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Yes we can!

Factor-Augmenting Technical Change without Cobb-Douglas and despite Uzawa*

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May 22, 2025

Abstract

The standard result in the literature states that balanced growth is only possible with labor-augmenting technological progress. If a more general form of technical progress is considered, the Cobb-Douglas production function is typically required (Uzawa, 1961). However, we show that this result is not universally true. A broader class of functions, known as Karamata functions or regularly varying functions, allows for both labor- and capital-augmenting technical progress. This generalizes the overlooked work of Sato and Beckmann (1970), who first challenged Uzawa's conclusion but in a more restrictive setting. Karamata functions, widely used in mathematical statistics, describe functions with predictable long-term growth while allowing for slow variations. Despite their suitability for scaling behavior in production, they have never been applied in economics. Our approach also aligns with the Houthakker-Jones framework and complements solutions to Uzawa's paradox. We demonstrate that factor-augmenting technical progress is fully compatible with balanced growth under general conditions, challenging the conventional view that Cobb-Douglas is the only valid form. We further provide some examples, as well as functions with controlled oscillations, compatible with balanced growth.

JEL: O40, O41, C60
Keywords: factor-augmenting technical progress, balanced growth theorem, Karamata functions.

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1 Introduction

The result that balanced growth is only possible in the presence of labor-augmenting technological progress, which is equivalent to Harrod neutrality, is well established in the literature. If one seeks a more general form of technological progress, one must resort to the Cobb-Douglas production function, which is simultaneously Harrod-neutral, Solow-neutral, and Hicks-neutral. This result dates back to Uzawa (1961).

I show that this result is not universally true, as there exists an entire class of functions, known as Karamata functions or regularly varying functions, that are compatible with both labor- and capital-augmenting technological progress. Quite surprisingly, I am not the first to establish this result. Indeed, the overlooked article by Sato and Beckmann (1970)[1] already indicated and demonstrated that the traditional result is incorrect. They constructed a class of production functions that allow for capital-augmenting technical progress while still ensuring the existence of a balanced growth path within the neoclassical framework. However, the class of production functions they obtain is actually a subset of the more general class that we propose. We thus introduce the Karamata class of production functions into the economics literature.

The Karamata class of functions, introduced by Karamata (1933), provides a general framework for studying the asymptotic behavior of functions that exhibit regular variation at large scales. Intuitively, these functions describe phenomena whose growth or decay is predictable over the long term, while still allowing for finer adjustments or slow variations. A key feature of functions in this class is their asymptotic homogeneity: at sufficiently large scales, they behave similarly to homogeneous functions, making them particularly useful for modeling scaling properties and equilibrium dynamics in complex systems.

To the best of my knowledge, Karamata functions have never been used in economics, despite their elegant fit with the properties of neoclassical production functions. However, they are widely used in mathematical statistics, particularly in the analysis of heavy-tailed distributions (such as Pareto distributions and, more generally, all distributions within the Fréchet domain of attraction).

Following the seminal work of Houthakker (1955), Jones (2005) provides deeper microfoundations for the neoclassical aggregate production function. He considers a framework in which firms have access to a finite number of complementary technologies, either Walras-Leontief or CES with low elasticity of substitution, and producing more requires the discovery of new ideas. These ideas are drawn from a Pareto distribution, as in Kortum (1997), and the resulting aggregate production function is Cobb-Douglas. The Karamata class of pro-

[1] This article has not been cited at all!

duction functions provides the ideal framework to formalize the Houthakker-Jones approach, as we demonstrate. Our approach is also consistent with the recent contribution by Jones (2023), who, using a theorem that connects extreme value maxima to the number of draws and the shape of the upper tail of probability distributions, shows that exponential growth can emerge without being tightly linked to any specific distribution, such as the Pareto distribution. In the same spirit, we show in a very general way that if the production function belongs to the Karamata class, a balanced growth path remains feasible–even if technical progress is not purely labor-augmenting.

Several alternative approaches have been proposed to explain balanced growth, despite Uzawa's theorem. For instance, Acemoglu (2003) uses endogenous growth models, where profit-maximizing firms carry out innovations, to show that technical progress can initially be factor-augmenting along the transition path, but eventually becomes purely labor-augmenting asymptotically. He generalizes and provides microfoundations for the induced-innovation models proposed by Fellner (1961), Kennedy (1964), and Samuelson (1965), which were criticized by Nordhaus (1973) for their lack of microfoundations. Under relatively general conditions, factor-augmenting technological progress is possible at least during the transition, and labor-augmenting technological progress emerges asymptotically along the balanced growth path.

More recently, Grossman *et al.* (2017) extended Uzawa's theorem by introducing human capital accumulation, where human capital is endogenously determined and capital is more complementary with human capital than with raw labor. They obtained a class of production functions for which the neoclassical growth model remains compatible with capital-augmenting technological progress, while also allowing for endogenous human capital growth, ensuring the existence of a balanced growth path. Léon-Ledesma and Satchi (2018) use the shape of the technological frontier and adjustment costs in firms' technology choices to distinguish between short-run and long-run elasticities of substitution. When their technological frontier becomes log-linear, the long-run production function reduces to a Cobb-Douglas form, even though the short-run elasticity of substitution remains below one. Our approach is fully compatible with theirs, since our production function also exhibits an elasticity of substitution different from one in the short run, while converging to one in the long run. However, such a production function is not necessarily Cobb-Douglas.

However, we show that the literature has misinterpreted Uzawa's Theorem: the presence of factor-augmenting technological progress is fully compatible with the standard neoclassical model, and the production function belongs to a very general class known as the Karamata class. We therefore begin by presenting the neoclassical framework in Section 2, where we restate the Balanced Growth Theorem, a well-known result. In particular, we prove it within

an asymptotic framework, highlighting subtle aspects emphasized by Acemoglu (2009).

Next, we discuss the structure of the production function: $Y(t) = F(B(t)K(t), A(t)L(t))$ which incorporates both capital- and labor-augmenting technical progress. We then establish the first key result of this paper, which is well known: defining the ratio of effective capital to effective labor as: $x(t) = \frac{B(t)K(t)}{A(t)L(t)}$ with the intensive production function given by: $y(t) = A(t)f(x(t))$, then: $x(t)$ grows at a constant rate and asymptotically tends to infinity, the growth rate of GDP (and other aggregate variables) is asymptotically constant, the capital share is determined by a combination of the growth rates of the variables.

The only assumption required to establish this result is that the labor share asymptotically converges to a non-degenerate constant, i.e., strictly between 0 and 1. Building on this result, the literature typically assumes that balanced growth is only possible if the production function is Cobb-Douglas, unless the growth of $B(t)$ is zero (meaning that technical progress is purely labor-augmenting and Harrod-neutral). Since the production function must ultimately be Cobb-Douglas, it automatically implies Harrod neutrality because Cobb-Douglas is simultaneously Harrod, Hicks, and Solow-neutral. We also note that our class of production functions can also be applied in the presence of investment-specific technical change, instead of a production function with factor-augmenting technology (Greenwood *et al.*, 1997).

However, in Section 3.1, we show that the class of production functions compatible with balanced growth, even in the presence of capital-augmenting technical progress, is more general than the Cobb-Douglas production function. Specifically, it belongs to the class of Karamata functions. We provide a first direct proof, followed by a second proof leveraging the Representation Theorem for this type of function. Additionally, we show that this class of functions automatically satisfies the Inada conditions.

In Section 3.2, Karamata's Theorem on the convergence of integrals allows for an elegant generalization of the results presented in Acemoglu (2009) regarding the asymptotic extension of Uzawa's Theorem. Specifically, Acemoglu (2009) shows that if the growth rates of capital and output converge to their steady-state values faster than $1/t$ (where t denotes times), then the relevant integrals converge to a constant value, and technical change becomes Harrod-neutral in the long run. However, if this condition is not satisfied, we show that a new time-dependent term appears in the production function. This term can be interpreted as capital-biased technical change. Notably, this form of technical change does not necessarily grow exponentially, but may take on various functional forms–still consistent, however, with recent empirical estimates provided since Klump *et al.* (2007). Indeed, according to the impossibility theorem of Diamond *et al.* (1978), it is not possible to jointly identify the elasticity of substitution and the bias of technical change toward labor and capital. Klump

et al. (2007) suggest using suggest using a sufficiently flexible functional form for capturing both types of technical progress, one that encompasses constant growth rates, as well as logarithmic or hyperbolic growth, through the use of a Box-Cox transformation to estimate the respective rates of technical change. Their results show that labor-augmenting technical progress is indeed exponential, whereas capital-augmenting technical progress follows a hyperbolic or logarithmic pattern. Their empirical findings are consistent with the potential forms that our production function can take. If one wishes to impose exponential growth for both types of technical progress, then it is necessary to impose the axiomatic framework of Phelps (1962) on the production function. We demonstrate that the Karamata's production function we propose satisfies this axiom and indeed constitutes the asymptotic production function that emerges along a balanced growth path with both forms of technical change.

We then discuss, in Section 3.3, the overlooked article by Sato and Beckmann (1970), which also constructs a class of production functions that allow for capital-augmenting technical progress. While the Cobb-Douglas function is a member of this class, it is merely a special case. Their construction is based on the elasticity of substitution between capital and labor. It is well known (see Grossman *et al.* (2017), for example) that balanced growth is only possible if the following relationship holds: $(1 - \sigma(K, L))g_B = 0$. This implies that either the elasticity of substitution $\sigma(K, L)$ is equal to 1, or capital-augmenting technical progress is absent ($g_B = 0$). The literature concludes that if $g_B > 0$, then the production function must be Cobb-Douglas. However, both Sato and Beckmann (1970)'s class of production functions and our class of Karamata functions demonstrate that even if the elasticity of substitution asymptotically equals 1, this does not necessarily imply that the production function is Cobb-Douglas. In this paper, we show that it belongs instead to the class of Karamata functions. Thus, we revive and generalize the results of Sato and Beckmann (1970).

In Section 4, we also generalize the results of Jones (2005). We demonstrate that if the production functions of firms belong to a specific class that includes, as special cases, the Walras-Leontieff production function and the CES production function with low elasticity of substitution, and if ideas are drawn from a heavy-tailed distribution (of which the Pareto distribution is a special case), then the aggregate production function belongs to the Karamata class. Indeed, Karamata functions are widely used in mathematical statistics to study extreme value distributions. Thus, our approach constitutes a natural generalization of this framework.

Moreover, in Jones's approach, the direction of technical progress implies that the production function becomes asymptotically Cobb-Douglas. We show that if the production function belongs to the Karamata class, then technical progress becomes asymptotically

neutral in both the Harrodian and Hicksian senses, because the elasticity of substitution asymptotically converges to 1. For this to occur, it is not necessary for the production function to be Cobb-Douglas, but rather for it to belong to the Karamata class.

