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Divisive By Design

Shaping Values In Optimal Mechanisms

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Francesco Nava

School of Business & Economics
Discussion Paper

Economics

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DIVISIVE BY DESIGN: SHAPING VALUES IN OPTIMAL MECHANISMS*

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Abstract

We study a principal who allocates a good to agents with private, independently distributed values through an optimal mechanism. The principal can strategically shape these value distributions by modifying the good’s features, which affect agents’ valuations. Our analysis reveals that optimal designs are frequently divisive—creating goods that appeal strongly to specific agents or agent types while being less valued by others. These divisive designs reduce information rents and increase total surplus, at the expense of competition. Even when total surplus is constrained, some divisiveness in designs remains optimal.

Keywords: Value Design, Mechanism Design, Differentiation

JEL Codes: D82, D46, L15

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1 Introduction

A bureaucrat is tasked with allocating a plot of land to one of several interested parties. The interested parties include a company seeking a new production site, an entrepreneur who wants to build a water skiing facility, and a school superintendent in need of a plot for a new elementary school. The bureaucrat has the authority to impose rules and regulations on this plot. For example, they can specify certain setbacks, enforce environmental restrictions, or approve the construction of a highway to improve access to the plot. The bureaucrat then allocates the land according to an optimal mechanism (Myerson (1981), Bulow and Roberts (1989)). The objective is to maximize transfers from the interested agents, either for the bureaucrat's benefit or for the community at large.¹ The question is: What are the optimal features that should be imposed on the plot of land?

We demonstrate that attributes that increase surplus while minimizing information rents are optimal, even if they reduce competition among bidders. Each feature influences the bidders' valuations in different ways. Some attributes may affect the valuation of one bidder without altering the others', while other features may impact the valuations of all bidders. We therefore consider a broad set of constraints, moment conditions, and distributional restrictions on how the features of a good shape valuations, and we characterize the optimal value distribution subject to these constraints.

Any combination of attributes that shapes a value design has three key implications: (i) it directly affects agents' valuations, and consequently the overall surplus; (ii) it alters agents' private information, thereby influencing the incentives required for agents' values to be truthfully revealed; and (iii) it impacts the allocation probability, which in turn affects the transfers in the optimal mechanism. We find that attributes leading to more differentiated valuations help the principal screen and increase revenue, as they are associated with an increase in surplus and/or a reduction in information rents.

Value distributions are determined by the characteristics of the good. For example, building restrictions imposed on a plot of land can have various implications for value distributions. Environmental restrictions may increase the value of the land to both the water ski entrepreneur and the company seeking a production site, as both entities value

¹While these transfers may constitute bribes, they could also be perfectly legal contributions through PACs. Moreover, the transfers do not necessarily have to be monetary, but can take some other form. It only matters that the bureaucrat is able to assign a price to what is given by every bidder. In this sense, a school superintendent may be able to provide some favors, not necessarily money, that the bureaucrat values.

green technologies, and the restrictions allow them to credibly signal this commitment to their customers. The school, on the other hand, may be unaffected by these restrictions. Approving highway access may lower the value of the land for the school superintendent due to increased pollution, while increasing its value for the production site due to improved transportation access. The water skiing facility may not be significantly impacted, as access has improved, but water quality would need to be more closely monitored.

These examples illustrate that the connection between features and the resulting value distributions can be complex and rather context-specific. It matters which markets we consider, the type of buyers, the characteristics that can be added to the good, just to name a few factors. Abstracting from these specifics, our goal is to find the optimal value designs, that is, the distributions the principal would like to achieve through the appropriate choice of attributes. To discipline our analysis, we consider settings where an increase in value for one (type of) agent is met by a decrease in value for another (type of) agent, as made precise in Section 3. Consequently, we think of *divisive designs* as those that lead to (i) some agents valuing the good highly while others value it little, or (ii) some types of agents valuing the good substantially while other types do not.²

In general, the principal benefits from adding features to the good that make it uniformly more desirable (thereby increasing surplus). However, we assume that the principal cannot generate valuations exceeding some arbitrarily large number, \bar{v} .

If this were the only constraint, the principal would select features that ensure at least one agent values the good exactly at \bar{v} . Having even a single agent with such a value distribution maximizes surplus by generating the highest willingness to pay for the good. This design simultaneously minimizes information rents by making the value fully predictable. The principal can set a price of \bar{v} , thereby extracting the entire surplus. Even if all other agents have a valuation of zero, it would still be optimal for the principal to choose features that result in such a design.

The emergence of atoms, if possible, is generally a feature of optimal designs, as shown in Section 4.1. This follows as the principal's revenue in the optimal mechanism is a convex function of the agents' value distributions. Therefore, any solution to the design problems we consider will be extreme, lying on the boundary of the constraint set. Put differently, value distributions that are convex combinations of other value designs are suboptimal.

²Note that both cases are observationally equivalent. Regardless of whether the change in values affects the types of an agent differentially or distinct agents, we would see agents with high and low values.

A second insight is that the principal’s revenue in the optimal mechanism is monotonically increasing in first-order stochastic dominance, as previously established by [Devanur, Huang, and Psomas \(2016\)](#) and [Hart and Reny \(2015\)](#). This implies that the principal can never benefit from uniformly reducing values.

While these two features of optimal revenue help narrow the set within which our optimal value distributions lie, they do not fully determine the optimal value distribution. Instead, the optimal distribution will depend on the specific constraints faced by the principal.

No Spillovers Suppose that the principal can select features that influence the value of the good for one agent, while leaving the value for the other agents unchanged. We begin in [Section 4.2](#) by considering settings in which the principal cannot raise the expected value beyond some constant $k < \bar{v}$, but can otherwise shape the distribution of values arbitrarily. Then, the principal will select features resulting in a binary distribution of values with mass at \bar{v} and zero, with mean k .

The principal selects attributes such that, on average, the agent values the good at k , the upper bound. Furthermore, in order to fully extract the gains from trade of the agent whose value is affected, the principal selects features that lead to a binary distribution with positive mass assigned to one strictly positive value. The designed value distribution not only makes the agent’s value predictable conditional on trade, but also affects the probability of allocating the good to competing agents. The principal aims to keep the chances of the competing agents at a maximum, as this increases their willingness to pay and, thus, revenue. They achieve this by minimizing the odds that the agent with the binary distribution receives the good. This agent only receives the good if their realized value is strictly positive, and the chances of this occurring are minimized when the strictly positive value is as high as possible.

The optimality of such binary distributions extends to the case where all agents respond to the features of the good in the same way, and the principal faces a constraint on the mean of the designed distributions. Contrary to the widespread belief that higher variance increases agents’ information rents, this result demonstrates that this is not necessarily the case. For instance, when the increase in variance allows the principal to better screen agents’ valuations, then the dispersion in values benefits the principal.

In sum, a principal prefers features that allow them to generate extreme distributions. These attributes are, with some probability, highly valued by agents, but they also carry

the risk that the agents may no longer value the good. This value design is particularly valuable when agents appear a priori identical, and the introduction of a feature allows the principal to segment them. To illustrate this, consider again the bureaucrat facing different companies. The bureaucrat would like to attract a company that values environmental standards, but cannot observe this, and, therefore, imposes a number of eco-friendly policies on the plot. For some companies, this aligns with their ethical standards and can make the brand more attractive to environmentally conscious consumers.³ For others, it is a negative, as their production standards do not meet these regulations. As a result, imposing such legislation can either increase or decrease the agents' values, and, at best, generate the binary distributions that are optimal in the absence of spillovers in valuations.

Spillovers We characterize optimal designs when the principal can select features of the good that affect the values of all agents simultaneously in Section 4.3. We focus on the economically relevant case where making the good more appealing to some agents necessarily reduces its appeal to others.

We first consider a constraint on the sum of agents' expected values, mirroring the mean-bound constraint from the setting without spillovers. Under this restriction, the principal would ideally design a good that perfectly caters to a single agent, making their value both as high and as predictable as possible. This extreme specialization simultaneously maximizes surplus generation and eliminates information rents entirely, since the favored agent's willingness to pay becomes perfectly known.

However, the feasibility of such extreme value distributions is questionable in practice. We therefore complement this analysis with a distributional constraint that more realistically bounds the principal's design ability.

We consider a scenario where the principal can design any value distributions for the two agents, provided that their cumulative distributions sum to at least $H(v)$ pointwise.⁴ In such a setting, the optimal design exhibits maximal asymmetry: one agent's values are concentrated below the median of H , while the other agent's values lie entirely above the median. Maximal differentiation serves a dual purpose: it increases total surplus while simultaneously reducing information rents as agents' valuations lie in a narrower set, thereby making them more predictable. This asymmetry decreases transfers from

³For some evidence that these types of considerations matter to firms, see <https://scmguides.com/10-essential-considerations-before-building-a-new-production-facility/>

⁴ H is a non-decreasing function with image in $[0, 2]$.

the disadvantaged agent, while it leads to an increase in transfers from the advantaged agent, which more than compensates. Therefore, maximally distinct value distributions are optimal.

This result has practical implications. When a principal faces competing entities and knows that a particular feature will increase one agent’s valuation while decreasing another’s, they should add the feature to maximize revenue.

Fixed Surplus With both spillovers and no spillovers, surplus maximization is a key driver of divisive designs. In Section 5, the principal can only add attributes that lead to a fixed surplus distribution G . This raises the question of whether divisive designs remain optimal. Now the revenue-maximizing design depends crucially on the structure of G and the lower bound of the value support. When the minimal value equals zero, maximally divisive designs—where only one agent values the good according to G —are revenue-minimizing among all designs generating the same surplus. Such a design is equivalent to a one-good, one-bidder setting and therefore, the optimal mechanism reduces to a posted price. However, other value distributions that assign some positive value to all agents generate the same surplus by design and allow for competition between agents for the good. This increases the principal’s revenue, establishing that maximally divisive designs perform worst.

Instead, threshold designs that segment agents into distinct value ranges can outperform both maximally and minimally divisive design alternatives. Threshold designs assign to one agent both mass at the minimal value and mass above some threshold, while the other agent receives all the mass below the threshold. These designs limit information rents, as they reduce the possibility to deviate. At the same time, they maintain competition between different agents. Notably, these threshold designs exhibit features reminiscent of designs that are optimal without spillovers: they spread one agent’s value distribution while concentrating the other agent’s distribution. Thus, even when surplus is fixed, some asymmetry in designs can remain optimal.

Applications While our primary example involves a bureaucrat or politician who allocates a good to one of several entities, our model can also provide insights in a variety of other settings. In the Supplementary Appendix Section A, we discuss how our model relates to celebrity auctions as well as the art market.

More generally, our model can be viewed as a product design model, where the principal decides what kind of product to sell and how to sell it. The principal benefits from

understanding what types of features agents value, motivating why market research that leads to better information about the customer base, what they value, and how they are affected by different attributes of the good can be extremely valuable.⁵ As described in Bloch (1995) p.26f, “Seeking the ‘ideal’ form for a product remains a significant goal for both designers and marketing management. The ideal form is a theoretical concept, [...], similar to a vanishing point in the distance.” Bloch (1995) continues by stating that the ideal form must be truly valued by consumers, evoke positive emotions, and be sympathetic to consumers’ tastes. “This ideal form must accomplish all of this, while simultaneously satisfying numerous design constraints.”, design constraints that depend on the specific setting. These are the considerations our model aims to formalize.

2 Related Literature

Our paper contributes to the literature on value and product design in allocation problems. Key insights in this area were pioneered by Johnson and Myatt (2006), who study environments where a price-setting monopolist influences market demand either by adding features to the product or through advertising. Their analysis focuses on reshaping demand through rotations and shifts, demonstrating that value distributions with minimal dispersion are optimal when the monopolist caters to a mass market, while dispersion is beneficial when the monopolist caters to different niches. While our approach is similar in spirit, our model differs fundamentally with implications for results. In the absence of market features in terms of niche versus mass, dispersion in values, both within and across buyers, generally increases revenue, due to higher surplus and lower information rents.⁶

The importance of increasing surplus becomes evident in light of Cantillon (2008). She shows that when surplus is fixed, revenue decreases with asymmetric value designs in first- and second-price auctions without exclusion. We complement her result by showing

⁵This described approach is followed in the segmenting-targeting-positioning framework in marketing. A firm aims to divide consumers according to their characteristics, the segmentation. It then targets the market through an advertising campaign and finally positions the product in the market so as to raise consumers’ values. See for instance Moutinho (2000) for an application of this concept to tourism.

⁶While not considering product design, Anderson and Renault (2006) analyzes how attributes of a given product are advertised. These attributes or characteristics are taken as given, and the seller can choose to highlight some or all of them. In contrast, we allow for the seller to create these attributes which are immediately visible. Despite these modeling differences, Anderson and Renault (2006) finds that it is beneficial to advertise characteristics that screen out low-value customers, which allows for a price increase for high-value customers.

that revenue can increase with asymmetric designs in the optimal mechanism, if there is exclusion.⁷

Value design has been previously explored in [Condorelli and Szentes \(2020, 2022\)](#). The former focuses on buyer-optimal value design with a single buyer, while the latter characterizes the optimal profit arising from different demand functions in a quantity competition setting. In contrast, we consider seller-optimal designs with multiple buyers and focus on optimal design strategies.

Finally, some of our results can be interpreted through the lens of information design.⁸ In these settings, the principal provides information about the product, as in [Bergemann and Pesendorfer \(2007\)](#), [Eso and Szentes \(2007\)](#), [Li and Shi \(2017\)](#), [Sorokin and Winter \(2018\)](#), and [Ganuza and Penalva \(2019\)](#). In contrast, we allow for the principal to design the product’s features, generating value for agents. Consequently, the influence of the principal on the distributions of valuations can be substantially greater. This leads us to consider a broader class of constraints than Bayes Plausibility.

But even in the case of a constraint that technically matches Bayes Plausibility, our results differ from [Bergemann and Pesendorfer \(2007\)](#) in terms of (i) the key trade-off, (ii) the result when the number of bidders differs from two, and (iii) the interpretation. In [Bergemann and Pesendorfer \(2007\)](#), the key trade-off is between the seller’s revenue and the bidders’ information rents. While such a trade-off is also present in our setting (as different value designs are associated with different informational content), there is additionally the role of competition to consider. More importantly, the optimal design simultaneously minimizes the information rents and maximizes the seller’s revenue, generating an alignment of [Bergemann and Pesendorfer \(2007\)](#)’s trade-off. As a result, even when the number of bidders exceeds two, the information rents remain zero, and every bidder has the same value design at the optimum (with the possible exception of the n -th bidder, where we obtain a multiplicity). In contrast, [Bergemann and Pesendorfer \(2007\)](#) finds that with three or more bidders, binary partitions are no longer optimal and the principal has to incur incentive costs.

Moreover, in [Bergemann and Pesendorfer \(2007\)](#) it is optimal to reveal no information when there is only one bidder. The principal sets the price equal to the expected value,

⁷In contrast to our focus on designing value distributions, [Deb and Pai \(2017\)](#) fix value designs and establish that symmetric mechanisms can heavily discriminate between agents when their value distributions differ.

⁸For instance, [Roesler and Szentes \(2017\)](#) and [Bobkova \(2019\)](#) study trade environments with a single buyer and seller, characterizing buyer-optimal information designs and the costs of information acquisition.

and extracts the entire surplus. In the context of value design, other designs can be optimal, such as any binary value distribution with the same expected value.

Finally, the interpretation of our result differs, which is most clearly evident from the case with one buyer. Revealing no information in the information design case corresponds to a good whose features are perfectly understood in the value design setting as all mass is placed at the mean. At the same time, with value design, but not information design, uncertainty about the value is irrelevant—as long as the distribution is binary with mass at one strictly positive value and zero.

Even though information design and value design are technically related, insights and results differ substantially.

3 A Model of Constrained Value Design

We begin by introducing the mechanism design framework, which forms the foundation for the value design approach in our analysis. We then proceed to discuss the design of optimal distributions. For clarity, we focus on the case of two agents, but extensions to N agents are considered throughout.

The Environment and the Optimal Mechanism A principal is tasked with allocating a single unit of a good to one of two competing agents, A and B . The value of the good to agent $i \in \{A, B\}$ is denoted by $v_i \in [\underline{v}, \bar{v}]$, where $\bar{v} > \underline{v} \geq 0$ are exogenous upper and lower bounds on the value of the good. Each agent is privately informed of their own value, and the values are independently distributed. The cumulative distribution function of the value v_i is denoted by F_i , with support $V_i \subseteq [\underline{v}_i, \bar{v}_i]$, where \underline{v}_i and \bar{v}_i represent the smallest and largest values in V_i , respectively. The distributions F_A and F_B are commonly known to both the agents and the principal. The utility of an agent is given by the private value of the good, v_i , if the agent is awarded the good, minus any transfer t_i made to the principal. The principal derives no value from the good itself and seeks to maximize revenue by selecting a mechanism to sell the good, with revenue defined as the sum of the transfers, $t_A + t_B$.

By the revelation principle, the principal’s problem can be solved by restricting attention to direct, incentive-compatible (IC), and individually rational (IR) mechanisms. A

direct mechanism (x, t) consists of an allocation rule x and a transfer rule t ,

$$(x, t) : V_A \times V_B \rightarrow \Delta_-^2 \times \mathbb{R}^2,$$

where agents are only asked to simultaneously report their types. The allocation rule $x(\mathbf{v}) = (x_A(\mathbf{v}), x_B(\mathbf{v}))$ specifies the probability with which agents A and B receive the good for any profile of reported values $\mathbf{v} \in V_A \times V_B$, and it belongs to the subsimplex Δ_-^2 , the set of allocation probabilities. The transfer rule $t(\mathbf{v}) = (t_A(\mathbf{v}), t_B(\mathbf{v}))$ specifies the transfer each agent must make to the principal for any profile of reported values $\mathbf{v} \in V_A \times V_B$. A direct mechanism is said to be IC and IR if, for any agent i and value v_i , we have that

$$v_i x_i(v_i) - t_i(v_i) \geq \max \{v_i x_i(z_i) - t_i(z_i), 0\} \text{ for any } z_i \in V_i.$$

This means that there exists a Bayesian Nash equilibrium in which all agents participate and report their types truthfully.⁹ Denote the optimal revenue for any pair of distributions (F_A, F_B) as

$$\begin{aligned} R(F_A, F_B) &= \max_{(x, t)} \mathbb{E}_F [t_A(\mathbf{v}) + t_B(\mathbf{v})] \\ \text{s. t. } &v_i x_i(v_i) - t_i(v_i) \geq \max \{v_i x_i(z_i) - t_i(z_i), 0\} \text{ for any } z_i, v_i \in V_i, \quad i \in \{A, B\}, \end{aligned}$$

where \mathbb{E}_F denotes the expectation over the joint distribution of values F . For values v_i at which F_i admits a density f_i , the *virtual value* of agent i is defined as

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)},$$

and we refer to the difference between the value and the virtual value as the *information rent*. For values v_i at which F_i does not admit a density, we define $\psi_i(v_i) = v_i$.¹⁰

⁹With a slight abuse of notation, we use the same operator to denote both the ex-post rules and the interim rules. The interim allocation probability of agent i with value v_i is $x_i(v_i) = \mathbb{E}_{F_{-i}} [x_i(\mathbf{v})] = \int_{\underline{v}_{-i}}^{\bar{v}_{-i}} x_i(\mathbf{v}) dF_{-i}(v_{-i})$. Similarly, the interim transfer is defined as $t_i(v_i) = \mathbb{E}_{F_{-i}} [t_i(\mathbf{v})] = \int_{\underline{v}_{-i}}^{\bar{v}_{-i}} t_i(\mathbf{v}) dF_{-i}(v_{-i})$. When cumulative distributions are discontinuous on the support, we use Riemann-Stieltjes integrals to calculate the expectations implicitly.

