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# Conjectures on the policy function in the presence of optimal experimentation

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# Conjectures on the policy function in the presence of optimal experimentation

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**Abstract**

In the economics literature there are two dominant approaches for solving models with optimal experimentation (also called active learning). The first approach is based on the value function and the second on an approximation method. In principle the value function approach is the preferred method. However, it suffers from the *curse of dimensionality* and is only applicable to small problems with a limited number of policy variables. The approximation method allows for a computationally larger class of models, but may produce results that deviate from the optimal solution. Our simulations indicate that when the effects of learning are limited, the differences may be small. However, when there is sufficient scope for learning, the value function solution is more aggressive in the use of the policy variable.

**Keywords**: design of fiscal policy, optimal experimentation, stochastic optimization, time-varying parameters, numerical experiments.

**JEL classification**: C63, E61, E62

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# 1 Introduction

In the last decade a large literature has emerged in economics on the subject of optimal or strategic experimentation also referred to as active learning. The seminal work on this subject in economics, stems from an early paper by MacRae (1972, 1975), followed by a range of theoretical papers like Easley and Kiefer (1988), Bolton and Harris (1999), Salmon (2001), Moscarini and Smith (2001) and applications like Buera et. al (2011).

There are two prevailing methods for solving a general class of models with optimal experimentation. The first method is based on the value function approach and the second on an approximation method. The value function approach uses dynamic programming for the full problem as used in studies by Prescott (1972), Taylor (1974), Easley and Kiefer (1988), Kiefer (1989), Kiefer and Nyarko (1989), Aghion et. al (1991) and more recently used in the work of Beck and Wieland (2002), Coenen et. al (2005), Levin et. al (2003) and Wieland (2000a, 2000b). A nice set of applications on optimal experimentation, using the value function approach, can be found in Willems (2012).

In principle, the value function is approach is the preferred method as it derives the optimal values for de policy variables through Bellman's (1957) dynamic programming. Unfortunately, it suffers from the *curse of dimensionality*, Bertsekas (1976), and is only applicable to small problems with one or two policy variables. This is caused by he fact that solution space needs to be discretized in such a fashion that it cannot be solved in feasible time. The approximation methods as described in Cosimano (2008) and Cosimano and Gapen (2005a, 2005b), Kendrick (1981) and Hansen and Sargent (2008) use approaches, that are applied in the neighborhood of the linear regulator problems.[1] [2] Because of this *local* nature with respect to the statistics of the model, the method is numerically far more tractable and allows for models of larger dimension. However, the verdict is still out as to how well it performs in terms of approximating the optimal solution derived through

[1] For consistency and clarity in the main text, we used the term approximation method in stead of adaptive or dual control. The adaptive or dual control approach in MacRae (1975), see Kendrick (1981), Amman (1996) and Tucci (2004), uses methods that draw on earlier work in the engineering literature by Bar-Shalom and Sivan (1969) and Tse (1973).

[2] There are difference between the approximation approaches in Cosimano (2008)) and in Kendrick (1981), which we will not discuss in detail here. Through out the paper we will use the approach in Kendrick(1981).

the value function. By the way, the approximation method described here, should not be mistaken for a *cautious* or *passive learning* method. Here we concentrate only on optimal experimentation - active learning - approaches.

Both solution methods consider dynamic stochastic models in which the control variables can be used not only to guide the system in desired directions but also to improve the accuracy of estimates of parameters in the models. Thus, there is a trade off in which experimentation of the policy variables early in time detracts from reaching current goals, but leads to learning or improved parameter estimates and thus improved performance of the system later in time. Ergo, the dual nature of the control. For this reason, we concentrate in Section 4 on the policy function in the initial period. Usually most of the experimentation - active learning - is done in the beginning of the time interval, and therefore, the largest difference between results obtained with the two methods may be expected in this period.

