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Super-inertial interest rate rules are not solutions of Ramsey optimal policy

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Abstract

This paper demonstrates that the equilibrium determined by the commitment of a Central Bank to a non-stationary ("super-inertial") interest rate rule (where the sum of the parameters of the lags of the interest rate exceeds one and does not depend on the persistence of shocks) does not correspond to the unique bounded solution and the stable equilibrium of Ramsey optimal policy for the new-Keynesian model. It always destabilizes inflation because of the rounding errors and the measurement errors of the parameters of the monetary policy transmission mechanism. By contrast, the commitment of a Central Bank to a stationary interest rate rule (where the sum of the parameters of lags of the interest rate is strictly below one and depends on the persistence of shocks) corresponds to the unique bounded solution and the stable equilibrium of Ramsey optimal policy.

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Keywords: new-Keynesian model, Ramsey optimal policy, interest-rate smoothing, super-inertial rule, stability.

Titre: Les règles de taux d'intérêt super-inertielles ne sont pas des solutions des politiques optimales de Ramsey.

Mots-Clés: modèle nouveau Keynésien, Politique monétaire optimale de Ramsey, lissage du taux d'intérêt, règle super-inertielle, stabilité

Résumé: Cet article démontre que l'équilibre déterminé par l'engagement d'une Banque Centrale à suivre un règle de taux d'intérêt directeur non stationnaire "super-inertielle" (où la somme des paramètres des taux d'intérêt passés dépasse un et ne dépend pas de la persistance des chocs) ne correspond pas à une solution de l'équilibre stable d'une politique monétaire optimale de Ramsey. Elle déstabilise toujours l'inflation du fait des erreurs d'arrondis et des erreurs de mesure des paramètres du mécanisme de transmission de la politique monétaire. En revanche, l'engagement d'une Banque Centrale à suivre une règle de taux d'intérêt directeur stationnaire où la somme des

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paramètres des taux d'intérêt passé est strictement plus petite que un et dépend de la persistance de chocs correspond à une solution de l'équilibre stable d'une politique monétaire optimale de Ramsey.

1 Introduction

Ramsey optimal stabilization policy serves as a reference for determining the values of pre-determined and non-pre-determined macroeconomic policy targets, such as inflation and output, and the policy maker's instruments for a given policy transmission mechanism of the private sector. The resulting welfare is usually higher than for alternative stabilization policies, except in the case of a social planner (Ljungqvist and Sargent (2012)).

Optimizing private agents have pre-determined state variables (such as their stock of wealth) and non-pre-determined decision variables (such as their flow of consumption). The policy maker sets policy instruments, such as interest rates or tax rates. A social planner maximizes welfare, choosing jointly and simultaneously the values of the private sector's non-pre-determined variables and the policy maker's policy instruments.

Ramsey optimal policy, however, is a Stackelberg dynamic-game equilibrium with the policy maker as the Stackelberg leader and the private sector as the Stackelberg follower. The private sector's pre-determined *and* non-pre-determined variables are the policy maker's targets. Observing the private sector's behavior as the leader, a policy maker can improve private agents' welfare by choosing current and future values of his own policy instruments (which are not under control of private agents). The policy maker performs two optimizations which may lead to time-inconsistency if he re-optimizes in the future (Simaan and Cruz (1973)).

In this paper we compare two different policy transmission mechanisms. In the first model, the new-Keynesian Phillips curve is taken as the transmission mechanism; inflation is the policy target; and the output gap serves as the policy instrument. In this setting, Clarida, Gali, and Gertler (1999) and Gali (2015) found an auto-regressive stationary representation of a Ramsey optimal policy rule with the sum of auto-regressive parameters strictly below one. In addition, one policy-rule parameter depends on the auto-correlation of an exogenous cost-push shock, for example, foreign energy prices. It is optimal to increase the sensitivity of the policy instrument to a persistent exogenous shock in order to decrease the sensitivity of the policy target (inflation) to this cost-push shock whenever the shock is more persistent (Chatelain and Ralf (2022)).

The second transmission mechanism is modeled as a new-Keynesian Phillips curve together with an Euler consumption equation. There are two policy targets, namely inflation and the output gap; and the funds rate serves as the policy instrument. Giannoni (2001, p.34), Giannoni (2014, section 4.1 and appendix A.5), and Giannoni and Woodford (2003), henceforth GW, found an auto-regressive non-stationary representation of a Ramsey optimal policy rule such that the sum of the auto-regressive parameters is strictly larger than one. They label this representation a "super-inertial rule". A super-inertial rule is thus defined as a non-stationary rule with an eigenvalue larger than one which violates the condition that all eigenvalues of the dynamic system of Ramsey optimal policy should be lower than one.

This implies a counter-intuitive advice for stabilization policy: "*Because of a root larger*

than one, the optimal policy requires an explosively growing response of the interest rate to deviations of inflation and the output gap from the target" (Giannoni, 2001, p.36). Furthermore, the parameters of this super-inertial policy rule do not depend on the auto-correlations of a stationary cost-push shock or of a productivity shock included in the new-Keynesian model.

However, "super-inertial rules" spread the idea that a destabilizing non-stationary interest-rate rule is a result of Ramsey optimal stabilization policy. For example, Creel and Hubert (2015) and Kara (2007) present a super-inertial policy rule as a benchmark equation for their estimations of Taylor rules. Super-inertial policy rules are mentioned in textbooks, such as Villieu (2015, p.354) and taught to graduate students. Simulations assuming super-inertial optimal funds-rate rules are commonly done for other models of transmission mechanisms in several central bank models (Adolfson et al. (2014), Chung et al. (2015)).

This paper explains why a super-inertial policy rule is not the mathematical solution of the optimal program written by GW. According to Ljungqvist and Sargent (2012), Clarida Gali Gertler (1999) and Gali (2015), Ramsey optimal policy under commitment implies that the dynamics of the system have a stable equilibrium. Because GW destabilizing super-inertial rule implies an unstable equilibrium, it is not the solution of Ramsey optimal policy.

More precisely, even with the unbelievable assumption of the perfect knowledge of structural parameters, one cannot avoid rounding errors for computing the optimal initial jumps of non-predetermined variables which are non-linear functions of these structural parameters with square roots and fractions. In our simulations, GW super-inertial stops being a valid approximation of Ramsey optimal policy after six quarters for a 7 decimal rounding error and after thirty-six quarters for a 16-decimal rounding error. For a plausible 10% error on the auto-correlation parameter of the cost-push shock, the path of policy targets explodes after one to three quarters. Then, following GW super-inertial rule leads very rapidly to large loss relative to the Ramsey allocation.

As a remark, Hansen and Sargent (2007) robust Ramsey optimal policy under commitment implies more structure than the stability of the equilibrium required for Ramsey optimal policy under commitment. It involves selecting the optimal feedback rule parameters from a Stackelberg dynamic game between the policy maker and an evil agent.

To solve the model, we use the following algorithm to determine the optimal policy rule based on Ljungqvist and Sargent (2012, chapter 19), Chatelain and Ralf (2019 and 2020), Hansen and Sargent (2007, chapter 5).

- In step 1, a Hamiltonian system of equations with the policy transmission mechanism equations of the policy maker's state variable and the Euler equations for the policy maker's Lagrange multipliers of the state variables is derived. This system of dynamic equations is related to an unstable equilibrium. Its Hamiltonian symplectic matrix \mathbf{H} has a saddle-point equilibrium property, with half of its eigenvalues inside the unit circle and the other half outside the unit circle.

- In step 2, an optimal-policy proportional feedback rule for the current and future periods is found. To this end, one has to seek the stable subspace of the Hamiltonian matrix \mathbf{H} , solving a Riccati equation for the unique optimal welfare matrix \mathbf{P} , which determines a linear relation between the costate and the state variables. Using \mathbf{P} , one determines the unique optimal feedback matrix of policy rule parameters \mathbf{F} , related to the benchmark representation of optimal policy rules where the policy instruments respond to the current

values of the policy maker's state variables including auto-regressive forcing variables. *This paper highlights that this key step is missing in GW.* Let us cite Ljungqvist and Sargent (2012, p.781):

"We seek a "stabilizing" solution of (19.3.11) [the linear quadratic regulator Hamiltonian system], i.e., one that satisfies:

$$\sum_{t=0}^{t=+\infty} \beta^t y_t' y_t < +\infty.$$

[where policy target variables are in vector y_t , taking into account a discount factor β in the loss function]. A stabilizing solution satisfies $\mu_0 = Py_0$ [that the Lagrange multipliers μ_0 are a unique linear combination to the policy targets y_0], where P solves the matrix Riccati equation (19.3.5). The solution for μ_0 replicates itself over time in the sense that $\mu_t = Py_t$."

- In step 3, the optimal initial conditions for non-predetermined state variables are computed as linear functions of predetermined variables using blocks of the matrix \mathbf{P} .¹

- In the optional step 4, a representation of optimal policy rules is looked for, where the policy instruments respond to lags or leads of state variables, the policy instruments, or other variables, using linear substitutions of current values of state variables.

Reasons for the contradicting results of Ramsey optimal policy and super-inertial rules are then:

(a) GW omit step 2 (seeking a proportional feedback rule solving a Riccati equation) so that the dynamics of the state variables evolve around an unstable equilibrium.

(b) Using step 3, the optimal initial values of inflation and the output gap are found by multiplying a matrix of non-linear functions (with square roots and fractions) of all parameters with the vector of initial values of the auto-regressive shocks.

