{\max}} < 1 < \frac{1 + \kappa_{\max}}{\beta} < \lambda(0, \varepsilon) < \frac{1}{\beta} + \frac{1}{\beta} \frac{\kappa_{\max}}{\alpha_L}.$$

(ii) *The optimal inflation persistence decreases with the policy maker's credibility measured by the probability of not reneging commitment $q \in]0, 1]$:*

$$\frac{\partial \lambda(q, \varepsilon)}{\partial q} < 0.$$

(iii) *The inflation eigenvalue decreases (respectively increases) with the elasticity of substitution ε for quasi-commitment ($q \in]0, 1]$) (respectively for discretion ($q = 0$)):*

$$\frac{\partial \lambda(q, \varepsilon)}{\partial \varepsilon} < 0 < \frac{\partial \lambda(0, \varepsilon)}{\partial \varepsilon} = \frac{\kappa}{\beta}$$

Proof. For (i), we first seek the limits of $\kappa(\varepsilon)\varepsilon$ which is an increasing function of $\varepsilon \in]1, +\infty[$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 1^+} \kappa(\varepsilon)\varepsilon &= \left(\sigma + \frac{\varphi + \alpha_L}{1 - \alpha_L} \right) \frac{(1 - \theta)(1 - \beta\theta)}{\theta} (1 - \alpha_L) = \kappa_{\max} \\ \lim_{\varepsilon \rightarrow +\infty} \kappa(\varepsilon)\varepsilon &= \left(\sigma + \frac{\varphi + \alpha_L}{1 - \alpha_L} \right) \frac{(1 - \theta)(1 - \beta\theta)}{\theta} \frac{(1 - \alpha_L)}{\alpha_L} = \frac{\kappa_{\max}}{\alpha_L} \text{ with } 0 < \alpha_L < 1 \\ &\Rightarrow \kappa_{\max} < \kappa(\varepsilon)\varepsilon < \frac{\kappa_{\max}}{\alpha_L}. \end{aligned}$$

For discretion, the inflation eigenvalue is an increasing affine function of $\kappa\varepsilon$ with boundaries:

$$1 < \frac{1}{\beta} < \frac{1}{\beta} + \frac{1}{\beta} \kappa_{\max} < \lambda(0, \varepsilon) = \frac{1}{\beta} + \frac{1}{\beta} \kappa\varepsilon < \frac{1}{\beta} + \frac{1}{\beta} \frac{\kappa_{\max}}{\alpha_L}.$$

For quasi-commitment, $\lambda(q, \varepsilon)$ is obtained solving a linear quadratic regulator model so that the inflation eigenvalue is necessarily within the range $]0, 1[$. It is a decreasing function of βq , of q (according to (i)) of $\kappa\varepsilon$ and of ε with this upper bound:

$$\lim_{q \rightarrow 0^+} \lim_{\varepsilon \rightarrow 1^+} \lambda(q, \varepsilon) = \frac{1}{2} \left(1 + \frac{1}{\beta q} + \frac{\kappa\varepsilon}{\beta q} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta q} + \frac{\kappa\varepsilon}{\beta q} \right)^2 - \frac{1}{\beta q}} = \frac{1}{1 + \kappa_{\max}} < 1$$

which is true because:

$$\lim_{q \rightarrow 0^+} \frac{1}{2} \left(1 + \frac{1}{\beta q} + \frac{1}{\beta q} \kappa_{\max} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta q} + \frac{1}{\beta q} \kappa_{\max} \right)^2 - \frac{1}{\beta q}} = \frac{1}{1 + \kappa_{\max}} < 1$$

and because when $q \rightarrow 0^+$:

$$\lambda(q, \varepsilon) \sim \frac{1 + \kappa}{2\beta q} \left(1 - \sqrt{1 - \frac{4\beta q}{(1 + \kappa)^2}} \right) \sim \frac{1 + \kappa}{2\beta q} \frac{1}{2} \frac{4\beta q}{(1 + \kappa)^2} = \frac{1}{1 + \kappa}.$$

