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Proper solutions for Epstein–Zin stochastic differential utility

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Abstract

This article considers existence and uniqueness of infinite-horizon Epstein–Zin stochastic differential utility (EZ-SDU) for the case that the coefficients R of relative risk aversion and S of elasticity of intertemporal complementarity (the reciprocal of elasticity of intertemporal substitution) satisfy $\vartheta := \frac{1-R}{1-S} > 1$. In this sense, this paper is complementary to (Herdegen et al., Finance Stoch. 27, pp. 159–188). The main novelty of the case $\vartheta > 1$ (as opposed to $\vartheta \in (0, 1)$) is that there is an infinite family of utility processes associated to every nonzero consumption stream. To deal with this issue, we introduce the economically motivated notion of a *proper* utility process, where, roughly speaking, a utility process is proper if it is nonzero whenever future consumption is nonzero. We proceed to show that for a very wide class of consumption streams C, there exists a proper utility process V associated to C. Furthermore, for a wide class of consumption streams C, the proper utility process V is unique. Finally, we solve the optimal investment–consumption problem for an agent with preferences governed by EZ-SDU who invests in a constant-parameter Black–Scholes–Merton financial market and optimises over right-continuous consumption streams that have a unique proper utility process associated to them.

Keywords Epstein–Zin stochastic differential utility · Lifetime investment and consumption · Proper solutions

Mathematics Subject Classification 60H20 · 91B16 · 91G10 · 93E20

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1 Introduction

In an infinite-horizon investment–consumption problem, the goal is to maximise the discounted expected utility of consumption, where maximisation takes place over consumption streams that can be financed by trading in a financial market (such that the resulting wealth process is nonnegative for all time). Merton [22] solved this problem for a constant-parameter financial market and constant relative risk aversion (CRRA) utility. In this article, we build on the prior works of the authors [12, 13] and analyse the investment–consumption problem for Epstein–Zin stochastic differential utility (EZ-SDU). EZ-SDU is a generalisation of time-additive CRRA utility that allows a disentanglement of the agent's risk aversion parameter from their temporal variance aversion preferences; see for example Duffie and Epstein [8], Schroder and Skiadas [24] and Herdegen et al. [12].

For SDU, the utility process is given in implicit form as the solution of a backward stochastic differential equation (BSDE), and before we can attempt to optimise over consumption streams, we must first decide how to associate a utility process to a given consumption stream. The issue is complicated by the fact that there is no terminal condition for the BSDE since we work with an infinite horizon. For some parameter combinations, Herdegen et al. [13] show that there exists a unique utility process for every consumption stream (perhaps taking values in $\overline{\mathbb{R}}$). However, in the parameter combinations studied in the present paper, uniqueness fails. Thus to make progress, we must first decide which utility process to associate to a given consumption stream; only then can we attempt to optimise over investment–consumption pairs.

The goals of the paper are as follows: first, to illustrate non-uniqueness using a class of consumption streams given by geometric Brownian motion as an example, showing that non-uniqueness is generic and not an aberration; second, to introduce the economically motivated concept of a *proper* solution and argue that it gives the "true" utility process to associate to a consumption stream; third, to prove that for a wide class of consumption streams, there exists a (unique) proper utility process; and fourth, to show that if we restrict attention to consumption streams for which there exists a unique proper utility process, we can prove a verification argument for the investment–consumption problem in a constant-parameter Black–Scholes–Merton market, and that the candidate optimal strategy is indeed optimal.

EZ-SDU is parametrised by two coefficients R and S which take values in $(0, 1) \cup (1, \infty)$; they correspond to the agent's risk aversion and their elasticity of intertemporal complementarity (the reciprocal of which is better known as elasticity of intertemporal substitution). The parameter $\vartheta := \frac{1-R}{1-S}$ is critical. First, it is argued in [12] that $\vartheta > 0$ is necessary for the Epstein–Zin SDU equation to have a meaningful solution over the infinite horizon. Second, the mathematics of the problem is vastly different depending on whether $\vartheta \in (0, 1)$ or $\vartheta \in (1, \infty)$. (The boundary case of $\vartheta = 1$, or equivalently $R = S$, is CRRA utility.)

The parameter combinations leading to $\vartheta \in (0, 1]$ are studied in [13] over the infinite horizon and by Kraft et al. [19], Seiferling and Seifried [25], Kraft and Seifried [17] and Matoussi and Xing [20] over a finite time horizon. In all these

papers, the case $\vartheta > 1$ is avoided due to mathematical difficulties. One (highly desirable) property of EZ-SDU when $\vartheta \in (0, 1]$ is that all evaluable consumption streams are uniquely evaluable – if there exists a solution to the EZ-SDU equation, it is necessarily unique. This follows from a comparison theorem (see for example [13, Theorem 5.8]) which shows that a *subsolution* to the EZ-SDU equation always lies below a *supersolution*; since a solution is both a subsolution and a supersolution, applying the comparison theorem twice to two such solutions yields uniqueness.

This paper deals with the case $\vartheta > 1$. In this case, the requirements of the comparison theorem in [13] are not met. It is tempting to hope that this is a technical issue and that by being smarter, it is possible to extend the comparison theorem to the case $\vartheta > 1$, thus resolving issues of uniqueness. However, this is not the case – the problem with $\vartheta > 1$ is fundamentally different to the problem with $\vartheta < 1$. When $\vartheta > 1$, it is not just that the comparison theorem fails, but rather that non-uniqueness is endemic to the problem. (Indeed, the only right-continuous consumption stream with a unique utility process is the zero process.) Note that the same issue of non-uniqueness arises in finite-horizon EZ-SDU as well, unless a nonzero bequest function is added at the terminal time. The main goals of this paper are to illuminate *how* non-uniqueness occurs, to provide an economically motivated criterion for selecting an economically meaningful solution, to show that such economically meaningful solutions exist and are unique, and finally to solve the investment–consumption problem in a Black–Scholes–Merton market.

We begin by studying under EZ-SDU the utility processes associated to consumption streams given by a geometric Brownian motion. In this case, an explicit, time-homogeneous utility process can be found. However, we show in Sect. 3.2 that this solution is not the only solution and there exists an infinite family of (equally explicit, but time-inhomogeneous) utility processes.

It is clear that to be able to formulate Merton's optimal investment–consumption problem for EZ-SDU, there must be a rule which assigns a particular utility process to each consumption stream over which we maximise. Various candidates for this assignation rule are plausible. Perhaps the most obvious choice is the *maximal* utility process. The rationale behind this would be that the agent gets to choose which utility process they associate to a given consumption stream, and so they naturally choose the best one. However, when $R > 1$, the maximal utility process associated to any consumption stream is the zero process, rendering the problem degenerate. An alternative choice might be the "game-theoretic" or minimax version of the Merton problem, where the agent maximises over the worst utility process associated to each consumption stream. However, when $R < 1$, the minimal utility process associated to any consumption stream is the zero process, again rendering the problem degenerate. Instead, one of the key contributions of the paper is to introduce the notion of a *proper* solution.

It will follow from the discussion in Sect. 3.2 below that if C is a consumption stream given by a geometric Brownian motion (and if $\mathbb{E}[\int_0^\infty e^{-\delta\vartheta s} C_s^{1-R} ds] < \infty$, where δ is the discount parameter), then for each $T \in [0, \infty]$, there exists a utility process associated to C which is nonzero for $t < T$ and zero for $t \geq T$. (In particular, zero is a solution, and corresponds to $T = 0$.) Economically, this may be interpreted as saying that the amount consumed after time T has no effect on the agent's utility.

This may be considered to be undesirable and motivates the definition of a proper solution: a solution $V = (V_t)_{t\geq 0}$ (of the defining equation for an EZ-SDU utility process, see (2.2) below) is proper if $\mathbb{E}[\int_t^\infty e^{-\delta\vartheta s} C_s^{1-R}\,\mathrm{d}s \mid \mathcal{F}_t] > 0$ implies that $(1-R)V_t > 0$.

The main results of the paper for the case $\vartheta > 1$ are as follows (more precise statements follow as Theorems 4.7, 4.9 and 7.5, respectively).

Main Result 1 *For a very wide class of consumption streams C, there exists a proper utility process V associated to C.*

Main Result 2 *For a wide class of consumption streams, the proper utility process is unique.*

Main Result 3 *In the constant-parameter Black–Scholes–Merton model, if we maximise over attainable consumption streams which admit a unique proper utility process, then the investment–consumption problem is solved by a constant proportional investment–consumption strategy (whose parameters may be identified in terms of the parameters of the EZ-SDU and the financial market).*

These results can be compared with those of Melnyk et al. [21], where we restrict the comparison to the case $\vartheta > 1$ – the subject of the present paper. The main focus of [21] is to understand the impact of transaction costs on the investment–consumption problem under EZ-SDU. However, the frictionless case is also covered and [21] presents some of the most complete results in the current literature. Melnyk et al. [21] do not prove any existence results and instead choose to optimise over the abstract class of consumption streams for which a solution exists. When $\vartheta > 1$, they further restrict to consumption streams $C = (C_t)_{t\geq 0}$ whose utility process $V = (V_t)_{t\geq 0}$ satisfies $(1-R)V_t \geq e^{-\delta\vartheta t} C_t^{1-R}$ for all $t \geq 0$ as well as the transversality condition $\lim_{t\to\infty} e^{-\gamma t}\mathbb{E}[|V_t|] = 0$ for a well-chosen discount factor γ. It is unclear exactly how large this class is, but there are many consumption streams which we show to have a unique proper solution and which do not lie in this class. In particular, the bound on V and the transversality condition together rule out the candidate optimal strategy for some parameter combinations, forcing [21] to impose additional restrictions on the parameter values [21, Assumption 3.3]. Finally, their approach only works when $R > 1$. A more thorough comparison of our results with those in [21] is provided in Appendix A.2.

The main results of the present paper may also be compared with the prior results of the authors for the case $\vartheta < 1$. In that setting, in [13], the authors show:

[13, Theorem 10.8] *For every consumption stream C, there exists a unique utility process V associated to C, if we allow the utility process to take values in $[-\infty, \infty]$.*

[13, Theorem 11.1] *In the constant-parameter Black–Scholes–Merton model, if we maximise over attainable consumption streams, the investment–consumption problem is solved by a constant proportional strategy (whose parameters may be identified in terms of the parameters of the EZ-SDU and the financial market).*

Note that the results of the present paper are less complete than those of [13]. We do not show existence (and uniqueness) of a utility process for *every* consump-

tion stream, but only for a wide class; similarly, when we consider the investment–consumption problem, we cannot optimise over *every* consumption stream, but only over a wide class. But this lack of completeness must be set against the additional complexity of the problem we consider here – the non-uniqueness of the utility process is an unavoidable and major issue.

The remainder of the paper is structured as follows. In Sect. 2, we introduce EZ-SDU. In Sect. 3, we give examples to show that in the case $\vartheta > 1$, utility processes under EZ-SDU are not unique. In particular, we show that to consumption streams given by a geometric Brownian motion, we may associate a family of EZ-SDU utility processes indexed by $T \in [0, \infty]$, each of which corresponds to ignoring consumption from time T onwards.

In Sect. 4, we give a precise definition of a proper solution and restate the main existence and uniqueness results precisely. Then in Sect. 5, we describe a change of coordinates which is very useful in simplifying the problem, and we introduce some additional solution concepts of a more mathematical nature. In Sect. 6, we are then in a position to give the main existence and uniqueness results (with some of the more technical material and proofs relegated to the appendices).

Finally, in Sect. 7, we apply our results on the existence and uniqueness of utility processes associated to consumption streams to the Merton investment–consumption problem for EZ-SDU in a Black–Scholes–Merton financial market. We derive a candidate solution to the Merton problem and then provide a verification theorem proving that the candidate solution is optimal within the class of all attainable consumption streams to which we may assign a unique proper solution, thus completing the study of the infinite-horizon investment–consumption problem for EZ-SDU.

Some mathematical material and some proofs are deferred to the appendices. In Appendix A, we compare our results in detail with the existing results of the authors [12, 13] and with Melnyk et al. [21]. In Appendix B, we introduce subsolutions and supersolutions to the EZ-SDU equation, which, roughly speaking, differ from solutions by replacing the equality in the EZ-SDU equation with an inequality. We then prove an important and useful comparison theorem which provides a sufficient criterion under which subsolutions are dominated from above by supersolutions. Appendix C proves results about two extra solution concepts, which are analytically easier to work with than proper solutions. Based on the results of Appendix C, Appendix D proves the results from Sect. 6 via some very technical results on the existence of proper solutions. Appendix E gives the proof of the final verification theorem.

2 Epstein–Zin stochastic differential utility

We work throughout on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $\mathcal{F}_0$ is $\mathbb{P}$-trivial and the filtration $\mathbb{F}$ is complete and *continuous*. Note that this is slightly stronger than the right-continuity of the filtration assumed in the usual conditions. However, it is a necessary assumption for the existence arguments in Appendix D to go through. Let $\mathscr{P}$ denote the set of progressively measurable processes, and let $\mathscr{P}_+$ and $\mathscr{P}_{++}$ be those processes in $\mathscr{P}$ that take nonnegative and strictly

positive values, respectively. Moreover, denote by $\mathscr{S}$ the set of all semimartingales. We identify processes in $\mathscr{P}$ or $\mathscr{S}$ that agree up to indistinguishability.

2.1 Stochastic differential utility

To understand Epstein–Zin stochastic differential utility (EZ-SDU), it is beneficial to introduce stochastic differential utility (SDU) in its more general form. We contrast SDU with *time-additive utility*. We consider infinite-horizon (lifetime) stochastic differential utility.

A time-additive (or expected) utility is characterised by a utility function

$$U : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{V},$$

where $\mathbb{V} \subseteq \overline{\mathbb{R}} = [-\infty, +\infty]$, is the range of U. A consumption stream $C = (C_t)_{t\geq 0}$ is a nonnegative progressively measurable process, i.e., an element of $\mathscr{P}_+$. The corresponding utility is given by $\mathbb{E}[\int_0^\infty U(t, C_t)\,\mathrm{d}t]$ (provided this expectation is well defined), and the *utility process* $V = (V_t)_{t\geq 0}$ – which measures the utility starting from a given time onward – is defined by $V_t = \mathbb{E}[\int_t^\infty U(s, C_s)\,\mathrm{d}s|\mathcal{F}_t]$. Under SDU, the utility function U is generalised to become an *aggregator* $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$. The SDU process $V^C = (V_t^C)_{t\geq 0}$ associated to a consumption stream C and an aggregator g is then the solution to

$$V_t = \mathbb{E}\left[\int_t^\infty g(s, C_s, V_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \qquad t \geq 0. \tag{2.1}$$

This creates a feedback effect in which the value at time t may depend in a nonlinear way on the value at future times and permits the modelling of a much wider range of preferences. However, in addition to issues about whether the conditional expectation is well defined, there are new issues concerning the existence and uniqueness of solutions to (2.1) which are not present for additive utilities.

Definition 2.1 An *aggregator* is a function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$. For $C \in \mathscr{P}_+$, define $\mathbb{I}(g, C) := \{V \in \mathscr{P} : \mathbb{E}[\int_0^\infty |g(s, C_s, V_s)|\,\mathrm{d}s] < \infty\}$. Further, let $\mathbb{UI}(g, C)$ be the set of elements of $\mathbb{I}(g, C)$ which are uniformly integrable. Then $V \in \mathbb{I}(g, C)$ is a *utility process* associated to the pair (g, C) if it has càdlàg paths and satisfies (2.1) for all $t \in [0, \infty)$.

Remark 2.2 It can be easily shown that all utility processes are uniformly integrable (see [12, Remark 3.2]).

2.2 Epstein–Zin stochastic differential utility

The Epstein–Zin stochastic differential utility (see Duffie and Epstein [8], Herdegen et al. [12], Kraft and Seifried [18], Schroder and Skiadas [24]) is parametrised by δ, R and S, where $\delta \in \mathbb{R}$ and $R, S \in (0, 1) \cup (1, \infty)$ and, with $\mathbb{V} = (1 - R)\mathbb{R}_+$, the Epstein–Zin aggregator $g_{\mathrm{EZ}} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ is defined as

$$g_{\mathrm{EZ}}(t, c, v) = e^{-\delta t}\frac{c^{1-S}}{1-S}\big((1-R)v\big)^{\frac{S-R}{1-R}}.$$

It is convenient to introduce the parameters $\vartheta = \frac{1-R}{1-S}$ and $\rho = \frac{S-R}{1-R} = \frac{\vartheta-1}{\vartheta}$ so that $g_{\mathrm{EZ}}(t,c,v) = e^{-\delta t}\frac{c^{1-S}}{1-S}((1-R)v)^{\rho}$. Note that when $S = R$, the aggregator reduces to the discounted CRRA utility function $U(t,c) = e^{-\delta t}\frac{c^{1-R}}{1-R}$. This case corresponds to $\vartheta = 1$ and $\rho = 0$.

Remark 2.3 One of the advantages of the aggregator g_{EZ} is that it is one-signed and hence the conditional expectation $\mathbb{E}[\int_t^\infty g(s,C_s,V_s)\,\mathrm{d}s|\mathcal{F}_t]$ is always well defined in $[-\infty,0]$ or $[0,\infty]$. Other authors have used a slightly different aggregator $g^{\Delta}_{\mathrm{EZ}}(c,v) := \frac{c^{1-S}}{1-S}((1-R)v)^{\rho} - \delta\vartheta v$; see e.g. Duffie and Epstein [8, Appendix C] or Melnyk et al. [21]. While the two formulations are essentially equivalent, our choice is slightly more general. Indeed, if V^{Δ} solves (2.1) for consumption C and aggregator g^{Δ}_{EZ}, then V given by $V_t = e^{-\delta\vartheta t}V^{\Delta}_t$ solves (2.1) for consumption C and aggregator g_{EZ}. However, the converse is not true: if V, for given C, solves (2.1) for the aggregator g_{EZ}, it need not follow that V^{Δ} given by $V^{\Delta}_t = e^{\delta\vartheta t}V_t$ solves (2.1) for the aggregator g^{Δ}_{EZ} because the integrals and expectations need not be well defined. A fuller discussion of this issue can be found in [12, Sect. 5.2].

Definition 2.4 A process $V^C = V = (V_t)_{t\geq 0}$ is an *EZ-SDU utility process associated to a consumption stream* $C \in \mathscr{P}_+$ if $\int_0^\infty e^{-\delta s}\frac{C_s^{1-S}}{1-S}((1-R)V_s)^{\rho}\,\mathrm{d}s \in L^1$ and if for each $t \geq 0$, it satisfies

$$V_t = \mathbb{E}\left[\int_t^\infty e^{-\delta s}\frac{C_s^{1-S}}{1-S}\big((1-R)V_s\big)^{\rho}\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]. \tag{2.2}$$

Implicit in the form of g_{EZ} is the fact that in order to define $((1-R)V)^{\rho}$, we must have $\operatorname{sgn}(V) = \operatorname{sgn}(1-R)$ and hence that $\mathbb{V} \subseteq (1-R)\overline{\mathbb{R}}_+$. Then for there to be equality in (2.2), we must at least have that the signs of the left- and right-hand sides of (2.2) agree, i.e., $\operatorname{sgn}(V) = \operatorname{sgn}(1-S)$. This forces $\operatorname{sgn}(1-R) = \operatorname{sgn}(1-S)$ or equivalently $\vartheta > 0$. Some authors, including Bansal and Yaron [1], have considered the case $\vartheta < 0$, but as argued in [12, Sect. 7.3], the solutions they find are utility bubbles and thus economically questionable. **For the rest of the paper**, we assume $\vartheta > 0$.

The case $0 < \vartheta < 1$ was considered in [13], and when $\vartheta = 1$, EZ-SDU reduces to the widely studied CRRA utility. In this paper, we consider parameter combinations leading to $\vartheta > 1$; either we have $R < S < 1$ or $1 < S < R$.

Standing Assumption 2.5 The parameters R and S are chosen in such a way that $\vartheta := \frac{1-R}{1-S} > 1$.

Remark 2.6 There is no universal agreement in the economics literature on the range of plausible values for the parameters R and S. Chetty [3] writes that "most economists believe based on introspection that [the relative risk aversion is in] (1, 5) while others contend that higher values [...] are reasonable", and Vissing–Jørgensen [26] adds (recall that the elasticity of intertemporal substitution (EIS) is the reciprocal of the elasticity of intertemporal complimentarity, as represented by

our parameter S) "[...], it seems fair to say that there is no consensus on whether [the elasticity of intertemporal substitution] is significantly above zero and, if so, what its value is," although implicit in her statement is the fact that the EIS is below 1. Indeed, [26] estimates the elasticity of intertemporal substitution to be between 0.3 and 0.4 for stockholders and between 0.8 and 1 for bondholders, both of which are below 1, and Guvenen [9] reports similar conclusions. In terms of our parameters, this means that there is general agreement that $R > 1$ and $S > 1$ so that $\vartheta > 0$, but no consensus on whether $S > R > 1$ ($0 < \vartheta < 1$) or $R > S > 1$ ($\vartheta > 1$). Prominent papers that make the assumption that $\vartheta > 1$ include Kandel and Stambaugh [16] and Campbell and Mankiw [2].

3 Non-uniqueness of utility processes when $\vartheta > 1$

In this section, we give a simple illustrative example highlighting a key issue with stochastic differential utility when $\vartheta > 1$ – the non-uniqueness of utility processes.