(c) These optimal initial conditions allow the GW super-inertial rule to approximate the Ramsey optimal path only for a small number of periods, because the GW Hamiltonian system of step 1 is locally unstable. Even if the policy maker knows all parameters with an infinite precision, because the initial conditions of step 3 are non-linear functions (with fractions and square roots) of these parameters, the probability of rounding errors is equal to one for numerical simulations. Therefore, numerical simulations of the super-inertial GW solution always imply to sub-optimal exploding paths after some periods. On two examples, we show that a 7-decimal rounding leads to exploding paths after 7 periods and a 16-decimal rounding leads to exploding paths after 33 periods. On two other examples, we consider a more realistic confidence interval allowing a 10% measurement error on the auto-correlation of the cost-push shocks by the policy maker. This 10% measurement error leads to exploding paths after 2 to 4 periods with GW super-inertial rule.

In practice, there always exist confidence intervals for the parameters of the policy transmission mechanism. Therefore, a commitment to a super-inertial policy rule would always

¹Re-optimizing step 3 next period may contradict next period's step 2 rule which was defined in the current period, so that the policy maker has to commit himself to a probability of re-optimizing in the future strictly below one (Schaumburg and Tambalotti (2006)).

lead policy makers to do exactly the opposite of their objective: to maximize a loss function to an infinite loss instead of minimizing it.

(d) By contrast, it is feasible to find a stationary representation of the Ramsey optimal policy rule found in step 2 where the parameters of the policy rule depend on the auto-correlation of shocks and where the sum of the parameters of the lags of the policy instrument are strictly below one (step 4).

A GW super-inertial solution (using the optimal initial conditions found in step 3 without solving the Riccati equation of step 2) can be used for any policy transmission mechanism for any policy instrument. It is not confined to the new-Keynesian two-equations model and an interest rate rule, which may have been irrelevant during the period of a binding negative lower bound for funds rate for some countries. For example, we show that it can be computed in the case of the new-Keynesian Phillips curve single equation with the output gap as the policy instrument (Clarida, Gali and Gertler (1999)).

The rest of the paper is organized as follows: Section 2 compares the stability properties of a super-inertial *versus* an inertial rule for the order-two transmission mechanism model of the new-Keynesian Phillips curve model with a cost-push shock (Clarida Gali and Gertler (1999)). Section 3 compares the stability properties of a super-inertial *versus* an inertial rule for the order-four transmission mechanism of the new-Keynesian model with consumption (GW). Section 4 concludes.

2 New-Keynesian Phillips Curve Transmission Mechanism

2.1 Ramsey optimal policy versus super-inertial rule

Clarida, Gali and Gertler (1999) and Gali (2015, chapter 5) solve Ramsey optimal policy for a new-Keynesian Phillips curve and an auto-regressive cost-push shock where inflation is the policy target and where the output gap is the (intermediate) policy instrument. We compute a super-inertial rule applying the GW method on this transmission mechanism. We then compare the stability properties of each equilibrium.

The monetary policy transmission mechanism is a new-Keynesian Phillips curve where current inflation is positively correlated with discounted expected inflation (discount factor β , $0 < \beta \leq 1$), current output gap (slope parameter $\kappa > 0$), and an auto-correlated cost-push shock u_t :

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t \Leftrightarrow E_t \pi_{t+1} = \frac{1}{\beta} \pi_t - \frac{\kappa}{\beta} x_t - \frac{1}{\beta} u_t, \quad (1)$$

where E_0 is the expectation operator at date 0 and t is a period index. Inflation in log deviation from its equilibrium value at date t is denoted π_t . Output in log deviation from its target at date t , i.e. the output gap, is denoted x_t . The exogenous stationary and predetermined cost-push shock u_t is auto-regressive of order one ($0 < \rho_u < 1$) with a given initial value. Disturbances $\varepsilon_{u,t}$ have zero-mean and are independent and identically normally distributed:

$$u_t = \rho_u u_{t-1} + \varepsilon_{u,t} \text{ where } \varepsilon_{u,t} \text{ is i.i.d. } N(0, s_u^2), u_0 \text{ given.} \quad (2)$$

This transmission mechanism is a dynamic system of order two, including two policy maker's state variables (π_t, u_t) with one lag. Subject to this dynamic system, a policy maker selects the output gap in order to control inflation, minimizing an expected discounted quadratic loss function:

$$E_0 \sum_{t=0}^{+\infty} \beta^t \left(\frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} \right) \quad (3)$$

where $\lambda_x > 0$ is the relative cost of changing the policy instrument x_t .

Denoting Lagrangian multipliers $\phi_{\pi,t}$ for the new Keynesian Phillips curve, the Lagrangian \mathcal{L} is:

$$\mathcal{L} = -E_0 \sum_{t=0}^{+\infty} \beta^t \left(\frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} + \phi_{\pi,t} [\pi_t - \beta\pi_{t+1} - \kappa x_t - u_t] \right). \quad (4)$$

Proposition 1 (i) *Ramsey optimal policy implies a stable equilibrium with two eigenvalues (δ, ρ) inside the unit circle.*

(ii) *The super-inertial equilibrium is unstable. It includes a third root $\frac{1}{\beta\delta}$ outside the unit circle besides the two eigenvalues (δ, ρ) of Ramsey optimal policy inside the unit circle.*

(iii) *Ramsey optimal policy allows two representations of the policy rule, namely, the benchmark proportional policy rule (step 2) and the stationary autoregressive policy rule (step 4), but it does not allow the non-stationary super-inertial representation.*

Proof. See Appendix. ■

The proof rests on solving a Riccati equation seeking a stable subspace of a Hamiltonian system including Euler equations (step 2) for Ramsey optimal policy. For the super-inertial rule the step solving the Riccati equation is omitted. In the latter case, the roots outside the unit circle of the Hamiltonian system are left unchanged in accordance with a non-stationary super-inertial policy rule. Table 1 summarizes the differences between the two different policy equilibria.

Table 1: Ramsey optimal policy versus a super-inertial rule for order two dynamics.

	Ramsey optimal policy	Super-inertial rule
4 steps algorithm	Includes Riccati step 2	Omits Riccati step 2
Step 2 proportional rule	$x_t = F_\pi \pi_t + F_u u_t$	None
System order	2	3
Stable roots	$0 < \delta < 1, 0 < \rho_u < 1$	$0 < \delta < 1, 0 < \rho_u < 1$
Unstable roots	None	$\frac{1}{\beta\delta} \geq 1$
Equilibrium	Stable	Unstable
Rule	Stationary $x_t = \delta x_{t-1} - \frac{\delta}{1-\delta\beta\rho_u} \frac{\kappa}{\lambda_x} u_t$	Non-stationary $x_t = x_{t-1} - \frac{\kappa}{\lambda_x} \pi_t$

Both equilibria have the same theoretical initial values of inflation and the output gap according to step 3. They are linear functions of the initial value of the cost-push shock u_0

and non-linear functions of all the parameters of the model: ρ_u , κ , β and λ_x :

$$\pi_0 = \frac{\delta}{1 - \delta\beta\rho_u}u_0 \text{ and } x_0 = -\frac{\kappa}{\lambda_x} \frac{\delta}{1 - \delta\beta\rho_u}u_0 \text{ with:}$$

$$0 < \delta = \frac{1}{2} \left(1 + \frac{1}{\beta} + \frac{\kappa^2}{\beta\lambda_x} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta} + \frac{\kappa^2}{\beta\lambda_x} \right)^2 - \frac{1}{\beta}} < 1.$$

The next section investigates the consequences of rounding errors and of the imperfect knowledge of at least one of the parameters of the policy transmission by the policy maker.

2.2 Instability and measurement errors

This section analyzes the sensitivity of impulse response functions of inflation and the output gap to a tiny measurement error of the auto-correlation ρ_u of the cost-push shock for a super-inertial rule *versus* Ramsey optimal policy. In table 2, the difference of calibration values for ρ_u between Gali (2015) and GW, equal to 0.45, is larger than the one for the slope of new-Keynesian Phillips curve, equal to 0.1037. New-Keynesian experts do not know exactly the parameters of their model. They have much less prior knowledge of the persistence ρ_u of the cost-push shock than of the slope κ of the new-Keynesian Phillips curve. In addition, the slope of the new-Keynesian Phillips curve depends on a number of other structural parameters (Gali (2015)).

We use Gali's (2015, chapter 5) calibrations and the calibrations in table 1 of Giannoni (2001), (2014) and Giannoni and Woodford (2003). Table 2 shows the disagreement on the three parameters κ , ρ_u and λ_x of both calibrations.

Table 2: Gali's (2015, chapter 5) calibration and GW calibration.

Authors	β	κ	ρ_u	λ_x	σ	ρ_v	λ_i
Gali	0.99	0.1275	0.8	0.02125	—	—	—
GW	0.99	0.0238	0.35	0.04800	1/0.1571	0.35	0.236

The parameters σ , the inverse of the intertemporal elasticity of substitution of consumption, ρ_v , the auto-correlation coefficient of a productivity shock, and λ_i , the relative cost of changing the funds rate will be introduced and used in the next section.

Gali (2015) computes households' welfare so that the relative cost of changing the policy instrument is endogenous: $\lambda_x = \frac{\kappa}{\varepsilon} = 0.02125$ with an elasticity of substitution between differentiated goods of 6. Gali (2015) and GW do not refer to any empirical estimations to ground the point estimates of their calibrations. For example, the elasticity of intertemporal substitution $\sigma = 6.3654$ is at the very top end of estimates of the cross-country meta-analysis done by Havranek (2015) who claims in his abstract that a "*calibration greater than 0.8 is inconsistent with the bulk of empirical evidence*". These large discrepancies among values chosen for calibration by new-Keynesian experts do not support the hypothesis of the perfect knowledge of parameters by policy makers.