Its lower bound is strictly positive:

$$\lim_{q \rightarrow 1^-} \lim_{\varepsilon \rightarrow +\infty} \lambda(q, \varepsilon) = \lambda_{\min} = \frac{1}{2} \left(1 + \frac{1}{\beta} + \frac{1}{\beta} \frac{\kappa_{\max}}{\alpha_L} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta} + \frac{1}{\beta} \frac{\kappa_{\max}}{\alpha_L} \right)^2 - \frac{1}{\beta}} > 0.$$

For (ii), see appendix. For claim (iii):

$$\begin{aligned} \text{sign} \frac{\partial \lambda(q, \varepsilon)}{\partial \varepsilon} &= \text{sign} \frac{\kappa}{\beta q} - \frac{\frac{1}{2} \left(1 + \frac{1}{\beta q} + \frac{\varepsilon \kappa}{\beta q} \right) \left(\frac{\kappa}{\beta q} \right)}{\sqrt{\left(1 + \frac{1}{\beta q} + \frac{\varepsilon \kappa}{\beta q} \right)^2 - \frac{4}{\beta q}}} \\ &= \text{sign} \sqrt{\left(1 + \frac{1}{\beta q} + \frac{\varepsilon \kappa}{\beta q} \right)^2 - \frac{4}{\beta q}} - \left(1 + \frac{1}{\beta q} + \frac{\varepsilon \kappa}{\beta q} \right) \\ &= \text{sign}(-2\lambda(q, \varepsilon)) < 0 \end{aligned}$$

■

4.2 Rule parameter

The inflation rule parameter F_π is an affine and decreasing function of the inflation eigenvalue λ :

$$F_\pi(q, \varepsilon) = \frac{1}{\kappa} - \frac{\beta q}{\kappa} \lambda(q, \varepsilon).$$

Proposition 5 *For any value of the elasticity of substitution between goods $\varepsilon > 1$, the inflation policy rule parameter $F_\pi(q, \varepsilon)$ is positive and increasing with the elasticity of substitution for quasi-commitment ($q \in]0, 1]$), whereas $F_\pi(0, \varepsilon)$ is negative, below -1 and decreasing with the elasticity of substitution $F_\pi(0, \varepsilon)$ for discretion:*

$$\begin{aligned} -\infty < F_\pi(0, \varepsilon) = -\varepsilon < -1 < 0 < F_\pi(q, \varepsilon) &= \frac{\lambda(q, \varepsilon)}{1 - \lambda(q, \varepsilon)} \varepsilon. \\ \frac{\partial F_\pi(0, \varepsilon)}{\partial \varepsilon} < 0 < \frac{\partial F_\pi(q, \varepsilon)}{\partial \varepsilon} \end{aligned}$$

Proof. For quasi-commitment, the policy rule parameter of the response to inflation is a decreasing function of credibility q and an increasing function of the elasticity of

substitution ε . To prove that the policy rule is positive, it is sufficient to prove:

$$\lim_{q \rightarrow 1^-} \lim_{\varepsilon \rightarrow 1^+} \frac{1}{\kappa} - \frac{\beta q}{\kappa} \left(\frac{1}{2} \left(1 + \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q} \right)^2 - \frac{1}{\beta q}} \right) > 0.$$

When $q = 1$ and when $\varepsilon \rightarrow 1^+$

$$F_\pi(q, \varepsilon) > \frac{1}{\kappa_{\max}} - \frac{\beta}{\kappa_{\max}} \left(\frac{1}{2} \left(1 + \frac{1}{\beta} + \frac{\kappa_{\max}}{\beta} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta} + \frac{\kappa_{\max}}{\beta} \right)^2 - \frac{1}{\beta}} \right) > 0.$$

In this case, one shows in the appendix that $F_\pi > 0$ is equivalent to $\kappa_{\max} + \beta > \beta$ which is true because $\kappa_{\max} > 0$. ■

5 IMPULSE RESPONSE FUNCTIONS AND WELFARE

5.1 Initial anchor of inflation

To ensure stability after a cost-push shock, inflation has to jump to the stable manifold. The size of this initial anchor depends on the elasticity of substitution between goods and the credibility of the policy maker.