¹⁰In the Supplementary Appendix Section B, we show why with such an adjustment to the definition of virtual values, virtual surplus still coincides with the principal's revenue.

The Value Design Problem We now describe the value design problem at the core of the analysis. Let any pair of distributions $(F_A, F_B) \in \Delta[\underline{v}, \bar{v}]^2$ be referred to as a *value design*. The principal’s value design problem is formulated as

$$\max_{F_A, F_B} R(F_A, F_B) \quad \text{subject to} \quad (F_A, F_B) \in K,$$

where $K \subset \Delta[\underline{v}, \bar{v}]^2$ represents an arbitrary constraint on the set of value designs. We summarize the constraints we consider in Table 1 for the baseline case of two agents.

A key feature common to all constraints we consider is the trade-off between value increases and value reductions. In the case of no spillovers, value can only be redistributed across the different types of agents. Increasing the mass allocated to a higher value necessarily involves reallocating mass to lower values. In the presence of spillovers, increasing the value for one agent leads to a corresponding reduction for the other. In the case of distributional constraints, this implies that a pointwise increase in value must be matched by a pointwise reduction, making the distributional constraints a natural extension of the moment condition.

Restriction Type	Moment Constraint	Distributional Constraint
No Spillovers	$\mathbb{E}_{F_i}[v] \leq k$	$\int_0^v G(t)dt \leq \int_0^v F_i(t)dt$
Spillovers	$\mathbb{E}_{F_A}[v] + \mathbb{E}_{F_B}[v] \leq k$	$H(v) \leq F_A(v) + F_B(v)$
Surplus Bound	–	$G(v) \leq F_A(v)F_B(v)$

Table 1: Design Constraints

Note: The parameter k is an arbitrary constant, H is a measure with mass 2, G is some cumulative distribution function. F_A and F_B are distributions chosen by the principal.

4 Optimal Value Design

We begin by characterizing general properties of optimal value distributions, before turning to the various constraints we consider. Results are proven in the Appendix, Corollaries are proven in Section F of the Supplementary Appendix, the proofs of remarks are omitted.

4.1 Optimal Revenue Properties

The first result establishes that the revenue from the optimal mechanism must be convex in the value design $(F_A, F_B) \equiv F$. This implies that any value design in the interior of the constraint set under consideration will necessarily be suboptimal. Therefore, revenue-maximizing designs must always be extremal and lie on the boundary of the constraint set.¹¹ This has significant implications throughout the analysis, highlighting why designers may prefer asymmetric designs that do not treat agents equally, and why dispersion in valuations can be advantageous.

Lemma 1 (Convexity of Revenue). *Optimal revenue is convex in the joint distribution of values. Specifically, for any $F, F', F'' \in \Delta[\underline{v}, \bar{v}]^2$ satisfying for some $a \in (0, 1)$*

$$F_A(v_A)F_B(v_B) \geq aF'_A(v_A)F'_B(v_B) + (1-a)F''_A(v_A)F''_B(v_B) \text{ for all } v \in [\underline{v}, \bar{v}]^2,$$

we have that $R(F) \leq aR(F') + (1-a)R(F'')$.

Intuitively, the result is true because revenue is the value function of the classical mechanism design problem. Given that this problem is linear in probabilities, its maximum must be convex.

Second, it can never be optimal to select features that induce distributions that are first-order stochastically dominated, a result that dates back to [Devanur et al. \(2016\)](#) and [Hart and Reny \(2015\)](#). A value design F' first-order stochastically dominates value design $F \in \Delta[\underline{v}, \bar{v}]^2$, denoted by $F' \succsim_1 F$, if

$$F'_i(v) \geq F_i(v) \text{ for all } v \in [\underline{v}, \bar{v}] \text{ and } i \in \{A, B\}.$$

Lemma 2 (FOSD ([Devanur, Huang, and Psomas, 2016](#); [Hart and Reny, 2015](#))). *For any $F, F' \in \Delta[\underline{v}, \bar{v}]^2$ such that $F' \succsim_1 F$, we have that $R(F) \leq R(F')$.*

To gain intuition for this result, which plays a key role in our analysis, consider an allocation with two value distributions, F_A and F_B , and an arbitrary mechanism. For simplicity, assume that F_B is supported on a finite set of values (v_1, \dots, v_n) with respective probabilities (p_1, \dots, p_n) . Now, consider an alternative distribution supported on values

¹¹Results in [Yang and Zentefis \(2024\)](#) demonstrate how to characterize extreme points within sets of monotone functions. Since revenue is convex, optimal designs will necessarily correspond to extreme points of the set being considered.

$(v_1, \dots, v_i + \epsilon, \dots, v_n)$, where the only change is that the realization of v_i is increased by ϵ , while the remaining probabilities stay unchanged. This alternative distribution first-order stochastically dominates the original value distribution.

Next, consider the revenue from maintaining the same mechanism as before, with the only modification being an ϵ -increase in the transfer paid by the i -th agent, conditional on winning. This mechanism remains incentive compatible, as no incentive constraint is violated, but it raises higher revenue for the principal. All first-order stochastic dominance shifts can be viewed as a sequence of such changes in a single value. This establishes that it cannot be optimal to select a distribution which is first-order stochastically dominated by some other feasible distribution.

Before proceeding to our constrained value design, observe that any design in which at least one agent values the good at \bar{v} with certainty is revenue-maximizing when the principal faces no design constraints—meaning that the constrained set of distributions is $K = \Delta[\underline{v}, \bar{v}]^2$. These value designs are optimal because surplus cannot exceed the maximal value for the good, \bar{v} . If an agent has this value with certainty, the principal can simply allocate the good to that agent while asking them to transfer \bar{v} , thereby extracting the maximal surplus.

Remark 1. *In any unconstrained optimal value design, at least one agent has a value distribution satisfying $F_i(v) = \mathbb{1}(v \geq \bar{v})$ for all values $v \in [\underline{v}, \bar{v}]$.*

This result highlights the three forces that emerge throughout our analysis. First, the principal aims to increase total surplus, as higher values lead to higher transfers. Second, the principal prefers agents' values to be as predictable as possible, since knowing the exact value reduces the information rents paid to the agents to ensure incentive compatibility. Third, it suffices to increase the value for one agent to \bar{v} , as the principal, knowing the value of that agent with certainty, is able to extract the full surplus from them without ever selling to the other agent. If the principal adjusts the distribution of both agents, then one agent obtains the good with some probability $p \in (0, 1)$, while the other agent receives it with the remaining probability, $1 - p$. This reduces the transfer of the agent who previously obtained the good with certainty, while simultaneously increasing the transfer of the agent who never received the good, by the same amount. Thus, the revenue is the same, regardless of whether the principal increases the value for one or more agents to \bar{v} . This demonstrates that inequality in valuations of different agents is irrelevant to the principal.

4.2 No Spillovers

We now turn to the analysis of the optimal distribution when the features of the good affect only agent A , while the valuation of agent B remains unchanged.

Moment Condition We begin by imposing a moment condition on the expected value, specifically that it cannot exceed some constant k . It immediately follows from Lemma 2 that this bound must hold with equality. If it did not, there would exist a value distribution that first-order stochastically dominates, yielding higher revenue. Therefore, for any optimal distribution, we must have $\mathbb{E}_F[v] = k$. From Lemma 1, it follows that the optimal distribution must be extremal. Since we only restrict the average value without imposing a pointwise constraint on the value distribution, this implies that atomic distributions will be optimal.

It is thus useful to define a class of binary distributions that take values at a point $h \in [k, \bar{v}]$ and \underline{v} , with a mean of k . Formally, for any $h \in [k, \bar{v}]$, let the probability distribution P^h satisfy

$$P^h(v) = \begin{cases} \frac{h-k}{h-\underline{v}} & \text{if } v = \underline{v}, \\ \frac{k-\underline{v}}{h-\underline{v}} & \text{if } v = h. \end{cases} \quad (1)$$

Here, $P^h(v)$ denotes the probability that the value equals v . Let F^h denote the cumulative distribution associated with P^h . The distribution F^k places all the mass at the mean k . The distribution $F^{\bar{v}}$ describes a distribution with mass at the maximal value \bar{v} and at zero. We refer to this distribution as the *maximally spread* distribution, and this turns out to be the optimal distribution for the principal.

Proposition 1. *Let $\underline{v} = 0$. The value design $(F_A^*, F_B) = (F^{\bar{v}}, F_B)$ is optimal among all value designs (F_A, F_B) satisfying $\mathbb{E}_{F_A}[v] \leq k$. This design is uniquely optimal when $F_B \neq F^{\bar{v}}$. For $F_B = F^{\bar{v}}$, any design with $F_A^* = F^h$, $h \in [k, \bar{v}]$ is also optimal.*

Our result depends on the value distribution of agent B . If B possesses some arbitrary distribution, but not the maximally spread distribution, then for agent A , the maximally spread distribution with mean k is the unique optimal distribution.

The optimal distribution maximizes surplus and reduces the information rents for agent A to zero. This holds for any bimodal distribution with an atom at one strictly positive value and mass at zero. The optimality of the maximal spread arises due to the

presence of the second agent, B .¹² Agent B also competes for the good, and the size of the transfer depends on the probability of receiving the good. The higher the allocation probability for the agent, the higher the transfer. The maximal spread for agent A maximizes the probability of receiving the good for agent B . Agent B receives the good if their virtual value is positive and exceeds the virtual value of A . The probability of exceeding A 's value increases with the probability of A having zero value. This probability is given by $\frac{h-k}{h}$, and it is increasing in h , the strictly positive value with an atom. Therefore, the bimodal distribution that assigns mass to the maximal value and zero, with constant mean, places the greatest probability on zero. This makes it the uniquely optimal distribution.

The mechanism associated with this value design is simple: first, offer the good to agent A at price \bar{v} , and if A rejects, offer the good to agent B .

If B 's distribution is the maximal spread, then multiple distributions are optimal. Any of these distributions places an atom at exactly one positive value (which must exceed the mean k) and at zero, such that the mean equals k . Thus, any of the probability distributions described in (1) for $h \geq k$ are optimal. Now, the principal awards the good to agent B if B 's value equals the maximal value \bar{v} . Otherwise, A receives the good. The principal can then extract the entire expected value from A for any of the described distributions, making them optimal.

Our result relies on the assumption that the minimal value \underline{v} equals zero. When the minimal value is zero, the principal never allocates the good to the agent. If the minimal value is strictly positive, it may also be optimal to allocate the good to the agent if their realized value is minimal. This fundamentally changes the considerations at play, as incentive compatibility of the mechanism needs to be ensured. The information rents are now strictly positive, and their extent depends on the difference between the maximal and the minimal value. Whether it is optimal to allocate the good to the agent if the realized value is minimal also depends on B 's distribution. If, for instance, B 's distribution assigns all mass to the mean, then the principal will only allocate the good to A if their value is maximal, and the bimodal distribution remains optimal.

From a mechanism design perspective, the result is perhaps surprising since it demonstrates that an optimal design can maximize dispersion in values. But by maximizing overall dispersion, the principal simultaneously minimizes the variation in valuations (and

¹²Indeed, without the second agent, any distribution that assigns mass to exactly one strictly positive value with expected value k is optimal.

consequently the information rents) for non-excluded types, who obtain the good with positive probability, thereby increasing revenue.

If the principal can add features to the good that independently affect the value distribution of all agents, then bimodal distributions with mass at the maximal value \bar{v} and mass at $\underline{v} \geq 0$ remain optimal.

Proposition 2. *A value design, (F_A^*, F_B^*) , is optimal among all designs satisfying $\mathbb{E}_{F_i}[v] \leq k$ for all i if and only if for agents $i \in \{A, B\}$ and $j \neq i$, we have that $(F_i^*, F_j^*) = (F^{\bar{v}}, F^h)$, where $h = k$ if $\underline{v} > 0$ and $h \in [k, \bar{v}]$ if $\underline{v} = 0$.*

In contrast to Proposition 1, this result holds for any minimum value \underline{v} . As the principal, by creating the appropriate good, can now affect the distributions of all agents independently, they select a good that induces the maximal spread for one agent, while the second agent's distribution places all mass at the mean. This is optimal for any minimal value \underline{v} . If the minimal value equals zero, then any bimodal distribution with mean k , and mass at a single strictly positive value and zero, is also optimal.

If the minimal value is positive, having two bimodal distributions is not optimal. To see this, suppose, to the contrary, that both agents possess the maximal spread. With probability $\left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)^2$, both agents have the minimal value, and with the remaining probability, at least one agent has the maximal value. The expected surplus is then

$$\left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)^2 \underline{v} + \left(1 - \left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)^2\right) \bar{v}.$$

Instead, if an agent i possesses the maximal spread, while the other agent j has all the mass at the mean, then agent i receives the good if their valuation is the maximal value. Otherwise, agent j attains the good. This results in the same surplus as if both agents had the maximal spread,

$$\left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right) k + \left(1 - \left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)\right) \bar{v} = \left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)^2 \underline{v} + \left(1 - \left(\frac{\bar{v}-k}{\bar{v}-\underline{v}}\right)^2\right) \bar{v}.$$

However, surplus does not equal revenue because incentive compatibility is crucial. If both agents possess a bimodal distribution, it is not possible for the principal to extract the entire surplus. To see this, observe that if the principal sells to all high-value types at a price equal to \bar{v} , they cannot sell the good when both buyers have a minimal valuation,

as this would violate incentive compatibility.¹³ Consequently, the seller cannot achieve a revenue that equals surplus with these distributions.

If the other agent has all the mass at the mean, the principal can set a different price for this agent, namely k , and extract the average value from them when they do not trade with the high-value agent. Therefore, the revenue is higher when one distribution is the maximal spread and the other places all mass at the mean, as compared to the case where both distributions are maximal spreads.

If the minimum value equals zero, the optimal strategy is for one agent to have a maximally spread distribution, while the other agent can have any bimodal distribution with mass at one strictly positive value and zero, such that the distribution's expected value is k . The principal can then sell the good in a manner similar to before. They award the good to the agent with the maximal spread if they have the maximal value (if both agents have the maximal value, they can simply choose one of them) at a price of \bar{v} . If the realized value is not maximal, the principal gives the good to the other agent at a price $h \in [k, \bar{v}]$, thereby extracting the entire expected value.

We consider the generalization of Proposition 2 to $n + 1$ agents. In any optimal design for these settings, n agents exhibit a maximally dispersed binary distribution, while the remaining agent's distribution follows the structure described in Proposition 2. This means that for the remaining agent all mass is optimally placed at the mean if $\underline{v} > 0$, while all the mass is placed at zero and exactly one strictly positive value when $\underline{v} = 0$.

Interestingly, it is possible to extract the expected value from each agent if $\underline{v} = 0$ and \bar{v} diverges to infinity. In this case, revenue from the optimal design approaches $(n + 1)k$. As $\underline{v} = 0$, the probability that an agent has value \bar{v} amounts to $p = k/\bar{v}$. For a large maximal value \bar{v} , p is close to 0. As a result, the likelihood that more than one agent has a high value \bar{v} is negligible and we only take into account the probability that one agent possesses the high value \bar{v} . Then, revenue is bounded below by $(n + 1)p\bar{v} = (n + 1)k$.¹⁴

In general, it is not obvious that the principal can implement features that affect exactly one agent differentially relative to the other agents. Rather, the principal may only select features that affect all agents in the same manner. If this is the case, the principal benefits from having all agents possess the maximally spread value distribution. As long as one of the agents has a realized value of \bar{v} , the principal extracts the entire

¹³Otherwise, every type would have an incentive to pretend to be a low-value type in order to receive the good at a lower price.

¹⁴To be specific, revenue amounts to $\sum_{s=1}^{n+1} \binom{n+1}{s} p^s (1-p)^{n+1-s} \bar{v}$ and converges to $(n + 1)k$ as $p \rightarrow 0$.

surplus by setting a price equal to \bar{v} . But the principal fails to sell the good when all agents have value \underline{v} , which happens with probability $P^{\bar{v}}(\underline{v})^{n+1}$. This probability converges to zero as $n \rightarrow \infty$. Therefore, for a sufficiently large number of potential buyers, the principal is only marginally hurt by selecting symmetric, but divisive designs.

Intuitively, the principal prefers risky value distributions for most agents. This is the case because the risk from an agent having a low value can be hedged by selling to another agent. This preference for risk was, to some extent, expected given the convexity of the principal's revenue over value designs. This convexity implies that the principal is risk-loving over value designs.

Second-Order Stochastic Dominance It is important to note that Proposition 1 also establishes that $F^{\bar{v}}$ is the optimal design relative to any distribution that is second-order stochastically dominated by an arbitrary distribution G with mean $\mathbb{E}_G[v] = k$. This follows from two insights. First, any distribution F_A that is second-order stochastically dominated by G satisfies

$$\int_0^v G(t)dt \leq \int_0^v F_A(t)dt \quad \text{for all } v \in [\underline{v}, \bar{v}], \quad (2)$$

and, therefore, has a lower mean than G . Therefore, F_A satisfies our earlier moment condition $\mathbb{E}_{F_A}[v] \leq k$. Second, $F^{\bar{v}}$ is second-order stochastically dominated by G . Thus, $F^{\bar{v}}$ is also the optimal distribution among all F_A satisfying the constraint in (2).

Corollary 1 (to Proposition 1). *Let $\underline{v} = 0$. The value design $(F_A^*, F_B) = (F^{\bar{v}}, F_B)$ is optimal among all value designs (F_A, F_B) satisfying (2). It is the unique optimal design if $F_B \neq F^{\bar{v}}$.*

The result hinges on the assumption that $\underline{v} = 0$. To see why we require this assumption, let $F_B = F^{\bar{v}}$. We know from Proposition 2 that in this case, for $\underline{v} > 0$, placing all the mass at the mean k for agent A would be optimal. This highlights that if the lowest value \underline{v} is strictly positive, then it may be optimal to choose a distribution that second-order stochastically dominates the given distribution G . In general, for a strictly positive minimal value $\underline{v} > 0$, the optimal distribution for A depends on B 's and cannot readily be pinned down.

However, for $\underline{v} = 0$ and $F_B = F^{\bar{v}}$, any distribution F^h that is bimodal and assigns mass to one strictly positive value and additionally, is second-order stochastically dominated

by G is optimal. If B 's distribution does not correspond to the maximal spread $F_B \neq F^{\bar{v}}$, while the minimal value \underline{v} equals zero, the maximal spread becomes the unique optimum for agent A .

In sum, in the absence of spillovers, the principal will aim to create divisive, bimodal value designs in which agents either highly value the good or do not value it at all.

4.3 Spillovers

We turn to settings where an attribute shapes the values of agents jointly. As was the case with no spillovers, if the principal could design a good that was more appealing to all agents, they would do so. We therefore focus on scenarios in which various designs reallocate value across agents—meaning that making the good more appealing for one agent will necessarily make the good less appealing for other agents.