In this paper we focus on comparing the policy function obtained by Beck and Wieland (2002), through the value function, to the one obtained through approximation methods. Note that the model used by Beck and Wieland (2002) closely resembles the model and framework presented in MacRae (1972, 1975), which has served as a bench mark in much of the literature.[3] In doing the comparison, we have derived an analytic form of the cost-to-go function for the approximation approach following the method outlined in the Amman and Kendrick (1995) paper and the extension of these results in Tucci, Kendrick and Amman (2010). After describing both of these methods in Section 3, we will present in Section 4. of the paper a comparison of the policy function results obtained through the value function approach and the approximation approach.

## 2 A general framework

As the MacRae model (1972, 1975) is used as a bench mark in Beck and Wieland (2002), from here on we will use the abbreviation *MBW* to refer to both sets of results in the literature. The general framework, Kendrick (1981), of this model starts with a linear difference equation for a finite time horizon

$$x_{t+1} = A_t(\theta_t)\, x_t + B_t(\theta_t)\, u_t + c_t(\theta_t) + \epsilon_t \tag{1}$$

[3] See Amman and Kendrick (2008).

where $t \in [0, N-1]$ is the time index, $x_t \in \Re^{(n\times 1)}$ the state vector, $u_t \in \Re^{(m\times 1)}$ the control vector, $\epsilon_t \in \Re^{(n\times 1)}$ a vector of system noise terms, $A_t(\theta_t) \in \Re^{(n\times n)}$ the state vector coefficient matrix, $B_t(\theta_t) \in \Re^{(n\times m)}$ the control vector coefficient matrix, $c_t(\theta_t) \in \Re^{(n\times 1)}$ the exogenous coefficient vector, $\theta_t \in \Re^{(s\times 1)}$ a vector containing the subset of the coefficients in $A_t(\theta_t)$, $B_t(\theta_t)$ and $c_t(\theta_t)$ that are stochastic. The measurement equation of the state $x_t$ has the shape

$$y_t = H_t x_t + \xi_t \tag{2}$$

where $y_t \in \Re^{(r\times 1)}$ is a measurement vector, $H_t \in \Re^{(r\times n)}$ is a known measurement coefficient matrix $\xi_t \in \Re^{(r\times 1)}$ a vector of measurement noise terms. The time-varying parameter equation for $\theta_t$ is

$$\theta_{t+1} = D_t \theta_t + \eta_t \tag{3}$$

where $D_t \in \Re^{(s\times s)}$ is a known matrix, $\eta_t \in \Re^{(s\times 1)}$ a vector of parameter-noise terms.[4] The a vector noise terms for the system, measurement and parameter evolution equations are normally distributed

$$\epsilon_t \overset{i.i.d.}{\sim} N\left(0, \Sigma^{\epsilon\epsilon}\right) \qquad \xi_t \overset{i.i.d.}{\sim} N\left(0, \Sigma^{\xi\xi}\right) \qquad \eta_t \overset{i.i.d.}{\sim} N\left(0, \Sigma^{\eta\eta}\right) \tag{4}$$

and the covariances matrices $\Sigma^{\epsilon\epsilon}, \Sigma^{\xi\xi}, \Sigma^{\eta\eta}$ are assumed to be known. The initial conditions for the state vector (1) and parameter vector (3), that are also normally, distributed

$$x_0 \sim N\left(x_0, \Sigma_0^{xx}\right) \qquad \theta_0 \sim N\left(\theta_0, \Sigma_0^{\theta\theta}\right) \tag{5}$$

with the covariances matrices $\Sigma_0^{xx} \in \Re^{(n\times n)}$, $\Sigma_0^{\theta\theta} \in \Re^{(s\times s)}$. Due to the linearity in (1), this normality is carried over in time. As both $\{\theta_t, x_t\}$ are stochastic, their (posterior) estimates $\{\hat{x}_{t|t}, \hat{\theta}_{t|t}\}$ will be obtained through Bayesian updating using the Kalman filter. The same holds for the corresponding estimated covariance matrices $\{\hat{\Sigma}_{t|t}^{xx}, \hat{\Sigma}_{t|t}^{\theta\theta}\}$.