3.1 An explicit utility process associated to geometric Brownian motion consumption streams

We begin by finding an explicit utility process associated to a consumption stream $C = (C_t)_{t\geq 0}$ given by a geometric Brownian motion. For the solution to exist and be finite, we need that $(e^{-\delta\vartheta t}C_t^{1-R})_{t\geq 0}$ – which is also a geometric Brownian motion – has negative drift. Explicitly, we assume that for some $\sigma \geq 0$ and $\phi > 0$,

$$e^{-\delta\vartheta t}C_t^{1-R} = C_0^{1-R}\exp\left(\sigma B_t - \left(\phi + \frac{\sigma^2}{2}\right)t\right). \tag{3.1}$$

In particular, since our analysis allows $\sigma = 0$, this assumption includes the case of constant consumption streams provided the agent's discount factor δ is strictly positive. Making the ansatz that the EZ utility process is of the form

$$V_t = \frac{A}{1-R}e^{-\delta\vartheta t}C_t^{1-R}$$

(see [12, Sect. 5.3] for why this is the correct ansatz), a solution to (2.2) is given by

$$\begin{aligned}\frac{A}{1-R}e^{-\delta\vartheta t}C_t^{1-R} &= \mathbb{E}\left[\int_t^\infty e^{-\delta s}\frac{C_s^{1-S}}{1-S}A^\rho e^{-\delta\rho\vartheta t}C_s^{\rho(1-R)}\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]\\ &= \frac{A^\rho}{1-S}\mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta s}C_s^{1-R}\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right],\end{aligned} \tag{3.2}$$

where we have used that $\rho\vartheta = \vartheta - 1$ and $\rho(1-R) = S - R$. Furthermore, since $\mathbb{E}[e^{-\delta\vartheta s}C_s^{1-R}|\mathcal{F}_t] = e^{-\delta\vartheta t}C_t^{1-R}e^{-\phi(s-t)}$, this reduces to

$$\frac{A}{1-R}e^{-\delta\vartheta t}C_t^{1-R} = \frac{A^\rho}{1-S}e^{-\delta\vartheta t}C_t^{1-R}\mathbb{E}\left[\int_t^\infty e^{-\phi(s-t)}\middle|\mathcal{F}_t\right]$$
$$= \frac{A^\rho}{\phi(1-S)}e^{-\delta\vartheta t}C_t^{1-R}. \tag{3.3}$$

Thus $A^{1-\rho} = \frac{\vartheta}{\phi}$, and since $1-\rho = \frac{1}{\vartheta}$, one utility process associated to a geometric Brownian motion $C = (C_t)_{t\geq 0}$ (such that $(e^{-\delta\vartheta t}C_t^{1-R})_{t\geq 0}$ has drift $-\phi < 0$) is given by

$$V_t = \left(\frac{\vartheta}{\phi}\right)^\vartheta \frac{e^{-\delta\vartheta t}C_t^{1-R}}{1-R}. \tag{3.4}$$

3.2 A family of utility processes indexed by absorption time

In this section, we show that when $\vartheta > 1$, for each consumption stream $C = (C_t)_{t\geq 0}$ such that C^{1-R} satisfies (3.1) with $\phi > 0$, there exists a family of associated EZ utility processes, parametrised by the first time they hit zero (and are absorbed).

We postulate a time-dependent form for the utility process $V = (V_t)_{t\geq 0}$ given by $V_t = \frac{A(t)}{1-R}e^{-\delta\vartheta t}C_t^{1-R}$ for a nonnegative function $A = (A(t))_{t\geq 0}$. Then as in (3.2) and (3.3), finding a utility process associated to C boils down to solving the integral equation

$$A(t)e^{-\delta\vartheta t}C_t^{1-R} = \vartheta\,\mathbb{E}\left[\int_t^\infty A(s)^\rho e^{-\delta\vartheta s}C_s^{1-R}\,\mathrm{d}s\middle|\mathcal{F}_t\right]$$
$$= \vartheta e^{-\delta\vartheta t}C_t^{1-R}\int_t^\infty A(s)^\rho e^{-\phi(s-t)}\,\mathrm{d}s.$$

Cancelling the $e^{-\delta\vartheta t}C_t^{1-R}$ terms and taking derivatives with respect to t shows that A solves the ODE

$$A'(t) = \phi A(t) - \vartheta A(t)^\rho. \tag{3.5}$$

Note that one solution is the constant solution $A(t) \equiv (\frac{\vartheta}{\phi})^\vartheta$, giving (3.4). More generally, the ODE (3.5) is separable and can be solved to give

$$A(t) = \left(\frac{\vartheta - (\vartheta - \phi A(0)^{1/\vartheta})e^{\frac{\phi}{\vartheta}t}}{\phi}\right)^\vartheta.$$

If we assume that $A(0) < (\frac{\vartheta}{\phi})^\vartheta$, then A hits zero at $t = T := \frac{\vartheta}{\phi}\log\frac{\vartheta}{\vartheta - \phi A(0)^{1/\vartheta}}$. Since $(A(t))_{t\geq T} \equiv 0$ is a solution on $[T,\infty)$, we can define a family of solutions to (3.5) and hence to (2.2), indexed by $A(0)$, such that

$$A(t) = \begin{cases} (\frac{\vartheta - (\vartheta - \phi A(0)^{1/\vartheta})e^{\frac{\phi}{\vartheta}t}}{\phi})^\vartheta & \text{for } t < T = \frac{\vartheta}{\phi}\log\frac{\vartheta}{\vartheta - \phi A(0)^{1/\vartheta}}, \\ 0 & \text{for } t \geq T. \end{cases}$$

(Note that if $A(0) > (\frac{\vartheta}{\phi})^{\vartheta}$, then A diverges to ∞, but this is not consistent with the fact that $\mathbb{E}[V_t] \to 0$.) Alternatively, the family of solutions can be thought of as indexed by T, where $T = \inf\{t \geq 0 : A(t) = 0\}$. Effectively, for the solution indexed by T, consumption after T does not yield any utility, and the utility process is zero thereafter. It is hard to argue, when considering the infinite time horizon, that this represents an economically meaningful solution of the problem. Hence intuitively the "correct" utility process should correspond to $T = \infty$ (i.e., $A(0) = (\frac{\vartheta}{\phi})^{\vartheta}$) and be given by (3.4).

Remark 3.1 The same issue also arises for finite-horizon EZ-SDU. A variant of the Merton problem for finite-horizon EZ-SDU and $\vartheta > 1$ is considered by Matoussi and Xing [20], Schroder and Skiadas [24], Seiferling and Seifried [25], Xing [28], among others. In [20, 25, 28], the issue of uniqueness is addressed by incorporating a strictly positive bequest function $U_\varepsilon(C_T) = \varepsilon \frac{C_T^{1-R}}{1-R}$ with $\varepsilon > 0$ at the finite time horizon T (and either restricting to consumption streams that are strictly positive or only considering the case $R > 1$), so that the EZ-SDU equation in this case is

$$V_t = \mathbb{E}\bigg[\int_t^T e^{-\delta s} \frac{C_s^{1-S}}{1-S} \big((1-R)V_s\big)^{\rho} \,\mathrm{d}s + e^{-\delta T} U_\varepsilon(C_T) \bigg| \mathcal{F}_t\bigg], \qquad 0 \leq t \leq T. \tag{3.6}$$

This is not a viable approach in the infinite-horizon case as a bequest "at infinity" has no meaning. In [24], the authors claim that EZ-SDU utility processes are unique by finding a solution to (3.6), letting $\varepsilon \searrow 0$ and then claiming that the limiting process – which is an EZ-SDU utility process for (3.6) with zero bequest – is the unique EZ-SDU utility process. As we have seen in this section, this is not the case.

The approach taken in the present paper is to embrace the existence of multiple utility processes and distinguish the *proper* utility process (which we introduce in the next section) from other utility processes. The definition of the proper utility process rules out solutions which ignore the utility gained from consumption from some finite time onwards.

4 Existence and uniqueness results

This section introduces some key definitions used throughout the paper and states the main existence (Theorem 4.7) and uniqueness (Theorem 4.9) results. Proofs of the key theorems are given in Sect. 6.

4.1 Proper solutions

The following definition of a *proper* solution to the EZ-SDU equation is motivated by the arguments of the previous section.

Definition 4.1 Let $C \in \mathscr{P}_+$ and suppose that $V = (V_t)_{t\geq 0}$ is a solution to (2.2). Then V is called a *proper solution* if up to nullsets,

$$(1-R)V_t > 0 \quad \text{on} \quad \bigg\{\mathbb{E}\bigg[\int_t^\infty C_s^{1-R}\,\mathrm{d}s \bigg| \mathcal{F}_t\bigg] > 0\bigg\} \qquad \text{for all } t \geq 0.$$

The notion of a proper solution immediately excludes the time-inhomogeneous utility processes found in Sect. 3.2.

Remark 4.2 Note that $\mathbb{E}[\int_t^\infty C_s^{1-R}\,\mathrm{d}s|\mathcal{F}_t] > 0$ on $\{(1-R)V_t > 0\}$ up to nullsets for all $t \geq 0$. Indeed, seeking a contradiction, suppose there are $t \geq 0$ and $D \in \mathcal{F}_t$ with $\mathbb{P}[D] > 0$ such that $(1-R)V_t > 0$ on D, but $\mathbf{1}_D\mathbb{E}[\int_t^\infty C_s^{1-R}\,\mathrm{d}s|\mathcal{F}_t] = 0$ $\mathbb{P}$-a.s. Then $\mathbb{E}[\mathbf{1}_D\int_t^\infty C_s^{1-S}\,\mathrm{d}s] = 0$, which yields $C = 0$ for $(\mathbb{P}\otimes\mathrm{d}t)$-almost every (ω, t) in $D\times[t,\infty)$. Consequently, $\mathbb{E}[\mathbf{1}_D V_t] = \mathbb{E}[\mathbf{1}_D\int_t^\infty g_{\mathrm{EZ}}(s, C_s, V_s)\,\mathrm{d}s] = 0$, and we arrive at a contradiction.

4.2 Self-order consumption streams

We proceed to introduce an important class of consumption streams that will play a key role throughout the paper.

Definition 4.3 Suppose that $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are nonnegative progressive processes. We say that X has the same order as Y (and write $X \overset{\mathbb{O}}{=} Y$, noting that $\overset{\mathbb{O}}{=}$ is an equivalence relation) if there exist constants $k, K \in (0,\infty)$ such that

$$0 \leq kY \leq X \leq KY. \tag{4.1}$$

The set of progressive processes with the same order as X is denoted by $\mathbb{O}(X) \subseteq \mathscr{P}_+$.

Definition 4.4 For each $X \in \mathscr{P}_+$, define the process $J^X = (J_t^X)_{t\geq 0}$ by setting $J_t^X = \mathbb{E}[\int_t^\infty X_s\,\mathrm{d}s|\mathcal{F}_t]$. Then the *self-order* consumption streams are given by

$$\mathbb{SO} := \left\{X \in \mathscr{P}_{++} : \mathbb{E}\left[\int_0^\infty X_t\,\mathrm{d}t\right] < \infty \text{ and } X \overset{\mathbb{O}}{=} J^X\right\}.$$

On some occasions, we need a slightly stronger condition. Define

$$\mathbb{SO}_\nu := \{X \in \mathbb{SO} : (e^{\nu t}X_t)_{t\geq 0} \in \mathbb{SO}\} \quad \text{for } \nu \geq 0,$$
$$\mathbb{SO}_+ := \bigcup_{\nu>0}\mathbb{SO}_\nu.$$

We proceed to show that a geometric Brownian motion with decreasing mean lies in $\mathbb{SO}$ and even in $\mathbb{SO}_+$. This class of processes turns out to be important in the study of the candidate optimal consumption in the constant-parameter investment–consumption problem; see Sects. 7.3 and 7.4.

Example 4.5 Let $X = (X_t)_{t\geq 0}$ satisfy $X_t = xe^{\sigma W_t - \mu t}$, where $\mu > \frac{1}{2}\sigma^2 \geq 0$. Note that we allow the case $\sigma = 0$, i.e., where X is an exponentially decreasing deterministic process. Then $\mathbb{E}[\int_0^\infty X_t\,\mathrm{d}t] < \infty$ and $X_t = kJ_t^X$ for $k = \frac{1}{\mu-\sigma^2/2}$. Taking $K = k$ in (4.1), we obtain $X \in \mathbb{SO}$. Moreover, it is straightforward to check that $X \in \mathbb{SO}_\nu$ for all $\nu \in [0, \mu - \sigma^2/2)$, whence $X \in \mathbb{SO}_+$.

Remark 4.6 There are two immediate and two subsidiary reasons why it is natural to work with the space $\mathbb{SO}$.

First, the definition of a utility process is such that it relates the current value of a utility process to the integrated future values of consumption. If the current consumption C_t is related to the current value of the utility process V_t, then we expect a relationship between the current consumption and the expected integrated future values of (a transformation of) consumption, i.e., $Z \in \mathbb{SO}$, where Z is some process derived from C. (The heuristics of Sect. 3.1 show that the appropriate transformation is $Z_t = e^{-\delta\vartheta t} C_t^{1-R}$.)

Second, in Sect. 7, we study utility processes in the context of the Merton investment–consumption problem. There, the optimal consumption rate process $C = (C_t)_{t\geq 0}$ is a constant multiple of the optimal wealth process, which is a geometric Brownian motion. In particular, if Z^C is defined by $Z_t^C = e^{-\delta\vartheta t} C_t^{1-R}$, then $Z_t^C = k J_t^{Z^C}$ so that we can take $k = K$ in (4.1).

Third, in principle, it is straightforward to check whether $(e^{-\delta\vartheta t} C_t^{1-R})_{t\geq 0}$ lies in $\mathbb{SO}$ (or $\mathbb{SO}_+$) — the condition relates to C alone, rather than say to C and some combination of the associated utility process, the investment strategy or the wealth process.

Fourth, and very pragmatically, using the space $\mathbb{SO}$, we can use the results of Herdegen et al. [13] to deduce the existence of a utility process associated to C; see Proposition 5.4.

4.3 Existence and uniqueness of proper solutions

Theorems 4.7 and 4.9 correspond to Main Results 1 and 2 presented in the introduction. Proofs of these are given in later sections. The first result shows that we may find a proper solution associated to a large class of consumption streams.

Theorem 4.7 *Suppose $C \in \mathscr{P}_+$ is a right-continuous consumption stream satisfying $e^{-\delta\vartheta t} C_t^{1-R} \leq Y_t$ for some $Y \in \mathbb{SO}_+$. Then there exists a proper solution to (2.2).*

Proper solutions are an economically meaningful concept that allow us to choose from the many solutions to the EZ-SDU equation, and Theorem 4.7 provides a large class of consumption streams which have proper solutions. However, we have not yet discussed their uniqueness. If the property of being proper does not provide a criterion for selecting a unique solution, then it does not help to overcome the issues of non-uniqueness intrinsic to EZ-SDU when $\vartheta > 1$. The following definition is therefore of great importance.

Definition 4.8 We say that $C \in \mathscr{P}_+$ is *uniquely proper* if there exists a unique proper solution to (2.2). Let $\mathbb{UP}$ denote the set of uniquely proper consumption streams.

Theorem 4.9 *Suppose that $C \in \mathscr{P}_+$ is such that $(e^{-\delta\vartheta t} C_t^{1-R})_{t\geq 0} \in \mathbb{SO}_+$. Then C is uniquely proper.*

4.4 Examples

In this section, we provide examples of consumption streams that are uniquely proper. We first consider examples of consumption streams that are of geometric Brownian motion type.

Proposition 4.10 *Suppose $C \in \mathscr{P}_+$ is bounded above and below by a constant multiple of a geometric Brownian motion $Z = (Z_t)_{t\geq 0}$ such that the process $(e^{-\delta\vartheta t} Z_t^{1-R})_{t\geq 0}$ has negative drift. Then C is uniquely proper.*

If we allow the geometric Brownian motion Z to have zero volatility, we immediately obtain the following corollary.

Corollary 4.11 *Suppose that $\delta > 0$. Suppose $C \in \mathscr{P}_+$ is bounded above and below by positive constants. Then C is uniquely proper.*

Remark 4.12 In Sect. 7, we study utility processes in the context of the Merton investment–consumption problem. There, the optimal consumption rate process $C = (C_t)_{t\geq 0}$ is a constant multiple of the optimal wealth process, which is a geometric Brownian motion. Further, if the problem is well posed, then $(e^{-\delta\vartheta t} C_t^{1-R})_{t\geq 0}$ has negative drift. Hence by Proposition 4.10, we can associate a unique proper solution to C.

We proceed to present a much wider class of "factor-model" consumption streams that are uniquely proper. To this end, suppose that $Y = (Y_t)_{t\geq 0}$ is a time-homogeneous, non-explosive, regular diffusion with state space $\mathbb{R}$ and dynamics

$$\mathrm{d}Y_t = a_Y(Y_t)\,\mathrm{d}B_t^Y + b_Y(Y_t)\,\mathrm{d}t,$$

where $a_Y, b_Y : \mathbb{R} \to \mathbb{R}$ are sufficiently regular and $B^Y = (B_t^Y)_{t\geq 0}$ is a Brownian motion. Suppose a consumption process $C \in \mathscr{P}_+$ satisfies

$$dC_t = C_t\big(\alpha(Y_t)\,\mathrm{d}B_t + \beta(Y_t)\,\mathrm{d}t\big), \tag{4.2}$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are sufficiently regular functions so that $C > 0$. Here $B = (B_t)_{t\geq 0}$ is another Brownian motion, and the covariation between B and B^Y satisfies $\langle B, B^Y\rangle_t = \int_0^t \rho(Y_s)\,\mathrm{d}s$, where $\rho : \mathbb{R} \to [-1, 1]$ is a sufficiently regular function. Then we have the following result.

Proposition 4.13 *Suppose that $a_Y, b_Y, \alpha, \beta : \mathbb{R} \to \mathbb{R}$ are continuous and bounded functions. Moreover, assume that a_Y is positive and bounded away from zero and there exists $\chi > 0$ such that for all $y \in \mathbb{R}$,*

$$\delta\vartheta - (1-R)\beta(y) + \frac{R(1-R)}{2}\alpha^2(y) \geq \chi.$$

Then C is uniquely proper with proper utility process $V^C = (V_t^C)_{t\geq 0}$ given by

$$V_t^C = e^{-\delta\vartheta t}\frac{C_t^{1-R}}{1-R}\Gamma(Y_t),$$

where $\Gamma \in C^2(\mathbb{R})$ *is the unique solution to the ODE*

$$\vartheta\Gamma^{-1/\vartheta} + b_Y\frac{\Gamma'}{\Gamma} + a_Y^2\frac{\Gamma''}{2\Gamma} + (1-R)a_Y\alpha\rho\frac{\Gamma'}{\Gamma} - \delta\vartheta + (1-R)\beta - \frac{R(1-R)}{2}\alpha^2 = 0 \tag{4.3}$$

in the class of positive functions that are bounded from above and away from zero and have a bounded derivative.

Remark 4.14 When considering the Merton investment–consumption problem for a factor model, the candidate optimal consumption process has dynamics of the form (4.2); see Sect. 7.5. Hence if the coefficients are sufficiently regular, Proposition 4.13 shows that the candidate optimal consumption process is uniquely proper.

5 A change of coordinates and additional solution concepts of a mathematical nature

The notion of a proper solution is based on the economic idea that a strictly positive bounded consumption should imply a nonzero utility. In this section, we introduce other notions of a solution which are of a more mathematical nature. These solution concepts are economically less meaningful, but aid in proving key results about proper solutions. First, however, to simplify the proofs in this and later sections, we introduce a change of coordinates.

5.1 A change of coordinates

Define the nonnegative processes $W = (W_t)_{t\geq 0}$ and $U = U^C = (U_t^C)_{t\geq 0}$ via $W_t = (1-R)V_t$ and $U_t = U_t^C = \vartheta e^{-\delta t}C_t^{1-S}$ and define the aggregator

$$h_{\mathrm{EZ}}(u, w) = uw^\rho. \tag{5.1}$$

Further define $J = J^{U^\vartheta}$ by

$$J_t = \mathbb{E}\left[\int_t^\infty U_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \qquad \text{for all } t \geq 0. \tag{5.2}$$

Then V solves (2.2) if and only if W solves

$$W_t = \mathbb{E}\left[\int_t^\infty h_{\mathrm{EZ}}(U_s, W_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \qquad t \geq 0. \tag{5.3}$$

Note that $V \in \mathbb{I}(g_{\mathrm{EZ}}, C)$ if and only if $W \in \mathbb{I}(h_{\mathrm{EZ}}, U^C)$. Hence V^C is a utility process associated to a consumption stream C with aggregator g_{EZ} if and only if $W = W^{U^C}$ is a utility process associated to the consumption stream U^C with aggregator h_{EZ}.

Remark 5.1 It is immediate that $V = (V_t)_{t\geq 0}$ is a proper solution associated to (g, C) if and only if $W = (W_t)_{t\geq 0}$ is a solution associated to (h_{EZ}, U) such that up to nullsets,

$$W_t > 0 \quad \text{on } \left\{ \mathbb{E}\left[\int_t^\infty U_s^\vartheta \, \mathrm{d}s \middle| \mathcal{F}_t \right] > 0 \right\} \qquad \text{for all } t \geq 0.$$

In a slight abuse of the definition, we then also refer to W as being proper. Similarly, we say $U \in \mathscr{P}_+$ is uniquely proper if there exists a unique proper solution W to (5.3). It follows that Theorem 4.9 is equivalent to the statement that if $U \in \mathbb{SO}_+$, then U is uniquely proper.