For $\rho_u = 0.8$, the predicted paths of all variables in the Ramsey optimal policy equilibrium are only approximated by the variables in the super-inertial rule equilibrium for a small number of periods (see appendix). For a long number of periods, the super-inertial policy diverges, because the initial values given by $\pi_0 = \frac{\delta}{1 - \delta\beta\rho_u}u_0 = -\frac{\lambda_x}{\kappa}x_0$ (step 3) face rounding

errors with probability one.

INSERT FIGURE 1 HERE

In figure 1, we consider the case where, although the true parameter is $\rho_u = 0.8$, the policy maker believes the cost-push shock to be more persistent: $\rho_u = 0.9$. This leads to a tiny change of the initial conditions for inflation $\pi_0^*(\rho_u)$ and the output gap $x_0^* = -\varepsilon\pi_0^*(\rho_u)$, identical for Ramsey optimal policy and the super-inertial rule. For Ramsey optimal policy, it also leads to a tiny change of the response of the output gap to the cost-push shock $F_u(\rho_u)$ in the proportional feedback-rule. In figure 1, Ramsey optimal policy converges back to the equilibrium. In the case of the super-inertial rule, the output gap diverges with respect to Ramsey optimal policy at period two and inflation diverges at period four. Following GW super-inertial rule leads very rapidly to large loss relative to the Ramsey allocation.

INSERT FIGURE 2 HERE

We now consider the zero-probability case in the real world where the policy maker has the exact knowledge of Gali's parameters according to table 2. GW implicitly bypass step 2 of optimal policy rule and only use the optimal initial values of inflation and output gap of step 3. However, even with the exact knowledge of parameters, these endogenous initial optimal values of inflation and output gap are never exactly known, because they are non-linear functions of these parameters (functions $\frac{\delta}{1-\delta\beta\rho_u}$ and $-\frac{\kappa}{\lambda_x}\frac{\delta}{1-\delta\beta\rho_u}$) with a numerical infinite stream of decimals which are rounded at most at the 16-decimal as shown in figure 2. Even the number of decimals for $\rho_u^t = 0.8^t$ increases with the number of periods: it is rounded at the 16-decimal on period $t = 16$. The output gap and inflation start diverging on period $t = 34$ (figure 2).

Although Clarida, Gali and Gertler (1999) and Gali (2015) did not mention the super-inertial rule equilibrium, it has been central in a model including the consumption Euler equation with the new-Keynesian Phillips curve, as will be detailed in the next section.

3 New-Keynesian Phillips Curve and Euler Consumption Equation Transmission Mechanism

3.1 Optimal program

Giannoni and Woodford (2003) apply Ramsey linear quadratic optimal policy to the new-Keynesian model, including not only the new-Keynesian Phillips curve and a cost push shock, but also the representative household's intertemporal substitution consumption Euler equation and its auto-regressive forcing variable v_t :

$$x_t = E_t x_{t+1} - \sigma (i_t - E_t \pi_{t+1}) + v_t \text{ where } \sigma > 0. \quad (5)$$

Current output gap x_t is positively correlated with the *expected* output gap and negatively correlated with the real rate of interest, equal to the nominal rate i_t minus *expected* inflation $E_t \pi_{t+1}$. We follow Giannoni and Woodford (2003) who use the notation σ for the intertemporal elasticity of substitution.²

²Giannoni ((2001), (2014)) uses the notation " σ^{-1} " for the intertemporal elasticity.

An exogenous predetermined shock v_t is auto-regressive of order one ($0 < \rho_v < 1$) with given initial value. Disturbances $\varepsilon_{v,t}$ have zero mean and are independently and identically normally distributed:

$$v_t = \rho_v v_{t-1} + \varepsilon_{v,t} \text{ where } \varepsilon_{v,t} \text{ is i.i.d. } N(0, s_v^2), v_0 \text{ given. } \Sigma_\varepsilon = \begin{pmatrix} s_v^2 & 0 \\ 0 & s_u^2 \end{pmatrix}. \quad (6)$$

Subject to these four equations with dynamics of order one (hence a dynamic system of order 4), the policy maker maximizes a quadratic utility function where the output gap is a second policy target $\lambda_x \geq 0$ besides inflation, and where the policy instrument is the interest rate with weight $\lambda_i > 0$:

$$-E_t \sum_{t=0}^{+\infty} \beta^t \left(\frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} + \lambda_i \frac{i_t^2}{2} \right). \quad (7)$$

Denoting Lagrangian multipliers $\phi_{x,t}$ for the consumption Euler equation, $\phi_{\pi,t}$ for the new-Keynesian Phillips curve, the Lagrangian \mathcal{L} is:

$$\mathcal{L} = -E_0 \sum_{t=0}^{+\infty} \beta^t \left\{ \begin{aligned} &\frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} + \lambda_i \frac{i_t^2}{2} + \phi_{x,t} [x_t - x_{t+1} + \sigma (i_t - \pi_{t+1}) - v_t] \\ &+ \phi_{\pi,t} [\pi_t - \beta_t \pi_{t+1} - \kappa x_t - u_t] \end{aligned} \right\}. \quad (8)$$

By comparison, in Clarida Gali and Gertler (1999) and Gali (2015, chapter 5) when deriving Ramsey optimal policy, the intertemporal substitution equation of consumption and its auto-correlated shock equation are set to zero, so that there is no interest rate smoothing: $i_t = x_t = v_t = 0$, $\lambda_i = 0$, $\phi_x = 0$. In their setting, the output gap x_t plays the role of the policy instrument in a “targeting rule”: $\lambda_x > 0$ and inflation is the only policy target.

This extended optimal program leads to proposition 2:

Proposition 2 (i) *Ramsey optimal policy implies a stable equilibrium with four eigenvalues δ_1 , δ_2 , ρ_u and ρ_v inside the unit circle.*

(ii) *The super-inertial equilibrium is unstable, with two other roots $\frac{1}{\beta\delta_1}$ and $\frac{1}{\beta\delta_2}$ outside the unit circle in addition to the four roots of Ramsey optimal policy inside the unit circle.*

(iii) *Ramsey optimal policy allows two representations of the policy rule, namely, the benchmark proportional policy rule (step 2) and a stationary autoregressive policy rule (step 4), but it does not allow the non-stationary super-inertial representation.*

Proof. See appendix. ■

The differences between the two different policy equilibria are summarized in the following table.

Table 3: Ramsey optimal policy versus a super-inertial rule for order two dynamics.

	Ramsey optimal policy	Super-inertial rule
4 steps algorithm	Includes Riccati step 2	Omits Riccati step 2
Step 2 proportional rule	$i_t = F_\pi \pi_t + F_x x_t + F_u u_t + F_v v_t$	None
System order	4	6
Stable roots	$ \delta_1 < 1 \quad \delta_2 < 1$ $0 < \rho_u < 1 \quad 0 < \rho_v < 1$	$ \delta_1 < 1 \quad \delta_2 < 1$ $0 < \rho_u < 1 \quad 0 < \rho_v < 1$
Unstable roots	None	$\left \frac{1}{\beta \delta_1} \right \geq 1, \left \frac{1}{\beta \delta_2} \right \geq 1$
Equilibrium	Stable	Unstable
Step 4 autoregressive rule	Stationary	Non-stationary

In what follows, we present the GW result within the four step algorithm of Ramsey optimal policy. The predicted path of all variables in the Ramsey optimal policy equilibrium are only approximated by the variables in the super-inertial rule equilibrium for a small number of periods because the initial values given by π_0^* and x_0^* are non-linear functions of all parameters and initial values of auto-regressive shocks (step 3) which face rounding errors with probability one.

First step: Hamiltonian system

In step 1 of the algorithm, the linear state-costate Hamiltonian system is found along with its boundary conditions from first order conditions of the Lagrangian. The Hamiltonian system (H) includes the four equations of the policy transmission mechanism, namely, the output gap equation, the new-Keynesian Phillips curve, the two auto-regressive equations for the exogenous shocks, the three first order conditions, and the boundary conditions for the initial and the final period:

$$(H) \left\{ \begin{array}{l} \text{4 transmission mechanism equations} \\ \frac{\partial \mathcal{L}}{\partial x_t} = 0 \Rightarrow \lambda_x x_t + \phi_{x,t} - \beta^{-1} \phi_{x,t-1} - \kappa \phi_{\pi,t} = 0 \\ \frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \pi_t - \beta^{-1} \sigma \phi_{x,t-1} + \phi_{\pi,t} - \phi_{\pi,t-1} = 0 \\ \frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \lambda_i i_t + \sigma \phi_{x,t} = 0 \\ \phi_{\pi,t=-1} = \phi_{x,t=-1} = 0 \text{ and } \lim_{t \rightarrow +\infty} \phi_{\pi,t} = \lim_{t \rightarrow +\infty} \phi_{x,t} = 0. \\ u_0 \text{ and } v_0 \text{ given, all equations for } t = 0, 1, 2, \dots \end{array} \right.$$

As we have one instrument for two targets the Hamiltonian system corresponds to a system of six equations that can be written recursively with a matrix that has four eigenvalues inside the unit circle and two eigenvalues outside the unit circle (see appendix).