The criterion function may be written as

$$J = E\left\{\delta^N L_N\left(x_N\right) + \sum_{t=0}^{N-1} \delta^t L_t\left(x_t, u_t\right)\right\} \tag{6}$$

[4] For more general forms of equation (3), including the *return to normality* model, see Tucci (2004) page 17.

where $J \in \Re$ is the criterion value, $E$ the expectations operator, $\delta \in [0, 1>$ the discount factor, $L_N \in \Re$ the criterion function for the terminal period $N$, $x_N \in \Re^{(n\times 1)}$ the state vector for the terminal period $N$, $L_t \in \Re$ the criterion function for period $t$, $x_t \in \Re^{(n\times 1)}$ the state vector for period $t$ and $u_t \in \Re^{(m\times 1)}$ the control vector for period $t$. The two terms on the right-hand side of equation (6) are defined as

$$L_N(x_N) = \frac{1}{2}(x_N - \tilde{x}_N)' W_N (x_N - \tilde{x}_N) \tag{7}$$

and

$$L_t(x_t, u_t) = \frac{1}{2}\Big[ (x_t - \tilde{x}_t)' W_t (x_t - \tilde{x}_t) + (x_t - \tilde{x}_t)' F_t (u_t - \tilde{u}_t) + (u_t - \tilde{u}_t)' \Lambda_t (u_t - \tilde{u}_t)\Big] \tag{8}$$

where $\tilde{x}_N \in \Re^{(n\times 1)}$ the desired state vector for terminal period $N$, $W_N \in \Re^{(n\times n)}$ the symmetric state variable penalty matrix for terminal period $N$, $\tilde{x}_t \in \Re^{(n\times 1)}$ the desired state vector for period $t$, $\tilde{u}_t \in \Re^{(m\times 1)}$ the desired control vector for period $t$, $W_t \in \Re^{(n\times n)}$ the symmetric state variable penalty matrix for period $t$, $F_t \in \Re^{(n\times m)}$ the penalty matrix on state-control variable deviations for period $t$, $\Lambda_t \in \Re^{(m\times m)}$ the symmetric control variable penalty matrix for period $t$.

This completes the description of the general framework that will be applied in Section 4. In the next Section we describe briefly how the policy function can be derived for the approximation method[5]

## 3 Solution methods for the general framework

The value function approach, though computational intensive, is quite straight forward, e.g. integrate the value $V$ function backward in time, starting at $N-1$

$$V(x_{N-1}, u_{N-1}) = \min_{u_{N-1}} \{L_{N-1}(x_{N-1}, u_{N-1}) + \delta E\, V(x_N | \mathcal{P}_{N-1})\} \tag{9}$$

[5]The general framework in this section is coded in two software packages *DualPC*, Amman and Kendrick (1990), and more recently in *Duali*, Kendrick et. al. (2006). The framework may also be extended for the situation with rational expectations, cf. Amman and Kendrick (2003) and Kendrick et. al (2013).

subject to equation (1). The set $\mathcal{P}_{N-1}$ contains, under the normality assumption, the means and variance of the stochastic elements in equations (1)-(5). If we ignore measurement error, this means that

$$\mathcal{P}_{N-1} = \{x_{N-1}, \Sigma^{xx}_{N-1}, \theta_{N-1}, \Sigma^{\theta\theta}_{N-1}\}$$

Equation (9) is solved using discretization, see Judd (1998) or Kendrick et. al (2006). Following Easley and Kiefer (1988) and Kiefer and Nyarko (1989) we can obtain for $N \to \infty$, the optimal stationary policy, $h_v$, from the value function

$$u^*_v = h_v(x_0) \tag{10}$$

where $u^*_v$ is the optimal response given the initial state $x_0$ in the value function approach.

