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The market price of jump risk for delivery periods: pricing of electricity swaps with geometric averaging

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Abstract
In this paper, we extend the *market price of risk for delivery periods* (MPDP) of electricity swap contracts by introducing a dimension for jump risk. As introduced by Kemper et al. [30], the MPDP arises through the use of geometric averaging while pricing electricity swaps in a geometric framework. We adjust the work by Kemper et al. [30] in two directions: First, we examine a Merton type model taking jumps into account. Second, we transfer the model to the physical measure by implementing mean-reverting behavior. We compare swap prices resulting from the arithmetic (approximated) average to the geometric weighted average. Under the physical measure, we discover a decomposition of the swap's market price of risk into the instantaneous market price of risk and the MPDP.

Keywords Electricity swaps · Delivery period · MPDP for diffusion and jump risk · Mean-reversion · Jumps · Samuelson effect · Seasonality

JEL Classification G130 · Q400

1 Introduction

With the turn of the millennium, pricing derivatives on electricity has become important through the liberalization of energy markets. Nowadays, new challenges appear due to the transition to a climate neutral energy system: Electricity generated from renewable energy sources, like wind and solar energy, clearly depends on the weather conditions of the season. Consequently, a rising share of renewable energy induces stronger intermittency and seasonality effects influencing especially delivery-dependent pricing effects. In electricity markets, such delivery-dependent futures contracts are the most important derivatives. They deliver the underlying over a period of time since electricity is not storable on a large scale.

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We therefore call them electricity *swaps*. The dependence on the delivery time affects the price dynamics, the pricing measure, and the swap's market price of risk for delivery periods (MPDP) introduced by Kemper et al. [30]. In this paper, we provide an extension of the MPDP. To do so, we adjust the model to a Merton type model taking jumps into account. In addition, under the physical measure, we identify a decomposition of the swap's market price of risk into the instantaneous market price of risk and the MPDP.

The delivery period is a unique feature of electricity markets that differs from other commodities such as oil, gas, or corn. In fact, it plays a crucial role in the pricing of electricity swaps. Following the market model approach, the electricity swap price results from averaging an instantaneous stream of futures with respect to the delivery time. This approach goes back to the famous model by Heath et al. [23]. It was firstly connected to energy-related derivatives by Clewlow and Strickland [15] and to electricity derivatives by Bjerksund et al. [9] followed by a row of works (see, e.g., Koekebakker and Ollmar [34], Benth and Koekebakker [5], Bjerksund et al. [9], Benth et al. [1], and Kemper et al. [30] for geometric settings, Hinderks et al. [25] for a structural model and Cuchiero et al. [17] measure-valued processes). One stream of literature investigates spot based price dynamics and derive electricity futures based on the spot price referring to the day ahead market (see, e.g., Cartea and Figueroa, [13], Cartea and Villaplana [14], Escribano et al. [19]). In this paper, we focus on a HJM-type approach modelling the futures market directly. That is we consider so-called *atomic* swap contracts inducing a delivery period of a month, that are used to price overlapping swap contracts delivering for example over a quarter or a year. We refer to Benth et al. [1] and Benth and Koekebakker [5] and Kemper et al. [30] for a construction of overlapping swap contracts based on atomic swaps.

The delivery period can be incorporated in different ways of averaging. We distinguish between three types of averaging: Arithmetic, approximated, and geometric averaging. *Arithmetic averaging* is the classical way to implement the swap's delivery period and is convenient for arithmetic price dynamics. In particular, continuous arithmetic averaging is applied by Benth et al. [3], Benth and Koekebakker [5] Benth et al. [1], Benth et al. [4], Benth et al. [7], Kleisinger-Yu et al. [32], Koekebakker and Ollmar [34], and Latini et al. [35], among others. For discrete arithmetic averaging, we refer to Lucia and Schwartz [37] and Burger et al. [12]. Instead, arithmetic averaging of geometric price dynamics is poorly suited since the resulting swap price dynamics are neither geometric nor Markovian. It requires, e.g., an approximation of the swap price volatility introduced by Bjerksund et al. [9] whenever we want to consider tractable swap price dynamics (see also Benth et al. [1], Benth and Koekebakker [5]). We call this procedure *approximated averaging*. *Geometric averaging*, instead, does not require any approximations whenever the price dynamics are of geometric type and lead to suitable geometric dynamics (see Kemper et al. [30]). Hence, the geometric average is tailor-made for relative growth rate models. Nevertheless, the geometric average does not preserve the martingale property. This issue is tackled by Kemper et al. [30] using a measure change with their MPDP. Usually, negative prices are not observable in the data of the futures prices, such that we stick to a geometric setting and compare the latter averaging procedures while adjusting the MPDP to a Merton type model.

Both papers, Kemper et al. [30] and Bjerksund et al. [9], investigate the modeling of the delivery period explicitly through a continuous weighted averaging approach for geometric futures prices. Both approaches lead to Markovian and geometric swap price dynamics. We discuss similarities and differences between these approaches and introduce a numéraire caused by the different averaging techniques in Sect. 2. In line with the market model approach, we base the averaging procedure on a continuous stream of futures contracts that is a martingale under the futures risk-neutral measure $\mathbb{Q}$. As the futures have instantaneous

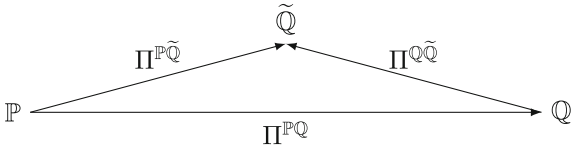

Fig. 1 Measure changes between the physical measure $\mathbb{P}$, the instantaneous risk-neutral measure $\mathbb{Q}$, and the swap's pricing measure $\widetilde{\mathbb{Q}}$ as well as their connections with the swap's market prices of risk $\Pi^{\mathbb{P}\mathbb{Q}}$, the instantaneous market price of risk $\Pi^{\mathbb{P}\widetilde{\mathbb{Q}}}$, and the MPDP denoted by $\Pi^{\mathbb{Q}\widetilde{\mathbb{Q}}}$

delivery, we refer to $\mathbb{Q}$ as the instantaneous risk neutral measure. The resulting swap price dynamics based on geometric averaging are not a martingale under $\mathbb{Q}$. We then define the MPDP of diffusion and jump risk and a new pricing measure $\widetilde{\mathbb{Q}}$, which can thus be used to price derivatives on the swap. We may refer to $\widetilde{\mathbb{Q}}$ as the "swap's" risk-neutral measure since the swap price is a $\widetilde{\mathbb{Q}}$-martingale without any approximations.

Therefore, we call $\mathbb{Q}$ also the "instantaneous" risk-neutral measure. It is a clear advantage that the approximated average preserves the martingale property of the swap under the measure $\mathbb{Q}$. A decomposition of the market price of risk for electricity swaps arises when turning to the physical measure $\mathbb{P}$. Figure 1 gives an overview over the connections between the different measures $\mathbb{P}$, $\mathbb{Q}$, and $\widetilde{\mathbb{Q}}$ and the swap's market price of risk $\Pi^{\mathbb{P}\mathbb{Q}}$, the instantaneous market price of risk $\Pi^{\mathbb{P}\widetilde{\mathbb{Q}}}$, and the MPDP denoted by $\Pi^{\mathbb{Q}\widetilde{\mathbb{Q}}}$.

Indeed, the MPDP is triggered by typical features of the electricity market entering the swap's volatility. In particular, delivery-dependent effects like seasonalities and term-structure effects play a crucial role. Fanelli and Schmeck [20] empirically identify seasonalities in the swap's delivery period by considering implied volatilities of electricity options. Renewable energy, like wind and solar energy, intensify especially the seasonal effects mentioned before.

An additional property of electricity and commodity markets is the Samuelson effect (see Samuelson [43]): The closer we reach the end of the maturity, the more effect the volatility has. Benth and Paraschiv [6] and Jaeck and Lautier [27] provide empirical evidence for the Samuelson effect in the volatility term-structure of electricity swaps. It can also be observed in the implied volatilities of electricity options, especially far out and in the money (see Kiesel et al. [31]). Kemper et al. [30] characterize the MPDP for such seasonalities and term-structure effects within a stochastic volatility model through the variance per unit of expectation of the delivery-dependent effects. We contribute to the literature by investigating the MPDP analytically, affected by seasonalities and the Samuelson effect. Moreover, we lay the foundation for the empirical analysis of the MPDP by specifying the model under the real world measure $\mathbb{P}$.

Further characteristics of the observerd electricity swap prices are mean-reversion and jump behavior. As mentioned by Latini et al. [35] and Kleisinger-Yu et al. [32] among others, mean-reversion is an important property of the electricity swap prices. Koekebakker and Ollmar [34] empirically validate that the short-term price varies around the long-term price, which confirms mean-reverting behavior. As Benth et al. [7], we face the problem of changing a mean-reverting process to the risk-neutral measure. We extend their measure change to the geometric setting. We even provide a proof for stochastic volatility settings in order to address models such as Kemper et al. [30] and Schneider and Tavin [44]. Besides mean-reversion, Benth et al. [7] include jumps as an outstanding characteristic of electricity prices. They consider compound Poisson processes under the physical measure in a mean-reverting, arithmetic setting. While adjusting the paper by Kemper et al. [30] to jumps, we

establish the *MPDP of jump risk* whenever the jump coefficient relies on delivery-dependent effects.

In this paper, we follow a so-called Heath-Jarrow-Morton approach to model forward markets, that is we define the swap with delivery period as average over an infinite dimensional stream of futures with instantaneous delivery. Note that these futures are not traded at the market, only the swaps with delivery period are traded. In particular, it is not possible to observe traded quotes of the futures and it is not possible to see if the spot price converges to the futures if time approaches maturity of a futures. In fact, this relationship typically does not hold true for the traded swaps either due to the delivery period (see e.g. Benth et al. [1]). Another approach to model the forward market is to start with a model for the spot and define the swap price as conditional expectation of the average spot during the delivery period, where the expectation is taken under some probability measure that is equivalent to the real world measure $\mathbb{P}$. Note that it is not necessary to have a martingale measure for the spot: as the underlying electricity is not storable on a large scale, it has to be consumed once purchased. In particular, it is not possible to set up buy- and hold-strategies, that are required in no-arbitrage portfolios. In this sense, the electricity spot is said to be "not tradable" (see e.g. Benth et al. [1]).

Our contribution to the literature is twofold: First, we adjust the paper by Kemper et al. [30] to the jump case under the instantaneous risk-neutral measure leading to an extended characterization of the MPDP regarding diffusion *and* jump risk. Second, we transfer the model to the physical measure $\mathbb{P}$. Under $\mathbb{P}$ we compare the swap prices resulting from geometric and approximated averaging as well as their risk-neutral measures revealing the decomposition of the swap's market price of risk into the instantaneous market price of risk and the MPDP. Consequently, the model lays the foundation for empirical investigations in the future.

The paper is organized as follows: Sect. 2 presents the geometric averaging approach under the instantaneous risk-neutral measure applied to the jump-type futures curve. In addition, it presents the MPDP of diffusion and jump risk. Section 4 introduces the model under the physical measure and identifies the decomposition of the swap's market price of risk. An example under the physical measure closes the section.

Finally, Sect. 5 concludes our main findings.

2 On the MPDP of diffusion and jump risk

We particularly focus on an electricity swap contract delivering 1 MWh of electricity during the agreed delivery period $(\tau_1, \tau_2]$. At a trading day $t \leq \tau_1$ before the contract expires, we denote the swap price by $F(t, \tau_1, \tau_2)$ settled such that the contract is entered at no cost. It can be interpreted as an average price of instantaneous delivery. Motivated by this interpretation, we consider a futures contract with price $f(t, \tau)$ that stands for instantaneous delivery at time $\tau \in (\tau_1, \tau_2]$. Note that such a contract does not exist on the market but it turns out to be useful for modeling purposes when considering delivery periods (see, e.g., Benth et al. [7] and Kemper et al. [30]).

Following the approach by Heath et al. [23], we derive the price of an electricity swap contract based on an instantaneous futures price model. More precisely, we compare two types of swap prices resulting from geometric and approximated averaging. The goal of this section is to investigate the pricing spread between both approaches in order to quantify the

consequences of the approximation and thus the effect of the precise geometric averaging procedure. As the pricing spread goes along with different risk-neutral measures, we additionally investigate the distance of both risk-neutral measures quantified by the MPDP. Moreover, we characterize the MPDP for specific volatility functions and different jump size distributions. Before doing so, we would like to repeat the main ideas of the MPDP introduced in Kemper et al. [30].

2.1 The idea of the MPDP

The procedure for the derivation of the swap's martingale measure used in this paper goes back to Kemper et al. [30] starting with a futures contract with instantaneous delivery. The dynamics for the futures price $f(t,\tau)$ is given by

$$df(t,\tau) = \sigma(\omega,t,\tau)f(t,\tau)dW_t^{\mathbb{Q}} . \tag{2.1}$$

Kemper et al. [30] investigate the gap between the swap's martingale measure associated with two methodologies for implementing the delivery period: arithmetic weighted average with approximation and the geometric weighted average without approximation. The geometric weighted average creates a drift term. More precisely, the swap price dynamics under $\mathbb{Q}$ resulting from geometric averaging without approximations evolve as

$$\frac{dF(t,\tau_1,\tau_2)}{F(t,\tau_1,\tau_2)} = -\frac{1}{2}\left(\mathbb{E}_U\left[\sigma(\omega,t,U)^2\right] - \mathbb{E}_U\left[\sigma(\omega,t,U)\right]^2\right)dt + \mathbb{E}_U\left[\sigma(\omega,t,U)\right]dW_t^{\mathbb{Q}} , \tag{2.2}$$

where the swap's volatility is given by

$$\mathbb{E}_U\left[\sigma(\omega,t,U)\right] = \int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)\sigma(\omega,t,u)du . \tag{2.3}$$

We refer to a more detailed discussion and motivation of the notation to Sect. 2.3 below. This additional drift term gives reason for the existence of the MPDP whenever the futures volatility depends on the delivery period. In particular, the MPDP is defined by

$$\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}} := -\frac{1}{2}\frac{\mathbb{V}_U\left[\sigma(\omega,t,U)\right]}{\mathbb{E}_U\left[\sigma(\omega,t,U)\right]} , \tag{2.4}$$

affecting the Brownian motion $W^{\mathbb{Q}}$. The MPDP introduced by Kemper et al. [30] is essential to change the measure to the swap's risk-neutral measure $\widetilde{\mathbb{Q}}$. While Kemper et al. [30] base the derivation of the electricity swap and the MPDP on a geometric dynamics with stochastic volatility in the spirit of Heston [24], this paper focuses on a geometric jump diffusion with deterministic volatility. Moreover, this paper investigates not only the connection between the futures' and swap's risk neutral measure as in Kemper et al. [30] but creates also a bridge to the physical measure.