Because inflation and the output gap are forward-looking variables, they are optimally chosen at the initial and the final date according to optimal initial and final conditions. The marginal values of the loss function with respect to the output gap $\phi_{x,t=-1}$ and to inflation $\phi_{\pi,t=-1}$ are equal to zero at the initial date (GW have chosen the initial date to be $t = -1$, while other authors prefer the initial date to be $t = 0$):

$$\phi_{\pi,t=-1} = \phi_{x,t=-1} = 0 \text{ and } \lim_{t \rightarrow +\infty} \phi_{\pi,t} = \lim_{t \rightarrow +\infty} \phi_{x,t} = 0. \quad (9)$$

Because the marginal value of the loss function with respect to the output gap at the initial date $t = -1$ is zero ($\phi_{x,t=-1} = 0$), the initial interest rate is set to zero: $i_{-1} = 0$.

First step (continued): Super-Inertial rule

In the case of an optimal commitment to Ramsey policy that has been in force since at least period $t - 2$ ($\phi_{x,t-3} = \phi_{\pi,t-3} = 0$ and $i_{t-3} = 0$) which corresponds to step 3 of the algorithm, GW eliminate Lagrange multipliers using linear substitutions between first order conditions to find their super-inertial rule where the funds rate is a function of inflation, of the output gap and of lags of funds rate and of the output gap:

"one can infer the value of the Lagrange multiplier $\phi_{x,t-1}$ from the value of i_{t-1} using Equation (first order condition for the interest rate), and similarly, the value of $\phi_{x,t-2}$ from the value of i_{t-2} :

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow i_t = -\frac{\sigma}{\lambda_i} \phi_{x,t} \text{ for } t = -2, -1, 0, 1, \dots \quad (10)$$

Then, substituting theses values into equation (first order condition for the output gap) for period $t - 1$, one can also infer the value of $\phi_{\pi,t-1}$ from the value of x_{t-1} . One can, of course, similarly solve for the period t Lagrange multipliers as functions of x_t, i_t and i_{t-1} :

$$\frac{\partial \mathcal{L}}{\partial x_t} = 0 \Rightarrow \phi_{\pi,t} = \frac{1}{\kappa} \lambda_x x_t - \frac{1}{\kappa} \frac{\lambda_i}{\sigma} i_t + \frac{\beta^{-1} \lambda_i}{\kappa \sigma} i_{t-1} \text{ for } t = -1, 0, 1, \dots \quad (11)$$

Using these expressions to substitute out the Lagrange multipliers in equation (first order condition for inflation), one obtains a linear relation among the endogenous variables $\pi_t, x_t, x_{t-1}, i_{t-1}, i_{t-1}$ and i_{t-2} that must hold for any period $t \geq 0$, for an optimal commitment for Ramsey policy that has been in force since at least period -2 :

$$i_t = \left(1 + \frac{\kappa\sigma}{\beta}\right) i_{t-1} + \frac{1}{\beta} \Delta i_{t-1} + \frac{\kappa\sigma}{\lambda_i} \pi_t + \frac{4\sigma\lambda_x}{\lambda_i} \frac{\Delta x_t}{4} \text{ for } t = 0, 1, 2, \dots \quad (12)$$

GW label this representation of the first order condition of inflation "super-inertial": the implied dynamics of the funds rate is explosive because the sum of parameters for the lags of the funds rate exceeds one: $1 + \frac{\kappa\sigma}{\beta} > 1$.

This equation is obtained using linear substitution across first order conditions *without including any restrictions in the stable subspace*. In other words, the system including the four equations of the transmission mechanism and the three first-order conditions which form the Hamiltonian system is exactly equivalent (if and only if) to a system of equations including additionally a super-inertial policy rule. Hence, the predicted path including the super-inertial rule for optimal initial conditions is equivalent to the predicted path of the Hamiltonian system for the same optimal initial conditions.

GW analysis stops at this step. However, the Hamiltonian dynamic system of a linear quadratic regulator always includes eigenvalues outside the unit circle (Hansen and Sargent (2007)), see appendix. Therefore, it is a locally unstable equilibrium. But Lagrange-multiplier first-order-condition dynamics is only an intermediate step of computation for

finding a representation of an optimal policy rule. Step 2 of the Ramsey optimal policy algorithm seeks an optimal policy rule solving a Riccati equation so that the dynamics of the state variables corresponds to a locally stable equilibrium.

2nd step: Riccati equation

Hansen and Sargent's (2007) section 3.5 and Chatelain and Ralf's (2020, 2021) solution of this discounted linear quadratic regulator, augmented with forcing variables (v_t, u_t) , is to solve a discrete algebraic Riccati equation for the matrix \mathbf{P} and then to compute the rule parameters (F_x, F_π) which depend on parameters (β, κ, σ) and preferences (λ_x, λ_i) but do not depend on the auto-correlation of shocks. It also solves a Sylvester equation for rule parameters (F_z, F_u) which depends on the auto-correlation of shocks (ρ_z, ρ_u) and of the other parameters of the transmission mechanism:

$$\phi_{x,t} = P_{xx}x_t + P_{x\pi}\pi_t + P_{xv}v_t + P_{xu}u_t, \quad (13)$$

$$\phi_{\pi,t} = P_{\pi x}x_t + P_{\pi\pi}\pi_t + P_{\pi v}v_t + P_{\pi u}u_t. \quad (14)$$

One finds the unique optimal proportional feedback interest rate rule:

$$i_t = F_x x_t + F_\pi \pi_t + F_v v_t + F_u u_t. \quad (15)$$

As soon as there are two endogenous policy targets, there is no longer a closed form solution for the parameters of matrices \mathbf{P} and \mathbf{F} . Numerical solutions are obtained using the linear quadratic regulator instruction in Matlab or Scilab.

3rd step: Optimal initial conditions

Optimal initial conditions are obtained using $\phi_{x,t} = \phi_{\pi,t} = 0$:

$$\begin{pmatrix} x_0 \\ \pi_0 \end{pmatrix} = - \begin{pmatrix} P_{xx} & P_{x\pi} \\ P_{\pi x} & P_{\pi\pi} \end{pmatrix}^{-1} \begin{pmatrix} P_{xv} & P_{xu} \\ P_{\pi v} & P_{\pi u} \end{pmatrix} \begin{pmatrix} v_0 \\ u_0 \end{pmatrix}. \quad (16)$$

4th step: Inertial policy rule

To find the inertial policy rule, one applies the operator $p(L) = (1 - \rho_u L)(1 - \rho_v L)$ (where L is the lag operator) to both sides of the proportional feedback rule. Ramsey optimal policy is determined by commitment from date $t = 0$ onwards to an inertial interest rate rule, where the sum of the parameters of lags of the interest rate is below one and depends on the auto-correlation of shocks,

$$i_t = (\rho_u + \rho_v) i_t - (\rho_v \rho_u) i_{t-2} + F_x p(L) x_t + F_\pi p(L) \pi_t + F_z (1 - \rho_u L) \varepsilon_{v,t} + F_u (1 - \rho_v L) \varepsilon_{u,t}. \quad (17)$$

This inertial policy rule is a representation of the policy rule which corresponds to the unique bounded solution of Ramsey optimal policy for the new-Keynesian model.

3.2 Robustness to measurement and rounding errors

We use GW calibration (table 2) to compute the sensitivity of impulse response functions of inflation, the output gap, and the funds rate for a super-inertial rule *versus* Ramsey optimal policy to a tiny measurement error of the auto-correlation ρ_u of a cost-push shock.

For $\rho_u = 0.35$, both, Ramsey optimal policy and the super-inertial rule, have the *same* optimal path for inflation, the output gap and the funds rate for a number of periods. However, if the initial conditions for inflation π_0^* and for the output gap x_0^* are numbers which have been rounded for a given finite number of decimals by a computer software, there is necessarily a date t where the super-inertial rule leads to unbounded inflation and both, an exploding output gap and an exploding funds rate, whereas Ramsey optimal policy still converges to its stable equilibrium.

INSERT FIGURE 3 HERE

Figure 3 shows impulse response functions for a cost-push shock of one unit, where, although the true parameter is $\rho_u = 0.35$, the policy maker believes it is $\rho_u = 0.4$. This leads to a tiny change of initial conditions for inflation $\pi_0^*(\rho_u)$ and the output gap $x_0^*(\rho_u)$, identical for Ramsey optimal policy and the super-inertial rule. For Ramsey optimal policy, it also leads to a tiny change of the response of the output gap to the cost-push shock $F_u(\rho_u)$ in the proportional feedback rule.

Ramsey optimal policy converges back to the equilibrium whereas, for the super-inertial rule, the funds rate, the output gap, and inflation diverge after at most six quarters. The policy maker anchors initial values believing the persistence of the cost-push shock is 0.4, whereas it is 0.35 for both, the super-inertial rule and Ramsey optimal policy. Super-Inertial policy diverges from Ramsey optimal policy for output gap (x) at period 3, for inflation (π) at period 6, and for the funds rate (i) at period 2. Following GW super-inertial rule leads very rapidly to large loss relative to the Ramsey allocation.

Coincidentally, excess inflation goes hand in hand with a negative output gap (recessions) in Gali and GW benchmark impulse response functions. The new-Keynesian model has to be interpreted as describing a stagflationist transmission mechanism when using Ramsey monetary policy. The new-Keynesian model is not usually advertised as describing stagflation by new-Keynesian authors.