Now lets turn to the approximation approach. The appraximation approach solves the cost-to-go function

$$J^*_N = \min_{\mathbf{u}_0} E\left\{ \ldots \min_{\mathbf{u}_{N-2}} E\left\{ \min_{\mathbf{u}_{N-1}} E\left\{C_N | \mathcal{Q}_{N-1}\right\} | \mathcal{Q}_{N-2} \right\} \cdots | \mathcal{Q}^0 \right\} \tag{11}$$

subject to (1) with

$$C_{N-j} = L_N(x_N) + \sum_{t=j}^{N-1} L_t(x_t, u_t) \tag{12}$$

for $j = 0, \ldots, N-1$ and $\mathcal{Q}_j$, like $\mathcal{P}_j$, is defined as the means and covariances of the stochastic elements at time $j$. The essential difference between (9) and (11) is in the way $\mathcal{P}_j$ and $\mathcal{Q}_j$ are treated in the optimization. With the value function, the stochastic elements are fully integrated backward in time over the full probability space.

In the approximation approach, first an *certainty equivalence* solution is derived, Bertsekas (1976), for the problem in (1)-(8) using dynamic programming. Then, $\forall k \in [t, .., N-1]$ the estimated stochastic elements

$$\{\hat{x}_{k|k}, \hat{\Sigma}^{xx}_{k|k}, \hat{\theta}_{k|k}, \hat{\Sigma}^{\theta\theta}_{k|k}\}$$

are projected forward in time in the neighbourhood of this certainty equivalence problem using Bayesian updating. Once a new set of estimates is obtained, the control set $\{u_k, .., u_{N-1}\}$ is recomputed. These steps are repeated until convergence is reached. Through this forward projection, the combinatorial explosion in (9) is avoided and the resulting algorithm can be solved in polynomial time with $O(n^3)$.[6]

Based on equations (11)-(12) and following Tucci et. al (2010), the cost-to-go $J_N$ can be divided into three terms, i.e.

$$J_N = J_D + J_C + J_P \tag{13}$$

$J_D$ being the deterministic component, $J_C$ the cautionary component and $J_P$ the probing component. The probing component is the component that represents the experimentation - active learning - part of $J_N$.The optimal is obtained by minimizing $J_N$[7]

$$u_a^* = \arg\min_u J_N(x_0, u) \tag{14}$$

which is similar to equation (10). and provides the possibility to (numerically) derive the policy function

$$u_a^* = h_a(x_0) \tag{15}$$

where $u_a^*$ is the optimal response given the initial state $x_0$ in the approximation approach. In the next section we will analyze the outcomes for both the policy functions (10) and (15).

## 4 Comparison of the MBW Policy Functions

In order to make a comparison between the results based on the value function and those of the approximation method, we used Wieland's software and his parameter set.[8] We can write the MBW model, using the framework of Section 2, in its scalar form as

[6] The dimension of the state $x_t$, $n$, is usually the largest dimension and matrix operations of the order $n^3$, define the degree of complexity in the algorithm.

[7] Once again we would like to stress the quite frequently, equation (13) may have multiple minima as in Amman and Kendrick (1995) and Kendrick (1978).

[8] The software is available through the web site http://www.volkerwieland.com. A detailed discussion on how to compute the value function is found in Kendrick et. al (2006).

Given the the difference equation

$$x_{t+1} = \alpha x_t - b u_t + c_t \epsilon_t \tag{16}$$

Find $\{u_0, .., u_{N-1}\}$ to minimize

$$\begin{aligned} J = \quad & \frac{1}{2}\delta^N w_N (x_N - \tilde{x}_N)^2 + \\ & \frac{1}{2}\sum_{t=0}^{N-1} \delta^t \{w_t (x_t - \tilde{x}_t)^2 + \lambda_t (u_t - \tilde{u}_t)_t^2 + \psi_t (x_t - \tilde{x}_t)(u_t - \tilde{u}_t)\} \end{aligned} \tag{17}$$