Proposition 6 *For any value of the elasticity of substitution between goods $\varepsilon > 1$,*

(i) The initial anchor (or jump) of inflation on the cost-push shock decreases with the elasticity of substitution between goods for both quasi-commitment and discretion.

(ii) The initial jump of inflation is an increasing function of the limited credibility q of the policy maker.

(iii) The initial anchor of near-zero credibility is always strictly smaller than the initial anchor in the case of zero credibility:

$$\pi_0(q, \varepsilon) = \frac{\lambda(q, \varepsilon)}{1 - \beta q \rho \lambda(q, \varepsilon)} u_0 \leq \pi_0(0, \varepsilon) = \frac{1}{1 - \beta \rho + \kappa(\varepsilon) \varepsilon} u_0.$$

Proof. (i) It is straightforward to check that:

$$\frac{\partial \pi_0(q, \varepsilon)}{\partial \varepsilon} < 0, \quad \frac{\partial \pi_0(0, \varepsilon)}{\partial \varepsilon} < 0$$

For discretion, the anchor of inflation is a decreasing function of $\kappa(\varepsilon) \varepsilon$ which is an increasing function of ε . As $\kappa_{\max} < \kappa(\varepsilon) \varepsilon < \frac{\kappa_{\max}}{\alpha_L}$, the zero credibility initial anchor of inflation (π_0/u_0) is bounded as follows:

$$0 < \frac{1}{1 - \beta \rho + \frac{\kappa_{\max}}{\alpha_L}} < \frac{1}{1 - \beta \rho + \kappa \varepsilon} < \lim_{\varepsilon \rightarrow 1} \frac{1}{1 - \beta \rho + \kappa \varepsilon} = \frac{1}{1 - \beta \rho + \kappa_{\max}}.$$

For limited credibility, the anchor of inflation is a decreasing function of $\kappa(\varepsilon) \varepsilon$ which is an increasing function of ε . As $\kappa_{\max} < \kappa \varepsilon < \frac{\kappa_{\max}}{\alpha_L}$, the non-zero credibility initial anchor

of inflation (π_0/u_0) upper bound is:

$$\lim_{q \rightarrow 1^-} \lim_{\varepsilon \rightarrow 1} \frac{\lambda}{1 - \beta q \rho \lambda} = \lim_{q \rightarrow 1^-} \lim_{\varepsilon \rightarrow 1} \frac{\lambda}{1 - \beta \rho \lambda} > 1,$$

with:

$$\lim_{q \rightarrow 1^-} \lim_{\varepsilon \rightarrow 1} \lambda(q, \varepsilon) = \frac{1}{2} \left(1 + \frac{1}{\beta} + \frac{\kappa_{\max}}{\beta} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta} + \frac{\kappa_{\max}}{\beta} \right)^2 - \frac{1}{\beta}} < 1.$$

(ii) It is straightforward to check that:

$$\frac{\partial \pi_0(q, \varepsilon)}{\partial q} > 0 \text{ for } q \in]0, 1]$$

(iii) The initial anchor of near-zero credibility is always strictly smaller than the initial anchor in the case of zero credibility. The gap tends to zero when the auto-correlation of the forcing variable tends to zero and when the elasticity of substitution tends to one: $\rho \rightarrow 0$ and $\varepsilon \rightarrow 1$.

$$\lim_{q \rightarrow 0^+} \frac{\lambda(q, \varepsilon)}{1 - \beta q \rho \lambda(q, \varepsilon)} = \lim_{q \rightarrow 0^+} \lambda(q, \varepsilon) \sim \frac{1}{1 + \kappa(\varepsilon)} < \frac{1}{1 - \beta \rho + \kappa(\varepsilon) \varepsilon}$$

For Gali's [2015] calibration ($\rho = 0.8$), for any elasticity of substitution $\varepsilon > 1$ and for any probability of not reneging commitment $q \in]0, 1]$, the zero credibility initial anchor of inflation is much higher than the one with minimal credibility. ■

5.2 Impulse response functions

The values of the parameters are taken from Gali's [2015] calibration: $\rho = 0.8$, $\beta = 0.99$, $\varepsilon = 6$, $\kappa = 0.1275$ obtained with $\theta = 2/3$, $1 - \alpha_L = 2/3$, $\sigma = 1$ and $\varphi = 1$. Expected impulse response functions are shown in figure 1 for four different degrees of credibility q : 0 (discretion (Gali [2015])), 10^{-7} (near-zero credibility), 0.5 (limited credibility), 1 (commitment, Gali [2015]).