INSERT FIGURE 4 HERE

In figure 4, we now consider the zero-probability case in the real world where the policy maker exactly knows GW parameters given in table 2. The non-linear functions of parameters providing optimal initial inflation, optimal initial output gap and optimal initial interest rate has a probability equal to one to face rounding errors. SCILAB default option for solving Riccati equations and optimal initial inflation, output gap and interest rate gives 7-decimal rounded numbers. Rounding errors implies that the impulse response functions of the super-inertial-rule equilibrium are such that the output gap and inflation start diverging in period $t = 8$ (figure 4).

4 Conclusion

In general, the result of Ramsey optimal policy is a proportional feedback rule which responds to auto-regressive shocks, where the sensitivity of the policy instruments to these auto-regressive shocks depends on the auto-correlation of these shocks (Chatelain and Ralf (2021)). It is not necessary to look for other representations of Ramsey policy rules. Seeking these alternative representations added a lot of confusion to the stabilization policy literature. Because there is always uncertainty on the parameter of policy transmission mechanism, stabilization policy aims to maintain the dynamic system of the policy targets as a locally stable equilibrium. This fundamental guideline of Ramsey optimal policy should be kept in mind.

References

- [1] Adolfson M., Laséen S., Lindé J., Svensson L. E. [2014], Monetary policy trade-offs in an estimated open-economy DSGE model, *Journal of Economic Dynamics and Control*, 42, 33-49.
- [2] Clarida R., Gali J., Gertler M. [1999], The science of monetary policy: a new Keynesian perspective, *Journal of economic literature*, 37(4), 1661-1707.
- [3] Chatelain J.B., Ralf K. [2019], A simple algorithm for solving Ramsey optimal policy with exogenous forcing variables, *Economics Bulletin*, 39(4), 2429-2440.
- [4] Chatelain J. B., Ralf K. [2020], The Welfare of Ramsey Optimal Policy facing Auto-regressive shocks, *Economics Bulletin*. 40(2), 1797-1803.
- [5] Chatelain J.B., Ralf K. [2021], Hopf bifurcation from new-Keynesian Taylor rule to Ramsey optimal policy, *Macroeconomic Dynamics*, 25(8), 2204-2236.
- [6] Chatelain J. B., Ralf K. [2022], Persistence-dependent Optimal Policy Rules, Research-gate working paper.
- [7] Chung H., Herbst E., Kiley M. T. [2015], Effective Monetary Policy Strategies in New Keynesian Models: A Reexamination, *NBER Macroeconomics Annual*, 29(1), 289-344.
- [8] Creel J., Hubert, P. [2015], Has inflation targeting changed the conduct of monetary policy? *Macroeconomic Dynamics*, 19(1), 1-21.
- [9] Gali J. [2015], *Monetary Policy, Inflation and the Business Cycle*, second edition, Princeton, Princeton University Press.
- [10] Giannoni M. P. [2001], Model Uncertainty and Optimal Monetary Policy, unpublished PhD thesis, Princeton University.
- [11] Giannoni M. P. [2014], Optimal interest-rate rules and inflation stabilization versus price-level stabilization, *Journal of Economic Dynamics and Control*, 41, 110-129.

- [12] Giannoni M. P., Woodford M. [2003], How forward-looking is optimal monetary policy?, *Journal of Money, Credit, and Banking*, 35(6), 1425-1469.
- [13] Hansen L.P., Sargent T. J. [2007], *Robustness*, Princeton University Press, Princeton.
- [14] Havránek T. [2015], Measuring intertemporal substitution: The importance of method choices and selective reporting, *Journal of the European Economic Association*, 13(6), 1180-1204.
- [15] Kara H. [2007], Monetary policy under imperfect commitment: Reconciling theory with evidence, *International Journal of Central Banking*, 3(1). 149-177.
- [16] Ljungqvist L., Sargent T.J. [2012], *Recursive Macroeconomic Theory*, Third edition, Cambridge (MA), MIT Press.
- [17] Schaumburg E., Tambalotti A. [2007], An investigation of the gains from commitment in monetary policy, *Journal of Monetary Economics*, 54(2), 302-324.
- [18] Simaan M., Cruz J. B. [1973], Additional aspects of the Stackelberg strategy in nonzero-sum games, *Journal of Optimization Theory and Applications*, 11(6), 613-626.
- [19] Villieu P. [2015], *Macroéconomie*, Paris, Economica.

Appendix

Proof of proposition 1

Step 1: The Hamiltonian system includes the new-Keynesian Phillips curve and the non-controllable dynamics of the cost-push shock. The first order condition on inflation is:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \pi_t + \phi_{\pi,t} - \phi_{\pi,t-1} = 0 \text{ for } t = 0, 1, 2, \dots \quad (18)$$

The marginal equation with respect to the single policy instrument (output gap) implies that the output gap is a linear increasing function of the Lagrange multiplier of inflation:

$$\frac{\partial \mathcal{L}}{\partial x_t} = 0 \Rightarrow \lambda_x x_t - \kappa \phi_{\pi,t} = 0 \text{ for } t = 0, 1, 2, \dots \Rightarrow x_t = \frac{\kappa}{\lambda_x} \phi_{\pi,t} \text{ or } \phi_{\pi,t} = \frac{\lambda_x}{\kappa} x_t. \quad (19)$$

With an initial natural boundary set to zero, the initial value of the policy instrument at -1 is also equal to zero:

$$\phi_{\pi,-1} = 0 \Rightarrow x_{-1} = -\frac{\kappa}{\lambda_x} \phi_{\pi,-1} = 0. \quad (20)$$

Step 1b: Super-inertial policy rule.

Using the above equation to substitute the Lagrange multiplier of inflation $\phi_{\pi,t}$ by the policy instrument x_t leads to the super-inertial rule:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow x_t = x_{t-1} - \frac{\kappa}{\lambda_x} \pi_t \text{ for } t = 0, 1, 2, 3, \dots \quad (21)$$

The policy instrument (the output gap) has non-stationary dynamics. The Hamiltonian can be written using the Lagrange multiplier dynamic Euler equation ($\frac{\partial \mathcal{L}}{\partial \pi_t} = 0$) or substituting the Lagrange multiplier by the policy instrument in the dynamic Euler equation. The Hamiltonian system is a dynamic system of order three. It includes the co-state (λ_x or equivalently x_t) Euler dynamic equation of order one besides the order-two dynamics of the state variables (π_t, u_t):

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\kappa}{\lambda_x} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} & -\frac{1}{\beta} \\ 0 & 1 & 0 \\ 0 & 0 & \rho_u \end{pmatrix} \begin{pmatrix} \pi_t \\ x_t \\ u_t \end{pmatrix} \quad (22)$$

$$\begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} & -\frac{1}{\beta} \\ -\frac{\kappa}{\beta \lambda_x} & \frac{\kappa^2}{\beta \lambda_x} + 1 & \frac{\kappa}{\beta \lambda_x} \\ 0 & 0 & \rho_u \end{pmatrix} \begin{pmatrix} \pi_t \\ x_t \\ u_t \end{pmatrix}.$$

The characteristic polynomial of the Hamiltonian matrix is:

$$\left(X^2 + \left(-\frac{1}{\beta} - \frac{\kappa^2}{\beta \lambda_x} - 1 \right) X + \frac{1}{\beta} \right) (X - \rho_u). \quad (23)$$

Two eigenvalues δ and ρ_u are inside the unit circle (for two predetermined variables $\phi_{\pi,t}$ and u_t) and the third one $\delta' = \frac{1}{\delta\beta}$ is *outside the unit circle*.

$$0 < \delta = \frac{1}{2} \left(1 + \frac{1}{\beta} + \frac{\kappa^2}{\beta \lambda_x} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta} + \frac{\kappa^2}{\beta \lambda_x} \right)^2 - \frac{1}{\beta}} < 1. \quad (24)$$

Step 2: Following Hansen and Sargent (2007, chapter 5), we have to derive the solution of the state-costate evolution equation that stabilizes the state-costate vector sequence *for any initial values of the policy maker's state variables* (π_0, z_0). The initial value of the costate takes the following form which is replicated over time:

$$\phi_{\pi,t} = P_{\pi\pi}\pi_t + P_{\pi u}u_t. \quad (25)$$

The elements of matrix \mathbf{P} can be found solving a Riccati equation and a Sylvester equation or a Riccati equation of higher order (Chatelain and Ralf (2020)). Using the relation between the costate and the policy instrument, it follows that this state-costate solution is implemented by the control law:

$$x_t = F_{\pi}\pi_t + F_u u_t. \quad (26)$$

We label this control law the *benchmark* representation of the policy rule where the current value of the policy instrument responds linearly to the current value of the state variables. It is a proportional negative-feedback rule. The super-inertial rule (co-state Euler dynamic equation of order one) is *substituted* by this *static* equation in the Hamiltonian system, in order to force dynamics to remain within the stable subspace of the Hamiltonian system. By so doing, the order of the dynamics decreases by a order equal to one. The order of the dynamics of the Hamiltonian system is *reduced to two*, including *only the two*

eigenvalues inside the unit circle: δ and ρ_u and excluding the non-stationary dynamics related to the remaining eigenvalue outside the unit circle of the Hamiltonian:

$$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} \delta = \frac{1}{\beta} - \frac{\kappa}{\beta} F_\pi & -\frac{1}{\beta} - \frac{\kappa}{\beta} F_u \\ 0 & \rho_u \end{pmatrix}^t \begin{pmatrix} \pi_0 \\ u_0 \end{pmatrix}. \quad (27)$$

This is the usual way to solve the linear quadratic regulator (Ljungqvist and Sargent (2012)). Solving a Riccati equation leads to the following unique values of optimal parameters for the new-Keynesian Phillips curve (Chatelain and Ralf 2019):

$$F_\pi = \frac{\kappa}{\lambda_x} \frac{\delta}{1 - \delta} \text{ and } F_u = \frac{-1}{1 - \beta \rho_u \delta} F_\pi. \quad (28)$$

Although there is a smoothing quadratic cost of changing the policy instrument, this benchmark representation does not include an auto-regressive term for the policy instrument, so it may not be called “inertial” but “benchmark optimal rule”. An increase in the cost of changing the policy instrument λ_x increases the persistence of inflation $\delta(\lambda_x)$ and reduces the absolute value of the policy rule parameter $|F_\pi|$. The control law is a counter example which shows that a policy-instrument-smoothing quadratic term in the loss function *does not imply* an auto-regressive term in the policy rule of Ramsey optimal policy.

