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Fischer, Marcel; Jankowski, Natascha

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Life cycle consumption and portfolio choice under real interest rate risk

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Marcel Fischer, Natascha Jankowski

Life cycle consumption and portfolio choice under real interest rate risk

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Life cycle consumption and portfolio choice under real interest rate risk[*†]

Marcel Fischer[‡] Natascha Jankowski[§]

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[‡]Copenhagen Business School, Department of Finance, Solbjerg Plads 3, DK-2000 Frederiksberg, Denmark and University of Konstanz, Department of Economics, Postbox 147, D-78457 Konstanz, Germany, phone: +49 7531 88 2645, e-mail: marcel.fischer@uni.kn.

[§]*Corresponding author.* University of Konstanz, Department of Economics, Postbox 147, D-78457 Konstanz, Germany, phone: +49 7531 88 4326, e-mail: natascha.jankowski@uni.kn.

Abstract

Life cycle consumption and portfolio choice under real interest rate risk

We set up a life cycle model with real interest rate risk to demonstrate that real interest rates have implications for optimal household consumption and investments. Lower interest rates lead to higher optimal stock investments and lower consumption. Ignoring the time-varying nature of real interest rates leads to overconsumption and underinvestment into stocks when interest rates are high and, ultimately, substantial welfare costs. Being exposed to an extended period of low interest rates even leads to substantial welfare losses when behaving optimally – particularly when being exposed to it at around retirement age when savings peak.

Key Words: consumption-savings decisions, real interest rate risk, life cycle model, household finance

JEL Classification Codes: E21, G11, G51

1 Introduction

Motivated by the high inflation rates and low interest rates observed in recent years, we are, to the best of our knowledge, the first to set up a life cycle model of optimal consumption and investment with stochastic time-varying real interest rates to explore how real interest rate risk affects optimal consumption and investment decisions. Our model predicts that when real interest rates are low, individuals should invest more in stocks and consume less. Ignoring the time-varying nature of real interest rates leads to suboptimal investment decisions, a tendency to overconsume and, ultimately, to welfare losses. Experiencing extended periods of low interest rates has welfare implications even when behaving optimally. These welfare effects peak at around retirement age when individuals' savings peak.

We show that individuals adjust their optimal consumption-investment decisions to the level of the interest rate. When real interest rates are low, individuals should invest more in the stock market as the present value of future labor income is higher compared to when interest rates are high. Since labor income is only weakly correlated with stock returns, a higher present value leads to higher stock holdings. When real interest rates are high, individuals should increase their consumption. With higher returns, they need less savings to attain the same future wealth level. This leaves more money for present consumption.

In a welfare analysis, we quantify the costs of ignoring the time-varying nature of interest rates. These costs can be decomposed into a level- and a risk-channel. Financial advisers' recommendations often build on the assumption of constant interest rates. Likewise, the financial advice given in many models assumes constant interest rates. This assumption, however, ignores both the level- and the risk-channel. Behaving optimally would increase their welfare by 0.8% of lifetime consumption. In a next step, we look at individuals who take the interest rate level into account. These individuals still do not understand the stochastic nature of interest rates (risk-channel) but observe the current interest rate and behave as if it stayed constant at that level (level-channel). We find that not accounting for the risk-channel leads to substantial overconsumption in high real interest environments and underconsumption in low real interest environments, thus, ultimately leading to welfare costs of 6.7% of lifetime consumption.

Our suboptimal behavioral rules are motivated by the empirical evidence on household inflation expectation formation, that documents that individuals' expectations often differ from those of fully informed rational agents (Coibion et al., 2018) and do not seem to be well-anchored in the data (Coibion et al., 2020; Binder and Rodrigue, 2018). As inflation expectations can be very heterogeneous across individuals (Weber et al., 2022), we allow for two different rules of how individuals determine their expectations about real interest rates in

our model. Especially in countries with low and stable inflation, households do not pay much attention to monetary policy and have generally little knowledge of what central banks do (Binder, 2017; Lamla and Vinogradov, 2019; Coibion et al., 2020). Thus, such individuals' inflation expectations are less influenced by monetary policy announcements, but rather by their own experiences. Realized inflation rates are a strong predictor of individuals' expectations of future inflation rates, i.e., if individuals observe high inflation, they also tend to believe that future inflation will be high (Jonung, 1981; Binder and Kamdar, 2022; Weber et al., 2023). Such individuals often use their own memories of past observations to form inflation expectations (Cavallo et al., 2017; D'Acunto et al., 2021). We capture this empirical phenomenon by assuming in one scenario that individuals take past real interest rates as representative for future real interest rates. Likewise, we account for individuals who do not account for monetary policy by exploring the consequences of behaving as if the real interest rate would permanently correspond to its long-run mean.

In a second welfare analysis, we also quantify the costs of being exposed to a low interest environment. From the individual's perspective, the interest environment can be seen as an exogenous event. Yet, experiencing low interest rates over an extended time period negatively affects the real growth rate of savings, thus, ultimately reducing consumption opportunities in the long run. We specifically investigate the implications of the low real interest environment following the financial crisis in 2007/2008. Even when individuals behave optimally, the exposure to this extended period of low interest rates leads to utility losses of up to 4.25% of remaining lifetime consumption compared to more regular interest rate fluctuations. The effect is strongest for individuals whose savings peak during the low interest rate environment, i.e., individuals experiencing the low interest environment shortly before and after retirement age.

There exists a huge life cycle literature, which does not explicitly model an interest rate process and instead assumes a constant interest rate. This literature focuses on, among others, the importance of accounting for labor income (Cocco et al., 2005; Catherine, 2022), owner-occupied homes as an asset class (Cocco, 2005; Yao and Zhang, 2005; Fischer and Stamos, 2013), low but constant interest rates (Horneff et al., 2019), or tax-deferred retirement saving (Dammon et al., 2004; Fischer and Gallmeyer, 2017). Our work contributes to this line of literature by, according to the best of our knowledge, setting up the first life cycle model with stochastic time-varying real interest rates to explore how real interest rate risk affects optimal consumption and investment decisions. In our model, utility-maximizing individuals choose real consumption as well as real investments into a stock market index and a money market account. Money market investments earn the real interest rate. Even though the money market account does not face default risk, its real return is subject to inflation

risk. We model the time-varying real interest rate by an Ornstein-Uhlenbeck process, which is the stochastic process underlying the interest rate model of Vasicek (1977).[1] This allows real interest rates to both become negative and exhibit mean reversion. In addition to financial income, individuals earn risky labor income as, e.g., in Cocco et al. (2005). Our model predicts that explicitly accounting for time-varying interest rates is important for two main reasons. First, time-varying interest rates affect optimal consumption-investment strategies. Lower interest rates lead to higher optimal stock investments and less consumption. Second, ignoring the time-varying nature of interest rates leads to significant welfare losses.

Another strand of literature examines optimal portfolio choice under stochastic interest rates and/or inflation. Munk et al. (2004) focus on determining an optimal asset allocation strategy while taking real interest rate risk into account and modeling mean-reverting stock returns. Deelstra et al. (2000) investigate portfolio choice when interest rates are stochastic. Contributing to this line of literature, that enhances our understanding of how interest rates affect optimal investment decisions, our work shows that optimal consumption decisions are also affected by time-varying real interest rates. Specifically, when interest rates are low, optimal consumption rates are also lower. Papers that model consumption decisions in the presence of time-varying (real) interest rates are the work of Campbell and Viceira (2001), Brennan and Xia (2002), Munk and Sørensen (2004), Munk and Sørensen (2010), and Aoki et al. (2024). These papers, however, have a different focus than our work. The early work by Campbell and Viceira (2001) and Brennan and Xia (2002) focuses on how to best hedge inflation risk in the portfolio. Munk and Sørensen (2004) extend this work by accounting for different interest rate dynamics. Munk and Sørensen (2010) focus on the valuation of labor income when income growth is an affine function in the short-term interest rate. Aoki et al. (2024) build a life cycle model in which money is held to reduce transaction frictions. They document that such frictions can help rationalize differences in observed stockholdings between richer homeowners with mortgage debt and poorer renters. Our work instead focuses on how time-varying interest rates affect optimal consumption-investment decisions. It also shows that ignoring the time-varying nature of interest rates leads to overconsumption and underinvestment in stocks when interest rates are high.

There are some theoretical studies that calculate the welfare cost of inflation on the macro level. For example, Cao et al. (2021) include financial innovations in an overlapping generations model. Auclert (2019) studies how unexpected inflation and changes in the real interest rate affect redistribution caused by monetary policy. Further, Meh et al. (2010) study welfare effects of inflation under different monetary policies. While these papers focus

[1]It is pretty common to use an Ornstein-Uhlenbeck process for the interest rate dynamics, see e.g. Brennan and Xia (2002), Campbell and Viceira (2001), among others.

on the macro level, our work focuses on the individual investor's level and illustrates how their optimal behavior changes with the level of the interest rate.

This paper proceeds as follows: Section 2 introduces our life cycle model and its calibration. Section 3 presents the policy functions and simulation results and assesses the impact of living in a low interest environment. Section 4 documents the robustness of our results with respect to several parameter choices. Section 5 concludes.

2 The model

We set up a realistically calibrated life cycle model of consumption and portfolio choice. Individuals enter the life cycle at age 20 and live up to a maximum age of 100 while facing mortality risk. During their lifetime individuals receive stochastic labor income similar to Cocco et al. (2005). From age 20 to 65 the labor income is stochastic, once individuals are retired at age 66 they receive a constant fraction (replacement ratio) of their last working income (at age 65). Individuals have to decide periodically on their consumption level and portfolio allocation in order to maximize their expected present discounted lifetime utility. To allow for a separation between the elasticity of intertemporal substitution and the degree of risk aversion, we assume the individual to have Epstein-Zin preferences (Epstein and Zin, 1989).

2.1 The capital market

Individuals can invest in a risky asset (represented by a stock market index) and in an asset that earns the real interest rate (represented by a money market account). Unlike the nominal interest rate, the real interest rate is not perfectly predictable, as it depends on the rate of inflation, which is not perfectly predictable. In the following, if not explicitly mentioned otherwise, we refer to the real interest rate as the interest rate for ease of notation.

The interest rate, r_t, follows an Ornstein-Uhlenbeck process as in the Vasicek (1977) model:

$$dr_t = \theta(\mu - r_t)dt + \sigma_r dW_t^r, \tag{1}$$

in which μ is the long-run average interest rate, σ_r is the standard deviation, θ is a parameter measuring the speed at which the interest rates revert to their mean, and W^r is a standard Brownian motion. The gross interest rate is denoted by $R_t = r_t + 1$. In its discretized form, the Ornstein-Uhlenbeck process is equivalent to an AR(1) process, so in our discrete model,

r_t follows the process

$$r_{t+1} = c + \phi r_t + \epsilon_{t+1}, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2). \tag{2}$$

The stock index gross return R_t^S over one period is assumed to be log-normally distributed, i.e.,

$$\ln(R_t^S) \sim N(\ln(R_t + \lambda) - \frac{\sigma_S^2}{2}, \sigma_S^2), \tag{3}$$

in which λ is the risk premium and σ_S the standard deviation. The log-return from t to $t+1$ can then be written as

$$\ln(R_t^S) = \ln(R_t + \lambda) - \frac{\sigma_S^2}{2} + \eta_t, \quad \eta_t \sim N(0, \sigma_S^2). \tag{4}$$

Shocks in stock returns, η_t, and shocks in interest rates, ϵ_t, can be correlated, reflecting that they may depend on common underlying factors, such as the evolution of the economy. We refer to the correlation coefficient as $\rho_{\epsilon\eta}$. Likewise, we allow for correlation of the growth rate of the labor income process with stock returns, $\rho_{u\eta}$, and changes in interest rates, $\rho_{u\epsilon}$.