INSERT FIGURE 1 HERE.

The parameters of the inflation dynamics change marginally between $q = 1$ and $q = 10^{-7}$. Inflation eigenvalue increases from $\lambda = 0.43$ to 0.57. Inflation sensitivity with a lagged cost-push shock shifts from -0.13 to -0.08 . Inflation initial anchor on a cost-push shock shifts from 0.65 to 0.57.

By contrast, the shifts from near-zero credibility $q = 10^{-7}$ to zero credibility $q = 0$ are large. Inflation eigenvalue increases from $\lambda = 0.57$ to 1.78 (multiplied by 3, crossing the bifurcation value 1). Inflation sensitivity with a lagged cost-push shock shifts from -0.08 to -1.01 (multiplied by 12). Inflation initial anchor on a cost-push shock shifts from 0.57 to 1.03 (multiplied by 1.8).

The impulse response function of inflation of zero credibility is markedly above the impulse response functions of inflation with limited credibility, including near-zero credibility.

To check the lack of robustness to misspecification of discretion, we compute two impulse response functions facing a $\pm 10\%$ error of the initial anchor of inflation. For quasi-commitment with near-zero credibility ($q = 10^{-7}$), the error gap of 10% with respect

to the perfect knowledge optimal path at the initial date is reduced to less than 1% after eight quarters (figure 2). For discretion ($q = 0$), the error gap of 10% with respect to the perfect knowledge optimal path at the initial date is increased to 110% after four quarters due to inflation and deflation spirals in sharp contrast with near-zero credibility $q = 10^{-7}$ paths with $\pm 10\%$ initial error also represented on figure 3.

INSERT FIGURE 2 AND FIGURE 3 HERE.

5.3 Welfare Losses

We denote the welfare of discretionary policy $W(0)$. It is usually computed using households' discount factor of $\beta = 0.99$ instead of policy maker's discount factor $\beta q = 0$:

$$W(0) = -\frac{1}{2} \sum_{t=0}^{t=+\infty} \beta^t \left(\pi_t^2 + \frac{\kappa}{\varepsilon} x_t^2 \right) = -\frac{1}{2} \left(1 + \frac{\kappa}{\varepsilon} \varepsilon^2 \right) \left(\frac{1}{1 + \kappa \varepsilon - \beta \rho} \right)^2 \sum_{t=0}^{t=+\infty} \beta^t (\rho^t u_0)^2$$

$$W(0) = -\frac{1}{2} \frac{1 + \kappa \varepsilon}{(1 + \kappa \varepsilon - \beta \rho)^2} \frac{u_0^2}{1 - \beta \rho^2} = -5.09.$$

In table 1, for comparison with the welfare of discretionary policy, the limited credibility welfare is computed using households' discount factor of $\beta = 0.99$ instead of policy maker's discount factor βq . We simulate the model over 200 periods in order to compute welfare for different elasticities and different levels of credibility. One can also compute welfare losses of Ramsey optimal policy solving a Riccati equation (Chatelain and Ralf [2020b]).

