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Consensus control of homogeneous networks of general second order discrete-time systems

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Abstract

We consider the problem of obtaining linear state feedback controllers to achieve consensus in a homogeneous network of general second-order discrete-time systems whose graph is directed. Consensus is achieved if certain polynomials associated with the closed-loop network are Schur stable. The coefficients of these polynomials depend on the eigenvalues of the network Laplacian matrix and can therefore be complex. This paper provides a new and simple proof and condition for a second-order polynomial with complex coefficients to be Schur. This condition was applied to obtain controllers which achieve consensus with a guaranteed rate of convergence. We show that consensus can always be achieved for marginally stable systems and discretized systems. Simple conditions for consensus achieving controllers are obtained when the Laplacian eigenvalues are all real. We also demonstrate how these results can be applied to some robust control problems.

Keywords Consensus · Discrete-time systems · Schur polynomials · Robust control

1 Introduction

Consensus control of a network of linear systems has been the subject of research for many years. Some of the early work (Fax and Murray 2004; Blondel et al. 2005; Kingston and Beard 2006; Moreau 2005) was motivated applications in different fields. The area of control in particular is driven by applications in synchronization, distributed averaging, formation flight, and the cooperative control of unmanned

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vehicles, to name a few (Ren et al. 2006; Olfati-Saber et al. 2007; Ren 2008; Zhu and Martínez 2010; Guan et al. 2012; Zhou et al. 2012).

Many systems can be approximated by linear second-order systems. Networks of such systems have been studied in both the continuous-time case (Ren et al. 2006; Wieland et al. 2008; Zhu et al. 2009; Yu et al. 2010), and the discrete-time case where the individual systems are double-integrators and results were obtained by applying a bilinear transformation (Casbeer et al. 2008; Sun et al. 2012; Gao et al. 2012; Xie and Wang 2012). A more general form of system dynamics is considered in Fruhnert and Corless (2015a). While they show that the general second-order discrete-time problem is still an open challenge, it is important for digital control (Liu et al. 2010; Guan et al. 2012).

To achieve consensus for a homogenous network of linear systems applying linear control, it is necessary and sufficient to ensure the stability of a bunch of polynomials associated with the closed-loop network (Wieland et al. 2008; Qu 2009; Ma and Zhang 2010). The coefficients of these polynomials can be complex since they depend on the possibly complex eigenvalues of the Laplacian matrix associated with the network graph. Hence, in discrete-time, we need conditions which guarantee that a polynomial with complex coefficients is Schur, that is, its roots have magnitude less than one. We note that necessary and sufficient conditions for Schur polynomials have already been established (Schur 1918; Cohn 1922; Jury 1988; Bistritz 1996). However, in this paper, we present a new and simple condition for a second-order polynomial with complex coefficients to be Schur. Our proof is independent of the Routh-Hurwitz and Schur-Cohn criteria.

The relationship between the Hurwitz and Schur stability tests is clear, and one could use such a relationship to transform results for the second-order continuous-time problem (Zhu et al. 2009; Fruhnert and Corless 2017) to the discrete-time case. However, the discrete-time problem has its own challenges and applying such transformations would yield implicit expressions requiring further manipulation. Instead, one can take the more direct route and use conditions for linear, n -dimensional, discrete-time SISO agents (You and Xie 2011). However, these conditions are given in the form of a modified algebraic Riccati inequality, which requires special solvers. Here, we use the newly developed simple condition for Schur polynomials and obtain necessary and sufficient conditions to achieve consensus with guaranteed rate of convergence. We derive explicit conditions for linear control gains and show that linear control is always sufficient to achieve consensus if systems are marginally stable. This is important if we wish to synchronize networks of oscillators or try to achieve consensus using digital control. The double-integrator model is a special case of a marginally stable system. Therefore, our results extend previous findings and model a broader class of systems. The general result is also useful to guarantee a desired rate of convergence, for example, double-integrators.

The paper is organized as follows. Section 2 formulates the consensus problem. The main result is presented in Sect. 3. Afterwards, Sect. 4 provides simple linear control strategies for special cases. Next, we present controllers which guaranteed a specified rate of convergence in Sect. 5. Then, the main result is proven in Sect. 6 and necessary and sufficient conditions for polynomials to be Schur are developed. Finally, the paper concludes with a numerical simulation in Sect. 7.

2 Problem formulation

Consider a homogeneous network of second-order discrete-time systems described by

$$\boxed{x_i(k+1) = A x_i(k) + B u_i(k) + u_0(k)} \quad (1)$$

for $i = 1, 2, \dots, N$ where $x_i(k) \in \mathbb{R}^2$ and $u_i(k) \in \mathbb{R}$ are the state and the control input for the i th system at time k , while $u_0(k) \in \mathbb{R}^2$ is a common reference input. The matrices $A \in \mathbb{R}^{2 \times 2}$ and $B \in \mathbb{R}^{2 \times 1}$ are constant.

The communication structure of the network is described by a weighted, directed graph $G = (V, E, W)$. The set $V = \{1, 2, \dots, N\}$ is called a **vertex set**, and there is a one-to-one correspondence between the vertices (elements of V) and the systems in the network. The **edge set** E is a subset of $V \times V$. An **edge** (j, i) is in E if system j can send information to system i ; in this case, we say that j is an **in-neighbor** of i . We let $N_i = \{j \mid (j, i) \in E\}$ be the set of all in-neighbors of i . Each system i assigns a weight $w_{ij} > 0$ to each of its in-neighbors j . Letting $w_{ij} = 0$ when j is not an in-neighbor of i yields the **weighting matrix** $W = \{w_{ij}\}$.

Control input u_i can only depend on the information available to system i . Assuming each system has access to its own state, control u_i can only be based on the states of system i and its in-neighbors. *We want to obtain feedback controllers for each $u_i(k)$ so that the closed-loop network achieves consensus in the following sense.*

Definition 1 A network of N discrete-time systems achieves **consensus** if, for every initial condition and all $i, j = 1, 2, \dots, N$:

$$\lim_{k \rightarrow \infty} x_j(k) - x_i(k) = 0.$$

To achieve consensus, the following linear controllers are considered for the i th system:

$$\boxed{u_i(k) = K \sum_{j \in N_i} w_{ij} [x_j(k) - x_i(k)]} \quad (2)$$

Note that the gain matrix $K \in \mathbb{R}^{1 \times 2}$ is common to all systems.