5.2 The order-ϑ solution

The first additional solution concept that will be useful is an order-ϑ solution. Proofs not given in this section can be found in Appendix C.

Definition 5.2 Let $U \in \mathscr{P}_+$ and suppose that $W = (W_t)_{t\geq 0}$ is a solution to (5.3). We say that W is an *order-ϑ solution* if $W \overset{\mathbb{O}}{=} J$, where J is defined by (5.2).

Remark 5.3 It follows directly from the definition of J in (5.2) that if W is an order-ϑ solution, then W is a proper solution.

We proceed to establish existence and uniqueness of order-ϑ solutions. To this end, define the operator $F_U : \mathbb{I}(h_{\mathrm{EZ}}, U) \to \mathscr{P}_+$ by

$$F_U(W)_t = \mathbb{E}\left[\int_t^\infty U_s W_s^\rho \, \mathrm{d}s \middle| \mathcal{F}_t \right] \qquad \text{for all } t \geq 0, \tag{5.4}$$

where we always choose a càdlàg version for the right-hand side of (5.4). Note that W is a solution associated to (h_{EZ}, U) if and only if it is a fixed point of the operator F_U. We want to show existence of a fixed point of F_U, but in order to also prove existence of a *maximal* solution associated to (h_{EZ}, U) (see Sect. 5.3), it will instead be beneficial to consider existence of a fixed point to a more general (perturbed) operator.

The following result is [13, Theorem B.2], which we recall for the convenience of the reader.

Proposition 5.4 *Let $\varepsilon \geq 0$ and let $U \overset{\mathbb{O}}{=} \Lambda$ for some $\Lambda \geq 0$ with $\Lambda^\vartheta \in \mathbb{SO}$. Then $F^\varepsilon_{U,\Lambda} : \mathbb{I}(h_{\mathrm{EZ}}, U) \to \mathscr{P}_+$ defined by*

$$F^\varepsilon_{U,\Lambda}(W)_t = \mathbb{E}\left[\int_t^\infty (U_s W_s^\rho + \varepsilon \Lambda_s^\vartheta) \, \mathrm{d}s \middle| \mathcal{F}_t \right] \tag{5.5}$$

has a fixed point $W \in \mathbb{I}(h_{\mathrm{EZ}}, U)$. It is the unique fixed point such that $W \overset{\mathbb{O}}{=} J$, where $J = (J_t)_{t\geq 0}$ is defined by (5.2).

Assuming that $U^\vartheta \in \mathbb{SO}$ and applying Proposition 5.4 with $\Lambda = U$, we obtain a fixed point W of the operator $F_U = F^0_{U,U}$ satisfying $W \overset{\mathbb{O}}{=} J \overset{\mathbb{O}}{=} U^\vartheta$. Moreover, this solution is the order-ϑ solution.

The following result is [13, Theorem 4.6], which we recall for the convenience of the reader.

Proposition 5.5 *Let $U \in \mathscr{P}_+$ be such that $U^\vartheta \in \mathbb{SO}$. Then there exists a unique order-ϑ solution to (5.3).*

5.3 The maximal solution

The second additional solution concept is that of a *maximal* solution. Such solutions frequently appear in BSDE theory; see e.g. Drapeau et al. [6, 7], Peng [23]. Proofs not given in this section can be found in Appendix C.

Definition 5.6 Let $U \in \mathscr{P}_+$ and suppose that $W = (W_t)_{t\geq 0}$ is a solution to (5.3). Then W is a *maximal solution* if $W \geq Y$ for any other solution $Y = (Y_t)_{t\geq 0}$.

Remark 5.7 If there exists a proper solution, then every maximal solution is proper. This follows from Remark 5.1 and Definitions 4.1 and 5.6: if Y is a proper solution and W is a maximal solution, then up to nullsets, for each $t \geq 0$,

$$W_t \geq Y_t > 0 \quad \text{on } \left\{ \mathbb{E}\left[\int_t^\infty U_s^{1-R}\, \mathrm{d}s \,\middle|\, \mathcal{F}_t \right] > 0 \right\}.$$

Proposition 5.8 *If a maximal solution exists, it is unique.*

Proof Suppose for contradiction that there are two maximal solutions W^1 and W^2. Then for all $t \geq 0$, $W^1_t \geq W^2_t$ since W^1 is maximal in the class of solutions and $W^2_t \geq W^1_t$ since W^2 is maximal, too. Thus $W^1_t = W^2_t$ for all $t \geq 0$. Since both W^1 and W^2 are càdlàg, they are indistinguishable. □

We now turn to the existence of a maximal solution associated to (h_{EZ}, U).

Proposition 5.9 *Suppose $U \in \mathscr{P}_+$ satisfies $U \leq \Lambda$ for Λ with $\Lambda^\vartheta \in \mathbb{SO}_+$. Then there exists a unique maximal solution associated to (h_{EZ}, U).*

The next result is a comparison result for maximal solutions.

Proposition 5.10 *Let h_{EZ} be the aggregator defined in (5.1) and suppose $U^1, U^2 \in \mathscr{P}_+$ satisfy $U^1 \leq U^2 \leq \Lambda$ for some Λ with $\Lambda^\vartheta \in \mathbb{SO}_+$. If W^1, W^2 are the maximal solutions associated to h_{EZ} and consumption U^1, U^2, respectively, then $W^1_t \leq W^2_t$ for all $t \geq 0$.*

5.4 Relating the different solution concepts

The next result, whose proof is given in Appendix C, shows that the solution found by a fixed point argument in Proposition 5.4 for $\varepsilon = 0$ is the unique maximal solution for (h_{EZ}, U).

Proposition 5.11 *Let h_{EZ} be the aggregator defined in (5.1). Suppose U is such that $U^{\vartheta} \in \mathbb{SO}_+$. Then the order-$\vartheta$ solution associated to (h_{EZ}, U) found in Proposition 5.4 (with $\varepsilon = 0$) is the maximal solution.*

Finally, we show that for U such that $U^{\vartheta} \in \mathbb{SO}_+$, all three solution concepts coincide. This result is useful in the proof of Theorem 7.5, the verification theorem in the Black–Scholes–Merton market.

Theorem 5.12 *Suppose that $U^{\vartheta} \in \mathbb{SO}_+$. Then the following three solutions to (5.3) all coincide and are unique*:

1) *the order-ϑ solution*;
2) *the maximal solution*;
3) *the proper solution*.

Proof Let W be the (unique) order-ϑ solution from Proposition 5.5. Then W is a maximal solution by Proposition 5.11 and a proper solution by Remark 5.3. Moreover, uniqueness in the three classes follows from Propositions 5.5 and 5.8 and Theorem 4.9 and Remark 5.1, respectively. □

6 Existence and uniqueness of proper solutions

6.1 Existence of proper solutions

The goal of this section is to prove Theorem 4.7. To this end, we exploit the change of coordinates described in Sect. 5.1. We first prove that there exists a proper solution W associated to the aggregator h_{EZ} and consumption stream U given by a discounted indicator function of a stochastic interval, i.e., $U_t = e^{-\gamma t}\mathbf{1}_{\{\sigma \le t < \tau\}}$ for σ and τ stopping times such that $\sigma \le \tau$. Since any right-continuous consumption stream can locally be approximated from below by (a scaled version of) these processes, we can show that there exists a proper solution associated to right-continuous processes. All proofs not included in this section can be found in Appendix D.1.

Proposition 6.1 *Let $\gamma > 0$ and σ and τ be stopping times such that $\sigma \le \tau$. Let $U = (U_t)_{t\ge 0}$ be given by $U_t := e^{-\gamma t}\mathbf{1}_{\{\sigma \le t < \tau\}}$. Then there exists a proper solution $W = (W_t)_{t\ge 0}$ associated to U, for which*

$$W_t \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}\left[e^{-\gamma(t\vee\sigma)} - e^{-\gamma(t\vee\tau)}\middle|\mathcal{F}_t\right]\right)^{\vartheta}.$$

To prove Theorem 4.7, we introduce two technical lemmas.

Lemma 6.2 *Suppose that W is a utility process associated to (h_{EZ}, U) and suppose there exists $t_0 \geq 0$ such that $U_t = 0$ for $t < t_0$. If $A \in \mathcal{F}_{t_0}$ and $\widetilde{U} = (\widetilde{U}_t)_{t\geq 0}$ is given by $\widetilde{U}_t = \mathbb{E}[\mathbf{1}_A|\mathcal{F}_t]U_t$, then $\widetilde{W}_t = \mathbb{E}[\mathbf{1}_A W_{t\vee t_0}|\mathcal{F}_t]$, $t \geq 0$, is a utility process associated to $(h_{\mathrm{EZ}}, (\widetilde{U}_t)_{t\geq 0})$.*

Lemma 6.3 *Suppose $U \in \mathscr{P}_+$ is right-continuous. Fix $t \geq 0$ and let*

$$A_t = \{J_t^{U^\vartheta} > 0\} \qquad \text{and} \qquad B_t = \bigcup_{\substack{T\geq t\\ T\in\mathbb{Q}}} \bigcup_{\substack{\varepsilon>0\\ \varepsilon\in\mathbb{Q}}} \{\mathbb{E}[\mathbf{1}_{\{U_T\geq\varepsilon\}}|\mathcal{F}_t] > 0\}.$$

Then $\mathbb{P}[A_t \setminus B_t] = 0$.

We may now prove Theorem 4.7. To show that W is proper, we must show that if $A_t^J = \{J_t^{U^\vartheta} > 0\}$ and $A_t^W = \{W_t > 0\}$, then $\mathbb{P}[A_t^J \setminus A_t^W] = 0$. By Lemma 6.3, since $A_t^J \subseteq B_t$ up to nullsets, we may instead prove that $\mathbb{P}[B_t \setminus A_t^J] = 0$.

Proof of Theorem 4.7 Since $e^{-\delta\vartheta t}C_t^{1-R} \leq Y_t$ for some $Y \in \mathbb{SO}_+$, it follows that $U := \vartheta C^{1-S} \leq \vartheta Y^{\frac{1}{\vartheta}}$. It is further easy to check that $Y \in \mathbb{SO}_+$ implies that $\vartheta^\vartheta Y \in \mathbb{SO}_+$. Therefore by Proposition 5.9, there exists a maximal solution W associated to (h_{EZ}, U). We now show that W is proper.

Fix $t^* \geq 0$. Set $A_{t^*}^J := \{J_{t^*}^{U^\vartheta} > 0\}$ and $A_{t^*}^W = \{W_{t^*} > 0\}$. By Lemma 6.3, it suffices to show that $\mathbb{P}[B_T^\varepsilon \setminus A_{t^*}^W] = 0$ for all rational $T \geq t^*$ and $\varepsilon > 0$, where $B_T^\varepsilon = \{\mathbb{E}[\mathbf{1}_{\{U_T\geq\varepsilon\}}|\mathcal{F}_{t^*}] > 0\}$. So fix rational $T \geq t^*$ and $\varepsilon > 0$. Define the stopping time $\tau_\varepsilon = \inf\{t \geq T : U_t \leq \frac{\varepsilon}{2}\}$ and note that $\tau_\varepsilon > T$ on $\{U_T \geq \varepsilon\}$ by right-continuity of U. Define the process $\widetilde{U} = (\widetilde{U}_t)_{t\geq 0}$ by $\widetilde{U}_t := \frac{\varepsilon}{2}e^{-\gamma t}\mathbb{E}[\mathbf{1}_{\{U_T\geq\varepsilon\}}|\mathcal{F}_t]\mathbf{1}_{\{t\in[T,\tau_\varepsilon)\}}$. Then $\widetilde{U}$ is dominated by U. Moreover, Proposition 6.1 for $\hat{U}_t = e^{-\gamma t}\mathbf{1}_{\{T\leq t<\tau_\varepsilon\}}$ with corresponding solution $\hat{W}$, Lemma 6.2 for $t_0 = T$ and $A = \{\hat{U}_T \geq \varepsilon\}$ and Jensen's inequality show that there exists a solution $\widetilde{W}$ associated to $\widetilde{U} = (\widetilde{U}_t)_{t\geq 0}$ such that for all $t \geq 0$,

$$\begin{aligned}\widetilde{W}_t &\geq \mathbb{E}\left[\mathbf{1}_{\{U_T\geq\varepsilon\}}\left(\frac{\varepsilon}{2\gamma\vartheta}\mathbb{E}[(e^{-\gamma(t\vee T)} - e^{-\gamma(t\vee\tau_\varepsilon)})|\mathcal{F}_t]\right)^\vartheta \middle| \mathcal{F}_t\right]\\ &\geq \left(\frac{\varepsilon}{2\gamma\vartheta}\mathbb{E}[\mathbf{1}_{\{U_T\geq\varepsilon\}}|\mathcal{F}_t]\mathbb{E}[(e^{-\gamma(t\vee T)} - e^{-\gamma(t\vee\tau_\varepsilon)})|\mathcal{F}_t]\right)^\vartheta.\end{aligned}$$

Since $\tau_\varepsilon > T$ on $\{U_T \geq \varepsilon\}$, it follows that $\tilde{W}_{t^*} > 0$ on $\{U_T \geq \varepsilon\}$. Now the claim follows from the fact that $W_{t^*} \geq \widetilde{W}_{t^*}$ by Proposition 5.10. □

6.2 Uniqueness of proper solutions

We now turn to the uniqueness of proper solutions. The aim of this section is to prove Theorem 4.9. The following two lemmas will be useful. Their proofs are given in Appendix D.2.

Lemma 6.4 *Fix $X \in \mathbb{SO}$ and let $J^X = (J_t^X)_{t\geq 0}$ be defined by*

$$J_t^X = \mathbb{E}\left[\int_t^\infty X_s \,\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad \text{for } t \geq 0.$$

Then there exists a martingale $M = (M_t)_{t\geq 0}$ with $J_t^X = M_t e^{-\int_0^t (X_s/J_s^X)ds}$.

Lemma 6.5 *Let $\alpha > 0$, $\beta \in (0,1)$ and let $G = (G_t)_{t\geq 0}$ be a càdlàg submartingale. Suppose that $X = (X_t)_{t\geq 0}$ is a right-continuous process such that $X_0 = \beta$ and $X_t \leq 1$ for all $t \geq 0$. Define $\tau = \inf\{t \geq 0 : X_t = 1\}$ and suppose that*

$$\mathrm{d}X_t \geq \alpha X_t \,\mathrm{d}t + \mathrm{d}G_t \qquad \text{for all } t < \tau, \tag{6.1}$$

where this expression (and similar ones below) are to be interpreted in integral form. Then there exist $\nu \in (0,1)$ and $T \in (0,\infty)$ such that $\mathbb{P}[\tau < T] > \nu$.

Proof of Theorem 4.9 We work in the (U,W)-coordinates in which case by Remark 5.1, it is sufficient to prove that if $U^\vartheta \in \mathbb{SO}_+$, then U is uniquely proper.

Fix $U \in \mathscr{P}_+$ such that $U^\vartheta \in \mathbb{SO}_+$ and recall that $J = (J_t)_{t\geq 0}$ is given by $J_t = \mathbb{E}[\int_t^\infty U_s^\vartheta \,\mathrm{d}s|\mathcal{F}_t]$. Suppose that $kJ \leq U^\vartheta \leq KJ$ and let $W \in \mathbb{O}(J)$ be the solution associated to (h_{EZ}, U) found in Proposition 5.4 (after setting $\varepsilon = 0$ and $U = \Lambda$), whose bounds are explicitly given in Corollary C.5. By Proposition 5.11, W is the unique maximal solution, and as we saw in the proof of Theorem 4.7, it is also proper. We now prove uniqueness.

For contradiction, assume that there exists a proper solution $Y = (Y_t)_{t\geq 0}$ such that $Y \neq W$. Since W is maximal, $Y \leq W$. Then since W is unique in the class $\mathbb{O}(J)$ by Proposition 5.4, it follows that $Y \overset{\mathbb{O}}{\neq} J$. Hence there exist $t \geq 0$, $B \in \mathcal{F}_t$ and $\varepsilon > 0$ such that $\mathbb{P}[B] > \varepsilon$ and $Y_t < kK^{-\vartheta} J_t$ on B. For ease of exposition, assume that $t = 0$, $B = \Omega$ and $Y_0 = y = (1-\varepsilon)kK^{-\vartheta} J_0$. The general case is similar.

Define $Z = (Z_t)_{t\geq 0}$ by $Z_t = Y_t (kK^{-\vartheta} J_t)^{-1}$ and note that Z is càdlàg by Remark B.3 and that $Z_0 = 1-\varepsilon$. Since $kK^{-\vartheta} JZ = Y$ is a utility process and since $U^\vartheta \geq kJ$, for all $t \leq T$,

$$\begin{aligned} kK^{-\vartheta} J_t Z_t = Y_t &= \mathbb{E}\left[\int_t^T U_s Y_s^\rho \,\mathrm{d}s + Y_T \middle| \mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[\int_t^T (kJ_s)^{\frac{1}{\vartheta}} (kK^{-\vartheta} J_s Z_s)^\rho \,\mathrm{d}s + kK^{-\vartheta} J_T Z_T \middle| \mathcal{F}_t\right]. \end{aligned}$$

By Lemma 6.4, we find that $J_t = M_t e^{-A_t}$ for $A_t = \int_0^t (U_s^\vartheta / J_s)\,\mathrm{d}s$. Hence dividing by $kK^{-\vartheta} M_t$ and collecting terms gives

$$e^{-A_t} Z_t \geq \frac{1}{M_t} \mathbb{E}\left[\int_t^T KM_s e^{-A_s} Z_s^\rho \,\mathrm{d}s + M_T e^{-A_T} Z_T \middle| \mathcal{F}_t\right].$$

Define $\widetilde{Z}_t = e^{-A_t} Z_t$ and consider the equivalent measure $\widetilde{\mathbb{P}}$ defined by $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = M_t$, so that

$$\widetilde{Z}_t \geq \widetilde{\mathbb{E}}\left[\int_t^T K e^{-(1-\rho)A_s} \widetilde{Z}_s^{\rho}\,\mathrm{d}s + \widetilde{Z}_T \,\middle|\, \mathcal{F}_t\right] \qquad \text{for all } t \leq T.$$

If we define O by $O_t = \widetilde{Z}_t + \int_0^t K e^{-(1-\rho)A_s} \widetilde{Z}_s^{\rho}\,\mathrm{d}s$, then O is a càdlàg $\widetilde{\mathbb{P}}$-supermartingale. By the Doob–Meyer decomposition, O can therefore be decomposed as $O = N + P$, where N is a local $\widetilde{\mathbb{P}}$-martingale and P a decreasing process with $P_0 = 0$, both of which are càdlàg. In particular, rearranging gives

$$\mathrm{d}\widetilde{Z}_t = -K e^{-(1-\rho)A_t} \widetilde{Z}_t^{\rho}\,\mathrm{d}t + \mathrm{d}N_t - \mathrm{d}P_t.$$

Let $\sigma := \inf\{t \geq 0 : Z_t \notin (0,1)\}$ and set $\hat{Z}_t = Z_{t\wedge\sigma}$, $\hat{N}_t = \int_0^{t\wedge\sigma} e^{A_s}\,\mathrm{d}N_s$ as well as $\hat{P}_t = \int_0^{t\wedge\sigma} e^{A_s}\,\mathrm{d}P_s$. Then applying the product rule to $\hat{Z}_t = e^{A_t}\widetilde{Z}_t$ up to $t \leq \sigma$ and noting that $\frac{\mathrm{d}A_t}{\mathrm{d}t} = \frac{U_t^{\vartheta}}{J_t} \leq K$, we obtain

$$\mathrm{d}\hat{Z}_t = \hat{Z}_t\,\mathrm{d}A_t - K\hat{Z}_t^{\rho}\,\mathrm{d}t + \mathrm{d}\hat{N}_t - \mathrm{d}\hat{P}_t \leq K(\hat{Z}_t - \hat{Z}_t^{\rho})\,\mathrm{d}t + \mathrm{d}\hat{N}_t \leq \mathrm{d}\hat{N}_t.$$

Since $\hat{N}_t \geq \hat{Z}_t - \hat{Z}_0 \geq -\hat{Z}_0$, $\hat{N}$ is a $\widetilde{\mathbb{P}}$-supermartingale.

Let $X_t = 1 - \hat{Z}_t^{1-\rho} \leq 1$. Then $X = (X_t)_{t\geq 0}$ is càdlàg and for $t < \sigma$,

$$\begin{aligned}
\mathrm{d}X_t &= -(1-\rho)\hat{Z}_t^{-\rho}\,\mathrm{d}\hat{Z}_t + \frac{\rho(1-\rho)}{2}\hat{Z}_t^{-(\rho+1)}\,\mathrm{d}\langle Z\rangle_t \\
&= -(1-\rho)(\hat{Z}_t^{1-\rho}\,\mathrm{d}A_t - K\,\mathrm{d}t) + \mathrm{d}L_t + \mathrm{d}Q_t \\
&\geq K(1-\rho)X_t\,\mathrm{d}t + \mathrm{d}L_t + \mathrm{d}Q_t \\
&\geq \mathrm{d}L_t,
\end{aligned}$$

where

$$\begin{aligned}
L_t &:= -(1-\rho)\int_0^t \hat{Z}_s^{-\rho}\,\mathrm{d}\hat{N}_s, \\
Q_t &:= (1-\rho)\int_0^t \hat{Z}_s^{-\rho}\,\mathrm{d}\hat{P}_s + \int_0^t \frac{\rho(1-\rho)}{2}\hat{Z}^{-(\rho+1)}\,\mathrm{d}\langle Z\rangle_s.
\end{aligned}$$

Since $L_t \leq X_t - X_0 \leq 1$, L is a (continuous) $\widetilde{\mathbb{P}}$-submartingale. Hence $G := L + Q$ is a continuous $\widetilde{\mathbb{P}}$-submartingale. The result that X explodes to 1 in finite time with positive probability follows from Lemma 6.5. This implies that Z hits zero in finite time and consequently that Y is not proper. □

7 The Merton investment–consumption problem

In this section, we consider a frictionless Black–Scholes–Merton financial market.