Table 1: Welfare loss in percentage of welfare loss with infinite horizon commitment ($w(q) = \frac{W(q)}{W(1)} - 1$, $\beta = 0.99$) when varying the elasticity of substitution ε and credibility q

-	-	-	$q = 1$	0.8	0.5	0.1	10^{-7}	0
ε	$\kappa(\varepsilon)$	$\frac{\kappa(\varepsilon)}{\varepsilon}$	$W(1)$	$w(q)$	$w(q)$	$w(q)$	$w(q)$	$w(q)$
3193	0.00032	10^{-7}	-2.119	2.8%	6.8%	10.8%	2.1%	73%
6	0.1275	0.02125	-2.688	3.2%	7.4%	10.9%	0.03%	89%
2.35	0.235	0.1	-3.489	3.6%	7.8%	10.2%	8.6%	111%
1.001	0.34	0.34	-7.971	4.1%	7.8%	7%	23.6%	141%

Because there is a wide gap between the large impulse response functions of zero-credibility $q = 0$ with respect to near zero credibility $q = 10^{-7}$, the welfare gap between near-zero versus zero credibility is also gigantic: from 71% if $\varepsilon = 3193$ to 117% when ε tends to one.

When considering only limited credibility cases, the losses with respect to infinite horizon commitment are at most an increase of 24% of welfare losses in the limit case of the elasticity of substitution tending to 1, (corresponding to a large relative weight on output gap in the loss function of 0.34) for all the range of non-zero probabilities of reneging commitment.

6 CONCLUSION

When the probability to renege commitment tends to zero, the quasi-commitment equilibrium is never the limit of the equilibrium under discretion where the probability to renege commitment is exactly zero. The discretion equilibrium is obtained by static optimization of the policy maker of an otherwise dynamics transmission mechanism, which implies inflation or deflation spirals as soon as the parameters of the economy are not perfectly known.

Following Once it is clear that the discretion equilibrium is not a relevant theory for stabilization policy, the empirical issue is the measurement of the sign of the slope of the new-Keynesian Phillips curve. If it turns out to be negative, the transmission mechanism corresponds to an accelerationist Phillips curve instead of the new-Keynesian Phillips curve. Unfortunately, for the U.S., nearly 50% of the estimates have a positive sign in a large number of estimations done by Mavroeidis et al. [2014]. Once this sign is known, the optimal response of the policy instrument to inflation will have the opposite sign under quasi-commitment.

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7 APPENDIX

The new-Keynesian Phillips curve can be written as a function of the Lagrange multiplier where $\kappa > 0$, $0 < \beta < 1$ and $0 < q < 1$:

$$E_t \pi_{t+1} + \frac{\kappa \varepsilon}{\beta q} \gamma_{t+1} = \frac{1}{\beta q} \pi_t - \frac{1}{\beta q} u_t - \frac{1-q}{q} E_t \pi_{t+1}^j$$

We keep the notation of Gali [2015] chapter 5 of the Lagrange multiplier with one step ahead subscript γ_{t+1} . The Hamiltonian system is:

$$\begin{pmatrix} 1 & \frac{\kappa \varepsilon}{\beta q} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ \gamma_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta q} & 0 & \frac{-1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ \gamma_t \\ u_t \end{pmatrix} + \begin{pmatrix} -\frac{1-q}{q} E_t \pi_{t+1}^j \\ 0 \\ 0 \end{pmatrix}.$$

This leads to:

$$\begin{pmatrix} \pi_{t+1} \\ \gamma_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q} & -\frac{\kappa \varepsilon}{\beta q} & -\frac{1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ \gamma_t \\ u_t \end{pmatrix} + \begin{pmatrix} -\frac{1-q}{q} E_t \pi_{t+1}^j \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic polynomial of the upper square matrix of the Hamiltonian system (with a determinant equal to $\frac{1}{\beta q}$) is:

$$\lambda^2 - \left(1 + \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q}\right) \lambda + \frac{1}{\beta q} = 0.$$

The Hamiltonian matrix has two stable roots ρ and λ and one unstable root $\frac{1}{\beta q \lambda}$:

$$\lambda = \frac{1}{2} \left(1 + \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q} - \sqrt{\left(1 + \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q}\right)^2 - \frac{4}{\beta q}} \right) < \sqrt{\frac{1}{\beta q}} < \frac{1}{\beta q \lambda}.$$