Remark 1 If the network achieves consensus, then in steady state, $u_i(k) = 0$ and the behavior of the i th system is governed by

$$x_i(k+1) = A x_i(k) + u_0(k).$$

3 Main result

3.1 Preliminaries

First, we require some concepts and results from graph theory. A directed path in G from a vertex j to a vertex i is a sequence (i_1, i_2, \dots, i_m) in V with $i_1 = j$, $i_m = i$, and $(i_k, i_{k+1}) \in E$ for $k = 1, \dots, m-1$. A graph G is said to contain a **spanning tree** if at least one vertex j^* in V has the following property: for every i in V , $i \neq j^*$, there is a directed path from j^* to i . If the systems are asymptotically stable, then consensus can be trivially achieved with $K = 0$. If the systems are not asymptotically stable, one can show that a spanning tree is necessary for consensus (Ren et al. 2005). Therefore, throughout the paper we assume that

G contains a spanning tree.

Associated with a weighted graph $G = (V, E, W)$ is its **Laplacian** matrix $L = \{l_{ij}\}$, defined by

$$l_{ii} = \sum_{j \neq i} w_{ij} \quad \text{and} \quad l_{ij} = -w_{ij} \quad \text{for} \quad i \neq j.$$

Note that, in terms of the Laplacian, controller (2) can be expressed as

$$u_i(k) = - \sum_{j=1}^N l_{ij} K x_j(k).$$

If $\mathbf{1}$ is the vector of all ones, then any non-zero multiple of this vector is an eigenvector of L corresponding to eigenvalue zero. If the graph G has a spanning tree, then there are no other eigenvectors corresponding to zero.

Fact 1 (Ren et al. 2005) A graph G contains a spanning tree if and only if the associated Laplacian matrix has only one eigenvalue at zero and all its other eigenvalues have a positive real part.

Now we introduce some parameters that are necessary to state the main result. The open-loop behavior of each system is determined by the characteristic polynomial of A , which is given by

$$p(s) = \det(sI - A) = s^2 + c_1 s + c_0. \quad (3)$$

Let

$$\gamma_0 = K(A + c_1 I)B, \quad \gamma_1 = KB. \quad (4)$$

Throughout the paper we assume that

(A, B) is controllable.

Then, the matrix $\begin{bmatrix} (A + c_1 I)B & B \end{bmatrix}$ is invertible and

$$K = \begin{bmatrix} \gamma_0 & \gamma_1 \end{bmatrix} \begin{bmatrix} (A + c_1 I)B & B \end{bmatrix}^{-1}. \quad (5)$$

If (A, B) is in controllable canonical form, then one can easily show that $K = [\gamma_0 \ \gamma_1]$.

3.2 A condition for consensus

The following theorem summarizes the necessary and sufficient conditions for the closed-loop network (1)–(2) to achieve consensus.

Theorem 1 The closed-loop network (1)–(2) achieves consensus if and only if for each non-zero eigenvalue μ of the graph Laplacian matrix,

$$|d_0|^2 + |d_1 - d_0 \bar{d}_1| < 1 \quad (6)$$

where

$$d_0 = c_0 + \mu \gamma_0, \quad d_1 = c_1 + \mu \gamma_1$$

and c_0, c_1 and γ_0, γ_1 are given by (3) and (4).

A proof of Theorem 1 is provided in Sect. 6. For a real Laplacian eigenvalue, condition (6) simplifies as described in the following remark.

Remark 2 If μ is real, then Corollary 1 tells us that condition (6) simplifies to

$$d_0 < 1 \quad \text{and} \quad |d_1| < 1 + d_0; \quad (7)$$

see Corollary 1.

Theorem 1 provides a necessary and sufficient condition on the gains γ_0 and γ_1 for consensus, namely (6). When (6) is satisfied, a consensus achieving gain matrix K is given by (5). We now use condition (6) to obtain explicit conditions on the controller gains.

3.3 Explicit conditions on controller gains

Inequality (6) is equivalent to

$$|d_2| < 1 - |d_0|^2 \quad \text{where} \quad d_2 := d_1 - d_0 \bar{d}_1. \quad (8)$$

For this to be satisfied, we must have

$$|d_0| < 1. \quad (9)$$

3.3.1 A preliminary result

As we shall see shortly, inequalities (8) and (9) are of the form

$$|\nu + \eta \gamma| < b, \quad (10)$$

where b and γ are real and ν and η can be complex. This is equivalent to

$$b > 0, \quad |\eta|^2 \gamma^2 + 2 \tilde{\alpha} \gamma + |\nu|^2 - b^2 < 0 \quad \text{where} \quad \tilde{\alpha} = \Re(\nu \eta)$$

or $p(\gamma) < 0$ where $p(\gamma) = |\eta|^2 \gamma^2 + 2 \tilde{\alpha} \gamma + |\nu|^2 - b^2$. Since $\lim_{|\gamma| \rightarrow \infty} p(\gamma) = \infty$, p must have two distinct real roots $\underline{\gamma} < \bar{\gamma}$ for the existence of γ satisfying $p(\gamma) < 0$; these roots are

$$\underline{\gamma}, \bar{\gamma} = \frac{-\tilde{\alpha} \pm \sqrt{b^2 |\eta|^2 - \tilde{\omega}^2}}{|\eta|^2} \quad \text{where} \quad \tilde{\omega} = \Im(\nu \eta). \quad (11)$$

Thus, necessary conditions for the existence of a real number γ satisfying inequality (10) are

$$b > 0, \quad |\tilde{\omega}| < b|\eta| \quad (12)$$

and inequality (10) holds if and only if

$$\underline{\gamma} < \gamma < \bar{\gamma}. \quad (13)$$

Special case: μ and η real. If μ and η are real, then (11) reduces to

$$\underline{\gamma}, \bar{\gamma} = \frac{-\nu \pm b}{\eta} \quad (14)$$

Remark 3 If $|\nu| < b$, then (10) holds with γ sufficiently small. If $|\nu| = b$, then

$$\underline{\gamma}, \bar{\gamma} = \frac{-\tilde{\alpha} \pm |\tilde{\alpha}|}{|\eta|^2}.$$

Hence, either $\underline{\gamma} = 0$ or $\bar{\gamma} = 0$ and inequality (10) can be satisfied by an arbitrarily small non-zero γ whose sign is opposite to the sign of $\tilde{\alpha}$. Thus, if one has a collection of inequalities of the form (10) for which each $\tilde{\alpha}$ has the same sign, then these inequalities can be simultaneously satisfied by an arbitrarily small non-zero γ whose sign is the opposite of the sign of the $\tilde{\alpha}$'s.