For comparison with the general framework in Section 2, this means that $A_t = \alpha = 1$, $c_t = 0$, $W_t = w_t = 1$, $\Lambda_t = \lambda_t = 0$, $F_t = \psi_t = 0$, $\tilde{x}_t = 0$, $\tilde{u}_t = 0$ $\forall\, t$ and $\Sigma^{\epsilon\epsilon} = \sigma_\epsilon^2 = 1$. The initial estimate of the stochastic parameter enters only the $B$ matrix, so $B(\hat{\theta}_{0|0}) = \hat{\theta}_{0|0} = \hat{b} = -0.3$, with estimated variance $\hat{\Sigma}_{0|0}^{\theta\theta} = \hat{\sigma}_b^2 = 0.25$. There is no measurement error, which means $H = 1$ and $\Sigma^{\xi\xi} = \sigma_\xi^2 = 0$ and no parameter evolution, hence $D = 1$ and $\Sigma^{\eta\eta} = \sigma_\eta^2 = 0$ and as a consequence $\forall t\ \hat{\theta}_{t+1|t} = \hat{\theta}_{t|t} = \hat{b}$. The initial estimate of the state $\hat{x}_{0|0} = \hat{x}_0 = x_0$ and its variance $\hat{\Sigma}_{0|0}^{xx} = \hat{\sigma}_x^2 = 0$, the discount factor $\delta = 0.95$, which completes the parameter set. Following Beck and Wieland (2002) we will focuss on the stationary outcome, hence $N \to \infty$.

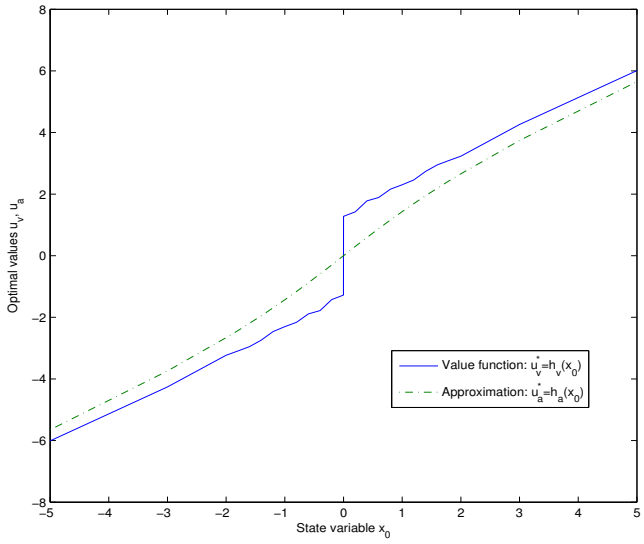
Since the MBW model has a single control variable and a single state variable it is possible to derive an analytic form of the cost-to-go for the approximation approach. This is based on earlier work done in Amman and Kendrick (1995) and Tucci et. al (2010), when we analyzed the question whether or not the cost-to-go function was characterized by non-convexities. The three components in equation (13) are[9]

[9]The derivation of equations (19)-(21) is rather lengthy. On request it can be provided by the corresponding author. The derivation is different from the one presented in Amman and Kendrick (1995) and Tucci et. al (2010), as we investigate the situation where $N \to \infty$ with $\delta \in [0, 1>$.

$$
\begin{aligned}
J_N &= J_P + J_C + J_D && (18)\\
J_D &= \frac{1}{2}\,\delta(x_0 + \hat{b}u)^2 && (19)\\
J_C &= \frac{1}{2}\,\frac{\delta[\hat{\sigma}_b^2(1-\delta)u^2 + \sigma_\epsilon^2]}{(1-\delta)} && (20)\\
J_P &= \frac{1}{2}\,\frac{\hat{\sigma}_b^2\delta^2[(\hat{\sigma}_b^2 - 1)u^2 + \sigma_\epsilon^2][x_0 + 2\hat{b}u]^2}{\hat{b}^2(1-\delta)(\hat{\sigma}_b^2 u^2 + \sigma_\epsilon^2)} && (21)
\end{aligned}
$$

By minimizing $J_N(u, x_0)$ we get the policy function $u_a^* = h_a(x_0)$ from (14) as the approximated optimal response, that we can compare with value function results, (10), $u_v^* = h_v(x_0)$, derived through numerical integration. For our simulations, we used a grid for $x_0 \in [-5, 5]$ and a line search for $u \in [-20, 20]$, analyzing the differences between the two approaches. We start with the baseline scenario using the above parameters. The results are shown in Figure 1