Moment Condition We begin by considering a constraint that restricts only the sum of means of the value distributions, mirroring the mean bound in the no spillover case. We first assume that the principal can achieve any value design (F_A, F_B) such that

$$\mathbb{E}_{F_A}[v] + \mathbb{E}_{F_B}[v] \leq k, \quad (3)$$

for some constant $k \in [2\underline{v}, 2\bar{v}]$. Since the value for each agent is bounded below by \underline{v} , k must exceed $2\underline{v}$. If the mean k were greater than $2\bar{v}$, then it would always be possible for the principal to allocate mass one to \bar{v} for both agents and extract the full surplus. For $k \in [2\underline{v}, 2\bar{v}]$, in any optimal design, the principal leaves one agent with the smallest possible value while allocating all remaining value to the other agent. The optimal design allows the principal to extract the entire value from the agents by eliminating their information rents. To state the result, let $F_{k-\underline{v}}^h$ denote a binary distribution with atoms at h and \underline{v} and with mean $k - \underline{v}$.¹⁵

Proposition 3. *The value design (F_A^*, F_B^*) is optimal among all designs satisfying (3):*

1. *For $k \in [2\underline{v}, \underline{v} + \bar{v})$ if and only if $F_i^*(v) = F_{k-\underline{v}}^h(v)$ with $h \in [k - \underline{v}, \bar{v}]$ for some agent $i \in \{A, B\}$, while $F_j^*(v) = \mathbb{1}(v \geq \underline{v})$ for all values $v \in [\underline{v}, \bar{v}]$ for agent $j \neq i$;*
2. *For $k \in [\underline{v} + \bar{v}, 2\bar{v}]$ if and only if $F_i^*(v) = \mathbb{1}(v \geq \bar{v})$ for all values $v \in [\underline{v}, \bar{v}]$ for some agent $i \in \{A, B\}$.*

¹⁵The probability distribution associated with $F_{k-\underline{v}}^h$ corresponds to the probability distribution defined in (1), except for the mean being $k - \underline{v}$, instead of k .

If $k \geq \underline{v} + \bar{v}$, the principal assigns the maximal value \bar{v} to agent i with certainty. The remaining value is assigned to agent j . However, it is completely irrelevant how the value is assigned to j , as agent j never receives the good. As in our benchmark case, where the principal could not exceed the valuation \bar{v} for each agent, the principal only needs one agent to have a high valuation with certainty. They can then extract the entire surplus, eliminating the information rents, by simply setting a price equal to \bar{v} and always assigning the good to the high-value agent.

If $k < \underline{v} + \bar{v}$, it is not feasible to assign the maximal value with certainty to one agent. Therefore, the principal prefers a design where agent i possesses an expected value of $k - \underline{v}$, while agent j 's expected value is reduced to the lower bound \underline{v} with certainty. For agent i , multiple distributions are optimal, as long as they are bimodal with mass at some $h \in [k - \underline{v}, \bar{v}]$ and \underline{v} . In these instances, full surplus extraction remains feasible. To see this, consider the following mechanism. The principal sets the price for agent i equal to h , and if i is willing to pay this price, awards the good to them. For agent j , the principal sets a price equal to \underline{v} and allocates the good to them if i did not pay the higher price. This extracts the entire surplus, as the expected revenue equals

$$\frac{k - \underline{v}}{h - \underline{v}}h + \left(1 - \frac{k - \underline{v}}{h - \underline{v}}\right)\underline{v} = k.$$

By the same logic as in Proposition 2, there cannot be any other distribution where revenue matches surplus, as all other distributions entail either a loss in surplus or a strictly positive information rent.

The principal induces sharp differences in how agents value the good—for instance, by designing goods that cater to only one of the agents. Similar insights would extend to an n -agent setting, where all the value is allocated to a single agent. The optimality of divisive value designs, in which the good caters to only one agent under this rather lax constraint, raises questions about whether divisive goods would also be optimal when more stringent constraints are imposed.

Distributional Constraints Therefore, we consider pointwise distributional constraints and begin with additive constraints.¹⁶ To define this class of constraints, consider an arbitrary measure defined on the interval $[\underline{v}, \bar{v}]$ with mass two on that interval. Let $H(v)$

¹⁶These constraints reflect a weak form of linearity of design costs in a dual costly design problem, as discussed in the Supplementary Appendix Section B.

denote the associated cumulative distribution.¹⁷ We ask how the principal would design the two value distributions (F_A, F_B) if their sum had to be bounded above pointwise by some cumulative measure H , i.e., if

$$F_A(v) + F_B(v) \geq H(v) \quad \text{for all } v \in [\underline{v}, \bar{v}]. \quad (4)$$

We find that the principal maximizes revenue by creating two maximally differentiated value distributions. To state the result, it is useful to define the median v^M for the measure H , as the smallest value \hat{v} such that $\int_{\underline{v}}^{\hat{v}} dH(v) \geq 1$.

Proposition 4. *Value design (F_A^*, F_B^*) is optimal among all designs satisfying (4) if and only if for some agent $i \in \{A, B\}$ and $j \neq i$,*

$$\begin{aligned} F_i^*(v) &= H(v) - 1 && \text{if } v \in [v^M, \bar{v}], \\ F_j^*(v) &= \min\{H(v), 1\} && \text{if } v \in [\underline{v}, v^M]. \end{aligned}$$

Distributions where one agent's values lie below the median of H and the other agent's values lie above the median of H yield higher revenue than any other design fulfilling the constraint. Such value designs have two benefits because they can accomplish at once the dual objectives of the principal—namely, surplus maximization and information rent minimization. Put differently, the two objectives are aligned at the optimal design, (F_A^*, F_B^*) . These distributions maximize surplus, $\mathbb{E}[\max\{v_A, v_B\}]$. Due to the convexity of the surplus operator, it is optimal to generate the highest possible expected value for one agent (again, one agent with a high value is all that the principal needs). Additionally, the design (F_A^*, F_B^*) minimizes agents' information rents by narrowing the support of both agents' distributions.

To establish the optimality of maximally divisive designs, we first establish that the constraint must bind at every point because revenue is increasing in first-order dominance—it is never optimal to destroy surplus. This is an immediate consequence of Lemma 2. Therefore, we can restrict attention to the setting where $F_A(v) + F_B(v) = H(v)$.

Note that we are not placing any restrictions on the properties of $H(v)$. It may be differentiable, contain atoms, or display gaps in the support. We begin by considering the subsets of the support of H where H is differentiable, assuming these exist. We show that on these subsets, F_A and F_B must have disjoint supports. This follows from

¹⁷Formally, let \mathcal{H} denote the measure and define $H(v) = \mathcal{H}([\underline{v}, v])$.

Lemma 1. Suppose, to the contrary, that there exist values v (with positive measure) for which the density $h(v) = \frac{dH(v)}{dv}$ is split between the two agents. Then, we can show that this split density is a convex combination of two other designs fulfilling (4) in which densities are more extreme. To see this, partition the set of values for which density is split into two subsets with equal measure, W' and W'' . For the subset W' , remove some density ϵ for each value from A and allocate it to B . For the other subset, W'' , remove the same density for each value from B and allocate it to A , such that both are still well-defined cumulative distributions. This leads to F'_A and F'_B . Now construct another two distributions, F''_A and F''_B , by increasing the density for A in the set W' by ϵ , and decreasing it in the set W'' . Consequently, the density of B must decrease for each value in the set W' , while it increases for all the values in W'' . We can then write F'_A and F''_A as

$$\begin{aligned} F'_A(v) &= F_A(v) - \epsilon(v - \hat{v}), \\ F''_A(v) &= F_A(v) + \epsilon(v - \hat{v}), \end{aligned}$$

where $v \in W'$ and \hat{v} is the lower bound of set W' . Then, $F_A(v) = \frac{1}{2}F'_A(v) + \frac{1}{2}F''_A(v)$. As the same argument extends to F_B , F'_B , and F''_B , and every value in W' and W'' , we have established that F_A and F_B are a convex combination of some other value designs. Consequently, by Lemma 1, F_A and F_B cannot be optimal, meaning that any value design that splits densities does not maximize revenue. Therefore, at the solution of this problem, the value distributions of the two agents must have supports that are disjoint.

We then establish that, among all value designs with disjoint supports, revenue is maximized by the most divisive design (even when the measure H is atomic). Consider a design with disjoint support where the support for B 's value distribution is $[\underline{v}, w_1] \cup [w_2, w_3]$, while that for A is $[w_1, w_2] \cup [w_3, \bar{v}]$. For such a design, the principal must ensure the incentive compatibility of the mechanism, which requires, for instance, deterring the agents from pretending to have a low value. Thus, the principal must prevent agent B from reporting \underline{v} as their value, even when their realized value v_B is well above the median. Similarly, the principal must deter agent A from reporting w_1 as their value, even when their realized value v_A is close to \bar{v} . However, deterring such large changes in values will require considerable compensation to the agents in the form of information rents.

As a result, the principal benefits from shrinking the supports: giving A all values above the median and giving B all values below the median. In this resulting design,

fewer deviations need to be deterred, since A 's largest deviation will be to $v^M > w_1$, and because B will no longer have values above the median. In such a design, the principal will pay less to elicit any given value. Nevertheless, A 's revenue contribution will increase, as they will pay all transfers associated with values above the median (which tend to be larger), while B 's revenue contribution will decrease.

The optimality of maximal value differentiation generalizes to n agents. In this case, only the distribution of the two most favored agents is pinned down. One agent receives all the highest values with measure one, while another agent attains the remaining highest values with measure one. Only these two agents, with the highest values, receive the good with positive probability. Therefore, the value designs for all other agents are arbitrary, subject to them fulfilling the constraint.

5 The Role of Surplus

All of the designs we considered so far have the feature that they increase surplus. This raises the question whether the optimal design is still divisive if it does not lead to a surplus gain. And can it ever be optimal to create designs that reduce surplus?

5.1 Fixed Surplus

We begin by limiting the distribution of surplus in the design. Namely, we fix a cumulative distribution function G and ask what the optimal design is if the resulting surplus distribution $F_A F_B$ is first-order stochastically dominated by G , formally

$$F_A(v)F_B(v) \geq G(v) \quad \text{for all } v \in [\underline{v}, \bar{v}]. \quad (5)$$

This constraint entirely shuts down the surplus generation motive of optimal designs. Therefore, the remaining trade-off for the designer is one between minimizing information rents and maximizing competition. The technical challenge with this problem lies in the fact that both the objective and the constraints are quasi-convex, precluding conventional optimality insights.

One previously established insight carries over, though. By Lemma 2, optimal designs must always generate full surplus. Formally, a design F generates the full surplus only if

$F_A(v)F_B(v) = G(v)$ for all non-excluded values $v \in [\underline{v}, \bar{v}]$.¹⁸

Corollary 2 (to Lemma 2). *There always exist optimal value designs (F_A^*, F_B^*) satisfying (5) in which $F_A^*(v)F_B^*(v) = G(v)$ for all $v \in [\underline{v}, \bar{v}]$. Furthermore, any other optimal design satisfies $F_A^*(v)F_B^*(v) = G(v)$ for all non-excluded values $v \in [\underline{v}, \bar{v}]$.*

The intuition is straightforward: if the constraint were slack for some design F at some non-excluded value v , then there exists a design F' that satisfies the constraint and first-order stochastically dominates F . Since revenue is monotonic in first-order dominance shifts, this would preclude the optimality of F . We can restrict attention to non-excluded values, as for excluded values, it is irrelevant whether constraint (5) holds with equality.

We therefore focus on the class of designs that generate full surplus. Given condition (5), the maximally divisive design F^+ satisfies $(F_i^+(v), F_j^+(v)) = (G(v), 1)$ for all $v \in [0, \bar{v}]$ and $j \neq i$. Whether maximal divisiveness remains optimal depends on the minimal value \underline{v} and the given distribution G .

Proposition 5. *Let $\underline{v} = 0$. Then, any maximally divisive design F^+ is revenue-minimizing among designs satisfying $F_A F_B = G$.*

Let $\underline{v} > 0$, $G(v) = p$ for $v \in [\underline{v}, \bar{v}]$, and $G(\bar{v}) = 1$. Then, the maximally divisive design F^+ is revenue-maximizing and uniquely optimal.

If the minimal value \underline{v} is strictly positive and the primitive surplus distribution G equals a binary distribution with mass at \underline{v} and \bar{v} , the maximally divisive design is optimal. The principal can extract the full surplus by offering the good to the agent with distribution G at price \bar{v} . If the agent does not purchase the good, then the principal sells the good to the other agent at price \underline{v} . In contrast, any other design requires that both agents can have value \bar{v} . But then full surplus extraction is not possible, as the principal now faces agents with two different, strictly positive values.¹⁹

In stark contrast, maximally divisive designs are revenue-minimizing if $\underline{v} = 0$. In this case, the maximally divisive design allocates to one agent zero value. It thus reduces the problem to one of a single buyer as only one buyer is left with a strictly positive expected valuation. In any single-item, single-agent setting, the optimal mechanism is a posted price. With a fixed surplus distribution, any posted price generates identical revenue

¹⁸A value $v \in [\underline{v}, \bar{v}]$ is non-excluded if at the optimal mechanism, $x_A(v|F^*) + x_B(v|F^*) > 0$.

¹⁹Either, the principal has to provide information rents to at least one agent, or they can choose to not allocate the good to agents with low value. Alternatively, they could pool them and set a price equal to \underline{v} . Regardless, they cannot extract the entire surplus.

regardless of the design chosen, since the principal sells whenever someone is willing to pay and this probability remains unchanged across all designs. However, when both agents possess some strictly positive expected value and have private information about their valuation for the good, posted prices cannot be optimal. In this case, the principal would want the agents to compete for the good. Therefore, the principal would select a different mechanism, which increases their revenue—making maximal differentiation revenue-minimizing.

Threshold Designs This raises the question what type of designs are now optimal. One potential candidate are minimally divisive designs, which assign to both agents the same distribution. Formally, a minimally divisive design F^- satisfies $(F_A^-(v), F_B^-(v)) = (\sqrt{G(v)}, \sqrt{G(v)})$ for all $v \in [\underline{v}, \bar{v}]$. Another candidate are threshold designs. A w -threshold design $F^{[w]}$ satisfies for some $w \in [\underline{v}, \bar{v}]$ and some $i \neq j$:

$$(F_i^{[w]}(v), F_j^{[w]}(v)) = \begin{cases} (G(v)/G(w), G(w)) & \text{if } v < w \\ (1, G(v)) & \text{if } v \geq w \end{cases}$$

The threshold design assigns agent i all possible low values (below threshold w), and to agent j value \underline{v} with probability $G(w)$ and all possible high values (above w), while ensuring the constraint $F_A F_B = G$ is satisfied. Intuitively, it captures an intermediate level of divisiveness. Agent j receives the high values of G and mass at \underline{v} , which generates a high level of dispersion in values. At the same time, low values are concentrated. This concentration for agent i in combination with allocating the high values to agent j (with the exception of the mass at \underline{v}) generates differentiation in values.

The dispersion and differentiation of these threshold designs offer two key advantages: First, they restrict possible deviations (agent i can only deviate to values below w while agent j can only deviate to values above w), reducing information rents. Second, they maintain competition between high types of agent i and low types of agent j , reducing the cost of truthful revelation.

Proposition 5 does not only show that for the minimal value \underline{v} equal to zero, maximal designs are no longer optimal, but also that the optimal designs depend on the exact surplus distribution G . We therefore consider two other specific distributions, namely (i) ternary distributions, that assign mass to exactly three values and (ii) the uniform distribution.

For ternary distributions and the minimal value \underline{v} equal to zero, w -threshold designs are always optimal.

Proposition 6. *Let $\underline{v} = 0$. When $G(v) = p$ for $v \in [\underline{v}, w)$, $G(v) = q > p$ for $v \in [w, \bar{v})$, and $G(\bar{v}) = 1$, a w -threshold design $F^{[w]}$ is uniquely optimal.*

Threshold designs are particularly attractive for ternary surplus distributions because by selecting a suitable design the principal can guarantee that agents have exactly one strictly positive value. This can be done by setting $F_A^{[w]}(0) = p/q$, $F_A^{[w]}(w) = 1$, while $F_B^{[w]}(0) = F_B^{[w]}(w) = q$, $F_B^{[w]}(\bar{v}) = 1$. Screening becomes inexpensive since agents can be asked to pay their full value conditional on trade.

Our result also implies that threshold designs can outperform any other proportional design, which for some $\gamma \in [0, 1]$ satisfies $(F_A(v_A), F_B(v_B)) = (G(v_A)^\gamma, G(v_B)^{1-\gamma})$. This has been the approach taken by [Cantillon \(2008\)](#). She shows that symmetric distributions generate higher revenue relative to asymmetric ones in the class of proportional designs, with a specific emphasis on uniform distributions. However, the symmetric distributions are not necessarily optimal if the mechanism can also be adjusted.

To see this, consider the uniform surplus distribution $G(v) = v$ on $[0, 1]$.²⁰ A maximally divisive design $F^+ = (v, 1)$ leads to a posted price of $1/2$ and revenue of $1/4$. For the minimally divisive design $F^- = (\sqrt{v}, \sqrt{v})$ the optimal mechanism is a second-price auction with reserve price $4/9$ and revenue of $43/162 \approx 0.265$. We contrast this revenue with the revenue achieved through a threshold design with $w = 1/2$,

$$(F_A^{[w]}(v), F_B^{[w]}(v)) = \begin{cases} (2v, 1/2) & \text{if } v < 1/2 \\ (1, v) & \text{if } v \geq 1/2. \end{cases}$$

This design generates revenue exceeding $5/16 \approx 0.313$. The principal achieves this revenue by offering the good to agent B at price $1/2$. If agent B is unwilling to pay, the principal sells to agent A at price $1/4$.²¹ Therefore, the threshold design outperforms minimally and maximally divisive designs, generating higher revenue.²²

²⁰We provide the following calculations and derivations in the Supplementary Appendix Section C.

²¹This mechanism is not optimal for the threshold design, but guarantees revenue of $5/16$. It does so by selling to B at price $1/2$ when the highest value exceeds $1/2$, while selling to A at a price of $1/4$ when the highest value is between $1/4$ and $1/2$. This leads to a revenue of $5/16$ since the probability that A wins the object amounts to $1 - G(1/2) = 1/2$, while the probability that B wins amounts to $G(1/2) - G(1/4) = 1/4$.

²²The optimal threshold design with $w \approx 0.648$ achieves a revenue of approximately 0.336 , which is substantially better than both maximally and minimally divisive alternatives.

The advantages of the threshold design that were evident for the ternary and uniform distributions apply more broadly. The following result establishes that when the minimally divisive designs lead to significant exclusion, threshold designs always outperform the minimally differentiated one—making some divisiveness optimal.

We require the minimally differentiated design to be *weakly regular*. Weak regularity holds if G is differentiable and the virtual value $\psi^-(v)$ of \sqrt{G} is increasing when non-negative, with

$$\psi^-(v) = v - \frac{2(\sqrt{G(v)} - G(v))}{g(v)}.$$

This condition relaxes the usual regularity assumption by requiring virtual values to increase only on a portion of their graph.²³ Namely, the virtual values must be increasing for non-excluded values.