In Section 5, we show that there exist production functions that belong to the Karamata class, and we provide some examples: CEDD (Constant Elasticity of Derived Demand) production functions or certain mixed CES-Cobb-Douglas functions belong to the Karamata class. We also provide an example with infinite oscillations, since Karamata-type functions are compatible with controlled forms of oscillations, in our case concerning the capital share and the elasticity of substitution, even along the balanced growth path.

2 Neoclassical Economics and Balanced Growth

Consider a neoclassical economy starting at time t_0, where time $t \in [t_0, \infty)$ is assumed to be continuous. A single good, referred to as the output, is produced using physical capital $K(t)$ and labor $L(t)$. In each period, the labor market clears, and the entire available workforce, which grows at an exogenous rate $n \geq 0$, is employed, such that $L(t) = L(t_0)e^{n(t-t_0)}$, with $L(t_0) > 0$ given. Physical capital accumulates as follows: $\dot{K}(t) = I(t) - \delta K(t)$, where $I(t)$ denotes gross investment and $0 \leq \delta < 1$ represents the depreciation rate of physical capital. There is a given initial level of capital $K(t_0) > 0$. Output is allocated either to consumption $C(t)$ or to investment $I(t)$. The resource constraint is written as $Y(t) = C(t) + I(t)$.

Given this standard neoclassical framework, we are ready to state the asymptotic version of Uzawa's Theorem (Uzawa, 1961), adapted from Schlicht (2006), Jones and Scrimgeour (2008), and Acemoglu (2009, Chapter 2).

Proposition 1 (Steady-State Growth).

Consider an economy starting at date t_0, where output, physical capital, consumption and gross investment grow at asymptotically constant rates, given respectively by:

$$\lim_{t\to+\infty} \frac{\dot{Y}(t)}{Y(t)} = g_Y > 0, \quad \lim_{t\to+\infty} \frac{\dot{K}(t)}{K(t)} = g_K > 0, \quad \lim_{t\to+\infty} \frac{\dot{C}(t)}{C(t)} = g_C > 0$$

$$\text{and} \quad \lim_{t\to+\infty} \frac{\dot{I}(t)}{I(t)} = g_I > 0$$

under the following conditions: (i) $I(t) > 0$, or $0 < C(t) < Y(t)$, for all $t \in [t_0, \infty)$ and (ii) $\lim_{t\to\infty} \frac{I(t)}{Y(t)} = s \in (0,1)$, or $\lim_{t\to\infty} \frac{C(t)}{Y(t)} = 1 - s \in (0,1)$. Then, all growth rates are identical: $g_Y = g_K = g_C = g_I$.

Proof See Appendix A.

■

As often noted in the literature, this theorem can be stated and proven without any prior reference to the production function or the presence of equilibrium conditions. In this article, we aim to demonstrate that balanced growth is achievable for a neoclassical production technology featuring both labor-augmenting and capital-augmenting technical progress, without restricting this technology to the Cobb-Douglas production function.

First, we will properly define this production function and then characterize the neoclassical economy in the presence of both types of technological progress. Let $F : (0, \infty)^2 \to (0, \infty)$ be a constant returns to scale production function, defined as:

$$Y(t) = F(B(t)K(t), A(t)L(t))$$

where $B(t)$ and $A(t)$ are technology parameters scaling capital $K(t)$ and labor $L(t)$, respectively. These are known as capital-augmenting and labor-augmenting technological progress. These parameters evolve over time as follows:

$$A(t) = A(t_0)e^{g_A(t-t_0)} \quad \text{and} \quad B(t) = B(t_0)e^{g_B(t-t_0)},$$

where $A(t_0), B(t_0) > 0$ are given initial values, and $g_A > 0$ and $g_B > 0$ denote, respectively, labor-augmenting and capital-augmenting technological progress. As is well-known, if $g_B = 0$, the technological progress is purely labor-augmenting and purely Harrod-neutral, and the production function becomes $Y(t) = F(K(t), A(t)L(t))$. Conversely, if $g_B \neq 0$, the technological progress is referred to as factor-augmenting technological change.

We assume that F exhibits strictly positive marginal productivities ($F_K > 0$ and $F_L > 0$) and strictly decreasing marginal productivities ($F_{KK} < 0$ and $F_{LL} < 0$).[2] From the property of constant returns to scale, the cross-derivatives can be shown to be strictly positive ($F_{KL} = F_{LK} > 0$) [3]

We are now ready to state the central assumption on the production function F. We will demonstrate that there exists a class of production functions, which includes the Cobb-Douglas production function, that leads to a balanced growth path even if technological progress is not purely labor-augmenting. Specifically, we finally assume that the factor shares never degenerate to 0 or 1, even asymptotically. This assumption is central to rigorously defining a strongly neoclassical technology and to avoiding situations where the production function asymptotically behaves like an AK-type technology. Such behavior would result

[2] To simplify the text, we will sometimes use the following abbreviations:

$$F_j := \frac{\partial F}{\partial j}(B(t)K(t), A(t)L(t)) \quad \text{and} \quad F_{jj} := \frac{\partial^2 F}{\partial j^2}(B(t)K(t), A(t)L(t)),$$

for $j \in \{K(t), L(t)\}$.

[3] These results are the well-known Wicksell's Laws (see Appendix A).

in growth driven not by technological progress but by the lower-boundedness of marginal productivities. We will revisit below the Inada conditions, which are commonly assumed to prevent such behavior. However, these conditions are inherently satisfied by the class of production functions we consider, as we will demonstrate below.

Hypothesis (Asymptotically Constant and Strictly Positive Factor Shares).

Under the assumption of the neoclassical growth model with perfect markets, the capital share α_K and the labor share $1-\alpha_K$ are given by:

$$\alpha_K := \frac{F_K \cdot K}{Y} \in (0,1)$$
$$1-\alpha_K := \frac{F_L \cdot L}{F} \in (0,1)$$

and we assume that the capital share (respectively the labor share) converges asymptotically to a constant value $\alpha \in (0,1)$ (respectively $1-\alpha \in (0,1)$), strictly bounded between 0 and 1.

We now wish to work with the intensive production function. As is standard in the literature, we define $y(t) := \frac{Y(t)}{L(t)}$ as the per-worker output, $k(t) := \frac{K(t)}{L(t)}$ as the per-worker physical capital, and $x(t) := \frac{B(t)}{A(t)}k(t)$ as the effective per-worker physical capital. Using the constant returns to scale property of the production function, we can easily show that there exists a function $f : (0,\infty) \to (0,\infty)$ such that the per-worker output is written as $y(t) = A(t)F(x(t),1) := A(t)f(x(t))$. The marginal productivity of capital is expressed as $F_K = B(t)f'(x)$, the marginal productivity of labor is given by $F_L = A(t)\left(f(x(t)) - x(t)f'(x)\right)$, and the output-capital ratio (i.e., the average productivity of capital) is written as $\frac{Y(t)}{K(t)} = B(t)\frac{f(x(t))}{x(t)}$. The function f inherits the following properties from the function F: $f'(x(t)) > 0$ (strictly increasing) and $f''(x(t)) < 0$ (strictly concave).

Furthermore, we define the following four quantities.

1. $\alpha(x(t)) := \frac{f'(x(t))x(t)}{f(x(t))}$ is the elasticity of output with respect to capital and equals the capital share under perfect markets. The assumption we made on the capital share implies that $\alpha(x(t)) \in (0,1)$ for all $x(t) \in \mathbb{R}_+^\star$ and $\lim\limits_{t\to\infty} \alpha(x(t)) = \alpha \in (0,1)$.

2. $\theta(x(t)) := \frac{\alpha(x(t))}{1-\alpha(x(t))}$ is the ratio of factors shares.

3. $\beta(x(t)) := \frac{f''(x(t))x(t)}{f'(x(t))}$ is the elasticity of marginal productivity of capital with respect to capital.

4. Finally, $\sigma(x(t)) := \dfrac{1-\alpha(x(t))}{-\beta(x(t))}$ is the elasticity of substitution between the factors of production, capital and labor.[4]

We now combine the production function we have constructed with the insights from the Balanced Growth Theorem to show that balanced growth is possible in the presence of factor-augmenting technological progress. Of course, the proof remains incomplete, as we must still characterize the production functions compatible with this result. The remainder of this article will be devoted to that task.

Proposition 2.

In a neoclassical economy, where asymptotically $g_Y = g_K = g_C = g_I$, and characterized by the intensive production function $y(t) = A(t)f(x(t))$, where $x(t) := \frac{B(t)}{A(t)}k(t)$ is the ratio of effective capital to effective labor, we have:

$$(i)\ \lim_{t\to\infty} x(t) = \infty$$

$$(ii)\ \lim_{t\to\infty} \frac{\dot{x}(t)}{x(t)} = \frac{1}{1-\alpha} g_B > 0$$

$$(iii) g_Y := \lim_{t\to\infty} \frac{\dot{Y}(t)}{Y(t)} = n + g_A + \frac{\alpha}{1-\alpha} g_B$$

$$(iv)\ \lim_{t\to\infty} \alpha(x) = \alpha = \frac{g_Y - n - g_A}{g_Y - n + g_B - g_A} \in (0,1)$$

Proof. See Appendix A.

■

It is immediately evident that in the absence of capital-augmenting technological progress (i.e., $g_B = 0$), output (and capital, consumption, and investment) grows at the rate $n + g_A$, $\lim_{t\to\infty} \dfrac{\dot{x}(t)}{x(t)} = 0$, and thus effective capital per worker, defined in our notation as $x(t) =$

[4] The elasticity of substitution can be written as follows:

$$\sigma(x) = \frac{F_L F_K}{F F_{KL}} = -\frac{L F_L F_K}{K F F_{KK}} = \frac{A(t)\left(f(x(t)) - x(t)f'(x(t))\right) B(t) f'(x(t))}{-A(t) L f(x(t)) \frac{B(t)}{L} x(t) f''(x(t))}$$
$$= -\frac{\left(f(x(t)) - x(t)f'(x(t))\right) f'(x(t))}{f(x(t)) x(t) f''(x(t))}.$$

The first equality is the definition of the elasticity of substitution under constant returns to scale. The second equality uses Wicksell's Laws. The third equality employs the quantities defined in the text. The last equality demonstrates that the elasticity of substitution can also be expressed solely as a function of x. Simplifying the fraction by $f(x)f'(x)$ yields the result.

$\frac{A(t)}{B(0)}\frac{K(t)}{L(t)}$, converges to a steady-state constant. This corresponds to the classical result under a production technology with purely labor-augmenting technological progress.