7.1 The financial market and attainable consumption streams

The market consists of a risk-free asset with interest rate $r \in \mathbb{R}$, whose price process $S^0 = (S^0_t)_{t\geq 0}$ is given by $S^0_t = S^0_0 e^{rt}$, and a risky asset given by geometric Brownian motion with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$ and initial value $S_0 = s_0 > 0$. Explicitly, the price process $S = (S_t)_{t\geq 0}$ of the risky asset is given by

$$S_t = s_0 \exp(\sigma B_t + (\mu - \frac{1}{2}\sigma^2))$$

for a Brownian motion $B = (B_t)_{t\geq 0}$.

At each time $t \geq 0$, the agent chooses to consume at a rate $C_t \in \mathbb{R}_+$ per unit time and to invest a proportion $\Pi_t \in \mathbb{R}$ of their wealth into the risky asset. The proportion they invest in the risk-free asset S^0 at time t is then given by $1 - \Pi_t$. It follows that the wealth process $X = (X_t)_{t\geq 0}$ of the agent satisfies the SDE

$$\mathrm{d}X_t = X_t \Pi_t \sigma \, \mathrm{d}B_t + \Big(X_t\big(r + \Pi_t(\mu - r)\big) - C_t\Big)\mathrm{d}t, \tag{7.1}$$

subject to the initial condition $X_0 = x$, where $x > 0$ is the agent's initial wealth.

Definition 7.1 Given $x > 0$, an *admissible investment–consumption strategy* is a pair $(\Pi, C) = (\Pi_t, C_t)_{t\geq 0}$ of progressively measurable processes, where Π is real-valued and C is nonnegative, such that the SDE (7.1) has a unique strong solution $X^{x,\Pi,C}$ that is $\mathbb{P}$-a.s. nonnegative. We denote the set of admissible investment–consumption strategies for $x > 0$ by $\mathscr{A}(x)$.

Since the value associated to a strategy only depends on consumption and not on the amount invested in each of the assets, we introduce the following definition.

Definition 7.2 A consumption stream $C \in \mathscr{P}_+$ is called *attainable* for initial wealth $x > 0$ if there exists a progressively measurable process $\Pi = (\Pi_t)_{t\geq 0}$ such that (Π, C) is an admissible investment–consumption strategy. Denote the set of attainable consumption streams for $x > 0$ by $\mathscr{C}(x)$.

7.2 The Merton investment–consumption problem for Epstein–Zin stochastic differential utility

The Merton investment–consumption problem for an agent with EZ-SDU preferences consists in choosing an admissible investment–consumption strategy so as to maximise their subjective utility. Naively, one might wish to consider the problem of finding

$$V^*(x) = \sup_{C\in\mathscr{C}(x)} V^C_0, \tag{7.2}$$

where $V^C = (V^C_t)_{t\geq 0}$ solves (2.2). However, in the case $\vartheta > 1$ considered in the present paper, there does not exist a unique solution to (2.2) and so the problem (7.2) is not well defined. Instead, we restrict to the set of right-continuous consumption streams for which there exists a unique proper solution.

Definition 7.3 Let $\mathbb{UP}^*$ be the restriction of $\mathbb{UP}$ to right-continuous processes.

The problem then becomes that of finding

$$V^*(x) = \sup_{C \in \mathscr{C}(x) \cap \mathbb{UP}^*} V_0^C,$$

where $V^C = (V_t^C)_{t \geq 0}$ is the unique proper solution associated to the consumption C.

7.3 A candidate optimal investment–consumption strategy

In this section, we give a quick derivation of the optimal strategy for the Merton problem with preferences given by Epstein–Zin stochastic differential utility. This is by no means a novel contribution, and a more detailed exposition can be found in [12, Sect. 5.3].

From the scaling properties of the problem, it is to be expected that the optimal strategy is to invest a constant proportion of wealth in the risky asset and to consume a constant proportion of wealth. Consider therefore the investment–consumption strategy $\Pi \equiv \pi \in \mathbb{R}$ and $C \equiv \xi X$ for $\xi \in \mathbb{R}_+$. Fixing a proportional strategy determined by (π, ξ) and using Itô's lemma and the dynamics of X given in (7.1), we find

$$X_t^{1-R} = x^{1-R} \exp\left(\pi\sigma(1-R)B_t + (1-R)\Big(r + (\mu - r)\pi - \xi - \frac{\pi^2\sigma^2}{2}\Big)t\right).$$

In particular, since $C_t = \xi X_t$, we have $\mathbb{E}[C_s^{1-R}|\mathcal{F}_t] = C_t^{1-R} e^{-H(\pi,\xi)(s-t)}$, where with $\lambda = \frac{\mu - r}{\sigma}$,

$$H(\pi, \xi) = (R-1)\left(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}R\right).$$

Provided that $\delta\vartheta + H(\pi, \xi) > 0$, $C = (C_t)_{t \geq 0}$ is a geometric Brownian motion such that $(e^{-\delta\vartheta t} C_t^{1-R})_{t \geq 0}$ has negative drift – the case considered in Sect. 3.1. Hence the proper utility process associated to it (a specific case of (3.4)) is given by

$$V_t = h(\pi, \xi) e^{-\delta\vartheta t} X_t^{1-R}, \qquad \text{where } h(\pi, \xi) = \frac{\xi^{1-R}}{1-R}\left(\frac{\vartheta}{\delta\vartheta + H(\pi, \xi)}\right)^{\vartheta}. \tag{7.3}$$

Furthermore, by taking derivatives of $h(\pi, \xi)$ with respect to each of its arguments, the maximum value of h is attained when $\pi = \hat{\pi} := \frac{\lambda}{\sigma R}$ and $\xi = \hat{\xi} := \eta$, where

$$\eta = \frac{1}{S}\left(\delta + (S-1)r + (S-1)\frac{\lambda^2}{2R}\right).$$

In this case, $V_t = \hat{V}(t, X_t)$, where $\hat{V} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$\hat{V}(t, x) = \eta^{-\vartheta S} e^{-\delta\vartheta t} \frac{x^{1-R}}{1-R}. \tag{7.4}$$

This section can be summarised by the following result, which is [12, Proposition 5.4].

Proposition 7.4 *Define* $D = \{(\pi, \xi) \in \mathbb{R} \times \mathbb{R}_{++} : \delta\vartheta + H(\pi, \xi) > 0\}$. *Consider constant proportional strategies with parameters* $(\pi, \xi) \in D$. *Suppose* $\vartheta > 0$ *and* $\eta > 0$. *Then*:

(i) *For* $(\pi, \xi) \in D$, *one EZ-SDU utility process* $V = (V_t)_{t\geq 0}$ *associated to the strategy* $(\pi, \xi X)$ *is given by* (7.3).

(ii) *The maximum of* $h(\pi, \xi)$ *over the set* D *is attained at* $(\pi, \xi) = (\frac{\lambda}{\sigma R}, \eta)$, *and the maximum value is* $\frac{\eta^{-\vartheta S}}{1-R}$.

(iii) *If we only consider constant proportional strategies and associate to each such strategy the utility process given in* (7.3), *then the optimal strategy within that class is* $(\hat{\pi}, \hat{\xi}) = (\frac{\lambda}{\sigma R}, \eta)$ *and satisfies* $V_0^{\hat{\pi},\hat{\xi}X} = \eta^{-\vartheta S}\frac{x^{1-R}}{1-R} = \hat{V}(0, x)$, *where* x *denotes initial wealth.*

7.4 Verification of the candidate optimal strategy

The following verification theorem gives conditions under which the candidate strategy given in Proposition 7.4 is indeed optimal in the class of all admissible strategies.

Theorem 7.5 *Suppose that* $\vartheta > 1$. *Let the initial wealth be* $x > 0$ *and suppose that* $\eta > 0$. *If* V^C *is the unique proper utility process associated to* $C \in \mathbb{UP}^*$ *and* $\hat{V}(0, x)$ *is the candidate utility given in* (7.4), *then*

$$\sup_{C\in\mathscr{C}(x)\cap\mathbb{UP}^*} V_0^C = V_0^{\hat{C}} = \hat{V}(0, x),$$

and the optimal investment–consumption strategy is $(\hat{\Pi}, \hat{C}) = (\hat{\pi}, \hat{\xi}X^{x,\hat{\pi},\hat{\xi}X})$, *where*

$$\hat{\pi} := \frac{\mu - r}{\sigma^2 R},$$
$$\hat{\xi} := \eta = \frac{1}{S}\left(\delta + (S-1)r + \lambda^2\frac{(S-1)}{2R}\right).$$

The proof of this theorem follows the approach detailed in [13] which has the following steps. First, we show that if $\hat{X} = X^{\hat{\Pi},\hat{C}}$ is the wealth process under the candidate optimal strategy, then $\hat{V}(\cdot, X + \varepsilon\hat{X})$ is a supersolution for (g_{EZ}, C). (In essence, V is a supersolution (for C and g) of an equation such as (2.1) if the *equality* in (2.1) is replaced by *greater than or equal to*. See Appendix B for a definition of sub- and supersolutions and useful results about them. The reasons that we perturb the input of $\hat{V} = \hat{V}(\cdot, X)$ by the candidate optimal wealth process are to ensure that we can apply Itô's lemma to $\hat{V}$ since the argument is now strictly positive, to make sure that the local martingale part of $\hat{V}$ is a supermartingale, and to ensure that $\hat{V}$ satisfies the supersolution transversality condition. This is explained in more detail in Herdegen et al. [11], [13, Sect. 8].) Second, we use a version of the comparison theorem (Corollary B.7) for sub- and supersolutions to conclude that $\hat{V}(0, x(1 + \varepsilon)) \geq V_0^C$. Third and finally, we let $\varepsilon \searrow 0$ to give $\hat{V}(0, x) \geq V_0^C$. Optimality follows since $V_0^{\hat{C}} = \hat{V}(0, x)$ by Proposition 7.4.

However, the approach is not quite this simple for two main reasons. The most pressing reason is that when $\vartheta > 1$, the utility process V^C fails to be unique. The optimisation therefore takes place over those attainable and right-continuous consumption streams for which there exists a unique proper solution to the EZ-SDU equation. As we have argued in Sect. 6, the proper solution is the economically meaningful solution, and we may only consider those consumption streams that have a unique proper solution associated to them. The assumption of right-continuity, which is necessary for the proof, is not overly restrictive.

The next issue is that the hypotheses of the relevant comparison theorem (Corollary B.7) are not satisfied, even for right-continuous consumption streams with a unique proper solution. To overcome this issue, one must approximate an arbitrary consumption stream in $\mathbb{UP}^*$ by a series of consumption streams satisfying the conditions and then take limits. The requirement of right-continuity ensures that we may choose right-continuous approximating consumption streams, which then have an associated proper solution V^n by Theorem 4.7. Since the limiting process $V = \lim_{n\to\infty} V^n$ is a proper solution associated to C, it must agree with the unique proper solution V^C. The full proof for Theorem 7.5 is given in Appendix E.

7.5 Beyond the frictionless, constant parameter case

Our goal in this paper was to highlight an important issue concerning non-uniqueness of utility functions/value processes in the context of stochastic differential utility which heretofore has been overlooked, and to propose a potential remedy, namely the concept of a proper solution.

We have shown that this concept is useful in the setting of the classical (infinite-horizon) Merton problem in a Black–Scholes financial market setting for Epstein–Zin stochastic differential utility. Indeed, in the well-posed case, the optimal strategy is to consume at a rate which is proportional to current wealth and to invest a constant fraction of wealth in the risky asset. Then the wealth process $X = X^{x,C,\Pi} = (X_t)_{t\geq 0}$ is a geometric Brownian motion, as is $(e^{-\delta\vartheta t} C_t^{1-R})_{t\geq 0}$, and one can check that the latter lies in $\mathbb{SO}$ and even in $\mathbb{SO}_+$, whence C is a uniquely proper solution.

A natural next question is to ask to what extent our approach (in particular the classes $\mathbb{SO}$ and $\mathbb{SO}_+$) can be used in more general settings. In some cases, it may be that the results we provide are already sufficient, while in others, it may be that new proofs and new ideas are required (in particular to extend the set of consumption streams which can be shown to have a unique proper solution).

Here, we briefly discuss two examples where our ideas may be applied more or less directly.

The first example is a factor model, in which the dynamics of the stock price are driven by a factor process. We began a discussion of this example in Sect. 4.4 and Proposition 4.13 where we saw that for a wide class of consumption processes C, there exists a unique proper utility process V^C associated to C. Now we embed this into a consumption–investment problem. Assume that the price $S = (S_t)_{t\geq 0}$ of the risky asset satisfies the SDE

$$\mathrm{d}S_t = S_t\sigma(Y_t)\,\mathrm{d}B_t + S_t\mu(Y_t)\,\mathrm{d}t,$$

where as before, the (regular, non-explosive, time-homogeneous diffusion) process $Y = (Y_t)_{t\geq 0}$ valued in $\mathbb{R}$ is given by the SDE

$$\mathrm{d}Y_t = a_Y(Y_t)\,\mathrm{d}B_t^Y + b_Y(Y_t)\,\mathrm{d}t$$

and $\mathrm{d}\langle B, B^Y\rangle_t = \rho(Y_t)\,\mathrm{d}t$. Then the wealth process $X = X^{x\Pi,C}$ of the agent satisfies (compare (7.1))

$$\mathrm{d}X_t = X_t\Pi_t\sigma(Y_t)\,\mathrm{d}B_t + \Big(X_t\Big(r + \Pi_t\big(\mu(Y_t) - r\big)\Big) - C_t\Big)\,\mathrm{d}t$$

subject to $X_0 = x$. We can define admissible strategies (Π, C) and attainable consumption streams C very much as in Definitions 7.1 and 7.2. Then using the ansatz $V_t = \frac{e^{-\delta\vartheta t}}{1-R}X_t^{1-R}A(Y_t)$ and $C_t = X_t\xi(Y_t)$, it is possible to derive a candidate optimal consumption process which is of the form given in (4.2), but where α and β depend on A. Then A satisfies an ODE which is similar to (4.3), except that the fact that α and β depend on A introduces further non-linearities. Nonetheless, if there is a solution A which is bounded above and away from zero, we can conclude that there exists a unique proper utility process associated with the candidate solution. In principle, the programme can then be completed by extending the verification theorem (see the proof of Theorem 7.5 in Appendix E) to show that the candidate solution is indeed optimal (amongst the class of uniquely proper, attainable consumption processes). The resolution of the remaining issues is an interesting topic for future research, and it would be insightful to compare and contrast the results with those given in Xing [28] for the finite-horizon problem.

The second example is an extension of the Black–Scholes financial market of Sect. 7 to include proportional transaction costs; see Melnyk et al. [21] and Herdegen et al. [14]. In that case, under stochastic differential utility, the candidate optimal investment strategy is to keep the two-dimensional process (cash wealth, value of risky assets) in a wedge, and the candidate optimal consumption strategy is to consume at a rate which is a stochastic multiple of wealth, where the stochastic multiple is a function of the fraction of wealth in the risky asset. The utility process in turn can be written as the product of a power of total wealth and a function of the fraction of wealth in the risky asset. Then at least for the candidate strategy, we can directly use the results of this paper to define the unique proper utility process associated with C. Of course, to carry out this programme rigorously and to give a complete verification argument requires some care – but all this can be done, see [14] for details. Of course, there are also other significant challenges in addressing the transaction cost problem, including identifying the boundaries of the no-transaction wedge and deciding when the problem is well posed.

Appendix A: Comparisons with the literature

It is interesting to contrast the results of this paper with those of Herdegen et al. [13] and Melnyk et al. [21].

A.1 A comparison with Herdegen et al

In [13], the authors consider existence and uniqueness of the utility process and the investment–consumption problem in the case $0 < \vartheta < 1$. The fundamental existence result [13, Theorem 4.6] is re-used in the present paper to give existence for $\vartheta > 1$ when the consumption stream satisfies a self-order condition. Thereafter, the methodology and approach of [13] and this paper differ considerably. In [13], the authors prove a comparison lemma [13, Theorem 5.8] whose use is threefold. First, it gives a simple proof of uniqueness of the solution to the equation for the utility process. Second, it can be used to prove a monotonicity result which can be used to generalise the existence result for consumption streams satisfying the self-order condition to all consumption streams. Third, the comparison theorem plays a crucial role in the verification lemma.

When $\vartheta > 1$, the hypotheses of the comparison lemma in [13] are not satisfied. Indeed, the result cannot be true when $\vartheta > 1$ as then uniqueness would follow – and we have seen that uniqueness fails in general. Hence we need to introduce a new concept to identify the economically relevant solution. In the present paper, we named these proper solutions. Our goal was then to prove existence and uniqueness of such solutions. This required a new and fundamentally different comparison lemma and new ideas to prove existence and uniqueness. Finally, some of the arguments used in the verification lemma need to be modified since we are discussing proper solutions. In particular, we need to prove an important result which says that we may approximate $C \in \mathbb{UP}$ by a sequence $(C^n)_{n\in\mathbb{N}}$ of consumption streams with associated proper solutions.

A.2 A comparison with Melnyk et al

It is also interesting to compare the results and techniques of the present paper with those of the recent and wide-ranging paper by Melnyk et al. [21]. The ultimate focus of [21] is to consider the investment–consumption problem for EZ-SDU with transaction costs. However, they first consider the frictionless case. They use a slightly different setup and a reduced set of parameter combinations – in particular, they require that $R > 1$.

The main result in [21] regarding the optimal investment–consumption strategy (Theorem 3.4) involves calculating the optimal strategy in a space $\mathscr{A}^0$ [21, Definition 3.1]. For a strategy to belong to this class, the wealth and investment process must satisfy some integrability conditions, and there must exist a unique utility process. (Although uniqueness of the utility process is considered via a comparison lemma, which is different in both statement and proof to those in the present paper and [12], existence is not considered in [21]). Finally, for the case $\vartheta > 1$, results are proved under three extra conditions. The first is that consumption streams $C = (C_t)_{t\geq 0}$ must be strictly positive; the second [21, Eq. (5)] is that the associated utility process $V = (V_t)_{t\geq 0}$ satisfies $(1-R)V_t \geq e^{-\delta\vartheta t}C_t^{1-R}$ for $t \geq 0$; the third [21, Eq. (3)] is that the utility process satisfies the transversality condition $\lim_{t\to\infty} e^{-\gamma t}\mathbb{E}[|V_t|] = 0$ for an appropriate $\gamma \in \mathbb{R}$. The second and third of these assumptions are rather ad-hoc conditions, but form a crucial element of the proof of the comparison theorem. Combining the first and second of the trio of assumptions ensures that $(1-R)V_t > 0$ for

all $t \geq 0$. This has the effect of identifying the proper solution within the class of utility processes; however, these extra assumptions rule out many consumption streams for which there exists a proper utility process. Furthermore, for certain parameter combinations, the second and third assumptions even rule out the candidate optimal strategy. As a consequence, the authors of [21] are forced to require additional restrictions on the parameters [21, Assumption 3] beyond those which are necessary to ensure that the problem is well posed.

Appendix B: Subsolutions and supersolutions

This section introduces subsolutions and supersolutions and then proves that under certain conditions, subsolutions are bounded above by supersolutions. This important result is key in many of the proofs given in the appendices. In particular, it is crucial in the construction of a utility process associated to the limit of a sequence of consumption streams, and in the verification theorem when we compare the utility process for an arbitrary consumption to that associated with the candidate optimal consumption.

We recall from [13, Definition 5.1] the definition of an aggregator random field (a generalisation of the aggregator that is allowed to depend on the state of the world $\omega \in \Omega$) as well as the definition of a sub- and supersolution.

Definition B.1 An *aggregator random field* $g : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ is a product-measurable map such that for each fixed $\omega \in \Omega$, $g(\cdot, \omega, \cdot, \cdot)$ is an aggregator, and for progressively measurable processes $C = (C_t)_{t\geq 0}$ and $V = (V_t)_{t\geq 0}$, the process $(g(t, \omega, C_t(\omega), V_t(\omega)))_{t\geq 0}$ is progressively measurable.

We need the more general notion of an aggregator random field since it permits us to stochastically perturb our Epstein–Zin aggregator; cf. [13, Definition 5.3].