Weak regularity ensures that allocating the good to the highest non-negative virtual value is always optimal. For the minimally divisive design, this optimal mechanism reduces to a second price auction with a reserve price r^- . The allocation rule associated to such a mechanism $(x(r^-), t(r^-))$ satisfies $x_i(v_i, v_j | r^-) = \mathbb{1}(v_i > \max\{v_j, r^-\})$ for any $i \neq j$. The reserve price in the optimal auction is unique and well-defined by weak regularity. It can be found by solving $\psi^-(r^-) = 0$.

To calculate the revenue in this auction, we require the distribution of the minimum value. For any design F satisfying $F_A F_B = G$, the distribution of the minimum value is given by:

$$J(v) = 1 - (1 - F_A(v))(1 - F_B(v)) = F_A(v) + F_B(v) - G(v).$$

Consequently, $J^-(v) = 2\sqrt{G(v)} - G(v)$ denotes the distribution of the minimum value in the minimally differentiated design.

Proposition 7. *If for some G , the minimally differentiated design is weakly regular, and for some w ,*

$$w(G(r^-) - G(w)) > \int_{r^-}^{\bar{v}} (v - r^-) dJ^-(v), \tag{6}$$

then the w -threshold design raises more revenue than the minimally divisive one.

²³Formally, a distribution F is weakly regular if it is differentiable and if for any $v \in [\underline{v}, \bar{v}]$ such that $\psi^-(v) \geq 0$ and any $v' > v$, we have that $\psi^-(v') > \psi^-(v)$.

If $G(r^-)$ is large, then there is substantial exclusion in the optimal mechanism for minimally divisive designs. However, minimally divisive designs generate revenue beyond the reserve price. If this additional gain in values, captured by the right-hand side of (6), is small, then the advantage of the minimally divisive design relative to the threshold design is limited. In contrast, the threshold design leads to reduced exclusion. It segments excluded agents and sells to them at different prices, thereby generating higher revenue.

Threshold designs are by their very nature asymmetric and so is the mechanism used to allocate the good. Suppose now that we fix the optimal mechanism for the minimally divisive design. Can another design then outperform the minimally divisive one? The answer turns out to be no.

Minimally divisive designs lead to a distribution of the minimum value that first-order stochastically dominates the distribution of minimal values in any other design. This immediately implies that when we fix the standard mechanism that is optimal for the minimally divisive design, then no other design yields higher revenue.

Proposition 8. *The distribution of the minimum value in the minimally divisive design, F^- , first-order stochastically dominates the distribution of the minimum in any other design satisfying $F_A F_B = G$:*

$$J^-(v) < J(v) \text{ for all } v \in [\underline{v}, \bar{v}].$$

In any standard auction $(x(r), t(r))$ with reserve price r , revenue from the minimally divisive design exceeds revenue from any other design satisfying $F_A F_B = G$.

Therefore, any gains from divisiveness must come from tailoring the mechanism to the specific design chosen by the principal. The appeal of threshold designs lies precisely in their ability to facilitate mechanisms that extract significant surplus from both agents while minimizing allocative inefficiencies. But these mechanisms will have to treat agents differently.

In sum, threshold designs, which introduce some asymmetries both between different agents and types, increase revenue if exclusion occurs. If there is no exclusion under minimal divisiveness, then other designs, including the minimally divisive one, can be optimal. Our result thus highlights that while some divisiveness increases revenue when surplus is constant, as evidenced by the threshold designs, the surplus generation motive is generally an important feature of designs.

5.2 The Surplus-Information Trade-Off

Even though increasing surplus is generally valuable to the principal, this does not mean that a design that reduces surplus cannot be optimal. A decrease in surplus can increase revenue— as long as the reduction in information rents is even higher. This is summarized in Corollary 3.

Corollary 3 (to Proposition 1). *Let $\underline{v} = 0$. Consider any value design (F_A, F_B) in which $\mathbb{E}_{F_A}[v] = \mathbb{E}_{F_B}[v] = k$, $F_B \neq F^{\bar{v}}$, and $\mathbb{E}_{F_A}[\max\{\psi_A(v), 0\}] < m < k$. If so, value design (F_A, F_B) raises less revenue than the value design $(F_m^{\bar{v}}, F_B)$.*

This result builds on Proposition 1 and establishes that the principal always benefits from reducing the mean of one agent’s distribution if they can make their value sufficiently predictable by selecting the binary distribution $F_m^{\bar{v}}$, with mean m and mass at zero and \bar{v} .²⁴

With the binary distribution $F_m^{\bar{v}}$, the principal can extract the expected value m from agent A .²⁵ For any other distribution, F_A , the principal is only able to extract the expected virtual value. By construction, the expected value of the binary distribution is higher than the expected virtual value associated with F_A and therefore, the reduction in surplus is profitable for the principal. This result relies on the minimum value \underline{v} being zero, and on F_B being some distribution other than the maximal spread, $F^{\bar{v}}$. If F_B is the maximal spread, multiple distributions F_A are optimal, as described in Proposition 1. Whether it is then optimal to reduce surplus depends on the given F_A . If, for instance, F_A is such that it only allocates mass at the mean, then no improvement can be achieved.

Therefore, reductions in surplus can benefit the principal if they are accompanied by considerable decreases in information rents.

6 Conclusion

Our results establish that the divisiveness of a good is generally a valuable feature. Divisiveness raises the value of a good for some agents, while diminishing it for others. This leads to an increase in surplus and simultaneously, a reduction in information rents, which both increase revenue and thus benefit the principal.

²⁴For the associated probability distribution, see expression (1), now with mean m instead of k .

²⁵This can be achieved through the same mechanism as outlined in the discussion after Proposition 1.

When surplus is fixed, then our results are more nuanced in terms of divisiveness. Divisiveness can still be optimal, but so can other designs, potentially even the minimally divisive one. This hints at the importance of surplus generation. However, it is not necessarily the case that a reduction in surplus is always suboptimal—it can be beneficial if it is exceeded by a reduction in information rents.

Our analysis has abstracted from design costs and correlations in values. In the Supplementary Appendix Sections [D](#) and [E](#), we show how to extend our insights to these settings, and explain why correlation in values would simplify our analysis, while confirming our results.

Finally, when it comes to product design, our optimal distributions can serve as guidance that can be applied in any context. A product designer should aim for attributes that make goods divisive.

A Proofs

Proof of Lemmas 1 and 2: Convexity and FOSD We begin by deriving a key revenue optimality property of ex-post optimal mechanisms. We then exploit this optimality property to show that revenue is monotone in FOSD order and convex in the joint distribution of values.

Optimality Fix a value design $F \in \Delta(V_A) \times \Delta(V_B)$. Consider any direct, IC, IR mechanism $(x, t) : V_A \times V_B \rightarrow \Delta_-^2 \times \mathbb{R}^2$, with $x_i(\mathbf{v})$ and $t_i(\mathbf{v})$ denoting respectively i 's likelihood of winning the good and i 's payment conditional on reports $\mathbf{v} \in V_A \times V_B$. Denote the ex-ante revenue of such a mechanism when the value design is F by

$$\bar{R}(x, t|F) = \int_V [t_A(\mathbf{v}) + t_B(\mathbf{v})] dF(\mathbf{v}).$$

Recall that the revenue in the optimal direct, IC, IR mechanism is given by

$$R(F) = \max_{x, t} \bar{R}(x, t|F) \text{ subject to } \int_{V_{-i}} x_i(\mathbf{v}) v_i - t_i(\mathbf{v}) dF_{-i}(v_{-i}) \geq \max \left\{ \int_{V_{-i}} x_i(v'_i, v_{-i}) v_i - t_i(v'_i, v_{-i}) dF_{-i}(v_{-i}), 0 \right\},$$

for all $v'_i, v_i \in V_i$ and all $i \in \{A, B\}$. Denote by $(x(F), t(F))$ an optimal mechanism for the selected design F , and note that

$$R(F) = \bar{R}(x(F), t(F)|F).$$

Among optimal mechanisms, select one $(x(F), t(F))$ in which players' payments are determined by Vickrey's rule

$$t_i(\mathbf{v}|F) = x_i(\mathbf{v}|F) v_i - \int_{\underline{v}_i}^{v_i} x_i(s, v_{-i}|F) ds.$$

This mechanism is ex-post incentive compatible and individually rational, since for any $v'_i \in V_i$, $\mathbf{v} \in V$, and all $i \in \{A, B\}$

$$\begin{aligned} x_i(\mathbf{v}|F) v_i - t_i(\mathbf{v}|F) &= \int_{\underline{v}_i}^{v_i} x_i(s, v_{-i}|F) ds \geq \\ x_i(v'_i, v_{-i}|F) v_i - t_i(v'_i, v_{-i}|F) &= x_i(v'_i, v_{-i}|F) (v_i - v'_i) + \int_{\underline{v}_i}^{v'_i} x_i(s, v_{-i}|F) ds. \end{aligned}$$

The inequality follows because $x_i(v_i, v_{-i})$ is increasing in v_i by incentive compatibility, meaning that

$$x_i(v'_i, v_{-i}|F)(v'_i - v_i) = \int_{v_i}^{v'_i} x_i(v'_i, v_{-i}|F)ds \geq \int_{v_i}^{v'_i} x_i(s, v_{-i}|F)ds.$$

The mechanism $(x(F), t(F))$ is ex-post incentive compatible and individually rational. Therefore, it remains incentive compatible and individually rational even when a different design, say $F' \in \Delta(V_A) \times \Delta(V_B)$, is selected.²⁶ This is because in ex-post incentive compatible and individually rational mechanisms, agents' incentives are unaffected by the distribution of values. Optimality must therefore imply the following key property

$$R(F') \geq \bar{R}(x(F), t(F)|F'), \quad (7)$$

with the inequality being strict whenever the distributions differ for non-excluded types.

FOSD Next, consider any design $F' \in \Delta(V_A) \times \Delta(V_B)$ that first-order stochastically dominates F , meaning that

$$F'_i(v_i) \leq F_i(v_i) \text{ for all } v_i \in V_i \text{ and all } i \in \{A, B\}. \quad (8)$$

Observe that by the assumption in expression (8), the joint cumulative distribution $F'_A(v_A)F'_B(v_B)$ first-order stochastically dominates the joint distribution $F_A(v_A)F_B(v_B)$. The monotonicity of revenue in the FOSD order then follows by collecting these insights as

$$\begin{aligned} R(F') &\geq \bar{R}(x(F), t(F)|F') = \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)]dF'(\mathbf{v}) \\ &\geq \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)]dF(\mathbf{v}) = R(F). \end{aligned}$$

The first inequality holds by (7). By the nature of the Vickrey rule, the aggregate transfer increases in the aggregate allocation probability, $x_A(\mathbf{v}) + x_B(\mathbf{v})$. In turn, the aggregate allocation probability increases in values \mathbf{v} by incentive compatibility. Therefore, the second inequality follows because the aggregate transfer $t_A(\mathbf{v}) + t_B(\mathbf{v})$ increases in values.

²⁶When the support of F differs from that of F' , we can extend the allocation rule, $x(F)$, to cover the entire support of F' while retaining ex-post incentive compatibility and individual rationality by adding constant portions to it.

Convexity Finally, consider any two designs $F', F'' \in \Delta(V_A) \times \Delta(V_B)$ and any number $a \in (0, 1)$ such that

$$aF'_A(v_A)F'_B(v_B) + (1 - a)F''_A(v_A)F''_B(v_B) \leq F_A(v_A)F_B(v_B) \text{ for all } \mathbf{v} \in V_A \times V_B. \quad (9)$$

As before, because the Vickrey mechanism $(x(F), t(F))$ remains ex-post incentive compatible and individually rational even when F' and F'' are selected as value designs, we have that

$$R(F') \geq \bar{R}(x(F), t(F)|F') \text{ and } R(F'') \geq \bar{R}(x(F), t(F)|F''). \quad (10)$$

Convexity then follows by collecting these insights because

$$\begin{aligned} aR(F') + (1 - a)R(F'') &\geq a\bar{R}(x(F), t(F)|F') + (1 - a)\bar{R}(x(F), t(F)|F'') \\ &= \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)](adF'(\mathbf{v}) + (1 - a)dF''(\mathbf{v})) \\ &\geq \int_V [t_A(\mathbf{v}|F) + t_B(\mathbf{v}|F)]dF(\mathbf{v}) = R(F), \end{aligned}$$

where the first inequality holds by averaging the inequalities in (10), while the second inequality follows by (9) as the aggregate transfer $t_A(\mathbf{v}) + t_B(\mathbf{v})$ increases in values. ■

Proof of Proposition 1: One-Agent Mean Bound We compare revenue under $F_A^* = F^{\bar{v}}$ to revenue when setting some other distribution F_A such that $\mathbb{E}_{F_A}[v] = k$. The change from F_A to F_A^* has two effects: (i) it affects the virtual valuation of agent A ; and (ii) it affects the optimal allocation rule for the good. We first establish that revenue under F_A^* is higher than revenue under F_A , even when the allocation rule is suitably fixed. This establishes that F_A^* is an optimal design as optimizing the mechanism could only increase revenue further. We conclude by establishing uniqueness for $F_B \neq F^{\bar{v}}$.

As discussed in the Supplementary Appendix, the virtual value for distribution F_A^* amounts to

$$\psi_A^*(v) = \begin{cases} \bar{v} & \text{if } v = \bar{v} \\ 0 & \text{if } v = \underline{v} = 0 \end{cases}.$$

Further by revenue equivalence, revenue under distribution F_A^* is given by

$$R(F_A^*, F_B) = \int_{V_A^*} \psi_A^*(v) x_A^*(v) dF_A^*(v) + \int_{V_B} \psi_B(v) x_B^*(v) dF_B(v),$$

while revenue under any alternative design F_A amounts to

$$R(F_A, F_B) = \int_{V_A} \psi_A(v) x_A(v) dF_A(v) + \int_{V_B} \psi_B(v) x_B(v) dF_B(v).$$

Next we want to express the allocation rule, x , that is optimal for value distributions (F_A, F_B) in the quantile space. To do so, recall that for an arbitrary cumulative distribution, F , the inverse is defined as $v_i(q) = \inf\{v \mid F_i(v) \geq q\}$ for all $q \in [0, 1]$. The quantile representation of allocation rule, x , is then

$$y(q_A, q_B) = x(v_A(q_A), v_B(q_B)) \text{ for all } (q_A, q_B) \in [0, 1]^2.$$

Now consider an allocation rule x^+ that under design (F_A^*, F_B) , leads to the same allocation rule y in the quantile space. Formally, define such an allocation rule as follows

$$x^+(v_A, v_B) = y(F_A^*(v_A), F_B(v_B)) \text{ for all } (v_A, v_B) \in V_A^* \times V_B.$$

By construction, allocation rule x^+ under value design (F_A^*, F_B) is feasible, since it coincides in the quantile space with allocation rule x under design (F_A, F_B) —given that $x^+(v_A^*(q_A), v_B(q_B)) = y(q_A, q_B)$. Further, by construction interim allocation rules for x^+ under design (F_A^*, F_B) and for x under design (F_A, F_B) must coincide in the quantile space, and amount to $y_i(q_i) = \int_0^1 y_i(q_i, q_j) dq_j$ for $i \neq j$.

Consider the Vickrey transfer, t^+ , associated with such an allocation rule. Next, we establish that

$$\bar{R}(x^+, t^+ | F_A^*, F_B) \geq R(F_A, F_B).$$

Showing this immediately delivers the result, because revenue would further increase when the mechanism is optimally tailored to the value design (F_A^*, F_B) . Because interim allocations are fixed in the quantile space and virtual values are unchanged for B in either the value or the quantile space, we have that

$$\int_{V_B} \psi_B(v) x_B^+(v) dF_A(v) = \int_{V_B} \psi_B(v) x_B(v) dF_A(v).$$

To prove the result, it therefore suffices to show that

$$\int_{V_A^*} \psi_A^*(v) x_A^+(v) dF_A^*(v) \geq \int_{V_A} \psi_A(v) x_A(v) dF_A(v).$$

To show this, first observe that the expected virtual value of agent A satisfies

$$\int_{V_A^*} \psi_A^*(v) dF_A^*(v) = \int_{V_A^*} v dF_A^*(v) = k = \int_{V_A} v dF_A(v) \geq \int_{V_A} \max\{\psi_A(v), 0\} dF_A(v),$$

where the inequality follows because $v \geq \max\{\psi_A(v), 0\}$ for all v . In the quantile space, the latter is equivalent to

$$\int_{[0,1]} \psi_A^*(v_A^*(q)) dq = k \geq \int_{[0,1]} \max\{\psi_A(v_A(q)), 0\} dq. \quad (11)$$

Since the good is only ever allocated to an agent with a non-negative ironed virtual value, we have that $\psi_A(v_A(q)) \in [0, \bar{v}]$ whenever $y_A(q) > 0$. Further $\psi_A^*(v_A^*(q))$ is increasing and satisfies $\psi_A^*(v_A^*(q)) \in \{0, \bar{v}\}$. Because the interim allocation rule y_A is non-decreasing by incentive compatibility, we have that

$$\begin{aligned} \int_{[0,1]} \psi_A^*(v_A^*(q)) dq &\geq \int_{[0,1]} \max\{\psi_A(v_A(q)), 0\} dq \\ \Rightarrow \int_{[0,1]} \psi_A^*(v_A^*(q)) y_A(q) dq &\geq \int_{[0,1]} \psi_A(v_A(q)) y_A(q) dq, \end{aligned}$$

where the implication follows because the allocation rule $y_A(q)$ places more weight on quantiles at which $\psi_A^*(v_A^*(q)) = \bar{v} > \psi_A(v_A(q))$ and less weight on quantiles at which $\psi_A^*(v_A^*(q)) = 0 < \psi_A(v_A(q))$ relative to a scenario in which all quantiles receive the same weight. Because the interim allocation rule was fixed in the quantile space, we can rewrite the final condition in the value space to obtain the desired conclusion, namely that

$$\int_{V_A^*} \psi_A^*(v) x_A^+(v) dF_A^*(v) \geq \int_{V_A} \psi_A(v) x_A(v) dF_A(v).$$

The latter establishes that it is never optimal to select a distribution for A with an expected virtual value which is strictly smaller than the expected value. There are other distributions that allow for the expected value to be equal to the expected virtual value. But, all these distributions can attach positive probability to at most one strictly positive value. Suppose to the contrary, the principal selected a value distribution in which the

agent had more than one strictly positive value. In such instances, the only way for the principal to extract the full surplus would require awarding the good with certainty to A whenever agent A has positive value. But then the principal would not be able to extract the full surplus from agent A , since they could only sell to A at the smallest of their positive values, yielding the contradiction.