To complete the proof, we must ensure the existence of a production function that simultaneously generates balanced growth, constant factor shares, a constant output-capital ratio, and a constant marginal productivity of capital (equal to the rate of return on capital in equilibrium in a competitive economy).

3 The implied asymptotic production function

3.1 The Karamata production function

From the result that the capital share must be asymptotically constant, we are inclined to show that the asymptotic production function compatible with this result is the Cobb-Douglas production function. This is because, with a constant capital share, it suffices to solve the following differential equation:[5]

$$\frac{f'(x)}{f(x)} = \frac{\alpha}{x}$$

This leads to $\ln f(x) = \alpha \ln x + \ln C$, where C is an integration constant. Naturally, this implies $f(x) = Cx^{\alpha}$ and thus, the production function must be Cobb-Douglas: $Y = C\left(BK\right)^{\alpha}\left(AL\right)^{1-\alpha}$. This is how the literature often concludes.[6]

However, production functions where the capital share asymptotically converges to a constant are not limited to the Cobb-Douglas form. Instead, they belong to a broader class of functions that satisfy the properties of neoclassical production and whose capital share converges to a constant strictly between 0 and 1. Before proving that neoclassical technology belongs to the Karamata class, we introduce the concept of regularly varying functions.

[5]From now on, throughout the remainder of this paper, we omit the time index unless ambiguity might arise.

[6]Barro and Sala-i-Martin (2003) consider the output-capital ratio $\frac{Y(t)}{K(t)} = B(t)\frac{f(x(t))}{x(t)}$, which must be constant in the steady state. If we differentiate with respect to time, we obtain $0 = g_B + (\alpha(x(t)) - 1)(g_B + g_K(t) - n - g_A)$. If on a balanced growth path $g_K(t) \to g_Y$, the differential equation from Barro and Sala-i-Martin (2003) is written, in our notation, as: $\frac{f'(x(t))}{f(x(t))} = \frac{1}{x(t)}\frac{g_Y - n - g_A}{g_Y - n + g_B - g_A} = \frac{\alpha}{x(t)}$ which is equivalent to the direct approach using the capital share.

Definition 1 (Regularly Varying Function, First Definition, Bingham *et al.*, 1987).

A positive measurable function f defined on some neighborhood $[x_0, \infty)$ of infinity is called regularly varying at infinity with index α if, for each $\lambda > 0$ and some $\alpha \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha \quad \forall \lambda > 0.$$

The real number α is called the index of regular variation.

Of course, the Cobb-Douglas production function is a regularly varying function, as:

$$\lim_{x\to\infty} \frac{C(\lambda x)^\alpha}{Cx^\alpha} = \lambda^\alpha,$$

where the parameter α is the index of regular variation. However, it is not the only one. The main result of this paper is that the asymptotic production function is not exclusively Cobb-Douglas, as often claimed, but part of the more general Karamata class. While the Cobb-Douglas function is a special case, this class includes many other forms capable of supporting balanced growth asymptotically, even with capital-augmenting technical progress.

Proposition 3 (Karamata Production Function).

Let $f : (0, \infty) \to (0, \infty)$ be a production function such that $f'(x) > 0$ (strictly increasing) and $f''(x) < 0$ (strictly concave), where $\alpha(x) := \dfrac{f'(x)x}{f(x)}$ is the capital share, with $\alpha(x) \in (0, 1)$ for all $x \in \mathbb{R}_+$ and $\lim\limits_{x\to\infty} \alpha(x) = \alpha \in (0, 1)$. Then, f belongs to the Karamata class of regularly varying functions.

We begin by providing an initial proof, echoing the classic argument that a production function with an asymptotically constant capital share must be Cobb-Douglas. However, we demonstrate (heuristically using a "$\delta - \varepsilon$" approach) that such a function is asymptotically equivalent to a more general class of functions. Later, we will present a more constructive proof of this proposition, offering deeper insight into the properties of these functions.

First Proof. See Appendix B.

■

A more useful approach for our purposes is to establish a more explicit connection between the neoclassical production function, the capital share, and to more precisely characterize regularly varying neoclassical production functions. To achieve this, we need to define the concept of slow variation:

Definition 2 (Slowly Varying Function, Bingham *et al.*, 1987, p.6).

> Let L be a positive measurable function, defined on some neighborhood $[x_0, \infty)$ of infinity, and satisfying:
>
> $$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1 \quad \forall \lambda > 0$$
>
> Then L is said to be slowly varying (in Karamata's sense).

Let us note that the neighbourhood $[x_0, \infty)$ in both definitions is of little importance and we may suppose L defined on $[0, \infty)$ without lost of generality.[7] The concept of a slowly varying function allows us to provide an alternative definition of a regularly varying function.

Definition 3 (Regularly Varying Function, Second Definition), Bingham *et al.*, 1987.

> A positive measurable function f defined on some neighborhood $[x_0, \infty)$ of infinity is called regularly varying at infinity with index α if it can be represented in the form:
>
> $$f(x) = x^{\alpha} L(x)$$
>
> where $L(x)$ is a slowly varying function.

With this definition, it is still possible to see why the Cobb-Douglas production function is a candidate for an asymptotic production function. If we take a constant function $L(x) = c$ for all x, the limit of c/c approaches 1, confirming that the Cobb-Douglas production function is regularly varying. Thus, it represents a potential asymptotic neoclassical technology compatible with a steady state. However, it is a very particular case.

The following Representation Theorem will provide deeper insight into the structure of slowly varying production functions and explicitly link them to the capital share:[8]

[7] The notation L is customarily used for such functions because of the first letter of the French word "*lentement*" which means "*slowly*". These functions were introduced and studied by Karamata (1933) in a pioneering paper, written in French, with continuity in place of measurability.

[8] The second definition of a regularly varying function makes it clear that a slowly varying function L is regularly varying with index $\alpha = 0$. Therefore, the set of slowly varying functions forms a subset of the set of regularly varying ones. However, as the following discussion will make clear, we will work with a set of production functions with indexes $\alpha \in (0, 1)$.

Representation Theorem (Bingham *et al.*, 1987).

> The function L is slowly varying if and only if it can be written in the form:
>
> $$L(x) = c(x)e^{\int_{x_0}^{x} \frac{\varepsilon(u)}{u} du}$$
>
> for $x \geq x_0 > 0$, where $c(x)$ is measurable and satisfies $\lim_{x\to\infty} c(x) = c > 0$, and $\varepsilon(x)$ is a measurable function such that $\lim_{x\to\infty} \varepsilon(x) = 0$.
> Moreover, when $c(x) = c > 0$ is a constant function, L is called normalized slowly varying.

Normalized slowly varying functions were introduced by Kohlbecker (1958). As claimed by Bingham *et al.* (1987, p.15), and as can be easily shown, when using a normalized slowly varying function, the elasticity of L is given by the function $\varepsilon(x)$ (almost everywhere). More formally:

$$\varepsilon(x) = \frac{xL'(x)}{L(x)}.$$

Conversely, given a function L with $\varepsilon(x) := \frac{xL'(x)}{L(x)}$ continuous and of order $o(1)$ at infinity, we may integrate it to obtain the representation:

$$L(x) = ce^{\int_{x_0}^{x} \frac{\varepsilon(u)}{u} du},$$

thereby showing that L is a normalized slowly varying function.

Using this elegant and intuitive property of normalized slowly varying functions, we can now provide the second proof of our Proposition and establishes that the neoclassical production function compatible with a steady state, where the capital share is constant and non-degenerate, belongs to the Karamata class. Indeed, from the previous calculations, we have:

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x} \frac{\alpha(u)}{u} du,$$

where $\phi(x) = \ln f(x)$. Taking the exponential of both sides, we obtain:

$$f(x) = f(x_0)e^{\int_{x_0}^{x} \frac{\alpha(u)}{u} du}.$$

Now, since by assumption $\lim_{x\to\infty} \alpha(x) = \alpha$, we can write $\alpha(u) = \alpha + \varepsilon(u)$, where $\lim_{u\to\infty} \varepsilon(u) = 0$. Thus, the integral in the exponent can be rewritten as follows:

$$\int_{x_0}^{x} \frac{\alpha(u)}{u} du = \alpha \ln\left(\frac{x}{x_0}\right) + \int_{x_0}^{x} \frac{\varepsilon(u)}{u} du.$$

Let us define $L(x)$, a slowly varying function, as:

$$L(x) = x_0^{-\alpha} f(x_0) e^{\int_{x_0}^{x} \frac{\varepsilon(u)}{u} du}.$$

Thus, we can express $f(x)$ as: $f(x) = x^{\alpha} L(x)$, which provides the representation of the function f. This confirms that f is indeed a regularly varying function.

We can further characterize regularly varying production functions and understand why they enable the characterization of a steady-state path. To do so, let us show that if the production function f is regularly varying with index $\alpha \in (0, 1)$, then its first derivative f' and second derivative f'' are also regularly varying functions, with indices $\alpha - 1$ and $\alpha - 2$, respectively.[9] More formally:

Proposition 4.

> Let f be a positive measurable function defined on some neighborhood $[x_0, \infty)$ of infinity and regularly varying at infinity with index $\alpha \in (0, 1)$. Then, its first derivative f' and its second derivative f'' are also regularly varying at infinity with indices $\alpha - 1$ and $\alpha - 2$, respectively.

Proof. See Appendix B

■

We can also establish the following corollary, which shows that the elasticity of output per worker with respect to x (i.e., the capital share, $\alpha(x)$), the elasticity of marginal productivity of capital with respect to x (i.e., $\beta(x)$), and the elasticity of substitution (i.e., $\sigma(x)$) are slowly varying functions.

Corollary 1.

> $\alpha(x)$, $\beta(x)$ and $\sigma(x)$ are slowly varying functions.

Proof. See Appendix B.

■

Until now, we have not yet addressed the Inada conditions. A production function is truly neoclassical (let's say strongly neoclassical) only if it satisfies the Inada conditions. Interestingly, regularly varying functions imply that the Inada conditions are satisfied. However, the converse is not necessarily true.

[9] Note that this result holds when $\alpha \in (0, 1)$, i.e., when the asymptotic capital share is not degenerate.

Proposition 5 (Inada Conditions).

Let f be a regularly varying function with index $\alpha > 0$. Then, the following limits hold:

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to 0} f(x) = 0, \quad \lim_{x\to\infty} f'(x) = 0, \quad \lim_{x\to 0} f'(x) = \infty.$$

Proof. See Appendix B.

■

3.2 The Phelps' axiomatic

We now analyze what could be the asymptotic production function, drawing inspiration from the axiomatic framework of production functions with factor-augmenting technological change proposed by Phelps (1965) in an unpublished paper.[10] In this section, we adopt the modern approach developed by Schlicht (2006) and further refined by Jones and Scrimgeour (2008) and by Acemoglu (2009) in the asymptotic case.