Definition B.2 Let $C \in \mathscr{P}_+$ and g an aggregator random field such that g takes only one sign, i.e., $\mathbb{V} \subseteq \overline{\mathbb{R}}_+$ or $\mathbb{V} \subseteq \overline{\mathbb{R}}_-$. A $\mathbb{V}$-valued làd optional process V is called

– a *subsolution* for the pair (g, C) if $\limsup_{t\to\infty} \mathbb{E}[V_{t+}] \leq 0$ and for all bounded stopping times $\sigma \leq \tau$,

$$V_\sigma \leq \mathbb{E}\left[V_{\tau+} + \int_\sigma^\tau g(s, \omega, C_s, V_s)\, \mathrm{d}s \,\middle|\, \mathcal{F}_\sigma \right];$$

– a *supersolution* for the pair (g, C) if $\liminf_{t\to\infty} \mathbb{E}[V_{t+}] \geq 0$ and for all bounded stopping times $\sigma \leq \tau$,

$$V_\sigma \geq \mathbb{E}\left[V_{\tau+} + \int_\sigma^\tau g(s, \omega, C_s, V_s)\, \mathrm{d}s \,\middle|\, \mathcal{F}_\sigma \right];$$

– a *solution* for the pair (g, C) if it is both a subsolution and a supersolution and $V \in \mathbb{I}(g, C)$.

Remark B.3 By taking the limit as $\tau \to \infty$ and using the (combined) transversality condition $\lim_{t\to\infty} \mathbb{E}[V_{t+}] = 0$, it is clear that a solution V for the pair (g, C) satisfies (2.1). We then choose a càdlàg version of V so that V is a utility process associated to (g, C). This implies in particular that if V is a solution for the pair (g, C), then

$$V_\sigma = \mathbb{E}\left[V_\tau + \int_\sigma^\tau g(s, \omega, C_s, V_s)\, \mathrm{d}s \,\middle|\, \mathcal{F}_\sigma \right]$$

for all bounded stopping times $\sigma \leq \tau$, since $V_\tau = V_{\tau+}$.

B.1 Comparison of subsolutions and supersolutions

It is shown in [13, Theorem 5.8] that when g is decreasing in its last argument, if V^1 is a subsolution and V^2 a supersolution (both associated to g and some $C \in \mathscr{P}_+$) and one of V^1, V^2 is in $\mathbb{UI}(g, C)$, then $V^1_\tau \leq V^2_\tau$ for all finite stopping times τ. However, when $\vartheta > 1$, the Epstein–Zin aggregator g_{EZ} is increasing in its last argument so that this theorem does not apply.

The next proposition shows that we may weaken the condition that the aggregator has a negative derivative with respect to its last argument, and instead assume that it has a derivative which is bounded above by some positive decreasing exponential.

We introduce the following condition on a pair (V^1, V^2) of stochastic processes which will be a requirement for the subsequent comparison theorems.

Condition B.4 Let $g : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ be an aggregator random field and $C \in \mathscr{P}_+$. The pair (V^1, V^2) satisfies Condition B.4 for the pair (g, C) if one of V^1, V^2 is in $\mathbb{UI}(g, C)$ and $(V^1 - V^2)^+$ is bounded in L^1.

Remark B.5 Note that for a subsolution V^1 and a supersolution V^2 associated to the pair (g, C), a sufficient condition for the pair (V^1, V^2) to satisfy Condition B.4 for the pair (g, C) is that $V^1, V^2 \in \mathbb{UI}(g, C)$. This is because for every $t \geq 0$,

$$\begin{aligned}\mathbb{E}[(V^1_t - V^2_t)^+] \leq |V^1_0| + |V^2_0| + \mathbb{E}\left[\int_0^\infty |g(s, \omega, C_s, V^1_s)|\, \mathrm{d}s \right] \\ + \mathbb{E}\left[\int_0^\infty |g(s, \omega, C_s, V^2_s)|\, \mathrm{d}s \right] < \infty.\end{aligned}$$

However, Condition B.4 is more general.

Proposition B.6 *Let $g : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ be an aggregator random field that is concave and nondecreasing in its last argument. Let $C \in \mathscr{P}_+$. Suppose V^1 is a subsolution and V^2 a nonzero supersolution for the pair (g, C) and that the pair (V^1, V^2) satisfies Condition B.4 for (g, C). Also assume that for $\mathbb{P}$-a.e. ω, we have $g_v(t, \omega, C_t(\omega), V^2_t(\omega)) \leq \kappa e^{-\nu t}$ for all $t \geq 0$ for some $\kappa, \nu > 0$. Here, we interpret g_v to be the right derivative of g with respect to v, which exists everywhere for $v \neq 0$ by the concavity assumption. Then $V^1_\sigma \leq V^2_\sigma$ $\mathbb{P}$-a.s. for all finite stopping times σ.*

Proof Seeking a contradiction, suppose there is a finite stopping time σ such that $\mathbb{P}[V^1_\sigma > V^2_\sigma] > 0$. By replacing σ with $\sigma \wedge T$ for T sufficiently large, we may assume without loss of generality that σ is bounded. Set $A := \{V^1_\sigma > V^2_\sigma\}$. Since V^1 and V^2 are làd, we may define the right-continuous processes $(V^1_{t+})_{t\geq 0}$ and $(V^2_{t+})_{t\geq 0}$. Further, define the stopping time $\tau := \inf\{t \geq \sigma : V^1_{t+} \leq V^2_{t+}\}$. Then $(V^1_{\tau+} - V^2_{\tau+})\mathbf{1}_{\{\tau<\infty\}} \leq 0$ by the right-continuity of $(V^1_{t+})_{t\geq 0}$ and $(V^2_{t+})_{t\geq 0}$.

First, we show that $\mathbb{P}[A \cap \{\sigma < \tau\}] > 0$. Indeed, otherwise if $\mathbf{1}_{\{\sigma=\tau\}\cap A} = \mathbf{1}_A$ $\mathbb{P}$-a.s., the definition of sub- and supersolutions yields

$$\begin{aligned}\mathbf{1}_A(V^1_\sigma - V^2_\sigma) &\leq \mathbb{E}[\mathbf{1}_A(V^1_{\sigma+} - V^2_{\sigma+})|\mathcal{F}_\sigma] \\ &= \mathbb{E}[\mathbf{1}_A(V^1_{\tau+} - V^2_{\tau+})\mathbf{1}_{\{\tau<\infty\}}|\mathcal{F}_\sigma] \leq 0,\end{aligned}$$

and we arrive at a contradiction.

Next, by the definition of sub- and supersolutions and Jensen's inequality for $f(x) = x^+ = \max\{x, 0\}$, we get for $t \leq T$ and $B_t = A \cap \{\sigma \leq t < \tau\}$ that

$$\begin{aligned}\mathbf{1}_{B_t}(V^1_t - V^2_t)^+ \leq\ & \mathbb{E}[\mathbf{1}_{B_t}(V^1_{(T\wedge\tau)+} - V^2_{(T\wedge\tau)+})^+|\mathcal{F}_t] \\ &+ \mathbb{E}\left[\mathbf{1}_{B_t}\int_t^{T\wedge\tau}\left(f(s,\omega,C_s,V^1_s) - g(s,\omega,C_s,V^2_s)\right)^+ \mathrm{d}s\,\middle|\,\mathcal{F}_t\right]\end{aligned}$$

where the right-hand side is well defined since one of V^1, V^2 is in $\mathbb{UI}(g, C)$. Taking expectations yields

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{B_t}(V^1_t - V^2_t)^+] \\ &\leq \mathbb{E}[\mathbf{1}_{B_t}(V^1_{(T\wedge\tau)+} - V^2_{(T\wedge\tau)+})^+] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{B_t}\int_t^{T\wedge\tau}\left(g(s,\omega,C_s,V^1_s) - g(s,\omega,C_s,V^2_s)\right)^+ \mathrm{d}s\right]. \qquad \text{(B.1)}\end{aligned}$$

Taking the lim sup as $T \to \infty$ and using $\mathbf{1}_{B_t}\mathbf{1}_{\{\tau\leq T\}}(V^1_{\tau+} - V^2_{\tau+})^+ = 0$ $\mathbb{P}$-a.s. for all $T \geq 0$ and the transversality condition of sub- and supersolutions gives

$$\begin{aligned}&\limsup_{T\to\infty}\mathbb{E}[\mathbf{1}_{B_t}(V^1_{(T\wedge\tau)+} - V^2_{(T\wedge\tau)+})^+] \\ &= \limsup_{T\to\infty}\mathbb{E}[\mathbf{1}_{B_t}\mathbf{1}_{\{T<\tau\}}(V^1_{T+} - V^2_{T+})^+] + \limsup_{T\to\infty}\mathbb{E}[\mathbf{1}_{B_t}\mathbf{1}_{\{\tau\leq T\}}(V^1_{\tau+} - V^2_{\tau+})^+] \\ &\leq \limsup_{T\to\infty}\mathbb{E}[(V^1_{T+} - V^2_{T+})^+] \leq \limsup_{T\to\infty}\mathbb{E}[(V^1_{T+})^+ + (V^2_{T+})^-] \ \leq\ 0,\end{aligned}$$

where the last inequality follows since either $\mathbb{V} = \mathbb{R}_+$ (and $(V^2_{T+})^- = 0$) along with the transversality condition for supersolutions, or $\mathbb{V} = \mathbb{R}_-$ (and $(V^1_{T+})^+ = 0$) along with the transversality condition for supersolutions. Thus taking in (B.1) the lim sup

as $T \to \infty$, using the positivity of the integrand, we obtain

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{B_t}(V_t^1 - V_t^2)^+] &\leq \mathbb{E}\bigg[\mathbf{1}_{B_t}\int_t^\tau \big(g(s,\omega,C_s,V_s^1) - g(s,\omega,C_s,V_s^2)\big)^+\,\mathrm{d}s\bigg]\\ &\leq \mathbb{E}\bigg[\int_t^\infty \mathbf{1}_{B_s} g_v(s,\omega,C_s,V_s^2)(V_s^1 - V_s^2)^+\,\mathrm{d}s\bigg]\\ &\leq \mathbb{E}\bigg[\int_t^\infty \kappa e^{-\nu s}\mathbf{1}_{B_s}(V_s^1 - V_s^2)^+\,\mathrm{d}s\bigg],\end{aligned}$$

where the middle line uses that g is concave and nondecreasing in its last argument and $V_s^2 \neq 0$. Setting $\Gamma(t) := \mathbb{E}[\mathbf{1}_{B_t}(V_t^1 - V_t^2)^+]$, it follows that $\Gamma = (\Gamma(t))_{t\geq 0}$ is a nonnegative function such that $\Gamma(t) \leq \int_t^\infty \kappa e^{-\nu s}\Gamma(s)\,\mathrm{d}s$. Note that we have $\Gamma(t) \leq \mathbb{E}[(V_t^1 - V_t^2)^+] \leq \gamma$ for some $\gamma > 0$ by the L^1-boundedness of $(V^1 - V^2)^+$. Therefore, since

$$\int_0^\infty \kappa e^{-\nu t}\,\mathrm{d}t = \frac{\kappa}{\nu}$$

and $\int_0^\infty \kappa e^{-\nu s}\Gamma(s)\,\mathrm{d}s \leq \frac{\gamma\kappa}{\nu}$, we can apply Gronwall's inequality for Borel functions ([10, Theorem 2.5] with $y(t) = \Gamma(-t)$ and $\mu(A) = \int_{A\cap\mathbb{R}_-} \kappa e^{\nu t}\,\mathrm{d}t$) to conclude that $\Gamma(t) = 0$ for all $t > 0$.

Note that $\mathbf{1}_{B_t}(V_t^1 - V_t^2)^+ \geq 0$ for each $t \geq 0$. Hence by Fatou's lemma,

$$0 \leq \mathbb{E}[\mathbf{1}_{B_t}(V_{t+}^1 - V_{t+}^2)^+] \leq \liminf_{s\downarrow\downarrow t} \mathbb{E}[\mathbf{1}_{B_s}(V_s^1 - V_s^2)^+] = 0. \tag{B.2}$$

Furthermore, since $\mathbf{1}_{B_t}(V_{t+}^1 - V_{t+}^2) = \mathbf{1}_{B_t}(V_{t+}^1 - V_{t+}^2)^+ \geq 0$ for each $t \geq 0$ by the definition of τ, it follows from (B.2) that $P_t = \mathbf{1}_{B_t}(V_{t+}^1 - V_{t+}^2) = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Since $(V_{t+}^1)_{t\geq 0}$, $(V_{t+}^2)_{t\geq 0}$ and $(\mathbf{1}_{B_t})_{t\geq 0}$ are right-continuous, the process $P = (P_t)_{t\geq 0}$ is right-continuous and therefore indistinguishable from zero. In particular, $\mathbf{1}_A\mathbf{1}_{\{\sigma<\tau\}}(V_{\sigma+}^1 - V_{\sigma+}^2) = \mathbf{1}_{B_\sigma}(V_{\sigma+}^1 - V_{\sigma+}^2) = 0$ $\mathbb{P}$-a.s. But then the definition of sub- and supersolutions implies that

$$\mathbf{1}_A\mathbf{1}_{\{\sigma<\tau\}}(V_\sigma^1 - V_\sigma^2) \leq \mathbb{E}[\mathbf{1}_A\mathbf{1}_{\{\sigma<\tau\}}(V_{\sigma+}^1 - V_{\sigma+}^2)|\mathcal{F}_\sigma] = 0, \tag{B.3}$$

and we arrive at a contradiction. □

Corollary B.7 *Let g_{EZ} be the Epstein–Zin aggregator, $R < 1$ and $C \in \mathscr{P}_+$. Suppose V^1 is a subsolution and V^2 a positive supersolution for the pair (g_{EZ}, C) and that the pair (V^1, V^2) satisfies Condition B.4 for (g_{EZ}, C). Also assume that $e^{-\delta\vartheta t}C_t^{1-R} \leq Ke^{-\gamma t}(1-R)V_t^2$ for some $K, \gamma > 0$ and all $t \geq 0$. Then $V_\sigma^1 \leq V_\sigma^2$ for all finite stopping times $\sigma \geq 0$.*

Proof Taking derivatives of g_{EZ} with respect to its second argument gives, for $v > 0$,

$$\frac{\partial g_{\mathrm{EZ}}}{\partial v}(t,c,v) = (\vartheta - 1)e^{-\delta t}c^{1-S}\big((1-R)v\big)^{-\frac{1}{\vartheta}} \geq 0, \tag{B.4}$$

$$\frac{\partial^2 g_{\mathrm{EZ}}}{\partial v^2}(t,c,v) = -\rho e^{-\delta t}(1-R)c^{1-S}\big((1-R)v\big)^{-(1+\frac{1}{\vartheta})} \leq 0.$$

Hence using (B.4), $\frac{\partial g_{\mathrm{EZ}}}{\partial v}(s, C_s, V_s^2) \leq (\vartheta - 1)K^{\frac{1}{\vartheta}}e^{-\frac{\gamma}{\vartheta}t}$ and the conditions of Proposition B.6 are met with $\kappa = (\vartheta - 1)K^{\frac{1}{\vartheta}}$ and $\nu = \frac{\gamma}{\vartheta}$. □

Note that Corollary B.7 is only stated in the case $R < 1$. Corollary B.7 is used in the proof of the verification theorem (Theorem 7.5), but in that setting, we use a separate argument for $R > 1$.

Appendix C: Order-ϑ and extremal solutions

In this section, we state and prove some results about order-ϑ and extremal solutions.

Lemma C.1 *For* $0 \leq \mu \leq \nu$, $\mathbb{SO}_\mu \supseteq \mathbb{SO}_\nu$.

Proof This is straightforward and hence omitted. □

Remark C.2 For each $\nu \geq 0$, let $(k(\nu), K(\nu))$ be the tightest interval such that

$$k(\nu)\mathbb{E}\left[\int_t^\infty e^{\nu(s-t)}X_s\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right] \leq X_t \leq K(\nu)\mathbb{E}\left[\int_t^\infty e^{\nu(s-t)}X_s\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]$$

for all $t \geq 0$. It is easy to see that $k(\nu)$ and $K(\nu)$ are decreasing in ν. Furthermore, since $X \in \mathbb{SO}$, we have $k(\nu) \leq k(0) < \infty$. By a symmetric argument, one can show that $K(\nu)$ is decreasing in ν as well.

Proof of Proposition 4.10 Let the process $G = (G_t)_{t\geq 0}$ be given by

$$G_t = e^{-\delta\vartheta t}Z_t^{1-R}.$$

Suppose G has drift $-\phi$ for $\phi > 0$. Then for $\nu \geq 0$, $G^{(\nu)}$ defined by $G_t^{(\nu)} = e^{\nu t}G_t$ is also a geometric Brownian motion and satisfies $\mathbb{E}[G_s^{(\nu)}|\mathcal{F}_t] = G_s^{(\nu)}e^{(\nu-\phi)(s-t)}$ for $s \geq t$. Let $\nu \in [0, \phi)$. Then $J^{G^{(\nu)}} = \frac{1}{\phi-\nu}G^{(\nu)}$ and $G^{(\nu)} \in \mathbb{SO}$.

Define $H^{(\nu)}$ by $H_t^{(\nu)} = e^{(\nu-\delta\vartheta)t}C_t^{1-R}$. Then $H^{(\nu)} \overset{\mathbb{O}}{=} G^{(\nu)}$ because $C \overset{\mathbb{O}}{=} Z$. Integrating from time T onward and taking conditional expectations, we obtain $J^{H^{(\nu)}} \overset{\mathbb{O}}{=} J^{G^{(\nu)}}$. Combining these and using that $\overset{\mathbb{O}}{=}$ is an equivalence relation gives

$$H^{(\nu)} \overset{\mathbb{O}}{=} G^{(\nu)} \overset{\mathbb{O}}{=} J^{G^{(\nu)}} \overset{\mathbb{O}}{=} J^{H^{(\nu)}}.$$

Taking $\nu = 0$ implies that $(e^{-\delta\vartheta t}C_t^{1-R})_{t\geq 0} \in \mathbb{SO}$, and then taking $\nu \in (0, \phi)$, we conclude that $(e^{-\delta\vartheta t}C_t^{1-R})_{t\geq 0} \in \mathbb{SO}_\nu \subseteq \mathbb{SO}_+$. The result of the proposition then holds by Theorem 4.9. □

Lemma C.3 *Let $\delta \in (0,1]$ and let $g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that g is bounded, h is positive and there are constants $0 < \underline{h} \leq \overline{h} < \infty$ with $h \in [\underline{h}, \overline{h}]$. Then the ODE*

$$y''(x) = g(x)y'(x) + h(x)^{-\delta}y(x) - y(x)^{1-\delta} \tag{C.1}$$

has a solution $y \in C^2(\mathbb{R})$ with bounded first derivative that satisfies $y \in [\underline{h}, \overline{h}]$.

Proof Set $E_n := [-n,n] \times [\underline{h}, \overline{h}] \times \mathbb{R}$, define the function $f_n : E_n \to \mathbb{R}$ by $f_n(x,u,v) := g(x)v + h(x)^{-\delta}u - u^{1-\delta}$ as well as the function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ by $\varphi(v) = (\underline{h})^{-\delta}\overline{h} + (\overline{h})^{1-\delta} + \sup_{x\in\mathbb{R}} |g(x)|v$. Then $|f_n(t,u,v)| \leq \varphi(|v|)$ for all $(x,u,v) \in E_n$. Set $r_n := \frac{\overline{h}-\underline{h}}{2n}$ and let $R > 0$ be such that $\int_{r_1}^R \frac{s}{\varphi(s)} ds \geq \overline{h} - \underline{h}$. Note that this implies that $\int_{r_n}^R \frac{s}{\varphi(s)} ds \geq \overline{h} - \underline{h}$ for all $n \in \mathbb{N}$. Define the functions $\alpha_n, \beta_n : [-n,n] \to \mathbb{R}$ by $\alpha_n := \underline{h}$ and $\beta_n := \overline{h}$. Then

$$\alpha_n''(x) = 0 \geq f_n\big(x, \alpha_n(x), \alpha_n'(x)\big) = \underline{h}\big(h(x)^{-\delta} - (\underline{h})^{-\delta}\big) \geq \underline{h}\big((\overline{h})^{-\delta} - (\underline{h})^{-\delta}\big),$$
$$\beta_n''(x) = 0 \leq f_n\big(x, \beta_n(x), \beta_n'(x)\big) = \overline{h}\big(h(x)^{-\delta} - (\overline{h})^{-\delta}\big) \leq \overline{h}\big((\underline{h})^{-\delta} - (\overline{h})^{-\delta}\big).$$

It follows from De Coster and Habets [4, Theorem 4.3 and proof of Proposition 4.2] that the ODE (C.1) on $[n,n]$ with Neumann boundary conditions has a solution

$$y_n \in C^2([n,n])$$

satisfying $y_n \in [\underline{h}, \overline{h}]$ and $|y_n'(x)| \leq R$ for all $x \in [-n,n]$.

Let $(u_\infty, v_\infty) \in [\underline{h}, \overline{h}] \times [-R,R]$ be an accumulation point of the sequence

$$((y_n(0), y_n'(0))_{n\in\mathbb{N}}.$$

Consider the initial value problem for the ODE (C.1) with $y(0) = u_\infty$ and $y'(0) = v_\infty$. Then for each $m \in \mathbb{N}$ and $\varepsilon > 0$, by the stability result for initial value problems (see e.g. Walter [27, Theorem 13.X]) and the fact that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}((y_{n_k}(0), y_{n_k}'(0)) = (u_\infty, v_\infty)$, there exists $k \in \mathbb{N}$ such that $|y(x) - y_{n_k}(x)| \leq \varepsilon$ for all $x \in [-m,m]$, which letting $\varepsilon \to 0$ implies that $y \in [\underline{h}, \overline{h}]$ on $[-m,m]$. Now the claim follows by letting $m \to \infty$. □

Proof of Proposition 4.13 First we note that by Lemma C.3, the ODE (4.3) admits a positive solution Γ that is bounded from above and away from zero and has a bounded derivative.