To show that F_A^* is the unique optimum when $F_B \neq F^{\bar{v}}$, consider any other binary distribution F_A with mean k and a single atom on positive values at \bar{v}_A , meaning that

$$F_A(v) = \begin{cases} 1 & \text{if } v = \bar{v}_A \\ 1 - \frac{k}{\bar{v}_A} & \text{if } v < \bar{v}_A \end{cases}.$$

Letting $p_A = k/\bar{v}_A$ denote the probability of having value \bar{v}_A , revenue can be written as

$$\begin{aligned} R(F_A, F_B) &= (1 - p_A) \int_{V_B} \max\{\psi_B(v_B), 0\} dF_B(v_B) + p_A \int_{V_B} \max\{\psi_B(v_B), \bar{v}_A\} dF_B(v_B) \\ &= (1 - p_A) \int_{V_B} \max\{\psi_B(v_B), 0\} dF_B(v_B) + \int_{V_B} \max\{p_A \psi_B(v_B), k\} dF_B(v_B), \end{aligned} \quad (12)$$

where the equality follows because the optimal mechanism allocates the good to B when $v_A = 0$ and B 's virtual valuation is positive, or $v_A = \bar{v}_A$ and B 's virtual valuation exceeds \bar{v}_A .²⁷

Expression (12) is differentiable in p_A . Next, we differentiate (12) with respect to p_A and establish that revenue decreases in p_A , meaning that the optimal distribution will set p_A to be as small as possible, or equivalently \bar{v}_A as large as possible. Letting $V_+(w) = \{v_B \in [0, \bar{v}] | \psi_B(v_B) \geq w\}$ for $w \geq 0$ denote the set of values for which B 's virtual valuation weakly exceeds w , we find that

$$\frac{\partial R(F_A, F_B)}{\partial p_A} = - \int_{V_+(0)} \psi_B(v_B) dF_B(v_B) + \int_{V_+(\bar{v}_A)} \psi_B(v_B) dF_B(v_B) \leq 0,$$

where the inequality holds, since $V_+(\bar{v}_A) \subseteq V_+(0)$. Moreover, the inequality is strict whenever $\psi_B(v_B) \in (0, \bar{v}_A)$ for some $v_B \in V_B$. Thus, it is optimal to minimize p_A which is accomplished by setting $\bar{v}_A = \bar{v} \geq \bar{v}_B$. This establishes the uniqueness result when $F_B \neq F^{\bar{v}}$. To prove the optimality of designs F^h for $h \in [k, \bar{v}]$ when $F_B = F^{\bar{v}}$, it suffices to show that any design F^h yields the same revenue as F_A^* , which is immediate and thus

²⁷If F_B was not regular, the previous expression for revenue would still apply. In such scenarios, F_B would denote the ironed distribution of values yielding the same revenue, rather than F_B itself—see [Hartline \(2013\)](#), Theorem 3.14, p.78.

omitted. ■

Proof of Proposition 2: Two-Agent Mean Bound The proof proceeds in three steps. First, we demonstrate that the surplus is convex in the agents' realized values, and we leverage this convexity to derive value designs that maximize surplus. Second, we argue that the value designs specified in the proposition simultaneously maximize both surplus and revenue. Third, we establish that no alternative value design can achieve optimality.

Convexity in Values Consider any design (F_A, F_B) fulfilling the mean bound. Total surplus amounts to

$$S(F_A, F_B) = \int_{\underline{v}_A}^{\bar{v}_A} \int_{\underline{v}_B}^{\bar{v}_B} \max\{v_A, v_B\} dF_B(v_B) dF_A(v_A),$$

where Riemann-Stieltjes integrals are used to calculate the expectations implicitly. Conditional on any realized value of $v_A \in [\underline{v}_B, \bar{v}_B]$, surplus amounts to

$$\begin{aligned} \hat{S}(v_A) &= \int_{\underline{v}_B}^{\bar{v}_B} \max\{v_A, v_B\} dF_B(v_B) = \int_{\underline{v}_B}^{v_A} v_A dF_B(v_B) + \int_{v_A}^{\bar{v}_B} v_B dF_B(v_B) \\ &= v_A F_B(v_A) + [v_B F_B(v_B)]_{v_A}^{\bar{v}_B} - \int_{v_A}^{\bar{v}_B} F_B(v_B) dv_B \\ &= \bar{v}_B - \int_{v_A}^{\bar{v}_B} F_B(v_B) dv_B, \end{aligned}$$

where the second equality is algebraic, the third follows from calculations and integration by parts, and the fourth equality simply factors redundant terms.

For $v_A \geq \bar{v}_B$, surplus amounts to $\hat{S}(v_A) = v_A$ since $\max\{v_A, \bar{v}_B\} = v_A$. For $v_A \leq \underline{v}_B$, surplus amounts to $\hat{S}(v_A) = \underline{v}_B$ since $\max\{v_A, \underline{v}_B\} = \underline{v}_B$. Moreover, $\hat{S}(v_A)$ is differentiable in $v_A \in [\underline{v}_A, \bar{v}_A]$ and it is increasing since for $v_A \in [\underline{v}_B, \bar{v}_B]$,

$$\hat{S}'(v_A) = -\frac{d}{dv_A} \int_{v_A}^{\bar{v}_B} F_B(v_B) dv_B = F_B(v_A)$$

by the Leibniz rule. The derivative equals one, $\hat{S}'(v_A) = 1$, for $v_A > \bar{v}_B$, while it is equal to zero, $\hat{S}'(v_A) = 0$, for $v_A < \underline{v}_B$.

Finally, $\hat{S}(v_A)$ is convex since $\hat{S}(v_A) - \hat{S}(v'_A) \geq (v_A - v'_A) \hat{S}'(v'_A)$ for any values $v_A, v'_A \in$

$[\underline{v}_A, \bar{v}_A]$. To show this, assume without loss that $v_A \geq v'_A$. Observe that when $v'_A > \bar{v}_B$,

$$\hat{S}(v_A) - \hat{S}(v'_A) = v_A - v'_A = (v_A - v'_A)\hat{S}'(v'_A),$$

given that A 's values always exceed B 's value. When $\bar{v}_B \geq v_A \geq v'_A \geq \underline{v}_B$,

$$\begin{aligned} \hat{S}(v_A) - \hat{S}(v'_A) &= \int_{v'_A}^{v_A} F_B(v_B) dv_B \geq \int_{v'_A}^{v_A} F_B(v'_A) dv_B \\ &= (v_A - v'_A)F_B(v'_A) = (v_A - v'_A)\hat{S}'(v'_A), \end{aligned} \tag{13}$$

where the inequality follows because F_B is increasing. When $v_A > \bar{v}_B \geq v'_A \geq \underline{v}_B$,

$$\begin{aligned} \hat{S}(v_A) - \hat{S}(v'_A) &= v_A - \bar{v}_B + \hat{S}(\bar{v}_B) - \hat{S}(v'_A) \geq v_A - \bar{v}_B + (\bar{v}_B - v'_A)\hat{S}'(v'_A) \\ &\geq (v_A - v'_A)\hat{S}'(v'_A), \end{aligned}$$

where the equality follows because $\hat{S}(v_A) = v_A$ for $v_A > \bar{v}_B$ and $\hat{S}(\bar{v}_B) = \bar{v}_B$, the first inequality follows from (13), and the second inequality follows from $\hat{S}'(v'_A) \leq 1$ and $v_A > \bar{v}_B$. When $v_A > \bar{v}_B$ and $\underline{v}_B > v'_A$,

$$\hat{S}(v_A) - \hat{S}(v'_A) = v_A - \underline{v}_B \geq 0 = (v_A - v'_A)\hat{S}'(v'_A),$$

where the inequality follows as $v_A > \bar{v}_B$ and the final equality follows as $\hat{S}'(v'_A) = 0$.

When $\bar{v}_B \geq v_A \geq \underline{v}_B > v'_A$,

$$\hat{S}(v_A) - \hat{S}(v'_A) \geq 0 = (v_A - v'_A)\hat{S}'(v'_A),$$

where the inequality follows because surplus is increasing and the equality as $\hat{S}'(v'_A) = 0$.

Finally, for $\underline{v}_B > v_A$, we have that

$$\hat{S}(v_A) - \hat{S}(v'_A) = (v_A - v'_A)\hat{S}'(v'_A) = 0,$$

given that B 's values always exceed A 's. Thus, surplus conditional on v_A , $\hat{S}(v_A)$, is globally convex in v_A given that \hat{S} is differentiable and $\hat{S}(v_A) - \hat{S}(v'_A) \geq (v_A - v'_A)\hat{S}'(v'_A)$.

Surplus Maximizing Designs The surplus maximizing design F_A given F_B must solve

$$\max_{F_A} \int_{\underline{v}}^{\bar{v}} \hat{S}(v_A) dF_A(v_A) \text{ subject to } \int_{\underline{v}}^{\bar{v}} v_A dF_A(v_A) \leq k.$$

Because the surplus is convex and increasing in v_A while the constraint is linear, surplus maximization requires maximally spreading the distribution of v_A while keeping the mean at k —meaning that setting $F_A = F^{\bar{v}}$ maximizes surplus for any F_B . Thus, the convexity of surplus in realized values implies that for any distribution of values F_j , surplus will be maximized by any design (F_A, F_B) satisfying $F_i = F^{\bar{v}}$ for some $i \in \{A, B\}$ and $\int_{\underline{v}}^{\bar{v}} v_j dF_j(v_j) = k$ for $j \neq i$. In any such design, surplus amounts to

$$\begin{aligned} S^* = S(F_A, F_B) &= P^{\bar{v}}(\bar{v}) \int_{\underline{v}_j}^{\bar{v}_j} \max\{\bar{v}, v_j\} dF_j(v_j) + P^{\bar{v}}(\underline{v}) \int_{\underline{v}_j}^{\bar{v}_j} \max\{\underline{v}, v_j\} dF_j(v_j) \\ &= P^{\bar{v}}(\bar{v}) \int_{\underline{v}_j}^{\bar{v}_j} \bar{v} dF_j(v_j) + P^{\bar{v}}(\underline{v}) \int_{\underline{v}_j}^{\bar{v}_j} v_j dF_j(v_j) = \frac{k - \underline{v}}{\bar{v} - \underline{v}} \bar{v} + \frac{\bar{v} - k}{\bar{v} - \underline{v}} k, \end{aligned}$$

where the first equality follows from the definition of surplus, the second follows because $v_j \in [\underline{v}, \bar{v}]$, and the third as $\int_{\underline{v}_j}^{\bar{v}_j} v_j dF_j(v_j) = k$.

Revenue Optimality For any value design, surplus must bound revenue from above. This follows because inefficient optimal allocations may result in less surplus being produced for particular value realizations, and because the utility of any player i is non-negative by individual rationality, $U_i(F_A, F_B) = \int_{\mathbf{V}} [v_i x_i(\mathbf{v}|F) - t_i(\mathbf{v}|F)] dF(\mathbf{v}) \geq 0$. Therefore,

$$S(F_A, F_B) \geq R(F_A, F_B) + U_A(F_A, F_B) + U_B(F_A, F_B) \geq R(F_A, F_B),$$

and revenue is no larger than surplus in any value design. But for any proposed optimal design in the statement of the result, we have that $R(F_A, F_B) = S^*$, and thus these designs must indeed be optimal.

To see this, consider any surplus maximizing design satisfying $F_i = F^* = F^{\bar{v}}$ for some

$i \in \{A, B\}$ and $\int_{\underline{v}}^{\bar{v}} v_j dF_j(v_j) = k$ for $j \neq i$. Revenue for any such design amounts to

$$\begin{aligned} R(F_A, F_B) &= P^{\bar{v}}(\bar{v})\psi_i(\bar{v}) + P^{\bar{v}}(\underline{v}) \int_{\underline{v}_j}^{\bar{v}_j} \max\{\psi_j(v_j), \psi_i(\underline{v}), 0\} dF_j(v_j) \\ &= \frac{k - \underline{v}}{\bar{v} - \underline{v}} \psi_i(\bar{v}) + \frac{\bar{v} - k}{\bar{v} - \underline{v}} \int_{\underline{v}_j}^{\bar{v}_j} \max\{\psi_j(v_j), \psi_i(\underline{v}), 0\} dF_j(v_j). \end{aligned}$$

But then we must have that $R(F_A, F_B) = S^*$ for the following reasons. First, for any of these designs, we have that $\psi_i(\bar{v}) = \bar{v}$. Moreover, when $\underline{v} > 0$, in the unique optimal design F^k , we have that

$$\int_{\underline{v}_j}^{\bar{v}_j} \max\{\psi_j(v_j), \psi_i(\underline{v}), 0\} dF_j(v_j) = \int_{\underline{v}_j}^{\bar{v}_j} \max\{k, \psi_i(\underline{v}), 0\} dF_j(v_j) = k,$$

where the first equality follows as $\psi_j(v_j) = v_j = k$, and the second equality follows since $k \geq \underline{v} \geq \max\{\psi_i(\underline{v}), 0\}$. Finally, when $\underline{v} = 0$, in any optimal design F^h , we have that

$$\begin{aligned} \int_{\underline{v}_j}^{\bar{v}_j} \max\{\psi_j(v_j), \psi_i(\underline{v}), 0\} dF_j(v_j) &= \int_{\underline{v}_j}^{\bar{v}_j} \max\{\psi_j(v_j), 0\} dF_j(v_j) \\ &= P^h(h)h + P^h(0)0 = k. \end{aligned}$$

where the first equality follows as $\psi_i(\underline{v}) \leq \underline{v} = 0$, the second one holds as $\psi_j(\bar{v}_j) = \bar{v}_j = h$ and $\psi_j(0) \leq 0$, while the third one holds by definition.

No other surplus-maximizing design can be optimal though, because the only way to secure a revenue equal to S^* entails guaranteeing that $\psi_i(\bar{v}) = \bar{v}$ and

$$\int_{\underline{v}_j}^{\bar{v}_j} \max\{\psi_j(v_j), \psi_i(\underline{v}), 0\} dF_j(v_j) = \int_{\underline{v}_j}^{\bar{v}_j} v_j dF_j(v_j) = k.$$

But for the latter to hold, F_j can have either a single atom at the mean k , or only two atoms with one atom at 0 and the other above the mean k —which is only possible if $\underline{v} = 0$. In any other surplus-maximizing design, the designer would need to pay information rents to bidder j to get them to reveal their type truthfully. ■

Proof of Proposition 3: Reallocating Value, Fixed Mean The proof establishes the result for $k \in [2\underline{v}, \underline{v} + \bar{v}]$, since the other cases are immediate. To begin, we characterize the optimal designs when $\mathbb{E}_{F_i}[v] = k_i$ for some $i \in \{A, B\}$ and $\mathbb{E}_{F_j}[v] = k - k_i$ for $j \neq i$. We will then let the designer choose $k_i \in [\underline{v}, k - \underline{v}]$ to maximize revenue.

As in the first part of the proof of Proposition 2, the surplus maximizing design F_i given F_j must solve

$$S^+(k_i) = \max_{F_i} \int_{\underline{v}}^{\bar{v}} \hat{S}(v_i) dF_i(v_i) \text{ subject to } \int_{\underline{v}}^{\bar{v}} v_i dF_i(v_i) = k_i.$$

As before, because \hat{S} is convex and increasing in v_i while the constraint is linear, there exists a surplus-maximizing value design F_i which is maximally spread and has mean k_i . In particular, surplus will be maximized by designs (F_A, F_B) satisfying $F_i(v) = (\bar{v} - k_i)/(\bar{v} - \underline{v})$ for all $v < \bar{v}$ and some $i \in \{A, B\}$ and $\int_{\underline{v}}^{\bar{v}} v_j dF_j(v_j) = k - k_i$ for $j \neq i$. In these designs, surplus amounts to

$$S^+(k_i) = \frac{k_i - \underline{v}}{\bar{v} - \underline{v}} \bar{v} + \frac{\bar{v} - k_i}{\bar{v} - \underline{v}} (k - k_i).$$

Surplus S^+ is convex in k_i , decreasing for $k_i < k/2$, and increasing thereafter. Thus when selecting k_i to maximize surplus, optimality requires k_i to either be equal to \underline{v} or to $k - \underline{v}$. For both of these values of k_i , surplus is maximized and $S^+(\underline{v}) = S^+(k - \underline{v}) = k - \underline{v}$.

This completes the proof as for any of the designs in the statement of the result $R(F_A^*, F_B^*) = k - \underline{v}$, given that the principal either trades with j at price h or sells to i at price \underline{v} . ■

Proof of Proposition 4: Reallocating Value, Fixed Distribution We want to show that the maximally divisive design

$$\begin{aligned} F_B^*(v) &= \min\{H(v), 1\} & \text{if } v \in [\underline{v}, v^M) \\ F_A^*(v) &= H(v) - 1 & \text{if } v \in [v^M, \bar{v}], \end{aligned}$$

is optimal. To do so, we first argue that the constraint in (4) must bind almost surely. We then show that in an optimal design F , the supports of the two value distributions must have disjoint interiors. We conclude by establishing that the maximally divisive design $(H - 1, \min\{H, 1\})$ leads to higher revenue than any other value design with disjoint interiors.

Binding Constraint To see why the constraint must bind almost surely in some optimal design, posit that some optimal design F satisfies $F_A(v) + F_B(v) > H(v)$ on some

set W with positive measure. If so, there exists a value design $F' \succsim_1 F$ that satisfies constraint (4) and first-order stochastically dominates the original design F —implying that $R(F') \geq R(F)$.

Disjoint Interiors Next, we show that because revenue is convex in the joint distribution of values, and the set of value designs meeting the constraint $F_A + F_B = H$ is convex, optimal value distributions for the two agents must have essentially disjoint supports. For an arbitrary set V , denote its interior by \mathring{V} . Formally, this part of the proof establishes that in any optimal design the intersection between the interior of the two supports must be empty,

$$\mathring{V}_A \cap \mathring{V}_B = \emptyset.$$

To prove this, proceed by contradiction and assume that there exists an optimal design $F = (F_A, F_B)$ that satisfies $F_A + F_B = H$ and $W = \mathring{V}_A \cap \mathring{V}_B \neq \emptyset$. If so, for any $v \in W$, we have that $F_A(v), F_B(v) \in (0, 1)$, since v belongs to the interior of both supports. Furthermore, for any $v, v' \in W$ such that $v' > v$, we have that $F_i(v') - F_i(v) > 0$, because cumulative distributions strictly increase in the interior of the support.

This step of the proof hinges on redistributing measure between F_A and F_B on the set W . To ensure that the resulting new measures are proper distributions, the measure redistributed must be capped. To do so, construct a function $\varepsilon : W \rightarrow \mathbb{R}_+$ that satisfies for any $v, v' \in W$ such that $v' > v$

$$\varepsilon(v') - \varepsilon(v) \in (-\min_i\{F_i(v') - F_i(v)\}, \min_i\{F_i(v') - F_i(v)\}). \quad (14)$$

Condition (14) requires that any changes in the function ε are smaller in absolute value than changes in either of the cumulative distributions. It is always possible to find a function ε fulfilling (14) because $F_i(v') > F_i(v)$ in the interior of the support.

Now construct two alternative designs F' and F'' as follows. Let

$$F'_A(v) = \begin{cases} F_A(v) + \varepsilon(v) & \text{if } v \in W \\ F_A(v) & \text{if } v \in V \setminus W \end{cases} \quad \text{and} \quad F''_A(v) = \begin{cases} F_A(v) - \varepsilon(v) & \text{if } v \in W \\ F_A(v) & \text{if } v \in V \setminus W \end{cases},$$

and set $F'_B = H - F'_A$ and $F''_B = H - F''_A$. Condition (14) ensures that all cumulative distributions remain increasing in both value designs F' and F'' , because the changes in

ε are small in absolute value relative to the changes in F_i . Further we have that

$$F_A(v_A)F_B(v_B) \geq F_A(v_A)F_B(v_B) - \varepsilon(v_A)\varepsilon(v_B) = \frac{F'_A(v_A)F'_B(v_B)}{2} + \frac{F''_A(v_A)F''_B(v_B)}{2}.$$

The convexity of revenue then delivers a contradiction, because F cannot be optimal since

$$R(F) < R(F')/2 + R(F'')/2 < \max\{R(F'), R(F'')\},$$

where the inequalities are strict by selecting $\varepsilon(v)$ to be strictly positive on W .

