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**Locally Greedy Characterizations
of Interim Allocations**

by Charles Z. Zheng

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Locally Greedy Characterizations of Interim Allocations^{*}

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Abstract

The received characterizations of feasible interim allocations are mostly in the spirit of Border (1991): Fix a family of sets, each containing some player-types, and test the interim allocation under consideration against all these sets. In the published literature, such Border-like characterizations are known to be valid only in the paramodularity framework, which rules out combinatorial complexities such as matchings. This paper presents a necessity and sufficiency test for Border-like characterizations with or without paramodularity. It implies that the validity of the characterizations requires that any interim allocation about to become infeasible be locally greedy: that its domain be covered by a family of subsets within each of which the underlying ex post allocation follows some greedy algorithm. I prove impossibility of Border-like characterizations in the knapsack model of sharing economies and prove Border-like characterizations in a matching model that allows each player to have arbitrary numbers of types, and in a ranked-item auction model with a group-specific quota constraint.

JEL Classification: C61, D44, D82

Keywords: Auctions, assignment, matching, greedy algorithm, multiple objects, reduced forms, implementability, interim allocation, paramodularity, knapsack

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1 Introduction

Characterizing the set of feasible interim allocations has long been a difficult and often unavoidable problem in the literature of mechanism design. The problem is to find necessary and sufficient conditions for an interim allocation to be the reduced form of some feasible ex post allocation, the former being a function of only individual players' types, while the latter a function of the profile of types across all players. The problem is often unavoidable because in models with multiple objects or nonlinear preferences, the interim allocation, rather than its ex post allocation counterpart, is the choice variable that could be tractable to the mechanism designer. The importance of this problem was first recognized by Maskin and Riley [22] and Matthews [23] in their studies of risk-averse bidders and continues to current research such as the recent studies of Gershkov et al. [10] on auctions with endogenous valuations and Haghpahan et al. [13] on procurements from a group of heterogeneous sellers.

The literature has located specific conditions about the primitives that guarantee for this problem a relatively tractable solution, in the form of Border [3] or its generalization such as Che et al. [7]. Such Border-like solutions are relatively tractable because they require only linear inequalities, and the smallest set thereof, to test against any interim allocation under consideration. However, it is unknown whether such solutions are still valid outside the sufficient conditions that the literature has located. This issue is particularly salient because these conditions are mostly confined to a paramodularity framework, which restricts the environment to either a single item for sale or multiple items that have no nontrivial combinatorial complication. Matching and assignments are outside the framework. The lack of Border-like characterizations in combinatorially nontrivial multiple-object models causes a bottleneck in research so that only single-agent cases are being considered in such models (cf. Daskalakis et al. [9]). Gopalan et al. [12] cast a doubt on the possibility of Border-like solutions outside the paramodularity framework, claiming that such solutions exist only if a particular computational problem that is “widely believed” to be hard turns out to be easy.

This paper addresses the issue with a necessary and sufficient condition for Border-like solutions to hold with or without the paramodularity assumption (Theorem 1). The condition brings about both negative and positive news. On the negative side, I prove impossibility of Border-like solutions in the knapsack model, where a resource can be shared among multiple players provided that there is room for them to fit in subject to the integer constraint

that each player is either all in or all out (Proposition 2). On the positive side, I prove that the Border-like solution obtains in ranked-item matchings despite a group-specific quota constraint (Proposition 3), and in a matching model between N objects and two bidders such that each bidder can have any finite number of types, each possibly multidimensional (Proposition 4). Proposition 2 is the first impossibility observation for a general model in the literature.¹ Proposition 3 differs from the results in the matching literature recently unified by the majorization method of Kleiner et al. [18] in allowing for asymmetric mechanisms, driven by the group-specific quota, whereas the majorization method is based on symmetric models. Proposition 4 is the first Border-like characterization outside the mainstream, paramodularity framework without restricting each player to at most two types.

These findings are relevant to several active research areas. Knapsack problems are deemed applicable to numerous industrial sectors including transportation, logistics, telecommunication, etc. (Fifi and M’Hallah’s [16]). Proposition 2’s impossibility observation about exact characterizations makes it compelling to adopt approximation methods, such as what is recently proposed by Akbarpour et al. [1], for the mechanism design of knapsack problems. The group-specific quota constraint in Proposition 3 requires that at least a certain number of the top-quality items be allocated to the members of a certain group. Incorporating such group-specific preferential treatments in a matching model, the proposition provides a building block for future mechanism-design studies on the group-based policies in labor markets and college admissions. Much of the matching literature beyond the aforementioned ranked-item model is confined to the approach of finding a mechanism satisfying some properties that are assumed good rather than designing an optimal mechanism subject to constraints. According to Budish [5], part of the reason for this restriction is the lack of tools to characterize the constraints in matching problems. Proposition 4 is a step toward the direction of bringing the mechanism design approach into the matching literature.