We proceed to show that the process $V = (V_t)_{t\geq 0}$ defined by

$$V_t = e^{-\delta\vartheta t}\frac{C_t^{1-R}}{1-R}\Gamma(Y_t)$$

is a uniquely proper utility process. This in turn automatically yields uniqueness of Γ in the class of positive functions that are bounded from above and away from zero and have a bounded derivative.

We first show that V is a utility process for C. Set $\tilde{V} := (1-R)V$. It suffices to show that $\tilde{V}$ solves, for $0 \le t \le T < \infty$,

$$\tilde{V}_t = \mathbb{E}\left[\tilde{V}_T + \int_t^T \vartheta e^{-\delta u} C_u^{1-S} \tilde{V}_u^{\rho}\, \mathrm{d}u \,\middle|\, \mathcal{F}_t\right] \tag{C.2}$$

as well as $\lim_{t\to\infty} \mathbb{E}[\tilde{V}_T] = 0$. The latter condition follows from the fact that by the dynamics (4.2) of C and the fact that α is bounded, for each $T \ge 0$,

$$\begin{aligned}\mathbb{E}[\tilde{V}_T] &= \mathbb{E}\left[\exp\left(-\int_0^T \left(\delta\vartheta - (1-R)\beta(Y_t) + \frac{R(1-R)}{2}\alpha^2(Y_t)\right)\mathrm{d}t\right)\right] \\ &\le e^{-\chi T}.\end{aligned} \tag{C.3}$$

To establish (C.2), it suffices to show that the process $M = (M_t)_{t\ge 0}$ given by

$$M_t = \tilde{V}_t + \int_0^t \vartheta e^{-\delta u} C_u^{1-S} \tilde{V}^{\rho}\, \mathrm{d}u \tag{C.4}$$

is a martingale. Itô's formula, the identity $e^{-\delta t} C_t^{1-S} \tilde{V}_t^{\rho} = \tilde{V}_t \Gamma^{\rho}(Y_t)$ and the fact that Γ solves the ODE (4.3) give

$$\begin{aligned} dM_t &= \tilde{V}_t\Bigg(\vartheta\Gamma^{-\frac{1}{\vartheta}}(Y_t) - \delta\vartheta + \frac{\Gamma'(Y_t)}{\Gamma(Y_t)} b_Y(Y_t) + \frac{\Gamma''(Y_t)}{2\Gamma(Y_t)} a_Y(Y_t)^2 + (1-R)\beta(Y_t) \\ &\qquad - \frac{R(1-R)}{2}\alpha(Y_t)^2 + (1-R)\alpha_Y(Y_t)\rho(Y_t)\alpha(T_t)\frac{\Gamma'(Y_t)}{\Gamma(Y_t)}\Bigg)\mathrm{d}t \\ &\quad + e^{-\delta\vartheta t} C_t^{1-R}\left(\Gamma'(Y_t) a_Y(Y_t)\,\mathrm{d}B_t^Y + (1-R)\Gamma(Y_t)\alpha(Y_t)\,\mathrm{d}B_t\right) \\ &= \tilde{V}_t\left(\frac{\Gamma'(Z_t)}{\Gamma(Y_t)} a_Y(Y_t) dB_t^Z + (1-R)\alpha(Y_t)\,\mathrm{d}B_t\right), \end{aligned}$$

which shows that M is a local martingale. Using that $\mathbb{E}[\tilde{V}_t^2] \le (1-R)^2 \exp(\xi t)$ for some $\xi \ge 0$ by the dynamics of C and boundedness of α and β, it follows from the properties of Γ and the fact that a_Y is bounded that M is even a square-integrable martingale.

Finally, fix $\nu \in (0, \chi \wedge \vartheta\gamma^{-1/\vartheta})$, where γ is an upper bound for Γ. To show that C is uniquely proper, we show that $\tilde{V} \in \mathbb{SO}_\nu$, and then the result follows by Theorem 4.9.

Define the process $M^{(\nu)} = (M_t^{(\nu)})_{t\ge 0}$ by $M_t^{(\nu)} = \int_0^t e^{\nu s} dM_s$, where M is defined in (C.4). Then $M^{(\nu)}$ is like M a square-integrable martingale, and using that

$$\mathrm{d}(e^{\nu t}\tilde{V}_t) = \mathrm{d}M_t^{(\nu)} + e^{\nu t}\tilde{V}_t\left(\nu - \vartheta\Gamma(Y_u)^{-1/\vartheta}\right)\mathrm{d}t,$$

it follows that

$$e^{\nu t}\tilde{V}_t = \mathbb{E}\left[e^{\nu T}\tilde{V}_T + \int_t^T e^{\nu u}\tilde{V}_u\left(\vartheta\Gamma(Z_u)^{-1/\vartheta} - \nu\right)\mathrm{d}u \,\middle|\, \mathcal{F}_t\right].$$

Letting $T \to \infty$, the definition of ν, monotone convergence and the fact that

$$\lim_{t\to\infty} \mathbb{E}[e^{\nu T}\tilde{V}_T] = 0$$

by (C.3) give

$$e^{\nu t}\tilde{V}_t = \mathbb{E}\left[\int_t^\infty e^{\nu u}\tilde{V}_u\left(\vartheta\,\Gamma(Z_u)^{-1/\vartheta} - \nu\right)\mathrm{d}u \,\middle|\, \mathcal{F}_t\right].$$

Since $\vartheta\,\Gamma(Z_u)^{-1/\vartheta} - \nu$ is bounded above and below by positive constants and

$$\mathbb{E}[\int_0^\infty e^{\nu u}\tilde{V}_u du] < \infty$$

by (C.3), it follows that $\tilde{V} \in \mathbb{SO}_\nu$. □

The following result shows that there is a natural ordering for subsolutions and supersolutions to (5.5) for different values of ε and U. In particular, since a solution is both a subsolution and a supersolution, it follows that for $\varepsilon > 0$, the solution found in Proposition 5.4 is the unique solution, and not just the unique solution of order J. Recall Condition B.4 of Appendix B which is an integrability condition on pairs of stochastic processes.

Corollary C.4 *Fix $\nu > 0$. Suppose $\Lambda^\vartheta \in \mathbb{SO}_\nu$ and define the perturbed aggregator*

$$h_{\mathrm{EZ}}^{\varepsilon,\nu,\Lambda}(t,\omega,u,w) = uw^\rho + \varepsilon e^{\nu t}\Lambda_t^\vartheta(\omega), \qquad \text{for } \varepsilon > 0. \tag{C.5}$$

Fix $\varepsilon_2 > 0$ and $0 \le \varepsilon_1 \le \varepsilon_2$. Let $U^1, U^2 \in \mathscr{P}_+$ satisfy $U^1 \le U^2 \le \Lambda$. Suppose that W^1 is a subsolution for the pair $(h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}, U^1)$ and W^2 is a supersolution for $(h_{\mathrm{EZ}}^{\varepsilon_2,\nu,\Lambda}, U^2)$ such that W^1, W^2 take values in $[0,\infty)$ and (W^1, W^2) satisfies Condition B.4 for the pair $(h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}, U^1)$. Then $W_\sigma^1 \le W_\sigma^2$ for all finite stopping times $\sigma \ge 0$.

Proof First note that W^2 is a supersolution for $(h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}, U^1)$ since

$$\begin{aligned} W_\sigma^2 &\ge \mathbb{E}\left[\int_\sigma^\tau h_{\mathrm{EZ}}^{\varepsilon_2,\nu,\Lambda}(U_s^2, W_s^2)\,\mathrm{d}s + W_{\tau+}^2 \,\middle|\, \mathcal{F}_t\right] \\ &\ge \mathbb{E}\left[\int_\sigma^\tau h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}(U_s^1, W_s^2)\,\mathrm{d}s + W_{\tau+}^2 \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

As $\Lambda^\vartheta \in \mathbb{SO}_\nu$, there is K_Λ with $W_t^2 \ge \mathbb{E}[\int_t^\infty \varepsilon_2 \mathrm{e}^{\nu s}\Lambda_s^\vartheta\,\mathrm{d}s|\mathcal{F}_t] \ge \frac{\varepsilon_2}{K_\Lambda}e^{\nu t}\Lambda_t^\vartheta$. Therefore, since $U^1 \le \Lambda$ and $W^2 > 0$,

$$\frac{\partial h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}}{\partial w}(t,\omega,U_t^1,W_t^2) = \rho U_t^1 (W_t^2)^{-\frac{1}{\vartheta}} \le \rho\left(\frac{K_\Lambda}{\varepsilon_2}\right)^{\frac{1}{\vartheta}} e^{-\frac{\nu}{\vartheta}t}.$$

Furthermore, $\frac{\partial^2 h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}}{\partial w^2}(u,w) = -\frac{\vartheta-1}{\vartheta^2}uw^{-(1+\frac{1}{\vartheta})} \leq 0$ for $w > 0$ so that $h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}$ is concave. Since (W^1, W^2) satisfies Condition B.4 for the pair $(h_{\mathrm{EZ}}^{\varepsilon_1,\nu,\Lambda}, U^1)$, the assumptions of Proposition B.6 are met and the conclusion follows. □

Denote by $\mathbb{O}(Y; k, K)$ the set of processes such that (4.1) holds for pre-specified k and K. Let

$$\mathbb{SO}(k,K) = \left\{ X \in \mathscr{P}_{++} : \mathbb{E}\left[\int_0^\infty X_t \,\mathrm{d}t\right] < \infty \text{ and } 0 \leq kJ^X \leq X \leq KJ^X \right\}.$$

The following result gives explicit bounds $\hat{k}, \hat{K} > 0$ such that the fixed point W found in Proposition 5.4 satisfies $\hat{k}J \leq W \leq \hat{K}J$.

Corollary C.5 *Let $\varepsilon \geq 0$ and suppose that $U^\vartheta \in \mathbb{SO}(k,K)$. For $\varepsilon > 0$, suppose that A and B solve*

$$A = K^{-1}(A^\rho + \varepsilon), \qquad B = k^{-1}(B^\rho + \varepsilon), \tag{C.6}$$

and if $\varepsilon = 0$, set $A = K^{-\vartheta}$ and $B = k^{-\vartheta}$ (the positive solution to (C.6)). Then the fixed point W of $F_{U,U}^\varepsilon$ found in Proposition 5.4 is in $\mathbb{O}(J; kA, KB)$.

Proof We first show that $F_{U,U}^\varepsilon$ maps $\mathbb{O}(U^\vartheta; A, B)$ to itself. We only prove the upper bound as the lower bound is symmetric. Suppose that $W \leq BU^\vartheta$. Then

$$\begin{aligned}
F_{U,U}^\varepsilon(W)_t &= \mathbb{E}\left[\int_t^\infty (U_s W_s^\rho + \varepsilon U_s^\vartheta)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\int_t^\infty (U_s B^\rho U_s^{\vartheta\rho} + \varepsilon U_s^\vartheta)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= (B^\rho + \varepsilon)\mathbb{E}\left[\int_t^\infty U_s^\vartheta \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= (B^\rho + \varepsilon)J_t \leq \frac{1}{k}(B^\rho + \varepsilon)U_t^\vartheta = BU_t^\vartheta.
\end{aligned}$$

The proof of Proposition 5.4 given in [13, Theorem B.2] first shows that the mapping $F_{U,U}^\varepsilon : \mathbb{O}(U^\vartheta) \to \mathbb{O}(U^\vartheta)$ is a contraction and then uses Banach's fixed point theorem. Hence if we choose an initial process $W^0 \in \mathbb{O}(U^\vartheta; A, B)$, then repeated application of $F_{U,U}^\varepsilon$ yields a fixed point $W^* \in \mathbb{O}(U^\vartheta; A, B)$. Since the fixed point W found in Proposition 5.4 is unique in the class $\mathbb{O}(U^\vartheta)$, we get $W = W^* \in \mathbb{O}(U^\vartheta; A, B)$. Finally, since $kJ \leq U^\vartheta \leq KJ$, it follows that $\mathbb{O}(U^\vartheta; A, B) \subseteq \mathbb{O}(J; kA, KB)$. □

Proof of Proposition 5.9 Since $\Lambda^\vartheta \in \mathbb{SO}_+$, there exists $\nu > 0$ such that $\Lambda^\vartheta \in \mathbb{SO}_{\nu\vartheta}$. For such ν, let $\Lambda^{(\nu)} = (\Lambda_t^{(\nu)})_{t\geq 0}$ be given by $\Lambda_t^{(\nu)} = e^{\nu t}\Lambda_t$. It follows that $(\Lambda^{(\nu)})^\vartheta \in \mathbb{SO}$. For each $n \in \mathbb{N}$, let $U^n := \max\{U, \frac{1}{n}\Lambda^{(\nu)}\}$. Then $U^n \overset{\mathbb{O}}{=} \Lambda^{(\nu)}$ as $U^n \leq \Lambda \leq \Lambda^{(\nu)}$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ with $\varepsilon_n \searrow 0$.

By Proposition 5.4, for each $n \in \mathbb{N}$, there exists a solution W^n associated to $(h_{\mathrm{EZ}}^{\varepsilon_n,\nu\vartheta,\Lambda}, U^n)$, where $h_{\mathrm{EZ}}^{\varepsilon_n,\nu\vartheta,\Lambda}$ is defined in (C.5). Furthermore, W^n is decreasing in n by Corollary C.4 and $U^n(W^n)^\rho$ is dominated by $U^1(W^1)^\rho$. Hence by the dominated convergence theorem, we find that $W := \lim_{n\to\infty} W^n$ satisfies

$$W_t = \lim_{n\to\infty} \mathbb{E}\left[\int_t^\infty \left(U_s^n (W_s^n)^\rho + \varepsilon_n e^{\nu\vartheta s}\Lambda_s^\vartheta\right) \mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty U_s W_s^\rho \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right],$$

so that $W \in \mathbb{I}(h_{\mathrm{EZ}}, U)$ is a solution associated to (h_{EZ}, U).

Suppose that $W' \in \mathbb{I}(h_{\mathrm{EZ}}, U)$ is a solution (or an L^1-bounded subsolution) associated to (h_{EZ}, U). Then (W', W^n) satisfies Condition B.4 for the pair (h_{EZ}, U), since $W^n \in \mathbb{UI}(h_{\mathrm{EZ}}^{\varepsilon_n,\nu\vartheta,\Lambda}, U^n) \subseteq \mathbb{UI}(h_{\mathrm{EZ}}, U)$ and $(W' - W^n)^+ \le W'$, where W' is L^1-bounded. Hence $W^n \ge W'$ for each $n \in \mathbb{N}$ by Corollary C.4 and $W \ge W'$ is a maximal (resp. L^1-bounded sub-)solution. Uniqueness in the class of maximal solutions follows from Proposition 5.8. □

Proof of Proposition 5.10 Let ν be such that $\Lambda^\vartheta \in \mathbb{SO}_{\nu\vartheta}$. Define

$$\Lambda^{(\nu)} = (\Lambda_t^{(\nu)})_{t\ge 0}$$

by $\Lambda_t^{(\nu)} = e^{\nu t}\Lambda_t$ and for $n \in \mathbb{N}$ and $i \in \{1, 2\}$, define $U^{i,n} = \max\{U^i, \frac{1}{n}\Lambda^{(\nu)}\}$ and $\varepsilon_n = \frac{1}{n}$. Then by Proposition 5.4, there exists a solution $W^{i,n}$ associated to

$$(h_{\mathrm{EZ}}^{\varepsilon_n,\nu\vartheta,\Lambda}, U^{i,n}).$$

Furthermore, for all $n \in \mathbb{N}$ and $t \ge 0$, $W_t^{1,n} \le W_t^{2,n}$ by Corollary C.4.

As in Proposition 5.9, the unique maximal solution associated to U^i is given by $W^i := \lim_{n\to\infty} W^{i,n}$ for $i \in \{1, 2\}$. Thus we may conclude that for all $t \ge 0$, we have $W_t^1 = \lim_{n\to\infty} W_t^{1,n} \le \lim_{n\to\infty} W_t^{2,n} = W_t^2$. □

Proof of Proposition 5.11 Fix $\vartheta > 1$. Since $U^\vartheta \in \mathbb{SO}_+$, there exists $\hat\nu > 0$ such that $U^\vartheta \in \mathbb{SO}_{\hat\nu\vartheta}$. By Lemma C.1, it follows that $U^\vartheta \in \mathbb{SO}_{\nu\vartheta}$ for all $\nu \le \hat\nu$. Moreover, for each $\nu \le \hat\nu$, define $U_t^{(\nu)} = e^{\nu t}U_t$ and $J_t^{(\nu)} = \mathbb{E}[\int_t^\infty e^{\nu\vartheta s}U_s^\vartheta \,\mathrm{d}s|\mathcal{F}_t]$. Remark C.2 then gives $0 < k(\nu) \le K(\nu)$ with $k(\nu)J^{(\nu)} \le (U^{(\nu)})^\vartheta \le K(\nu)J^{(\nu)}$ and the estimates $0 < k(\hat\nu) \le \lim_{\nu\to 0} k(\nu) = k(0) =: k < \infty$ as well as

$$0 < K(\hat\nu) \le \lim_{\nu\to 0} K(\nu) = K(0) =: K < \infty,$$

where both limits are decreasing in ν.

Moreover, for each $\varepsilon > 0$ and $0 < \nu \le \hat\nu$, there exists by Proposition 5.4 a solution $W^{\varepsilon,\nu}$ associated to $(h_{\mathrm{EZ}}^{\varepsilon,\nu\vartheta,\Lambda}, U^{(\nu)})$. Furthermore, by Corollary C.5, we have $W_t^{\varepsilon,\nu} \le K(\nu)B^{\varepsilon,\nu}J_t^{(\nu)}$, where $B = B^{\varepsilon,\nu}$ solves $B = k(\nu)^{-1}(B^\rho + \varepsilon)$. By Proposition 5.9, the unique maximal solution associated to U is given by $W := \lim_{\varepsilon\to 0} W^{\varepsilon,\nu}$. Since $\lim_{\varepsilon\to 0} B^{\varepsilon,\nu} = B^{0,\nu} = k(\nu)^{-\vartheta}$, it follows that $W_t \le K(\nu)k(\nu)^{-\vartheta}J^{(\nu)}$ for all $\nu \le \hat\nu$. Now taking the limit as $\nu \searrow 0$ gives $W_t \le Kk^{-\vartheta}J^{(0)} =: Kk^{-\vartheta}J$.

Similarly, maximality of W and the lower bound found in Corollary C.5 give $W \ge kK^{-\vartheta}J$. Hence the maximal solution is in $\mathbb{O}(J)$. Since the solution from Proposition 5.4 is unique in $\mathbb{O}(J)$, it is equal to the maximal solution. □

Appendix D: The proper solution: proofs

D.1 Proving Proposition 6.1

This rather intricate section is dedicated to proving Proposition 6.1. In this section, we emphasise the role that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ plays in determining the utility process $V = (V_t)_{t\geq 0}$ associated to a pair (g, C). To this end, if $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration used in Definition 2.1, we refer to V as the utility process associated to *the triple* $(g, C, \mathbb{F})$.

Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{T} = \{t_0, t_1, t_2, \ldots, t_n\}$ a finite ordered set. We assume without loss of generality that $t_0 = 0$ and $t_n = \infty$.

Definition D.1 An $(\mathbb{F}, \mathbb{T})$-*stopping time* is an $\mathbb{F}$-stopping time with values in $\mathbb{T}$.

Throughout this section, for an $(\mathbb{F}, \mathbb{T})$-stopping time $\tau = \sum_{i=0}^{n} t_i \mathbf{1}_{A_i}$ for some family $(A_i)_{i\in\{0,\ldots,n\}}$ of disjoint sets such that $\mathbb{P}[\bigcup_{i=0}^{n} A_i] = 1$ and $A_i \in \mathcal{F}_{t_i}$, we define $B_i^{\mathbb{T},\tau} := \{\tau > t_i\} = \bigcup_{j=i+1}^{n} A_j$ and $i(t;\mathbb{T}) := \max\{i : t_i \leq t\}$. When it is clear which $\mathbb{T}$ and τ we are referring to, we drop the extra indices $\mathbb{T}$ and τ and write $i(t) = i(t;\mathbb{T})$ and $B_i = B_i^{\mathbb{T},\tau}$. Note that $\{\tau > t\} = \{\tau > t_{i(t;\mathbb{T})}\}$ for all $t \geq 0$.

For the first results in this section, we crucially need that the filtration is constant between the points in $\mathbb{T}$.

Condition D.2 The pair $(\mathbb{F}, \mathbb{T})$ satisfies Condition D.2 if $\mathcal{F}_t = \mathcal{F}_{t_{i(t;\mathbb{T})}}$ for all $t \geq 0$.

We first prove the existence of a proper solution associated to $(h_{\mathrm{EZ}}, U, \mathbb{F})$, where $U = (U_t)_{t\geq 0}$ is given by $U_t = e^{-\gamma t}\mathbf{1}_{\{t<\tau\}}$. Here $\gamma > 0$, and τ is an $(\mathbb{F}, \mathbb{T})$-stopping time, where $(\mathbb{F}, \mathbb{T})$ satisfies Condition D.2.