2.2 The model

Consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,\tau]},\mathbb{Q})$, where the filtration satisfies the usual conditions. We first model the solution of a futures contract and then derive the corresponding dynamics to avoid lacks of existence in the presence of jumps (see Papapantoleon

[40]). At time $t \leq \tau$, let the logarithmic price process of the futures contract be defined as

$$\ln f(t,\tau) = \ln f(0,\tau) + \int_0^t \sigma(s,\tau) dW_s^{\mathbb{Q}} + \int_0^t \eta(s,\tau) d\widetilde{J}_s^{\mathbb{Q}} - \int_0^t c^{\mathbb{Q}}(s,\tau) ds \ , \quad (2.5)$$

with initial non-random conditions $f(0,\tau) > 0$. Moreover, $W^{\mathbb{Q}}$ is a one-dimensional standard Brownian motion under $\mathbb{Q}$ independent of the jump process $\widetilde{J}^{\mathbb{Q}}$. In particular, $\widetilde{J}^{\mathbb{Q}}$ is a compound compensated jump process defined through the compensated Poisson random measure $\widetilde{N}^{\mathbb{Q}}(dt,dz) = N(dt,dz) - \ell^{\mathbb{Q}}(dz)dt$:

$$\widetilde{J}_t^{\mathbb{Q}} = \int_0^t \int_{\mathbb{R}} z \widetilde{N}^{\mathbb{Q}}(ds,dz) \ , \quad (2.6)$$

with Lévy measure $\ell^{\mathbb{Q}}(dz) = \lambda^{\mathbb{Q}} G(dz)$, which is independent of the delivery time, and where $\lambda^{\mathbb{Q}} > 0$ indicates the jump intensity and $G(dz)$ the jump size distribution. The last term in Equation (2.5) defines the compensator of the logarithmic return under the current measure $\mathbb{Q}$:

$$c^{\mathbb{Q}}(t,\tau) = \frac{1}{2}\sigma^2(t,\tau) + \psi^{\mathbb{Q}}(i\eta(t,\tau)) \ , \quad (2.7)$$

where $\psi^{\mathbb{Q}}(ir)$ is the integrand of the Lévy-Khintchine exponential defined through the moment generating function

$$\psi^{\mathbb{Q}}(r) := \int_{\mathbb{R}} \left(e^{rz} - 1 - rz\right) \ell^{\mathbb{Q}}(dz) \ . \quad (2.8)$$

We assume that the futures price volatility and jump coefficients, $\sigma(t,\tau)$ and $\eta(t,\tau)$, are deterministic and that the futures price $f(t,\tau)$ is $\mathcal{F}_t$-adapted for $t \in [0,\tau]$. We further assume that they satisfy suitable integrability and measurability conditions (see Assumption 1 in Appendix A for details) to ensure that the process in Equation (2.5) is a $\mathbb{Q}$-martingale, and that Equation (2.5) gives the unique solution to the process evolving as

$$\frac{df(t,\tau)}{f(t-,\tau)} = \sigma(t,\tau) dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} \left(e^{\eta(t,\tau)z} - 1\right) \widetilde{N}^{\mathbb{Q}}(dt,dz) \ . \quad (2.9)$$

As $\sigma(t,\tau)$ depends on both, trading time t and delivery time τ, we allow for volatility structures as the Samuelson effect or seasonalities in the delivery time, which are addressed in Examples 3.1 and 3.2.

2.3 Implementing the delivery period

Following the Heath-Jarrow-Morton approach to price futures and swaps in electricity markets, the swap price is usually defined as the *arithmetic weighted average* of futures prices (see, e.g., Benth et al. [1], Bjerksund et al. [9], and Benth et al. [7]):

$$F^A(t,\tau_1,\tau_2) := \int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2) f(t,u) du, \quad (2.10)$$

for a general weight function

$$w(u,\tau_1,\tau_2) := \frac{\hat{w}(u)}{\int_{\tau_1}^{\tau_2} \hat{w}(v) dv} \ , \quad \text{for } u \in (\tau_1,\tau_2] \ , \quad (2.11)$$

where $\hat{w}\colon (\tau_1, \tau_2] \to \mathbb{R}_0^+$ is the corresponding settlement function which is deterministic, integrable and non-negative. Note that w defines a probability density function with support on $(\tau_1, \tau_2]$ since it is positive and integrates to one, that is $\int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2)du = 1$. Hence, we denote U as a random delivery variable with density $w(u, \tau_1, \tau_2)$ (see also Kemper et al. [30]). The most popular example is given by a constant settlement type $\hat{w}(u) = 1$, such that the density becomes $w(u, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1}$ and $U \sim \mathcal{U}((\tau_1, \tau_2])$ is uniformly distributed over the delivery period. This corresponds to a one-time settlement. A continuous settlement over the time interval $(\tau_1, \tau_2]$ is covered by a continuous discount function $\hat{w}(u) = e^{-ru}$, where r is the constant interest rate (see, e.g., Benth et al. [1]).

The arithmetic average of the futures price as in Equation (2.10) leads to tractable dynamics for the swap as long as one assumes an arithmetic structure of the futures prices as well. This is based on the fact that arithmetic averaging is tailor-made for absolute growth rate models. Nevertheless, if one defines the futures price as a geometric process as in Equation (2.9), one can show that the dynamics of the swap price F^A defined through Equation (2.10) are given by

$$\begin{aligned}\frac{dF^A(t, \tau_1, \tau_2)}{F^A(t-, \tau_1, \tau_2)} &= \left[\sigma(t, \tau_2) - \int_{\tau_1}^{\tau_2} \frac{\partial \sigma}{\partial u}(t, u) \frac{w(\tau, \tau_1, \tau_2)}{w(\tau, \tau_1, u)} \frac{F^A(t, \tau_1, u)}{F^A(t, \tau_1, \tau_2)} du\right] dW_t^{\mathbb{Q}} \\ &+ \int_{\mathbb{R}} \left(e^{\eta(t, \tau_2)z} - 1 - \int_{\tau_1}^{\tau_2} \frac{\partial e^{\eta(s,u)z}}{\partial u} \frac{w(\tau, \tau_1, \tau_2)}{w(\tau, \tau_1, u)} \frac{F^A(t, \tau_1, u)}{F^A(t, \tau_1, \tau_2)} du\right) \widetilde{N}^{\mathbb{Q}}(dz, dt),\end{aligned} \tag{2.12}$$

for any $\tau \in (\tau_1, \tau_2]$ (see Benth et al. [1], cf. Chapter 6.3.1). Thus, the dynamics of the swap price is neither a geometric process nor Markovian, which makes it unhandy for further analysis. To overcome this issue, Bjerksund et al. [9] suggest an approximation in the setup without jumps, which we call *approximated averaging* since it is the arithmetic average of approximated logarithmic returns. Approximated averaging maintains the martingale property meaning that the swap is a martingale whenever f is a martingale. If we transfer the approximated averaging procedure to our jump setting, we can define the swap price process based on approximated averaging by

$$\frac{dF^a(t, \tau_1, \tau_2)}{F^a(t-, \tau_1, \tau_2)} := \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2) \frac{df(t, u)}{f(t-, u)} du\ . \tag{2.13}$$

In contrast, *geometric averaging* originates from the arithmetic average of logarithmic returns without any need for approximations. Hence, in line with Kemper et al. [30], we define the swap price originating from geometric averaging by

$$F(t, \tau_1, \tau_2) := e^{\int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2) \ln f(t,u) du}\ , \tag{2.14}$$

(see also Kemna and Vorst [29]). Assume that the volatility and jump coefficients satisfy further integrability conditions (see Assumption 2 in Appendix A). It turns out, that the resulting swap price dynamics is a geometric process with a non-zero drift term:

Lemma 2.1 *(The Swap Price under $\mathbb{Q}$) Let Assumption 2 in Appendix A be satisfied. Under the instantaneous pricing measure $\mathbb{Q}$, the dynamics of the swap price process $F(\cdot, \tau_1, \tau_2)$, defined in Equation (2.14), are given by*

$$\begin{aligned}\frac{dF(t, \tau_1, \tau_2)}{F(t-, \tau_1, \tau_2)} &= \mathbb{E}[\sigma(t, U)]\, dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right) \widetilde{N}^{\mathbb{Q}}(dt, dz) \\ &- \left(\frac{1}{2}\mathbb{V}[\sigma(t, U)] + \mathbb{E}[\psi^{\mathbb{Q}}(\eta(t, U))] - \psi^{\mathbb{Q}}(\mathbb{E}[\eta(t, U)])\right) dt,\end{aligned} \tag{2.15}$$

where U denotes the random delivery variable with density $w(u, \tau_1, \tau_2)$.

Proof Plugging the integral representation of the futures rate process from Equation (2.5) into Equation (2.14) gives us $F(t, \tau_1, \tau_2) = F(0, \tau_1, \tau_2)e^{\bar{X}(t,\tau_1,\tau_2)}$, where an application of the stochastic Fubini Theorem (see Protter [42], cf. Theorem 65, Chapter IV.6) leads to

$$\bar{X}(t, \tau_1, \tau_2) = \int_0^t \mathbb{E}\Big[\sigma(s, U)\Big] dW_s^{\mathbb{Q}} + \int_0^t \mathbb{E}[\eta(s, U)] d\widetilde{J}_s^{\mathbb{Q}} - \frac{1}{2}\int_0^t \mathbb{E}\Big[\sigma^2(s, U)\Big] ds - \int_0^t \mathbb{E}[\psi^{\mathbb{Q}}(\eta(s, U))] ds \,. \tag{2.16}$$

Then, Equation (2.15) follows using Itô's formula (see, e.g., Øksendal and Sulem [39]). □

Having presented the three procedures of continuous time averaging that are used to derive the swap from an underlying futures curve, we would like to compare them: Arithmetic averaging, defined by Equation (2.10), is tractable for arithmetic futures curves, whereas approximated averaging, defined by Equation (2.13), and geometric averaging, defined by Equation (2.14), are well suited for geometric futures curves. In line with a series of literature (see Koekebakker and Ollmar [34], Benth and Koekebakker [5], Bjerksund et al. [9], Benth et al. [1], and Kemper et al. [30]), we follow the geometric approach. Our goal throughout this paper is to investigate the pricing spread between geometric and approximated averaging analytically.

3 The MPDP

Although the futures price f and the approximated F^a are martingales under the pricing measure $\mathbb{Q}$, the swap price F is not a $\mathbb{Q}$-martingale: Indeed, the swap price process under $\mathbb{Q}$ has a negative drift term consisting of two parts given by the swap's variance and the difference between the averaged Lévy-Khintchine integrand and the Lévy-Khintchine integrand of the averaged jump coefficient. Hence, using geometric averaging leads to a new interpretation of risk related to the delivery period as we will analyze in the following.

Analogous to Kemper et al. [30], we derive the corresponding risk-neutral measure $\widetilde{\mathbb{Q}}$ under which the electricity swap price F is a martingale. For deriving the swap's risk-neutral measure, we thus define the MPDP extended to jumps in the following.

Definition 3.1 [(The MPDP)] At time $t \in [0, \tau_1]$, the *market price of diffusion and jump risk for delivery periods* associated to the delivery period $(\tau_1, \tau_2]$ is defined by $\Pi^{\mathbb{Q}\widetilde{\mathbb{Q}}} := (\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}, \Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}})$, where

$$\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) := -\frac{1}{2}\frac{\mathbb{V}[\sigma(t, U)]}{\mathbb{E}[\sigma(t, U)]}\,, \tag{3.1}$$

$$\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) := -\frac{\int_{\mathbb{R}} \mathbb{E}[e^{\eta(t,U)z}] - e^{\mathbb{E}[\eta(t,U)]z}\ell^{\mathbb{Q}}(dz)}{\int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right)\ell^{\mathbb{Q}}(dz)}\,. \tag{3.2}$$

In general, the MPDP does not coincide with the market price of risk. In fact, it is an additional risk that has to be taken into account whenever approximated averaging is conducted. Technically speaking, the MPDP characterizes the distance between the martingale measure of the swaps, F and F^a, resulting from geometric and approximated averaging. In particular, Π_1 refers to the additional diffusion risk, which is measurable and $\mathcal{F}_t$-adapted as $\sigma(t, u)$ is.

It can be interpreted as the trade-off between the weighted average variance of a stream of futures, on the one hand, and the variance of the swap, on the other hand (see also Kemper et al. [30] for an elaboration of the MPDP $\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}$ and a detailed interpretation). $\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}$ is the additional jump risk, which is the difference between the Lévy-Khintchine integrands standardized by the swap's jump coefficient.

Remark 3.1 (*i*) Note that $\Pi^{\mathbb{Q}\tilde{\mathbb{Q}}}$ would be zero, whenever the volatility and jump coefficients are independent of delivery time. For this reason, we call $\Pi^{\mathbb{Q}\tilde{\mathbb{Q}}}$ the *market price of risk for delivery periods* (MPDP).

(*ii*) The MPDP of diffusion and jump risk is strengthened by delivery-dependent effects within the volatility and jump coefficients. For example, pronounced term-structure effects or seasonalities in the delivery period within these coefficients capture a distinct dependence on the delivery period and, consequently, lead to a high MPDP (see also Example 3.1 and 3.2 for the MPDP of diffusion risk and Examples 3.3 and 3.4.

(*iii*) $\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}$ is in line with the MPDP for diffusion risk found in Kemper et al. [30], where a stochastic volatility scenario is considered.

In a next step, we would like to characterize the MPDP of diffusion risk $\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}$ more explicitly. The MPDP of diffusion risk arises through delivery-dependent volatility effects such as seasonality in delivery periods and the Samuelson effect (see Kemper et al. [30]). We state the corresponding MPDP in the following two examples while assuming a one-time settlement such that $w(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1}$.

Example 3.1 (Seasonal Volatility) Inspired by Fanelli and Schmeck [20], we capture seasonality in the delivery period by incorporating a trigonometric function into the futures volatility $\sigma(t, u) = S_1(u)$ in (see Equation (3.1)) given by

$$S_1(u) := a + b\cos(2\pi(u + c)) \,, \tag{3.3}$$

for $a > b > 0$ and $c \in [0, 1)$. Note that $S_1(u)$ is constant over trading time and bounded by $0 < (a - b) \leq S_1(u) \leq a + b \leq 2a$. Additionally, we consider a finite trading and delivery horizon, so that Assumptions 1 and 2 in Appendix A are satisfied. According to Definition 3.1, this leads to a MPDP of diffusion risk of the following form

$$\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}(t, \tau_1, \tau_2) = -\frac{1}{2}\frac{\mathbb{V}[S_1(U)]}{\mathbb{E}[S_1(U)]} \,, \tag{3.4}$$

where

$$\mathbb{E}[S_1(U)] = a + \frac{b}{2\pi(\tau_2 - \tau_1)}\Big[\sin(2\pi(u + c))\Big]_{u=\tau_1}^{u=\tau_2} \,, \tag{3.5}$$

$$\begin{aligned}\mathbb{E}[S_1(U)^2] = a^2 + \frac{b^2}{2} + \frac{ab}{\pi(\tau_2 - \tau_1)}\Big[\sin(2\pi(u + c))\Big]_{u=\tau_1}^{u=\tau_2} \\ + \frac{b^2}{8\pi(\tau_2 - \tau_1)}\Big[\sin(4\pi(u + c))\Big]_{u=\tau_1}^{u=\tau_2} \,.\end{aligned} \tag{3.6}$$

Example 3.2 [Term-Structure Volatility] We implement the Samuelson effect as in Schneider and Tavin [44] into the futures volatility $\sigma(t, u) = S_2(u - t)$ (see Equation (3.1)) through an exponential function with exponential damping factor $\Lambda > 0$ and terminal volatility $\bar{\lambda} > 0$ given by

$$S_2(u - t) := \bar{\lambda} e^{-\Lambda(u-t)} \,. \tag{3.7}$$

Since $S_2(u-t)$ is deterministic and bounded by $0 < S_2(u-t) \leq \bar{\lambda}$ and we consider a finite trading and delivery horizon with $t \leq u$, Assumptions 1 and 2 in Appendix A are satisfied.

According to Definition 3.1, this leads to a MPDP of diffusion risk of the following form

$$\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2) = -\frac{1}{2}\frac{\bar{\bar{\Lambda}} - \bar{\Lambda}^2}{\bar{\Lambda}} e^{-\Lambda(\tau_1 - t)} \,, \tag{3.8}$$

for constant parameters $\bar{\Lambda} := \frac{\bar{\lambda}(1-e^{-\Lambda(\tau_2-\tau_1)})}{\Lambda(\tau_2-\tau_1)}$ and $\bar{\bar{\Lambda}} := \frac{\bar{\lambda}^2(1-e^{-2\Lambda(\tau_2-\tau_1)})}{2\Lambda(\tau_2-\tau_1)}$ implicitly depending on the delivery period.