Let us now assume that, at date T, the production function takes the form:

$$Y(T) = G(K(T), L(T), T)$$

The production function G can be rewritten as follows:

$$e^{-\int_T^t g_Y(s)ds} Y(t) = G\left[e^{-\int_T^t g_K(s)ds} K(t), e^{-n(t-T)}L(t), T\right]$$

That is, by multiplying both sides by $e^{\int_T^t g_Y(s)ds}$ and using the property of constant returns to scale:

$$Y(t) = G\left[e^{\int_T^t (g_Y(s) - g_K(s))\,ds} K(t), e^{\int_T^t (g_Y(t) - n)\,ds} L(t), T\right]$$

We rewrite the function G as follows:

$$y(t) = A(t)G\left[\frac{B(t)}{A(t)}k(t), 1, T\right] = A(t)g\left[\frac{B(t)}{A(t)}k(t), T\right]$$

[10] Parallel to Phelps (1965), Sato and Beckmann (1968), as well as Sato (1970), proposed an alternative axiomatic framework to justify the presence of factor-augmenting technological progress, based on the elasticity of substitution.

where we define: $A(t) := A(0)e^{\int_T^t (g_Y(t)-n)\,ds}$ and $\frac{B(t)}{A(t)} := \frac{B(0)}{A(0)}e^{-\int_T^t (g_K(s)-n)\,ds}$.
To understand the asymptotic behavior of the production function, we need to study the asymptotic behavior of the following integrals:

$$\int_T^t (g_Y(s)-n) = \int_T^t (g-n)\,ds + \int_T^t (g_Y(s)-g)\,ds = (g-n)(T-t) + \int_T^t (g_Y(s)-g)\,ds$$
$$\int_T^t (g_K(s)-n) = \int_T^t (g-n)\,ds + \int_T^t (g_K(s)-g)\,ds = (g-n)(T-t) + \int_T^t (g_K(s)-g)\,ds$$

or, the integrals:

$$\int_T^t (g_Y(s)-g)\,ds \quad \text{and} \quad \int_T^t (g_K(s)-g)\,ds$$

To develop our intuition, let us assume that the deviations between the growth rates $g_Y(t)$ and $g_K(t)$ and their common asymptotic values g behave asymptotically as follows:

$$g_Y(t) - g \sim t^{\beta_Y} L_Y(t) \quad \text{and} \quad g_K(t) - g \sim t^{\beta_K} L_K(t)$$

where $L_Y(t)$ and $L_K(t)$ are slowly varying functions. Consequently, the asymptotic behavior of the integrals is given by:

$$\int_T^t (g_Y(s)-g)\,ds \sim \int_T^t s^{\beta_Y} L_Y(s)ds \quad \text{and} \quad \int_T^t (g_K(s)-g)\,ds \sim \int_T^t s^{\beta_K} L_K(s)ds$$

This behavior strongly depends on the parameters β_Y and β_K, as well as on the fact that $L_Y(t)$ and $L_K(t)$ are slowly varying functions. Thus, it is straightforward to see that if $\beta_Y, \beta_K < -1$, then $g_Y(t)$ and $g_K(t)$ converge to g at a rate faster than $\frac{1}{t}$ and therefore the above integrals converge to a constant. The production function can then be written as $Y(t) = F(K(t), A(t)L(t))$, with $A(t) = A(0)e^{\int_T^t (g-n)\,ds}$ and $\frac{B(t)}{A(t)} := \frac{B(0)}{A(0)}e^{-\int_T^t (g-n)\,ds} = \frac{B(0)}{A(t)}$ which implies that $B(t) = B(0)$ is constant, and technical progress is purely labor-augmenting. This result is also demonstrated in Acemoglu (2009).

However, we adopt an approach, which also allows us to consider the case where $g_Y(s)$ and $g_K(s)$ converge to g at a rate slower than $\frac{1}{t}$, a case not addressed in the literature. For that, the theorem of Karamata provides a rigorous framework for analyzing the convergence of integrals involving slowly varying functions. It can be stated as follows:

Karamata's Theorem (Bingham *et al.*, 1987, p.26).

> Let L be a slowly varying and locally bounded function in $[x_0, \infty)$ for some $x_0 \geq 0$. Then
> $$\int_{x_0}^{x} t^{\alpha} L(t)dt \sim \frac{x^{\alpha+1}}{\alpha+1} L(x)$$
> for $\alpha > -1$ and when $x \to \infty$.

If $-1 < \beta_Y, \beta_K < 0$, then $g_Y(t)$ and $g_K(t)$ converge to g at a rate slower than $\frac{1}{t}$. The integral grows slowly (or diverges sub-exponentially), as shown by the theorem of Karamata, and we have:

$$\int_T^t (g_Y(s) - g_K(s))\, ds \sim \frac{t^{1-\beta_j} L_j(t)}{1-\beta_j}$$

where $j = \min\{\beta_Y, \beta_K\}$. This integral is dominated by the slower rate of convergence.

If $\beta_Y, \beta_K = -1$, the integral grows logarithmically as $\int_T^t s^{\beta_j} L_j(s)ds \sim L_j(t) \ln t$, for $j = \{Y, K\}$ and we have:

$$\int_T^t (g_Y(s) - g_K(s))\, ds \sim (L_Y(t) - L_K(t)) \ln t$$

Thus, in cases where $-1 \leq \beta_Y, \beta_K < 0$, the production function can be written as $Y(t) = F(B(t)K(t), A(t)L(t))$ with factor-augmenting technical progress. However, the growth rate of $B(t)$ is not constant, as $B(t)$ is an arbitrary function of time. This does not guarantee a balanced growth path with a constant capital share. Here, I propose to identify the specific condition that aligns with the axiomatic framework proposed by Phelps (1965) in an unpublished paper, which has been overlooked in the literature. His goal was to establish the necessary and sufficient conditions, in the same spirit as the result by Uzawa (1961), under which a production function exhibits factor-augmenting technical progress compatible with a constant capital share (and thus a constant labor share as well).

According to Phelps (1965), *if there exists a function $B(t)$ that depends only on time, such that the average productivity of capital $Y(t)/K(t)$ increases proportionally to $B(t)$, and the marginal productivity of capital also increases proportionally to $B(t)$, then there exists a balanced growth path along which the shares of the factors of production remain constant.*

In our approach, this implies that $\beta_Y, \beta_K = 0$, and, $g_Y(t) - g \sim L_Y(t)$ and $g_K(t) - g \sim L_K(t)$. Consequently, the asymptotic behavior of the integral is linear, given respectively by

$$\int_T^t (g_Y(s) - g)\, ds \sim \int_T^t L_Y(s)ds \text{ et } \int_T^t (g_K(s) - g)\, ds \sim \int_T^t L_K(s)ds$$

Assuming, $L_Y(t)$ and $L_K(t)$ converge asymptotically to constants L_Y and L_K, the integral becomes asymptotically linear:

$$\int_T^t (g_Y(s) - g_K(s))\, ds \sim (L_Y - L_K)\,(t - T) = g_B(t - T)$$

where we denote: $g_B := L_Y - L_K$.

Thus, although this result aligns with the axiomatic framework of Phelps (1965) and generates a path on which the factor shares are asymptotically constant, the condition leads to a contradiction with the result established in Proposition 1, namely that the average productivity is asymptotically constant, and thus $g_Y = g_K$. This apparent contradiction can once again be resolved if we assume that the production function belongs to the Karamata class.

Indeed, if g is regularly varying, recalling that $x(t) := \frac{B(t)}{A(t)}k(t)$, then we can write (using the Representation Theorem)

$$Y(t) = A(t)L(t)g\left[\frac{B(t)}{A(t)}k(t), T\right] = cA(t)L(t)x(t)^{\alpha}e^{\int_{x_0}^{x(t)} \frac{\varepsilon(x(s))}{s} ds}$$

or

$$Y(t) = ce^{(1-\alpha)\int_T^t (L_Y(t) + g - n)\, ds + \alpha \int_T^t (L_Y(t) - L_K(t))\, ds + \int_{x_0}^{x(t)} \frac{\varepsilon(x(s))}{s} ds} K(t)^{\alpha} L(t)^{1-\alpha}$$

In this case, the asymptotic production function can be written as: $Y(t) = F(B(t)K(t), A(t)L(t))$ where $A(t) = A(T)e^{\int_T^t (L_Y(t) + g - n)\, ds}$ and $B(t) = B(T)e^{\int_T^t (L_K(t) - L_Y(t))\, ds}$. The integrals respectively converge to $g_A(t - T)$ and $g_B(t - T)$ where, $g_A := L_Y + g - n$ and $g_B := L_Y - L_K$.

Finally, we require that, along the balanced growth path, both the average and marginal productivity of capital remain constant. Indeed, as we have already shown, since the production function can be written in intensive form as $y(t) = A(t)f(x(t))$, the average productivity of capital is given by: $\frac{Y(t)}{K(t)} = B(t)\frac{f(x(t))}{x(t)}$. For its growth rate to be zero, the following condition must hold:

$$g_B + (\alpha - 1)\frac{\dot{x}(t)}{x(t)} = 0$$

Similarly, the marginal productivity of capital is given by $F_K = B(t)f'(x(t))$, and its growth rate is zero if the following condition is satisfied:

$$g_B + f''(x(t))\frac{\dot{x}(t)}{x(t)} = g_B + (\alpha - 1)\frac{\dot{x}(t)}{x(t)} = 0$$

Now, since we have shown that asymptotically $\lim_{x(t)\to\infty} f''(x(t)) = 1-\alpha$ and $\lim_{x(t)\to\infty} \frac{\dot{x}(t)}{x(t)} = \frac{g_B}{1-\alpha}$, these two growth rates are asymptotically zero. Thus, once again, if the production function belongs to the Karamata class, it is possible to reconcile the existence of a balanced growth path with the presence of factor-augmenting technical progress, and a constant average and marginal productivity of capital.

Note that up to this point, we have worked with a production function where technical progress is factor-augmenting. However, we could have considered the case of an investment-specific technological change, such that $Y(t) = C(t) + I(t)/B(t)$, where g_B represents an embodied technical change that allows for less forgone consumption (see Greenwood *et al.* (1997) or Solow (1960)). If the production function is written as $Y(t) = F(K(t), A(t)L(t))$, then, by defining $x(t) := \frac{K(t)}{A(t)L(t)}$, such that $y(t) = A(t)f(x(t))$, if $g_B > 0$, we would have $g_K = n + g_B + \frac{1}{1-\alpha} g_B$ and $g_Y = n + g_B + \frac{\alpha}{1-\alpha} g_B$, with $\alpha = \frac{g_Y - n - g_A}{g_Y - n - g_A + g_B}$. Note then that we have $g_Y = g_K - g_B$ (see also Grossmann *et al.* (2017)). In this case, the average productivity of capital is no longer asymptotically constant since $g_Y - g_K = -g_B$. For the capital (and labor) share to remain constant, it then becomes necessary, and even sufficient, that the marginal productivity of capital also grows asymptotically at the rate $-g_B$. This is precisely the axiomatic framework proposed by the unpublished and overlooked paper of Phelps (1965) to show the conditions under which factor-augmenting technical progress is possible. With $K(t)$ growing at the rate $g_Y + g_B$, $x(t) \to \infty$ and our framework can be used with investment-specific technological change.