Policy rule parameter function of $\lambda(\varepsilon)$ and ε :

$$(1 - \lambda) \left(1 - \frac{1}{\beta q \lambda} \right) = -\frac{\kappa \varepsilon}{\beta q} \implies \left(\frac{1 - \lambda}{\beta q \lambda} \right) \left(\frac{\beta q \lambda - 1}{\kappa} \right) = -\frac{\varepsilon}{\beta q} \implies F_\pi = \frac{1 - \beta q \lambda}{\kappa} = \left(\frac{\lambda}{1 - \lambda} \right) \varepsilon.$$

The Hamiltonian system can be written as a function of the stable eigenvalue λ , after eliminating ε :

$$\begin{pmatrix} \pi_{t+1} \\ \gamma_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda + \frac{1}{\beta q \lambda} - 1 & 1 + \frac{1}{\beta q} - \lambda - \frac{1}{\beta q \lambda} & -\frac{1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ \gamma_t \\ u_t \end{pmatrix}.$$

Proposition 7 Rule parameters P_u and P_z of the response of the Lagrange multiplier on

inflation to exogenous variables:

$$\gamma_t = P_\pi \pi_t + P_u u_t \quad (8)$$

$$P_\pi = \frac{1}{1-\lambda} > 0, \quad P_u = \frac{1}{1-\lambda} \frac{\frac{1}{\beta q}}{\rho - \frac{1}{\beta q \lambda}} = \frac{1}{1-\lambda} \frac{\lambda}{\beta q \lambda \rho - 1} < 0. \quad (9)$$

Proof. The solution stabilizes the state-costate vector for any initial value of inflation π_0 and of the exogenous variables u_0 in a stable subspace of dimension two within a space of dimension three (π_t, γ_t, u_t) of the Hamiltonian system. We seek a characterization of the Lagrange multiplier γ_t of the form:

$$\gamma_t = P_\pi \pi_t + P_u u_t.$$

To deduce the control law associated with vector (P_π, P_u) , we substitute it into the Hamiltonian system:

$$\begin{pmatrix} \pi_{t+1} \\ P_\pi \pi_{t+1} + P_u u_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta q} - (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) & (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) & -\frac{1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ P_\pi \pi_t + P_u u_t \\ u_t \end{pmatrix}.$$

We write the last two equations in this system separately:

$$\begin{aligned} P_\pi \pi_{t+1} + P_u u_{t+1} &= (P_\pi - 1) \pi_t + P_u u_t \\ u_{t+1} &= \rho u_t. \end{aligned}$$

It follows that:

$$\pi_{t+1} = \frac{P_\pi - 1}{P_\pi} \pi_t + \frac{(1-\rho) P_u}{P_\pi} u_t.$$

The first equation is such that:

$$\pi_{t+1} = \left[\frac{1}{\beta q} - (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) \right] \pi_t + (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) (P_\pi \pi_t + P_u u_t) - \frac{1}{\beta q} u_t.$$

Factorizing:

$$\pi_{t+1} = \left[\frac{1}{\beta q} - (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) + (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) P_\pi \right] \pi_t + \left[(1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) P_u - \frac{1}{\beta q} \right] u_t.$$

The method of undetermined coefficients implies for the first term:

$$\begin{aligned} \frac{P_\pi - 1}{P_\pi} &= \frac{1}{\beta q} + (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) (P_\pi - 1), \\ P_\pi &= \frac{1}{1-\lambda}. \end{aligned}$$

For the second term:

$$\begin{aligned}\frac{(1-\rho)P_u}{P_\pi} &= (1-\lambda) \left(1 - \frac{1}{\beta q \lambda}\right) P_u - \frac{1}{\beta q} \Rightarrow \\ \frac{1}{\beta q} &= \left(1 - \frac{1}{\beta q \lambda} - 1 + \rho\right) (1-\lambda) P_u \Rightarrow \\ P_u &= \frac{1}{1-\lambda} \frac{\frac{1}{\beta q}}{\rho - \frac{1}{\beta q \lambda}} \Rightarrow \frac{P_u}{P_\pi} = \frac{\frac{1}{\beta q}}{\rho - \frac{1}{\beta q \lambda}} = \frac{-\lambda}{1 - \lambda \beta q \rho}.\end{aligned}$$