Hence, the MPDP of diffusion risk is constant for a fixed contract in Example 3.1, whereas the Samuelson effect remains still visible in Example 3.2. For a detailed investigation of the volatility term structure, we refer to Kemper et al. [30].

The MPDP of jump risk is triggered by delivery-dependent jump effects. We choose $\eta(t,u)$ independent of trading time thinking e.g. of seasonal effects that can be described by $\eta(u) = S_1(u)$. In the following examples, we characterize the MPDP of jump risk and the spread based on four suitable jump size distributions: Normal, exponential, negative exponential, and gamma. We also state corresponding moments of the distributions following Gray and Pitts [22] (cf. Chapter 2).

Example 3.3 (Normal Jump Sizes) If the jump sizes are normally distributed with $Z \sim \mathcal{N}(\mu_J, \sigma_J^2)$ for $|\mu_J| < \infty, \sigma_J^2 < \infty$, then the moment generating function is given by

$$M_Z(\eta) = e^{\mu_J \eta + \frac{1}{2}\sigma_J^2 \eta^2} \,, \tag{3.9}$$

such that the MPDP and the spread in Equations (3.2) and (4.34) are given by

$$\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = -\frac{\mathbb{E}[e^{\frac{1}{2}\eta^2(U)\sigma_J^2 + \eta(U)\mu_J}] - e^{\frac{1}{2}\mathbb{E}[\eta(U)]^2\sigma_J^2 + \mathbb{E}[\eta(U)]\mu_J}}{e^{\frac{1}{2}\mathbb{E}[\eta(U)]^2\sigma_J^2 + \mathbb{E}[\eta(U)]\mu_J} - 1} \,. \tag{3.10}$$

The fourth moment is attained by $\mathbb{E}[Z^4] = \mu_J^4 + 6\mu_J^2\sigma_J^2 + 3\sigma_J^4$. Since η is finite and the moment generating function and fourth moment of the jump sizes exist, Assumptions 1 and 2 in Appendix A are satisfied.

Example 3.4 (Exponential Jump Sizes) If the jump sizes are exponentially distributed with $Z \sim \mathcal{E}xp(\lambda_J)$, for $\lambda_J > 0$, then the moment generating function is given by

$$M_Z(\eta) = \left(1 - \frac{\eta}{\lambda_J}\right)^{-1} = \frac{\lambda_J}{\lambda_J - \eta} \,, \tag{3.11}$$

for $\eta < \lambda_J$, such that the MPDP and the spread in Equations (3.2) and (4.34) are given by

$$\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = -\frac{\lambda_J}{\mathbb{E}[\eta(U)]}\left(1 - \mathbb{E}\left[\frac{\lambda_J - \mathbb{E}[\eta(U)]}{\lambda_J - \eta(U)}\right]\right) \,, \tag{3.12}$$

defined for $\eta(U) < \lambda_J$ and $\mathbb{E}[\eta(U)] < \lambda_J$. The n-th moment is attained by $\mathbb{E}[Z^n] = \frac{n!}{\lambda_J^n}$ for $n \in \mathbb{N}$.

Since η is finite and the moment generating functions as well as the n-th moment of jump sizes exist, Assumptions 1 and 2 in Appendix A are satisfied.

Example 3.5 (Negative Exponential Jump Sizes) If the jump sizes follow a negative exponential distribution with $(-Z) \sim \mathcal{E}xp(\lambda_J^-)$, for $\lambda_J^- > 0$, i.e. the density function is given by $f_{-Z}(z) = \lambda e^{\lambda z}$ for $z \leq 0$, then the moment generating function is given by

$$M_{-Z}(\eta) = \frac{\lambda_J^-}{\lambda_J^- + \eta}, \tag{3.13}$$

for $\lambda_J^- + \eta > 0$, such that the MPDP and the spread in Equations (3.2) and (4.34) are given by

$$\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(\tau_1, \tau_2) = -\frac{\lambda_J^-}{\mathbb{E}[\eta(U)]}\left(1 - \mathbb{E}\left[\frac{\lambda_J^- + \mathbb{E}[\eta(U)]}{\lambda_J^- + \eta(U)}\right]\right), \tag{3.14}$$

defined for $\lambda_J^- + \eta(U) > 0$ and $\lambda_J^- + \mathbb{E}[\eta(U)] > 0$. The n-th moment is attained by $\mathbb{E}[Z^n] = (-1)^n \frac{n!}{(\lambda_J^-)^n}$ for $n \in \mathbb{N}$. Since η is finite and the moment generating functions as well as the n-th moment of jump sizes exist, Assumptions 1 and 2 in Appendix A are satisfied.

Example 3.6 (Gamma Jump Sizes) If the jump sizes follow a Gamma distribution with $Z \sim Gam(\alpha, \beta)$, for $\alpha, \beta > 0$, then the moment generating function is given by

$$M_Z(\eta) = \left(1 - \frac{\eta}{\beta}\right)^{-\alpha} \tag{3.15}$$

for $\eta < \beta$, such that the MPDP and the spread in Equations (3.2) and (4.34) are given by

$$\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(\tau_1, \tau_2) = -\frac{\mathbb{E}\left[\left(1 - \frac{\eta(U)}{\beta}\right)^{-\alpha}\right] - \left(1 - \frac{1}{\beta}\mathbb{E}[\eta(U)]\right)^{-\alpha}}{\left(1 - \frac{1}{\beta}\mathbb{E}[\eta(U)]\right)^{-\alpha} - 1}, \tag{3.16}$$

defined for $\eta(U) < \beta$ and $\mathbb{E}[\eta(U)] < \beta$. The n-th moment is attained by $\mathbb{E}[Z^n] = \frac{1}{\beta^n}\frac{\Gamma(\alpha)}{\Gamma(\alpha)}$ for $n \in \mathbb{N}$ for $n > 0$. Since η is finite and the moment generating functions as well as the n-th moment of jump sizes exist, Assumptions 1 and 2 in Appendix A are satisfied.

3.1 On the swap's martingale measure

We define a new pricing measure $\widetilde{\mathbb{Q}}$, such that the swap price process $F(\cdot, \tau_1, \tau_2)$ is a martingale. Following Øksendal and Sulem [39], define the Radon-Nikodym density through

$$Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) = \prod_{j=1}^{2} Z_j^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2), \tag{3.17}$$

where

$$Z_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) := e^{-\int_0^t \Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2) d\widetilde{W}^{\mathbb{Q}}(s) - \frac{1}{2}\int_0^t \Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)^2 ds}, \tag{3.18}$$

$$Z_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) := e^{\int_0^t \int_{\mathbb{R}} \ln(1-\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z)\widetilde{N}^{\mathbb{Q}}(ds,dz) + \int_0^t \int_{\mathbb{R}} \left(\ln(1-\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z) + \Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z\right)\ell^{\mathbb{Q}}(dz)ds}. \tag{3.19}$$

Assume that

$$\mathbb{E}_{\mathbb{Q}}[Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}(\tau_1, \tau_1, \tau_2)] = 1 \ , \tag{3.20}$$

which means that $Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}(\cdot, \tau_1, \tau_2)$ is indeed a martingale for the entire trading time. We will show later that the martingale property is satisfied for suitable models such that Equation (3.20) holds true. We then define the new measure $\widetilde{\mathbb{Q}}$ through the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}(\tau_1, \tau_1, \tau_2) \ , \tag{3.21}$$

which clearly depends on the delivery period $(\tau_1, \tau_2]$. Girsanov's theorem states that if we define the process $W^{\widetilde{\mathbb{Q}}}$ and the random measure $\widetilde{N}^{\widetilde{\mathbb{Q}}}(dt, dz)$ by

$$dW_t^{\widetilde{\mathbb{Q}}} = dW_t^{\mathbb{Q}} + \Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2)dt \ , \tag{3.22}$$

$$\widetilde{N}^{\widetilde{\mathbb{Q}}}(dt, dz) = \widetilde{N}^{\mathbb{Q}}(dt, dz) + \Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2)\ell^{\mathbb{Q}}(dz)dt \ , \tag{3.23}$$

then $W^{\widetilde{\mathbb{Q}}}$ is a Brownian motion under $\widetilde{\mathbb{Q}}$ and $\widetilde{N}^{\widetilde{\mathbb{Q}}}(\cdot, \cdot)$ is the $\widetilde{\mathbb{Q}}$-compensated Poisson random measure of $N(\cdot, \cdot)$ with compensator $\left(1 - \Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s, \tau_1, \tau_2)\right)\ell^{\mathbb{Q}}(dz)$. Under some further assumptions, ensuring that $Z_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ stays positive and that $Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is a true martingale (see Assumption 3 in Appendix A), a straightforward valuation leads to the following result:

Proposition 3.1 *(The Swap Price under $\widetilde{\mathbb{Q}}$) Let Assumption 3 in Appendix A be satisfied. The swap price process $F(\cdot, \tau_1, \tau_2)$, defined in (2.14), is a martingale under $\widetilde{\mathbb{Q}}$. The swap price dynamics are given by*

$$\frac{dF(t, \tau_1, \tau_2)}{F(t-, \tau_1, \tau_2)} = \mathbb{E}[\sigma(t, U)]dW_t^{\widetilde{\mathbb{Q}}} + \int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right)\widetilde{N}^{\widetilde{\mathbb{Q}}}(dt, dz) \ , \tag{3.24}$$

where $W^{\widetilde{\mathbb{Q}}}$ is a Brownian motion under $\widetilde{\mathbb{Q}}$ and $\widetilde{N}^{\widetilde{\mathbb{Q}}}(\cdot, \cdot)$ is the compound compensated Poisson random measure under $\widetilde{Q}$ with Lévy measure $\left(1 - \Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2)\right)\ell^{\mathbb{Q}}(dz)$ for $t \in [0, \tau_1]$.

Proof We know by definition that $\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is a continuous adapted process that is square-integrable and $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is deterministic and càdlàg in time. Hence, all processes are predictable. Following Øksendal and Sulem [39] (cf. Theorem 1.35), we need to show that Equation (3.20) is satisfied, so that $Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is a true martingale. Considering the dynamics of $Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ using Itô's formula, we have

$$\begin{aligned} &dZ^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) \\ &\quad = Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t-, \tau_1, \tau_2)\left[-\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2)dW_t^{\mathbb{Q}} - \int_{\mathbb{R}} \Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2)z\widetilde{N}^{\mathbb{Q}}(dt, dz)\right] \ , \end{aligned}$$

so that $Z^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is a local $\mathbb{Q}$-martingale, where $W^{\mathbb{Q}}$ and $\widetilde{N}^{\mathbb{Q}}(\cdot, \cdot)$ are independent of each other. Hence, it is enough to show, that $Z_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ and $Z_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ are true martingales. We can prove Novikov's condition regarding the continuous part (see, e.g., Protter [42], cf. Theorem 41, Chapter III.8) as $\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}$:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\frac{1}{2}\int_0^{\tau_1} \Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)dW_s^{\mathbb{Q}}}\right] = e^{\frac{1}{2}\int_0^{\tau_1} \Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)^2 ds} < \infty \ .$$

Hence, $Z_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is a true martingale. Moreover, $Z_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is a true martingale under $\mathbb{Q}$ since

$$\mathbb{E}_{\mathbb{Q}}[Z_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[e^{\lambda^{\mathbb{Q}}\int_0^{\tau_1}\ln(1-\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z)\widetilde{N}^{\mathbb{Q}}(ds,dz)}\right] e^{\int_0^{\tau_1}\int_{\mathbb{R}}\left(\ln(1-\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z)+\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z\right)\ell^{\mathbb{Q}}(dz)ds} = 1\,,$$

where the last equality follows from the Lévy-Khintchine representation and Assumption 3 in Appendix A. Hence, we can apply Girsanov's Theorem (see, e.g., Øksendal and Sulem [39], cf. Theorem 1.35) and the assertion follows. □

Note that the MPDP of diffusion and jump risk, $\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ and $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$, are negative following from Jensen's inequality. Hence, the geometric averaging technique induces less risk than the application of the approximated arithmetic average for which we need to pay a cost of approximation risk.

As the Assumption 3 in Appendix A for the measure change in Proposition 3.1 is a bit technical, we would like to give sufficient conditions for it in the next lemma:

Lemma 3.1 (i) *The Novikov condition in Assumption 3* (i) *is satisfied, if* $\sigma(t,u)$ *is deterministic and finite.*

(ii) *Assumption 3* (ii) *is satisfied under the following conditions:*

a) the jump size distribution takes only positive or negative values,
b) $\mathbb{E}_U[\eta(t,U)]$ *is strictly positive and finite, and*
c) the moment generating function $M_Z(\eta) := \mathbb{E}\left[e^{\eta Z}\right]$ *exists.*

Proof (i) The result directly follows, when applying the conditions on σ to the market price of diffusion risk for the delivery period in Eq. 3.1.

(ii) In order to prove that $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2)z \leq 1$, positive and negative jumps are considered separately.

Case 1. Let us assume that the support of the jump size distribution is on $z \in (0,\infty)$, so that condition a) is satisfied. To prove that the inequality holds true, numerator and denominator of $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ are further investigated. By Jensen's inequality for convex functions, it follows for all $z \in \mathbb{R}$, that

$$e^{\mathbb{E}[\eta(t,U)]z} \leq \mathbb{E}\left[e^{\eta(t,U)z}\right]\,.$$

Hence, considering all jumps together yields a positive numerator in $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ given by

$$\int_{\mathbb{R}} \mathbb{E}[e^{\eta(t,U)z}] - e^{\mathbb{E}[\eta(t,U)]z}\ell^{\mathbb{Q}}(dz) \geq 0\,,$$

which exists under condition c). Under condition b), it follows that $e^{\mathbb{E}[\eta(t,U]z} > 1$ for all $z > 0$. Considering all jumps together, yields a positive denominator in $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ given by

$$\int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right)\ell^{\mathbb{Q}}(dz) > 0\,.$$

By definition of the market price of jump risk for delivery periods in Eq. 3.2, it follows that $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is negative.

Case 2. Let us assume that the support of the jump size distribution is on $z \in (-\infty, 0)$, so that condition a) is satisfied. To prove that the inequality holds true, numerator and denominator of $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ are again further investigated. The numerator of $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ stays positive as in Case 1, since Jensen's inequality is unaffected by the support of the jump size distribution. Under condition b), it follows that $e^{\mathbb{E}[\eta(t,U]z} < 1$ for all $z < 0$. Considering all jumps together, yields a positive denominator in $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ given by

$$\int_{\mathbb{R}^+} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right) \ell^{\mathbb{Q}}(dz) < 0 \,.$$

By definition of the market price of jump risk for delivery periods in Eq. 3.2, it follows that $\Pi_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is again negative.

The remaining part of the assumption directly follows.

□

Note, that Example 3.3 is not applicable under the assumptions of Lemma 3.1, while all other examples in this section satisfy the conditions in Lemma 3.1.