In addition to the above implications, Theorem 1 also uncovers a deep connection between Border-like solutions and the greedy algorithm in combinatorial optimization. A well-known economic example of greedy algorithms is Myerson’s [25] optimal auction: Score every bidder-type by its virtual utility; for every profile of realized types across bidders, rank the bidders by their scores that the bidders with their associated realized types have; then

¹Che et al. [7] have noted impossibility of Border-like characterizations for three examples based on unpublished computations.

one-by-one in descending order of the rank, allocate to each bidder the maximum expected quantity down to either the last bit of resource or the last bidder who has a positive score. Border [3] uses a similar greedy algorithm (with different scoring rules) to obtain his solution. While the role of greedy algorithms is not explicit in Che et al. [7], they rely on Hassin’s [15] theorem of network flows based on the paramodularity assumption, which Hassin relates to greedy algorithms explicitly. Thus, there might be a sense that Border-like solutions are somehow related to greedy algorithms, but the logical connection is unclear. Moreover, outside the paramodularity framework, it is known that greedy solutions need not be optimal, and hence the role that greedy algorithms could play in Border-like solutions is unknown.

Theorem 1 fills the void with a “local greediness” implication. The validity of Border-like solutions requires that every boundary point of the set of feasible interim allocations be locally greedy: that the domain of all interim allocations be covered by a family of restricted domains within each of which the underlying ex post allocation for the boundary point acts like a greedy solution with respect to some ranking among the elements within (Proposition 1). The logical connection between paramodularity and Border-like solutions is that paramodularity guarantees the existence of a total ranking on the domain of all interim allocations that validates a single greedy algorithm globally. Without paramodularity, such a total ranking need not exist, yet there is a family of partial orderings each of which rationalizes the boundary point as a greedy solution locally if Border-like solutions are valid. In addition to contributing an economic insight to a mostly mathematical topic, the local greediness implication is instrumental in the search for the construction that validates the Border-like solutions in the aforementioned applications.

The literature of characterizing feasible interim allocations is initiated by Border [3] for a single-unit symmetric auction model, significantly extended by Border [4], Manelli and Vincent [21], Mierendorff [24], Goeree and Kushnir [11],² and further generalized to multiunit models by Che et al. [7] and Cai et al. [6]. These works except Cai et al. [6] are all within the paramodularity framework mentioned before. As pointed out by Che et al. [7, Supplemental Appendix E.1], the (non-Border-like) characterization provided by Cai et al. [6], though without the paramodularity assumption, requires a continuum of constraints (despite finite type space) to test an interim allocation. A recent discovery in the literature is the equivalence between Border-like solutions and a majorization condition within a framework

²Goeree and Kushnir also consider social choice models.

that assumes paramodularity and restricts attention to symmetric monotone mechanisms, such as Hart and Reny [14], Kleiner et al. [18] and Kolesnikov et al. [19]. Different from the literature, Vohra et al. [28] provide a non-Border-like characterization for a Bayesian persuasion problem through a direct meticulous calculation.

Contemporarily, Lang and Yang [20] provide a Border-like characterization in a total unimodularity model, which is outside the paramodularity framework and applies generally to matching models, though it is unknown whether the model applies to cases where a player has more than two types. My Proposition 4 complements their work with a Border-like characterization in a specific matching model that allows each player to have any number of types. Furthermore, I show that Theorem 1 leads to a Border-like characterization in a “decomposability” model that theirs belongs to (Proposition 5).

Recently, Valenzuela-Stookey [27] proposes a novel approach to approximate the set of feasible interim allocations with a superset and a subset related to greedy solutions. My impossibility Proposition 2 supports his approach by showing that there is a model where exact characterizations are impossible. The local greediness implication of my main result also complements the overall relevance of greedy algorithms.

The next section defines the primitives and motivates the Border-like characterization for the general audience. Section 3 presents the necessary and sufficient condition for the validity of the Border-like characterization and the local greediness implication of the necessary condition. Section 4 presents the three applications described above. Appendix A contains an application to a total unimodularity model and the extension of part of the paper to infinite or continuum type spaces. Appendix B contains all the proofs.