3.3.2 Conditions on γ_0

Let

$$\mu = \alpha + j\omega \quad (15)$$

It follows from (12) and (13) that (9) holds if and only if

$$(c_0^2 - 1)\omega^2 < \alpha^2 \quad \text{and} \quad \underline{\gamma}_0 < \gamma_0 < \bar{\gamma}_0 \quad (16)$$

where

$$\underline{\gamma}_0, \bar{\gamma}_0 = \frac{-c_0 \alpha \pm \sqrt{|\mu|^2 - c_0^2 \omega^2}}{|\mu|^2}. \quad (17)$$

If $|c_0| < 1$, then (16) holds for all non-zero μ if $\gamma_0 = 0$. If $|c_0| = 1$, then (16) is satisfied by all non-zero μ if γ_0 is sufficiently small and its sign equals that of $-c_0$.

If μ is real, then $\omega = 0$, $\mu = \alpha > 0$ and

$$\boxed{\underline{\gamma}_0, \bar{\gamma}_0 = (-c_0 \pm 1)/\mu} \quad (18)$$

3.3.3 Conditions on γ_1

Here we use condition (8) to obtain conditions on γ_1 . With $\mu = \alpha + j\omega$, one may readily compute that

$$\begin{aligned} d_2 &= (1 - c_0 - \gamma_0 \alpha - j\gamma_0 \omega)c_1 + [(1 - c_0)\alpha - |\mu|^2 \gamma_0 + j(1 + c_0)\omega] \gamma_1 \\ b &:= 1 - |d_0|^2 = 1 - c_0^2 - 2\alpha\gamma_0 c_0 - |\mu|^2 \gamma_0^2 \end{aligned}$$

Case 1: $c \neq -1$: If $c_0 \neq -1$, let

$$\gamma_2 := (1 + c_0)\gamma_1 - c_1\gamma_0 \quad (19)$$

Then

$$\gamma_1 = \frac{c_1 \gamma_0 + \gamma_2}{1 + c_0} \quad (20)$$

and

$$d_2 = \nu + (\beta + j\omega)\gamma_2$$

where

$$\nu = \frac{c_1 b}{1 + c_0}, \quad \beta = \frac{(1 - c_0)\alpha - |\mu|^2 \gamma_0}{1 + c_0}$$

It follows from (12) and (13) that (8) holds if and only if

$$(c_2^2 - 1)\omega^2 < \beta^2 \quad \text{and} \quad \underline{\gamma}_2 < \gamma_2 < \bar{\gamma}_2 \quad (21)$$

where $c_2 = \frac{c_1}{1+c_0}$ and

$$\underline{\gamma}_2, \bar{\gamma}_2 = \frac{-c_2 \beta \pm \sqrt{\beta^2 + (1 - c_2^2) \omega^2}}{\beta^2 + \omega^2} b$$

From this, one can show that

$$\underline{\gamma}_1, \bar{\gamma}_1 = \frac{\tilde{a} c_1 (\gamma_0 \alpha + c_0 - 1) + \gamma_0 c_1 (1 + c_0) \omega^2 \pm b \sqrt{\tilde{a}^2 + [(1 + c_0)^2 - c_1^2] \omega^2}}{\tilde{a}^2 + (1 + c_0)^2 \omega^2} \quad (22)$$

where

$$\tilde{a} = (1 + c_0) \beta = (1 - c_0) \alpha - |\mu|^2 \gamma_0$$

If μ is real, then $\omega = 0$, (21) implies that $\beta \neq 0$ and

$$\underline{\gamma}_1, \bar{\gamma}_1 = \frac{-c_1}{\mu} \pm \frac{b}{\tilde{a}} \quad (23)$$

which can be expressed as

$$\underline{\gamma}_1, \bar{\gamma}_1 = [-c_1 \pm (1 + c_0 + \mu \gamma_0)] / \mu \quad (24)$$

Remark 4 If $|c_1| < |1 + c_0|$, that is, $|c_2| < 1$ then (21) holds for all non-zero μ if γ_2 is sufficiently small. If $|c_1| = |1 + c_0|$ and β has the same sign for all μ , then (21) can be satisfied by all non-zero μ if γ_2 is sufficiently small and its sign equals that of $-c_2 \beta$. If $|c_0| \leq 1$, then γ_0 can be chosen arbitrarily small; hence it can be chosen so that the sign of β is the same as that of $1 - c_0$ for all non-zero μ . In this case γ_1 can be chosen arbitrarily small.

Case 2: $c_0 = -1$: If $c_0 = -1$, then

$$d_2 = (2 - \gamma_0 \alpha) c_1 - J \gamma_0 \omega c_1 + \eta \gamma_1 \quad (25)$$

$$b = \eta \gamma_0, \quad \eta := 2 \alpha - |\mu|^2 \gamma_0 \quad (26)$$

It follows from (12) and (13) that (8) holds if and only if

$$c_1^2 \omega^2 < \eta^2 \quad \text{and} \quad \underline{\gamma}_1 < \gamma_1 < \bar{\gamma}_1 \quad (27)$$

where

$$\underline{\gamma}_1, \bar{\gamma}_1 = \frac{(\alpha \gamma_0 - 2) c_1 \pm \gamma_0 \sqrt{\eta^2 - \omega^2 c_1^2}}{\eta}. \quad (28)$$

which corresponds to (22) when $c_0 = -1$.

If μ is real, then $\omega = 0$, $\mu = \alpha$, $\eta = \alpha(2 - \alpha\gamma_0)$ and

$$\underline{\gamma}_1, \bar{\gamma}_1 = \frac{-c_1}{\mu} \pm \gamma_0.$$

that is the bounds on γ_1 are also given by (24). If $c_1 = 0$, then

$$\underline{\gamma}_1, \bar{\gamma}_1 = \pm\gamma_0.$$

With $\gamma_0 \neq 0$ this can be satisfied by sufficiently small γ_1 .

Remark 5 If systems (1) are asymptotically stable, then consensus will be achieved even if no feedback is applied. For $\gamma_0 = \gamma_1 = 0$, condition (6) reduces to

$$c_0 < 1 \quad \text{and} \quad |c_1| < 1 + c_0,$$

which are the conditions for open-loop stability.

3.4 Limitations

In the continuous-time analog of the problem considered here consensus can always be achieved by appropriate choice of K (Fruhnert and Corless 2017). However, that is not the case in discrete-time. We have already seen limitations on the imaginary part of a Laplacian eigenvalue μ in (16), (21) and (27). The next result puts limitations on the real part of μ . A proof of this result is given in Sect. 6. To state this result we need

$$\alpha_m = \min\{\Re(\mu) \mid \mu \in \Lambda_L\} \quad \alpha_M = \max\{\Re(\mu) \mid \mu \in \Lambda_L\} \quad (29)$$

where Λ_L is the set of non-zero eigenvalues of the Laplacian matrix L .