Step 3: The optimal initial value of the policy target and the policy instrument are obtained using the initial date transversality condition $\phi_{\pi,-1} = 0$ and results from the solution of the Riccati equation:

$$\phi_{\pi,0} = P_{\pi\pi}\pi_0 + P_{\pi u}u_0 = 0 \Rightarrow \pi_0 = -P_{\pi\pi}^{-1}P_{\pi u}u_0 = \frac{\delta}{1 - \delta\beta\rho_u}u_0 \text{ and} \quad (29)$$

$$x_0 = -\frac{\kappa}{\lambda_x}\pi_0 = -\frac{\delta}{1 - \delta\beta\rho_u} \frac{\kappa}{\lambda_x}u_0. \quad (30)$$

Knowing these initial conditions and the matrix solution of the Riccati equation, one can compute welfare (Chatelain and Ralf (2020)).

Step 4: One substitutes inflation by a lag of the policy instrument in order to obtain an observationally equivalent “inertial” policy rule to the benchmark policy rule of step 2. The proof uses the step 2 policy rule, $\pi_t = \frac{1}{F_\pi}x_t - Au_t$ with $A = \frac{-1}{1 - \beta\rho_u\delta}$, and substitutes inflation by the output gap in the transmission mechanism of step 2. One has the equivalence between the system of step 2 and the system of step 4 with Clarida, Gali and Gertler’s (1999) inertial

rule, for any initialization of the policy maker's state variables (π_0, z_0) :

$$\Leftrightarrow \begin{cases} \begin{pmatrix} u_{t+1} \\ \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ (1-\rho)A\delta & \delta \end{pmatrix} \begin{pmatrix} u_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \\ x_t = F_\pi \pi_t + AF_\pi u_t \text{ so that:} \\ \begin{pmatrix} u_t \\ x_t \end{pmatrix} = \mathbf{N} \begin{pmatrix} u_t \\ \pi_t \end{pmatrix} \text{ with } \mathbf{N} = \begin{pmatrix} 1 & 0 \\ AF_\pi & F_\pi \end{pmatrix} \\ \begin{pmatrix} u_{t+1} \\ x_{t+1} \end{pmatrix} = \mathbf{N}^{-1}(\mathbf{A} + \mathbf{BF})\mathbf{N} \begin{pmatrix} u_t \\ x_t \end{pmatrix} + \mathbf{N}^{-1} \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \\ \pi_t = \frac{1}{F_\pi}x_t - Au_t \text{ so that:} \\ \begin{pmatrix} u_t \\ \pi_t \end{pmatrix} = \mathbf{N}^{-1} \begin{pmatrix} u_t \\ x_t \end{pmatrix}. \end{cases} \quad (31)$$

One has:

$$\mathbf{N}^{-1}(\mathbf{A} + \mathbf{BF})\mathbf{N} = \begin{pmatrix} \rho_u & 0 \\ (1-\delta)F_\pi A\rho_u & \delta \end{pmatrix}, \quad (32)$$

where the bottom row of the matrix provides a representation of inertial the policy rule (Clarida, Gali and Gertler (1999) and Gali (2015)):

$$x_t = \delta x_{t-1} - \frac{\delta}{1-\delta\beta\rho_u} \frac{\kappa}{\lambda_x} \rho_u u_{t-1} \text{ or } x_t = \delta x_{t-1} - \frac{\delta}{1-\delta\beta\rho_u} \frac{\kappa}{\lambda_x} u_t \text{ for } t = 1, 2, \dots \quad (33)$$

The representation of the inertial optimal policy rule for the output gap has an autoregressive component δ which is strictly lower than one. It also depends on the auto-correlation of the cost push shock ρ_u and on the shock u_t .

Because of the exact equivalence of the two dynamic systems, with identical optimal initial conditions, the paths predicted for inflation and the output gap for a given path of the cost-push shock are identical. The system of step 2 equations is equivalent to the system of step 4 equations for expected values, with dynamics of order two:

$$\begin{cases} x_t = F_\pi \pi_t + AF_\pi u_t \\ \begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} \delta = \frac{1}{\beta} - \frac{\kappa}{\beta} F_\pi & -\frac{1}{\beta} - \frac{\kappa}{\beta} F_u \\ 0 & \rho \end{pmatrix}^t \begin{pmatrix} \frac{\delta}{1-\delta\beta\rho_u} \\ 1 \end{pmatrix} u_0 \end{cases} \quad (34)$$

$$\Leftrightarrow \begin{cases} \pi_t = \frac{1}{F_\pi}x_t - Au_t \\ \begin{pmatrix} x_t \\ u_t \end{pmatrix} = \begin{pmatrix} \delta & -\frac{\delta}{1-\delta\beta\rho_u} \frac{\kappa}{\lambda_x} \\ 0 & \rho \end{pmatrix}^t \begin{pmatrix} -\frac{\kappa}{\lambda_x} \frac{\delta}{1-\delta\beta\rho_u} \\ 1 \end{pmatrix} u_0. \end{cases} \quad (35)$$

The two eigenvalues (δ, ρ) are inside the unit circle. The order two dynamics is stable.

Step 1B equation for expected values with step 3 optimal initial conditions is:

$$\begin{pmatrix} \pi_t \\ x_t \\ u_t \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} & -\frac{1}{\beta} \\ -\frac{\kappa}{\beta\lambda_x} & \frac{\kappa^2}{\beta\lambda_x} + 1 & \frac{\kappa}{\beta\lambda_x} \\ 0 & 0 & \rho_u \end{pmatrix}^t \begin{pmatrix} \frac{\delta}{1-\delta\beta\rho_u} \\ -\frac{\kappa}{\lambda_x} \frac{\delta}{1-\delta\beta\rho_u} \\ 1 \end{pmatrix} \varepsilon_{u,0}.$$

Its dynamics is related to *intermediate computation dynamics* (step 1 with state-costate

evolution of the Hamiltonian system) of a higher order (three) than the order (two) of the dynamics of the policy maker's state variables. Two eigenvalues δ and ρ_u are inside the unit circle and **one eigenvalue is outside the unit circle** $\frac{1}{\delta\beta}$. *A non-stationary "super-inertial" policy rule destabilizes the dynamic system obtained for Ramsey optimal policy.*

Simulations

We replicate the Ramsey optimal policy impulse response function using Gali's (2015) calibration: $\beta = 0.99$, $\kappa = 0.1275$, $\rho_u = 0.8$. The impulse response function is the same as the one for the super-inertial rule (figure 1). In the linear proportional rule, the parameter of the response of the output gap to inflation is $F_\pi = 4.51$ and the value of the related root is $\delta = \frac{1}{\beta} - \frac{\kappa}{\beta}F_\pi = 0.429$.

We now consider a change in the persistence of the cost-push shock which is erroneously believed to be $\rho = 0.9$.

Table 4:

ρ_u	0.8	0.9
$\pi_0 = \frac{\delta}{1-\delta\beta\rho_u}$	0.650	0.694
$x_0 = -6.\pi_0$	-3.900	-4.169
$-\frac{1}{\beta} - \frac{\kappa}{\beta}F_u$	-0.13003	-0.0694
F_u	-6.8335	-7.301

For the super-inertial rule, there is only a change of the optimal initial conditions, which are erroneous if the policy maker believes it is $\rho_u = 0.9$ whereas the true model is $\rho_u = 0.8$.

$$\begin{pmatrix} \pi_t \\ x_t \\ u_t \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} & -\frac{1}{\beta} \\ -\frac{\kappa}{\beta\lambda_x} & \frac{\kappa^2}{\beta\lambda_x} + 1 & \frac{\kappa}{\beta\lambda_x} \\ 0 & 0 & 0.8 \end{pmatrix}^t \begin{pmatrix} \frac{\delta}{1-\delta\beta.(0.9)} \\ -\frac{\kappa}{\lambda_x} \frac{\delta}{1-\delta\beta.(0.9)} \\ 1 \end{pmatrix} \varepsilon_{u,0}.$$

For Ramsey optimal policy there is a change of initial conditions (which is erroneous) and a change of the sensitivity of future inflation to the cost-push shock in the transition matrix (which decreases if the persistence ρ_u increases):

$$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} x_t = F_\pi\pi_t + F_u(0.9)u_t \\ \delta = \frac{1}{\beta} - \frac{\kappa}{\beta}F_\pi & -\frac{1}{\beta} - \frac{\kappa}{\beta}F_u(0.9) \\ 0 & \rho = 0.8 \end{pmatrix}^t \begin{pmatrix} \frac{\delta}{1-\delta\beta(0.9)} \\ 1 \end{pmatrix} u_0.$$