2.2 Optimization problem

In every period t, individuals have to decide on how much they consume, C_t, and what fraction of their wealth they invest in stocks, α_t. $f(t)$ denotes the individual's probability to survive from time t to $t+1$, β denotes its time preference factor, γ its degree of risk aversion, and Ψ its elasticity of intertemporal substitution.
The individuals' optimization problem at time t can then be written as

$$\max_{\{C_t, \alpha_t\}} V_t(X_t) = \left((1-\beta) \cdot C_t^{1-\frac{1}{\Psi}} + \beta \cdot \left(f(t) \cdot \mathbb{E}_t \left[V_{t+1}(X_{t+1})^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\Psi}}{1-\gamma}} \right)^{\frac{1}{1-\frac{1}{\Psi}}} \tag{5}$$

subject to

$$W_{t+1} = L_{t+1} + (W_t - C_t) \cdot (\alpha_t R_t^S + (1-\alpha_t) R_t) \tag{6}$$

$$C_t > 0, C_t \leq W_t, \tag{7}$$

$$\alpha \in [0,1], \tag{8}$$

and Equations (2) and (4), in which $X_t = [t, L_t, W_t, R_{t-1}]$ denotes the vector of state variables. Equation (6) is the individual's budget constraint. Equation (7) restricts consumption to be positive and prevents borrowing against future labor income. Further, as private in-

Table 1
Estimation results

Coefficient	Value	Standard Error
Interest rate process		
Intercept c	0.0008	0.0036
AR coefficient ϕ	0.6705***	0.0551
Interest rate standard deviation σ_ϵ	0.0283***	0.0012
Stock return process		
Risk premium λ	0.0495**	0.0206
Stock return standard deviation σ_η	0.1880***	0.0199

This table summarizes the coefficient estimates from the regressions. *, **, and *** indicate significance of the estimates on the 10%, 5%, and 1% level, respectively.

vestors are typically unable to short-sell stocks or buy them on margin, we impose Equation (8). Except for the chosen equity exposure, results for investors that do not face borrowing or short-selling constraints do not differ much from those reported in this manuscript and can be found in Figure 4 of the Online Appendix. To reduce the number of state variables by one, we normalize the optimization problem by W_t. We explain details on the solution of this optimization problem in Appendix A.

2.3 Calibration

In this section, we present the calibration of our model. The coefficients for the labor income process are taken from Fischer and Koch (2025) who update those of Cocco et al. (2005) to include more recent data. We estimate the parameters for the assets of the capital market using historical data. For the interest rate process, we use yearly nominal returns on 30-day T-Bill rates from the WRDS database from 1926 to 2021 and adjust them for inflation using the changes in the consumer price index (CPI) for the corresponding year. For the stock price index, we use the nominal returns on the S&P 500, using data from Robert Shiller's homepage (`http://www.econ.yale.edu/~shiller/data.htm`) from 1926 to 2021 and again adjust for changes in the CPI.

Both the interest rate process from Equation (2) and the stock return process from Equation (3) are estimated using maximum likelihood estimation. Table 1 summarizes the

Table 2
Base case calibration

Description	Parameter	Value	Source
Degree of risk aversion	γ	10	own calibration
Time preference factor	β	0.94	own calibration
Elasticity of intertemporal substitution	Ψ	0.2	own calibration
Residual correlation stock return and interest rate	$\rho_{\epsilon\eta}$	-0.0720	own estimation
Residual correlation labor income and stock return	$\rho_{u\eta}$	0.0735	own estimation
Residual correlation labor income and interest rate	$\rho_{u\epsilon}$	-0.0838	own estimation
Replacement ratio	-	0.7444	own estimation

This table summarizes the parameters for the base case of our model.

estimated coefficients.[2] For the real interest rate process, maximum likelihood estimation results in an insignificant intercept, revealing that the average real interest rate in the data is close to zero. Therefore, we set $c = 0$. The autoregression coefficient is $\phi = 0.6705$, the error term standard deviation is $\sigma_\epsilon = 0.0283$. The risk premium is $\lambda = 0.0495$, i.e., around 5 percentage points, and the stock return error terms exhibit a standard deviation of $\sigma_\eta = 0.1880$. Except for the average real interest rate, c, all estimates are significant at the 5% level.

Correlation between labor income, stock returns, and the interest rate is calculated using PSID data from 1980 to 2019 for the labor income process.[3] The residual correlation between stock returns and the real interest rate is $\rho_{\epsilon\eta} = -0.0720$, between labor income and stock returns $\rho_{u\eta} = 0.0735$, and between labor income and the real interest rate $\rho_{u\epsilon} = -0.0838$. We also calculate the replacement ratio from the PSID data by taking the ratio of the income directly before retirement (at age 65) and the average income during retirement (from age 66 to age 100). We find that during retirement individuals receive 74.44% of their working

[2]To account for the possibility of a unit root in the time series of the real interest rate, we formally test for it using the Dickey-Fuller test (Dickey and Fuller, 1979). The test rejects the null hypothesis of a unit root at the 1% level. Likewise, we do not find empirical support for correlation between the risk premium and the risk-free rate.

[3]Ideally, all parameters would be estimated from a consistent data set spanning the same time period for all observations. Some observations in the PSID data required for our estimation are, however, unfortunately not available prior to 1980. Also estimating the other parameters using data from 1980 onwards leads to a reduction of the number of observations by more than 50% and thus an increased estimation error. We therefore decided to estimate parameters based on the longest time series available for all parameters to keep the estimation error as low as possible. Our results are, however, robust to estimating our parameters on a consistent data set as can be seen in Figures 1 and 2 in the Online Appendix.

income at age 65 and therefore set the replacement ratio to 0.7444.

In our model, individuals can live to a maximum of age 100. Before age 100, we use the male life table from the Center of Disease Control and Prevention (Arias and Xu, 2018) to calculate survival probabilities.

We choose the preference parameters such that the growth of wealth from age 20 to retirement age, 65, matches the empirically observed one in the PSID data. We find that a time preference factor of $\beta = 0.94$, a degree of risk aversion of $\gamma = 10$,[4] and an elasticity of intertemporal substitution of $\Psi = 0.2$, jointly provide a good fit. Our parameter estimates are summarized in Table 2. Appendix B provides the more technical details of the parameter estimation.

3 Results

3.1 Level- and risk-channel

In this section, we illustrate the importance of accounting for the time-varying nature of real interest rates by documenting how optimal consumption and investment decisions vary with the level of the interest rate. We compare these results with two counterfactuals, in which we ask how individuals would behave if they did not account for the time-varying nature of real interest rates and/or their level. More precisely, we split the impact of real interest rate risk into two channels: the level-channel and the risk-channel. Treating these two channels separately allows us to explore, which part of real interest rate risk has the largest implications for individuals' behavior and welfare. People recognizing neither the level-channel nor the risk-channel assume the interest rate stays constant at its long-term mean μ, i.e.,

$$r_{t+1} = \mu, \ \forall t. \tag{9}$$

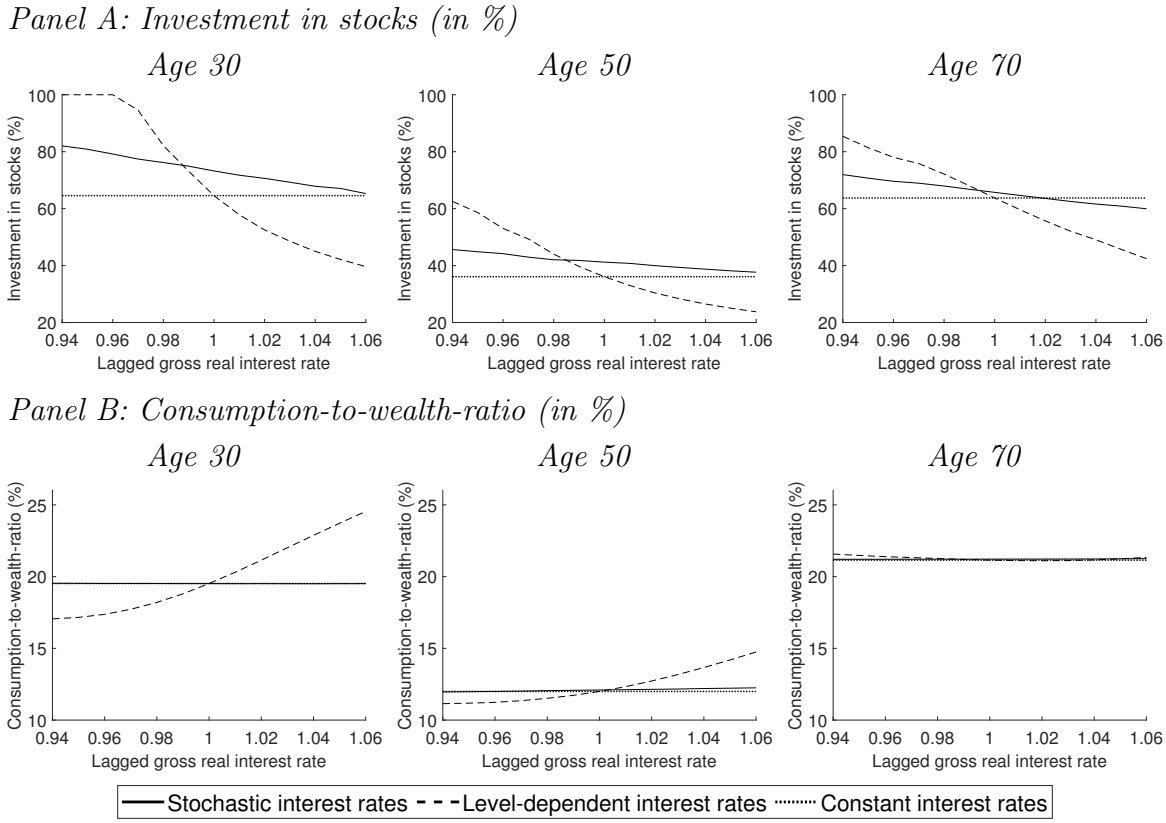
Individuals who consider the level-channel but not the risk-channel observe the current interest rate level, but assume it stays constant at the currently observed level, i.e.,

$$r_{t+i} = r_t, \ \forall t, i. \tag{10}$$

Financial advice given to individuals often (implicitly) builds on the assumption of constant interest rates. Likewise, many theoretical models assume constant interest rates. Under this counterfactual, individuals only take the current interest level into account, but ignore

[4]Section 4.2 demonstrates the qualitative robustness of our results to other degrees of risk aversion.

Figure 1
Policy functions



This figure depicts policy functions for 30-, 50-, and 70-year-old individuals with labor-to-wealth ratios of 27.5%, 11.8%, and 13.7%, respectively. It compares the optimal policy function of the base case, in which interest rates are stochastic (level- and risk-channel), with two counterfactuals in which only the level of the lagged interest rate (dashed lines) and the long-run mean of the interest rate (dotted lines) are taken into account, respectively. Panel A shows policy functions for the proportion of wealth invested in stocks. Panel B shows optimal consumption-to-wealth ratios.

the stochastic nature of the interest rate. They do not consider that the interest rate is time-varying and is subject to changes in the future.

Figure 1 shows policy functions for individuals aged 30, 50, and 70 with labor-to-wealth ratios of 27.5%, 11.8%, and 13.7%, respectively, roughly corresponding to the average labor-to-wealth ratio of individuals of that age on our simulated paths.[5] The solid line shows the policy function of base case individuals who consider both level- and risk-channel. They take stochastic interest rates into account, so they make their decisions based on the interest rate process following Equation (2). The dashed line shows the policy function of individuals

[5]Our results are qualitatively robust to other ages and labor-to-wealth ratios.

who only consider the level-channel. They make optimal decisions assuming the interest rate process would follow Equation (10). The dotted line shows the policy of individuals who assume the interest rate stays constant at its long-term mean. Independent of the interest rate, these individuals make decisions as if the interest rate would constantly stay at its long-term mean (Equation (9)), which, from Table 1, is zero. Hence, under this counterfactual, individuals ignore both the level- and the risk-channel.

The top panels of Figure 1 depict the share of wealth invested into stocks as a function of the level of the lagged gross real interest rate, which we vary between 0.94 and 1.06, roughly corresponding to the empirically observed post world war two levels of the real interest rate. From the solid lines, the optimal share decreases in the level of the interest rate. That is, when interest rates are low, individuals should invest more into the risky stock market index. This result makes intuitive economic sense as the present value of non-financial income (labor income during working age and retirement benefits during retirement age), whose payments are similar to that of a coupon-bearing bond and are only weakly correlated with stocks, increases. This in turn encourages investors to hold a larger share of their financial wealth in stocks. Even though the mean-reverting nature of the interest rate process suggests that the current period's real interest rate should be higher than its lagged value when the lagged value is below the long-run mean, the current period's real interest rate is to be expected below its long-run mean. Hence, investors should invest more into stocks when interest rates are low.