Lemma 1 If there exists a matrix K such that the closed-loop network (1)–(2) achieves consensus, then

$$\kappa |c_0| < 1 \quad \text{and} \quad \kappa |c_1| < 2 \quad (30)$$

where

$$\kappa = \frac{\alpha_M - \alpha_m}{\alpha_M + \alpha_m}. \quad (31)$$

In the next section, we look at some important classes of networked systems and provide easily verifiable conditions for consensus.

4 Special cases

Identifying gains that satisfy (6) for all $\mu \neq 0$ can be challenging. Here we consider some special cases and obtain simpler conditions for the existence of feasible parameters γ_0 and γ_1 .

4.1 Marginally stable systems

Here we claim that consensus can always be achieved for networks of marginally stable systems. If the systems are marginally stable, then $c_0 \leq 1$ and $|c_1| \leq 1 + c_0$ and equality holds for at least one of the inequalities. This is a special case of

$$|c_0| \leq 1 \quad \text{and} \quad |c_1| \leq |1 + c_0|. \quad (32)$$

Lemma 2 If (32) holds, then consensus in the closed-loop network (1)–(2) can always be achieved with arbitrarily small γ_0 and γ_1 .

Proof If $c_0 \neq -1$, then the result follows the results in Sect. 3.3.2 and Remark 4. If $c_0 = -1$, then we must have $c_1 = 0$, and the result follows from the results in Sect. 3.3.2 and case 2 of section Sect. 3.3.3. \square

4.2 Laplacian eigenvalues in a disc

From Lemma 1, we see that if either $|c_0| > 1$ or $|c_1| > 2$, then the range of the eigenvalues of the Laplacian matrix L must be restricted to achieve consensus. Here we consider the non-zero eigenvalues of the Laplacian to be constrained to a disc of radius R with center $C > 0$. First, we have the following result, which provides simpler sufficient conditions for achieving consensus; Sect. 6 contains a proof.

Lemma 3 The closed-loop network (1)–(2) achieves consensus if, for each non-zero eigenvalue μ of the graph Laplacian matrix,

$$|c_0 + \mu \gamma_0| + |c_1 + \mu \gamma_1| < 1. \quad (33)$$

Lemma 4 Suppose that

$$|\mu - C| \leq R \quad (34)$$

for all non-zero eigenvalues μ of the Laplacian. Then, the closed-loop network (1)–(2) achieves consensus if

$$|c_0 + \gamma_0 C| + |c_1 + \gamma_1 C| + (|\gamma_0| + |\gamma_1|) R < 1. \quad (35)$$

There exist γ_0, γ_1 satisfying (35) if and only if:

1) $R/C \geq 1$:

$$|c_0| + |c_1| < 1. \quad (36)$$

In this case, (35) holds with $\gamma_0 = \gamma_1 = 0$.

2) $R/C < 1$:

$$(|c_0| + |c_1|) R/C < 1. \quad (37)$$

In this case, (35) holds with

$$\gamma_0 = -c_0/C \quad \text{and} \quad \gamma_1 = -c_1/C. \quad (38)$$

Proof From Lemma 3, consensus is achieved if, for each non-zero eigenvalue μ of the Laplacian,

$$|d_0| + |d_1| < 1 \quad (39)$$

where $d_i = c_i + \mu \gamma_i$ for $i = 0, 1$. If $|\mu - C| \leq R$, then $\mu = C + r(\cos \psi + j \sin \psi)$ where $0 \leq r \leq R$ and $\psi \in \mathbb{R}$. Thus,

$$|d_i|^2 = (c_i + \gamma_i C)^2 + 2(c_i + \gamma_i C) \gamma_i r \cos \psi + \gamma_i^2 r^2$$

and

$$\begin{aligned} \max_{|\mu - C| \leq R} |d_i|^2 &= (c_i + \gamma_i C)^2 + 2|c_i + \gamma_i C| |\gamma_i| R + \gamma_i^2 R^2 \\ &= (|c_i + \gamma_i C| + |\gamma_i| R)^2. \end{aligned}$$

Hence,

$$\max_{|\mu - C| \leq R} |d_i| = |c_i + \gamma_i C| + |\gamma_i| R$$

and (35) implies (39).

We note that $f_i(\gamma_i) := |c_i + C \gamma_i| + |\gamma_i| R$ is continuous and piecewise linear. Therefore, its minimum will occur at one of the two switching points $\gamma_i = 0$ and $\gamma_i = -c_i/C$; hence the minimum is the smallest of

$$f_i(0) = |c_i| \quad \text{and} \quad f_i(-c_i/C) = |c_i| R / C.$$

If $R/C \geq 1$, then the minimum is $|c_i|$ and (35) can be satisfied if and only if (36) holds. If $R/C < 1$, then $\gamma_i = -c_i/C$ is the minimizer of f_i , and (35) can be satisfied if and only if (37) holds. \square

Remark 6 (Robustness) The conditions in Lemma 4 can be used to guarantee robustness with respect to changes in the graph; the only knowledge needed about the graph is that the non-zero eigenvalues of the graph Laplacian lie within a disc of radius R and center C .

To obtain robustness with respect to plant parameters suppose that, for some \bar{c}_0, \bar{c}_1 and Δc , the parameters satisfy

$$|c_0 - \bar{c}_0| + |c_1 - \bar{c}_1| \leq \Delta c. \quad (40)$$

If $R/C \geq 1$, then (35) holds with $\gamma_0 = \gamma_1 = 0$ for all c_0 and c_1 satisfying (40) if and only if

$$|\bar{c}_0| + |\bar{c}_1| + \Delta c < 1.$$

If $R/C < 1$, then (35) holds with

$$\gamma_0 = -\bar{c}_0/C \quad \text{and} \quad \gamma_1 = -\bar{c}_1/C$$

for all c_0 and c_1 satisfying (40) if and only if

$$(|\bar{c}_0| + |\bar{c}_1|)R/C + \Delta c < 1.$$

4.3 Real Laplacian eigenvalues

Here we consider weighted graphs whose Laplacian eigenvalues are all real. This occurs, for example, if the graph is undirected and the weighting matrix is symmetric or if the graph is a string. For this important special case, we can obtain easily verifiable necessary and sufficient conditions for consensus control as outlined in the following result.