Convex Support By the previous part of the proof, we know that in any optimal design F , supports must have disjoint interiors. Next, we restrict attention to value designs F in which the designed distributions have supports $\hat{V}_B = [\underline{v}, w_1] \cup [w_2, w_3]$ and $\hat{V}_A \in [w_1, w_2] \cup [w_3, \bar{v}]$, for $\underline{v} < w_1 < w_2 < w_3 \leq \bar{v}$. We establish that neither of these designs yields higher revenue than the maximally divisive design. We then conclude by arguing that if such split supports are suboptimal, splitting the supports further cannot be optimal either. Note that by the definition of the median value, we must have that $w_1 < v^M < w_3$.

Recall that by using Riemann-Stieltjes integrals and by positing that at any point of discontinuity of a cumulative F_i , virtual values are equal to values, we can write revenue as

$$\begin{aligned} R(F) &= \int_{V_A} \psi_A(v_A)x_A(v)dF_A(v) + \int_{V_B} \psi_B(v_B)x_B(v)dF_B(v) \\ &= \int_{V_A} x_A(v)d[v(1 - F_A(v))] + \int_{V_B} x_B(v)d[v(1 - F_B(v))]. \end{aligned}$$

Integrating each component by parts, treating $v_i(1 - F_i(v_i))$ and $x_i(v_i)$ as the two parts, and letting D_i denote the set of values at which the allocation rule of i is non-differentiable, yields

$$\begin{aligned} R(F) &= \sum_{i \in \{A, B\}} \left[\underline{v}_i x_i(\underline{v}_i) + \int_{V_i \setminus D_i} v(1 - F_i(v)) dx_i(v) \right] \\ &= \sum_{i \in \{A, B\}} \left[\underline{v}_i x_i(\underline{v}_i) + \int_{V_i \setminus D_i} v(1 - F_i(v)) x'_i(v) dv + \sum_{v \in D_i} v(1 - F_i(v)) [x_i^+(v) - x_i^-(v)] \right]. \end{aligned}$$

As the supports of the two distributions are disjoint by the previous argument, we can redefine variables so that

$$\begin{aligned} (\hat{F}_A(v), d\hat{x}_A(v)) &= \begin{cases} (F_A(v), dx_A(v)) & \text{if } v \geq v_M \text{ and } v \in V_A \\ (F_B(v), dx_B(v)) & \text{if } v \geq v_M \text{ and } v \in V_B \\ (0, 0) & \text{if } v < v_M \end{cases} , \\ (\hat{F}_B(v), d\hat{x}_B(v)) &= \begin{cases} (F_A(v), dx_A(v)) & \text{if } v < v_M \text{ and } v \in V_A \\ (F_B(v), dx_B(v)) & \text{if } v < v_M \text{ and } v \in V_B \\ (0, 0) & \text{if } v \geq v_M \end{cases} . \end{aligned}$$

With this labeling, revenue can be rewritten as

$$R(F) = \underline{v}x_B(\underline{v}) + w_1x_A(w_1) + \int_{\underline{v}}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) + \int_{v^M}^{\bar{v}} v(1 - \hat{F}_A(v))d\hat{x}_A(v).$$

For some non-negative number M , consider the alternative interim allocation rule

$$\begin{aligned} \hat{x}_B(v) &= x_B(\underline{v}) + \int_{\underline{v}}^v d\hat{x}_B(v) \text{ for } v \in [\underline{v}, v^M], \\ \hat{x}_A(v) &= M + \int_{v^M}^v d\hat{x}_A(v) \text{ for } v \in [v^M, \bar{v}]. \end{aligned}$$

This allocation rule is incentive compatible because all differentials are positive by incentive compatibility of the original allocation rule. Further it is interim feasible for the maximally divisive design $F^* = (F_A^*, F_B^*)$ as we establish in a separate part of the proof below. Thus to prove that the maximally divisive design raises more revenue than $R(F)$, it suffices to show that

$$\begin{aligned} &\underline{v}\hat{x}_B(\underline{v}) + v^M\hat{x}_A(v^M) + \int_{\underline{v}}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) + \int_{v^M}^{\bar{v}} v(1 - F_A^*(v))d\hat{x}_A(v) \\ &> \underline{v}x_B(\underline{v}) + w_1x_A(w_1) + \int_{\underline{v}}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) + \int_{v^M}^{\bar{v}} v(1 - \hat{F}_A(v))d\hat{x}_A(v). \end{aligned} \tag{15}$$

Showing that (15) holds would establish the result by the usual logic because $R(F^*)$, the revenue accruing to the principal when the design is F^* and the optimal mechanism is selected, must exceed the left-hand side of (15) which amounts to revenue under some other incentive compatible mechanism.

We consider three distinct cases: (i) $w_2 \leq v^M$, $w_1 > \underline{v}$, $w_3 < \bar{v}$; (ii) $w_2 > v^M$, $w_1 > \underline{v}$,

$w_3 < \bar{v}$; (iii) $w_3 = \bar{v}$.

Case (i): In this scenario, inequality (15) can be amended to

$$\begin{aligned} & v^M M + \int_{w_1}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) + \int_{v^M}^{w_3} v(1 - F_A^*(v))d\hat{x}_A(v) \\ & > w_1 x_A(w_1) + \int_{w_1}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) + \int_{v^M}^{w_3} v(1 - \hat{F}_A(v))d\hat{x}_A(v). \end{aligned} \quad (16)$$

This follows because the distributions are identical below w_1 and above w_3 and $v^M \in V_B$. Further, we have that $\hat{F}_A(v) = F_B(v) = H(v) - H(w_2) + H(w_1)$ and $F_A^*(v) = H(v) - 1$ for all $v \in [v^M, w_3]$, and thus

$$\begin{aligned} & \int_{v^M}^{w_3} v(1 - F_A^*(v))d\hat{x}_A(v) - \int_{v^M}^{w_3} v(1 - \hat{F}_A(v))d\hat{x}_A(v) = \int_{v^M}^{w_3} v(\hat{F}_A(v) - F_A^*(v))d\hat{x}_A(v) \\ & = (1 - H(w_2) + H(w_1)) \int_{v^M}^{w_3} v d\hat{x}_A(v). \end{aligned}$$

We also have that $\hat{F}_B(v) = F_B(v) = H(v) - H(w_2) + H(w_1)$ for $v \in [w_2, v^M]$, $\hat{F}_B(v) = F_A(v) = H(v) - H(w_1)$ for $v \in [w_1, w_2]$, while $F_B^*(v) = H(v)$ for $v \in [w_1, v^M]$, and thus

$$\begin{aligned} & \int_{w_1}^{v^M} v(1 - F_B^*(v))d\hat{x}_B(v) - \int_{w_1}^{v^M} v(1 - \hat{F}_B(v))d\hat{x}_B(v) = \int_{w_1}^{v^M} v(\hat{F}_B(v) - F_B^*(v))d\hat{x}_B(v) \\ & = \int_{w_2}^{v^M} v(F_B(v) - H(v))d\hat{x}_B(v) + \int_{w_1}^{w_2} v(F_A(v) - H(v))d\hat{x}_B(v) \\ & = -(H(w_2) - H(w_1)) \int_{w_2}^{v^M} v d\hat{x}_B(v) - H(w_1) \int_{w_1}^{w_2} v d\hat{x}_B(v). \end{aligned}$$

In light of the last two identities, condition (16) can be rewritten as

$$\begin{aligned} & v^M M - w_1 x_A(w_1) + (1 - H(w_2) + H(w_1)) \int_{v^M}^{w_3} v d\hat{x}_A(v) \\ & - (H(w_2) - H(w_1)) \int_{w_2}^{v^M} v d\hat{x}_B(v) - H(w_1) \int_{w_1}^{w_2} v d\hat{x}_B(v) > 0. \end{aligned}$$

As $H(w_1), H(w_2) - H(w_1) < 1$, it suffices to show that

$$v^M M - w_1 x_A(w_1) - \int_{w_1}^{v^M} v d\hat{x}_B(v) > 0.$$

Integrating by parts, this is equivalent to

$$\begin{aligned} v^M M - w_1 x_A(w_1) + \int_{w_1}^{v^M} \hat{x}_B(v) dv &> [\hat{x}_B(v) v]_{w_1}^{v^M} \\ &= v^M [x_B(v^M) - x_B(w_2) + x_A(w_2) - x_A(w_1) + x_B(w_1)] - w_1 x_B(w_1). \end{aligned} \quad (17)$$

Further because allocation rules are increasing, we have that $\int_{w_1}^{v^M} \hat{x}_B(v) dv > (v^M - w_1) x_B(w_1)$. Consequently equation (17) holds as long as

$$v^M (M - x_B(v^M) - x_B(w_2) + x_A(w_2)) + (v^M - w_1) x_A(w_1) > 0,$$

which holds for $M = x_B(v^M) - x_B(w_2) + x_A(w_2)$ —thus establishing the result.

Case (ii): As in case (i), we need to establish inequality (15). As in case (i), we can establish that

$$\int_{w_2}^{w_3} v(1 - F_A^*(v)) d\hat{x}_A(v) - \int_{w_2}^{w_3} v(1 - \hat{F}_A(v)) d\hat{x}_A(v) = (1 - H(w_2) + H(w_1)) \int_{w_2}^{w_3} v d\hat{x}_A(v),$$

because $\hat{F}_A(v) = F_B(v) = H(v) - H(w_2) + H(w_1)$ and $F_A^*(v) = H(v) - 1$ for $v \in [w_2, w_3]$.

Moreover, we have that

$$\int_{v^M}^{w_2} v(1 - F_A^*(v)) d\hat{x}_A(v) - \int_{v^M}^{w_2} v(1 - \hat{F}_A(v)) d\hat{x}_A(v) = (1 - H(w_1)) \int_{v^M}^{w_2} v d\hat{x}_A(v),$$

because $\hat{F}_A(v) = F_A(v) = H(v) - H(w_1)$ and $F_A^*(v) = H(v) - 1$ for $v \in [v^M, w_2]$. Finally, we have that

$$\int_{w_1}^{v^M} v(1 - F_B^*(v)) d\hat{x}_B(v) - \int_{w_1}^{v^M} v(1 - \hat{F}_B(v)) d\hat{x}_B(v) = -H(w_1) \int_{w_1}^{v^M} v d\hat{x}_B(v),$$

because $\hat{F}_B(v) = F_A(v) = H(v) - H(w_1)$ and $F_B^*(v) = H(v)$ for $v \in [w_1, v^M]$. In light of the last identities, condition (15) can be rewritten as

$$\begin{aligned} v^M M - w_1 x_A(w_1) + (1 - H(w_2) + H(w_1)) \int_{w_2}^{w_3} v d\hat{x}_A(v) \\ + (1 - H(w_1)) \int_{v^M}^{w_2} v d\hat{x}_A(v) - H(w_1) \int_{w_1}^{v^M} v d\hat{x}_B(v) > 0. \end{aligned}$$

As $H(w_1), H(w_2) - H(w_1) < 1$, this again reduces to

$$v^M M - w_1 x_A(w_1) - \int_{w_1}^{v^M} v d\hat{x}_B(v) > 0,$$

which from integration by parts, again reduces to

$$v^M M - w_1 x_A(w_1) + \int_{w_1}^{v^M} \hat{x}_B(v) dv > v^M (x_A(v^M) - x_A(w_1) + x_B(w_1)) - w_1 x_B(w_1). \quad (18)$$

Because $\int_{w_1}^{v^M} \hat{x}_B(v) dv > (v^M - w_1) x_B(w_1)$, equation (18) holds as long as

$$v^M (M - x_A(v^M)) + (v^M - w_1) x_A(w_1) > 0,$$

which holds for $M = x_A(v^M)$ —thus establishing the result.

Case (iii): In this scenario, inequality (15) can be amended to

$$\begin{aligned} & v^M M + \int_{w_1}^{v^M} v(1 - F_B^*(v)) d\hat{x}_B(v) + \int_{v^M}^{w_2} v(1 - F_A^*(v)) d\hat{x}_A(v) \\ & > w_1 x_A(w_1) + \int_{w_1}^{v^M} v(1 - \hat{F}_B(v)) d\hat{x}_B(v) + \int_{v^M}^{w_2} v(1 - \hat{F}_A(v)) d\hat{x}_A(v). \end{aligned} \quad (19)$$

As before, we have that

$$\int_{v^M}^{w_2} v(1 - F_A^*(v)) d\hat{x}_A(v) - \int_{v^M}^{w_2} v(1 - \hat{F}_A(v)) d\hat{x}_A(v) = (1 - H(w_1)) \int_{v^M}^{w_2} v d\hat{x}_A(v),$$

because $\hat{F}_A(v) = F_A(v) = H(v) - H(w_1)$ and $F_A^*(v) = H(v) - 1$ for $v \in [v^M, w_2]$. Finally, we have that

$$\int_{w_1}^{v^M} v(1 - F_B^*(v)) d\hat{x}_B(v) - \int_{w_1}^{v^M} v(1 - \hat{F}_B(v)) d\hat{x}_B(v) = -H(w_1) \int_{w_1}^{v^M} v d\hat{x}_B(v),$$

because $\hat{F}_B(v) = F_A(v) = H(v) - H(w_1)$ and $F_B^*(v) = H(v)$ for $v \in [w_1, v^M]$. But then, condition (19) can be rewritten as

$$v^M M - w_1 x_A(w_1) + (1 - H(w_1)) \int_{v^M}^{w_2} v d\hat{x}_A(v) - H(w_1) \int_{w_1}^{v^M} v d\hat{x}_B(v) > 0. \quad (20)$$

which can be shown to hold following the same steps as in case (ii).

Interim Feasibility: Concluding this step of the proof still requires establishing that

the proposed allocations are interim feasible. Define V_i^* as the support of distribution F_i^* . Theorem 1 in [Border \(1991\)](#) implies that our earlier interim allocation rules (\hat{x}_A, \hat{x}_B) are feasible for the maximally divisive design F^* if and only if for any $(v_A, v_B) \in V_A^* \times V_B^*$

$$\begin{aligned} \int_{\underline{v}}^{v_B} \hat{x}_B(v) dH(v) + \int_{v^M}^{v_A} \hat{x}_A(v) dH(v) &\leq 1 - (1 - F_B^*(v_B))(1 - F_A^*(v_A)) \\ &= 1 - (1 - H(v_B))(2 - H(v_A)). \end{aligned} \quad (21)$$

To show that this condition holds, denote the difference between the left- and right-hand sides of (21) by

$$\Gamma(v_A, v_B) = \int_{\underline{v}}^{v_B} \hat{x}_B(v) dH(v) + \int_{v^M}^{v_A} \hat{x}_A(v) dH(v) - 1 + (1 - H(v_B))(2 - H(v_A)).$$

We want to show that $\Gamma(v_A, v_B) \leq 0$ for all $(v_A, v_B) \in V_A^* \times V_B^*$. Observe that $\Gamma(\bar{v}, v^M) \leq 0$ must hold, since

$$\Gamma(\bar{v}, v^M) = \int_{\underline{v}}^{v^M} \hat{x}_B(v) dH(v) + \int_{v^M}^{\bar{v}} \hat{x}_A(v) dH(v) - 1 = \int_{\underline{v}}^{\bar{v}} x_A(v) + x_B(v) dH(v) - 1 \leq 0,$$

where the second equality holds by definition of the new allocation rule and the disjoint support assumption, and where the inequality holds because the aggregate probability of trade in the original mechanism had to be smaller than 1.

Because $\Gamma(\bar{v}, v^M) \leq 0$, we also have that $\Gamma(\bar{v}, v_B) \leq \Gamma(\bar{v}, v^M) \leq 0$ for all $v_B \leq v^M$. Further, we have that for $v_A = v^M$ and all $v_B \leq v^M$,

$$\Gamma(v^M, v_B) = \int_{\underline{v}}^{v_B} \hat{x}_B(v) dH(v) - H(v_B) \leq 0,$$

where the inequality holds as B can receive the good with probability no higher than 1.

To conclude, observe that Γ is single-dip in v_A . To see this, note that

$$\frac{d\Gamma(v_A, v_B)}{dv_A} = dH(v_A)(\hat{x}_A(v_A) - 1 + H(v_B)).$$

But $d\Gamma(v_A, v_B)/dv_A$ can be negative only when $\hat{x}_A(v_A) < 1 - H(v_B)$ and must remain positive once it becomes larger than 0, given that \hat{x}_A is increasing by incentive compatibility.

Because of this, (21) must hold, since for all $(v_A, v_B) \in V_A^* \times V_B^*$,

$$\Gamma(v_A, v_B) \leq \max\{\Gamma(v^M, v_B), \Gamma(\bar{v}, v_B)\} \leq 0,$$

where the first inequality holds since $\Gamma(v_A, v_B)$ is single-dip in v_A by the previous argument, and the second one holds since $\Gamma(v^M, v_B) \leq 0$ and $\Gamma(\bar{v}, v_B) \leq 0$.

This establishes the optimality of the maximally divisive design for the three scenarios considered. Analogous arguments also apply for any other value designs with disjoint interiors, thereby establishing the optimality of the maximally divisive design. ■

Proof of Proposition 5: Maximal Divisiveness and Surplus Bounds To prove the result, let $\underline{v} = 0$ and consider the maximally divisive value design $F^+ = (G, 1)$. By the optimality of posted prices in single-buyer settings, the optimal mechanism for value design F^+ has a simple structure. It requires selling to bidder A at some fixed price r^+ , whenever A is willing to purchase at r^+ , while not selling the good otherwise—for instance, when G is regular, $r^+ = \psi_G^{-1}(0)$. The same revenue though can also be obtained with any other value design (\hat{F}_A, \hat{F}_B) satisfying $\hat{F}_A(v)\hat{F}_B(v) = G(v)$ for all v . This can be accomplished by selecting a mechanism (x, t) in which the principal sets again r^+ as the price of the good and sells to one of the agents willing to pay said amount. Revenue from such a mechanism, (x, t) , coincides with the optimal revenue for the maximally divisive design F^+ , since

$$R(F^+) = r^+(1 - G(r^+)) = r^+ \Pr(\max\{v_A, v_B\} \geq r^+) = \bar{R}(x, t | \hat{F}) \leq R(\hat{F}),$$

where the inequality follows as posted price mechanisms need not be optimal, when both agents can have positive value for the good.