Turning to the counterfactual under which individuals do not account for the risk-channel (dashed lines), but trade as if interest rates would remain at their observed lagged values, the effects from our base case scenario are amplified at all ages as the mean-reverting nature and its implications for the current and future periods' interest rates are ignored.

Under the counterfactual in which both the level- and the risk-channel are closed (dotted lines) and individuals trade as if the interest rate would correspond to its long-run mean, the optimal share of wealth invested in the stock market index is constant for all three ages. It is typically lower than in the case in which the individual accounts for the stochastic nature of the interest rate, reflecting the more desirable risk profile of investments into the money market account. When the interest rate is assumed to be constant, the investment into the money market account appears as a risk-free investment. Once the stochastic nature of the interest rate is accounted for, this is no longer the case. Hence, with constant assumed interest rates, the money market account appears more desirable than it actually is, leading to higher investments into it and lower ones into stocks. The order of magnitude of this effect declines with increasing (lagged) interest rates, reflecting that the reward per unit of interest rate risk overtaken increases for higher levels of the interest rates, thus rendering

the money market account more attractive.

Also with stochastic interest rates, the money market account remains less risky than investments into stocks. That is, even if the return on the money market account is risky, investing more into it decreases the overall level of portfolio risk overtaken. During retirement age, in the absence of labor income risk and with a lower present value of labor income, this effect becomes quantitatively so important, that the optimal exposure to stocks turns even lower with stochastic interest rates than without to keep the overall risk-return-tradeoff at its optimal level. With stochastic interest rates, the overall level of portfolio risk is higher. To nevertheless keep it at an acceptable level, the investor invests more in the money market account. With constant interest rates, on the other hand, the investor can invest more into stocks without ending up with a too high level of total portfolio risk.

The share of wealth invested into equity varies over the life cycle, reflecting that payments from labor income better substitute for bonds than for stocks. At age 30, individuals typically have not built up larger savings, which is reflected in a high labor-to-wealth ratio. Consequently, they optimally choose relatively high equity exposures. At age 50, individuals have typically already built up more savings, leading to lower labor-to-wealth ratios and thus a lower optimal equity exposure. At age 70, individuals have already started to decumulate their savings, thus again leading to increasing labor-to-wealth ratios and an increase in the optimal equity exposure.

The lower panels of Figure 1 plot the share of wealth spent on consumption (consumption-to-wealth ratio) against the lagged interest rate. From the solid lines, individuals accounting for both the risk- and the level-channel slightly increase their consumption as the interest rate increases, reflecting that with a higher interest rate, a lower savings rate suffices to attain the same future wealth level. Hence, a larger share of wealth can be spent on present consumption. Similarly, when interest rates are low, individuals consume less and save a larger part of their wealth. This is consistent with the empirical evidence in, e.g., Beaudry et al. (2024), who document increasing wealth-to-income ratios with declining interest rates.

For $\Psi = 0.2$, the income effect outweighs the substitution effect. Qualitatively, the relationship between the interest rate and the consumption-to-wealth ratio shows the expected sign. Quantitatively, the effects are relatively small for the optimal consumption-to-wealth ratios, reflecting that the interest rate process is mean-reverting, i.e., deviations from the mean are only short-lived. Hence, individuals' optimal reactions to changes in real interest rates are much weaker than for individuals that regard these changes as permanent.[6]

[6]In our base case parameter setting, with $\gamma = 10$ and $\Psi = 0.2$, the individual's aversion to risk dominates the aversion to time fluctuations and an early resolution of uncertainty is preferred. A lower degree of $\Psi < 0.1$, which leads to the aversion to time fluctuations dominating risk aversion, leads to qualitatively similar results. Yet, quantitatively the consumption-investment policies – particularly when only ignoring

For the counterfactuals in which the risk-channel is ignored (dashed lines) and individuals behave as if the observed lagged real interest rate would remain constant, the consumption effects are substantially amplified, reflecting that individuals believe an above-average interest rate to persist. Under this assumption, a higher present consumption is possible, as the thereby forgone savings can be made up for by the anticipated high present and future returns. At retirement age, the consumption effects are quantitatively much smaller, reflecting that mortality rates are substantially higher and the expected remaining investment horizon over which the individual can benefit from a higher interest rate is thus significantly shorter than for a 30- or 50-year-old individual.

For the counterfactual in which both the risk- and the level-channel are ignored (dotted lines) and individuals behave as if the interest rate permanently corresponded to its long-run mean, the consumption-to-wealth ratio is constant and roughly corresponds to that of a fully rationally acting investor.

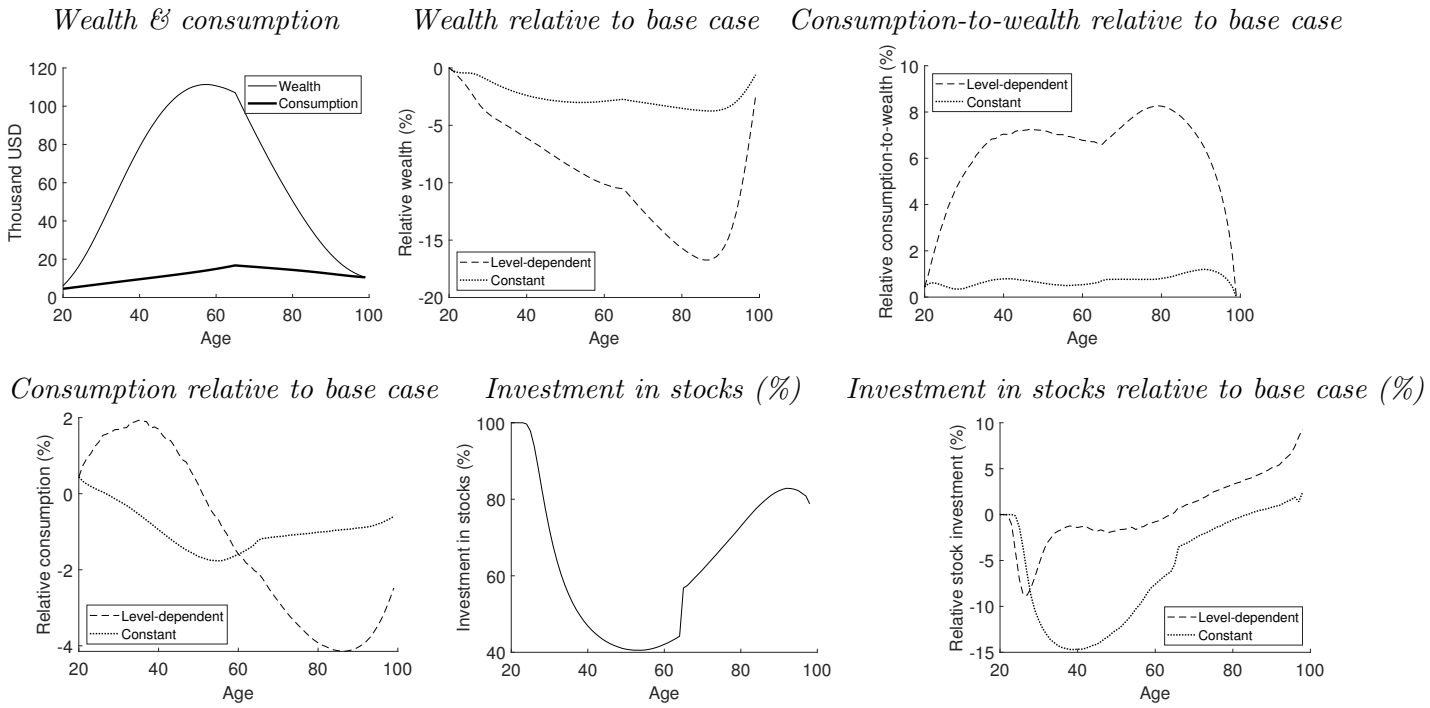
Overall, our results in this section reveal that optimal consumption-investment decisions vary with the level of the past observed real interest rate. Ignoring the risky nature of the real interest rate in consumption-investment decision-making leads to rather extreme consumption-investment decisions as the mean-reverting nature of real interest rates is not taken into account. Ignoring both the risk- and the level-channel on the other hand leads to underinvestment into stocks as the predictive component of the interest rate process is not optimally exploited.

3.2 Unconditional policies

Our results in Section 3.1 give us a good understanding of how the level of the interest rate affects optimal consumption-investment decisions. They do, however, not account for how likely the different levels of the interest rates are. We therefore simulate 50,000 paths over the life cycle to better understand the implications of stochastic interest rates for individuals' consumption-investment decisions. In the simulation, individuals start with a wealth of 6,140 USD, the empirically observed median wealth at the age of 20 in the PSID data, and a labor-to-wealth ratio of 1. The initial interest rate at age 20 is set to its long-run mean, i.e., a value of the gross interest rate of 1. To investigate the welfare effects of ignoring the risk-channel or both channels, we solve the optimization problem from Section 2.2, subject to Equation (9), or Equation (10) instead of Equation (2) to obtain policy functions. We then simulate paths over the life cycle on which the true interest rate process follows Equation (2), but the consumption-investment decisions are those from the before-mentioned optimization

the risk-channel – vary a bit more with the level of the real interest rate as can be seen from Figure 3 in the Online Appendix.

Figure 2
Results for individuals following different policies



This figure depicts the evolution of wealth, consumption, and stock investment over the life cycle. The top left graph shows results for consumption and wealth in our base case parameter setting when individuals make optimal decisions, the thin line shows the evolution of wealth and the bold line consumption. The bottom middle graph shows results for stock investment in our base case. The top middle, top right, and bottom left and right graphs show how wealth, the consumption-to-wealth ratio, the absolute consumption level, and stock investment change relative to the base case if individuals ignore the risk-channel (dashed lines) or the risk- and the level-channel (dotted lines). All results are based on 50,000 simulated paths.

problems.

Figure 2 depicts the evolution of wealth and consumption over the life cycle. The upper left and lower middle graph shows results for our base case parameter setting when individuals make optimal decisions. The upper middle and right graphs as well as the lower left and right graphs show how wealth, the consumption-to-wealth ratio, the absolute consumption level , and the investment in stocks are affected relative to the base case when individuals ignore the risk-channel (dashed lines) or the risk- and the level-channel and behave as if the interest rate permanently corresponded to its long-run mean (dotted lines).

The upper left graph in Figure 2 shows the well-known hump-shaped evolution of wealth (thin line) over the life cycle. This pattern reflects the need to build up savings during working life, which can then gradually be decumulated during retirement age. From the

consumption-smoothing motive, the evolution of consumption (bold line) over the life cycle is much smoother.

Not accounting for the risk- and/or the level-channel in interest rates has important implications for consumption, investments and savings. When abstracting away from the risk- and/or level channel, from the upper right panel of Figure 2 individuals overconsume, reflecting that they are ignoring the need for precautionary savings caused by interest rate risk. This risk is particularly pronounced for individuals only ignoring the risk- but acknowledging the level-channel, i.e., individuals who behave as if observed lagged interest rates would remain constant at that level. From Figure 1, when observing a high lagged interest rate, these individuals overconsume substantially. As a consequence of this overconsumption, their savings grow less, which, ultimately also leads to lower consumption at older age. Consistent with Figure 1, the effects are weaker for individuals ignoring both the risk- and the level-channel, behaving as if interest rates would permanently correspond to their long-run mean. From the upper middle panel, the higher consumption-to-wealth ratios translate into lower wealth levels and lower absolute consumption levels (lower left panel).

The lower middle panel of Figure 2 depicts the evolution of the optimal share of wealth invested into stocks (investment into stocks) over the life cycle. Consistent with earlier work, such as Cocco et al. (2005), it shows an inverse hump-shaped pattern. At younger age, individuals' labor-to-wealth ratios are relatively high. As the payments from labor income are more similar to those of a coupon-bearing bond than to those of a stock, younger investors allocate a higher share of their financial wealth to stocks. At retirement age, the share of wealth allocated to stocks jumps up, reflecting that during retirement age non-financial income comes as retirement income and is, unlike labor income during working age, not subject to labor income shocks. With the lower non-financial income risk, investors need less risk-free precautionary savings and can invest more into stocks. During retirement age, individuals decumulate their savings and thus increase their non-financial-income-to-wealth ratios again. Consequently, along with the decrease in wealth, individuals increase their investments into stocks.