Lemma 5 If all the eigenvalues of the Laplacian matrix are real, then there exists a matrix K such that the closed-loop network (1)–(2) achieves consensus if and only if

$$\kappa |c_0| < 1 \quad \text{and} \quad \kappa (|1 - c_0| + |c_1|) < 2 \quad (41)$$

where κ is given by (31). Such K will be given by (5) where γ_0 and γ_1 satisfy

$$\gamma_{0m} < \gamma_0 < \gamma_{0M} \quad (42)$$

$$-\gamma_0 + \gamma_{1m} < \gamma_1 < \gamma_0 + \gamma_{1M} \quad (43)$$

with

$$\begin{aligned} \gamma_{0m} &= (\gamma_{1m} - \gamma_{1M})/2 \\ \gamma_{0M} &= \beta_1 (1 - c_0) + \beta_2 |1 - c_0| \\ \gamma_{1m} &= -\beta_1 (1 + c_0 + c_1) + \beta_2 |1 + c_0 + c_1| \\ \gamma_{1M} &= \beta_1 (1 + c_0 - c_1) - \beta_2 |1 + c_0 - c_1| \end{aligned}$$

and

$$\beta_1 = \frac{\alpha_M + \alpha_m}{2\alpha_M\alpha_m}, \quad \beta_2 = \frac{\alpha_M - \alpha_m}{2\alpha_M\alpha_m}$$

where α_m and α_M are defined in (29).

Proof Using Theorem 1 and recalling (18) and (24) we see that consensus is achieved if and only if

$$(-c_0 - 1)/\mu < \gamma_0 < (-c_0 + 1)/\mu \quad (44)$$

$$(-c_1 - 1 - c_0)/\mu - \gamma_0 < \gamma_1 < (-c_1 + 1 + c_0)/\mu + \gamma_0 \quad (45)$$

for all $\lambda \in \Lambda_L$, that is, all non-zero eigenvalues μ of the graph Laplacian. Recall that these eigenvalues are positive. Noting that

$$\beta_1 + \beta_2 = 1/\alpha_m \leq \frac{1}{\mu} \leq 1/\alpha_M = \beta_1 - \beta_2$$

for all $\lambda \in \Lambda_L$, we see that, for any real number c ,

$$\min_{\mu \in \Lambda_L} c/\mu = \beta_1 c - \beta_2 |c|, \quad \max_{\mu \in \Lambda_L} c/\mu = \beta_1 c + \beta_2 |c|$$

Hence, (44) holds for all $\mu \in \Lambda_L$ if and only if

$$\gamma_{0m,2} < \gamma_0 < \gamma_{0M}$$

where

$$\gamma_{0m,2} = \max_{\mu \in \Lambda_L} \frac{-1 - c_0}{\mu} = -\beta_1 (1 + c_0) + \beta_2 |1 + c_0| \quad (46)$$

$$\gamma_{0M} = \min_{\mu \in \Lambda_L} \frac{1 - c_0}{\mu} = \beta_1 (1 - c_0) - \beta_2 |1 - c_0| \quad (47)$$

Similarly, (45) holds for all non-zero μ if and only if (43) holds.

Now, there exists γ_1 satisfying (43) if and only if $2\gamma_0 > \gamma_{1m} - \gamma_{1M} = 2\gamma_{0m}$, that is,

$$\gamma_0 > \gamma_{0m}$$

For any two real numbers a and b ,

$$|a + b| + |a - b| = 2 \max\{|a|, |b|\}. \quad (48)$$

Hence,

$$\begin{aligned} \gamma_{0m} &= (\gamma_{1m} - \gamma_{1M})/2 \\ &= -\beta_1(1 + c_0) + \beta_2(|1 + c_0 + c_1| + |1 + c_0 - c_1|)/2 \\ &= -\beta_1(1 + c_0) + \beta_2 \max\{|1 + c_0|, |c_1|\}. \end{aligned}$$

Recalling (46), we see that $\gamma_{0m} \geq \gamma_{0m,2}$.

Finally, there exists γ_0 satisfying (42) if and only if $\gamma_{0m} - \gamma_{0M} < 0$, that is,

$$-2\beta_1 + \beta_2 (\max\{|1 - c_0|, |c_1|\} + |1 + c_0|) < 0$$

or, noting that $\kappa = \beta_2/\beta_1$,

$$\kappa (|1 - c_0| + \max\{|1 + c_0|, |c_1|\}) < 2. \quad (49)$$

It follows from (48) that $|1 - c_0| + |1 + c_0| = 2 \max\{|c_0|, 1\}$. Since $\kappa < 1$, inequality (49) is equivalent to (41). \square

Remark 7 From the proof of the above lemma, we see that if γ_0 satisfies (42), then there exists γ_1 satisfying (43). Thus, provided (41) holds, one can simply obtain a controller K by first choosing γ_0 to satisfy (42), then choosing γ_1 to satisfy (43), and letting K be given by (5).

Remark 8 (Robust control) The above result can be applied to robust control problems. Consider an uncertain second-order system described by

$$x(k+1) = Ax(k) + \mu Bu(k) \quad (50)$$

where (A, B) is controllable and μ is real and satisfies

$$0 < \underline{\mu} \leq \mu \leq \overline{\mu}$$

The using Lemma 5 one can obtain a controller which stabilizes the corresponding closed loop system for all μ provided (41) hold with

$$\alpha_m = \underline{\mu}, \quad \alpha_M = \overline{\mu}.$$

4.4 Discretized systems

From Fruhnert and Corless (2015b), consensus can always be achieved by appropriate choice of gains if linear control is applied to a similar setup in continuous-time where $\dot{x}_{c,i} = A_c x_{c,i} + B_c u_{c,i}$. On the other hand, Lemma 1 shows that this is not always the case in discrete-time. If a continuous-time system ($\dot{x} = A_c x_c + B_c u_c$) is discretized with time constant $h > 0$, then the corresponding discrete-time system is characterized by

$$A = e^{A_c h} \quad \text{and} \quad B = \int_0^h e^{A_c \tau} d\tau B_c. \quad (51)$$

Lemma 6 A closed-loop network (1)–(2) of discretized systems (51) can always achieve consensus if (A_c, B_c) is controllable, the graph contains a spanning tree, the discretization time constant $h > 0$ is chosen small enough, and gains are chosen appropriately.