3.2 On the pricing spread

We would like to compare the approximated swap price F^a under $\mathbb{Q}$ (see Equation (2.13)) with the swap price F under $\widetilde{\mathbb{Q}}$ (see Equation (2.14)). The diffusion part of the swap price dynamics coincides since we consider a deterministic volatility structure. The only differences are located in the compensator of the compound compensated Poisson process and the jump coefficient. If the jump coefficient is independent of delivery time, the distribution of F^a under $\mathbb{Q}$ and the distribution of F under $\widetilde{\mathbb{Q}}$ are the same. For differences in a stochastic volatility setting, we refer to Kemper et al. [30]. For the swap prices F, F^A, and F^a, we have the following result under the instantaneous measure $\mathbb{Q}$:

Corollary 3.1 (i) *The swap price F is always smaller or equal than F^A.*
(ii) *The pricing spread between F and F^a is attained by*

$$F^a(t,\tau_1,\tau_2) - F(t,\tau_1,\tau_2) = F^a(t,\tau_1,\tau_2)\Big[1 - D(t,\tau_1,\tau_2)\Big] \,, \tag{3.25}$$

$$D(t,\tau_1,\tau_2) = e^{-\frac{1}{2}\int_0^t \mathbb{V}[\sigma(s,U)]ds - \int_0^t \int_{\mathbb{R}} \left(\ln \mathbb{E}[e^{\eta(s,U)z}] - \mathbb{E}[\eta(s,U)]z\right) N(ds,dz)} \,. \tag{3.26}$$

(iii) *If the jump coefficient, $\eta(t,u)$, is independent of the delivery time, then the swap price F is smaller or equal than F^a.*

Proof (i) The continuous arithmetic weighted average is greater than the geometric one, which directly follows from Jensen's inequality. More precisely,

$$F(t,\tau_1,\tau_2) \overset{(2.14)}{=} e^{\mathbb{E}_U[\ln f(t,U)]} \overset{\text{Jensen}}{\leq} e^{\ln \mathbb{E}_U[f(t,U)]} = \mathbb{E}_U\left[f(t,U)\right] \overset{(2.10)}{=} F^A(t,\tau_1,\tau_2) \,.$$

(ii) Using Equation (2.13), we find that $F^a(t,\tau_1,\tau_2) = e^{\bar{X}^a(t,\tau_1,\tau_2)}$, where $\bar{X}^a(t,\tau_1,\tau_2)$ is the solution of the following arithmetic Brownian motion:

$$d\bar{X}^a(t,\tau_1,\tau_2) = \mathbb{E}[\sigma(t,U)]dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} \ln \mathbb{E}[e^{\eta(t,U)z}]\widetilde{N}^{\mathbb{Q}}(dt,dz)$$

$$- \left(\frac{1}{2} \mathbb{E}[\sigma(t,U)]^2 + \int_{\mathbb{R}} \mathbb{E}[e^{\eta(t,U)z}] - 1 - \ln \mathbb{E}[e^{\eta(t,U)z}] \right) dt \ .$$

Taking Eq. 2.16 into account, the pricing spread is given by

$$F^a(t,\tau_1,\tau_2) - F(t,\tau_1,\tau_2) = F^a(t,\tau_1,\tau_2)\,(1 - D(t,\tau_1,\tau_2)) \ ,$$

such that $F(t,\tau_1,\tau_2) = F^a(t,\tau_1,\tau_2) D(t,\tau_1,\tau_2)$, where

$$D(t,\tau_1,\tau_2) = e^{\bar{X}(t,\tau_1,\tau_2) - \bar{X}^a(t,\tau_1,\tau_2)} = e^{-\frac{1}{2}\int_0^t \mathbb{V}[\sigma(s,U)]ds - \int_0^t \int_{\mathbb{R}} \ln \mathbb{E}[e^{\eta(s,U)z}] - \mathbb{E}[\eta(s,U)]z N(ds,dz)} \ .$$

(*iii*) If $\eta(t,u) \perp\!\!\!\perp u$, then

$$D(t,\tau_1,\tau_2) = e^{-\frac{1}{2}\int_0^t \mathbb{V}[\sigma(s,U)]ds} \ .$$

Since $\mathbb{V}[\sigma(\cdot,U)] \geq 0$ by Jensen, it follows that $D(t,\tau_1,\tau_2) \in (0,1]$ and thus $F \leq F^a$. □

We conclude that arithmetic and in specific cases approximated averaging lead to higher swap prices than the geometric average. We would like to stress that D in Equation (3.26) is not affected by measure changes since it is characterized by a drift component and a pure jump component exclusively (see also Equation (3.28)). Moreover, note that D can be seen as stochastic discount factor, which can be used to derive the swap price F given F^a. Vice versa, consider

$$F^a(t,\tau_1,\tau_2) = F(t,\tau_1,\tau_2) D^{-1}(t,\tau_1,\tau_2) \ . \tag{3.27}$$

The exponential part of D^{-1} can be interpreted as a price (premium) per share, which we pay for an imprecise averaged swap. Moreover, we can see D as the price process of a non-dividend paying asset evolving as

$$\frac{dD(t,\tau_1,\tau_2)}{D(t-,\tau_1,\tau_2)} = -\frac{1}{2}\mathbb{V}[\sigma(t,U)]dt + \int_{\mathbb{R}} \left(\frac{e^{\mathbb{E}[\eta(t,U)]z}}{\mathbb{E}[e^{\eta(t,U)z}]} - 1 \right) N(dt,dz) \ , \tag{3.28}$$

such that we can interpret D as a numéraire. If F^a is a martingale, then $\frac{F}{D}$ is also a martingale. If F is a martingale, then $F^a D$ is a martingale (see, e.g., Shreve [46], cf. Theorem 9.2.2). We can thus use it to price options and other derivatives on the swap. In the subsequent section, we introduce the model under its physical measure $\mathbb{P}$.

Remark 3.2 (Delivery-Dependent Intensity) Let us consider an adjusted version of the futures price under the instantaneous pricing measure $\mathbb{Q}$ similar to Equation (2.9) given by

$$\frac{df(t,\tau)}{f(t-,\tau)} = \sigma(t,\tau) dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} \left(e^{\eta z} - 1 \right) \widetilde{N}^{\tau}(dt,dz) \ , \tag{3.29}$$

where $\eta \in \mathbb{R}$ and the compensated Poisson random measure is defined by $\widetilde{N}^{\tau}(dt,dz) := N(dt,dz) - \lambda^{\mathbb{Q}}(\tau) G(dz) dt$, with a jump intensity adjusted to a deterministic, positive, and bounded function of the *delivery time*.

(*i*) Analogous to Lemma 2.1, the dynamics of the swap price, defined by geometric averaging, are given by

$$\begin{aligned} \frac{dF(t,\tau_1,\tau_2)}{F(t-,\tau_1,\tau_2)} &= \mathbb{E}[\sigma(t,U)]\, dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} \left(e^{\eta z} - 1 \right) N(dt,dz) \\ &\quad - \left(\frac{1}{2}\mathbb{V}\left[\sigma(t,U)\right] + \mathbb{E}[\lambda^{\mathbb{Q}}(U)] \int_{\mathbb{R}} \left(e^{\eta z} - 1 \right) G(dz) \right) dt, \end{aligned} \tag{3.30}$$

where U is the random delivery variable with density $w(u, \tau_1, \tau_2)$.

(*ii*) If the volatility is independent of delivery time, then the approximated and geometric average coincide and so their risk-neutral pricing measure. However, the resulting swap price process in Equation (3.30) is not a martingale under $\mathbb{Q}$ since the intensity is affected by the averaging procedure.

(*iii*) In the case of delivery-dependent volatility, the MPDP from Definition 3.1 adjusts to $(\Pi_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}, 0)$. Hence, the MPDP associated to the Brownian motion stays the same and its second dimension becomes zero since the jump coefficient η is independent of delivery time. However, under the assumption that the intensity is delivery-dependent, the swap price is in general not a martingale under $\widetilde{\mathbb{Q}}$.

(*iv*) The swap price is a $\widetilde{\mathbb{Q}}-$martingale, only if the swap's jump intensity under $\widetilde{\mathbb{Q}}$ is given by $\mathbb{E}[\lambda^{\mathbb{Q}}(U)]$, i.e., if $\widetilde{N}^{\tau_1,\tau_2}(dt, dz) := N(dt, dz) - \mathbb{E}[\lambda^{\mathbb{Q}}(U)]G(dz)dt$, is a compensated Poisson random measure under $\widetilde{\mathbb{Q}}$.

4 The real-world model

A typical feature of electricity prices beyond seasonalities and the Samuelson effect is the mean-reverting behavior (see, e.g., Benth et al. [1] and Benth et al. [7]). In order to implement the drift feature, we derive the futures under the physical measure $\mathbb{P}$. Note that we will include mean-reversion at the futures and thus the swap's *rate* level. We then consider the resulting market prices of risk transferring to the instantaneous and the swap's risk-neutral measure.

4.1 The swap price under the physical measure

We now derive the price of a swap contract that delivers one unit of electricity during the fixed delivery period $(\tau_1, \tau_2]$, similar to Sect. 2 but now under the physical measure $\mathbb{P}$. Hence, starting from the physical measure $\mathbb{P}$, the logarithmic futures price process from Equation (2.5), given by

$$\begin{aligned}\ln f(t,\tau) = \; & e^{-\int_0^t \kappa(s)ds} \ln f(0,\tau) + \int_0^t e^{-\int_v^t \kappa(q)dq} \mu(v,\tau)dv \\ & + \int_0^t e^{-\int_v^t \kappa(q)dq} \sigma(v,\tau)dW_v^{\mathbb{P}} + \int_0^t e^{-\int_v^t \kappa(q)dq} \eta(v,\tau)d\widetilde{J}_v^{\mathbb{P}},\end{aligned} \tag{4.1}$$

where $W^{\mathbb{P}}$ is a Brownian motion under the physical measure $\mathbb{P}$ independent of the compound compensated jump process $\widetilde{J}^{\mathbb{P}}$. In particular, $\widetilde{J}^{\mathbb{P}}$ is defined through the $\mathbb{P}$-compensated Poisson random measure $\widetilde{N}^{\mathbb{P}}(dt, dz) = N(dt, dz) - \ell^{\mathbb{P}}(dz)dt$ with Lévy measure $\ell^{\mathbb{P}}(dz) = \lambda^{\mathbb{P}}G(dz)$ that is independent of delivery time. Note that $\lambda^{\mathbb{P}} > 0$ indicates the jump intensity under the physical measure and $G(dz)$ is the jump size distribution.

In order to characterize the futures price in more detail, we introduce the following lemma.

Lemma 4.1 *We assume that the coefficients satisfy suitable integrability and measurability conditions (see Assumption 4 in Appendix A) such that*

Equation 4.1 is the unique strong solution to the dynamics

$$\frac{df(t,\tau)}{f(t-,\tau)} = \sigma(t,\tau)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \left(e^{\eta(t,\tau)z} - 1\right) \widetilde{N}^{\mathbb{P}}(dt, dz) + c^{\mathbb{P}}(t, \tau, \ln f(t,\tau))dt, \tag{4.2}$$

where the drift-term is characterized by

$$c^{\mathbb{P}}(t,\tau,Y) = \mu(t,\tau) - \kappa(t)Y + \frac{1}{2}\sigma(t,\tau)^2 + \psi^{\mathbb{P}}(\eta(t,\tau)) \ . \tag{4.3}$$

Hence, the logarithmic futures evolves as

$$d\ln f(t,\tau) = (\mu(t,\tau) - \kappa(t)\ln f(t,\tau))\,dt + \sigma(t,\tau)dW_t^{\mathbb{P}} + \eta(t,\tau)d\widetilde{J}_t^{\mathbb{P}} \ . \tag{4.4}$$

Proof The unique strong solution follows from Benth et al. [1] (cf. Proposition 3.1). Applying Ito's formula leads to the desired dynamics (see Øksendal and Sulem [39], cf. Theorem 1.16). □

Note that the assumption behind the model induces a finite second moment as well as a finite moment generating function of the jump size distribution. In Examples 3.3 to 3.6, we consider suitable distributions for these jump sizes.

Remark 4.1 In the literature, we sometimes find the application of lognormal distributed jump sizes (see, e.g., Borovkova and Permana [10] and Borovkova and Permana [11]). This distribution, however, is not suitable for our setting since its moment generating function $\mathbb{E}[e^{\eta Z}]$ is not finite at any positive value η (see, e.g., Gray and Pitts [22], cf. Chapter 2.2.6). Hence, the lognormal distribution contradicts the integrability assumption in Assumption 4 (i) under the physical measure in Appendix A.

As in the previous section, we now derive the swap prices resulting from geometric and approximated averaging.

Lemma 4.2 *[The Swap Price under $\mathbb{P}$] Let Assumption 4 and 5 in Appendix A be satisfied. Then,*

the swap price based on geometric averaging evolves as

$$\begin{aligned}\frac{dF(t,\tau_1,\tau_2)}{F(t-,\tau_1,\tau_2)} &= \mathbb{E}[\sigma(t,U)]dW_t^{\mathbb{P}} \\ &+ \int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right)\widetilde{N}^{\mathbb{P}}(dt,dz) + \widetilde{c}^{\mathbb{P}}(t,\tau_1,\tau_2,\ln F(t,\tau_1,\tau_2))dt,\end{aligned} \tag{4.5}$$

where the drift term is given by

$$\widetilde{c}^{\mathbb{P}}(t,\tau_1,\tau_2,\bar{Y}) = \mathbb{E}[\mu(t,U)] - \kappa(t)\bar{Y} + \frac{1}{2}\mathbb{E}[\sigma(t,U)]^2 + \psi^{\mathbb{P}}(\mathbb{E}[\eta(t,U)]) \ . \tag{4.6}$$

Proof Following the considerations in the previous section, the swap price is defined by the geometric average in Equation (2.14). Using the integral representation of Eq. 4.4 and the stochastic Fubini theorem (see Protter [42], cf. Theorem 65), we can introduce the dynamics of the swap's logarithmic return by

$$\begin{aligned}&d\ln F(t,\tau_1,\tau_2) \\ &= (\mathbb{E}[\mu(t,U)] - \kappa(t)\ln F(t,\tau_1,\tau_2))\,dt + \mathbb{E}[\sigma(t,U)]dW_t^{\mathbb{P}} + \mathbb{E}[\eta(t,U)]d\widetilde{J}_t^{\mathbb{P}}.\end{aligned} \tag{4.7}$$

An application of Ito's formula (see Øksendal and Sulem [39], cf. Theorem 1.16) yields the desired swap dynamics. □

Note that the speed of mean-reversion $\kappa(t)$ has to be independent of the delivery time. This assumption ensures that $\ln F$, in Equation (4.7), is again an Ornstein-Uhlenbeck process and that the swap's price dynamics in Equation (4.5) stay tractable. This is also in line with the

findings in Benth et al. [7] (cf. Proposition 2.2) and Latini et al. [35]. In particular, the mean-reverting effect comprises the jump component as well, even if we implement it through a measure change of the Brownian part. More precisely, mean-reversion connected to jumps covers indeed a unique feature of electricity markets known as spikes: Spikes are large jumps quickly returning to the "normal" level (see, e.g., Klüppelberg et al. [33]). They arise as electricity is not storable on a large scale and since the electricity demand is not elastic (see Borovkova and Schmeck [11]).

Let us now investigate the swap price under the physical measure resulting from approximated averaging (see Eq. 2.13 in order to compare the pricing spread between both approaches.

Lemma 4.3 *Let Assumption 4 and 5 in Appendix A be satisfied. Then, the swap price dynamics based on approximated averaging evolve as*

$$\begin{aligned}&\frac{dF^a(t,\tau_1,\tau_2)}{F^a(t-,\tau_1,\tau_2)}\\&= \mathbb{E}[\sigma(t,U)]dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \left(\mathbb{E}[e^{\eta(t,U)z}]-1\right)\widetilde{N}^{\mathbb{P}}(dt,dz) + \mathbb{E}_U[c^{\mathbb{P}}(t,U,\ln f(t,U))]dt,\end{aligned} \tag{4.8}$$

with $\mathbb{E}_U$ denoting the expectation with respect to the random delivery variable U having density $w(u,\tau_1,\tau_2)$.