Proposition D.3 *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{T}$ a finite ordered set such that $(\mathbb{F}, \mathbb{T})$ satisfies Condition D.2. Let τ be an $(\mathbb{F}, \mathbb{T})$-stopping time and define $U = (U_t)_{t\geq 0}$ by $U_t = e^{-\gamma t}\mathbf{1}_{\{t<\tau\}}$. Then there exists a proper solution $W = (W_t)_{t\geq 0}$ associated to $(h_{\mathrm{EZ}}, U, \mathbb{F})$ such that*

$$W_t \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t]\right)^{\vartheta} \qquad \text{for all } t \geq 0. \tag{D.1}$$

The proof of Proposition D.3 relies on the following lemma.

Lemma D.4 *Suppose that $\tau = \sum_{i=0}^{n} t_i \mathbf{1}_{A_i}$ is an $(\mathbb{F}, \mathbb{T})$-stopping time. Then*

$$\begin{aligned}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t] = \mathbf{1}_{B_{i(t)}}(e^{-\gamma t} - e^{-\gamma t_{i(t)+1}}) \\ + \sum_{j=i(t)+1}^{n-1} \mathbb{E}[\mathbf{1}_{B_j}|\mathcal{F}_t](e^{-\gamma t_j} - e^{-\gamma t_{j+1}}).\end{aligned}$$

Proof Using the definition of τ and B_j, the fact that $B_{i(t)} \in \mathcal{F}_t$ and rearranging the telescoping sum gives

$$\begin{aligned}
&\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t] \\
&= \sum_{j=i(t)+1}^{n} (e^{-\gamma t} - e^{-\gamma t_j})\mathbb{P}[A_j|\mathcal{F}_t] \\
&= e^{-\gamma t}\mathbb{P}[B_{i(t)}|\mathcal{F}_t] - \sum_{j=i(t)+1}^{n} e^{-\gamma t_j}(\mathbb{P}[B_{j-1}|\mathcal{F}_t] - \mathbb{P}[B_j|\mathcal{F}_t]) \\
&= \mathbf{1}_{B_{i(t)}}(e^{-\gamma t} - e^{-\gamma t_{i(t)+1}}) + \sum_{j=i(t)+1}^{n-1} \mathbb{E}[\mathbf{1}_{B_j}|\mathcal{F}_t](e^{-\gamma t_j} - e^{-\gamma t_{j+1}}).
\end{aligned}$$

This establishes the claim. □

Proof of Proposition D.3 Define $W = (W_t)_{t\geq 0}$ recursively backwards by $W_\infty = 0$ $\mathbb{P}$-a.s. and $W_t = \mathbf{1}_{B_{i(t)}} w(t, \xi_{t_{i(t)+1}})$ for $t < \infty$, where

$$\begin{aligned}
w(t, y) &= \left(y^{\frac{1}{\vartheta}} + \frac{1}{\gamma\vartheta}(e^{-\gamma t} - e^{-\gamma t_{i(t)+1}})\right)^{\vartheta}, \\
\xi_{t_{i(t)+1}} &= \mathbb{E}[W_{t_{i(t)+1}}|\mathcal{F}_{t_{i(t)}}].
\end{aligned} \tag{D.2}$$

We first show that W is a solution associated to $(h_{\mathrm{EZ}}, U, \mathbb{F})$. We then show that (D.1) holds and W is proper.

First, note that

$$F_U(W)_\infty := \lim_{t\to\infty} F_U(W)_t \leq \lim_{t\to\infty} \int_t^\infty e^{-\gamma s}\left(\frac{\exp(-\gamma\vartheta s)}{\gamma s}\right)^{\rho} \mathrm{d}s = 0 = W_\infty.$$

We now show that $W_t = F_U(W)_t = \mathbb{E}[\int_t^\infty U_s W_s^\rho \,\mathrm{d}s|\mathcal{F}_t]$ for all $t \geq 0$ by backward induction. For the inductive step, fix $k \in \{0, \ldots, n-1\}$ and assume that W satisfies $W_{t_{k+1}} = F_U(W)_{t_{k+1}}$ and that $t_k \leq t < t_{k+1}$. By the definition of $W_{t_{k+1}}$, since $\mathbf{1}_{B_k}\mathbf{1}_{B_{k+1}} = \mathbf{1}_{B_{k+1}}$ and by Condition D.2,

$$\mathbb{E}[W_{t_{k+1}}|\mathcal{F}_t] = \mathbb{E}[\mathbf{1}_{B_k} W_{t_{k+1}}|\mathcal{F}_t] = \mathbf{1}_{B_k}\mathbb{E}[W_{t_{k+1}}|\mathcal{F}_{t_k}] = \mathbf{1}_{B_k}\xi_{t_{k+1}}.$$

Hence combining this with the induction hypothesis yields

$$\begin{aligned}
F_U(W)_t &= \mathbb{E}\left[\int_t^{t_{k+1}} U_s W_s^\rho \,\mathrm{d}s + W_{t_{k+1}}\middle|\mathcal{F}_t\right] \\
&= \mathbf{1}_{B_k}\left(\int_t^{t_{k+1}} e^{-\gamma s}\big(w(s, \xi_{t_{k+1}})\big)^\rho \,\mathrm{d}s + \xi_{t_{k+1}}\right).
\end{aligned}$$

If $\omega \in B_k^c$, then clearly $W_t(\omega) = 0 = F_U(W)_t(\omega)$. If $\omega \in B_k$, then $\xi := \xi_{t_{k+1}}(\omega)$ is known. Using the fact that $\lim_{t\nearrow t_{k+1}} w(t, \xi) = \xi$ and $\frac{\partial w}{\partial t}(t, \xi) = -e^{-\gamma t}(w(t, \xi))^\rho$

for $t_k \le t < t_{k+1}$ and integrating yields

$$w(t,\xi) = \int_t^{t_{k+1}} e^{-\gamma s}\big(w(s,\xi)\big)^\rho \,\mathrm{d}s + \xi.$$

In particular,

$$W_t(\omega) = w(t,\xi) = \int_t^{t_{k+1}} e^{-\gamma s}\big(w(s,\xi)\big)^\rho \,\mathrm{d}s + \xi = F_U(W)_t(\omega).$$

Consequently, $W_t = F_U(W)_t$ for $t_k \le t \le t_{k+1}$, and hence for all $t \ge 0$ by induction.

We now show that $W = (W_t)_{t\ge 0}$ defined in (D.2) is proper by proving the following statement by backward induction for $k \in \{0,\dots,n-1\}$: If $t_k \le t < t_{k+1}$, then

$$W_t^{\frac{1}{\vartheta}} \ge \frac{\mathbf{1}_{B_k}}{\gamma\vartheta}\Bigg(\sum_{j=k+1}^{n-1} \mathbb{E}[\mathbf{1}_{B_j}|\mathcal{F}_t](e^{-\gamma t_j} - e^{-\gamma t_{j+1}}) + e^{-\gamma t} - e^{-\gamma t_{k+1}}\Bigg). \tag{D.3}$$

Here, we define $\sum_n^{n-1} a = 0$ for arbitrary $a \in \mathbb{R}$. Hence the statement holds true for $k = n-1$ by the definition of W in (D.2). For the induction step, assume that it holds true for $k+1$ and let $t_k \le t < t_{k+1}$ so that $i(t) = k$. Using the definition of W given in (D.2), Jensen's inequality, the inductive hypothesis and the fact that $\mathbf{1}_{B_i}\mathbb{E}[\mathbf{1}_{B_j}|\mathcal{F}_{t_i}] = \mathbb{E}[\mathbf{1}_{B_i}\mathbf{1}_{B_j}|\mathcal{F}_{t_i}] = \mathbb{E}[\mathbf{1}_{B_j}|\mathcal{F}_{t_i}]$ for $i \le j$, we obtain

$$\begin{aligned}
W_t^{\frac{1}{\vartheta}} &= \mathbf{1}_{B_k}\Bigg((\mathbb{E}[W_{t_{k+1}}|\mathcal{F}_t])^{\frac{1}{\vartheta}} + \frac{e^{-\gamma t} - e^{-\gamma t_{k+1}}}{\gamma\vartheta}\Bigg) \\
&\ge \frac{\mathbb{E}[\mathbf{1}_{B_{k+1}}|\mathcal{F}_t]}{\gamma\vartheta}\Bigg(\sum_{j=k+2}^{n-1} \mathbb{E}[\mathbf{1}_{B_j}|\mathcal{F}_t](e^{-\gamma t_j} - e^{-\gamma t_{j+1}}) + e^{-\gamma t_{k+1}} - e^{-\gamma t_{k+2}}\Bigg) \\
&\quad + \mathbf{1}_{B_k}\frac{e^{-\gamma t} - e^{-\gamma t_{k+1}}}{\gamma\vartheta} \\
&= \frac{1}{\gamma\vartheta}\sum_{j=k+1}^{n-1} \mathbb{E}[\mathbf{1}_{B_j}|\mathcal{F}_t](e^{-\gamma t_j} - e^{-\gamma t_{j+1}}) + \frac{\mathbf{1}_{B_k}}{\gamma\vartheta}(e^{-\gamma t} - e^{-\gamma t_{k+1}}).
\end{aligned}$$

Hence (D.3) holds for all t with $t_k \le t < t_{k+1}$, and therefore by induction for all $t < t_n = \infty$.

As the right-hand side of (D.3) is equal to $\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t]$ by Lemma D.4, equation (D.1) holds and W is proper. □

We now show that for any continuous filtration $\mathbb{F}$ and $\mathbb{F}$-stopping time τ, we can find a proper utility process associated to $U_t = e^{-\gamma t}\mathbf{1}_{\{t<\tau\}}$ by approximating $(\mathbb{F},\mathbb{T})$ by a monotone sequence of pairs $(\mathbb{F}^n,\mathbb{T}^n)$ satisfying Condition B.

Lemma D.5 *Let $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ be a continuous filtration and τ an $\mathbb{F}$-stopping time. Let $\mathbb{T}^n = \{k2^{-n} : k = 0,1,\dots,n2^n\} \cup \{\infty\}$ and define $\mathbb{F}^n = (\mathcal{F}_t^n)_{t\ge 0}$ by*

$$\mathcal{F}_t^n = \mathcal{F}_{(2^{-n}\lfloor 2^n t\rfloor)\wedge n}$$

for $t \geq 0$. Then

$$\tau_n := 2^{-n}\lceil 2^n \tau \rceil \mathbf{1}_{\{\tau \leq n\}} + \infty \mathbf{1}_{\{\tau > n\}}$$

is an $(\mathbb{F}^n, \mathbb{T}^n)$-stopping time. Furthermore, $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies Condition D.2 for each $n \in \mathbb{N}$, $\tau_n \searrow \tau$ and $\mathcal{F}^n_t \nearrow \mathcal{F}_t$ for all $t \geq 0$.

Proof Note that τ_n takes values in $\mathbb{T}^n$ and

$$\{\tau_n \leq t\} = \{\tau \leq n\} \cap \{2^{-n}\lceil 2^n \tau \rceil \leq t\} = \{\tau \leq (2^{-n}\lfloor 2^n t \rfloor) \wedge n\} \in \mathcal{F}^n_t$$

so that τ_n is an $(\mathbb{F}^n, \mathbb{T}^n)$-stopping time. In addition, for each $n \in \mathbb{N}$ and $t \geq 0$, we have $t_{i(t;\mathbb{T}_n)} = (2^{-n}\lfloor 2^n t \rfloor) \wedge n$. Hence $\mathcal{F}^n_t = \mathcal{F}_{t_{i(t;\mathbb{T}_n)}} = \mathcal{F}^n_{t_{i(t;\mathbb{T}_n)}}$ so that $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies Condition D.2. It is easily checked that $\tau_n \searrow \tau$ and $\mathcal{F}^n_t \nearrow \mathcal{F}_t$ for all $t \geq 0$. □

To prove Proposition 6.1, we need the following result, which is a variant of Hunt's lemma and the reverse Fatou lemma. Its proof is standard and hence omitted.

Lemma D.6 *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ a sequence of random variables bounded in absolute value by an integrable random variable Y. Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be an increasing family of σ-algebras and $\mathcal{G}_\infty := \sigma(\bigcup_{n=1}^\infty \mathcal{G}_n)$. Then*

$$\limsup_{n\to\infty} \mathbb{E}[X_n \mid \mathcal{G}_n] \leq \mathbb{E}[\limsup_{n\to\infty} X_n \mid \mathcal{G}_\infty]\ \mathbb{P}\text{-a.s.}$$

Proposition D.7 *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a continuous filtration, τ an $\mathbb{F}$-stopping time and $\gamma > 0$. Let $U = (U_t)_{t\geq 0}$ be given by $U_t = e^{-\gamma t}\mathbf{1}_{\{t<\tau\}}$. Then there exists a proper solution $W = (W_t)_{t\geq 0}$ associated to the triple $(h_{\mathrm{EZ}}, U, \mathbb{F})$ such that*

$$W_t \geq \Big(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t]\Big)^\vartheta$$

for all $t \geq 0$.

Proof By Lemma D.5, we may choose a sequence $(\mathbb{F}^n = (\mathcal{F}^n_t)_{t\geq 0}, \mathbb{T}^n)_{n\in\mathbb{N}}$ such that $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies Condition D.2 for each $n \in \mathbb{N}$ and an $(\mathbb{F}^n, \mathbb{T}^n)$-stopping time τ_n such that $\tau_n \searrow \tau$ and $\mathcal{F}^n_t \nearrow \mathcal{F}_t$ for $t \geq 0$. Since the pair $(\mathbb{F}^n, \mathbb{T}^n)$ satisfies for each $n \in \mathbb{N}$ the conditions of Proposition D.3, there exists for $U^n = (U^n_t)_{t\geq 0}$ defined by $U^n_t = e^{-\gamma t}\mathbf{1}_{\{t<\tau_n\}}$ a proper solution $W^n = (W^n_t)_{t\geq 0}$ associated to the triple $(h_{\mathrm{EZ}}, U^n, \mathbb{F}^n)$ such that

$$W^n_t \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau_n)}|\mathcal{F}^n_t]\right)^\vartheta. \tag{D.4}$$

Since W^n is a solution (and therefore càdlàg by Remark B.3), for all bounded stopping times $\sigma \leq \tau$,

$$W^n_\sigma = \mathbb{E}\left[\int_\sigma^\tau U^n_s (W^n_s)^\rho \,\mathrm{d}s + W^n_\tau \middle| \mathcal{F}^n_\sigma\right]. \tag{D.5}$$

Consider $\overline{u} = (\overline{u}(t))_{t\geq 0}$ defined by $\overline{u}(t) = e^{-\gamma t}$ for $t \geq 0$. Then by taking derivatives, one finds that $W^{\overline{u}} = (W_t^{\overline{u}})_{t\geq 0}$ defined by $W_t^{\overline{u}} = \frac{e^{-\gamma\vartheta t}}{\gamma^\vartheta \vartheta^\vartheta}$ is a solution associated to $(h_{\mathrm{EZ}}, \overline{u})$ (and any filtration). Furthermore, using $J_t^{\overline{u}^\vartheta} = \int_t^\infty (\overline{u}(s))^\vartheta \,\mathrm{d}s = \frac{e^{-\gamma\vartheta t}}{\gamma\vartheta}$, it follows that $W^{\overline{u}} \overset{\mathbb{O}}{=} J^{\overline{u}^\vartheta} = (J_t^{\overline{u}^\vartheta})_{t\geq 0}$ and $W^{\overline{u}}$ is the maximal solution associated to $\overline{u}$ by Proposition 5.11. Therefore, since $U_t^n \leq \overline{u}(t)$ for $t \geq 0$, Proposition 5.10 gives $W_t^n \leq W_t^{\overline{u}} \leq W_0^{\overline{u}} = \frac{1}{\gamma^\vartheta \vartheta^\vartheta} < \infty$ for all $t \geq 0$ and W^n is bounded. Similarly, $U_t(W_t^n)^\rho \leq \overline{u}(t)(W_t^{\overline{u}})^\rho$ and $\mathbb{E}[\int_0^\infty \overline{u}(t)(W_t^{\overline{u}})^\rho \,\mathrm{d}t] = W_0^{\overline{u}} < \infty$.

Define $W_t^* = \limsup_{n\to\infty} W_t^n$ for each $t \geq 0$. We will show that $W^* = (W_t^*)_{t\geq 0}$ is a subsolution associated to $(h_{\mathrm{EZ}}, U, \mathbb{F})$. Taking the lim sup in (D.5) and using Lemma D.6 gives

$$\begin{aligned} W_\sigma^* &= \limsup_{n\to\infty} \mathbb{E}\bigg[\int_\sigma^\tau U_s^n (W_s^n)^\rho \,\mathrm{d}s + W_\tau^n \bigg| \mathcal{F}_\sigma^n\bigg] \\ &\leq \mathbb{E}\bigg[\int_\sigma^\tau U_s (W_s^*)^\rho \,\mathrm{d}s + W_\tau^* \bigg| \mathcal{F}_\sigma\bigg]. \end{aligned} \tag{D.6}$$

Let $Y_t = W_t^* + \int_0^t U_s (W_s^*)^\rho \,\mathrm{d}s$. Then by (D.6), $Y = (Y_t)_{t\geq 0}$ is an optional strong submartingale. It is therefore làdlàg (see Dellacherie and Meyer [5, Theorem A1.4]) and by the strong submartingale property, $Y_\tau \leq \mathbb{E}[Y_{\tau+}|\mathcal{F}_\tau]$ for all stopping times τ. Consequently W^* is làdlàg and

$$W_\tau^* = Y_\tau - \int_0^\tau U_s (W_s^*)^\rho \,\mathrm{d}s \leq \mathbb{E}[Y_{\tau+}|\mathcal{F}_\tau] - \int_0^\tau U_s (W_s^*)^\rho \,\mathrm{d}s = \mathbb{E}[W_{\tau+}^*|\mathcal{F}_\tau].$$

Thus $W_\sigma^* \leq \mathbb{E}[\int_\sigma^\tau U_s (W_s^*)^\rho \,\mathrm{d}s + W_{\tau+}^*|\mathcal{F}_\sigma]$. In addition, the transversality condition for subsolutions holds since $\limsup_{t\to\infty} \mathbb{E}[W_{t+}^*] \leq \lim_{t\to\infty} W_t^{\overline{u}} = 0$, so that W^* is a subsolution for $(h_{\mathrm{EZ}}, U, \mathbb{F})$.

Since W^* is nonnegative and $\mathbb{P}$-a.s. bounded above by $W_0^{\overline{u}} < \infty$, it is bounded and a fortiori L^1-bounded. Moreover, (D.4), the fact that $\tau_n \geq \tau$ for all $n \in \mathbb{N}$ and the L^1-martingale convergence theorem give

$$\begin{aligned} W_t^* &\geq \lim_{n\to\infty} \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau_n)}|\mathcal{F}_t^n]\right)^\vartheta \\ &\geq \left(\frac{1}{\gamma\vartheta} \lim_{n\to\infty} \mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t^n]\right)^\vartheta \\ &= \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t]\right)^\vartheta. \end{aligned} \tag{D.7}$$

Let $W^U = (W_t^U)_{t\geq 0}$ be the *maximal* solution associated to $(h_{\mathrm{EZ}}, U, \mathbb{F})$. Then by Proposition 5.9, W^U is the maximal L^1-bounded subsolution. Combining with (D.7), this gives that W^U is a proper solution and

$$W_t^U \geq W_t^* \geq \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t]\right)^\vartheta \qquad \text{for all } t \geq 0.$$

This ends the proof. □

Finally, we remove the assumption that $\sigma = 0$ and prove Proposition 6.1.