2 The Characterization Problem

2.1 The Primitives

Let I be the set of players, and J the set of objects, both assumed finite. For any $i \in I$, let T_i be the set of the types of player i . Let $T := \prod_{i \in I} T_i$, and write its generic element in the form of $t := (t_i)_{i \in I}$, briefly called *type profile*, representing the profile of realized types across all players, t_i being the one for player i . Let μ be the probability measure on T that has T as the support, and let $\mu(t) := \mu(\{t\})$ for any singleton $\{t\}$. For each $i \in I$, let μ_i

be the marginal measure of μ onto T_i ; let $T_{-i} := \prod_{k \in I \setminus \{i\}} T_k$ and let $\mu_{-i}(\cdot|t_i)$ denote the conditional measure on T_{-i} according to μ conditional on t_i . Assume that T is a finite set.³

Let \mathcal{R} denote either the set \mathbb{R} of real numbers or a discrete subset of \mathbb{R} (e.g., \mathbb{Z} of integers), and let \mathcal{R}_+ be the set of nonnegative elements of \mathcal{R} . Let $X \subseteq \mathcal{R}^I$ be the set of allocation outcomes that are feasible.⁴ An element of X is in the form of $x := ((x_i^j)_{i \in I})_{j \in J}$ such that x_i^j is the quantity of object j allocated to player i . (Thus, quantities are divisible if $\mathcal{R} = \mathbb{R}$, and indivisible if \mathcal{R} is discrete.) Let $\text{cv}(X)$ denote the convex hull of X . Assume that X is nonempty and compact in $\mathbb{R}^{I \times J}$.

As in any standard mechanism design model, by the revelation principle the designer can restrict attention to direct revelation mechanisms, the chief component of which are ex post allocations. An *ex post allocation* is a profile $((q_i^j)_{i \in I})_{j \in J}$ of functions $q_i^j : T \rightarrow \mathbb{R}$ ($\forall (i, j) \in I \times J$) such that $((q_i^j(t))_{i \in I})_{j \in J} \in \text{cv}(X)$ for any $t \in T$. Thus, an ex post allocation can randomize on the feasible allocation outcomes, and its range can be a non-discrete subset of $\mathbb{R}^{I \times J}$ even when $X \subseteq \mathbb{Z}^{I \times J}$.

Note that an ex post allocation is a function of the profile of types across all players, while an individual player knows only his own type during the interim stage where he makes decisions. From the perspective of player i of type t_i , the ex post allocation q reduces to a profile of interim probabilities

$$Q_i^j(t_i) = \sum_{t_{-i} \in T_{-i}} q_i^j(t_i, t_{-i}) \mu_{-i}(t_{-i}|t_i) \quad (1)$$

across objects $j \in J$. In general, an *interim allocation* is any profile $Q := ((Q_i^j)_{i \in I})_{j \in J}$ of functions $Q_i^j : T_i \rightarrow \mathbb{R}$ ($\forall (i, j) \in I \times J$).

Since $(Q_i^j)_{j \in J}$ is the chief instrument to incentivize player i , designing a mechanism often amounts to designing an interim allocation. For an interim allocation to be a feasible choice variable, it needs to satisfy (1) always. In general, an interim allocation $Q := ((Q_i^j)_{i \in I})_{j \in J}$ is said to be *feasible*,⁵ or the *reduced form* of some ex post allocation $q := ((q_i^j)_{i \in I})_{j \in J}$, if and only if Eq. (1) holds for all $(i, j) \in I \times J$ and all $t_i \in T_i$ (q referred to as the *underlying* ex post allocation for Q). Let \mathcal{Q} denote the set of feasible interim allocations.

³Appendix A.2 extends some results in this paper to infinite or continuum type spaces.

⁴Much of the paper can be extended trivially to the case where the set X depends on the type profile t .

⁵Often called “implementable” instead of “feasible” in the literature.

2.2 Test Sets

Characterizing feasible interim allocations is to provide a necessity and sufficiency test for $((Q_i^j)_{i \in I})_{j \in J} \in \mathcal{Q}$ by some condition about the functions Q_i^j . Since Q_i^j is a function defined only on T_i , any such a condition should be about the behavior of Q_i^j on various subsets of T_i , with (i, j) ranging in $I \times J$. That is, the test should be a condition of $((Q_i^j)_{i \in I})_{j \in J}$ with respect to various profiles $((S_i^j)_{i \in I})_{j \in J}$ of sets across player-object pairs such that $S_i^j \subseteq T_i$ for every $(i, j) \in I \times J$.