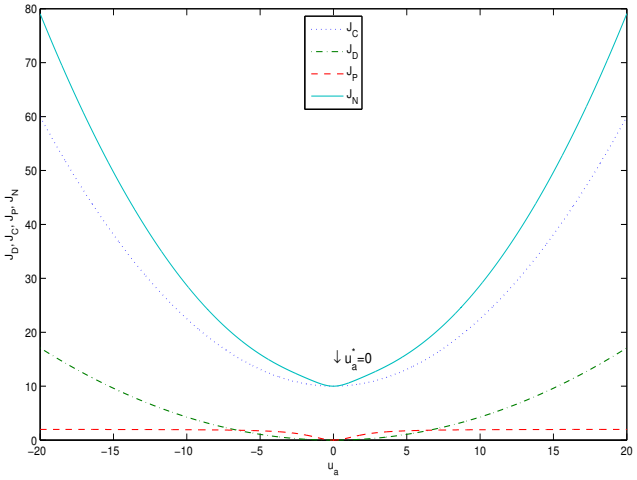
Figure 1: Baseline
$\delta = 0.95$, $\hat{b} = -0.30$, $\hat{\sigma}_b^2 = 0.25$, $\sigma_\epsilon^2 = 1.0$



With this baseline parameter set, a high discount rate and high variances, the value function solution has more aggressive control values than those in the approximation approach. This was to be expected as the value function, in principle, covers to whole solution space in comparison to the local approximation approach.[10]

This is highlighted in Figure 2, where we plotted the three components of the of the cost-to-go for $x_0 = 0$. Especially for a sufficiently small $\varepsilon > 0$, $\{0 < |x_0| < \varepsilon\}$, the neighborhood of the stationary solution, the cautionary component $J_C$ dominates the cost-to-go in the approximation and the difference between the two methods are the largest.

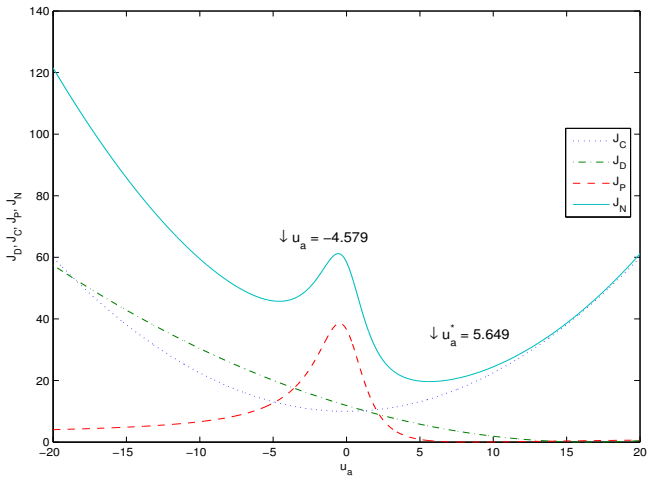
Figure 2: cost-to-go
$\delta = 0.95$, $\hat{b} = -0.30$, $\hat{\sigma}_b^2 = 0.25$, $\sigma_\epsilon^2 = 1.0$, $x_0 = 0$




As shown in Figure 3, in the 'tails' of the plot of Figure 1, $\{0 < |x_0 - 5| < \varepsilon\}$, the experimentation, component $J_P$ is dominant and the two methods tend to convergence. Actual, the cost-to-go is non-convex and has a local minimum at $u_a = -4.579$.

[10] Note that in Figure 1, that for a sufficiently small $\varepsilon > 0$, $\{0 < |x_0| < \varepsilon\}$, the solution for $u_v^*$ is somewhat jagged, which is caused by the discretization and has no economic interpretation. The same occurs for $\delta$ close to 1, causing numerical instability when integrating the value function.