Proof of Proposition 6.1 Let $U_t = e^{-\gamma t}\mathbf{1}_{\{\sigma \le t < \tau\}}$ and $\widehat{U}_t = e^{-\gamma t}\mathbf{1}_{\{t<\tau\}}$. Then by Proposition D.7, there exists a proper solution $\widehat{W}$ associated to the pair $(h_{\mathrm{EZ}}, \widehat{U})$ such that $\widehat{W}_t \ge (\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t])^{\vartheta}$. First, consider the event $\{t \ge \sigma\}$. Then $U_s = \widehat{U}_s$ for all $s \ge t$, and hence the (unique) maximal solutions W and $\widehat{W}$ associated to U and $\widehat{U}$ coincide at t. Hence

$$W_t\mathbf{1}_{\{t\ge\sigma\}} = \widehat{W}_t\mathbf{1}_{\{t\ge\sigma\}} \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma t} - e^{-\gamma(t\wedge\tau)}|\mathcal{F}_t]\right)^{\vartheta}\mathbf{1}_{\{t\ge\sigma\}}.$$

Next, consider the event $\{t < \sigma\}$. Using that

$$W_\sigma = \widehat{W}_\sigma \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}[(e^{-\gamma\sigma} - e^{-\gamma\tau})|\mathcal{F}_\sigma]\right)^{\vartheta},$$

Jensen's inequality and the tower property of conditional expectations give

$$\begin{aligned} W_t\mathbf{1}_{\{t<\sigma\}} &= \mathbf{1}_{\{t<\sigma\}}\mathbb{E}[W_\sigma|\mathcal{F}_t] \\ &\ge \mathbf{1}_{\{t<\sigma\}}\mathbb{E}\left[\left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma\sigma} - e^{-\gamma\tau}|\mathcal{F}_\sigma]\right)^{\vartheta}\middle|\mathcal{F}_t\right] \\ &\ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma\sigma} - e^{-\gamma\tau}|\mathcal{F}_t]\right)^{\vartheta}\mathbf{1}_{\{t<\sigma\}}. \end{aligned}$$

Combining the above inequalities yields

$$W_t \ge \left(\frac{1}{\gamma\vartheta}\mathbb{E}[e^{-\gamma(t\vee\sigma)} - e^{-\gamma(t\vee\tau)}|\mathcal{F}_t]\right)^{\vartheta}.$$

This ends the proof. □

D.2 Further proofs for Sect. 6

Proof of Lemma 6.2 First, suppose that $t \ge t_0$. Then since W is a utility process associated to U, we obtain

$$\widetilde{W}_t = \mathbf{1}_A W_t = \mathbf{1}_A^{1+\rho} W_t = \mathbb{E}\left[\int_t^\infty \mathbf{1}_A U_s(\mathbf{1}_A W_s)^\rho \,\mathrm{d}s\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty \widetilde{U}_s\widetilde{W}_s^\rho \,\mathrm{d}s\middle|\mathcal{F}_t\right].$$

Conversely, suppose that $t < t_0$. Then since $\widetilde{U}_s = 0$ for $s < t_0$ and both $\widetilde{W}_{t_0} = \mathbf{1}_A W_{t_0}$ and $\widetilde{W}_{t_0} = \mathbb{E}[\int_{t_0}^\infty \widetilde{U}_s\widetilde{W}_s^\rho \,\mathrm{d}s|\mathcal{F}_{t_0}]$, we get

$$\widetilde{W}_t = \mathbb{E}[\mathbf{1}_A W_{t_0}|\mathcal{F}_t] = \mathbb{E}\left[\int_{t_0}^\infty \widetilde{U}_s\widetilde{W}_s^\rho \,\mathrm{d}s\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\infty \widetilde{U}_s\widetilde{W}_s^\rho \,\mathrm{d}s\middle|\mathcal{F}_t\right].$$

This ends the proof. □

Proof of Lemma 6.3 Seeking a contradiction, suppose $C_t := A_t \setminus B_t$ has positive measure. Then $\mathbb{E}[\mathbf{1}_{C_t} J_t^U] > 0$ by the definition of A_t. For each rational $T \geq t$, the definition of B_t gives $\mathbb{E}[\mathbf{1}_{C_t} U_T | \mathcal{F}_t] \leq \mathbb{E}[\mathbf{1}_{B_t^c} U_T | \mathcal{F}_t] = 0$ which yields $\mathbf{1}_{C_t} U_T = 0$ $\mathbb{P}$-a.s. Since U is right-continuous, $\mathbf{1}_{C_t} U_T = 0$ for all $T \geq t$ $\mathbb{P}$-a.s. Taking expectations yields $\mathbb{E}[\mathbf{1}_{C_t} J_t^U] = \mathbb{E}[\int_t^\infty \mathbf{1}_{C_s} U_s \, \mathrm{d}s] = 0$ and we arrive at a contradiction. □

Proof of Lemma 6.4 Let $N = (N_t)_{t \geq 0}$ be the uniformly integrable martingale given by $N_t = \mathbb{E}[\int_0^\infty X_s \, \mathrm{d}s | \mathcal{F}_t]$. Then $J_t^X = N_t - \int_0^t X_s \, \mathrm{d}s$. Define the increasing process A by $A_t = \int_0^t (X_s / J_s^X) \, \mathrm{d}s$. Since $X \in \mathbb{SO}$, we have $0 < X_t \leq K J_t^X$ for some K and hence $0 < A_t \leq Kt$.

Define M via $M_t = e^{A_t} J_t^X$. Then $dM_t = e^{A_t} dJ_t^X + X_t e^{A_t} dt = e^{A_t} dN_t$, and all that remains to show is that the local martingale M is a martingale. Since A is increasing and $A_t \leq Kt$, we have $\mathbb{E}[\|e^A\|_T | N_T|] \leq \mathbb{E}[(e^{KT} - 1) | N_T|] < \infty$ for $T \geq 0$, where $\|e^A\|_T$ is the total variation of $e^A = (e^{A_t})_{t \geq 0}$ at time T. Hence M is a martingale by Herdegen and Muhle-Karbe [15, Lemma A.1]. □

Proof of Lemma 6.5 First note that X is a (local) submartingale bounded above by 1 and so converges almost surely to an $\mathcal{F}_\infty$-measurable random variable $X_\infty \leq 1$ by the martingale convergence theorem.

Fix $\xi \in (0, \beta)$. Let $\sigma = \inf\{t \geq 0 : X_t \notin (\xi, 1)\} \leq \tau$. From the dynamics of X given in (6.1),

$$X_{t \wedge \sigma} \geq X_0 + \int_0^{t \wedge \sigma} \alpha X_s \, \mathrm{d}s + G_{t \wedge \sigma} - G_0 \geq \beta + \alpha \xi (t \wedge \sigma) + G_{t \wedge \sigma} - G_0.$$

Then using that G is a càdlàg submartingale and the optional sampling theorem,

$$\mathbb{E}[X_{t \wedge \sigma}] \geq \beta + \alpha \xi \mathbb{E}[t \wedge \sigma]. \tag{D.8}$$

Since $X \leq 1$, taking the lim sup and using the reverse Fatou lemma on the left-hand side of (D.8) and the monotone convergence theorem on the right-hand side gives

$$1 \geq \mathbb{E}[X_\sigma] = \mathbb{E}[\mathbf{1}_{\{\sigma < \infty\}} X_\sigma] + \mathbb{E}[\mathbf{1}_{\{\sigma = \infty\}} X_\infty] \geq \beta + \alpha \xi \mathbb{E}[\sigma] \geq \beta.$$

Therefore $\mathbb{E}[\sigma] \leq \frac{1-\beta}{\alpha \xi}$ and $\mathbb{P}[\sigma = \infty] = 0$. Consequently, since X is right-continuous, X_σ has values in $(-\infty, \xi] \cup \{1\}$ $\mathbb{P}$-a.s. and

$$1 - (1 - \xi)\mathbb{P}[X_\sigma \leq \xi] = \mathbb{P}[X_\sigma = 1] + \xi \mathbb{P}[X_\sigma \leq \xi] \geq \mathbb{E}[X_\sigma] \geq \beta.$$

In particular, $\mathbb{P}[X_\sigma \leq \xi] \leq \frac{1-\beta}{1-\xi}$ and $\mathbb{P}[\sigma = \tau] = \mathbb{P}[X_\sigma = 1] \geq 1 - \frac{1-\beta}{1-\xi} = \frac{\beta - \xi}{1-\xi}$. Furthermore,

$$\mathbb{P}[\sigma \geq T, \sigma = \tau] \leq \mathbb{E}\left[\frac{\sigma}{T} \mathbf{1}_{\{\sigma \geq T\}} \mathbf{1}_{\{\sigma = \tau\}}\right] \leq \frac{1}{T} \mathbb{E}[\sigma] \leq \frac{1-\beta}{\alpha \xi T} \qquad \text{for all } T \geq 0,$$

as well as

$$\mathbb{P}[\tau < T] \geq \mathbb{P}[\sigma < T, \sigma = \tau] = \mathbb{P}[\sigma = \tau] - \mathbb{P}[\sigma \geq T, \sigma = \tau] \geq \frac{\beta - \xi}{1 - \xi} - \frac{1 - \beta}{\alpha \xi T}.$$

Choose $\nu = \frac{1}{2} \frac{\beta - \xi}{1 - \xi}$ and $T = \frac{1-\beta}{\alpha \xi \nu}$. Then $\mathbb{P}[\tau < T] \geq \nu$. □

Appendix E: Proof of Theorem 7.5

To prove Theorem 7.5, we first give analogues of the definitions of order-ϑ and maximal solutions in the original coordinates of discounted consumption streams and the aggregator g_{EZ}.

Definition E.1 Let $C \in \mathscr{P}_+$ and suppose that $V = (V_t)_{t\geq 0}$ is a solution to (2.2). We say that V is a *CRRA-order solution* if there exist $0 < k \leq K < \infty$ such that

$$k\mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta s} C_s^{1-R}\,\mathrm{d}s \middle| \mathcal{F}_t\right] \leq V_t \leq K\mathbb{E}\left[\int_t^\infty e^{-\delta\vartheta s} C_s^{1-R}\,\mathrm{d}s \middle| \mathcal{F}_t\right].$$

Let $C \in \mathscr{P}_+$ and suppose that $V = (V_t)_{t\geq 0}$ is a solution to (2.2). We say that V is an *extremal* solution if $(1-R)V \geq (1-R)Y$ for any other solution $Y = (Y_t)_{t\geq 0}$.

Given the relationships between solutions to (2.2) and solutions to (5.3) and the different coordinate systems, V is a CRRA-order solution (for C and g_{EZ}) if and only if W is an order-ϑ solution (for U^C and h_{EZ}), and V is extremal if and only if W is maximal. The names *CRRA-order* and *extremal* are chosen to be suggestive in the original coordinates – when we know the sign of $1-R$, we may also call a solution V minimal if $R < 1$ or maximal if $R > 1$. The name CRRA-order arises from the correspondence to the valuation problem under additive CRRA utility.

Applying Propositions 5.5, 5.9 and 5.10 in the setting of the original coordinate system, we obtain the following two results, the first of which also appears in [13, Theorem 4.6].

Proposition E.2 *Suppose that* $(e^{-\delta\vartheta t} C_t^{1-R})_{t\geq 0} \in \mathbb{SO}$. *Then there exists a unique CRRA-order solution to* (2.2).

Proposition E.3 *For each* $C \in \mathscr{P}_+$ *such that* $e^{-\delta\vartheta t} C_t^{1-R} \leq Y_t$ *for some* $Y \in \mathbb{SO}_+$, *there exists a unique extremal solution* V^C *to* (2.2). *Furthermore,* V^C *is increasing in* C.

It follows from the proof of Proposition 5.9 that the unique maximal solution associated to (h_{EZ}, U) is maximal in the class of L^1-bounded subsolutions. We can restate this in the original coordinates.

Corollary E.4 *Let* $C \in \mathscr{P}_+$ *be such that* $e^{-\delta\vartheta t} C_t^{1-R} \leq Y_t$ *for some* $Y \in \mathbb{SO}_+$. *Then the extremal solution for* (g_{EZ}, C) *is the maximal* L^1*-bounded subsolution when* $R < 1$, *and the minimal* L^1*-bounded supersolution when* $R > 1$.

The final result in this section is a direct analogue of Theorem 5.12.

Theorem E.5 *Suppose that* $(e^{-\delta\vartheta t} C_t^{1-R})_{t\geq 0} \in \mathbb{SO}_+$. *Then the following three solutions to* (2.2) *all coincide and are unique*:

1) *the CRRA-order solution*;
2) *the extremal solution*;
3) *the proper solution*.

To prove Theorem 7.5, we use the following result which is proved as an intermediate part of [13, Theorem 8.1]. The result in [13] is written for the case $0 < \vartheta < 1$ and $\delta = 0$, but it is not difficult to check that the argument extends to the case $\vartheta > 1$ and arbitrary $\delta \in \mathbb{R}$.

Lemma E.6 *Let $\varepsilon > 0$ and let $\hat{X} = X^{\hat{\Pi},\hat{C}}$ denote the wealth process under our candidate optimal strategy. Fix (Π, C) and let $X = X^{\Pi,C}$ denote the corresponding wealth process. Then $\hat{V}(\cdot, X + \varepsilon\hat{X})$ is a supersolution for the pair $(g_{\mathrm{EZ}}, C + \eta\varepsilon\hat{X})$.*

We may then prove Theorem 7.5. Note that $\hat{V}(\cdot, \hat{X})$ is a solution for the pair $(g_{\mathrm{EZ}}, \eta\hat{X})$ and by scaling, $\hat{V}(\cdot, \varepsilon\hat{X})$ is a solution for the pair $(g_{\mathrm{EZ}}, \eta\varepsilon\hat{X})$. We expect that $\hat{V}(\cdot, X^{\Pi,C})$ is a supersolution for (g_{EZ}, C), but when $R > 1$, the transversality condition might not hold. Furthermore, the conditions required for Proposition B.6 to hold may be impossible to verify. However, as we show in the proof below, by considering the perturbed problem, the transversality condition is guaranteed and the comparison theorem can be applied.

Proof of Theorem 7.5 It follows from Proposition 7.4 that $\hat{V}(\cdot, \hat{X})$ is a utility process associated to the candidate optimal strategy $(\hat{\Pi}, \hat{C})$. Since $\hat{V}(\cdot, \hat{X})$ is a CRRA-order solution to (2.2), it is the unique proper solution to (2.2) by Theorem E.5. Hence $V_0^{\hat{C}} = \hat{V}(0, x)$. It therefore only remains to show that $V_0^C \leq \hat{V}(0, x)$ for all $C \in \mathscr{C}(x) \cap \mathbb{UP}^*$. Fix an arbitrary $C \in \mathscr{C}(x) \cap \mathbb{UP}^*$ and let $\Pi = (\Pi_t)_{t\geq 0}$ be an associated investment process.

We first prove the result when $R > S > 1$. (Note that in this case Corollary B.7 does not hold. However, it is not needed as the proof relies instead upon the minimality of the unique proper solution to give the comparison result.) By Lemma E.6, for each $\varepsilon > 0$, $\hat{V}(\cdot, X^{C,\Pi} + \varepsilon\hat{X})$ is a supersolution associated to $C^\varepsilon = C + \eta\varepsilon\hat{X}$. Since $(C^\varepsilon)^{1-S} \leq (\eta\varepsilon)^{1-S}\hat{X}^{1-S}$, there exists by Proposition E.3 a unique extremal solution V^{C^ε} associated to C^ε, and V^{C^ε} is increasing in ε. It is the minimal supersolution by Corollary E.4. Hence by minimality, $V_t^{C^\varepsilon} \leq \hat{V}(\cdot, X_t^{C,\Pi} + \varepsilon\hat{X}_t) < 0$ for all $t \geq 0$.

Let $V^* = \lim_{\varepsilon\to 0} V^{C^\varepsilon}$. Then $V_0^* \leq \hat{V}(0, x)$. Consequently, since g_{EZ} is increasing in its last two arguments and C^ε and V^{C^ε} are increasing in ε, $g_{\mathrm{EZ}}(\cdot, C^\varepsilon, V^{C^\varepsilon})$ is increasing in ε. Applying the monotone convergence theorem for conditional expectations yields

$$\begin{aligned} V_t^* = \lim_{\varepsilon\to 0} V_t^{C^\varepsilon} &= \lim_{\varepsilon\to 0} \mathbb{E}\left[\int_t^\infty g_{\mathrm{EZ}}(s, C_s^\varepsilon, V_s^{C^\varepsilon})\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^\infty g_{\mathrm{EZ}}(s, C_s, V_s^*)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

Therefore V^* is a solution associated to (g_{EZ}, C). Moreover, for each $t \geq 0$, one has $V_t^* < 0$ $\mathbb{P}$-a.s. due to the fact that $V_t^{C^\varepsilon} < 0$ $\mathbb{P}$-a.s. for each ε and V^{C^ε} is increasing in ε. Thus V^* is proper. It therefore agrees with the unique proper solution V^C so that $V_0^C \leq \hat{V}(0, x)$.

We now prove the result when $R < S < 1$. Fix an arbitrary $C \in \mathscr{C}(x) \cap \mathbb{UP}^*$ with associated investment process $\Pi = (\Pi_t)_{t\geq 0}$. First, in order to apply a comparison theorem in the form of Corollary B.7, let $0 < \zeta < \eta\frac{S}{1-S}$ and define $\widetilde{X}_t = e^{\zeta t} X_t^{C,\Pi}$, $Y_t = e^{\frac{\zeta}{S}t}\hat{X}_t$ and $\widetilde{C}_t = e^{\zeta t} C_t$ for $t \geq 0$. Note that if $r_\zeta = r + \zeta$ and $\mu_\zeta = \mu + \zeta$, then

$$\mathrm{d}\widetilde{X}_t = \widetilde{X}_t \Pi_t \sigma \,\mathrm{d}B_t + \Big(\widetilde{X}_t\big(r_\zeta + \Pi_t(\mu_\zeta - r_\zeta)\big) - \widetilde{C}_t\Big)\mathrm{d}t.$$

We may think of $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ as the wealth process associated to the strategy $(\Pi, \widetilde{C} = (\widetilde{C}_t)_{t\geq 0})$ in a more favourable financial market with risk-free rate r_ζ, drift μ_ζ of the risky asset and well-posedness parameter

$$\eta_\zeta = -\frac{1-S}{S}\left(r_\zeta + \frac{\lambda^2}{2R}\right) = \eta - \frac{1-S}{S}\zeta \in (0, \eta).$$

The volatility is unchanged. Furthermore, since

$$\frac{dY_t}{Y_t} = \frac{\lambda}{R}\,\mathrm{d}B_t + \left(\left(r + \frac{\lambda^2}{R} - \eta\right) + \frac{\zeta}{S}\right)\mathrm{d}t = \frac{\lambda}{R}\,\mathrm{d}B_t + \left(r_\zeta + \frac{\lambda^2}{R} - \eta_\zeta\right)\mathrm{d}t,$$

$Y = (Y_t)_{t\geq 0}$ is the wealth process under the optimal strategy in the new financial market with parameters r_ζ and μ_ζ. Define $\hat{V}^\zeta(t,x) = \eta_\zeta^{-\vartheta S} e^{-\delta\vartheta t}\frac{x^{1-R}}{1-R}$. Then $\hat{V}^\zeta(\cdot, \widetilde{X} + \varepsilon Y)$ is a supersolution for $(g_{\mathrm{EZ}}, \widetilde{C} + \eta\varepsilon Y)$ by Lemma E.6 and then also for (g_{EZ}, C) since $\widetilde{C} + \eta\varepsilon Y \geq C$ and g_{EZ} is increasing in c.

Second, let $C^n := C \wedge n\hat{X}$. Then there exists an extremal solution V^{C^n} associated to C^n which is increasing in n by virtue of Proposition E.3. Moreover, $\hat{V}^\zeta = \hat{V}^\zeta(\cdot, \widetilde{X} + \varepsilon Y)$ is a fortiori a supersolution for (g_{EZ}, C^n) since $C \geq C^n$ and g_{EZ} is increasing in c. Furthermore, since C is right-continuous, C^n is right-continuous. The extremal solution V^{C^n} associated to C^n is therefore proper by Theorem 4.7 and Remark 5.7.

Third, using that $(C^n)^{1-R} \leq n^{1-R}\hat{X}^{1-R}$, we obtain

$$\begin{aligned}(1-R)\hat{V}^\zeta(t, \widetilde{X}_t + \varepsilon Y_t) &\geq \eta_\zeta^{-\vartheta S} e^{-\delta\vartheta t}(\varepsilon Y_t)^{1-R} \\ &= \eta_\zeta^{-\vartheta S}\varepsilon^{1-R} e^{(\frac{\zeta(1-R)}{S} - \delta\vartheta)t}\hat{X}^{1-R} \\ &\geq \frac{\eta_\zeta^{-\vartheta S}\varepsilon^{1-R}}{n^{1-R}} e^{(\frac{\zeta(1-R)}{S} - \delta\vartheta)t}(C_t^n)^{1-R}.\end{aligned}$$

Furthermore, since $V^{C^n} \in \mathbb{UI}(g_{\mathrm{EZ}}, C^n)$ by Remark 2.2 and

$$\begin{aligned}\mathbb{E}[(V_t^{C^n} - \hat{V}_t^\zeta)^+] \leq \mathbb{E}[V_t^{C^n}] &= \mathbb{E}\left[\int_t^\infty g_{\mathrm{EZ}}(s, C_s^n, V_s^{C^n})\,\mathrm{d}s\right] \\ &\leq \mathbb{E}\left[\int_0^\infty g_{\mathrm{EZ}}(s, C_s^n, V_s^{C^n})\,\mathrm{d}s\right] < \infty,\end{aligned}$$

Corollary B.7 gives $\hat{V}_t^\zeta \geq V_t^{C^n}$ for all $t \geq 0$. In particular, $\hat{V}_0^\zeta \geq V_0^{C^n}$.

Fourth, set $V^* = \lim_{n\to\infty} V^{C^n}$. Using that g_{EZ} is increasing both in c and v, monotone convergence gives

$$V_t^* = \lim_{n\to\infty} V_t^{C^n} = \lim_{n\to\infty} \mathbb{E}\left[\int_t^\infty g_{\mathrm{EZ}}(s, C_s^n, V_s^{C^n})\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$$
$$= \mathbb{E}\left[\int_t^\infty g_{\mathrm{EZ}}(s, C_s, V_s^*)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \qquad t \geq 0.$$

Hence V^* is a solution for (g_{EZ}, C). It is a proper solution since for each $t \geq 0$, $V_t^* > 0$ if $V_t^{C^n} > 0$ for some n, and $V_t^{C^n} > 0$ on $\{J_t^{(C^n)^{1-R}} > 0\} = \{J_t^{C^{1-R}} > 0\}$ up to nullsets by the fact that each V^{C^n} is proper and $\hat{X}$ is strictly positive. Therefore, V^* must agree with the unique proper solution V^C for C.

Finally, combining the above yields

$$\hat{V}_0^\zeta = \hat{V}^\zeta\big(0, x(1+\varepsilon)\big) \geq \lim_{n\to\infty} V_0^{C^n} = V_0^* = V_0^C,$$

and taking $\zeta, \varepsilon \searrow 0$ gives $\hat{V}(0, x) \geq V_0^C$. □

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