It is often convenient to “unpack” the sets S_i^j in any such a profile $((S_i^j)_{i \in I})_{j \in J}$ and put the elements into the same “bag.” That is, for every such a profile $((S_i^j)_{i \in I})_{j \in J}$, let

$$S := \{(j, i, t_i) \mid (i, j) \in I \times J; t_i \in S_i^j\}. \quad (2)$$

The intuition of this unpacking operation is illustrated below for a case with two players ($I = \{1, 2\}$) and two objects ($J = \{a, b\}$):

$$\begin{array}{ccccccc} S_1^a = \{t_1, t'_1\} & S_1^b = \{t_1, t''_1, t'''_1\} & S_2^a = \{t_2, t'_2\} & S_2^b = \emptyset \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \{(a, 1, t_1), (a, 1, t'_1)\} & \{(b, 1, t_1), (b, 1, t''_1), (b, 1, t'''_1)\} & \{(a, 2, t_2), (a, 2, t'_2)\} & \emptyset \\ \downarrow & & & \\ S = \{(a, 1, t_1), (a, 1, t'_1), (b, 1, t_1), (b, 1, t''_1), (b, 1, t'''_1), (a, 2, t_2), (a, 2, t'_2)\}. \end{array}$$

Obviously the unpacking operation is reversible, so that any set S containing some object-player-type triples corresponds to a unique profile $((S_i^j)_{i \in I})_{j \in J}$. Thus, I will treat the profile $((S_i^j)_{i \in I})_{j \in J}$ and the mixed bag S interchangeably and will refer to both as a *test set*.

Note that any interim allocation $((Q_i^j)_{i \in I})_{j \in J}$ is just a function of object-player-type triples, with $(j, i, t_i) \mapsto Q_i^j(t_i)$ for all $(i, j) \in I \times J$ and all $t_i \in T_i$. Thus,

$$\mathcal{Z} := \bigcup_{(i,j) \in I \times J} (\{(j, i)\} \times T_i)$$

is the domain of interim allocations. A test set is simply a subset of the domain \mathcal{Z} .

2.3 Heuristic Derivation of Border-Like Characterizations

How should the feasibility condition for an interim allocation $((Q_i^j)_{i \in I})_{j \in J}$ look like with respect to a test set S ? Let us start with the simplest case where S is singleton, say

$S = \{(a, 1, \tau_1)\}$ for some object a , player 1 and his type τ_1 . Suppose that $((Q_i^j)_{i \in I})_{j \in J}$ is the reduced form of some ex post allocation $((q_i^j)_{i \in I})_{j \in J}$. The allocation outcome $((q_i^j(t))_{i \in I})_{j \in J}$ is a random vector determined by the stochastic type profile $t := (t_i)_{i \in I}$. From the perspective of $(a, 1, \tau_1)$, namely, the pairing of object a with player 1 of type τ_1 , the random vector registers a projection $q_1^a(\tau_1, t_{-1})$ if the realized type profile $(t_i)_{i \in I}$ happens to have $t_1 = \tau_1$, and it registers nothing if $t_1 \neq \tau_1$. When the type profile t ranges randomly in T , the projection (or lack thereof) that $((q_i^j((t_i)_{i \in I}))_{i \in I})_{j \in J}$ registers in the dimension of $(a, 1, \tau_1)$ is equal to $Q_1^a(\tau_1)$ in expectation due to (1). Meanwhile, since the random vector is bounded within the convex hull $\text{cv}(X)$ of the set X of feasible allocation outcomes, all these projections in the dimension of $(a, 1, \tau_1)$ are bounded between $\min_{x \in X} x_1^a$ and $\max_{x \in X} x_1^a$. Since the random vector registers a projection on $(a, 1, \tau_1)$ iff $t_1 = \tau_1$, it follows that

$$\sum_{t \in T} \left(\min_{x \in X} x_1^a \right) \chi_{\{\tau_1\}}(t_1) \mu(t) \leq Q_1^a(\tau_1) \leq \sum_{t \in T} \left(\max_{x \in X} x_1^a \right) \chi_{\{\tau_1\}}(t_1) \mu(t),$$

where χ_A denotes the characteristic function of set A .