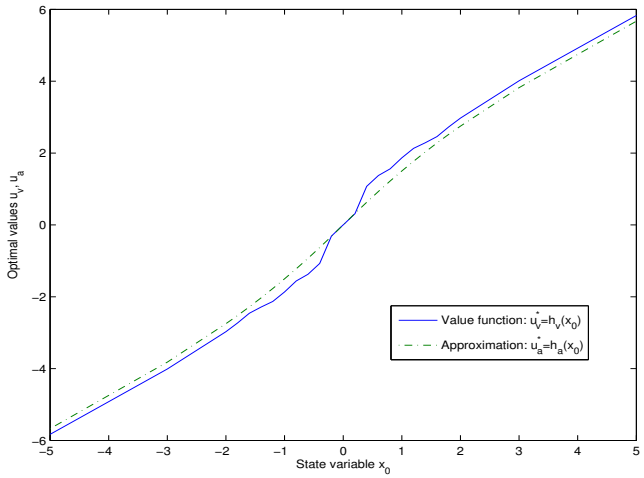
Figure 3: cost-to-go
$\delta = 0.95$, $\hat{b} = -0.30$, $\hat{\sigma}_b^2 = 0.25$, $\sigma_\epsilon^2 = 1.0$, $x_0 = 5$




On a more fundamental level, you see the 'local nature' of the approximation method reflected in Figure 3. As it searches around the certainty equivalence path, in this case the equalibrium state, the experimentation is centered arond $u = 0$.

To get more insight into the nature of the differences, we lowered the discount factor $\delta$ as shown in Figure 4. A lower $\delta$ means that the value of future experimentation - learning - has a lower value and leads to less experimentation. As a consequence, the solutions are closer than in the baseline scenario.

Figure 4: Lower discount factor $\delta$.
$\delta = 0.80$, $\hat{b} = -0.30$, $\hat{\sigma}^2_\theta = 0.25$, $\sigma^2_\epsilon = 1.0$




A similar effect takes place in Figure 5 and 6, in which we simulated what happens to the policy variable if we lower and increase the noise in system $\sigma^2_\epsilon$. As a consequence there is less or more need for experimentation to bring down the uncertainty in the system. Once again, the value function solution is more aggressive near the stationary point 0 than the approximation solution.

Figure 5: Lowering noise
$\delta = 0.95$, $\hat{b} = -0.30$, $\hat{\sigma}_b^2 = 0.25$, $\sigma_\epsilon^2 = 0.10$

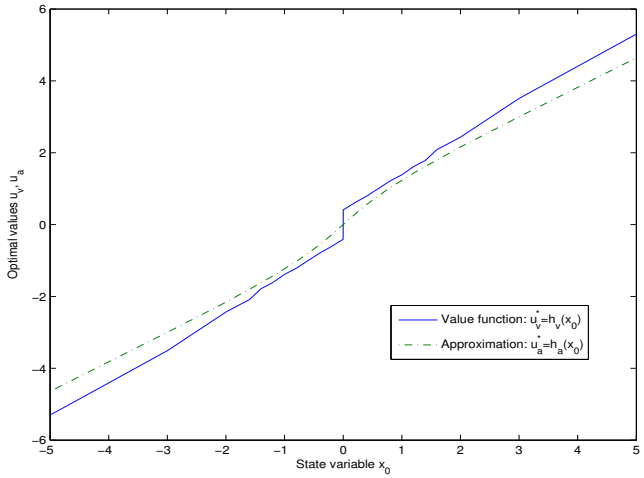
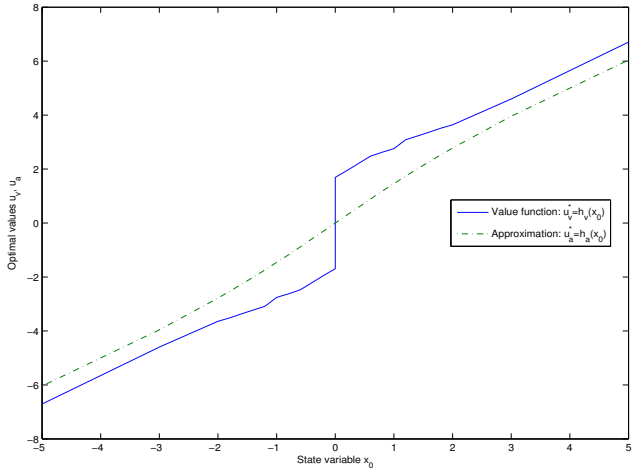