■

Proposition 8 *Optimal policy rule parameters are given by:*

$$\begin{aligned}F_\pi &= \varepsilon (P_\pi - 1) = \lambda \varepsilon P_\pi = \varepsilon \frac{\lambda}{1-\lambda} = \frac{1-\beta\lambda}{\kappa}, \\ F_u &= \varepsilon P_u = \varepsilon P_\pi \frac{\lambda}{\beta\lambda\rho - 1} = \varepsilon \frac{1}{1-\lambda} \frac{\lambda}{\beta\lambda\rho - 1}, \\ \frac{F_u}{F_\pi} &= A = \frac{1}{\lambda} \frac{P_u}{P_\pi} = \frac{1}{\beta\lambda\rho - 1} = \frac{P_u}{P_\pi - 1} = -1 + \beta\rho \frac{P_u}{P_\pi}.\end{aligned}$$

Proof. The first order condition relates the Lagrange multiplier to the policy instrument:

$$\begin{aligned}x_t &= \varepsilon \gamma_{t+1} = \varepsilon (\gamma_t - \pi_t) \\ x_t &= F_\pi \pi_t + F_u u_t = \varepsilon (\gamma_t - \pi_t) = \varepsilon (P_\pi \pi_t + P_u u_t - \pi_t) \Rightarrow \\ F_\pi &= \varepsilon (P_\pi - 1), F_u = \varepsilon P_u.\end{aligned}$$

■

Furthermore, we can find the following lower bound for the policy rule parameter.

$$\begin{aligned}\text{If } 1 - \frac{1}{2}(\beta q + 1 + \kappa) + \sqrt{\frac{1}{4}(\beta q)^2 \left(1 + \frac{1}{\beta q} + \frac{\kappa}{\beta q}\right)^2 - \beta q} &> 0 \Leftrightarrow \sqrt{\frac{1}{4}(1 + \beta q + \kappa)^2 - \beta q} > \frac{1}{2}(-1 + \beta q) \\ \frac{1}{4}(1 + \beta q + \kappa)^2 - \beta q &> \frac{1}{4}(-1 + \beta q + \kappa)^2 \\ (1 + \beta q + \kappa)^2 - (-1 + \beta q + \kappa)^2 &= 4(\kappa + \beta q) > 4\beta q \\ \kappa + \beta q &> \beta q \text{ which is true.}\end{aligned}$$

Proof of proposition 4 (ii): The larger the discount rate, the lower the discount factor, the less we weight the present, the larger the speed of convergence, the lower the inflation persistence eigenvalue, which is the root of the characteristic polynomial:

$$\begin{aligned}\lambda^2 - \left(1 + \frac{1}{\beta q} + \frac{\kappa \varepsilon}{\beta q}\right) \lambda + \frac{1}{\beta q} &= 0. \\ \lambda^2 - (1 + bP) \lambda + P &= 0.\end{aligned}$$

With our notations:

$$S > 1, b = 1 + \kappa\varepsilon > 1, P = \frac{1}{\beta q} \geq 1$$

$$\Delta > 0 \Leftrightarrow (1 + bP)^2 > 4P$$

Let's prove that the inflation eigenvalue is an increasing function of the product of the two roots, which is an inverse function of the discount factor βq . When proven true, the inflation eigenvalue is a decreasing function of the discount factor βq .

$$\begin{aligned} \text{sign} \{ \lambda' (P) \} &= \text{sign} \left\{ b - \frac{2b(Pb + 1) - 4}{2\sqrt{(Pb + 1)^2 - 4P}} \right\} \\ &= \text{sign} \left\{ \sqrt{(Pb + 1)^2 - 4P} - (Pb + 1) + \frac{2}{b} \right\} \\ &= \text{sign} \left\{ \sqrt{(Pb + 1)^2 - 4P} - (Pb + 1) + \frac{2}{b} \right\} \\ &= \text{sign} \left\{ -2\lambda(P) + \frac{2}{b} \right\} \\ \lambda'(P) > 0 &\Leftrightarrow b\lambda(P) < 1 \text{ for } b > 0 \text{ and } \lambda(P) > 0 \end{aligned}$$

We use the classic functional inequality:

$$\begin{aligned} \sqrt{1+x} &< 1 + \frac{1}{2}x \text{ for } x \geq -1 \Rightarrow \\ b\lambda &= b \frac{1+bP}{2} \left(1 - \sqrt{1 - \frac{4P}{(1+bP)^2}} \right) \\ &\leq b \frac{1+bP}{2} \frac{1}{2} \frac{4P}{(1+bP)^2} = \frac{bP}{bP+1} < 1 \end{aligned}$$

QED.