Proof We use the approximated averaging methodology (see Equation (2.13)) in order to derive the swap price evolution and apply the stochastic Fubini theorem (see Protter [42], cf. Theorem 65) leading to Equation (4.8). □

4.2 The risk premium

We would like to close this section with a discussion of the risk premium in our setting. The risk premium usually represents the difference between the forward price and the spot price prediction at delivery time (cf. for instance Benth et al. [2]). In the case of swaps, Benth et al. [7] extend the definition of the risk premium as the difference between the swap price and the expected value of the spot price weighted over the delivery period. Inspired by the extended definition of the risk premium to swaps by Benth et al. [7], we consider the risk premium as the differences between the swap price and the expected value of the geometrically averaged futures price weighted over the delivery period. Additionally, we consider the risk premium as a pricing spread (similar to in Corollary 3.1) resulting from different pricing methodologies, i.e. the difference between two prices under the same measure.

Corollary 4.1 *(Risk Premium)*

(*i*) *The risk premium between the swap price F and F^A is always non-positive under the physical measure, i.e.,*

$$RP^{F,F^A}(t,\tau_1,\tau_2) := F(t,\tau_1,\tau_2) - F^A(t,\tau_1,\tau_2) \leq 0. \tag{4.9}$$

(*ii*) *The risk premium between the swap price F and F^a under the physical measure is determined by*

$$RP^{F,F^a}(t,\tau_1,\tau_2) := F(t,\tau_1,\tau_2) - F^a(t,\tau_1,\tau_2) = F^a(t,\tau_1,\tau_2)\,(D(t,\tau_1,\tau_2)-1)\,, \tag{4.10}$$

where $D(t,\tau_1,\tau_2)$ is defined in Equation (3.26). Hence, the distance between F and F^a is not affected by the measure change.

(*iii*) *The risk premium of the geometric swap price between the physical and the swap's risk neutral measure is defined by*

$$RP^F(t,\tau_1,\tau_2) := F(t,\tau_1,\tau_2) - \mathbb{E}_{\mathbb{P}}\left[F(\tau_1,\tau_1,\tau_2)|\mathcal{F}_t\right] \tag{4.11}$$

and is explicitly determined by

$$RP^F(t,\tau_1,\tau_2) = F(t,\tau_1,\tau_2)\,(1 - D_{RP}(t,\tau_1,\tau_2))\ , \tag{4.12}$$

where

$$\begin{aligned} &D_{RP}(t,\tau_1,\tau_2)\\ &:= \exp\Bigg\{ \int_0^{\tau_1} e^{-\int_s^{\tau_1}\kappa(v)dv}\mathbb{E}\left[\mu(s,U)\right]ds + \frac{1}{2}\int_t^{\tau_1}\mathbb{E}\left[\sigma(s,U)\right]^2\left(e^{-2\int_s^{\tau_1}\kappa(v)dv}+1\right)ds\\ &+ \int_t^{\tau_1}\psi^{\mathbb{P}}\left(\mathbb{E}\left[\eta(s,U)\right]e^{-\int_s^{\tau_1}\kappa(v)dv}\right) + \psi^{\widetilde{\mathbb{Q}}}\left(\mathbb{E}\left[\eta(s,U)\right]\right)ds\\ &+ \int_0^t e^{-\int_s^t\kappa(v)dv}\mathbb{E}\left[\sigma(s,U)\right]dW_s^{\mathbb{P}} - \int_0^t\mathbb{E}\left[\sigma(s,U)\right]dW_s^{\widetilde{\mathbb{Q}}}\\ &+ \int_0^t e^{-\int_s^t\kappa(v)dv}\mathbb{E}\left[\eta(s,U)\right]d\widetilde{J}_s^{\mathbb{P}} - \int_0^t\mathbb{E}\left[\eta(s,U)\right]d\widetilde{J}_s^{\widetilde{\mathbb{Q}}}\Bigg\}. \end{aligned} \tag{4.13}$$

Proof (*i*) F is always smaller or equal than F^A analogous to Corollary 3.1 (i) since Jensen's inequality also holds under the physical measure.

(*ii*) Analogous to Corollary 3.1 (ii).

(*iii*) A straightforward evaluation of $\mathbb{E}_{\mathbb{P}}\left[F(\tau_1,\tau_1,\tau_2)|\mathcal{F}_t\right]$ leads to

$$\mathbb{E}_{\mathbb{P}}\left[F(\tau_1,\tau_1,\tau_2)|\mathcal{F}_t\right] = F(t,\tau_1,\tau_2)D_{RP}(t,\tau_1,\tau_2). \tag{4.14}$$

So that Equation (4.12) directly follows.

□

Hence, the spread between the swap prices F and F^a under the physical measure $\mathbb{P}$ coincides with the pricing spread from Corollary 3.1 (ii) under the instantaneous risk measure $\mathbb{Q}$, as the numéraire in Equation (3.28) is not affected by a change of measure.

4.3 The swap price *F* under its risk-neutral measure $\widetilde{\mathbb{Q}}$

In order to derive the swap's martingale measure $\widetilde{\mathbb{Q}}$, we introduce the *swap's* market price of risk for the swap price resulting from geometric averaging in the next definition:

Definition 4.1 We define the swap's market price of risk by $\Pi^{\mathbb{P}\widetilde{\mathbb{Q}}} := (\Pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}}, \Pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}})$, where

$$\Pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2) := \frac{\mathbb{E}[\mu(t,U)] - \kappa(t)\ln F(t,\tau_1,\tau_2) + \frac{1}{2}\mathbb{E}[\sigma(t,U)]^2}{\mathbb{E}[\sigma(t,U)]}\ , \tag{4.15}$$

$$\Pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2) := 1 - \int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)\frac{\mathbb{E}[\eta(t,U)]}{\int_{\mathbb{R}}\left(e^{\mathbb{E}[\eta(t,U)]z}-1\right)\ell^{\mathbb{P}}(dz)}\,. \tag{4.16}$$

Note that the market price of risk does not enter the jump size distribution since we restrict $\Pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}$ to depend on trading time and delivery period. Hence, the market price of jump risk affects the jump intensity only.

We follow the methodology of Benth et al. [7] to change the measure from the physical measure $\mathbb{P}$ to the swap's risk-neutral measure $\widetilde{\mathbb{Q}}$. Therefore, let $\pi = (\pi_1, \pi_2)$ be a predictable process satisfying

$$\mathbb{E}\left[\int_0^{\tau_1}\|\pi(s,\tau_1,\tau_2)\|^2\,ds\right] < \infty\,. \tag{4.17}$$

We define a new process $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}$ being the unique strong solution of

$$dZ^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2) = Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(t-,\tau_1,\tau_2)dH(t,\tau_1,\tau_2)\,, \tag{4.18}$$

such that $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(0,\tau_1,\tau_2) = 1$, where

$$dH(t,\tau_1,\tau_2) = \pi_1(t,\tau_1,\tau_2)dW_t^{\mathbb{P}} + \pi_2(t,\tau_1,\tau_2)d\widetilde{J}_t^{\mathbb{P}}\,. \tag{4.19}$$

If π_j satisfies Equation (4.17), then H is a well-defined square integrable martingale. Note that the process $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}$ is known as the Doléans-Dade exponential of H that is explicitly given by

$$Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2) = e^{H(t,\tau_1,\tau_2)-\frac{1}{2}\int_0^t\pi_1(s,\tau_1,\tau_2)^2ds}\prod_{0<s\le t}(1+\Delta H(s,\tau_1,\tau_2))\,e^{-\Delta H(s,\tau_1,\tau_2)}\,. \tag{4.20}$$

If $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}$ is a strictly positive martingale, then we can define the equivalent probability measure $\widetilde{\mathbb{Q}}$ by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} = Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)\,, \tag{4.21}$$

where $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}$ functions as the Radon-Nikodym derivative. If we further assume that $\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)] = 1$, then Girsanov's theorem (see Øksendal and Sulem [39], cf. Theorem 1.35) states for $\pi := -\Pi^{\mathbb{P}\widetilde{\mathbb{Q}}}$ that

$$W_t^{\widetilde{\mathbb{Q}}} = W_t^{\mathbb{P}} + \int_0^t \Pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)ds\,, \tag{4.22}$$

is a Brownian motion with respect to $\widetilde{\mathbb{Q}}$ and

$$\widetilde{N}^{\widetilde{\mathbb{Q}}}(dt,dz) = \widetilde{N}^{\mathbb{P}}(dt,dz) + \Pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2)\ell^{\mathbb{P}}(dz)dt\,, \tag{4.23}$$

is a $\widetilde{\mathbb{Q}}$-compensated Poisson random measure of $N(\cdot,\cdot)$.

Under the above assumptions specified later a straightforward valuation leads to the following result:

Proposition 4.1 *The swap price process F defined in Equation (2.14) is a martingale under $\widetilde{\mathbb{Q}}$ given by*

$$\frac{dF(t,\tau_1,\tau_2)}{F(t-,\tau_1,\tau_2)} = \mathbb{E}[\sigma(t,U)]dW_t^{\widetilde{\mathbb{Q}}} + \int_{\mathbb{R}}\left(e^{\mathbb{E}[\eta(t,U)]z}-1\right)\widetilde{N}^{\widetilde{\mathbb{Q}}}(dt,dz)\,. \tag{4.24}$$

We would like to investigate the consequences of our previous assumptions.

Remark 4.2 (*i*) The Doléans-Dade exponential in Equation (4.20) is positive if $\pi_2(s-)\Delta J > -1$, i.e., if $\Pi_2^{\mathbb{P}\tilde{\mathbb{Q}}}\Delta J < 1$. Hence, similar to Benth et al. [7], we need to assume that the market price of jump risk is bounded and deterministic over the entire time period such that $\Pi_2^{\mathbb{P}\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2)z < 1$ for $\ell^{\mathbb{P}}$-a.e. $z \in \mathbb{R}$ and for each $t \in [0,\tau_1]$.

(*ii*) If $\ln f$, and so $\ln F$, is driven by a compensated Poisson process only, then the swap's market price of risk is attained by $\Pi^{\mathbb{P}\tilde{\mathbb{Q}}} := (0, \Pi_2^{\mathbb{P}\tilde{\mathbb{Q}}})$, where

$$\Pi_2^{\mathbb{P}\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2) := 1 - \frac{\mathbb{E}[\eta(t,U)]\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)}{\int_{\mathbb{R}}\left(e^{\mathbb{E}[\eta(t,U)]z}-1\right)\ell^{\mathbb{P}}(dz)} + \frac{\mathbb{E}[\mu(t,U)]-\kappa(t)\ln F(t,\tau_1,\tau_2)}{\int_{\mathbb{R}}\left(e^{\mathbb{E}[\eta(t,U)]z}-1\right)\ell^{\mathbb{P}}(dz)}\,. \tag{4.25}$$

In this setting, we need to require that $\kappa(t) \equiv 0$.

Note that a positive local martingale is a supermartingale. Hence, in order to prove that the Radon-Nikodym density $Z^{\mathbb{P}\tilde{\mathbb{Q}}}$ is a true martingale, it is sufficient to verify that $\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\tilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)] = 1$ is satisfied, which is proven in the next proposition.

Proposition 4.2 *Under Assumption 6 in Appendix A, the process $Z^{\mathbb{P}\tilde{\mathbb{Q}}}$ defined by Equation (4.18) is a strictly positive true martingale.*

Proof In Appendix B, we prove this proposition even in a stochastic volatility framework. □

4.4 The approximated swap price F^a under the instantaneous risk-neutral measure

We introduce the *instantaneous* market price of risk for the approximated swap price in the next definition.

Definition 4.2 We define the instantaneous market price of risk for the approximated swap by $\Pi^{\mathbb{P}\mathbb{Q}} := (\Pi_1^{\mathbb{P}\mathbb{Q}}, \Pi_2^{\mathbb{P}\mathbb{Q}})$, where

$$\Pi_1^{\mathbb{P}\mathbb{Q}}(t,\tau_1,\tau_2) := \frac{\mathbb{E}[\mu(t,U)]-\kappa(t)\ln F(t,\tau_1,\tau_2)+\frac{1}{2}\mathbb{E}[\sigma^2(t,U)]}{\mathbb{E}[\sigma(t,U)]}\,, \tag{4.26}$$

$$\Pi_2^{\mathbb{P}\mathbb{Q}}(t,\tau_1,\tau_2) := 1 - \int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)\frac{\mathbb{E}[\eta(t,U)]}{\int_{\mathbb{R}}\left(\mathbb{E}[e^{\eta(t,U)z}]-1\right)\ell^{\mathbb{P}}(dz)}\,. \tag{4.27}$$

Note that we assume that the market price of jump risk affects the jump intensity only. The market price of risk does not enter the jump size distribution since we restrict $\Pi_2^{\mathbb{P}\mathbb{Q}}$ to depend on trading and delivery period.

Similar to the last subsection, we can define the equivalent (instantaneous) probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z^{\mathbb{P}\mathbb{Q}}(\tau_1,\tau_1,\tau_2)\,, \tag{4.28}$$

where $Z^{\mathbb{P}\mathbb{Q}}$ functions as the Radon-Nikodym derivative characterized by $\pi := -\Pi^{\mathbb{P}\mathbb{Q}}$. If we further assume that $\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\mathbb{Q}}(\tau_1,\tau_1,\tau_2)] = 1$, then Girsanov's theorem (see Øksendal and Sulem [39], cf. Theorem 1.35) states that

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \Pi_1^{\mathbb{P}\mathbb{Q}}(s,\tau_1,\tau_2)ds\,, \tag{4.29}$$

is a Brownian motion with respect to $\mathbb{Q}$ and

$$\widetilde{N}^{\mathbb{Q}}(dt, dz) = \widetilde{N}^{\mathbb{P}}(dt, dz) + \Pi_2^{\mathbb{P}\mathbb{Q}}(t, \tau_1, \tau_2)\ell^{\mathbb{P}}(dz)dt \ , \tag{4.30}$$

is a $\mathbb{Q}$-compensated Poisson random measure of $N(\cdot, \cdot)$.

Under the above assumptions a straightforward valuation leads to the following result:

Proposition 4.3 *The approximated swap price process F^a defined in Equation (2.13) is a martingale under $\mathbb{Q}$ given by*

$$\frac{dF^a(t, \tau_1, \tau_2)}{F^a(t-, \tau_1, \tau_2)} = \mathbb{E}[\sigma(t, U)]dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} \left(\mathbb{E}[e^{\eta(t,U)z}] - 1\right) \widetilde{N}^{\mathbb{Q}}(dt, dz) \ . \tag{4.31}$$

We refer to Sect. 4.3 for the consequences of the assumptions made above.

4.5 The decomposition of the market price of risk

From the previous subsections, we know the corresponding market prices of risk for the swap price resulting from geometric averaging (see Definition 4.1) and from approximated averaging (see Definition 4.2). In this subsection, we identify a clear distinction between both market prices of risk leading to a specific decomposition that is strongly connected to the MPDP.

We now introduce the *decomposition* of the swap's market price of risk, from Definition 4.1, which finally connects the instantaneous market price of risk, specified in Definition 4.2, and the MPDP, defined in Definition 3.1. The decomposition result is stated in the next proposition.