3.3 The elasticity of substitution

We now demonstrate how the work of Sato and Beckmann (1970) aligns with our approach. Sato and Beckmann (1970) also highlight a common misconception in the literature, namely that the Cobb-Douglas production function is the only production function compatible with balanced growth and the presence of factor-augmenting technical change that is not exclusively labor-augmenting. They point out that even if the elasticity of substitution asymptotically converges to a unitary value, this does not imply that the asymptotic production function must be Cobb-Douglas. Based on this observation, they construct a family of production functions that asymptotically behave like a Cobb-Douglas function but can exhibit very different functional forms. We extend their approach by further generalizing and precisely characterizing the class of production functions that encompasses the family they proposed.

First, they show that if the capital share $\alpha(x)$ asymptotically converges to a constant value

strictly between 0 and 1, then the elasticity of substitution also asymptotically converges to 1. Thus, this convergence of the elasticity of substitution is a necessary condition for the existence of balanced growth in the presence of factor-augmenting technical change that is not exclusively labor-augmenting. Interestingly, this necessary condition directly follows from the properties of the class of production functions we proposed, as demonstrated by the following proposition:

Proposition 6 (Asymptotic Elasticity of Substitution).

> If the capital share converges to a non-degenerate value, i.e., $\lim_{x\to\infty} \alpha(x) = \alpha \in (0,1)$, then the elasticity of substitution asymptotically tends to 1, i.e., $\lim_{x\to\infty} \sigma(x) = 1$.

Proof.

$$\lim_{x(t)\to\infty} \sigma(x) = \lim_{x(t)\to\infty} \frac{1-\alpha(x)}{-\beta(x)} = \frac{1-\alpha}{-(\alpha-1)} = 1$$

■

We have used the results regarding the slow variations of the capital share $\alpha(x)$ and the elasticity of the marginal productivity of capital $\beta(x)$. However, this result is already well-known. For example, Grossman *et al.* (2017) show that a steady-state equilibrium is achieved if the following condition is satisfied: $[1-\sigma(x)]g_B = 0$. They conclude that this holds if $g_B = 0$ or if $\sigma(x) = 1$, which corresponds to the Cobb-Douglas production function. However, we refine this result by noting that $\lim_{x\to\infty}[1-\sigma(x)]g_B = 0$. Assuming $g_B > 0$, this relation is satisfied if the production function admits an asymptotically unitary elasticity of substitution.

We have previously discussed how to characterize Karamata-type production functions based on the capital share. Now, we will explore an alternative approach based on the elasticity of substitution. To do so, we present the following necessary and sufficient condition, derived from Sato and Beckmann (1970):

Proposition 7 (Asymptotic Elasticity of Substitution).

> A necessary and sufficient condition for the capital share to converge to a non-degenerate value, i.e., $\lim_{x\to\infty} \alpha(x) = \alpha \in (0,1)$, is that the following relation holds:
>
> $$\lim_{x\to\infty} \int_{x_0}^{x} \frac{\sigma(u)-1}{\sigma(u)} \frac{du}{u} = C,$$
>
> where $C \in \mathbb{R}$ is a finite real constant.

Proof. See Appendix C.

■

As Sato and Beckmann (1970) rightly pointed out, such a condition is too general to fully understand the role of the elasticity of substitution within the class of production functions we are investigating. Ideally, we would identify the set of elasticity functions (as a function of x) that can satisfy this necessary and sufficient condition. This ideal set can be approximated using the following sufficient condition provided by Sato and Beckmann (1970):

Proposition 8 (Sufficient Condition).

> If there exists a constant N such that
> $$\lim_{x\to\infty}\left|\frac{\sigma(x)-1}{\sigma(x)}\right|(\ln x)^k = N,$$
> with $0 \leq N < \infty$ and $k > 1$, then $\lim_{x\to\infty}\alpha(x) = \alpha \in (0,1)$.

Proof. See Appendix C.

■

Using this sufficient condition, we can construct an entire family of production functions that satisfy it. Specifically, it is enough for their elasticity of substitution to asymptotically satisfy:

$$\sigma(x) = \begin{cases} \dfrac{1}{1+\dfrac{N}{(\ln x)^k}} := \bar{\sigma}(x), & \text{if } \sigma(x) > 1, \\ \dfrac{1}{1-\dfrac{N}{(\ln x)^k}} := \underline{\sigma}(x), & \text{if } \sigma(x) < 1. \end{cases}$$

Sato and Beckmann (1970) claimed that the most general form of this class of production functions, that they can found was (their equation 20, page 396, in our notations)

$$f(x) = \begin{cases} Ce^{\displaystyle\int_{x_0}^{x} \frac{1}{1+B^{-1}e^{\frac{N}{k-1}(\ln u)^{1-k}}}\frac{du}{u}}, & \text{if } \bar{\sigma}(x) = \frac{1}{1+\frac{N}{(\ln x)^k}}, \\ Ce^{\displaystyle\int_{x_0}^{x} \frac{1}{1+B^{-1}e^{-\frac{N}{k-1}(\ln u)^{1-k}}}\frac{du}{u}}, & \text{if } \underline{\sigma}(x) = \frac{1}{1-\frac{N}{(\ln x)^k}}. \end{cases}$$

with $B > 0$, defined as

$$B = \begin{cases} \theta(x_0)e^{\frac{(\ln x_0)^{1-k}}{k-1}}, , & \text{if } \bar{\sigma}(x) = \frac{1}{1+\frac{N}{(\ln x)^k}}, \\ \theta(x_0)e^{-\frac{(\ln x_0)^{1-k}}{k-1}}, & \text{if } \underline{\sigma}(x) = \frac{1}{1-\frac{N}{(\ln x)^k}}. \end{cases}$$

However, the family of production functions they propose is a subset of the family of

production functions we have constructed. Indeed, note that $\lim_{x\to\infty}\theta(x) = B \in \mathbb{R}_+^\star$, and therefore:

$$\lim_{x\to\infty}\alpha(x) = \lim_{x\to\infty}\frac{\theta(x)}{1+\theta(x)} = \frac{B}{1+B} := \alpha \in (0,1)$$

Moreover, by the Representation Theorem, we can write:

$$\varepsilon(x) = \alpha(x) - \alpha = \frac{\theta(x)}{1+\theta(x)} - \frac{B}{1+B}$$

and therefore, their family of production functions belongs to the Karamata class.

4 The direction of technical progress

At this point, we can ask is the following: does our class of production functions have a microeconomic foundation? More specifically, generalizing the work of Houthakker (1955), Jones (2005) demonstrates that if we consider a local Leontief technology or a local CES technology with an elasticity of substitution less than 1, it is possible to show that the aggregated production function becomes Cobb-Douglas if firms develop new ideas randomly drawn from a Pareto distribution. We leverage the class of Karamata functions, which are used in mathematics to study heavy-tailed probability distributions (extreme values), of which the Pareto distribution is a special case.

Thus, we generalize the approach of Jones (2005) by providing the same microeconomic foundations for our regularly varying production functions. To do so, we construct the aggregate function defined as the upper envelope $F(x;N)$ of production functions using local technologies given by $y_i = a_i f\left(\frac{b_i}{a_i}k\right) = a_i f(x_i)$. Jones (2005) obtains an analytical result by assuming that f_i takes a Walras-Leontief form (i.e., with complementary factors corresponding to an elasticity of substitution approaching 0), and he provides numerical results by assuming that the local production function is CES with an elasticity of substitution less than 1.[11]

Interestingly, these production functions are slowly varying functions. Indeed, it is straightforward to verify that:

$$\lim_{x_i\to\infty}\frac{\min\{\lambda x_i, 1\}}{\min\{x_i, 1\}} = 1,$$

$$\lim_{x_i\to\infty}\frac{\left(\beta\left(\lambda x_i\right)^\sigma + (1-\beta)^\sigma\right)^{\frac{1}{\sigma}}}{\left(\beta\left(x_i\right)^\sigma + (1-\beta)^\sigma\right)^{\frac{1}{\sigma}}} = 1.$$

The first generalization we propose is to assume that $f(x_i)$ is an arbitrary slowly varying

[11] It is worth noting a contradiction in Jones's assumptions, as his local function is assumed to satisfy the Inada conditions, whereas neither the Walras-Leontief function nor the CES function satisfy them everywhere. However, this does not affect the validity of his results.

function, which encompasses the functions proposed by Jones (2005). Note that $f(x_i)$ could also be regularly varying, i.e., $f(x_i) \sim x_i^{\alpha} L(x_i)$, where $L(x_i)$ is slowly varying, but this is not required for the following argument.

The second generalization we propose is that the terms a_i and b_i are drawn from arbitrary heavy-tailed distributions, not just Pareto distributions, where heavy-tailed distributions are probability distributions whose tails decay more slowly than those of exponential distributions, meaning their tails remain significant even for large values (e.g., Pareto or Cauchy distributions). The essence of Jones's (2005) demonstration is to show that the resulting aggregate production function lies in the basin of attraction of the Fréchet distribution.

In general, a non-negative random variable X and its distribution are said to be regularly varying with index $\alpha \geq 0$ if the survival function $\bar{G}(x) := 1 - G(x)$ is regularly varying with index $-\gamma$ (see Bingham *et al.*, 1987) for instance. Indeed, all distributions belonging to the domain of attraction of the Fréchet distribution can be rewritten as

$$G(x) = 1 - x^{-1/\gamma} L(x)$$

where $L(x)$ is a slowly varying function. The parameter γ directly controls the tail behavior of the survival function, and thus that of extreme values. The survival function $\bar{G}(x) := 1 - G(x) = x^{-1/\gamma} L(x)$ is a regularly varying function with index $-1/\gamma$.

Specifically, the variables a_i and b_i are drawn from heavy-tailed distributions and with survival functions:

$$\mathbb{P}(a_i > a) \sim L_a(a) a^{-\alpha}, \quad \mathbb{P}(b_i > b) \sim L_b(b) b^{-\beta},$$

where $L_a(a)$ and $L_b(b)$ are slowly varying functions, and $\alpha, \beta > 0$. These distributions have heavy tails, characterized by a power-law decay modulated by the slowly varying functions L_a and L_b. As in Jones (2005), we assume that a single technique is used at each point in time.