From the lower right panel of Figure 2, not accounting for the risk- and level-channel has large implications for the chosen investment into stocks. Particularly at younger age, the investments into stocks remain significantly under their optimal values, reflecting that from Figure 1, except for very low interest rates, individuals ignoring the risk- and/or the level-channel typically choose lower investments into stocks than would be optimal. The effect is particularly big for individuals ignoring both channels at around age 40 when the investments into stocks remain about 15% below their optimal value.

Taken together, even though both not accounting for the risk-channel or not accounting

for the risk- and level-channel lead to suboptimal consumption-investment decisions, our results in Figures 1 and 2 reveal remarkable differences for the accumulation of wealth. Individuals only ignoring the risk-channel that account for the level of the interest rate but not its time-varying nature, tend to substantially overconsume at younger age in states with high interest rates, thus building up too few savings relative to individuals behaving optimally. Ultimately, the lower level of wealth then also turns into a lower absolute level of consumption.

From Figure 2, individuals ignoring the risk- and the level-channel also slightly overconsume, reflecting that they do not build up sufficient precautionary savings. Yet, this effect is quantitatively weaker. From the lower right panel of Figure 2 and Figure 1, however, this individual systematically underinvests into equity. Consequently, it forgoes the equity premium, further contributing to a slower accumulation of wealth. The underinvestment into stocks is larger when ignoring both the risk and the level channel.

Not correctly accounting for the stochastic time-varying nature of the interest rate process leads to suboptimal consumption-investment decisions and is therefore also associated with welfare effects. Ignoring the risk- but accounting for the level-channel leads to higher welfare costs due to overconsumption in states with higher interest rates. The welfare cost from not accounting for the risk-channel and behaving as if the interest rate would permanently correspond to its lagged observed value is as high as 6.7% of lifetime consumption, of which 6.3 percentage points or 94% can be attributed to the suboptimal consumption policy[7].

Individuals not accounting for the risk- and level-channel behave as if interest rates permanently corresponded to their long-run mean and thus forego the information about future changes in the interest rates as well as the opportunities of higher interest rates in present consumption. Their lifetime consumption opportunities could increase by only 0.8% when accounting for the risk- and level-channel in their consumption-investment decisions, of which 0.6 percentage points or 75% can – despite only small but accumulating differences between optimal and suboptimal consumption policy – be attributed to the suboptimal consumption policy. That is, for these investors suboptimal investment decisions play a more important role than for individuals only ignoring the risk channel and account for 25% of welfare costs. Yet, welfare costs from suboptimal consumption decisions are also more important for these individuals, suggesting once the amount invested is smaller due to overconsumption in the past, the exact choice between these two assets, which both have a reasonable risk-return profile, becomes less important than the problem of persistent

[7]The 6.3 percentage points are computed from a scenario in which only the consumption is chosen ignoring the level-channel while the investment strategy follows the optimal investment strategy accounting for the true stochastic time-varying nature of interest rates

overconsumption.

The welfare effects reported above are particularly useful for private investors. For institutional investors, other characteristics, such as the out-of-sample Sharpe Ratio of an investment strategy are more useful to get an impression of whether conditioning the optimal equity exposure on the level of the real interest rate may be useful. For that purpose, we compute out-of-sample Sharpe Ratios from our simulated data for all investment strategies considered. The realized Sharpe Ratios are 0.230 when behaving optimally, 0.227 when only the level of the interest rate is taken into account (risk-channel ignored), and 0.226 when behaving as if the real interest rate always corresponded to its long-run mean (risk- and level-channel ignored). That is, not accounting for the risk- and/or the level-channel does not only lead to welfare costs, but is also reflected in a slightly lower Sharpe Ratio.

Accounting for 0.25% transaction costs, the Sharpe Ratios drop to 0.218 when behaving optimally, 0.216 when only the level of the interest rate is taken into account, and 0.213 when behaving as if the real interest rate always corresponded to its long-run mean. Again, the Share Ratio is highest when following the optimal strategy.

Consistent with our welfare effects from above, the Sharpe Ratios also reveal that conditioning the optimal equity exposure on the level of the real interest rate is useful. Quantitatively, the effects are an order of magnitude larger for the welfare effects compared to the Share Ratios, suggesting that conditioning the optimal equity exposure on the level of the real interest rate is particularly useful for private investors in bad states of the world, where the marginal utilities of consumption are highest.

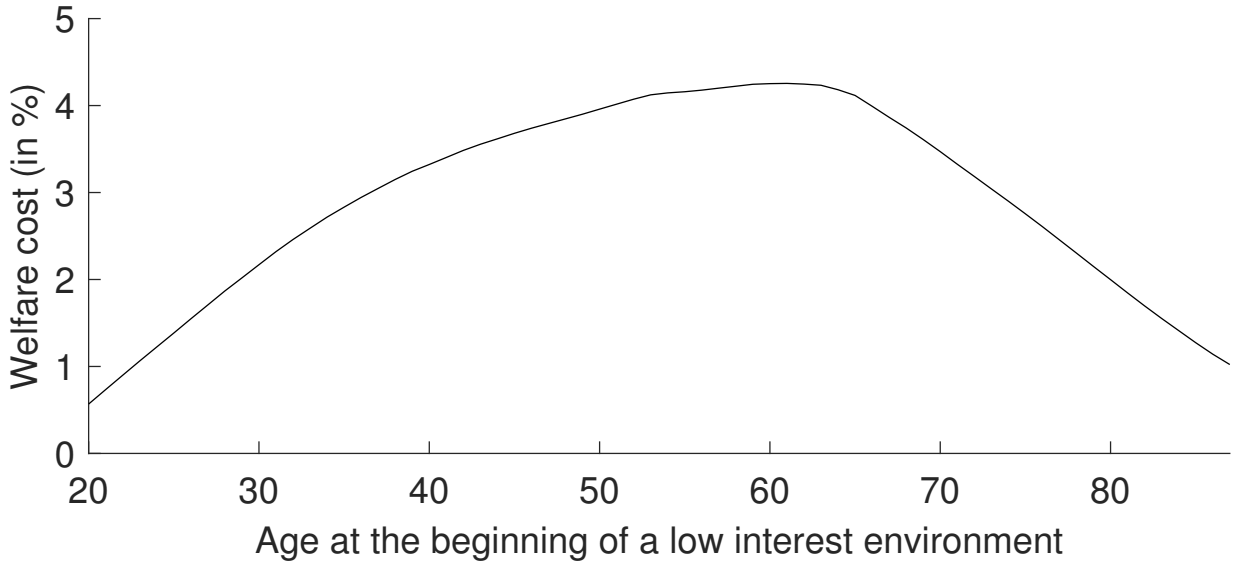
Overall, our results in this section show that accounting for both the risk- and the level-channel is important in consumption-investment decisions to avoid overconsuming at younger ages at the expense of being forced to accept lower levels of consumption at older ages.

3.3 Consequences of a low interest environment

Over the past decades, individuals have been facing historically low interest rates. Being exposed to a period of low real interest rates over an extended period of time should negatively affect the accumulation of wealth and optimal consumption-investment strategies. That is, individuals being exposed to low-interest environments may not be able to reach the same standard of living as individuals facing more attractive interest rates for their savings.

In this section, we explore the consequences of being exposed to a low interest rate environment over an extended period within the framework of our model. To measure the welfare effects of a low interest environment, we compare attainable welfare levels from consumption from 50,000 simulated paths of individuals being exposed to a low interest

Figure 3
Welfare cost of low interest rate environments



This figure displays the average realized loss in remaining lifetime consumption (welfare cost) of individuals living in a low interest environment relative to individuals who do not, as a function of the age at which individuals start facing the low interest environment.

environment to those individuals that are not.

To explore whether the consequences of being exposed to a low interest environment vary with age, we allow individuals in our model framework to be exposed to the low interest environment at different ages. More technically, individuals face the interest rate process from Equation (2) prior to age τ. From age τ to $\tau + \tilde{\tau}$ individuals face a low interest environment, in which we use the empirically observed real interest rates from 2009 to 2021. More specifically, we use the yearly returns on 30-day T-bill rates adjusted for inflation. That is $\tilde{\tau} = 12$. During this time period, real interest rates have been constantly below zero with an average real interest rate of -1.7%. After time $\tau + \tilde{\tau}$, real interest rates are again determined by the interest rate process from Equation (2).

In Figure 3, we report the average loss in remaining lifetime consumption from 50,000 simulations at the age of entering the low interest environment. We show the loss of being exposed to the low interest environment versus not being exposed to it as a function of the age, from which on the individuals face the low interest environment.

From Figure 3, welfare costs are hump-shaped and peak around the age of 60 with 4.25%. That is, welfare costs tend to be highest when individuals have accumulated the largest savings. This result makes intuitive economic sense, as absolute losses from negative

real interest rates are highest when individuals' savings peak.

At very young ages, savings are typically small, implying that the consequences of being in a low interest environment should also be relatively small. Likewise, at very high ages, individuals have already decumulated a significant share of their savings, thus being less affected by low real interest rates. Savings peak at around retirement age, which is why welfare costs tend to be highest, if individuals face low real interest rates during the last working years as well as the first years during their retirement period, i.e., when the low real interest rate environment starts at around the age of 60.

To explore the implications of facing a low real interest environment over an extended period, we plot how wealth and consumption are affected under the low interest environment relative to a normal interest environment, in which real interest rates follow the interest rate process from Equation (2), in Figure 4. We show the implications for individuals who enter the low interest environment at the age of 60 (right graphs), roughly corresponding to the peak of welfare costs from Figure 3. We deliberately choose to report results when welfare effects peak as effects for wealth and consumption should be highest at these ages. We also report results for individuals who enter the low interest environment at the age of 40 (left graphs) to give the reader an intuition for effects at younger age. Panel A depicts the evolution of wealth relative to the base case (change of wealth in percent), Panel B the evolution of consumption relative to the base case (change of consumption in percent).

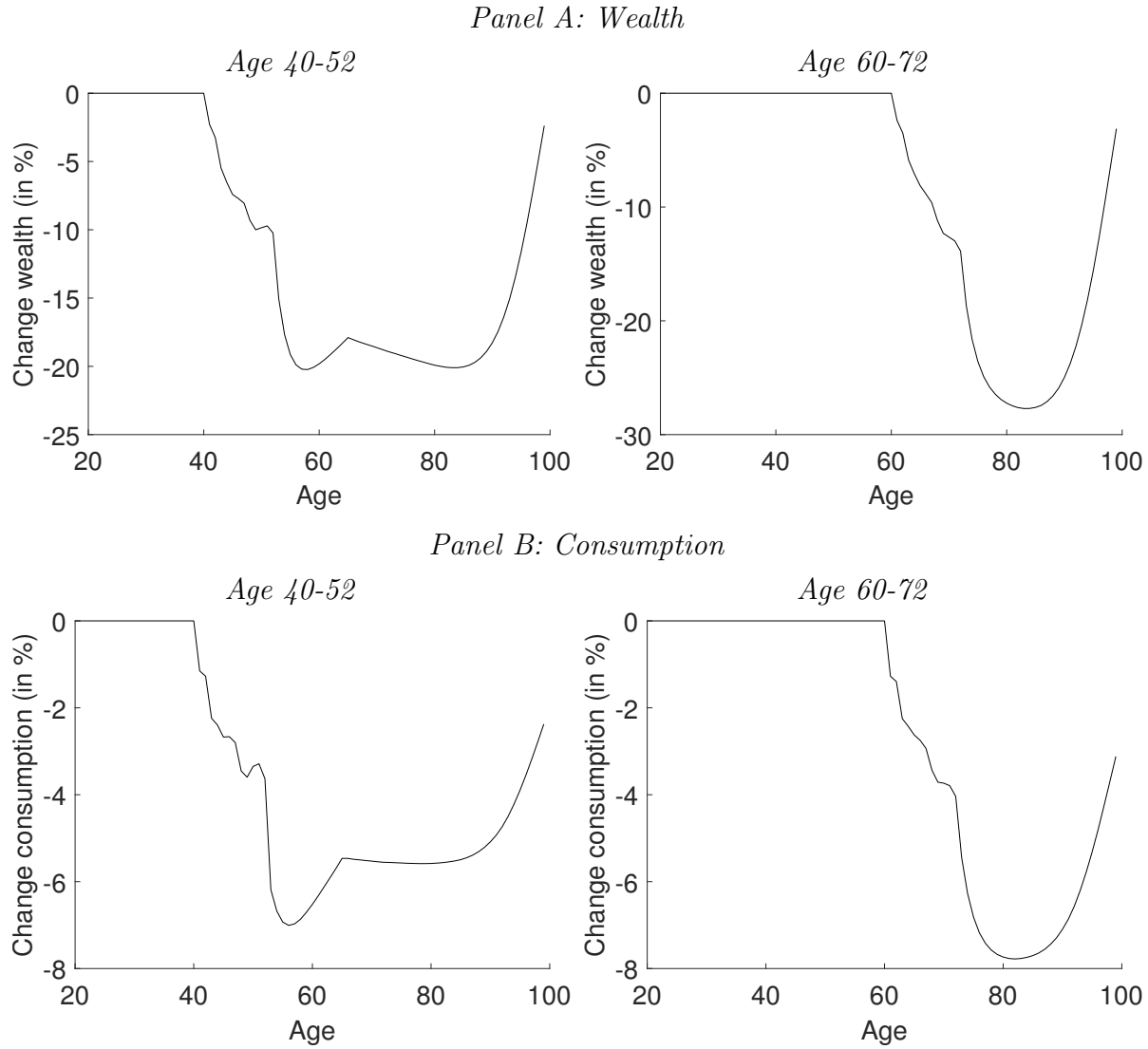
From Figure 4, by construction, there are no differences in wealth and consumption until the age of 40/60, when real interest rates start differing between the two scenarios. As soon as being exposed to the low interest environment, individuals' wealth instantaneously drops and continues to drop for an extended period. Simultaneously, individuals react to the drop in their savings by reducing their consumption levels. Both effects are long-lasting and continue to persist even after the end of the low interest rate environment and irrespective of whether the shock occurs at age 40 or 60.