Proof If (A_c, B_c) is controllable then (A, B) is controllable for $h > 0$ sufficiently small. If $h \rightarrow 0$, then $A \rightarrow I$. Thus, in the limit, A is marginally stable with $c_0 = 1$ and $c_1 = -2$. In this case the conditions in (16) and (21) for achieving consensus reduce to

$$\alpha^2 > 0 \quad \text{and} \quad \beta^2 > 0$$

respectively, where $\beta = -|\mu|^2\gamma_0/2$, which are clearly satisfied. Since c_0 and c_1 are continuous functions of h and the quantities in inequalities (16) and (21) are continuous in c_0 and c_1 , these inequalities hold for $h > 0$ sufficiently small. \square

5 Guaranteed rate of convergence

Sometimes we are interested not only in convergence, but convergence with a specified convergence rate, which is defined as follows.

Definition 2 A network of N discrete-time systems achieves consensus with convergence rate $0 < \rho < 1$ if, for every initial condition, there exists $c \geq 0$, so that

$$||x_j(k) - x_i(k)|| \leq c \rho^k$$

for all $i, j = 1, 2, \dots, N$ and $k \geq 0$ where $||x|| = \sqrt{x^T x}$.

Theorem 1 can be used to guarantee consensus with convergence rate ρ . To illustrate this, we introduce the following transformed systems

$$\tilde{x}_i(k+1) = \rho^{-1}A \tilde{x}_i(k) + B \tilde{u}_i(k) + B \tilde{u}_0(k) \quad (52)$$

with inputs $\tilde{u}_i(k) = u_i(k)/\rho^{k+1}$ and $\tilde{u}_0(k) = u_0(k)/\rho^{k+1}$. Open-loop behavior is determined by the characteristic polynomial of $\rho^{-1}A$, that is,

$$\begin{aligned} \tilde{p}(s) &= \det(sI - \rho^{-1}A) \\ &= \rho^{-2} \det(s\rho I - A) = \rho^{-2} p(\rho s) \end{aligned}$$

This yields $\tilde{p}(s) = s^2 + \tilde{c}_1 s + \tilde{c}_0$ where

$$\boxed{\tilde{c}_0 = c_0/\rho^2, \quad \tilde{c}_1 = c_1/\rho} \quad (53)$$

First, we present the following result.

Lemma 7 Network (1) achieves consensus with convergence rate ρ if the associated transformed network (52) achieves consensus.

Proof Consider any $0 < \rho < 1$. Let $\tilde{x}_i(k) := \rho^{-k}x_i(k)$ for all i . Then $x_i(k) = \tilde{x}_i(k)\rho^k$ and

$$||x_j(k) - x_i(k)|| = ||\tilde{x}_j(k) - \tilde{x}_i(k)|| \rho^k.$$

Hence, $x_j(k) - x_i(k)$ exponentially converges to zero with guaranteed rate ρ if $\tilde{x}_j(k) - \tilde{x}_i(k)$ converges to zero. The result is obtained by noting that when the behavior of $x_i(k)$ is governed by (1), the behavior of $\tilde{x}_i(k)$ is described by (52). \square

Lemma 7 holds for any type of controller. Here we consider linear controllers of the form given in (2) for network (52), that is,

$$\tilde{u}_i(k) = \tilde{K} \sum_{j \in N_i} w_{ij} [\tilde{x}_j(k) - \tilde{x}_i(k)].$$

Since $u_i(k) = \rho^{k+1} \tilde{u}_i(k)$ and $x_i(k) = \rho^k \tilde{x}_i(k)$ this is equivalent to

$$u_i(k) = K \sum_{j \in N_i} w_{ij} [x_j(k) - x_i(k)],$$

where $K = \rho \tilde{K}$. With $\rho^{-1}A + \tilde{c}_1 I = \rho^{-1}(A + c_1 I)$ and $\tilde{K} = \rho^{-1}K$, we proceed as in (4) and define

$$\tilde{\gamma}_0 = \rho^{-2} K (A + c_1 I) B, \quad \tilde{\gamma}_1 = K \rho^{-1} B \quad (54)$$

This yields

$$K = \begin{bmatrix} \tilde{\gamma}_0 & \tilde{\gamma}_1 \end{bmatrix} \begin{bmatrix} \rho^{-2} (A + c_1 I) B & \rho^{-1} B \end{bmatrix}^{-1} \quad (55)$$

Now, Theorem 1 and Lemma 7 yield the following result.

Theorem 2 The closed-loop network (1)–(2) achieves consensus with convergence rate $0 < \rho < 1$ if for each non-zero eigenvalue μ of the graph Laplacian,

$$|\tilde{d}_0|^2 + |\tilde{d}_1 - \tilde{d}_0 \bar{\tilde{d}}_1| < 1$$

where $\tilde{d}_0 = \tilde{c}_0 + \mu \tilde{\gamma}_0$, $\tilde{d}_1 = \tilde{c}_1 + \mu \tilde{\gamma}_1$, and \tilde{c}_0 , \tilde{c}_1 , $\tilde{\gamma}_0$, and $\tilde{\gamma}_1$ are given by (53)–(54).

Remark 9 If systems (1) are in controllable canonical form, then one can easily show that

$$K = \begin{bmatrix} \rho^2 \tilde{\gamma}_0 & \rho \tilde{\gamma}_1 \end{bmatrix}.$$

Remark 10 One cannot achieve consensus with arbitrary small rate ρ using linear controller (2) except in the trivial case in which all the non-zero eigenvalues of L are the same. To see this, suppose convergence with an arbitrary small rate ρ can be achieved and recall (53). Then Sect. 3.3.2 implies that $\omega = 0$ for every eigenvalue of L ; see (16). Lemma 1 implies that $\alpha_m = \alpha_M$. Thus all the non-zero eigenvalues of L are the same.

6 Proof of main result

6.1 A condition for consensus

The following result provides a useful condition for consensus of the closed loop system.

Lemma 8 (Ma and Zhang 2010) The closed-loop network (1)–(2) achieves consensus if for each non-zero eigenvalue μ of graph Laplacian, the matrix $A - \mu BK$ is Schur.

Lemma 9 The matrix $A - \mu BK$ is Schur if and only if

$$p(s) = s^2 + (c_1 + \gamma_1 \mu) s + (c_0 + \gamma_0 \mu) \quad (56)$$

is Schur, where c_0, c_1, γ_0 , and γ_1 are given by (3)–(4).