The function $F(x; N)$ is defined as the maximum of N terms:

$$F(x; N) = \max_{i=1,\ldots,N} a_i f\left(x_i\right)$$

Each term $y_i = a_i f\left(\frac{b_i}{a_i} k\right)$ depends on the distributions of a_i and b_i as well as the properties of f. To show that $F(x; N)$ is regularly varying, we analyze its asymptotic behavior as $N \to \infty$. Since $f(x_i)$ is slowly varying, for large x_i:

$$y_i \sim a_i L(x_i) x_i^{\rho},$$

where $\rho = 0$ for slowly varying $f(x_i)$, and $L(x_i)$ is the slowly varying component of $f(x_i)$. The heavy tails of a_i and b_i dominate the behavior of the maximum. For large N, the largest a_i and b_i satisfy:

$$\mathbb{P}(\max a_i > a) \sim N L_a(a) a^{-\alpha}, \quad \mathbb{P}(\max b_i > b) \sim N L_b(b) b^{-\beta}.$$

The term $a_i f\left(\frac{b_i}{a_i}k\right)$ inherits its asymptotics primarily from the largest a_i and b_i. Therefore:

$$F(x;N) \sim c_N x_i^{\rho} L(x_i),$$

where c_N is an appropriate scaling factor dependent on N, α and β.[12] For large x_i:

$$F(\lambda x;N) \sim c_N(\lambda x)^{\rho} L(\lambda x), \quad \text{and} \quad F(x;N) \sim c_N x^{\rho} L(x).$$

Using the slowly varying property of $L(x)$:

$$\frac{F(\lambda x;N)}{F(x;N)} \to \lambda^{\rho}, \quad \forall \lambda > 0.$$

This shows that $F(x;N)$ is regularly varying with index ρ. Therefore, even if $f(x)$ is only slowly varying, the function $F(x;N)$ becomes regularly varying due to the dominance of the heavy-tailed distributions of a_i and b_i. The asymptotic behavior of $F(x;N)$ reflects the power-law decay of the underlying distributions, ensuring regular variation with index ρ.

Thus, as we have just shown by generalizing the results of Jones (2005), the global production function is not solely Cobb-Douglas but belongs to the class of regularly varying functions, further justifying our approach. However, instead of delivers a Cobb-Douglas production function at any point in time, we provide a more general production of the Kamarata class. Consequently, if the production function belongs to the Karamata class, we have shown that $\sigma \to 1$, and thus technical progress will be asymptotically neutral in both the Hicksian and Harrodian senses. We can thus propose a form of neutrality based on these results:

Proposition (Karamata's neutrality).

> Inventions are neutral in Karamata's sense if the production function is regularly varying with an index $\alpha \in (0,1)$.

5 Examples

5.1 CEDD production functions

The first natural example is the CEDD (Constant Elasticity of Derived Demand) production function proposed by Sato (1970).[13] The CEDD, denoted E_k, is the elasticity of derived demand for capital per unit of labor (i.e., for a given number of labor units) and is defined

[12] Jones chooses a specific c_N for the Pareto distribution, which is common in this literature.

[13] Bruno (1968) also presented a special case of this function, calling it the Constant-marginal-shares production function, as it generates constant shares of production factors.

as:

$$E_k = -\frac{F_K}{KF_{KK}} = -\frac{Bf'(x)}{KBf''(x)\frac{B}{AL}} = -\frac{f'(x)}{xf''(x)} = -\frac{1}{\beta(x)}$$

Thus, the elasticity of substitution is expressed as:

$$\sigma(x) = (1 - \alpha(x))E_k(x)$$

The elasticity of substitution equals the product of the labor share and the elasticity of derived demand for capital per unit of labor. Asymptotically, this elasticity is constant since it is a regularly varying function with an index of $\frac{1}{1-\alpha}$. It is therefore natural to construct a specific production function within this class, called the CEDD, where $E_k := -\frac{1}{\beta} = \frac{\sigma(x)}{1-\alpha(x)}$ is constant for all x. In this case, if the elasticity of the marginal productivity of capital is constant, we have:

$$\frac{f'(x)}{xf''(x)} = -\frac{1}{\beta} \Rightarrow f'(x) = ax^{-\beta}$$

with $a > 0$. Integrating this function again gives:

$$f(x) = \frac{a}{1-\beta}x^{1-\beta} + c$$

where $c \in \mathbb{R}$ is the integration constant. The capital share can then be expressed as $\alpha(x) = (1-\beta)\frac{f(x)-c}{f(x)}$ and the elasticity of substitution becomes $\sigma(x) = 1 + \frac{1-\beta}{\beta}\frac{c}{f(x)}$.

This function belongs to the Karamata class, as it is regularly varying with an index of $1 - \beta = \alpha \in (0, 1)$ and the elasticity of substitution converges to 1 as $x \to \infty$. Consequently, the CEDD production function can be written (normalizing the first integration constant $a := \alpha$) as:

$$Y(t) = (B(t)K(t))^{\alpha}(A(t)L(t))^{1-\alpha} + c(A(t)L(t))$$

This is essentially a Cobb-Douglas production function with a linear term depending solely on effective labor.

It is worth noting that this type of CEDD function can take other forms, which may not be regularly varying. For instance, one can symmetrically define the elasticity of derived demand for labor per unit of capital as $E_l := \frac{F_L}{LF_{LL}}$. Following a similar reasoning, the resulting production function takes the form:

$$Y(t) = (B(t)K(t))^{\alpha}(A(t)L(t))^{1-\alpha} + c(B(t)K(t))$$

This function was used by Kurz (1968), and later by Jones and Manuelli (1990) (with $A(t)$ and $B(t)$ constant) to generate transitional dynamics in the AK endogenous growth model. However, the intensive form of this function, $f(x) = x^{\alpha} + cx$ is regularly varying but with an index of $\alpha = 1$ which means it does not belong to our class of production functions (which, moreover, implies Inada conditions are not satisfied by this type of production function).

Additionally, as pointed out by Sato (1970), if both types of elasticities are constant and

such that $\sigma = (1-\alpha)E_k = \alpha E_l$ is a constant, the function belongs to the CES production function class. These functions, however, are not regularly varying with a paramer $\alpha \in (0,1)$. Therefore, the CEDD-type functions that belong to our class of Karamata-type production functions are those where E_k is constant but not E_l.[14]

5.2 Mixed CES-Cobb Douglas production function

Another production function proposed in the literature, which belongs to the Karamata class, is a hybrid between the Cobb-Douglas and CES production functions. This type of production function originates from the unpublished work of Bruno (1962) and has been used in the empirical studies of Liu and Hildebrand (1965) and Nerlove (1967). It was also derived by Sato and Beckman (1970) as a special case within their class of production functions, which is compatible with the existence of balanced growth despite the presence of factor-augmenting technology. There are three ways to express this production function such that the capital share converges to a non-degenerate value. We begin by considering the following function:

$$f(x) = \left[\beta x^{(1-\alpha)\rho} + (1-\beta)\right]^{\frac{1}{\rho}}$$

where $0 < \alpha < 1$ and $\rho > 0$. The capital share is then given by:

$$\alpha(x) = (1-\alpha)\beta f(x)^{-\rho} x^{(1-\alpha)\rho} = (1-\alpha)\frac{\beta x^{(1-\alpha)\rho}}{\beta x^{(1-\alpha)\rho} + (1-\beta)} = \frac{1-\alpha}{1 + \frac{1-\beta}{\beta} x^{-(1-\alpha)\rho}}$$

Thus, as $x \to \infty$ the capital share converges to $1-\alpha \in (0,1)$. We note that if $\rho < 0$, the capital share converges to 0 and the function becomes only slowly varying. Finally, if $\rho = 1$ this production function corresponds to the CEDD function. Therefore, this production function naturally generalizes the CEDD function. The elasticity of substitution is given by:

$$\sigma(x) = \frac{1}{1 + \rho\frac{\alpha(x)-(1-\alpha)}{1-\alpha(x)}} = \frac{1}{1 - \rho\frac{1-\alpha}{1+\alpha\frac{\beta}{1-\beta}x^{(1-\alpha)\rho}}}$$

If $\rho > 0$ the elasticity of substitution converges to 1 which is consistent with the characteristics of our class of production functions. Note that ρ can be greater than $\rho(1-\alpha) < 1$ ensuring that $\sigma(0) = \frac{1}{1-\rho(1-\alpha)} > 0$. In this case, the elasticity of substitution starts above 1. This property may be undesirable if one seeks a production function where the elasticity of substitution remains below 1, at least along the transition path. To address this, we can

[14] It should also be noted that $E_k \neq 1$, otherwise, the production function is of the "Bernoulli" type (as named by A. Marshall), and it is expressed in this case (see Sato, 1970) as:

$$Y(t) = (A(t)L(t)) \ln\left(\frac{A(t)K(t)}{B(t)L(t)}\right) + c(A(t)L(t))$$

This Bernoulli-type production function is only slowly varying, i.e., with an index $\alpha = 0$.

consider the following symmetric production function:

$$f(x) = [\beta + (1-\beta)x^{\alpha\rho}]^{\frac{1}{\rho}}$$

where $0 < \alpha < 1$ and $\rho < 0$. The capital share in this case becomes:

$$\alpha(x) = \frac{\alpha + \frac{\beta}{1-\beta}x^{\rho(1-\alpha)}}{1 + \frac{\beta}{1-\beta}x^{\rho(1-\alpha)}}$$

and it converges to the constant $\alpha \in (0,1)$ as $x \to \infty$. The elasticity of substitution is then given by:

$$\sigma(x) = \frac{1}{1-\rho+\frac{\rho\alpha}{\alpha(x)}}$$

If $\rho < 0$, the elasticity of substitution also converges to 1. It starts at a value below 1, given by $\sigma(0) = \frac{1}{1-\rho+\rho\alpha} < 1$ and increases monotonically toward its limit of 1.

This type of production function also supports the approach proposed by Jones (2003, 2005). Indeed, we can propose combining the CES and Cobb-Douglas production functions in the following manner, by factoring out the Cobb-Douglas function from the CES function:

$$f(x) = \left[\beta\left(\frac{x_0}{x}\right)^{\rho} + 1 - \beta\right]^{1/\rho} x^{\alpha}$$

where $\rho < 0$. Here, x represents the long-term quantities of factors used, while x_0 epresents the short-term quantities of factors. This approach is fully compatible with the Karamata production function that we propose in this paper, which clearly distinguishes short-term production functions (in the context of transitional dynamics) from their asymptotic behavior.