Figure 6: Increasing noise
$\delta = 0.95$, $\hat{b} = -0.30$, $\hat{\sigma}_b^2 = 0.25$, $\sigma_\epsilon^2 = 2.00$




This is illustrated once more by Figures 7 and 8 where we lowered and

increased the initial estimated variance of the parameter, $\hat{\sigma}_b^2$.

Figure 7: Lowering parameter uncertainty
$\delta = 0.95$, $\hat{b} = -0.30$, $\hat{\sigma}_b^2 = 0.04$, $\sigma_\epsilon^2 = 1.0$

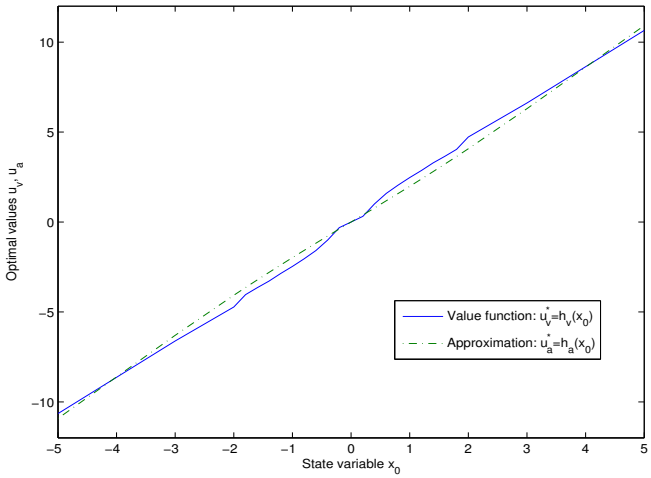
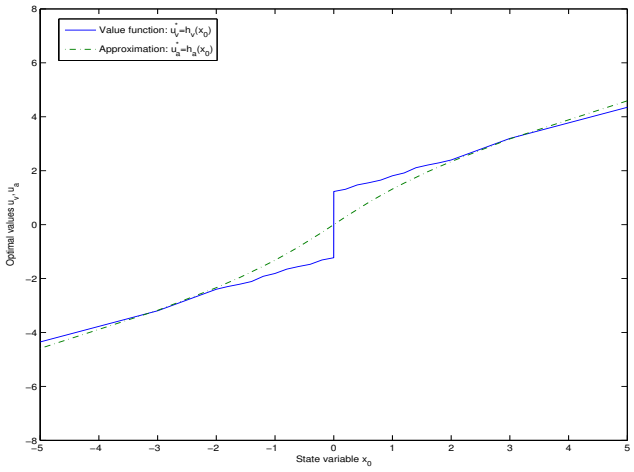



Figure 8: Increasing parameter uncertainty
$\delta = 0.95$, $\hat{b} = -0.30$, $\hat{\sigma}_b^2 = 0.49$, $\sigma_\epsilon^2 = 1.0$



The overall picture is that the approximation approach is a good alternative for the value function approach as long as the uncertainty in the model is low to modest.

# 5 Conclusions

Although, the above results are in line with our intuition, we need to be modest. In this short paper we provided some evidence, that for the MacRae-Beck-Wieland model, the approximation method produces a solution close enough to the value function approach. Moreover, we have been able to identify some elements of the model specifications which affect the difference between the value function and the approximation solutions. Our simulations indicate that when the effects of learning are limited, the differences are small.

This is by no means a proof that approximation methods will provide results that are close to those obtainable with value function methods in larger models. The verdict is still out on this issue and is likely to stay out in the foreseeable future given the computational inefficiency of the value function method and the complexity of the issue.

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