Figure 1: Impulse response function of inflation with $q=1, 0.5, 10^{-7}$ and $q=0$ top curve.

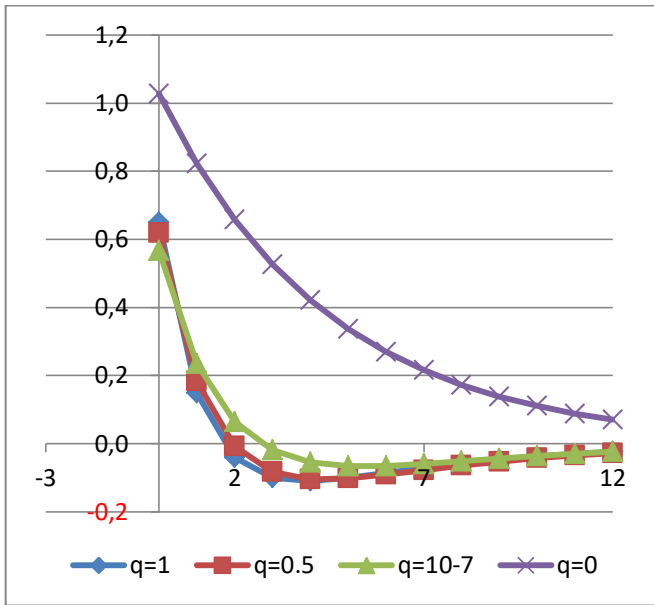


Figure 2: Impulse response of inflation: $q=10^{-7}$ and out of equilibrium path +/- 10% error.

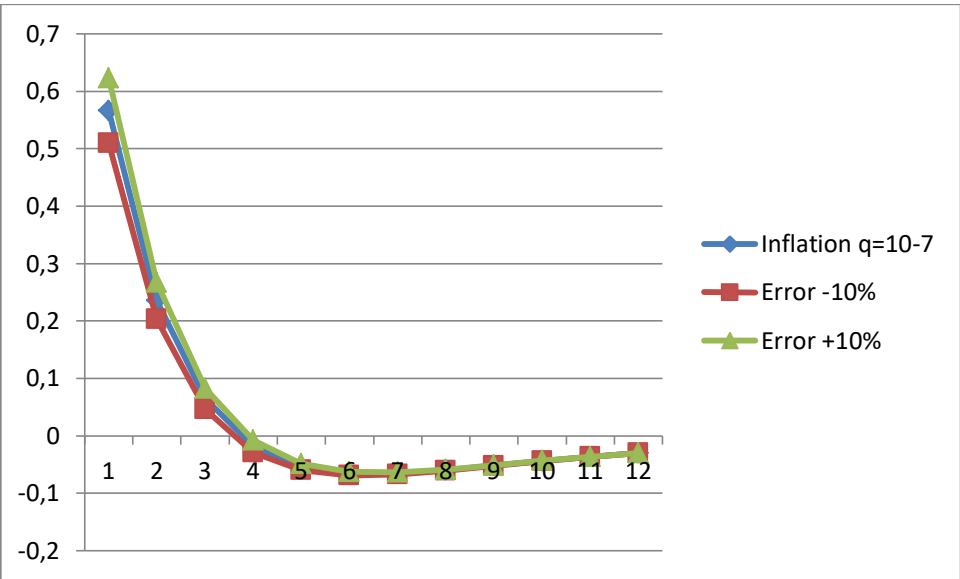


Figure 3: Impulse response of inflation: $q=0$ and out of equilibrium path +/- 10% error besides $q=10^{-7}$ and out of equilibrium path +/- 10% error