Overall, our results in this section show that being exposed to a low interest environment has lifetime implications for savings and consumption opportunities – even when individuals adjust their consumption-investment strategies to circumstances.

4 Robustness

In this section, we demonstrate the robustness of our results to various assumptions and parameter choices. In Section 4.1, we begin with the speed of mean-reversion of the interest rate process. Section 4.2 discusses the implications of a different degree of risk aversion. In Section 4.3, we show what explains the relatively high equity exposure in our dynamic life

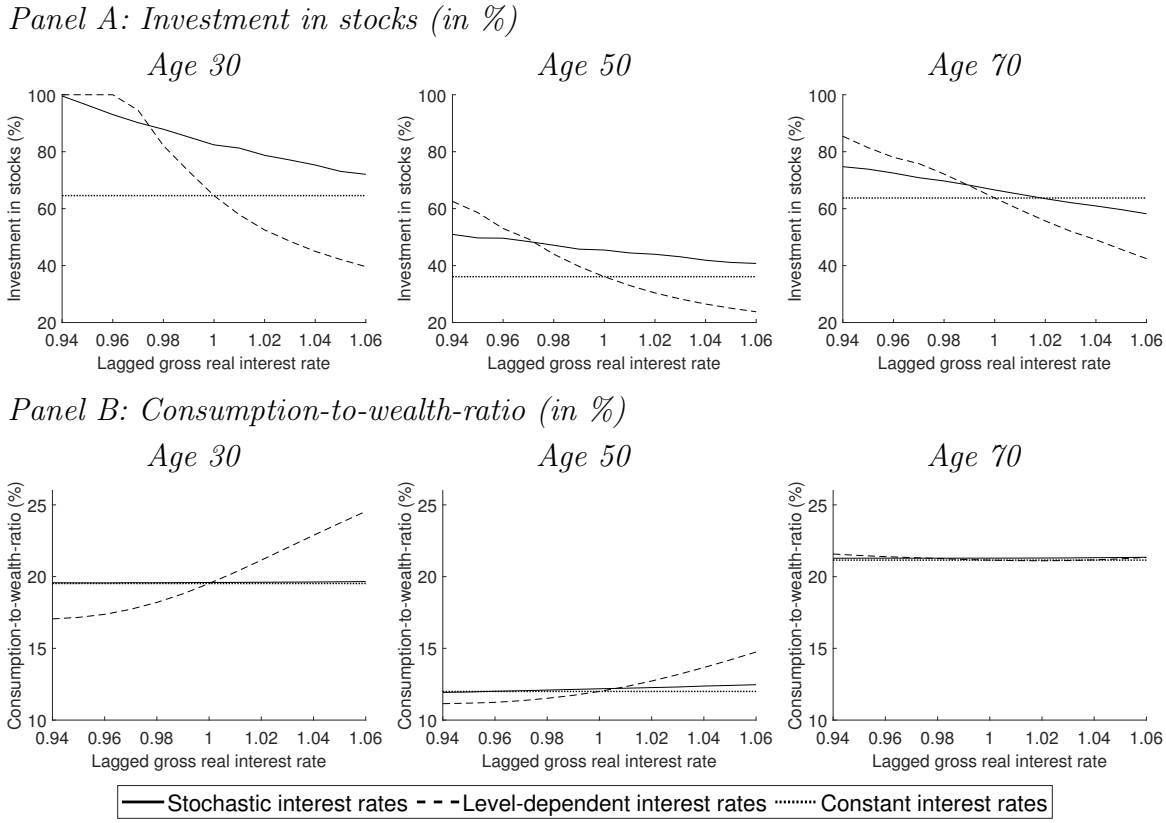
Figure 4
Comparison of the base case and a low interest environment



This figure depicts the average difference of wealth and consumption relative to our base case scenario in percent when individuals face the low interest environment from age 40 to 52 (left graphs) and age 60 to 72 (right graphs). Panel A depicts the evolution of wealth relative to the base case (change wealth), Panel B the evolution of consumption relative to the base case (change consumption).

cycle model relative to static models of portfolio choice.

Figure 5
Policy functions with $\phi = 0.8$



This figure depicts policy functions for 30-,50-, and 70-year-old individuals with labor-to-wealth ratios of 27.5%, 11.8%, and 13.7%, respectively. It compares the optimal policy function of the base case, in which interest rates are stochastic (level- and risk-channel), with two counterfactuals in which only the level of the lagged interest rate (dashed lines) and the long-run mean of the interest rate (dotted lines) are taken into account, respectively. Panel A shows policy functions for the proportion of wealth invested in stocks. Panel B shows optimal consumption-to-wealth ratios.

4.1 Speed of mean-reversion of interest rate process

In Table 1, we estimated the autocorrelation parameter to take a value of $\phi = 0.6705$. This parameter is important in our model, as from Equation (2) it determines the speed of mean-reversion of the interest rate process with higher (smaller) values implying a weaker (stronger) rate of mean-reversion. In this section, we ask how such a weaker rate of mean-reversion affects the optimal consumption-investment policies and welfare. For that purpose, we set $\phi = 0.8$ for the remainder of this section.

From Figure 5, the optimal equity exposure for a rational investor increases when the rate of mean-reversion becomes weaker, i.e., when ϕ increases. From Equation (2), with our

empirical estimate for $c = 0$, next period's expected interest rate is $\mathbb{E}[r_{t+1}] = \phi r_t$, implying that the expected change in the interest rate is $\mathbb{E}[r_{t+1} - r_t] = (\phi - 1) r_t$. For a reasonably-behaved interest rate process with $0 < \phi < 1$, the expected (absolute) change in the interest rate is smaller for larger values of ϕ. Higher values of ϕ then imply a lower risk of larger changes in the interest rate, i.e., the future investment opportunity set. That is, the future investment opportunity set is less likely to change a lot and there is therefore a lower need for safer investments to protect against adverse events in the future investment opportunity set. Hence, the optimal equity exposure increases. The investment strategies of investors ignoring the level- or both channels do not change as these investors, by assumption, do not account for changes in interest rates. Unlike optimal investments, the optimal consumption-to-wealth ratio is less affected by a change in the speed of mean-reversion than the optimal equity exposure, reflecting that a weaker rate of mean reversion of the interest rate process only affects consumption opportunities indirectly. When the speed of mean reversion is weaker, larger changes in interest rates (toward the mean) become less likely. Hence, rational investors need to build up less precautionary savings to protect themselves against the risk of changing interest rates, suggesting that they can afford a slightly higher consumption rate. For low levels of the interest rate, this increase is counteracted by the fact that with a weaker rate of mean reversion, the interest rate is more likely to remain at the undesirable lower level for an extended period of time, in turn suggesting a lower consumption rate. Quantitatively, the former effect outweighs the latter. For instance, at age 50, consumption rates with $\phi = 0.6705$ range from 11.96% with a lagged real interest rate of -6% to 12.25% with a lagged real interest rate of +6%. With $\phi = 0.8$, the corresponding values are 11.93% and 12.47%, respectively. That is, the speed of mean reversion does affect the optimal consumption rate. However, even for ϕ as high as 0.8, these effects are quantitatively small.[8] However, consistent with our results in Figure 1, the optimal consumption rate slightly increases in the level of the interest rate. With a higher interest rate, a higher present consumption is possible, as the forgone savings can be made up for by the anticipated higher present and future returns.

Welfare effects from ignoring both the level and the time-varying nature of interest rates increase to 2.2% as the speed of mean-reversion becoming slower implies that larger deviations from the mean become more likely, thus increasing the error of building an investment strategy on the assumption that the interest rate would correspond to its long-run mean. The welfare effects of only considering the level channel decrease to 5.1%. Not accounting

[8]For values of ϕ close to 1, which are not supported by the data, but imply a very weak rate of mean reversion, the consumption effects are quantitatively enhanced and can also be seen easily graphically. These results are available from the authors upon request.

for changes in future interest rates is less harmful, when changes tend to be smaller.

4.2 Different degree of risk aversion

The payments from labor income are more similar to those of interest-bearing assets than to those of stocks. Hence, models with labor income tend to predict higher optimal equity exposures than models without. A degree of risk aversion of $\gamma = 10$, as in our base case parameter setting, is therefore a common assumption in the life cycle literature with labor income (see, e.g. Cocco et al. (2005)). Yet, a degree of risk aversion of $\gamma = 10$ is at the upper end of degrees of risk aversion considered reasonable by Mehra and Prescott (1985).

In this section, we show that our results are qualitatively robust to varying the degree of risk aversion. For that purpose, we report results for $\gamma = 8$ in this section. Results for $\gamma = 6$, which are qualitatively robust can be found in the Online Appendix in Figure 5.

Figure 6 depicts policy functions for individuals aged 30, 50, and 70. Similar to Figure 1, in which results are reported for a degree of risk aversion of $\gamma = 10$, Figure 6 reports results for a degree of risk aversion of $\gamma = 8$. As can be seen from a comparison of Figure 6 and 1, our results remain qualitatively robust with the lower degree of risk aversion.

The fraction of wealth invested in stocks decreases as the real interest rate increases and the consumption-to-wealth ratio increases when the interest rate is higher. With a lower level of risk aversion, the percentage of wealth invested in stocks is higher since individuals are more willing to take on the higher risk of a stock investment. For ages before retirement, the consumption-to-wealth ratio also increases compared to the case with $\gamma = 10$ since with lower risk aversion, the precautionary savings motive is weaker. During retirement, when labor income is no longer subject to shocks and the need for precautionary savings is therefore weaker, the consumption-to-wealth ratios are basically identical to those with $\gamma = 10$.