5.3 Infinite oscillations

In Karamata's theory, infinite oscillations is possible. Indeed, the definition of a slowly varying function remains valid even when $L(x)$ exhibits infinite oscillations, including cases where:

$$\lim_{x\to\infty} \inf L(x) = 0 \quad \lim_{x\to\infty} \sup L(x) = \infty$$

Such behavior arises in functions which exhibit oscillations between 0 and ∞ while still satisfying the slow variation condition. The key requirement is the asymptotic stability under scaling, meaning that the relative growth between $L(tx)$ and $L(x)$ tends to 1, regardless of the absolute amplitude of oscillations. In contrast, when dealing with functions that exhibit more complex oscillations, we define generalized regular variation as:

$$0 < \lim_{x\to\infty} \inf L(x) \leq \lim_{x\to\infty} \sup L(x) < \infty$$

The strict inequalities here are crucial. They ensure that the function's growth is controlled: the ratio $\frac{f(tx)}{f(x)}$ does not collapse to zero or diverge to infinity, preserving the regularity of

the asymptotic behavior. Regular variation requires a more rigid structure, maintaining bounded oscillations to reflect predictable scaling behavior. Thus, while slowly varying functions can exhibit extreme oscillations, regularly varying functions demand controlled asymptotic behavior to ensure consistent growth across scales.

This property implies that the existence of a regular growth path is compatible not only with the presence of factor-augmenting technical progress but also with oscillations in both the factor shares and the elasticity of substitution. To illustrate this possibility, consider the capital share given by the following function:

$$\alpha(x) = \frac{1}{1 + e^{\frac{\cos(x)}{2}}}$$

which is a function bounded between 0 and 1, as the capital share fluctuates approximately between 0.38 and 0.62. Using the ratio of the capital share to the labor share, we obtain:

$$\theta(x) = e^{\frac{-\cos(x)}{2}}$$

which we can (log-)integrate to obtain:

$$\frac{\sigma(x) - 1}{\sigma(x)} = \frac{\sin(x)}{2}$$

Thus, the elasticity of substitution varies within the range $\left[\frac{2}{3}, 2\right]$, which is consistent with empirically estimated values in the literature. The corresponding production function is then given by the Representation Theorem:

$$f(x) = x^{\alpha} L(x)$$

where $\varepsilon(x) = \frac{1}{1+e^{\frac{\cos(x)}{2}}} - \alpha$.[15]

6 Conclusion

Do two functions with the same asymptotic limits necessarily coincide? The literature has traditionally answered yes to this fundamental question. However, a production function whose capital share is asymptotically constant and whose elasticity of substitution asymptotically equals 1 is not necessarily a Cobb-Douglas production function. This implies that it is possible to introduce a more general form of technical progress than labor-augmenting technological progress into a neoclassical growth model without jeopardizing the existence of a balanced growth path, as previously demonstrated by Acemoglu (2003) in the context of endogenous growth models. However, we establish this result within a much broader and more abstract neoclassical framework.

[15] However, one important point to note is that the derivative of f is always strictly positive, but its second derivative can change sign because of fluctuations, meaning that it is not always concave along the balanced growth path.

Nonetheless, while the Karamata production function we propose clarifies this important point, it does not prevent the elasticity of substitution from asymptotically converging to 1. Thus, if an economy follows a balanced growth path and empirical evidence suggests that the elasticity of substitution is different from 1, the standard neoclassical model is indeed insufficient to account for this fact, as well as for the declining labor share. However, if the variability of the elasticity of substitution and the declining labor share result from business cycle fluctuations, then the Karamata production function would not be theoretically incompatible with these observations.

Appendix

Appendix A.

Proof (Proposition 1). The condition $\lim_{t\to\infty} \frac{I(t)}{Y(t)} = s \in (0,1)$ represents the idea that, even asymptotically, the economy does not invest (and therefore does not consume) the entirety of the income generated by output $Y(t)$. This condition naturally implies that $g_I = g_C = g_Y$. Without this condition, it is possible to construct counterexamples that invalidate the balanced growth result (see Acemoglu (2005), Exercise 2.14 for a counterexample, and the proof below).

Let $\chi(t) := \frac{K(t)}{Y(t)}$ denote the capital-output ratio. Its growth rate is written as:

$$\frac{\dot{\chi}(t)}{\chi(t)} = \frac{\dot{K}(t)}{K(t)} - g_Y = \frac{I(t)}{K(t)} - \delta - g_Y = \frac{I(t)}{Y(t)}\frac{Y(t)}{K(t)} - \delta - g_Y$$

We have:

$$\lim_{t\to\infty} \frac{\dot{\chi}(t)}{\chi(t)} = g_K - g_Y = \lim_{t\to\infty} \frac{I(t)}{Y(t)} \lim_{t\to\infty} \frac{1}{\chi(t)} - (\delta + g_Y) = s \lim_{t\to\infty} \frac{1}{\chi(t)} - (\delta + g_Y)$$

Thus, finally:

$$\lim_{t\to\infty} \chi(t) = \lim_{t\to\infty} \frac{K(t_0)}{Y(t_0)} e^{(g_K - g_Y)t} = \frac{s}{\delta + g_K}$$

which is satisfied if $g_K = g_Y$. Therefore, we have shown that $g_Y = g_K = g_C = g_I$.

■

Proposition (Wicksell's Laws).

Given that the production function $F : (0,\infty)^2 \to (0,\infty)$ is homogeneous of degree 1, and that marginal productivities are strictly decreasing, the cross-partial derivatives are positive. Furthermore, the following relationships hold:

$$F_{LK} = -\frac{L}{K}F_{LL}, \quad F_{KL} = -\frac{K}{L}F_{KK} \quad \text{and} \quad F_{LL} = \left(\frac{K}{L}\right)^2 F_{KK}$$

Proof. Since the production function F exhibits constant returns to scale and is therefore homogeneous of degree 1, its partial derivatives, which define the marginal productivities, are homogeneous of degree 0. By applying Euler's theorem to the marginal productivities, we obtain the desired results:

$$0 = LF_{LL} + KF_{LK} \quad \Rightarrow \quad F_{LK} = -\frac{L}{K}F_{LL} > 0$$
$$0 = LF_{KL} + KF_{KK} \quad \Rightarrow \quad F_{KL} = -\frac{K}{L}F_{KK} > 0$$

Since $F_{KL} = F_{LK}$, it follows directly that: $F_{LL} = \left(\frac{K}{L}\right)^2 F_{KK}$.

■

Proof (Proposition 2). The growth rate of $x(t)$ is given by:

$$\frac{\dot{x}(t)}{x(t)} = g_B - g_A - n - \frac{\dot{K}(t)}{K(t)} = g_B - g_A - (n+\delta) - \frac{I(t)}{K(t)}.$$

Asymptotically, this growth rate becomes:

$$\lim_{t\to\infty}\frac{\dot{x}(t)}{x(t)} = g_B - g_A - (n+\delta) + \lim_{t\to\infty}\frac{I(t)}{K(t)}.$$

Using the Balanced Growth Theorem, we know:

$$\lim_{t\to\infty}\frac{I(t)}{K(t)} = \lim_{t\to\infty}\frac{I(t)}{Y(t)}\frac{Y(t)}{K(t)} = s\cdot\frac{\delta+g_Y}{s} = \delta + g_Y.$$

Thus, we obtain:

$$\lim_{t\to\infty}\frac{\dot{x}(t)}{x(t)} = g_B - g_A - (n+\delta) + \delta + g_Y = g_Y - n - g_A + g_B.$$

From the production function $Y(t) = F(B(t)K(t), A(t)L(t))$, we derive:

$$\begin{aligned}\frac{\dot{Y}(t)}{Y(t)} &= \alpha(x(t))\left(g_B + \frac{\dot{K}(t)}{K(t)}\right) + (1-\alpha(x(t)))(g_A+n)\\ &= n + g_A + \alpha(x(t))\left(\frac{\dot{K}(t)}{K(t)} - n - g_A + g_B\right).\end{aligned}$$

Taking the limit on both sides of the equation, we have:

$$\lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)} = n + g_A + \lim_{t\to\infty}\alpha(x(t))\left(\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} - n - g_A + g_B\right).$$

Recalling that $g_Y = g_K$, we establish point (iv):

$$\lim_{t\to\infty}\alpha(x(t)) = \alpha = \frac{g_Y - n - g_A}{g_Y - n - g_A + g_B}.$$

This allows us to derive the asymptotic growth rate of output (iii):

$$g_Y = n + g_A + \frac{\alpha}{1-\alpha}g_B.$$

From this, we deduce the growth rate of $x(t)$ (ii):

$$\lim_{t\to\infty}\frac{\dot{x}(t)}{x(t)} = \frac{1}{1-\alpha}g_B > 0.$$

Thus, $x(t) = \frac{B(t_0)K(t_0)}{A(t_0)L(t_0)}e^{\frac{1}{1-\alpha}g_B t}$, and therefore (i):

$$\lim_{t\to\infty} x(t) = \infty.$$

■

Appendix B

First Proof of Proposition 3. Let $\phi(x) := \ln f(x)$, so that $\phi'(x) = \frac{f'(x)}{f(x)}$. The capital share in income can then be expressed as:

$$\alpha(x) = \frac{f'(x)x}{f(x)} = x\phi'(x).$$

Thus:

$$\phi'(x) = \frac{\alpha(x)}{x}.$$

By integrating both sides between a constant $x_0 > 0$ and x, we obtain:

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x}\frac{\alpha(u)}{u}du.$$

Given that $\lim_{x\to\infty}\alpha(x) = \alpha$, for any $\varepsilon > 0$, there exists x_1 such that for all $x \geq x_1$, $|\alpha(x) - \alpha| < \varepsilon$. Thus, for $x \geq x_1$, the capital share satisfies: $\alpha - \varepsilon < \alpha(x) < \alpha + \varepsilon$.

For $x \geq x_1$, we can bound $\phi(x)$ as follows:

$$(\alpha - \varepsilon)\int_{x_1}^{x}\frac{du}{u} \leq \phi(x) - \phi(x_1) \leq (\alpha + \varepsilon)\int_{x_1}^{x}\frac{du}{u}.$$

After evaluating the integrals, we get:

$$(\alpha - \varepsilon)\ln\left(\frac{x}{x_1}\right) \leq \phi(x) - \phi(x_1) \leq (\alpha + \varepsilon)\ln\left(\frac{x}{x_1}\right).$$

Taking the exponential of both sides yields:

$$\left(\frac{x}{x_1}\right)^{\alpha-\varepsilon} e^{\phi(x_1)} \leq f(x) \leq \left(\frac{x}{x_1}\right)^{\alpha+\varepsilon} e^{\phi(x_1)}.$$

Now, consider the ratio $\frac{f(\lambda x)}{f(x)}$. Using the above bounds, we can write:

$$\left(\frac{\lambda x}{x}\right)^{\alpha-\varepsilon} \leq \frac{f(\lambda x)}{f(x)} \leq \left(\frac{\lambda x}{x}\right)^{\alpha+\varepsilon}.$$

Simplifying gives:

$$t^{\alpha-\varepsilon} \leq \frac{f(\lambda x)}{f(x)} \leq t^{\alpha+\varepsilon}.$$

Finally, taking the limit as $x \to \infty$ (with λ fixed) and noting that $\varepsilon > 0$ can be made arbitrarily small, we obtain:

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\alpha},$$

which proves that f is a regularly varying function.