Proof of proposition 2

Denoting Lagrangian multipliers $\phi_{x,t}$ for the consumption Euler equation, $\phi_{\pi,t}$ for the new-Keynesian Phillips curve, the Lagrangian \mathcal{L} is:

$$\begin{aligned}
\mathcal{L} &= -E_0 \sum_{t=0}^{+\infty} \beta^t \left\{ \frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} + \lambda_i \frac{i_t^2}{2} + \beta \phi_{x,t+1} [x_t + \sigma (i_t - \pi_{t+1}) - x_{t+1}] \right. \\
&\quad \left. + \beta \phi_{\pi,t+1} [\pi_t - \kappa x_t - \beta \pi_{t+1}] \right\} \\
&= -E_0 \sum_{t=0}^{+\infty} \beta^t \left\{ \frac{\pi_t^2}{2} + \lambda_x \frac{x_t^2}{2} + \lambda_i \frac{i_t^2}{2} + \beta \phi_{x,t+1} \left[\left(1 + \frac{\sigma \kappa}{\beta}\right) x_t - \frac{\sigma}{\beta} \pi_t + \sigma i_t - x_{t+1} \right] \right. \\
&\quad \left. + \beta \phi_{\pi,t+1} \left[-\frac{\kappa}{\beta} x_t + \frac{1}{\beta} \pi_t - \pi_{t+1} \right] \right\}.
\end{aligned}$$

Usual matrix notations for the linear quadratic regulator are:

$$\begin{aligned}
Q &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda_x \end{pmatrix}, R = \lambda_i \\
\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} &= \underbrace{\begin{pmatrix} 1 + \frac{\sigma \kappa}{\beta} & -\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix}}_{=\mathbf{A}_{yy}} \underbrace{\begin{pmatrix} x_t \\ \pi_t \end{pmatrix}}_{=\mathbf{y}_t} + \underbrace{\begin{pmatrix} \sigma \\ 0 \end{pmatrix}}_{=\mathbf{B}_y} i_t + \underbrace{\begin{pmatrix} -1 & \frac{\sigma}{\beta} \\ 0 & -\frac{1}{\beta} \end{pmatrix}}_{=\mathbf{A}_{yz}} \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{=\mathbf{z}_t}.
\end{aligned}$$

Step 1: Euler equations and Hamiltonian system:

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \lambda_i i_t + \sigma \beta \phi_{x,t+1} = 0 \Rightarrow i_t = -\frac{\sigma}{\lambda_i} \beta \phi_{x,t+1}.$$

We substitute the funds rate by the Lagrange multiplier for both equations of the transmission mechanism using the following matrix $\begin{pmatrix} \frac{\beta \sigma^2}{\lambda_i} & 0 \\ 0 & 0 \end{pmatrix}$.

The other first order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_t} = 0 &\Rightarrow \lambda_x x_t + \phi_{x,t} - \beta^{-1} \phi_{x,t-1} - \kappa \phi_{\pi,t} = 0 \\
\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 &\Rightarrow \pi_t - \beta^{-1} \sigma \phi_{x,t-1} + \phi_{\pi,t} - \phi_{\pi,t-1} = 0.
\end{aligned}$$

In matrix form:

$$\begin{aligned}
\beta \begin{pmatrix} 1 + \frac{\sigma \kappa}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{\sigma}{\beta} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \phi_{x,t+1} \\ \phi_{\pi,t+1} \end{pmatrix} &= - \begin{pmatrix} \lambda_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} \phi_{x,t} \\ \phi_{\pi,t} \end{pmatrix} \\
\begin{pmatrix} \phi_{x,t+1} \\ \phi_{\pi,t+1} \end{pmatrix} &= - \begin{pmatrix} \frac{1}{\beta} & \frac{\kappa}{\beta} \\ \frac{\sigma}{\beta} & \frac{1}{\beta} (\beta + \kappa \sigma) \end{pmatrix} \begin{pmatrix} \lambda_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} \frac{1}{\beta} & \frac{\kappa}{\beta} \\ \frac{\sigma}{\beta} & \frac{1}{\beta} (\beta + \kappa \sigma) \end{pmatrix} \begin{pmatrix} \phi_{x,t} \\ \phi_{\pi,t} \end{pmatrix}
\end{aligned}$$

We form the Hamiltonian matrix which is a particular case of the textbook matrix of the Hamiltonian system of a linear quadratic regulator.

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ \phi_{x,t+1} \\ \phi_{\pi,t+1} \\ v_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} 1 + \kappa \frac{\sigma}{\beta} + \frac{\sigma^2}{\lambda_i} \lambda_x & -\frac{\sigma}{\beta} + \kappa \frac{\sigma^2}{\lambda_i} & -\frac{\sigma^2}{\lambda_i} & -\kappa \frac{\sigma^2}{\lambda_i} & -1 & \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 & 0 & 0 & -\frac{1}{\beta} \\ -\frac{1}{\beta} \lambda_x & -\frac{\kappa}{\beta} & \frac{1}{\beta} & \frac{\kappa}{\beta} & 0 & 0 \\ -\frac{\sigma}{\beta} \lambda_x & -\left(1 + \frac{\kappa \sigma}{\beta}\right) & \frac{\sigma}{\beta} & 1 + \kappa \frac{\sigma}{\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_v & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_u \end{pmatrix}^{t+1} \begin{pmatrix} x_0^*(\rho_u) \\ \pi_0^*(\rho_u) \\ \phi_{x,0} = 0 \\ \phi_{\pi,0} = 0 \\ v_0 \\ u_0 \end{pmatrix}.$$

Giannoni's (2000) matrix M is the first sub-matrix with the intersection of the first four lines with the first four columns. The product of the four eigenvalues of M is its determinant equal to $\frac{1}{\beta^2}$. Such a discounted symplectic matrix has two eigenvalues (δ_1, δ_2) inside the unit circle and two "mirror" eigenvalue outside the unit circle $(1/\beta\delta_1, 1/\beta\delta_2)$. The two other eigenvalues inside the unit circle are the auto-regressive components of the shocks (δ_1, δ_2) .

Giannoni (2000, p.35) mentions that this matrix has the same eigenvalues that the matrix he constructed after substitution of the Lagrange multipliers $\phi_{x,t}$ and $\phi_{\pi,t}$ by the funds rate and its two lags. This means that he has constructed a mathematically equivalent system of equations to the above Hamiltonian system of equations. He finds optimal initial conditions (x_0^*, π_0^*) , Giannoni (2000, p.35) which correspond to step 3. However, he does not perform step 2, so that its impulse response functions are determined using optimal initial conditions (x_0^*, π_0^*) and the Hamiltonian system of equations which includes two roots outside the unit circle.

In Step 2, one solves a matrix Riccati equation to find matrix P and then the policy rule parameters F which depend on preferences and parameters of the transmission mechanism:

$$\begin{aligned} \phi_{x,t} &= P_{xx}x_t + P_{x\pi}\pi_t + P_{xv}v_t + P_{xu}u_t \\ \phi_{\pi,t} &= P_{\pi x}x_t + P_{\pi\pi}\pi_t + P_{\pi v}v_t + P_{\pi u}u_t \\ i_t &= F_x x_t + F_\pi \pi_t + F_v v_t + F_u u_t. \end{aligned}$$

In step 3, Optimal initial conditions are obtained using $\phi_{x,t} = \phi_{\pi,t} = 0$:

$$\begin{pmatrix} x_0^* \\ \pi_0^* \end{pmatrix} = - \begin{pmatrix} P_{xx} & P_{x\pi} \\ P_{\pi x} & P_{\pi\pi} \end{pmatrix}^{-1} \begin{pmatrix} P_{xv} & P_{xu} \\ P_{\pi v} & P_{\pi u} \end{pmatrix} \begin{pmatrix} v_0 \\ u_0 \end{pmatrix}, \quad (36)$$

so that optimal impulse response functions of Ramsey optimal policy are given by a system which has four eigenvalues inside the unit circle:

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ v_{t+1} \\ u_{t+1} \end{pmatrix} = \left(\begin{pmatrix} 1 + \frac{\sigma\kappa}{\beta} & -\frac{\sigma}{\beta} & -1 & \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 & -\frac{1}{\beta} \\ 0 & 0 & \rho_v & 0 \\ 0 & 0 & 0 & \rho_u \end{pmatrix} + \begin{pmatrix} \sigma \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} F_x & F_\pi & F_v & F_u \end{pmatrix} \right) \begin{pmatrix} x_t \\ \pi_t \\ v_t \\ u_t \end{pmatrix} \\
\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ v_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \sigma F_x + \frac{\kappa}{\beta} \sigma + 1 & \sigma F_\pi - \frac{1}{\beta} \sigma & \sigma F_v (\rho_u) - 1 & \frac{1}{\beta} \sigma + \sigma F_u (\rho_u) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 & -\frac{1}{\beta} \\ 0 & 0 & \rho_v & 0 \\ 0 & 0 & 0 & \rho_u \end{pmatrix}^{t+1} \begin{pmatrix} x_0^* (\rho_u) \\ \pi_0^* (\rho_u) \\ v_0 \\ u_0 \end{pmatrix}.$$

We highlighted which variables depends on ρ_u , which is the parameter which is not exactly known in our example.