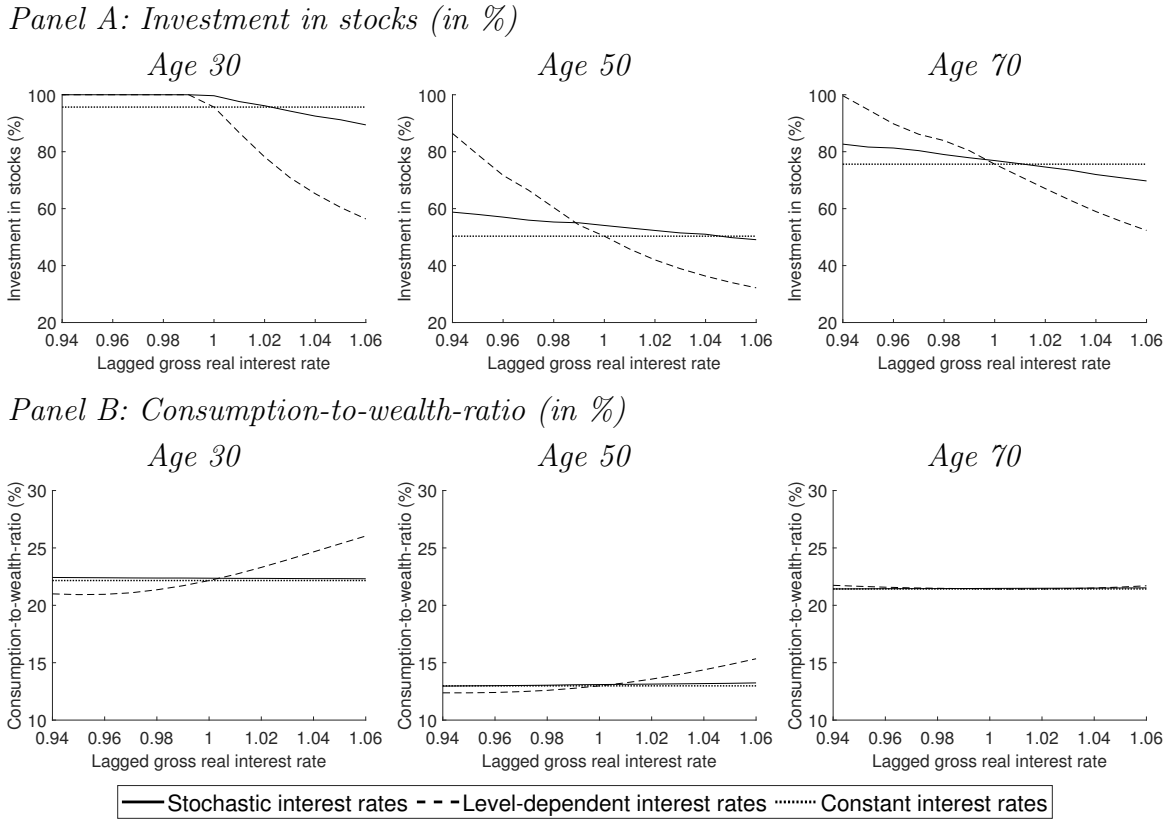
4.3 Decomposing the equity exposure

The optimal equity exposures in our dynamic life cycle model are relatively high compared to static models of portfolio choice – particularly for a degree of risk aversion of $\gamma = 10$. In the classical static framework of Samuelson (1969) and Merton (1969), the optimal fraction invested in the risky asset is constant and can be approximated by

$$\frac{1}{\gamma}\frac{\mathbb{E}[R_t^S - R_t]}{\mathrm{Var}(R_t^S)}. \tag{11}$$

With our parameters, the optimal demand for stocks from Equation (11) is approximately 12.5% when the interest rate is at its long-run mean. In this section, we argue that there

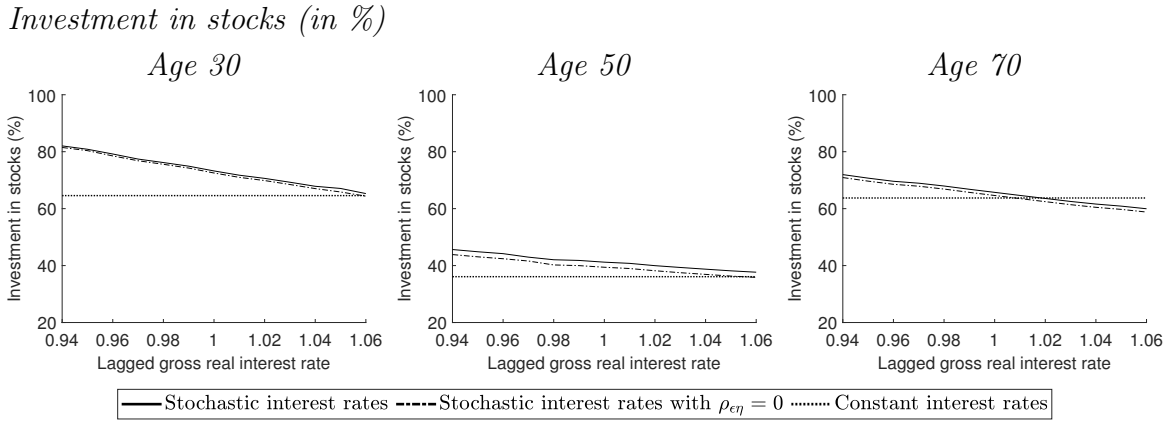
Figure 6
Policy functions for a risk aversion of $\gamma = 8$



This figure depicts policy functions for 30-,50-, and 70-year-old individuals with labor-to-wealth ratios of 27.5%, 11.8%, and 13.7%, respectively. It compares the optimal policy function of the base case, in which interest rates are stochastic (level- and risk-channel), with two counterfactuals in which only the level of the lagged interest rate (dashed lines) and the long-run mean of the interest rate (dotted lines) are taken into account, respectively. Panel A shows policy functions for the proportion of wealth invested in stocks. Panel B shows optimal consumption-to-wealth ratios.

are five factors, contributing to the deviation of the optimal equity exposure from the static portfolio choice model benchmark. First, there exists negative (residual) correlation between changes in interest rates and stock returns, creating a hedging demand. Second, in our work, the real interest rate is not constant, but stochastic, which renders the optimal equity exposure state-dependent. Third, in our model there is correlation between the growth of labor income and stock returns, causing a hedging demand. Fourth, our model accounts for mortality. When individuals face a risk of not benefiting from the future consumption opportunities from present saving, they alter their present consumption-savings decision. Fifth, the existence of labor income itself affects the optimal equity exposure. From the

Figure 7
Policy functions with no residual correlation between stocks and interest rates



This figure depicts policy functions for for the proportion of wealth invested in stocks for 30-, 50-, and 70-year-old individuals with labor-to-wealth ratios of 27.5%, 11.8%, and 13.7%, respectively. It compares the optimal policy function of the base case, in which interest rates are stochastic (level- and risk-channel), with two counterfactuals in which the interest rate is stochastic but the correlation between stock returns and the real interest rate $\rho_{\epsilon\eta}$ is zero (dash-dotted lines) and in which only the the long-run mean of the interest rate is taken into account (dotted lines).

work of Cocco et al. (2005), the payments from labor income are more similar to those of a bond than those of a stock, implying that with labor income, investors should increase their exposure to equity. In this section, we decompose the deviation of our model's predicted optimal equity exposure from a static model's benchmark into these five components by consecutively closing these five channels.

First, removing the residual correlation between the stochastic interest rates and returns on stocks (and thereby the hedging demand) slightly decreases the optimal equity exposure at the long-run mean interest rate from 73.2% to 72.5% at age 30, from 41.2% to 39.4% at age 50, and from 65.7% to 64.6% at age 70 as can be seen from Figure 7. Second, switching from a time-varying to a constant interest rate reduces the optimal equity exposure at the long-run mean interest rate from 72.5% to 64.6% at age 30, from 39.4% to 36.1% at age 50, and from 64.6% to 63.7% at age 70. Third, additionally removing the correlation structure between labor income and stock returns by reducing correlation from the empirically observed value of 0.0735 to zero increases the optimal equity exposure during working life from 64.6% to 85.8% at age 30 and from 36.1% to 45.9% at age 50, reflecting that a negative hedge position to offset the labor income risk in stock returns is no longer needed once correlation drops to zero. During retirement age, at age 70, the removal of the correlation does not alter the optimal equity exposure of 63.7% as retirement income is, by assumption, risk-free and

thus uncorrelated with stock returns. Fourth, further removing mortality risk by setting the probability of survival to 100% except for the last period (terminal condition) affects the optimal equity exposure during working age only very little. It increases from 85.8% to 86.4% at age 30 and from 45.9% to 46.7% at age 50. During retirement age, when mortality rates are higher, the removal of mortality increases the optimal equity exposure from 63.7% to 69.8%, reflecting that in the absence of mortality risk, the present value of future labor income, whose payments are more similar to those of bonds than of stocks, increases, thus allowing for a higher present investment into equity. Finally, additionally completely removing labor income from our model, i.e., requiring individuals to solely finance consumption from their initial wealth and financial income on it, reduces the optimal equity exposure from 86.4% at age 30, 46.7% at age 50, and 69.8% at age 70 to 13.5%, roughly corresponding to the Merton-Samuelson approximation from Equation (11). The Merton-Samuelson approximation from Equation (11) relies on a characterization of the risky asset's return by its first two moments. It is therefore only exact for distributions that can be fully characterized by their first two moments, such as the binomial or the normal distribution.[9] In our work, in which the return on the risky asset follows a lognormal distribution, the optimal share in the risky asset is only one percentage point higher, documenting that the Merton-Samuelson approximation is a very good approximation in our framework. The slightly higher equity exposure compared to the binomial or normal distribution in our framework with lognormally distributed returns reflects that returns are right-skewed, thus rendering states with very low returns less likely and thus equity investments more desirable.

Overall, the existence of labor income in our model accounts for the largest share of the deviation of the optimal equity exposures reported in Section 3 from the Merton-Samuelson approximation.

5 Conclusion

We contribute to the literature by being the first to set up a life cycle model with stochastic time-varying real interest rates to explore their implications for optimal consumption and investment decisions. Real interest rate risk materializes through two channels: a level-channel and a risk-channel. The level-channel entails that at different times, the interest rate can obtain a different level. Our model predicts that when real interest rates are low, individuals should invest more in stocks and consume less (wealth effect). The risk-channel takes the stochastic nature of real interest rates into account and triggers a precautionary

[9] With normally or binomially distributed returns (with the same mean and standard deviation), we get the exact equity exposure of 12.5% from Equation (11).

savings motive. Ignoring the time-varying nature of interest rates leads to overconsumption and underinvestment into stocks when interest rates are high and, ultimately, substantial welfare costs. Being exposed to a low real interest environment has substantial welfare consequences even if individuals behave rationally. When exposed to an extended period of low real interest rates, savings grow at a slower rate or even shrink when real interest rates are negative. Ultimately, lower savings turn into lower consumption opportunities. Welfare costs from being exposed to a low real interest rate environment tend to be highest around retirement age, where savings peak.

Extending beyond these predictions for individuals and households, our results also suggest implications for policy makers. As welfare effects from low interest rate environments hit individuals around retirement age hardest, government policies aiming at reducing the consequences of extended periods of low interest rates should pay particular attention to individuals at around retirement age.

Likewise, our results suggest implications for pension plans, particularly defined benefit plans, or funded pension systems, that aim at providing their participants given target levels of payouts as under some public pension systems found around the world. Our results suggest that these pension plans or systems should not only increase their stock investments in periods of lower interest rates, but - even though probably not very popular - may also have to raise contribution rates to attain pre-specified target levels of payoffs.

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A Solution of the optimization problem

To reduce the number of state variables by one, the optimization problem is normalized with wealth. Normalizing the value function $v_t(x_t) = V_t(X_t) \cdot \frac{1}{W_t}$ leads to the normalized optimization problem

$$\max_{\{c_t,\alpha_t\}} v_t(x_t) = \left((1-\beta) \cdot c_t^{1-\frac{1}{\Psi}} + \beta \cdot \left(f(t) \cdot \mathbb{E}_t \left[(v_{t+1}(x_{t+1}) \cdot \frac{W_{t+1}}{W_t})^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\Psi}}{1-\gamma}} \right)^{\frac{1}{1-\frac{1}{\Psi}}} \tag{12}$$

subject to

$$c_t > 0, c_t \leq 1, \tag{13}$$

$$\alpha_t \in [0,1], \tag{14}$$

$$\frac{W_{t+1}}{W_t} = \frac{L_t}{W_t} \cdot R_t^L + (1-c_t) \cdot (\alpha_t R_t^S + (1-\alpha_t) R_t) \tag{15}$$

in which $c_t = \frac{C_t}{W_t}$ is the consumption-to-wealth ratio and $x_t = [t, \frac{L_t}{W_t}, R_{t-1}]$ is the new vector of state variables. R_t^L denotes the gross growth rate of labor income from t to $t+1$.