Proof The matrix $A - \mu BK$ is Schur if and only if its characteristic polynomial $p(s) = s^2 + d_1 s + d_0$ is Schur.

$$\begin{aligned} p(s) &= \det(sI - A + \mu BK) \\ &= \det(sI - A) \det(I + \mu (sI - A)^{-1} KB) \\ &= \det(sI - A) (1 + \mu K(sI - A)^{-1} B) \end{aligned}$$

and $\det(sI - A)$ is given by (3). Considering the power series expansion $(sI - A)^{-1} = \frac{1}{s}I + \frac{1}{s^2}A + \dots$ and comparing the coefficients of s^0 and s on both sides of the equation yields

$$\begin{aligned} d_0 &= c_0 + \mu K(A + c_1 I)B = c_0 + \gamma_0 \mu \\ d_1 &= c_1 + \mu KB = c_1 + \gamma_1 \mu. \end{aligned}$$

□

The coefficients of p in (56) are not necessarily real, since μ can be complex. Thus we need conditions which guarantee that a second order polynomial with complex coefficients is Schur.

6.2 A simple characterization of second-order Schur polynomials

This section develops necessary and sufficient conditions for a second order polynomial with complex coefficients to be Schur. A polynomial is Schur if all its roots have magnitude less than one. Here we provide a new result which is simple proof for second-order Schur polynomials with real or complex coefficients.

Lemma 10 Suppose $p(\lambda) = \lambda^2 + d_1 \lambda + d_0$ where $d_0, d_1 \in \mathbb{C}$. Then p is Schur if and only if

$$\boxed{|d_0|^2 + |d_1 - d_0 \bar{d}_1| < 1} \quad (57)$$

Proof First, we express d_0 and d_1 in polar form, that is,

$$d_0 = |d_0| e^{j\delta_0}, \quad d_1 = |d_1| e^{j\delta_1}. \quad (58)$$

where δ_0 and δ_1 are real. The polynomial p is Schur if and only if \tilde{p} is Schur where $\tilde{p}(\lambda) = e^{-j\delta_0} p(\lambda e^{j\delta_0/2})$. Note that

$$\tilde{p}(\lambda) = \lambda^2 + |d_1| e^{j\phi} \lambda + |d_0| \quad (59)$$

where

$$\phi = \delta_1 - \delta_0/2. \quad (60)$$

If z_1 and z_2 are the roots of \tilde{p} , then

$$\tilde{p}(\lambda) = \lambda^2 - (z_1 + z_2) \lambda + z_1 z_2. \quad (61)$$

Comparing (59) and (61), we see that

$$z_1 z_2 = |d_0| \quad \text{and} \quad z_1 + z_2 = -|d_1| e^{j\phi}. \quad (62)$$

If $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then it now follows that $r_1 r_2 e^{j(\theta_1 + \theta_2)} = |d_0|$. Hence, $\theta_1 = -\theta_2 := \theta$ and (62) results in

$$r_1 r_2 = |d_0| \quad (63)$$

$$(r_1 + r_2) \cos \theta = -|d_1| \cos \phi \quad (64)$$

$$(r_1 - r_2) \sin \theta = -|d_1| \sin \phi. \quad (65)$$

The polynomial \tilde{p} is Schur if and only if the magnitudes of its roots are less than one, that is, $r_1, r_2 < 1$, which is equivalent to (assuming w.l.o.g. $r_1 \geq r_2 \geq 0$)

$$r_1 r_2 < 1 \quad (66)$$

$$(r_1 - 1)(r_2 - 1) > 0 \quad \text{or} \quad (r_1 - 1)(-r_2 - 1) > 0. \quad (67)$$

It follows from (63) that inequality (66) is equivalent to

$$|d_0| < 1. \quad (68)$$

With $r_1 r_2 < 1$, (67) is equivalent to

$$\frac{r_1 + r_2}{1 + r_1 r_2} < 1 \quad \text{or} \quad \frac{r_1 - r_2}{1 - r_1 r_2} < 1. \quad (69)$$

Since $1 = \cos^2 \theta + \sin^2 \theta$, $r_1 r_2 < 1$, and $r_1 \geq r_2$, inequality (69) holds if and only if

$$\left(\frac{r_1 + r_2}{1 + r_1 r_2} \right)^2 \cos^2 \theta + \left(\frac{r_1 - r_2}{1 - r_1 r_2} \right)^2 \sin^2 \theta < 1$$

which, recalling (63)–(65), is the same as

$$\left(\frac{|d_1|}{1 + |d_0|} \right)^2 \cos^2 \phi + \left(\frac{|d_1|}{1 - |d_0|} \right)^2 \sin^2 \phi < 1$$

or

$$(1 + |d_0|^2) |d_1|^2 - 2 |d_0| |d_1|^2 \cos 2\phi < (1 - |d_0|^2)^2.$$

It follows from (58) and (60) that $\bar{d}_0 d_1^2 = |d_0| |d_1|^2 e^{i2\phi}$; hence the above inequality can be rewritten as

$$(1 + |d_0|^2) |d_1|^2 - \bar{d}_0 d_1^2 - d_0 \bar{d}_1^2 < (1 - |d_0|^2)^2. \quad (70)$$

Noting that

$$\begin{aligned} & (1 + |d_0|^2) |d_1|^2 - \bar{d}_0 d_1^2 - d_0 \bar{d}_1^2 \\ &= \bar{d}_1 d_1 - \bar{d}_1 (d_0 \bar{d}_1) - (\bar{d}_0 d_1) d_1 + (\bar{d}_0 d_1) (d_0 \bar{d}_1) \\ &= (\bar{d}_1 - \bar{d}_0 d_1) (d_1 - d_0 \bar{d}_1) = |d_1 - d_0 \bar{d}_1|^2 \end{aligned}$$

(70) can be written as

$$|d_1 - d_0 \bar{d}_1|^2 < (1 - |d_0|^2)^2.$$

The above inequality and (68) are equivalent to

$$|d_1 - d_0 \bar{d}_1| < 1 - |d_0|^2$$

which is equivalent to (57). \square

Corollary 1 Suppose $p(\lambda) = \lambda^2 + d_1 \lambda + d_0$ where $d_0, d_1 \in \mathbb{R}$. Then p is Schur if and only if

$$\boxed{d_0 < 1, \quad |d_1| < 1 + d_0} \quad (71)$$

Proof When d_0 and d_1 are real, (57) can be expressed as $d_0^2 + |1 - d_0||d_1| < 1$, that is,

$$|1 - d_0||d_1| < (1 - d_0)(1 + d_0)$$

which is equivalent to (71). \square

Corollary 2 Suppose $p(\lambda) = \lambda^2 + d_1 \lambda + d_0$ where $d_0, d_1 \in \mathbb{C}$. Then p is Schur if

$$|d_0| + |d_1| < 1. \quad (72)$$

Also, if p is Schur, then we must have

$$|d_0| < 1 \quad \text{and} \quad |d_1| < 1 + |d_0|. \quad (73)$$

Proof Inequality (72) implies that

$$|d_1| < 1 - |d_0|$$

Hence

$$|d_1 - d_0 \bar{d}_1| \leq |d_1| (1 + |d_0|) < (1 - |d_0|) (1 + |d_0|) = 1 - |d_0|^2$$

that is

$$|d_1 - d_0 \bar{d}_1| < 1 - |d_0|^2 \quad (74)$$

Hence p is Schur.