In step 4, we apply the operator $p(L)$ on all terms of the the proportional policy rule. The auto-regressive components of the shocks v_t and u_t are substituted by the two first lags of the policy instrument: i_{t-1} and i_{t-2} :

$$\begin{aligned}
(1 - \rho_u L)(1 - \rho_v L) i_t &= (1 - (\rho_u + \rho_v) L + (\rho_v \rho_u) L^2) i_t = \\
i_t - (\rho_u + \rho_v) i_t + (\rho_v \rho_u) i_{t-2} &= F_x p(L) x_t + F_\pi p(L) \pi_t + F_z (1 - \rho_u L) \varepsilon_{v,t} + F_u (1 - \rho_v L) \varepsilon_{u,t}.
\end{aligned} \tag{37}$$

The sum of the parameters of the two lags of the dependent variables is the sum S less the product P of the two auto-correlation coefficients of shocks:

$$i_t = (\rho_u + \rho_v) i_t - (\rho_v \rho_u) i_{t-2} + F_x p(L) x_t + F_\pi p(L) \pi_t + F_z (1 - \rho_u L) \varepsilon_{v,t} + F_u (1 - \rho_v L) \varepsilon_{u,t}. \tag{38}$$

Because the two auto-correlation coefficients are between zero and one and are the roots of the polynomial p , $p(1) > 0$. Hence, the sum of the two parameters of the lags of the interest rate is strictly *below* one:

$$p(1) = 1 - S + P > 0 \Rightarrow S - P < 1. \tag{39}$$

Hence, this observationally equivalent representation of the benchmark optimal policy rule is inertial. It also depends on the auto-correlation of shocks.

Simulations of the order four model of the transmission mechanism

We replicate Ramsey optimal policy impulse response functions using GW's calibration. In the linear proportional rule, the parameter of the response of the output gap to inflation is $F_\pi = 1.71, F_x = -0.18, F_v = 0.15$. Table 4 lists changes if the persistence of the cost-push shock is erroneously believed to be $\rho_u = 0.4$.

Table 5: Changes of initial condition and sensitivity of the output gap to the cost-push shock:

ρ_u	0.35	0.4
x_0^*	-0.465	-0.455
π_0^*	1.315	1.405
$\frac{\sigma}{\beta} + \sigma F_u$	-6.922	-7.447
F_u	-2.098	-2.180

For the super-inertial rule, there is a change on the initial conditions x_0^* and π_0^* , which are erroneous if the policy maker believes it is $\rho_u = 0.4$ whereas the true model is $\rho_u = 0.35$.

For Ramsey optimal policy there is a change of the initial condition (which is erroneous) and a change of the sensitivity of the future output gap to the cost-push shock in the transition matrix.

Scilab Code

```

R=0.236;
Qpi=1; Qx =0.048; Qv=0 ; Qu=0 ;
beta1=0.99; gamma1=1/0.1571; kappa=0.0238;
rho1=0.35; rho2=0.4;
Ay=[1+(kappa*gamma1/beta1) -gamma1/beta1 ; -kappa/beta1 1/beta1] ;
Ayz=[ -1 gamma1/beta1 ; 0 -1/beta1 ]
Az=[rho1 0 ; 0 rho2 ];
Azy=zeros(Az);
A1=[ Ay Ayz ; Azy Az]
B1=[gamma1 ; 0 ; 0; 0]
A=sqrt(beta1)*A1
B=sqrt(beta1)*B1
q=[ Qx Qpi Qv Qu ]
Q=diag(q) ;
Big=sysdiag(Q,R)
[w,wp]=fullrf(Big)
C1=wp(:,1:4)
D12=wp(:,5:$)
[C1,D12]'*[C1,D12]
M=syslin('d',A,B,C1,D12)
[F,P]=lqr(M)
spec(A+B*F)
A1+B1*F
A1
Pyy=P(1:2, 1:2)
Pyz=P(1:2, 3:4)
v0=0
u0=1
z0=[v0; u0]
N=-inv(Pyy)*Pyz
Nz0=N*z0

```


$$yz0=[\text{ Nz0 ; z0}]$$

$$\text{Loss}=yz0'\text{P}yz0$$

Figure 1: Impulse response functions of inflation and the output gap, for 7 periods following a persistent cost-push shock of 1 unit ($\rho_u=0.8$ versus error $\rho_u=0.9$) for the new-Keynesian Phillips-curve model.

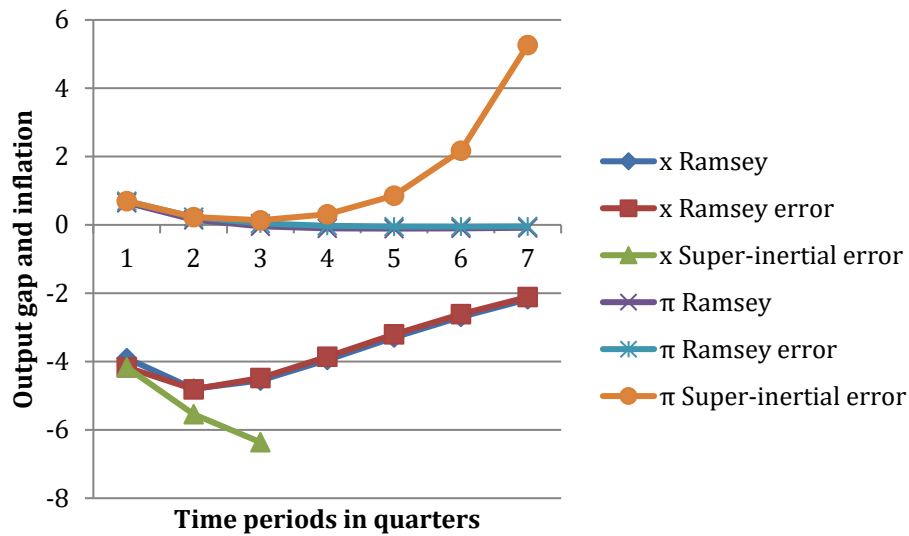


Figure 2: Super-inertial impulse response function, for 44 periods following a shock of 1 unit, with the exact knowledge of persistence $\rho=0.8$ (Gali 2015, chapter 5), with 16-decimal rounding errors on endogenous optimal initial inflation and optimal initial output gap.

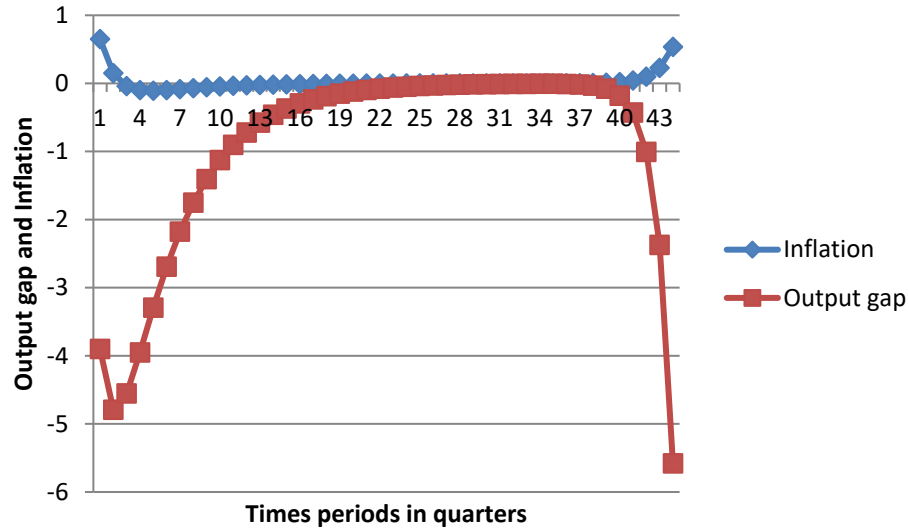


Figure 3: Paths of the funds rate, the output gap and inflation, for seven periods following a cost-push shock of one unit for the model with a new-Keynesian Phillips curve and an Euler consumption equation ($\rho_u=0.35$ versus error $\rho_u=0.4$).

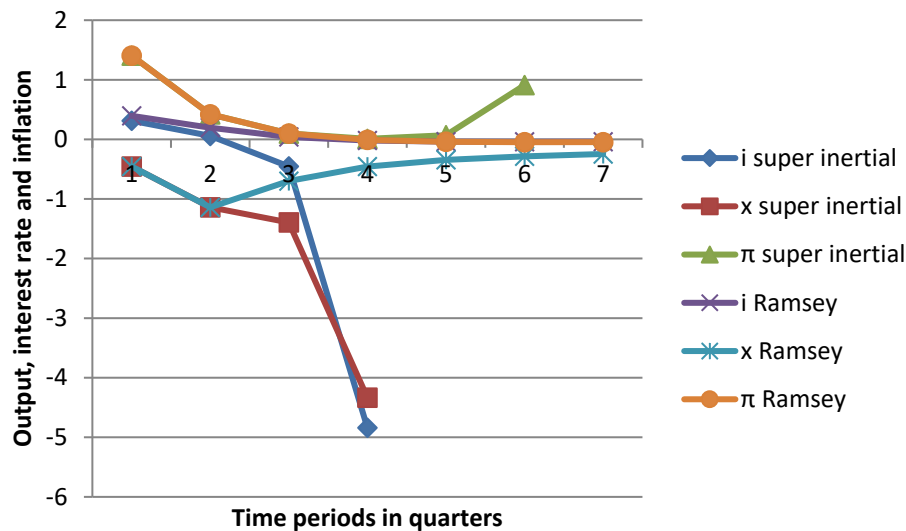


Figure 4: Super-inertial policy rule impulse-response functions, for 9 periods following a shock of 1 unit, with the exact knowledge of persistence $\rho=0.35$ for the model including the new-Keynesian Phillips curve and the Euler consumption equation, with 7-decimal rounding errors on endogenous optimal initial inflation, optimal initial inflation and optimal initial interest rate.