When the surplus distribution is binary ($G(v) = p$ for $v \in [\underline{v}, \bar{v})$ and $G(\bar{v}) = 1$), the allocation rule and the transfers in the optimal mechanism for the maximally divisive, $(F_i^+, F_j^+) = (G, 1)$, amount to

$$\begin{aligned} x_i(v_i, v_j | F^+) &= \mathbb{1}(v_i = \bar{v}) \text{ and } x_j(v_j, v_i | F^+) = \mathbb{1}(v_i = \underline{v}), \\ t_i(v_i, v_j | F^+) &= x_i(v_i, v_j | F^+) \bar{v} \text{ and } t_j(v_j, v_i | F^+) = x_j(v_j, v_i | F^+) \underline{v}. \end{aligned}$$

These rules award the good to i in case of high value at price \bar{v} , and sell the good to j at

price \underline{v} otherwise. This mechanism is incentive compatible since interim allocation rules are increasing. It is individually rational, as ex-post payoffs are non-negative. Further, it is optimal because revenue coincides with surplus $R(F^+) = S(F^+)$. When $\underline{v} > 0$, this design is uniquely optimal. To see this, consider any other value design \hat{F} in which $\hat{F}_A \hat{F}_B = G$. If so, $\hat{V}_i = \hat{V}_j = \{\underline{v}, \bar{v}\}$. By construction, the designs \hat{F} and F^+ generate the same surplus, $S(\hat{F}) = S(F^+)$. To show that $R(\hat{F}) < R(F^+)$, we argue that some agents must obtain a positive information rent in the optimal mechanism associated with value design \hat{F} . By incentive compatibility, if an agent reports the high value truthfully, it must be that

$$U_i(\bar{v}|\hat{F}) = x_i(\bar{v}|\hat{F})\bar{v} - t_i(\bar{v}|\hat{F}) \geq x_i(\underline{v}|\hat{F})\bar{v} - t_i(\underline{v}|\hat{F}).$$

If $x_A(\underline{v}) = x_B(\underline{v}) = 0$, the result follows because the mechanism does not generate full surplus, and thus $R(\hat{F}) < S(\hat{F})$. If instead $x_i(\underline{v}) > 0$ for at least some player i , by individual rationality, we further have that

$$U_i(\bar{v}|\hat{F}) \geq x_i(\underline{v})\bar{v} - t_i(\underline{v}) > x_i(\underline{v})\underline{v} - t_i(\underline{v}) = U_i(\underline{v}|\hat{F}) \geq 0.$$

If so, agent i 's ex-ante utility is strictly positive, $U_i(\hat{F}) = \hat{F}_i(\bar{v})U_i(\bar{v}|\hat{F}) + \hat{F}_i(\underline{v})U_i(\underline{v}|\hat{F}) > 0$. Thus, we have that

$$R(\hat{F}) \leq S(\hat{F}) - U_A(\hat{F}) - U_B(\hat{F}) < S(F^+) = R(F^+),$$

and divisive designs maximize revenue. ■

Proof of Proposition 6: Optimal Threshold Designs Consider any ternary distribution, $G(v) = p$ for $v \in [0, w)$, $G(v) = q > p$ for $v \in [w, \bar{v})$, and $G(\bar{v}) = 1$, and value design $F^{[w]}$. In such a value design, $F_i^{[w]}(0) = F_i^{[w]}(w) = q$, while $F_j^{[w]}(0) = p/q$, $F_j^{[w]}(w) = 1$. The optimal mechanism associated with this design amounts to

$$\begin{aligned} x_i(v_i, v_j|F^{[w]}) &= \mathbb{1}(v_i = \bar{v}) \text{ and } x_j(v_j, v_i|F^{[w]}) = \mathbb{1}(v_j = w \cap v_i = 0), \\ t_i(v_i, v_j|F^{[w]}) &= x_i(v_i, v_j|F^{[w]})\bar{v} \text{ and } t_j(v_j, v_i|F^{[w]}) = x_j(v_j, v_i|F^{[w]})w. \end{aligned}$$

These rules award the good to i in case of high value at price \bar{v} , and otherwise offer to sell the good to j at price w . This mechanism is incentive compatible since interim

allocation rules are increasing. It is individually rational, as ex-post payoffs are non-negative. Further, it is optimal because revenue coincides with surplus $R(F^{[w]}) = S(F^{[w]})$. It is uniquely optimal because as before, any other design for which the constraint binds would lead to an agent having two positive values and accruing some information rent. ■

Proof of Proposition 7: The Optimality of Divisive Designs If the minimally differentiated design is weakly regular, its revenue amounts to

$$R(F^-) = 2r^- \left(\sqrt{G(r^-)} - G(r^-) \right) + \int_{r^-}^{\bar{v}} v dJ^-(v) = (1 - G(r^-))r^- + \int_{r^-}^{\bar{v}} (v - r^-) dJ^-(v).$$

Now consider some number $w < r^-$, a threshold design $F^{[r^-]}$ in which B has low values, and a mechanism (x, t) such that

$$\begin{aligned} x_A(v_A, v_B) &= \mathbb{1}(v_A \geq r^-) \text{ and } x_B(v_A, v_B) = \mathbb{1}(v_B \geq w \cap v_A = \underline{v}), \\ t_A(v_A, v_B) &= \mathbb{1}(v_A \geq r^-)r^- \text{ and } t_B(v_B, v_A) = \mathbb{1}(v_B \geq w \cap v_A = \underline{v})w. \end{aligned}$$

In this mechanism, the good is sold to A at price r^- when their value exceeds r^- , and is offered to B at price w otherwise. The revenue from this mechanism for threshold design $F^{[r^-]}$ amounts to

$$\bar{R}(x, t | F^{[r^-]}) = (1 - G(r^-))r^- + (G(r^-) - G(w))w.$$

The result then follows since

$$R(F^{[r^-]}) \geq \bar{R}(x, t | F^{[r^-]}) > r^-(1 - G(r^-)) + \int_{r^-}^{\bar{v}} (v - r^-) dJ^-(v) = R(F^-),$$

given the assumptions invoked on G . ■

Proof of Proposition 8: Auctions and Surplus Bound To prove the result on second-order statistics, fix an arbitrary value $v \in [\underline{v}, \bar{v}]$, and consider the following relaxed problem (which does not account for the restriction requiring cumulative distributions to increase),

$$\min_{F_A(v), F_B(v)} F_A(v) + F_B(v) - G(v) \text{ s.t. } F_A(v)F_B(v) \geq G(v).$$

The objective function of this problem is linear, and the constraint set is quasi-convex. Taking first-order conditions implies that at the minimum

$$F_A(v) = F_B(v) = \sqrt{G(v)}.$$

Because $\sqrt{G(v)} \in [0, 1]$ and $\sqrt{G(v)}$ is increasing, for all $v \in [\underline{v}, \bar{v}]$, this procedure delivers a proper value design $(F_A^-, F_B^-) = (\sqrt{G}, \sqrt{G})$. Further by construction, we have that for this value design

$$J^-(v) = 2\sqrt{G(v)} - G(v) < F_A(v) + F_B(v) - G(v) = J(v).$$

Thus, the distribution of the second-order statistic J^- associated with the minimally divisive design (F_A^-, F_B^-) first-order stochastically dominates the distribution of the second-order statistic J associated with any other design (F_A, F_B) satisfying $F_A F_B = G$.

To prove the result on standard auctions, consider any standard auction with reserve price r —meaning that

$$x_i(\mathbf{v}|r) = \mathbb{1}(v_i > \max\{v_j, r\}).$$

Select any incentive compatible payment rule for such an auction, for instance $t_i(\mathbf{v}|r) = \max\{v_j, r\} \mathbb{1}(v_i > \max\{v_j, r\})$. By revenue equivalence, the revenue in this auction is maximized by the minimally divisive design, since

$$\begin{aligned} \bar{R}(x(r), t(r)|F_A, F_B) &= r(J(r) - G(r)) + \int_r^{\bar{v}} v dJ(v) = \int_r^{\bar{v}} \max\{v, r\} dJ(v) - rG(r) \\ &\leq \int_r^{\bar{v}} \max\{v, r\} dJ^-(v) - rG(r) = \bar{R}(x(r), t(r)|F_A^-, F_B^-), \end{aligned}$$

where the inequality follows since $J^- \succsim_1 J$ and $\max\{r, v\}$ is increasing in v . ■

References

- Anderson, S. P. and R. Renault (2006). Advertising content. *American Economic Review* 96(1), 93–113.
- Bergemann, D. and M. Pesendorfer (2007). Information structures in optimal auctions. *Journal of economic theory* 137(1), 580–609.
- Bloch, P. H. (1995). Seeking the ideal form: Product design and consumer response. *Journal of marketing* 59(3), 16–29.
- Bobkova, N. (2019). Information choice in auctions. Technical report, Working paper.

- Border, K. C. (1991). Implementation of reduced form auctions: A geometric approach. *Econometrica: Journal of the Econometric Society*, 1175–1187.
- Bulow, J. and J. Roberts (1989). The simple economics of optimal auctions. *The Journal of Political Economy* 97, No. 5, 1060–1090.
- Cantillon, E. (2008). The effect of bidders’ asymmetries on expected revenue in auctions. *Games and Economic Behavior* 62(1), 1–25.
- Condorelli, D. and B. Szentes (2020). Information design in the holdup problem. *Journal of Political Economy* 128(2), 681–709.
- Condorelli, D. and B. Szentes (2022). Surplus sharing in cournot oligopoly. *Theoretical Economics* 3(17), 955–975.
- Deb, R. and M. M. Pai (2017). Discrimination via symmetric auctions. *American Economic Journal: Microeconomics* 9(1), 275–314.
- Devanur, N. R., Z. Huang, and A. Psomas (2016). The sample complexity of auctions with side information. *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, 426–439.
- Eso, P. and B. Szentes (2007). Optimal information disclosure in auctions and the handicap auction. *The Review of Economic Studies* 74(3), 705–731.
- Ganuza, J.-J. and J. Penalva (2019). Information disclosure in optimal auctions. *International Journal of Industrial Organization* March(63), 460–479.
- Hart, S. and P. J. Reny (2015). Maximal revenue with multiple goods: nonmonotonicity and other observations. *Theoretical Economics* 10(3), 893–922.
- Hartline, J. D. (2013). Mechanism design and approximation. *Book draft. October* 122.
- Johnson, J. P. and D. P. Myatt (2006). On the simple economics of advertising, marketing, and product design. *American Economic Review* 96(3), 756–784.
- Li, H. and X. Shi (2017). Discriminatory information disclosure. *American Economic Review* 107(11), 3363–3385.
- Moutinho, L. (2000). Segmentation, targeting, positioning and strategic marketing. *Chapter 5*, 121–166.
- Myerson, R. B. (1981). Optimal auction design. *Mathematics of operations research* 6(1), 58–73.
- Roesler, A.-K. and B. Szentes (2017). Buyer-optimal learning and monopoly pricing. *American Economic Review* 107(7), 2072–80.
- Sorokin, C. and E. Winter (2018). Pure information design in classical auctions. Technical report, Working paper.
- Yang, K. H. and A. K. Zentefis (2024). Monotone function intervals: Theory and applications. *American Economic Review* 114(8), 2239–70.

DIVISIVE BY DESIGN: SUPPLEMENTARY APPENDIX

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A Applications

Consider a celebrity auction. In these auctions, celebrities offer fans the opportunity to spend time with them for a specific activity.¹ The activity, which serves as a feature of the good, can either increase, decrease, or have no effect on the value of certain agents. To make this example more concrete, imagine a famous tennis player participating in such an auction. The tennis player could offer a casual tennis match, an ambitious training session, or, alternatively, a dinner with the auction winner. The type of good—whether a celebrity plus casual tennis, a celebrity plus intense training session, or a celebrity plus dinner—can significantly influence valuations and has the potential to help screen bidders’ preferences. Some bidders may prefer an intense training session, others may simply want to meet their favorite tennis player and play a casual round, while some might favor the dinner to have the opportunity for more personal interaction. Our results suggest that the activity chosen should cater to the preferences of specific bidders, thereby increasing surplus and facilitating better screening.

Similarly, our model applies to the art market. An artist creating an object with the goal of selling it should aim to incorporate features that increase surplus while minimizing information rents. The artist can select the art form (e.g., sculpture or painting), the materials used, and the size of the object. Each of these decisions has the potential to influence bidders’ preferences and, in turn, their bids (Bocart, Gertsberg, and Pownall, 2022). For example, choosing the color blue in painting has been associated with higher revenues (Ma, Noussair, and Renneboog, 2022). Similarly, larger paintings generate more income, even after accounting for the increased material costs.² The artist’s goal is then to create a piece that increases the value for buyers and, additionally, better screens their valuations. This helps explain why art often polarizes, with some customers valuing it substantially while others may feel alienated.

¹For example, this can be booked through <https://www.charitybuzz.com/>.

²See, for instance, https://www.gallerytoday.com/blog/122_what-determines-the-price-of-a-painting-does-size-matter.html, which argues that this is due to larger paintings being seen as more prestigious.

B Designing Mechanisms with General Distributions

Our analysis in the main paper relies on optimal value designs that frequently feature atomic distributions with mass points at specific values. Since standard mechanism design results typically assume smooth, continuous distributions, we demonstrate how to apply classical revenue equivalence insights when agents' value distributions, F_A and F_B , may contain atoms or other discontinuities. This extension ensures that our theoretical results remain valid for the discrete and mixed distributions that emerge as optimal in our setting.

Since the revelation principle holds irrespective of distributional assumptions, we begin by considering a direct mechanism $(x, t) : [\underline{v}, \bar{v}]^2 \rightarrow \Delta(A, B, 0) \times \mathbb{R}^2$. Define the interim allocation probability and the expected transfer of agent i reporting value z_i , when others are sincere, as follows

$$x_i(z_i) = \int_{V_{-i}} x_i(z_i, v_{-i}) dF_{-i}(v_{-i}),$$

$$t_i(z_i) = \int_{V_{-i}} t_i(z_i, v_{-i}) dF_{-i}(v_{-i}).$$

When the distributions are not everywhere differentiable or continuous on the support, we use Riemann-Stieltjes integrals to keep the notation compact. This means that the differential accounts for jumps at points of discontinuity of F_{-i} and neglects points where the cumulative distribution is continuous but not differentiable. Formally, for a cumulative distribution F with support V and a function $u : V \rightarrow \mathbb{R}$, we have

$$\int_V u(\mathbf{v}) dF(\mathbf{v}) = \int_S u(\mathbf{v}) \frac{\partial F(\mathbf{v})}{\partial \mathbf{v}} d\mathbf{v} + \sum_{\mathbf{v} \in D} u(\mathbf{v}) [F_+(\mathbf{v}) - F_-(\mathbf{v})],$$

where $D \subseteq V$ denotes the set of discontinuity points of F , $S \subseteq V$ denotes the set of points where F is differentiable, and $F_+(\mathbf{v})$ and $F_-(\mathbf{v})$ denote the right and left limits of F at \mathbf{v} , respectively.

With this notation, the expected payoff of type v_i claiming to be z_i is given by

$$v_i x_i(z_i) - t_i(z_i).$$

As usual, a mechanism (x, t) is incentive compatible on $[\underline{v}, \bar{v}]$ when truth-telling is optimal for all agents

$$v_i x_i(v_i) - t_i(v_i) = \max_{z_i \in [\underline{v}, \bar{v}]} [v_i x_i(z_i) - t_i(z_i)] \equiv U_i(v_i).$$

Even when value distributions are not well-behaved, incentive compatibility can be restated as requiring that the interim probability of winning the object increases in the agent's type, as the following lemma shows.

Lemma: A direct mechanism is incentive compatible on $[\underline{v}, \bar{v}]$ for agent i if and only if $x_i(v_i)$ is non-decreasing and $U'_i(v_i) = x_i(v_i)$ for all $v_i \in [\underline{v}, \bar{v}]$.

Proof (if part): If incentive compatibility holds for agent i , for any two values $v_i, z_i \in [\underline{v}, \bar{v}]$, we have that

$$U_i(z_i) \geq z_i x_i(v_i) - t_i(v_i) = U_i(v_i) + (z_i - v_i) x_i(v_i).$$

The same inequality holds when switching v_i and z_i , so we obtain

$$(z_i - v_i) x_i(z_i) \geq U_i(z_i) - U_i(v_i) \geq (z_i - v_i) x_i(v_i). \quad (1)$$

Condition (1) also implies that

$$(z_i - v_i) (x_i(z_i) - x_i(v_i)) \geq 0,$$

which is equivalent to $x_i(v_i)$ being non-decreasing. Dividing both sides of (1) by $(z_i - v_i)$ and taking the limit as $z_i \rightarrow v_i$ yields

$$\lim_{z_i \rightarrow v_i} \frac{U_i(z_i) - U_i(v_i)}{z_i - v_i} = U'_i(v_i) = x_i(v_i).$$

Proof (only if part): If $U'_i(v_i) = x_i(v_i)$, the expected payoff depends only on the allocation rule, since:

$$U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} x_i(s) ds.$$

Thus, for any v_i and any z_i , we have

$$U_i(z_i) - U_i(v_i) = \int_{\underline{v}}^{z_i} x_i(s) ds - \int_{\underline{v}}^{v_i} x_i(s) ds \geq (z_i - v_i) x_i(v_i),$$

where the inequality holds because $x_i(v_i)$ is non-decreasing. This inequality is equivalent to incentive compatibility. ■

We can use this restatement of incentive compatibility to derive the classical revenue equivalence result for this setting.

Revenue Equivalence Theorem: In any incentive compatible, direct mechanism (x, t) , the interim transfer of agent i with value v_i is given by:

$$t_i(v_i) = t_i(\underline{v}) - \underline{v}x_i(\underline{v}) + v_ix_i(v_i) - \int_{\underline{v}}^{v_i} x_i(s) ds.$$

Proof: Since $U_i(v_i) = v_ix_i(v_i) - t_i(v_i)$ and $U_i(\underline{v}) = \underline{v}x_i(\underline{v}) - t_i(\underline{v})$, incentive compatibility implies that

$$v_ix_i(v_i) - t_i(v_i) = \underline{v}x_i(\underline{v}) - t_i(\underline{v}) + \int_{\underline{v}}^{v_i} x_i(s) ds.$$

Thus, interim transfers in any two incentive compatible mechanisms with the same allocation rule must coincide up to a constant. ■

Revenue equivalence also implies that even when value distributions are discontinuous or non-differentiable, the designer's problem coincides with the maximization of virtual surplus. To see this, recall that a mechanism is individually rational if

$$U_i(v_i) \geq 0 \quad \text{for all } v_i.$$

When incentive compatibility holds, individual rationality (IR) simplifies to

$$U_i(\underline{v}) \geq 0 \quad \text{or} \quad t_i(\underline{v}) \leq \underline{v}x_i(\underline{v}).$$

By revenue equivalence, we also know that in an incentive-compatible and direct mechanism, the expected payment of agent i is given by

$$\mathbb{E}[t_i(v_i)] = -U_i(\underline{v}) + \int_{\underline{v}}^{\bar{v}} v_ix_i(v_i) dF_i(v_i) - \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v_i} x_i(s) ds dF_i(v_i).$$

Exploiting Riemann-Stieltjes integrals, we can, as usual, change the order of integration to obtain

$$\int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v_i} x_i(s) ds dF_i(v_i) = \int_{\underline{v}}^{\bar{v}} \int_s^{\bar{v}} dF_i(v_i) x_i(s) ds = \int_{\underline{v}}^{\bar{v}} (1 - F_i(v_i)) x_i(v_i) dv_i.$$

Define the virtual valuation as

$$\psi_i(v_i) = \begin{cases} v_i - \frac{1-F_i(v_i)}{f_i(v_i)} & \text{if } v_i \in S_i \\ v_i & \text{if } v_i \notin S_i \end{cases}$$

where S_i denotes the set of points in V_i where F_i is differentiable.³ With this notation, the principal's expected revenue is again equal to the expected virtual surplus generated by the mechanism (x, t) , up to a constant:

$$R(x, t | F_A, F_B) = \sum_{i \in N} \left[-U_i(\underline{v}) + \int_{\underline{v}}^{\bar{v}} \psi_i(v_i) x_i(v_i) dF_i(v_i) \right].$$

As with non-atomic distributions, the optimal mechanism can therefore be found by solving

$$\max_{(x, t)} \sum_{i \in N} (t_i(\underline{v}) - \underline{v} x_i(\underline{v})) + \int_V \left[\sum_{i \in N} \psi_i(v_i) x_i(\mathbf{v}) \right] dF(\mathbf{v}),$$

subject to incentive compatibility

$$(z_i - v_i)(x_i(z_i) - x_i(v_i)) \geq 0 \quad \text{for all } z_i, v_i \in V_i,$$

and individual rationality

$$t_i(\underline{v}) \leq \underline{v} x_i(\underline{v}) \quad \text{for all } i \in \{A, B\}.$$

Alternatively, a feasible interim allocation rule is optimal for suitable transfers if and only if it solves

$$\max_x \sum_{i \in N} \int_{V_i} [\psi_i(v_i) x_i(v_i)] dF_i(v_i),$$

$$\text{subject to } (z_i - v_i)(x_i(z_i) - x_i(v_i)) \geq 0 \quad \text{for all } z_i, v_i \in V_i,$$

since individual rationality binds at the optimum. The latter corresponds to the maximization problem used in two of our results.