Proposition 4.4 *The swap's market price of risk, $\Pi^{\mathbb{P}\widetilde{\mathbb{Q}}}$, resulting from geometric averaging (see Definition 4.1), decomposes into*

$$\Pi_j^{\mathbb{P}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) = \Pi_j^{\mathbb{P}\mathbb{Q}}(t, \tau_1, \tau_2) + \bar{\Pi}_j^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) \ , \quad \textit{for } j = 1, 2 \ , \tag{4.32}$$

where $\Pi_j^{\mathbb{P}\mathbb{Q}}$ is specified in Definition 4.2 and $\bar{\Pi}_j^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ defines the spread of diffusion and jump risk. More precisely,

$$\bar{\Pi}_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) = -\frac{1}{2}\frac{\mathbb{V}[\sigma(t, U)]}{\mathbb{E}[\sigma(t, U)]} \ , \tag{4.33}$$

$$\bar{\Pi}_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) = -\mathbb{E}[\eta(t, U)] \int_{\mathbb{R}} zG(dz) \frac{\int_{\mathbb{R}} \mathbb{E}[e^{\eta(t,U)z}] - e^{\mathbb{E}[\eta(t,U)]z} G(dz)}{\int_{\mathbb{R}} \left(\mathbb{E}[e^{\eta(t,U)z}] - 1\right) G(dz) \int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right) G(dz)} \ , \tag{4.34}$$

where $\bar{\Pi}_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}$ is independent of the jump intensity.

Proof The result is attained by subtracting the swap's market price of risk $\Pi^{\mathbb{P}\widetilde{\mathbb{Q}}}$ defined in Definition 4.1 from the instantaneous market price of risk $\Pi^{\mathbb{P}\mathbb{Q}}$ defined in Definition 4.2. □

Hence, we found a representation of the swap's market price of risk of the swap price F, characterized by the instantaneous market price of risk of the approximated swap F^a and the spread $\bar{\Pi}^{\mathbb{Q}\widetilde{\mathbb{Q}}} = \left(\bar{\Pi}_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}, \bar{\Pi}_2^{\mathbb{Q}\widetilde{\mathbb{Q}}}\right)$. We further investigate the spread in the next lemma.

Lemma 4.4 *(i) The spread of diffusion risk, $\bar{\Pi}_1^{\mathbb{Q}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2)$, is negative for all trading times $t \in [0, \tau_1]$.*

(*ii*) *If the average jump size is positive, i.e., if* $\int_{\mathbb{R}} zG(dz) > 0$*, then the spread of jump risk,* $\bar{\Pi}_2^{\mathbb{Q}\tilde{\mathbb{Q}}}$*, is negative.*

(*iii*) *If the average jump size is zero, i.e., if* $\int_{\mathbb{R}} zG(dz) = 0$*, then the spread of jump risk,* $\bar{\Pi}_2^{\mathbb{Q}\tilde{\mathbb{Q}}}$*, is zero.*

(*iv*) *If the volatility is independent of the delivery, i.e., if* $\sigma(t,u) \perp\!\!\!\perp u$*, then the spread of diffusion risk is zero, i.e.,* $\bar{\Pi}_1^{\mathbb{Q}\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2) \equiv 0$.

(*v*) *If the jump coefficient is independent of the delivery, i.e., if* $\eta(t,u) \perp\!\!\!\perp u$*, then the spread of jump risk is zero, i.e.,* $\bar{\Pi}_2^{\mathbb{Q}\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2) \equiv 0$.

Proof The results in (*i*) and (*ii*) follow directly from Jensen's inequality. The results in (*iii*) and (*iv*) follow from the fact that the numerator becomes zero whenever the delivery period disappears. □

As a result, whenever the spread $\bar{\Pi}_j^{\mathbb{Q}\tilde{\mathbb{Q}}}$ is negative for $j = 1, 2$, then the approximated swap induces more risk than the swap price based on geometric averaging. In particular, the considered spread has the same properties as the MPDP (see Kemper et al. [30]). Indeed, a comparison with our previous considerations in Sect. 2 gives the following insights:

Remark 4.3 (*i*) The spread of diffusion risk, $\bar{\Pi}_1^{\mathbb{Q}\tilde{\mathbb{Q}}}$, coincides with the MPDP of diffusion risk, $\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}$, in Equation (3.1) from Sect. 2.

(*ii*) The spread of jump risk, $\bar{\Pi}_2^{\mathbb{Q}\tilde{\mathbb{Q}}}$, does not coincide with the MPDP of jump risk, $\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}$, from Equation (3.2) but with $\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}(1 - \Pi_2^{\mathbb{P}\mathbb{Q}})$. This connection occurs naturally by the change of measure.

(*iii*) The condition in Lemma 4.4(*iii*) holds true, for example, when jump sizes follow are standard normal distribution.

Hence, starting from the physical measure, we can find the swaps true martingale measure based on the swap's market price of risk defined in Definition 4.1. If we would like to adjust already existing models using the instantaneous market price of risk, we can easily adjust the model through the spread defined in Proposition 4.4 that is strongly connected to the MPDP defined in Definition 3.1.

4.6 An example based on short-term long-term evolution

In this section, we give an example in the spirit of the popular short-term long-term models based on Gibson and Schwartz [21] and Schwartz and Smith [45]. On the spot level, these models typically divide the price evolution into a non-stationary Gaussian component, ginving the long term mean of spot prices. It is influenced e.g. by political or regulatory decisions. Furthermore, a stationary mean reverting component describes short term price fluctuations due to imbalances in supply and demand. This mean reverting short term component leads to a Samuelson effect in the futures price dynamics, see e.g. Benth and Schmeck [8]. Then, the futures prices evolve according to

$$\begin{aligned}\frac{df(t,\tau)}{f(t-,\tau)} &= \sigma dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \left(e^{\eta(t,\tau)z} - 1\right) \widetilde{N}^{\mathbb{P}}(dt,dz) \\ &\quad + \left(\mu - \kappa \ln f(t,\tau) + \frac{1}{2}\sigma^2 + \psi^{\mathbb{P}}(\eta(t,\tau))\right) dt, \end{aligned} \tag{4.35}$$

where $\eta(t,\tau) = e^{-\Lambda(\tau-t)}$ with $\Lambda > 0$ captures the term structure effect in the spirit of Samuelson (cf. the volatility in Example 3.2), and $\widetilde{N}^{\mathbb{P}}$ is a compund Poisson process. As we want to give an example under the physical measure, we have added a drift term similar to Equation (4.3) and assume constant drift and mean-reversion parameters μ and $\kappa > 0$ for simplicity. Assuming a gradual inflow of renewables, jumps are more likely downward pointing as opposed to upward pointing jumps in power systems with more nuclear, hydro, gas, and temperature dependent demand (cf. Paraschiv et al. [41] and Hinderks and Wagner [26]). To capture the renewable effect within the jump size distribution, we assume for simplicity a negative jump size distribution captured by the Dirac measure assigned to a jump size of -1.

Under Assumptions 4 and 5 in Appendix A, the swap price dynamics under the physical measure $\mathbb{P}$ based on geometric averaging evolve according to Lemma 4.2 as

$$\begin{aligned}\frac{dF(t,\tau_1,\tau_2)}{F(t-,\tau_1,\tau_2)} &= \sigma dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right) \widetilde{N}^{\mathbb{P}}(dt,dz) \\ &\quad + \left(\mu - \kappa \ln F(t,\tau_1,\tau_2) + \frac{1}{2}\sigma + \psi^{\mathbb{P}}(\mathbb{E}[\eta(t,U)])\right) dt,\end{aligned} \tag{4.36}$$

where the swap's jump coefficient for a constant settlement type function $w(u,\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}$ is given by

$$\mathbb{E}[\eta(t,U)] = \frac{1-e^{-(\tau_2-\tau_1)\Lambda}}{(\tau_2-\tau_1)\Lambda}\eta(t,\tau_1)\ . \tag{4.37}$$

Note that the Assumptions 4 and 5 are satisfied, since μ, κ, and σ are constant, $\eta(t,u)$ is deterministic and finite for $t \in [0,\tau_1]$ and $u \in [\tau_1,\tau_2]$ for a finite trading horizon $\tau_1 < \infty$, and the second moment of the negative exponential distribution as well as the moment generating function exist. To turn to the swap's risk neutral measure $\widetilde{\mathbb{Q}}$, the swap's market price of volatility and jump risk are given according to Definition 4.1 by

$$\Pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2) := \frac{\mu - \kappa \ln F(t,\tau_1,\tau_2) + \frac{1}{2}\sigma^2}{\sigma}\ , \tag{4.38}$$

$$\Pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2) := 1 + \frac{\mathbb{E}[\eta(t,U)]}{\lambda_J^- \int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right) G(dz)}\ . \tag{4.39}$$

Thinking in the spirit of the short-term long-term framework, the Gaussian part of the dynamics stands for long term evolutions. Thus, here Π_1 captures effects connected to the incorporation of the delivery period that have a long term character. On the other hand, the Jump component stands for short-term evolutions due to imbalances in supply and demand. Π_2 captures delivery dependend effects connected to expectations on e.g. negative spikes in the underlying.

Under Assumption 6, the swap price process F is a martingale under the risk neutral measure $\widetilde{\mathbb{Q}}$ evolving as

$$\frac{dF(t,\tau_1,\tau_2)}{F(t-,\tau_1,\tau_2)} = \sigma dW_t^{\widetilde{\mathbb{Q}}} + \int_{\mathbb{R}} \left(e^{\mathbb{E}[\eta(t,U)]z} - 1\right) \widetilde{N}^{\widetilde{\mathbb{Q}}}(dt,dz)\ , \tag{4.40}$$

where $W_t^{\widetilde{\mathbb{Q}}}$ and $\widetilde{N}^{\widetilde{\mathbb{Q}}}(dt,dz)$ are as in Equations (4.22) and (4.23). Note that the Assumption 6 (i) is satisfied whenever

$$0 \geq 2e^{-\mathbb{E}[\eta(t,u)]} + \mathbb{E}[\eta(t,u)] - 2\ , \tag{4.41}$$

which is for example true for $t = 0, \tau_1 = \frac{1}{12}, \tau_2 = \frac{2}{12}$, and a Samuelson parameter of $\Lambda = \log 2$. This corresponds to monthly delivery periods and fast mean reverting spikes in the underlying that mean revert back half way within two days. The remaining assumptions in Assumption 6 are satisfied, since μ, κ, and σ are constant, $\eta(t, u)$ is deterministic and finite for $t \in [0, \tau_1]$ and $u \in [\tau_1, \tau_2]$ for a finite trading horizon $\tau_1 < \infty$, and the fourth moment as well as the moment generating function exist for the Dirac measure.

5 Conclusion

In the framework of jump diffusions, we develop the new framework of the so-called market price of risk for the delivery periods (MPDP) for modelling the dynamics of the term structure of swap prices. The main advantage is that this approach allows for a more precise pricing of electricity swap contracts, while avoiding any kind of approximations when implementing the delivery period in a geometric modelling setting.

We adjust the Heston type setting of Kemper et al. [30] to a jump framework of Merton type leading to the MPDP for diffusion *and* jump risk. We thus identify the MPDP for the jump component, which turns out to be negative as it is the case for the diffusion component. In addition, we transfer the model to the physical measure under which we allow for mean-reversion and delivery-dependent effects such as seasonalities and term-structure effects. A comparison of the risk-neutral measures of the swap resulting from geometric and approximated averaging, $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$, offers the decomposition of the "swap's" market price of risk comprising the "instantaneous" market price of risk and the MPDP for jump and diffusion risk. We may refer $\widetilde{\mathbb{Q}}$ to the "swap's" risk-neutral measure since the swap price is a $\widetilde{\mathbb{Q}}$-martingale without any approximations. In contrast, the "instantaneous" risk-neutral measure, $\mathbb{Q}$, results from an approximation of the swap price leading in general not to the "swap's" pricing measure. Consequently, any pricing methodology based on approximated averaging can easily be turned to the "swap's" risk-neutral measure by an application of our MPDP.

We compare swap prices resulting from geometric averaging with swaps based on approximated averaging in line with with Kemper et al. [30] and Bjerksund et al. [9]. We find that different averaging techniques lead to a pricing spread that stays untouched by measure changes. In particular, the swap price based on geometric averaging turns out to be smaller than the one resulting from approximated averaging. The spread itself can be characterized by a change of measure based on the MPDP as introduced by Kemper et al. [30]. As the MPDP leads to the swap's pricing measure, $\widetilde{\mathbb{Q}}$, the spread remediates the approximated swap price and adjusts it downwards to the correct price of the swap contract.

We finally investigate the model under the physical measure. To this end, we consider two types of models characterized, on the one hand, by seasonality in the delivery time (see Fanelli and Schmeck [20]) and, on the other hand, by the Samuelson effect (see Samuelson [43]). We adapt them to a jump setting, and provide the corresponding discretized swap price models. Seasonal delivery dependence causes a MPDP that is constant over trading time and seasonal in delivery time. In contrast, term-structure dependence analytically induces a decreasing behavior of the MPDP over trading time. Hence, the closer we reach the expiration date, the more pronounced the MPDP, and the larger the pricing spread. Consequently, the MPDP reduces risk caused by approximated averaging especially when the end of the maturity approaches.

To conclude, we expand the MPDP to the jump setting and investigate the MPDP influenced by typical characteristics of the electricity market. We expect that a higher market share of renewables cause higher delivery-dependent seasonalities in the volatility and consequently leading to a growing MPDP. This applies especially for Germany, having ambitious plans for future investments in renewable energy. This increases the importance of the MPDP of diffusion risk, which has to be taken into account to ensure an accurate pricing procedure. However, this is a questions for future research.

Appendix

A Technical Requirements

Assumption 1 For the model (2.5), we make the following assumptions to apply Itô's formula (see Øksendal and Sulem [39], cf. Theorem 1.16):

(*i*) For $\mathcal{A} := \{(t,\tau) \in [0,\tau_2]^2 : t \leq \tau\}$ the functions $\sigma : \mathcal{A} \to \mathbb{R}^+$ and $\eta : \mathcal{A} \to \mathbb{R}$ are adapted such that the integrals exist, meaning that $\mathbb{Q}[\int_0^t \sigma^2(s,\tau) + \int_{\mathbb{R}} |(e^{\eta(s,\tau)z} - 1)|\ell^{\mathbb{Q}}(dz)ds < \infty] = 1$ for all $0 \leq t \leq \tau$.

In order to ensure existence and uniqueness of solutions to Equation (2.9) (see Øksendal and Sulem [39], cf. Theorem 1.19), we further assume:

(*ii*) (At most linear growth) There exists a constant $C_1 < \infty$ such that

$$|\sigma(t,\tau)x|^2 + \int_{\mathbb{R}} |(e^{\eta(t,\tau)z} - 1)x|^2 \ell^{\mathbb{Q}}(dz) \leq C_1(1 + |x|^2) , \quad \forall x \in \mathbb{R} . \tag{A.1}$$

(*iii*) (Lipschitz continuity) There exists a constant $C_2 < \infty$ such that

$$|\sigma(t,\tau)x - \sigma(t,\tau)y|^2 + \int_{\mathbb{R}} |(e^{\eta(t,\tau)z} - 1)x - (e^{\eta(t,\tau)z} - 1)y|^2 \ell^{\mathbb{Q}}(dz) \leq C_2(|x-y|^2) , \quad \forall x, y \in \mathbb{R} . \tag{A.2}$$

Hence, by Øksendal and Sulem [39] (cf. Theorem 1.19), it follows that $\mathbb{E}_{\mathbb{Q}}[|f(t,\tau)|^2] < \infty$ for all $t \in [0,\tau]$. By the Itô-Lévy Isometry (see Øksendal and Sulem [39], cf. Theorem 1.17) part (*iii*) implies that

$$\mathbb{E}_{\mathbb{Q}}[f^2(t,\tau)] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \sigma^2(v,\tau) f^2(v,\tau) + f^2(v,\tau) \int_{\mathbb{R}} (e^{\eta(v,\tau)z} - 1)^2 \ell^{\mathbb{Q}}(dz)dv\right] < \infty , \tag{A.3}$$

so that the square-integrability conditions are satisfied implying that f is a true martingale under $\mathbb{Q}$.