Now suppose p is Schur. Then, it follows from (74) that $|d_0| < 1$ and

$$|d_1| (1 - |d_0|) \leq |d_1 - d_0 \bar{d}_1| < 1 - |d_0|^2 = (1 + |d_0|)(1 - |d_0|)$$

□

6.3 Proof of Theorem 1, Lemma 1, and Lemma 3

Theorem 1 is a consequence of Lemmas 8, 9, and 10.

Lemma 1 is a consequence of Corollary 2. Corollary 2 requires that $|d_0| < 1$ which holds for some $\omega > 0$ if and only if it holds for $\omega = 0$ since $|d_0|^2 = |\alpha \gamma_0 + c_0|^2 + \omega^2 \gamma_0^2$. From the proof of Lemma 5, $|d_0| < 1$ holds for a range of real valued μ if and only if $\gamma_{0m,2} < \gamma_{0M}$ where $\gamma_{0m,2}$ and γ_{0M} are given by (46)–(47). Applying (48) and recalling that $\kappa = \beta_2/\beta_1$ yields

$$2 \beta_2 \max \{ |1|, |c_0| \} < 2 \beta_1$$

that is,

$$\kappa \max \{ |1|, |c_0| \} < 1$$

Thus, $\kappa |c_0| < 1$ is a necessary condition. Corollary 2 also requires that $|d_1| < 1 + |d_0|$; hence $|d_1| < 2$. Using the arguments in considering $|d_0| < 1$ we obtain that $\kappa |c_1| < 2$ is also a necessary condition.

Lemma 3 is a consequence of Corollary 1.

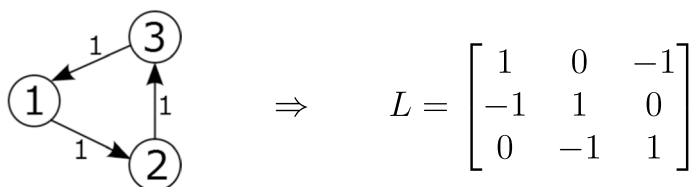


Fig. 1 Communication network with assigned edge weights

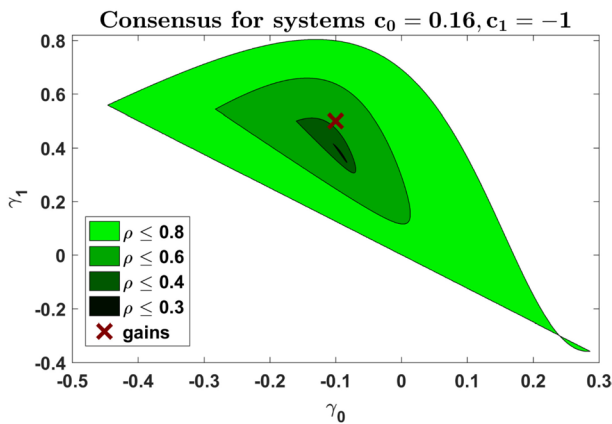


Fig. 2 Regions of gains achieving consensus with guaranteed rate of convergence

7 Example

Consider a network of three systems with

$$A = \begin{bmatrix} 0 & 1 \\ -0.16 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and common input $u_0(k) = 0.5 \cos k$. Figure 1 describes the communication network and shows its associated Laplacian matrix L which has non-zero eigenvalues $1.5 \pm j\sqrt{3}/2$.

Since A has eigenvalues 0.2 and 0.8, the open loop systems are stable and consensus with 0.8 as a convergence rate can be achieved with $u_i = 0$. To achieve consensus with a convergence rate smaller than 0.8 we use our consensus controllers. Controller gains satisfying the conditions in Corollary 2 are shown in Fig. 2. We failed to identify gains for $\rho < 0.29$, and we picked $\gamma_0 = -0.1$ and $\gamma_1 = 0.5$ to guarantee convergence with at least $\rho \leq 0.6$. Figure 3 compares the behaviour of the open loop systems to the closed loop systems. The initial states are the same in each case and were chosen randomly with uniform distribution. We observe that consensus was achieved more quickly for the closed loop systems.

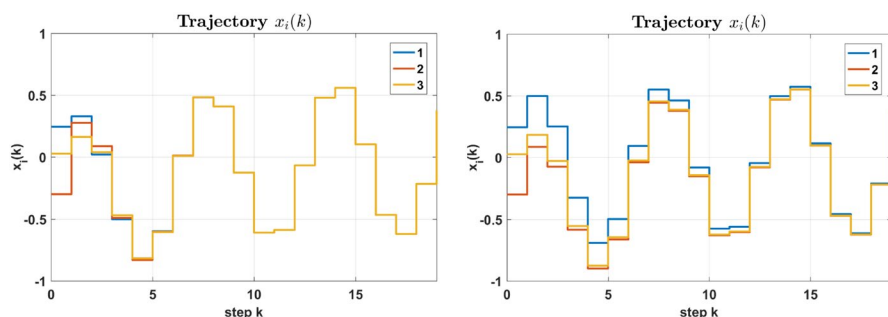


Fig. 3 Simulation of three systems achieving consensus with (left) and without (right) control u_i . Note that only $x_{i,1}(k)$ is shown since $x_{i,2}(k) = x_{i,1}(k + 1)$

8 Conclusion

This paper provided a new and simple proof for general second-order polynomials with complex coefficients to be Schur. The result obtained was used to develop necessary and sufficient conditions so that a homogeneous network of linear second-order discrete-time systems achieves consensus with guaranteed rate of convergence. Linear feedback control was applied, explicit inequalities for the control gains given, and complex eigenvalues of the Laplacian matrix were considered. It was shown that consensus can always be achieved if systems are marginally stable. We provided simple conditions for the existence of a linear controller if all eigenvalues of the Laplacian matrix are real. We also demonstrated how these results can be applied to some robust control problems. The results were illustrated with an example.

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