■

Proof (Proposition 4). Using the representation of the production function $f(x) = x^{\alpha}L(x)$, where $L(x)$ is slowly varying, we compute the first derivative:

$$f'(x) = \alpha x^{\alpha-1}L(x) + x^{\alpha}L'(x).$$

The ratio of the first derivative to $x^{\alpha-1}L(x)$ is:

$$\frac{f'(x)}{x^{\alpha-1}L(x)} = \alpha + \frac{xL'(x)}{L(x)} = \alpha(x).$$

Since $L(x)$ is slowly varying, $\frac{xL'(x)}{L(x)} = \varepsilon(x) \to 0$ as $x \to \infty$. Thus, $\alpha(x) \to \alpha$, and asymptotically, the marginal productivity behaves as: $f'(x) \sim \alpha x^{\alpha-1}L(x)$. This implies that $f'(x)$ is regularly varying with index $\alpha - 1$.

Next, consider the second derivative:

$$f''(x) = \alpha(\alpha-1)x^{\alpha-2}L(x) + 2\alpha x^{\alpha-1}L'(x) + x^{\alpha}L''(x).$$

The ratio of the second derivative to $x^{\alpha-2}L(x)$ is:

$$\frac{f''(x)}{x^{\alpha-2}L(x)} = \alpha(\alpha-1) + 2\alpha\frac{xL'(x)}{L(x)} + \frac{x^2L''(x)}{L(x)}.$$

Since $L(x)$ is slowly varying, the last two terms $\frac{xL'(x)}{L(x)}$ and $\frac{x^2L''(x)}{L(x)}$ converge to 0 as $x \to \infty$. Thus, the second derivative asymptotically behaves as: $f''(x) \sim \alpha(\alpha-1)x^{\alpha-2}L(x)$. This implies that $f''(x)$ is regularly varying with index $\alpha - 2$.

■

Proof (Corrolary 1). Since f, f', and f'' are regularly varying functions, we have

$f(x) \sim x^{\alpha} L(x)$, $f'(x) \sim \alpha x^{\alpha-1} L(x)$, and $f''(x) \sim \alpha(\alpha-1)x^{\alpha-2} L(x)$. Therefore:

$$\alpha(x) := \frac{x f'(x)}{f(x)} \sim \frac{x \alpha x^{\alpha-1} L(x)}{x^{\alpha} L(x)} = \alpha,$$
$$\beta(x) := \frac{x f''(x)}{f'(x)} \sim \frac{x \alpha(\alpha-1) x^{\alpha-2} L(x)}{\alpha x^{\alpha-1} L(x)} = \alpha - 1.$$

Thus, the coefficient $\beta(x)$ asymptotically converges to the negative of the labor share, $-(1-\alpha)$. It is then straightforward to see that:

$$\lim_{x \to \infty} \frac{\alpha(tx)}{\alpha(x)} = \frac{\alpha}{\alpha} = 1,$$
$$\lim_{x \to \infty} \frac{\beta(tx)}{\beta(x)} = \frac{\alpha - 1}{\alpha - 1} = 1.$$

Therefore, $\alpha(x)$ and $\beta(x)$ are slowly varying functions. Finally, $\sigma(x)$ is a slowly varying function, since the ratio of two slowly varying functions is also a slowly varying function (Bingham *et al.*, 1987).

■

Proof (Proposition 5, Inada Conditions). 1. $\lim_{x \to \infty} f(x) = \infty$. Since $f(x)$ is regularly varying with index $\alpha > 0$, it satisfies:

$$f(x) \sim x^{\alpha} L(x),$$

where $L(x)$ is a slowly varying function. The term x^{α} grows unbounded as $x \to \infty$ because $\alpha > 0$. The function $L(x)$ is positive and slowly varying, meaning it does not grow faster than any power of $\ln(x)$. Therefore, $L(x)$ remains positive and grows sufficiently slowly compared to x^{α}. Thus:

$$f(x) \to \infty \quad \text{as} \quad x \to \infty.$$

2. $\lim_{x \to 0} f(x) = 0$. Again, using the asymptotic representation:

$$f(x) \sim x^{\alpha} L(x),$$

where $\alpha > 0$ and $L(x)$ is slowly varying. As $x \to 0^{+}$, $x^{\alpha} \to 0$ because $\alpha > 0$. The term $L(x)$ remains positive and slowly varying. Specifically, $L(x)$ grows very slowly (e.g., logarithmic growth) or remains constant as $x \to 0^{+}$. Since x^{α} dominates $L(x)$ near zero, we conclude:

$$f(x) \to 0 \quad \text{as} \quad x \to 0^{+}.$$

3. $\lim_{x\to\infty} f'(x) = 0$. From the derivative of $f(x) = x^\alpha L(x)$, we have:

$$f'(x) = \frac{d}{dx}\left(x^\alpha L(x)\right) = \alpha x^{\alpha-1}L(x) + x^\alpha L'(x).$$

The term $\alpha x^{\alpha-1}L(x)$: Since $\alpha - 1 < 0$ (because $\alpha > 0$), $x^{\alpha-1} \to 0$ as $x \to \infty$. $L(x)$ is slowly varying and remains positive, so this term tends to 0. The term $x^\alpha L'(x)$: For a slowly varying function $L(x)$, $L'(x)$ satisfies $\lim_{x\to\infty} xL'(x) = 0$. Therefore, $x^\alpha L'(x) \to 0$ as $x \to \infty$. Combining these results:

$$f'(x) \to 0 \quad \text{as} \quad x \to \infty.$$

4. $\lim_{x\to 0} f'(x) = \infty$. Using the same derivative expression:

$$f'(x) = \frac{d}{dx}\left(x^\alpha L(x)\right) = \alpha x^{\alpha-1}L(x) + x^\alpha L'(x).$$

The term $\alpha x^{\alpha-1}L(x)$: Since $\alpha - 1 > 0$ (because $\alpha > 0$), $x^{\alpha-1} \to \infty$ as $x \to 0^+$. $L(x)$ is slowly varying and remains positive, so this term dominates and grows unbounded. The term $x^\alpha L'(x)$: For a slowly varying function $L(x)$, $L'(x)$ remains bounded as $x \to 0^+$. Thus, $x^\alpha L'(x) \to 0$ as $x \to 0^+$. Since the first term dominates and grows unbounded:

$$f'(x) \to \infty \quad \text{as} \quad x \to 0^+.$$

■

Appendix C

Proof (Proposition 7). Let $\theta(x)$ denote the elasticity of output per worker $y(t)$ with respect to the ratio of effective capital to effective labor $x(t)$, which is equal to the ratio of the capital and labor shares:

$$\theta(x) = \frac{\alpha(x)}{1-\alpha(x)}.$$

This measure has proven very useful in empirical studies as well as in the proof of Uzawa's theorem (see Uzawa (1961) or Jones and Scrimgeour (2004) for example). We can show that:

$$\frac{\theta'(x)}{\theta(x)} = \frac{1}{x}\frac{\sigma(x)-1}{\sigma(x)}.$$

This linear differential equation can be solved from $x_0 > 0$ to x as follows:

$$\theta(x) = \theta(x_0)e^{\int_{x_0}^{x} \frac{\sigma(u)-1}{\sigma(u)}\frac{du}{u}}.$$

Furthermore, note that $\theta(x)$ is a slowly varying function since $\lim_{x\to\infty} \frac{\sigma(u)-1}{\sigma(u)} = 0$, according to the necessary condition. We now establish the following equivalence:

$$\lim_{x\to\infty} \alpha(x) = \alpha \in (0,1) \Leftrightarrow \lim_{x\to\infty} \theta(x) = \theta := \frac{\alpha}{1-\alpha} \in \mathbb{R}_+^\star.$$

Next, since $\theta > 0$, we have:

$$\lim_{x\to\infty} \ln(\theta(x)) = \ln\theta \in \mathbb{R},$$

where we substitute $\theta(x)$ as computed above:

$$\lim_{x\to\infty} \ln\theta(x_0) + \lim_{x\to\infty} \int_{x_0}^{x} \frac{\sigma(u)-1}{\sigma(u)} \frac{du}{u} = \ln\theta \in \mathbb{R}.$$

This directly leads to the desired result:

$$\lim_{x\to\infty} \int_{x_0}^{x} \frac{\sigma(u)-1}{\sigma(u)} \frac{du}{u} = C,$$

where $C := \ln\frac{\theta}{\theta_0} \in \mathbb{R}$ (a finite real constant).

■

Proof (Proposition 8). Let us perform the following change of variables: $u = e^t$, so that $t = \ln u$. This implies $\eta(t) := \frac{\sigma(e^t)-1}{\sigma(e^t)}$ and $du = e^t dt$. Updating the bounds, when $u = x_0$, $t = \ln x_0 := t_0$, and when $u = x$, $t = \ln x$. We have:

$$\lim_{t\to\infty} \int_{t_0}^{t} \eta(u)du = C.$$

This equation converges if there exists a number $0 \leq N < +\infty$ such that:

$$\lim_{t\to\infty} \frac{|\eta(t)|}{t^\beta} = N,$$

with $\beta < -1$.

Indeed, the condition on the function $\eta(t)$ implies that for sufficiently large t, $|\eta(t)| \sim Nt^\beta$, where N is a finite constant. This means that $\eta(t)$ decays as a power of t as $t \to \infty$. Consequently, the integral becomes asymptotically equivalent to:

$$\int_{t_0}^{\infty} |\eta(u)|du \sim \int_{t_0}^{\infty} Nu^\beta du = \lim_{t\to\infty} \frac{t^{\beta+1}}{\beta+1} - \frac{(\ln x_0)^{\beta+1}}{\beta+1}.$$

For this integral to converge, we must have $t^{\beta+1} \to 0$ as $t \to \infty$, which holds if $\beta < -1$.

Thus, we have:

$$\lim_{t\to\infty} \frac{|\eta(t)|}{t^\beta} = N.$$

Substituting $t = \ln x$, $k = -\beta$, and $\eta(t) = \frac{\sigma(x)-1}{\sigma(x)}$, we recover the stated condition.

■

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