Assumption 2 For the geometric weightening approach in Equation (2.14), we need to apply the stochastic Fubini Theorem (see Protter [42], cf. Theorem 65, Chapter IV. 6). Therefore, we assume that

(*i*) $\sigma(\cdot,\tau)$ and $\eta(\cdot,\tau)$ are $\mathcal{P} \times \mathcal{B}((\tau_1,\tau_2])$ measurable, where $\mathcal{P}$ is the predictable σ-algebra making all adapted, càglàd processes measurable,

(*ii*) $\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1} \int_{\tau_1}^{\tau_2} \sigma^2(t,u) w(u,\tau_1,\tau_2) du\, dt\right] < \infty$,

(*iii*) $\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1}\int_{\tau_1}^{\tau_2}\eta(t,u)w(u,\tau_1,\tau_2)du\,dt\right]<\infty$,

(*iv*) $\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1}\int_{\mathbb{R}}\int_{\tau_1}^{\tau_2}(e^{\eta(t,u)z}-1)^2w(u,\tau_1,\tau_2)du\,\ell^{\mathbb{Q}}(dz)dt\right]<\infty$,

such that the integrals still exist and linear growth and Lipschitz continuity are satisfied (see Assumption 1).

Assumption 3 To apply Girsanov's Theorem (see Øksendal and Sulem [39], cf. Theorem 1.35), we assume that $\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}$ and $\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}$ are predictable, satisfying

(*i*) $\mathbb{E}_{\mathbb{Q}}[\int_0^{\tau_1}\Pi_1^{\mathbb{Q}\tilde{\mathbb{Q}}}(s,\tau_1,\tau_2)^2ds]<\infty$, such that $Z_1^{\mathbb{Q}\tilde{\mathbb{Q}}}$ is a true martingale, and

(*ii*) $\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2)z \le 1$ for $\ell^{\mathbb{Q}}$-a.e. $z\in\mathbb{R}$ and all $t\in[0,\tau_1]$ and $\mathbb{E}_{\mathbb{Q}}[\int_0^{\tau_1}\exp\left\{\ln(1-\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z)+\Pi_2^{\mathbb{Q}\tilde{\mathbb{Q}}}(s,\tau_1,\tau_2)z\right\}ds]<\infty$, such that $Z_2^{\mathbb{Q}\tilde{\mathbb{Q}}}$ is a true martingale.

Assumption 4 For the model in Equation (4.1), we make the following assumptions to apply Itô's formula (see Øksendal and Sulem [39], cf. Theorem 1.16):

(*i*) The functions $\mu\colon\mathcal{A}\to\mathbb{R}, \kappa\colon[0,\tau_1]\to\mathbb{R}^+, \sigma\colon\mathcal{A}\to\mathbb{R}^+$, and $\eta\colon\mathcal{A}\to\mathbb{R}^+$ are adapted such that the integrals exist, meaning that for all $0\le t\le\tau$, we have $\mathbb{P}[\int_0^t\mu^2(v,\tau)+\kappa^2(v)+\sigma^2(v,\tau)+\int_{\mathbb{R}}\eta^2(v,\tau)z^2+e^{2\eta(v,\tau)z}\ell^{\mathbb{P}}(dz)dv<\infty]=1$.

In order to ensure existence and uniqueness of solutions to Equation (4.4) (see Øksendal and Sulem [39], Theorem 1.19), we further assume:

(*ii*) (At most linear growth) There exists a constant $C_1<\infty$ such that $\forall x\in\mathbb{R}$:

$$|\mu(t,\tau)|^2+|\kappa(t)x|^2+|\sigma(t,\tau)|^2+|\eta(t,\tau)|^2\int_{\mathbb{R}}|z|^2\ell^{\mathbb{P}}(dz)$$
$$+\int_{\mathbb{R}}|e^{\eta(t,\tau)z}|^2\ell^{\mathbb{P}}(dz)\le C_1(1+|x|^2)\,. \tag{A.4}$$

(*iii*) (Lipschitz continuity) There exists a constant $C_2<\infty$ such that

$$\kappa^2(t)|x-y|^2\le C_2(|x-y|^2)\,,\quad\forall x,y\in\mathbb{R}\,. \tag{A.5}$$

Assumption 5 For the geometric weightening approach in Equation (2.14) applied in Sect. 4, we apply the stochastic Fubini Theorem (see Protter [42], cf. Theorem 65, Chapter IV. 6). Therefore, we assume that

(*i*) κ,μ,σ,η are $\mathcal{P}\times\mathcal{B}((\tau_1,\tau_2])$ measurable, where $\mathcal{P}$ is the predictable σ-algebra making all adapted, càglàd processes measurable,

(*ii*) $\mathbb{E}_{\mathbb{P}}\left[\int_0^{\tau_1}\int_{\tau_1}^{\tau_2}\mu(t,u)w(u,\tau_1,\tau_2)du\,dt\right]<\infty$,

(*iii*) $\mathbb{E}_{\mathbb{P}}\left[\int_0^{\tau_1}\int_{\tau_1}^{\tau_2}\sigma^2(t,u)w(u,\tau_1,\tau_2)du\,dt\right]<\infty$,

(*iv*) $\mathbb{E}_{\mathbb{P}}\left[\int_0^{\tau_1}\int_{\mathbb{R}}\int_{\tau_1}^{\tau_2}(e^{\eta(t,u)z}-1)^2w(u,\tau_1,\tau_2)du\,\ell^{\mathbb{P}}(dz)dt\right]<\infty$,

(*v*) $\mathbb{E}_{\mathbb{P}}\left[\int_0^{\tau_1}\int_{\tau_1}^{\tau_2}\eta(t,u)w(u,\tau_1,\tau_2)du\,dt\right]<\infty$.

Assumption 6 To prove that $Z^{\mathbb{P}\tilde{\mathbb{Q}}}$ is a true martingale, we assume that κ,μ,σ,η are deterministic and that

(*i*) $\Pi_2^{\mathbb{P}\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2)z\le 1$ for $\ell^{\mathbb{P}}$-a.e. $z\in\mathbb{R}$ and each $t\in[0,\tau_1]$,

(*ii*) $\ell^{\mathbb{P}}$ has fourth moment, that is $\int_{\mathbb{R}} z^4 \ell^{\mathbb{P}}(dz) < \infty$,
(*iii*) $\int_0^{\tau_1} \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2)\mu^2(t, u)du\, dt < \infty$,
(*iv*) $\int_0^{\tau_1} \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2)\eta^2(t, u)du\, dt < \infty$,
(*v*) $\int_0^{\tau_1} \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2)\sigma^4(t, u)du\, dt < \infty$,
(*vi*) $\int_0^{\tau_1} \kappa^2(t)dt < \infty$.

B Proof of Proposition 4.2

Inspired by Benth et al. [7], we prove that $\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1, \tau_1, \tau_2)] = 1$ and expand their Theorem 3.5 to a geometric setting with stochastic volatility in order to address settings as in Kemper et al. [30]. For the scope of the proof, we consider the swap price F from Lemma 4.2 characterized by stochastic volatility of the form

$$\sigma(t, \tau)\sqrt{\nu(t)}\,, \tag{B.1}$$

where $\sigma(t, \tau)$ is deterministic and ν is the stochastic volatility that is modeled as a Cox-Ingersoll-Ross process evolving as

$$d\nu(t) = \kappa_\nu\,(\theta_\nu - \nu(t))\,dt + \sigma_\nu\sqrt{\nu(t)}dB_t^{\mathbb{P}}\,, \tag{B.2}$$

for $\nu(0) = \nu_0 > 0$, where $B^{\mathbb{P}}$ and $\widetilde{J}^{\mathbb{P}}$ are independent of each other and $B^{\mathbb{P}}$ and $W^{\mathbb{P}}$ are correlated. In particular, we assume a correlation structure $d\langle W^{\mathbb{P}}, B^{\mathbb{P}}\rangle_t = \rho dt$ where $\rho \in (-1, 1)$ such that we can rewrite $B^{\mathbb{P}} = \rho W^{\mathbb{P}} + \sqrt{1-\rho^2}\bar{B}^{\mathbb{P}}$ for $\bar{B}^{\mathbb{P}} \perp\!\!\!\perp W^{\mathbb{P}}$. Moreover, we assume that $\kappa_\nu, \theta_\nu, \sigma_\nu > 0$ satisfy the extended Feller condition, i.e., $\sigma_\nu^2 < \kappa_\nu\theta_\nu$, to ensure that $\mathbb{E}_{\mathbb{P}}[\nu^{-2}(t)]$ is bounded on the entire trading time $t \in [0, \tau_1]$ (see Dereich et al. [18], cf. Chapter 3). Note, that the extended Feller implies the classical Feller condition (see Karatzas and Shreve [28], cf. Chapter 5) ensuring that the volatility stays positive.

We proceed in the following steps:

1. Derivation of a new risk-neutral measure $\widetilde{\mathbb{Q}}^n$ through a stopping time $\hat{\tau}_n$.
2. Proof that $\mathbb{E}_{\widetilde{\mathbb{Q}}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1, \tau_1, \tau_2)]$ is lower boundend, i.e.,

$$\begin{aligned}&\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1, \tau_1, \tau_2)\right]\\ &\quad\geq 1 - \frac{1}{n}\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_s \ln F(s, \tau_1, \tau_2)\right] - \frac{1}{n}\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_s \nu^{-1}(s)\right] - \frac{1}{n}\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_s \nu(s)\right]\,.\end{aligned}$$

3. Proof that there exist upper boundaries for $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}[\sup_s \ln F(s, \tau_1, \tau_2)]$, $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}[\sup_s \nu^{-1}(s)]$, and $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}[\sup_s \nu(s)]$, that are independent of n.

1. Derivation of $\widetilde{\mathbb{Q}}^n$. Similar to Benth et al. [7], we set $g(z) := (1+z)\log(1+z) - z$ and define the predictable compensator of $\frac{1}{2}\langle H^c, H^c\rangle + \sum_{t\leq\cdot} g(\Delta H(t))$ by

$$\begin{aligned}C(t, \tau_1, \tau_2) := \frac{1}{2}\int_0^t \pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}}(s, \tau_1, \tau_2)^2 + \pi_\nu^{\mathbb{P}\widetilde{\mathbb{Q}}}(s, \tau_1, \tau_2)^2 ds\\ + \int_0^t \int_{\mathbb{R}} g(\pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}(s, \tau_1, \tau_2)z)\ell^{\mathbb{P}}(dz)ds\,,\end{aligned}$$

where H from Equation (4.19) now embraces stochastic volatility such that

$$H(t, \tau_1, \tau_2) := \int_0^t \pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}}(s, \tau_1, \tau_2)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}(s, \tau_1, \tau_2)z\widetilde{N}^{\mathbb{P}}(ds, dz)$$

$$+\int_0^t \pi_\nu^{\mathbb{P}\widetilde{\mathbb{Q}}}(s,\tau_1,\tau_2)d\bar{B}_s^{\mathbb{P}}\,.$$

Note that this stochastic volatility setting covers a three-dimensional market price of risk $\pi := (\pi_1, \pi_2, \pi_\nu)$ for all independent random parts $W^{\mathbb{P}}$, $\widetilde{J}^{\mathbb{P}}$, $\bar{B}^{\mathbb{P}}$. As we are in an incomplete setting, we choose the market price of volatility risk, π_ν, such that the market price of risk admits the same structure as in the Heston model, i.e., $\rho\pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}} + \sqrt{1-\rho^2}\pi_\nu^{\mathbb{P}\widetilde{\mathbb{Q}}} = \frac{\delta_\nu}{\sigma_\nu}\sqrt{\nu(t)}$ (see Heston [24]). Now let us define a sequence of stopping times

$$\hat{\tau}_n := \inf\left\{t \in [0,\tau_1] : |\ln F(t,\tau_1,\tau_2)| \geq n, \text{ or } |\nu^{-1}(t)| \geq n, \text{ or } |\nu(t)| \geq n\right\}, \tag{B.3}$$

and observe that for every $n \in \mathbb{N}$, the stopped process $C(t \wedge \hat{\tau}_n, \tau_1, \tau_2)$ is bounded. Hence, by by Lépingle and Mémin [36] (cf. Theorem III.1), we know that $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(t \wedge \hat{\tau}_n, \tau_1, \tau_2)$ is a uniformly integrable martingale such that we can define the probability measure $\widetilde{\mathbb{Q}}^n$ by

$$\frac{d\widetilde{\mathbb{Q}}^n}{d\mathbb{P}} := Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1 \wedge \hat{\tau}_n, \tau_1, \tau_2)\,. \tag{B.4}$$

2. Proof of lower boundary of $\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1)]$. First, $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}$ is a positive local martingale by the assumption that $\pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}(t,\tau_1,\tau_2) \geq -1$ for all $t \in [0,\tau_1]$. Hence, it is a supermartingale, so that we know the upper boundary for $\tau_1 \geq 0$:

$$\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)] \leq \mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(0,\tau_1,\tau_2)] = 1\,.$$

Next, we consider the lower boundary, following Benth et al. [7]:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)] &\geq \mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)\mathbb{1}_{\hat{\tau}_n>\tau_1}]\\ &= \mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1\wedge\hat{\tau}_n,\tau_1,\tau_2)\mathbb{1}_{\hat{\tau}_n>\tau_1}] = \widetilde{\mathbb{Q}}^n[\hat{\tau}_n > \tau_1]\,,\end{aligned}$$

where the last equality follows from the change of measure defined in Step 1 (see Equation (B.4)). By definition of the stopping time $\hat{\tau}_n$ (see Equation (B.3)), we deduce

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)] \geq& 1-\widetilde{\mathbb{Q}}^n[\hat{\tau}_n \leq \tau_1]\\ \geq& 1-\left(\widetilde{\mathbb{Q}}^n\left[\sup_{s\in[0,\tau_1]}\ln F(s,\tau_1,\tau_2)\geq n\right]+\widetilde{\mathbb{Q}}^n\left[\sup_{s\in[0,\tau_1]}\nu^{-1}(s)\geq n\right]\right.\\ &\left.+\widetilde{\mathbb{Q}}^n\left[\sup_{s\in[0,\tau_1]}\nu(s)\geq n\right]\right)\\ \geq& 1-\frac{1}{n}\left(\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\ln F(s,\tau_1,\tau_2)\right]\right.\\ &\left.+\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\nu^{-1}(s)\right]+\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\nu(s)\right]\right)\,,\end{aligned}$$

where the last inequality follows from Markov's inequality. If we show that the expectations on the right hand side have upper boundaries that are independent of $n \in \mathbb{N}$, then $\mathbb{E}_{\mathbb{P}}[Z^{\mathbb{P}\widetilde{\mathbb{Q}}}(\tau_1,\tau_1,\tau_2)] = 1$, which is addressed in the third step.