³Virtual values need not be defined outside the support V_i .

C Threshold Design Example: $G(v) = v$

To illustrate the profitability of threshold designs, we work through a concrete example where values are uniformly distributed, so $G(v) = v$. This setting offers clean analytical solutions while capturing the essential economic forces at play.

The Minimally Differentiated Benchmark Consider first the symmetric case where both agents have identical value distributions: $F_A^-(v) = F_B^-(v) = \sqrt{v}$. The virtual values in this design are:

$$\psi_i^-(v) = 3v - 2\sqrt{v}.$$

These virtual values turn positive when $v > 4/9$, establishing the optimal reserve price. Since virtual values are strictly increasing for $v \geq 4/9$ (with derivative $3 - 1/\sqrt{v} \geq 3/2 > 0$), the optimal mechanism allocates the object to the highest-value bidder above this threshold. This yields revenue of:

$$R(F^-) = \int_{4/9}^1 (3v - 2\sqrt{v}) dv = 43/162 \approx 0.265.$$

The Maximally Differentiated Alternative At the other extreme, consider a maximally differentiated design where only agent A values the good: $F_A^+(v) = v$ and $F_B^+(v) = 1$. This reduces the problem to a simple monopoly pricing problem with a single buyer. The optimal mechanism becomes a posted price of $1/2$, generating revenue of exactly $R(F^+) = 1/4 = 0.25$.

Notably, maximal divisiveness performs *worse* than the symmetric design. This may appear to be a striking reversal of the typical advantage of asymmetric designs we see in other settings. But the optimal design is not a symmetric one; instead it is a threshold design aimed at extracting surplus from types that would otherwise be excluded.

The Threshold Design Solution Now consider a w -threshold design for $w \in [0, 1]$. This design assigns to agent:

- A a uniform distribution on $[0, w]$: $F_A^{[w]}(v) = \min\{v/w, 1\}$;
- B value 0 with probability $G(w)$ and uniform distribution on $[w, 1]$: $F_B^{[w]}(v) = \max\{w, v\}$.

This creates a distinct segmentation: agent A is assigned to low values while agent B ends up with the high values and an atom at 0. The virtual values become

$$\psi_A^w(v) = 2v - w \text{ for } v \leq w \quad \text{and} \quad \psi_B^w(v) = 2v - 1 \text{ for } v \geq w.$$

Since these are increasing, the optimal mechanism allocates to the agent with the highest non-negative virtual value. This generates revenue:

$$R(F^{[w]}) = w \int_0^w \max\{2v_A - w, 0\} d\frac{v_A}{w} + \int_0^w \int_w^1 \max\{2v_A - w, 2v_B - 1, 0\} dv_B d\frac{v_A}{w},$$

where the two terms amount respectively to case in which B has value 0 and the case in which they value above w .

Computing the Optimal Threshold The analysis splits into two cases based on whether w exceeds $1/2$, as depicted in Figure 1.

Case 1: $w > 1/2$ In this scenario, there never is exclusion. After working through the integration regions where different agents have higher virtual values, revenue becomes

$$R(F^{[w]}) = \frac{w^2}{4} + \frac{(1-w)(25w^2 - 2w + 1)}{24w}.$$

Case 2: $w \leq 1/2$ In these scenario, it is possible that the good is not allocated to either agent. Revenue calculations are further complicated by exclusion, but yield

$$R(F^{[w]}) = \frac{w^2}{4} + \frac{6 + 3w - 5w^2}{24}.$$

Revenue Performance Figure 2 shows that revenue peaks at $w^* \approx 0.648$, achieving approximately 0.336—a substantial improvement over both benchmarks. The threshold design outperforms the symmetric design by 27% and the maximally differentiated design by 34%.

The threshold design maintains competitive pressure between agents (unlike the posted price solution) while creating specialized market segments that reduce information rents. Agent A faces competition only from low-value realizations of agent B , while agent B

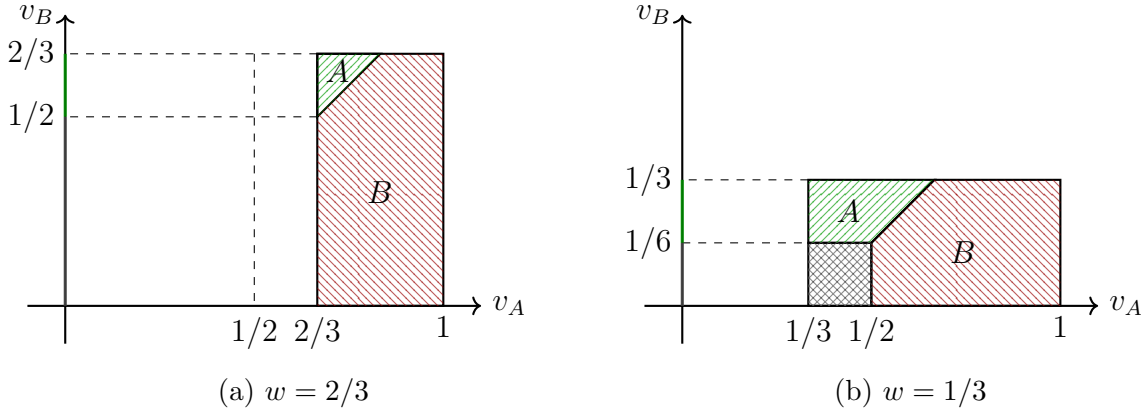


Figure 1: Integration regions for w -threshold designs when $G(v) = v$. Panels (a)–(b) depict integration regions for different values of w : in green north-east lines, value profiles for which A wins the good; in red north-west lines, value profiles for which B wins the good; and in gray crosshatch, value profiles for which the good is not allocated.

competes primarily in the high-value range where the surplus gains justify the information rents.

This example demonstrates that when surplus is constrained, optimal designs remain divisive but require nuanced market segmentation rather than extreme differentiation or symmetry.

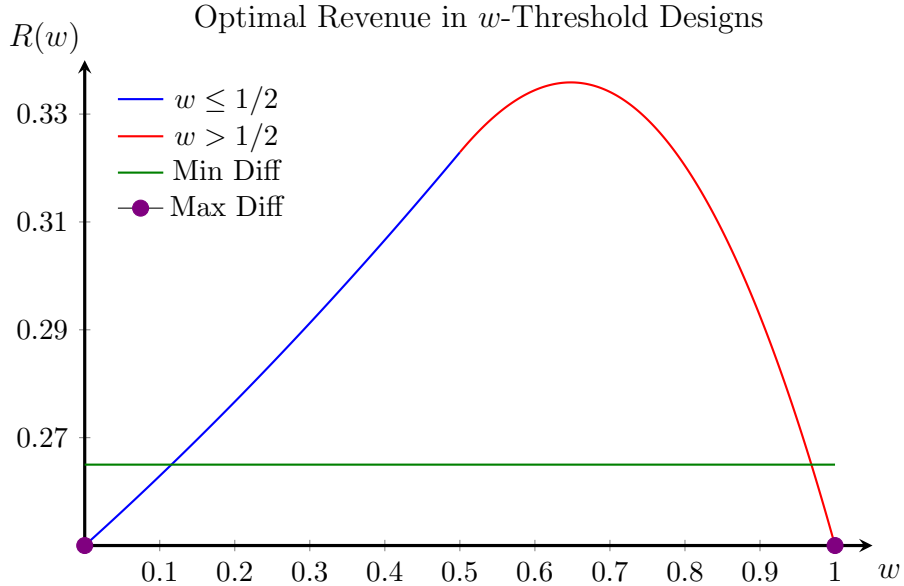


Figure 2: The plot depicts optimal revenue as a function of w , as well as optimal revenue in both the maximally and minimally divisive designs.

D Correlated Private Values

Maintaining the independence of value distributions across design problems is a natural assumption if one believes that agents should not be able to infer anything about their competitors' realized values based on their own. This assumption is not only plausible but also makes the problem more challenging, as it requires accounting for agents' information rents, rather than focusing solely on surplus design. However, many of our results extend to settings where signals are not independent, and bidders have some information about the realized values of their competitors.

Consider a correlated information setting, as discussed by [Cr  mer and McLean \(1985\)](#), where revenue generically coincides with surplus, $\mathbb{E}[\max\{v_A, v_B\}]$. To provide some insight while staying close to the core of our analysis, we allow the principal to design marginal value distributions for both bidders, but not the correlation structure between their values, which is fixed. Additionally, we assume that bidders know their own value and have some information about their competitors' values. This approach mirrors our baseline analysis, where the principal designs the marginal value distributions for both agents but cannot influence the independence of their signals.

To fix the correlation structure, consider any copula $W : [0, 1]^2 \rightarrow [0, 1]$, where $W(F_A(v_A), F_B(v_B)) = \Pr(V_A \leq v_A, V_B \leq v_B)$ identifies the joint probability of an event given the marginal distributions F_A and F_B . By Sklar's Theorem, copulas can be used to capture arbitrary correlation structures—such as independence, perfect positive correlation (concordance), and perfect negative correlation (discordance). Assume that W determines the underlying correlation of tastes across agents and cannot be influenced by the principal, as was the case in the original setup where $W(F_A(v_A), F_B(v_B)) = F_A(v_A)F_B(v_B)$. Instead, let the principal design the two marginal value distributions, $F_A(v_A)$ and $F_B(v_B)$. This approach is suitable for settings in which the features of the goods can be determined, but the correlation in tastes across agents must be taken as given due to factors beyond the principal's control.

No Spillovers When designing marginal distributions subject to the mean-bound constraint $\mathbb{E}_{F_i}[v_i] \leq k$ for all i , extreme bimodal designs remain optimal. This follows from the fact that variance increases the expected surplus, $\mathbb{E}[\max\{v_A, v_B\}]$, due to the convexity of the maximum operator. As an example, consider the case where values are discordant,

such that $W(F_A, F_B) = \min\{F_A, F_B\}$ for any pair of probabilities (F_A, F_B) . Additionally, assume $\bar{v} = 2$ and $k = 1$. In this case, surplus, or equivalently revenue, is maximized by a value design (F_A, F_B) in which both agents value the good at $v_i = 2$ or $v_i = 0$ with equal probability. This design corresponds to the maximally divisive design described in Proposition 2. Under such a maximally spread two-atom distribution, surplus is exactly equal to 2, since at least one agent must value the good at 2 due to the discordance. Thus, revenue also equals 2, and the principal secures this surplus by awarding the good with certainty to the agent with the realized value equal to 2. No other value design can lead to higher surplus, as values never exceed 2. Therefore, designs in which both distributions are maximally spread are optimal when values are negatively correlated.

With concordance $W(F_A, F_B) = \max\{0, F_A + F_B - 1\}$, the design in which one agent always values the good at the mean, $k = 1$, while the other agent's value is maximally spread, as described in Proposition 2, yields a surplus of 1.5. This exceeds the surplus from the design where both agents are maximally spread, which results in a surplus of 1. Similarly, if both agents have all mass at the mean, the surplus equals 1. Consequently, with positively correlated values, the principal is better off when one agent has all mass at the mean, while the other agent's value is maximally spread, in line with our finding when values are independent.

Spillovers When designing marginal distributions subject to the linear constraint, $F_A + F_B = H$, maximally divisive designs remain optimal. This follows because such designs increase surplus, $\mathbb{E}[\max\{v_A, v_B\}]$, by minimizing the chance of having two agents with high values.

As an example, consider the concordant copula discussed above, and assume that $H(v) = v$ for all $v \in [0, 2]$. In this case, surplus in the maximally divisive design, $(F_A, F_B) = (H - 1, H)$, coincides with the expected value, $\mathbb{E}_{H-1}[v] = 1.5$, of the high value agent A . Thus, revenue also amounts to 1.5. This revenue can be achieved by never allocating the good to agent B , but always awarding it to agent A . To extract full surplus, the transfer of the high value agent A is set to the value associated to the quantile reported by agent B , i.e., $t_A(\mathbf{v}) = \inf\{v \mid H(v) - 1 \geq H(v_B)\}$, while B never pays anything. In such a setting, revenue under the minimally divisive design, where both agents draw values from $H/2$, simply amounts to the expected value of one of the two agents, $\mathbb{E}_{H/2}[v] = 1$. This highlights that the optimality of maximally divisive value

designs, as discussed in Proposition 4, does not depend on the independence of value distributions, but holds in more general settings. However, other designs may also be optimal for specific correlation structures.

In the same example, when the copula is discordant, the minimally divisive design yields the same surplus as the maximally divisive one. Further, discordance generally increases surplus since differentiation in values increases gains from trade.

E Cost of Value Designs

It is natural in many contexts to assume that attributes can be added for free, as it is far from obvious whether and how much the addition of features costs. For completeness, we explore how our results change if we allow for costs, with $C(F)$ representing the cost of a particular value design F .

A natural starting point is to consider costs that increase with the “quality” of the design, measured by first-order stochastic dominance. Under such cost structures, the principal would first select a distribution according to costs, which subsequently serves as the boundary distribution that cannot be exceeded. This transforms our constrained optimization problems into cost-benefit trade-offs, where the principal balances the revenue gains from better designs against their implementation costs.

We characterize the types of cost functions that yield the same designs as those derived for our various constraints, demonstrating the robustness of our main insights.

No Spillovers If an attribute affects each agent individually, but there are no spillovers, then this corresponds to a setting with separable cost functions, i.e., $C(F) = c(F_A) + c(F_B)$. Designs that are optimal for $\mathbb{E}_{F_i}[v] \leq k$ would naturally arise if the separable cost functions depend solely on the mean of the value distribution chosen for each agent. Formally, this would be represented as $C(F) = c(\mathbb{E}_{F_A}[v]) + c(\mathbb{E}_{F_B}[v])$ for some increasing function $c : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}_+$. In such environments, the designer would first select the mean based on cost considerations, and then choose the shape of the optimal design by maximizing revenue subject to the constraint $\mathbb{E}_{F_i}[v] \leq k$. Thus, even in costly design settings, the principal would still choose features that result in bimodal value designs.

Spillovers With spillovers, a cost that depends on the sum of average values for the good will still yield the result that it is optimal to assign all value to one agent, in the manner described in Proposition 3.

For the additive constraint, $H(v) = F_A(v) + F_B(v)$, cost functions must satisfy a weak linearity property to ensure the optimality of the maximal split distribution. This requires that any two value designs with the same sum incur the same cost. Formally, this would be represented as $C(F) = C(F')$ if $F_A + F_B = F'_A + F'_B$. This assumption is satisfied by common design cost functions, such as entropy, and would hold for integrable cost functions of the form:

$$\int_0^{\bar{v}} c(v) d[F_A(v) + F_B(v)]$$

for some function $c : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}_+$.

Finally, designs that are optimal for the multiplicative constraint, $G(v) = F_A(v)F_B(v)$, would also be optimal in costly design settings where design costs depend solely on the distribution of surplus and increase according to first-order stochastic dominance.

F Proofs of Corollaries

Proof of Corollary 1: Second-Order Stochastic Dominance We show that F^* is second-order stochastically dominated by G . The proof then follows because any distribution that is second-order stochastically dominated by G must have a mean no higher than k , and because F^* was optimal among all distributions with mean no higher than k .

If $G = F^*$, then F^* is second-order stochastically dominated by G trivially. So, suppose that $G \neq F^*$. If so, for some $\hat{v} < \bar{v}$ in the support of G and all $v \in [0, \hat{v})$,

$$\Delta(v) = G(v) - F^*(v) < 0,$$

because F^* maximizes the probability that the value is equal to 0 among all distributions with mean equal to k . Moreover $\Delta(v)$ is non-decreasing for any $v < \bar{v}$, because G is non-decreasing and F^* is constant for $v < \bar{v}$. Finally observe that at \bar{v} , Riemann-Stieltjes

integration by parts yields

$$\int_0^{\bar{v}} \Delta(s) ds = [s(G(s) - F^*(s))]_0^{\bar{v}} - \int_0^{\bar{v}} s d(G(s) - F^*(s)) = \mathbb{E}_{F^*}[v] - \mathbb{E}_G[v] = 0,$$

where the final equality holds because both distributions have mean k . But if so, F^* is second-order stochastically dominated by G , since $\int_0^v \Delta(s) ds = \int_0^v G(s) - F^*(s) ds < 0$ for all $v < \bar{v}$. ■

Proof of Corollary 2: No Value Destruction with Surplus Bound We establish the result by contradiction. Suppose by contradiction that there exists an optimal design (F_A^*, F_B^*) , in which $F_A^*(w)F_B^*(w) > G(w)$ for some non-excluded value w . If so, an alternative value design (\hat{F}_A, \hat{F}_B) exists that first-order stochastically dominates the original design, $\hat{F} \succsim_1 F^*$, and that still satisfies the constraint

$$F_A^*(w)F_B^*(w) > \hat{F}_A(w)\hat{F}_B(w) \geq G(w).$$

Because revenue is increasing in first-order stochastic dominance, the optimal revenue under design F^* is no larger than the optimal revenue under design \hat{F} . Further, revenue strictly increases when value w is not excluded and the mechanism is fixed to $(x(F^*), t(F^*))$, because higher transfers will be paid more often in the \hat{F} design. ■

Proof of Corollary 3: Profitable Surplus Reduction The observation follows directly from the proof of Proposition 1, but noting that equation (11) must now hold by the assumptions invoked on F_A and m . ■

References

- Bocart, F. Y., M. Gertsberg, and R. A. Pownall (2022). An empirical analysis of price differences for male and female artists in the global art market. *Journal of Cultural Economics* 46(3), 543–565.
- Cr  mer, J. and R. P. McLean (1985). Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. *Econometrica* 53, 345–361.
- Ma, M. X., C. N. Noussair, and L. Renneboog (2022). Colors, emotions, and the auction value of paintings. *European Economic Review* 142, 104004.

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