We discretize the continuous state variables and then use backward induction to numerically solve the normalized optimization problem. We use Gaussian quadrature to calculate the expectation in Equation (12). The speed of the optimization is increased by using parallel computation. With 12 cores solving subproblems in parallel, it takes around 8 minutes to solve for the optimal consumption-investment strategy at every age.

B Estimation of base case parameters

Nominal interest rates and stock returns are transformed into real rates using the Fisher equation. We estimate the real interest rate process parameters by standard maximum likelihood for an AR(1) process. The stock risk premium and variance are also calculated using maximum likelihood with the log-likelihood function

$$\ln L(\lambda, \sigma_S^2; x_t) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln(\sigma_S^2) - \frac{1}{2\sigma_S^2}\sum_{t=1}^{T}\left(\ln(R_t^S) - (\ln(R_t + \lambda) - \frac{\sigma_S^2}{2})\right)^2. \quad (16)$$

$\rho_{\epsilon\eta}$ is then calculated as the residual correlation.

To obtain the correlation with labor income shocks, we calculate residuals for the log labor income process, denoted by $\log(Y_{it}^*)$, in which i is the individual and t the time, following exactly the procedure of Fischer and Koch (2025) and Cocco et al. (2005). Then we follow Cocco et al. (2005) and calculate return residuals (corresponding to the labor return error term u_t) by

$$z_{id,t} = \log(Y_{i,t+d}^*) - \log(Y_{i,t}^*) \quad (17)$$

with $d = 1$ if the observation is from before the year 1997 and $d = 2$ if the year is 1997 or later (starting 1997 the data is only available bi-annually). The 2-year returns are annualized. As in Cocco et al. (2005) the shock of the labor income is decomposed into an aggregated shock, ξ_t, and an idiosyncratic component, ω_{it}:

$$z_{i1,t} = \xi_t + \omega_{it}. \quad (18)$$

Averaging over all individuals yields

$$\bar{z}_{1t} = \xi_t. \quad (19)$$

Correlation with stock and interest rate shocks are calculated as respective residual corre-

lations $\rho_{\xi\eta}$ and $\rho_{\xi\epsilon}$, respectively. We assume that there is no correlation between the idiosyncratic component and both, the stock and interest rate shocks. In total, the correlation between labor income and stock return residuals is then given by

$$\rho_{u\eta} = \frac{\rho_{\xi\eta} \cdot \sigma_{\xi}}{\sigma_u}. \tag{20}$$

We proceed correspondingly for interest rate shocks. We calculate σ_{ξ} by taking the standard deviation of $\bar{z}_{1t}$. σ_u is calculated analogous as in Cocco et al. (2005).

To calibrate the preference parameters, we use PSID data from the waves after the financial crisis, i.e., 2011, 2013, 2015, 2017, 2019, and 2021. Wealth is adjusted for inflation to 2019 values using the CPI published by the U.S. Bureau of Labor Statistics.

We restrict wealth of the individuals to be positive and calculate the growth of median wealth from individuals aged 20 to individuals aged 65. In the data, median wealth grows by a factor of 14.07. Using our calibrated parameters in the model, median wealth in our model grows by a factor of 14.81, roughly corresponding to the growth rate observed in the data.

C Online Appendix

This Online Appendix to the manuscript "Life cycle consumption and portfolio choice under real interest rate risk" contains further robustness results. In a similar fashion to Table 1, Table 1 provides the coefficient estimates from Equations (2) and (4) for the time period from 1980 to 2019, the longest time span for which we have data on stocks, bonds, labor income, and household wealth in the PSID data. Similarly, Table 2 provides the re-estimated correlation coefficients from Table 2. Overall, the differences between Tables 1 and 1 as well as Tables 2 and 2 are only very small. Signs do not switch and all coefficients are roughly the same order of magnitude. Consequently, the implications for consumption-investment strategies are only very small as a comparison of Figures 1 and 2 with Figures 1 and 2 shows.

In our base case parameter setting in the main manuscript, we assumed a degree of risk aversion of $\gamma = 10$ and an elasticity of intertemporal substitution of $\Psi = 0.2$, implying that the individual's attitude towards risk dominates the aversion to time fluctuations. Figure 3 reports in the same fashion as Figure 1 results for $\Psi = 0.05$, i.e., a setting in which the attitude towards risk is dominated by the aversion to time fluctuations. From a comparison of Figures 1 and 3, the change in the elasticity of intertemporal substitution only affects our results quantitatively, but not qualitatively. In particular, our key insights on how the interest rate affects optimal consumption-investment decisions remain robust.

In the main manuscript, we focus on an investor who cannot borrow or short sale assets. In Figure 4, we report in a similar fashion to Figure 1 results when these constraints are relaxed. From a comparison of Figures 4 and 1, our results are not much affected by whether borrowing and short selling is allowed or not. Only at younger age and for low levels of the interest rate, investors want to lever up their investments into stocks when trading as if the interest rate would permanently remain at the currently observed level.

Finally, completing our robustness results on the impact of risk aversion from Section 4.2 in the main manuscript, Figure 5 depicts in a similar fashion as Figure 6 results for a degree of risk aversion of $\gamma = 6$. It documents that consistently with the comparison between Figures 6 and 1 a lower degree of risk aversion increases the optimal equity exposure. Our key results are qualitatively robust to this modification. In particular, our results on how the real interest rate affects optimal consumption-investment strategies remain unaltered.

Table 1
Estimation results for the time horizon 1980-2019

Coefficient	Value	Standard Error
Interest rate process		
Intercept c	0.0031	0.0035
AR coefficient ϕ	0.7119***	0.1369
Interest rate standard deviation σ_ϵ	0.0169***	0.0016
Stock return process		
Risk premium λ	0.0582**	0.0268
Stock return standard deviation σ_η	0.1554***	0.0175

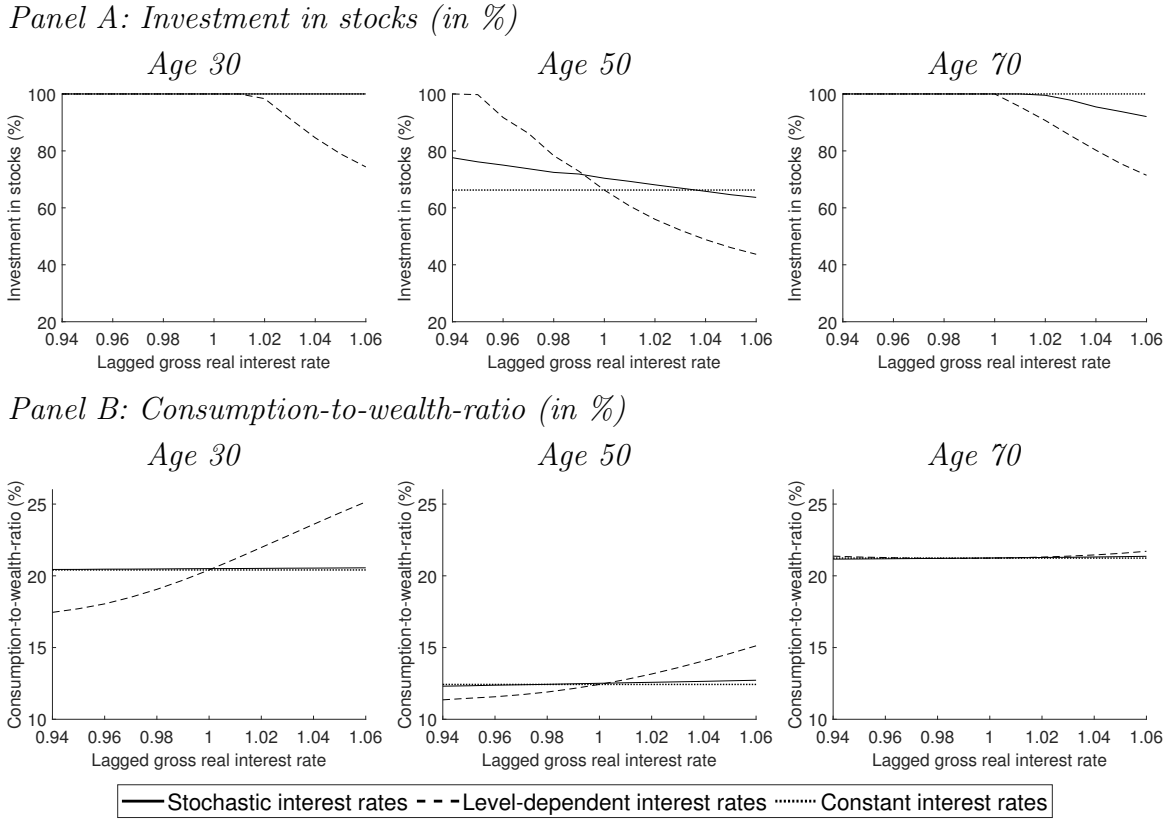
This table summarizes the coefficient estimates from the regressions using data from 1980 to 2019. *, **, and *** indicate significance of the estimates on the 10%, 5%, and 1% level, respectively.

Table 2
Calibration for the time horizon 1980-2019

Description	Parameter	Value	Source
Degree of risk aversion	γ	10	own calibration
Time preference factor	β	0.94	own calibration
Elasticity of intertemporal substitution	Ψ	0.2	own calibration
Residual correlation stock return and interest rate	$\rho_{\epsilon\eta}$	-0.1119	own estimation
Residual correlation labor income and stock return	$\rho_{u\eta}$	0.0734	own estimation
Residual correlation labor income and interest rate	$\rho_{u\epsilon}$	-0.0827	own estimation
Replacement ratio	-	0.7444	own estimation

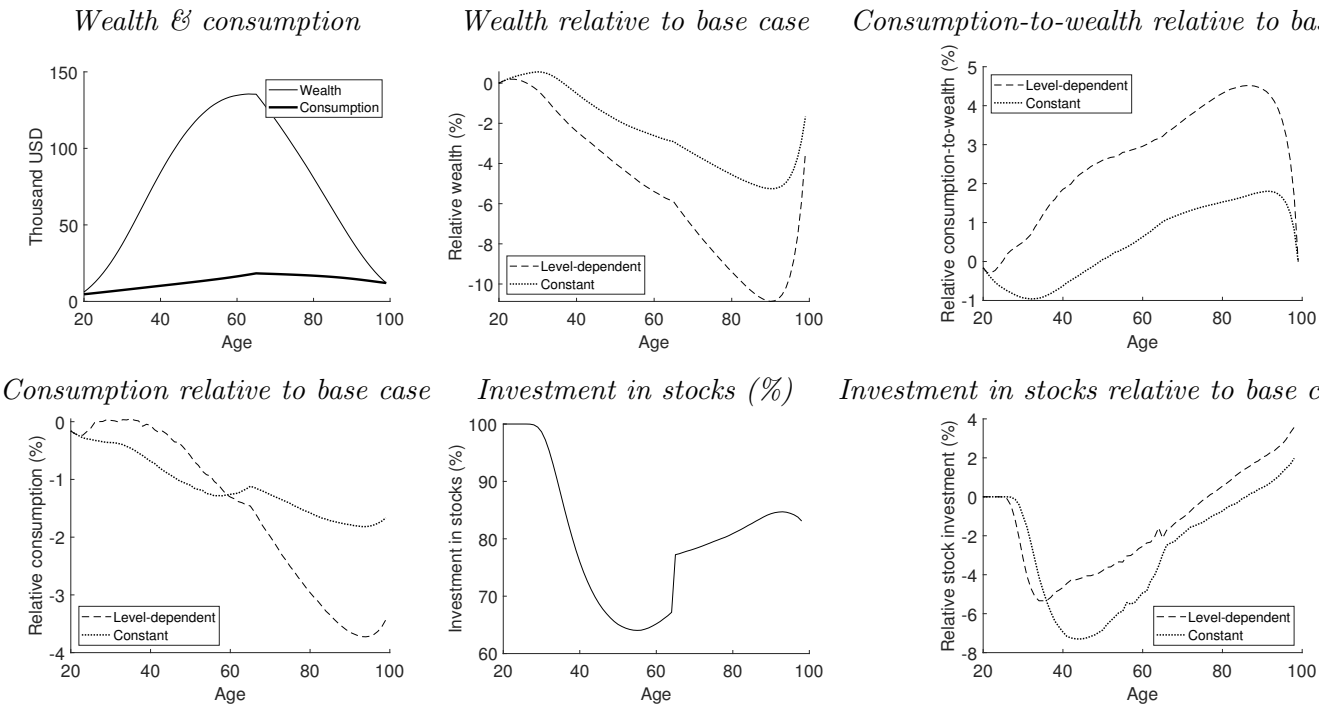
This table summarizes the parameters for the case with an estimation horizon 1980-2019.