3. Proof of upper boundaries. In order to identify upper boundaries under the measure $\widetilde{\mathbb{Q}}^n$ defined in Equation (B.4), we need to derive the dynamics of $\ln F$, ν^{-1}, and ν under

$\widetilde{\mathbb{Q}}^n$. We apply Girsanov's theorem (see Øksendal and Sulem [39], cf. Theorem 1.35) to Equations (4.7) and (B.2), where

$$W_t^{\widetilde{\mathbb{Q}}^n} = W_t^{\mathbb{P}} + \int_0^t \Pi_1^{\mathbb{P}\widetilde{\mathbb{Q}}}(s, \tau_1, \tau_2) \mathbb{1}_{[0,\hat{\tau}_n]}(s) ds \ ,$$
$$B_t^{\widetilde{\mathbb{Q}}^n} = B_t^{\mathbb{P}} + \int_0^t \frac{\delta_\nu}{\sigma_\nu} \sqrt{\nu(s)} \mathbb{1}_{[0,\hat{\tau}_n]}(s) ds \ ,$$

are correlated standard Brownian motions under $\widetilde{\mathbb{Q}}^n$ and

$$\widetilde{N}^{\widetilde{\mathbb{Q}}^n}(dt, dz) = \widetilde{N}^{\mathbb{P}}(dt, dz) + \Pi_2^{\mathbb{P}\widetilde{\mathbb{Q}}}(t, \tau_1, \tau_2) \mathbb{1}_{[0,\hat{\tau}_n]}(t) \ell^{\mathbb{P}}(dz) dt \ ,$$

is the $\widetilde{\mathbb{Q}}^n$-compensated Poisson random measure. Moreover, by Ito's formula, we find

$$d\nu^{-1}(t) = \nu^{-1}(t) \left(\kappa_\nu + \delta_\nu \mathbb{1}_{[0,\hat{\tau}_n]}(t) - \nu^{-1}(t)(\kappa_\nu \theta_\nu - \sigma_\nu^2)\right) dt - \sigma_\nu \nu^{-\frac{3}{2}}(t) dB_t^{\widetilde{\mathbb{Q}}^n} \ .$$

Hence, we can show

$$\begin{aligned}
&\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]} |\nu^{-1}(s)|\right] \\
&\overset{(\star)}{\leq} \frac{1}{\nu_0} + \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]} \int_0^s \nu^{-1}(t) \left(\kappa_\nu + \delta_\nu \mathbb{1}_{[0,\hat{\tau}_n]}(t) - \nu^{-1}(t)(\kappa_\nu \theta_\nu - \sigma_\nu^2)\right) dt\right] \\
&+ \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]} \int_0^s \sigma_\nu \nu^{-\frac{3}{2}}(t) dB_t^{\widetilde{\mathbb{Q}}^n}\right] \\
&\overset{(\star\star)}{\leq} \frac{1}{\nu_0} + (\kappa + |\delta_\nu|)\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1} \nu^{-1}(t) dt\right] \\
&+ (\kappa_\nu \theta_\nu - \sigma_\nu^2)\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1} \nu^{-2}(t) dt\right] + \sigma_\nu n^{-\frac{3}{2}} \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]} \int_0^s dB_t^{\widetilde{\mathbb{Q}}^n}\right] \\
&\overset{(\star\star\star)}{=} \frac{1}{\nu_0} + (\kappa + |\delta_\nu|) \int_0^{\tau_1} \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\nu^{-1}(t)\right] dt \\
&+ (\kappa_\nu \theta_\nu - \sigma_\nu^2) \int_0^{\tau_1} \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\nu^{-2}(t)\right] dt \ .
\end{aligned}$$

Inequality $(\star)$ follows from the integral representation of ν^{-1} and the triangle inequality. Inequality $(\star\star)$ results from the fact, that the extended Feller condition is satisfied (i.e., $\sigma_\nu^2 < \kappa_\nu \theta_\nu$) and that $\nu^{-1} \leq n$ under $\widetilde{\mathbb{Q}}^n$. Since both processes ν^{-1} and ν^{-2} are positive, the supremum disappears in the first two cases and the upper boundary is used. Equality $(\star\star\star)$ is reached by stochastic Fubini to the first two integrals and the last term disappears. From Dereich et al. [18] (cf. Chapter 3), we know that the expectations of the inverse and the inverse quadratic stochastic volatility, $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\nu^{-1}(t)\right]$ and $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\nu^{-2}(t)\right]$, can be characterized explicitly and are bounded independently of n, as long as the extended Feller condition $\sigma_\nu^2 < \kappa_\nu \theta_\nu$ is satisfied. Hence, $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]} |\nu^{-1}(s)|\right] \leq c_1 \perp\!\!\!\perp n$.

Moreover, we can show that $|\nu|^2$ is uniformly integrable:

$$\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]} |\nu(s)|^2\right] = \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]} \left(\nu_0 + \int_0^s \kappa_\nu \theta_\nu - (\kappa_\nu + \delta_\nu \mathbb{1}_{[0,\hat{\tau}_n]}(t))\nu(t) dt\right.\right.$$

$$
\begin{aligned}
&+\int_0^s \sigma_\nu \sqrt{\nu(t)} dB^{\widetilde{\mathbb{Q}}^n}(t)\Big)^2\Big] \\
&\overset{(\star)}{\leq} 4\Big(v_0^2 + \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\sup_{s\in[0,\tau_1]}\Big(\int_0^s \kappa_\nu\theta_\nu dt\Big)^2 + \sup_{s\in[0,\tau_1]}\Big(\int_0^s (\kappa_\nu + \delta_\nu \mathbb{1}_{[0,\hat{\tau}_n]}(t))\nu(t)dt\Big)^2 \\
&\quad + \sup_{s\in[0,\tau_1]}\Big(\int_0^s \sigma_\nu\sqrt{\nu(t)}dB_t^{\widetilde{\mathbb{Q}}^n}\Big)^2\Big]\Big) \\
&\overset{(\star\star)}{\leq} 4\Big(v_0^2 + 4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\Big(\int_0^{\tau_1}\kappa_\nu\theta_\nu dt\Big)^2\Big] \\
&\quad + 4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\Big(\int_0^{\tau_1}(\kappa_\nu + \delta_\nu\mathbb{1}_{[0,\hat{\tau}_n]}(t))\nu(t)dt\Big)^2\Big] \\
&\quad + 4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\Big(\int_0^{\tau_1}\sigma_\nu\sqrt{\nu(t)}dB_t^{\widetilde{\mathbb{Q}}^n}\Big)^2\Big]\Big) \\
&\overset{(\star\star\star)}{\leq} 4\Big(v_0^2 + 4\tau_1\int_0^{\tau_1}\kappa_\nu^2\theta_\nu^2 dt + 4\tau_1(\kappa_\nu + |\delta_\nu|)^2\int_0^{\tau_1}\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\sup_{s\in[0,t]}\nu(s)^2\Big]dt \\
&\quad + 4\sigma_\nu^2\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\int_0^{\tau_1}\nu(t)dt\Big]\,,\Big)
\end{aligned}
$$

where the first equality represents the integral version of ν. Inequality $(\star)$ results from the Cauchy-Schwartz inequality to the sum and an application of the triangle inequality. We apply Doob's inequality to all expectations in Inequality $(\star\star)$. In Inequality $(\star\star\star)$, we apply the Cauchy-Schwartz inequality to the first and second integral and apply Ito's isometry to the last summand. We finish with the stochastic Fubini to the second integral while making the integrand even bigger. Note that for the last summand, we have $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1}\nu(t)dt\right] \leq \tilde{c}_\nu \perp\!\!\!\perp n$ since we can find explicit expressions in Cont and Tankov [16] (cf. Chapter 15). Setting $c_\nu := 4v_0^2 + 16\tau_1^2\kappa_\nu^2\theta_\nu^2 + 16\sigma_\nu^2\tilde{c}_\nu$, then, by Gronwall, we receive $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}|\nu(s)|^2\right] \leq c_\nu e^{16(\kappa_\nu+|\delta_\nu|)^2\tau_1^2} =: c_2 \perp\!\!\!\perp n$.

Next, we show that $|\ln F|^2$ is uniformly integrable:

$$
\begin{aligned}
&\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\sup_{s\in[0,\tau_1]}|\ln F(s,\tau_1,\tau_2)|^2\Big] \\
&\quad = \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\sup_{s\in[0,\tau_1]}\Big(\ln F(0,\tau_1,\tau_2) + \int_0^s \left(1 - \mathbb{1}_{[0,\hat{\tau}_n]}(t)\right)\left(\mathbb{E}[\mu(t,U)] - \kappa(t)\ln F(t,\tau_1,\tau_2)\right)dt \\
&- \int_0^s \frac{1}{2}\mathbb{E}[\sigma(t,U)]^2\nu(t)\mathbb{1}_{[0,\hat{\tau}_n]}(t)dt \\
&\quad + \int_0^s \mathbb{E}[\sigma(t,U)]\sqrt{\nu(t)}dW_t^{\widetilde{\mathbb{Q}}^n} + \int_0^s \mathbb{E}[\eta(t,U)]d\widetilde{J}_t^{\widetilde{\mathbb{Q}}^n} \\
&\quad - \int_0^s \mathbb{E}[\eta(t,U)]\left(1 - \frac{\mathbb{E}[\eta(t,U)]\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)}{\int_{\mathbb{R}} e^{\mathbb{E}[\eta(t,U)]z} - 1\ell^{\mathbb{P}}(dz)}\right)\mathbb{1}_{[0,\hat{\tau}_n]}(t)\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)dt\Big)^2\Big] \\
&\overset{(\star)}{\leq} 7\Big(\ln F(0,\tau_1,\tau_2)^2 + \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\Big[\sup_{s\in[0,\tau_1]}\Big(\int_0^s \left(1 - \mathbb{1}_{[0,\hat{\tau}_n]}(t)\right)\mathbb{E}[\mu(t,U)]dt\Big)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\left(\int_0^s \frac{1}{2}\mathbb{E}[\sigma(t,U)]^2\nu(t)\mathbb{1}_{[0,\hat{\tau}_n]}(t)dt\right)^2\right]\\
&+\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\left(\int_0^s \left(1-\mathbb{1}_{[0,\hat{\tau}_n]}(t)\right)\kappa(t)\ln F(t,\tau_1,\tau_2)dt\right)^2\right]\\
&+\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\left(\int_0^s \mathbb{E}[\sigma(t,U)]\sqrt{\nu(t)}dW_t^{\widetilde{\mathbb{Q}}^n}\right)^2\right]+\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\left(\int_0^s \mathbb{E}[\eta(t,U)]d\widetilde{J}_t^{\widetilde{\mathbb{Q}}^n}\right)^2\right]\\
&+\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sup_{s\in[0,\tau_1]}\left(\int_0^s \mathbb{E}[\eta(t,U)]\left(1-\frac{\mathbb{E}[\eta(t,U)]\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)}{\int_{\mathbb{R}} e^{\mathbb{E}[\eta(t,U)]z}-1\ell^{\mathbb{P}}(dz)}\right)\mathbb{1}_{[0,\hat{\tau}_n]}(t)\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)dt\right)^2\right]\Bigg)\\
&\overset{(\star\star)}{\leq} 7\Bigg(\ln F(0,\tau_1,\tau_2)^2+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\left(1-\mathbb{1}_{[0,\hat{\tau}_n]}(t)\right)\mathbb{E}[\mu(t,U)]dt\right)^2\right]\\
&+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\frac{1}{2}\mathbb{E}[\sigma(t,U)]^2\nu(t)\mathbb{1}_{[0,\hat{\tau}_n]}(t)dt\right)^2\right]\\
&+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\left(1-\mathbb{1}_{[0,\hat{\tau}_n]}(t)\right)\kappa(t)\ln F(t,\tau_1,\tau_2)dt\right)^2\right]\\
&+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]\sqrt{\nu(t)}dW_t^{\widetilde{\mathbb{Q}}^n}\right)^2\right]+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\mathbb{E}[\eta(t,U)]d\widetilde{J}_t^{\widetilde{\mathbb{Q}}^n}\right)^2\right]\\
&+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\mathbb{E}[\eta(t,U)]\left(1-\frac{\mathbb{E}[\eta(t,U)]\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)}{\int_{\mathbb{R}} e^{\mathbb{E}[\eta(t,U)]z}-1\ell^{\mathbb{P}}(dz)}\right)\mathbb{1}_{[0,\hat{\tau}_n]}(t)\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)dt\right)^2\right]\Bigg)\\
&\overset{(\star\star\star)}{\leq} 7\Bigg(\ln F(0,\tau_1,\tau_2)^2+4\tau_1\int_0^{\tau_1}\mathbb{E}[\mu(t,U)]^2dt\\
&+4\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^4dt\,\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1}\nu(t)^2dt\right]+4\int_0^{\tau_1}\kappa(t)^2dt\,\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1}\ln F(t,\tau_1,\tau_2)^2dt\right]\\
&+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]\sqrt{\nu(t)}dW_t^{\widetilde{\mathbb{Q}}^n}\right)^2\right]+4\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\left(\int_0^{\tau_1}\mathbb{E}[\eta(t,U)]d\widetilde{J}_t^{\widetilde{\mathbb{Q}}^n}\right)^2\right]\\
&+4\int_0^{\tau_1}\mathbb{E}[\eta(t,U)]^2dt\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\\
&\left[\int_0^{\tau_1}\left(1-\mathbb{E}[\eta(t,U)]^2\frac{\left(\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)\right)^2}{\left(\int_{\mathbb{R}} e^{\mathbb{E}[\eta(t,U)]z}-1\ell^{\mathbb{P}}(dz)\right)^2}\right)\left(\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)\right)^2dt\right]\Bigg)\\
&\leq 7\Bigg(\ln F(0,\tau_1,\tau_2)^2+4\tau_1\int_0^{\tau_1}\mathbb{E}[\mu(t,U)]^2dt+4c_2\tau_1\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^4dt\\
&+4\int_0^{\tau_1}\kappa(t)^2dt\,\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\\
&\left[\int_0^{\tau_1}\sup_{s\in[0,t]}\ln F(s,\tau_1,\tau_2)^2dt\right]\\
&+4\sqrt{\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^4dt}\sqrt{\tau_1c_2}+4\int_0^{\tau_1}\mathbb{E}[\eta(t,U)]^2\int_{\mathbb{R}} z^2\ell^{\widetilde{\mathbb{Q}}^n}(dz)dt\\
&+4\int_0^{\tau_1}\mathbb{E}[\eta(t,U)]^2dt\int_0^{\tau_1}\left(\left(\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)\right)^2+\mathbb{E}[\eta(t,U)]^2\frac{\left(\int_{\mathbb{R}} z\ell^{\mathbb{P}}(dz)\right)^4}{\left(\int_{\mathbb{R}} e^{\mathbb{E}[\eta(t,U)]z}-1\ell^{\mathbb{P}}(dz)\right)^2}\right)dt\Bigg),\\
&=: c_Y+28\int_0^{\tau_1}\kappa(t)^2dt\,\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1}\sup_{s\in[0,t]}\ln F(s,\tau_1,\tau_2)^2dt\right].
\end{aligned}
$$

The first equality represents the integral version of $\ln F$. Inequality $(\star)$ results from the Cauchy-Schwartz inequality to the sum and an application of the triangle inequality. We apply Doob's inequality to all expectations in Inequality $(\star\star)$. In Inequality $(\star\star\star)$, we apply the Cauchy-Schwartz inequality to the first three integrals. We finish with Itô-Lévy Isometry (see Øksendal and Sulem [39], cf. Theorem 1.17) to the last summand and an application of the stochastic Fubini theorem to the fourth summand (including $\ln F$) while making the integrand even bigger. By the previous considerations, we know that $\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^2\nu(t)dt\right] \leq \sqrt{\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^4dt}\mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\sqrt{\int_0^{\tau_1}\nu(t)^2dt}\right] \leq \sqrt{\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^4dt}\sqrt{\tau_1 c_2}$ is bounded independently of n and that $\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^4dt\ \mathbb{E}_{\widetilde{\mathbb{Q}}^n}\left[\int_0^{\tau_1}\nu(t)^2dt\right] \leq c_2\tau_1\int_0^{\tau_1}\mathbb{E}[\sigma(t,U)]^4dt$ is independent of n. By the choice of c_Y, an application of Gronwall's inequality yields $\mathbb{E}_{\mathbb{P}^n}\left[\sup_{s\in[0,\tau_1]}|\ln F(s,\tau_1,\tau_2)|^2\right] \leq c_Y e^{28\int_0^{\tau_1}\kappa(t)^2dt} =: c_3 \perp\!\!\!\perp n$, such that we have shown, that $Z^{\mathbb{P}\widetilde{\mathbb{Q}}}$ is indeed a true martingale.

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Declarations

Conflict of interest No potential competing interest was reported by the authors.

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