Figure 1
Policy functions for estimates from 1980-2019



This figure depicts policy functions for 30-, 50-, and 70-year-old individuals with labor-to-wealth ratios of 27.5%, 11.8%, and 13.7%, respectively. It compares the optimal policy function of the base case, in which interest rates are stochastic (level- and risk-channel), with two counterfactuals in which only the level of the lagged interest rate (dashed lines) and the long-run mean of the interest rate (dotted lines) are taken into account, respectively. Panel A shows policy functions for the proportion of wealth invested in stocks. Panel B shows optimal consumption-to-wealth ratios.

Figure 2
Results for individuals following different policies for estimates from 1980-2019



This figure depicts the evolution of wealth, consumption, and stock investment over the life cycle. The top left graph shows results for consumption and wealth in our base case parameter setting when individuals make optimal decisions, the thin line shows the evolution of wealth and the bold line consumption. The bottom middle graph shows results for stock investment in our base case. The top middle, top right, and bottom left and right graphs show how wealth, the consumption-to-wealth ratio, the absolute consumption level, and stock investment change relative to the base case if individuals ignore the risk-channel (dashed lines) or the risk- and the level-channel (dotted lines). All results are based on 50,000 simulated paths.

Figure 3
Policy functions with an elasticity of intertemporal substitution of $\Psi = 0.05$

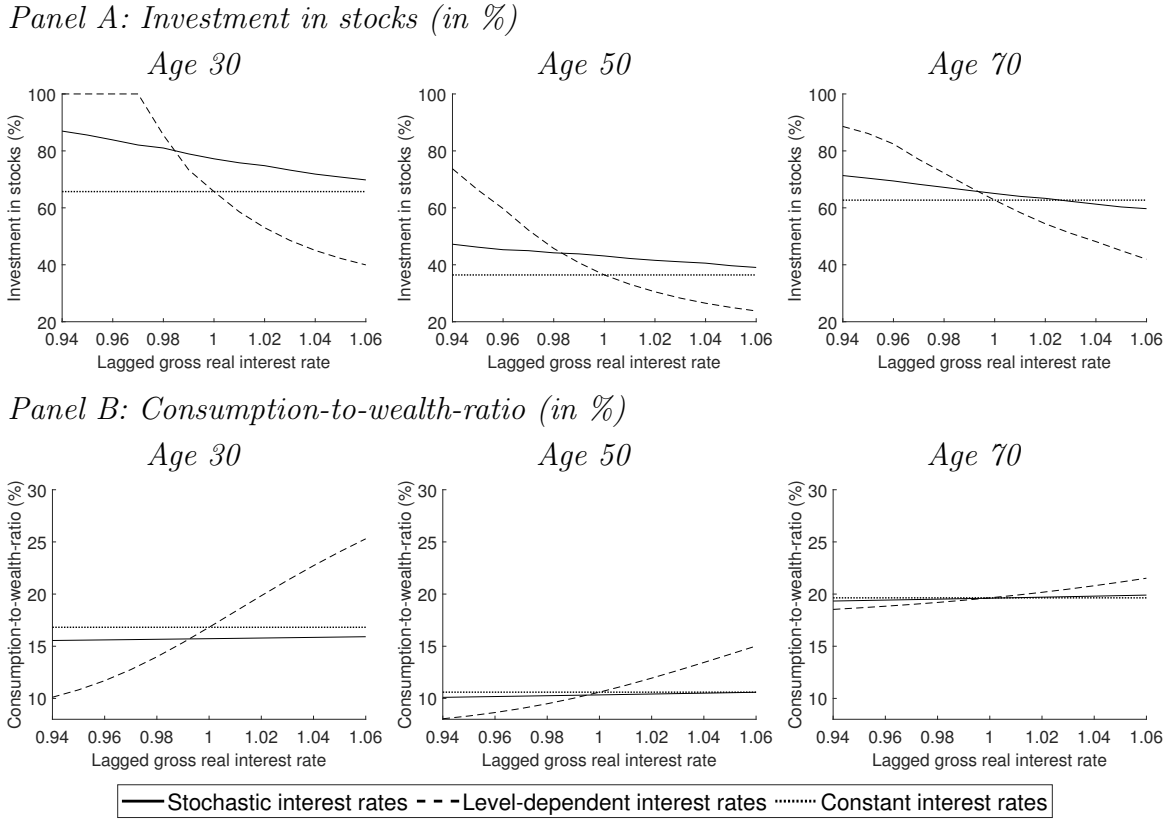


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Figure 4
Policy functions without borrowing and short selling constraints

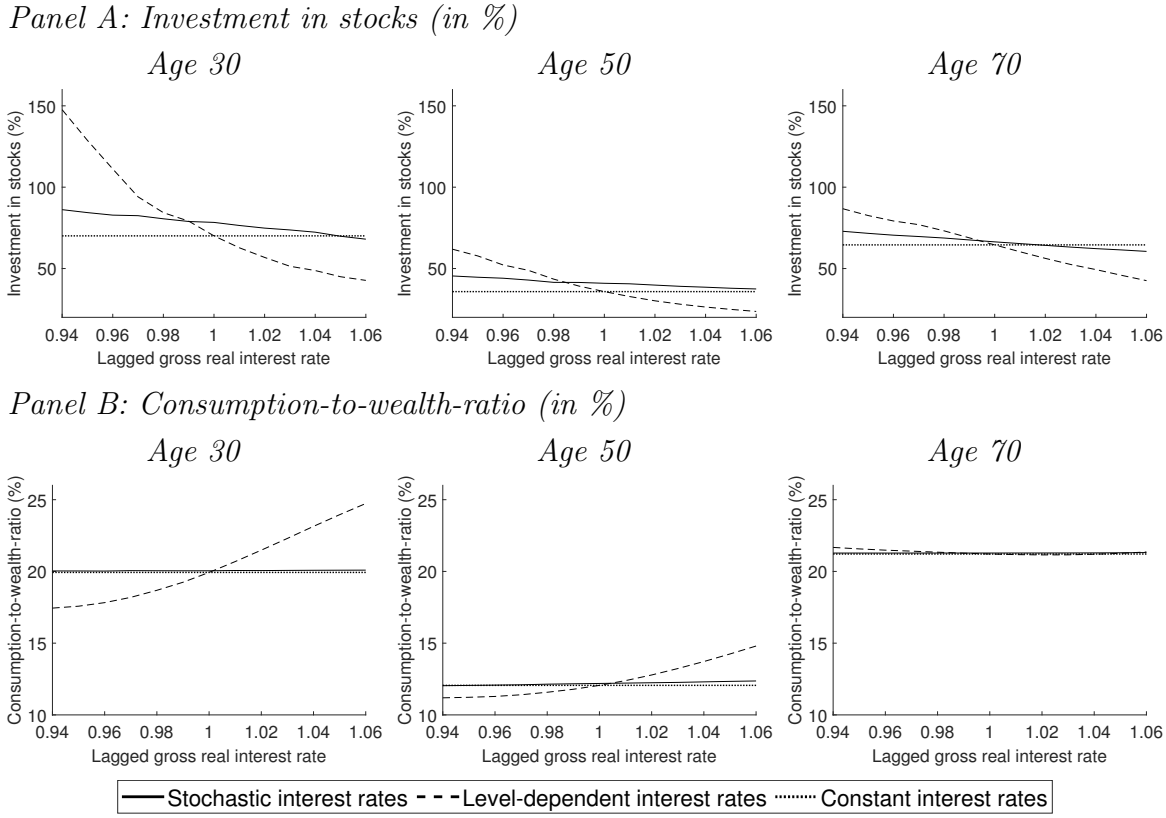

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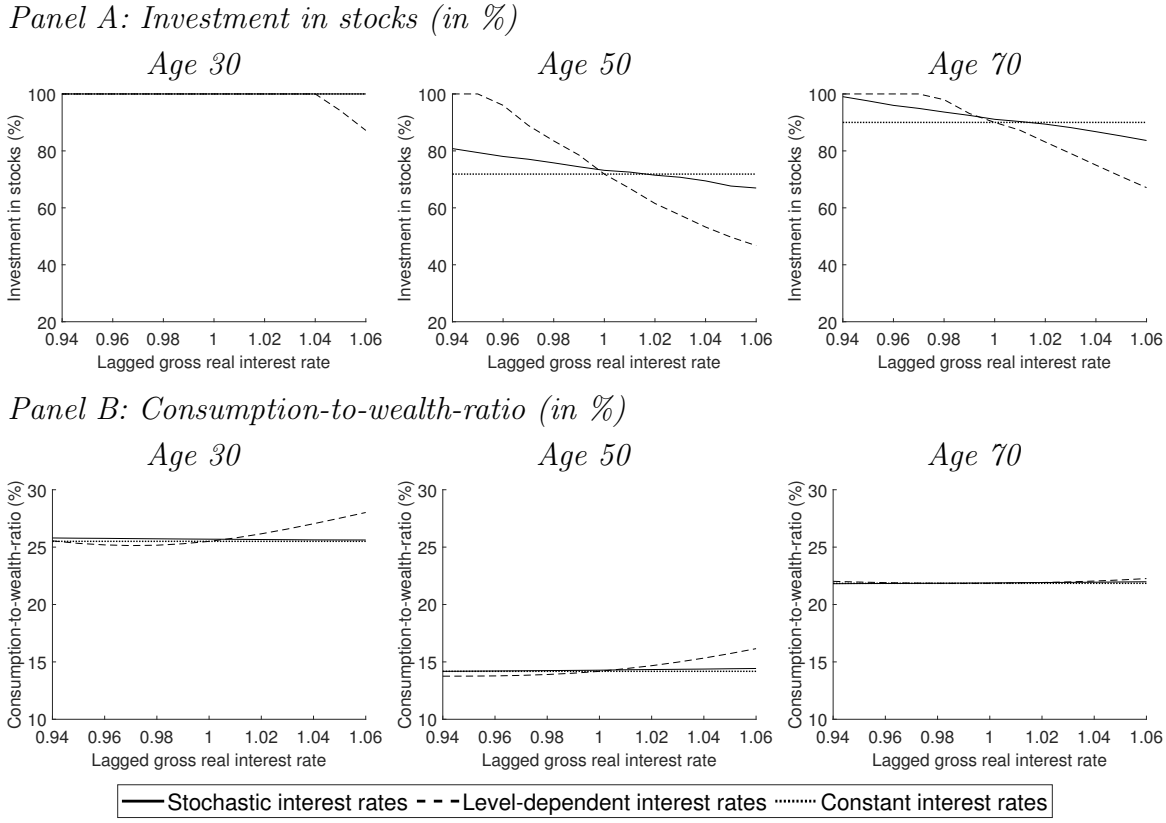
Figure 5
Policy functions for a risk aversion of $\gamma = 6$



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Impressum:
Arbeitskreis Quantitative Steuerlehre, arqus, e.V.
Vorstand: Prof. Dr. Ralf Maiterth (Vorsitzender),
Prof. Dr. Kay Blaufus, Prof. Dr. Dr. Andreas Löffler
Sitz des Vereins: Berlin

Herausgeber: Kay Blaufus, Jochen Hundsdoerfer, Martin Jacob, Dirk Kiesewetter, Rolf J. König, Lutz Kruschwitz, Andreas Löffler, Ralf Maiterth, Heiko Müller, Jens Müller, Rainer Niemann, Deborah Schanz, Sebastian Schanz, Caren Sureth-Sloane, Corinna Treisch

Kontaktadresse:
Prof. Dr. Dr. h.c. Dr. h.c. Caren Sureth-Sloane,
Universität Paderborn, Fakultät für Wirtschaftswissenschaften,
Warburger Str. 100, 33098 Paderborn,
www.arqus.info, Email: info@arqus.info

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