Now consider the case where S has two elements, say $S = \{(a, 1, \tau_1), (b, 2, \tau_2)\}$. As in the singleton case, we are to collect the projections that the random allocation outcome $((q_i^j((t_i)_{i \in I}))_{i \in I})_{j \in J}$ registers on $(a, 1, \tau_1)$ or $(b, 2, \tau_2)$. If we add all such projections up and divide the sum by the total number of all instances of the type profile, then in average we should get $Q_1^a(\tau_1) + Q_2^b(\tau_2)$. Meanwhile, at every instance of the type profile $(t_i)_{i \in I}$, the random vector $((q_i^j((t_i)_{i \in I}))_{i \in I})_{j \in J}$ registers either (i) a single projection $q_1^a(\tau_1, t_{-1})$ when $t_1 = \tau_1$ and $t_2 \neq \tau_2$, or (ii) a single projection $q_2^b(\tau_2, t_{-2})$ when $t_1 \neq \tau_1$ and $t_2 = \tau_2$, or (iii) two projections simultaneously, $q_1^a(\tau_1, t_{-1}) + q_2^b(\tau_2, t_{-2})$, when $t_1 = \tau_1$ and $t_2 = \tau_2$, else (iv) makes no projection at all. Since the random vector is bounded within $\text{cv}(X)$, the projection in Case (i), as in the previous singleton S case, is bounded within $[\min_{x \in X} x_1^a, \max_{x \in X} x_1^a]$. Similarly, the projection in Case (ii) is in $[\min_{x \in X} x_2^b, \max_{x \in X} x_2^b]$. In Case (iii), we get the sum of the two projections that register simultaneously, and so analogously the sum $q_1^a(\tau_1, t_{-1}) + q_2^b(\tau_2, t_{-2})$ is within $[\min_{x \in X} (x_1^a + x_2^b), \max_{x \in X} (x_1^a + x_2^b)]$. Consequently, $Q_1^a(\tau_1) + Q_2^b(\tau_2)$ is bounded between the expected values of these upper and lower bounds.

It would be cumbersome to write down the expected values of the upper and lower bounds explicitly, which would require collecting all the instances of the cases listed above, each corresponding to a particular expression of the upper or lower bound. For succinct expressions of the expected values, I adopt a notation from Che et al [7]: For any test set

S	$(a, 1, t_1)$	$(a, 1, t'_1)$	$(b, 1, t_1)$	$(b, 1, t''_1)$	$(b, 1, t'''_1)$	$(a, 2, t_2)$	$(a, 2, t'_2)$
t_1	$(a, 1, t_1)$		$(b, 1, t_1)$				
t_2						$(a, 2, t_2)$	
	$(a, 1)$		$(b, 1)$			$(a, 2)$	
$I(S, (t_1, t_2))$	$(1, a)$		$(1, b)$			$(2, a)$	

Table 1: type profile (t_1, t_2) selects those $(j, i, \tau_i) \in S$ for which $\tau_i = t_i$

$S \subseteq \mathcal{X}$ and any type profile $t := (t_i)_{i \in I} \in T$, define

$$I(S, t) := \{(i, j) \in I \times J \mid (j, i, t_i) \in S\}. \quad (3)$$

The elements of $I(S, t)$ correspond to those in S that simultaneously receive projections from the random allocation outcome $((q_i^j(t))_{i \in I})_{j \in J}$ at one single instance where the type profile is t . For example, in the two-element S described previously, $I(S, t)$ is $\{(1, a)\}$ in Case (i), $\{(2, b)\}$ in Case (ii), $\{(1, a), (2, b)\}$ in Case (iii), and empty in Case (iv). Put differently, a type profile t filters out the elements (j, i, t'_i) of S whose types t'_i do not match t , thereby producing the set $I(S, t)$ of relevant player-object pairs under t . Table 1 illustrates that with a two-player two-object example.

From the standpoint of $I(S, t)$, the upper and lower bounds motivated previously are defined for any test set $S \subseteq \mathcal{X}$ and any type profile $t \in T$:

$$f(S, t) := \max_{x \in X} \sum_{(i, j) \in I(S, t)} x_i^j \quad (4)$$

$$g(S, t) := \min_{x \in X} \sum_{(i, j) \in I(S, t)} x_i^j. \quad (5)$$

For example, in the two-element case of $S = \{(a, 1, \tau_1), (b, 2, \tau_2)\}$, the lower bound in all cases can all be expressed as $g(S, t)$, and the upper bound in all cases, $f(S, t)$, with t ranging in T randomly. Then, as motivated previously, it is conceivable that a feasible interim allocation $((Q_i^j)_{i \in I})_{j \in J}$ should satisfy

$$\sum_{t \in T} g(S, t) \mu(t) \leq \sum_{(j, i, t_i) \in S} Q_i^j(t_i) \mu_i(t_i) \leq \sum_{t \in T} f(S, t) \mu(t) \quad (6)$$

for every test set $S \subseteq \mathcal{X}$.

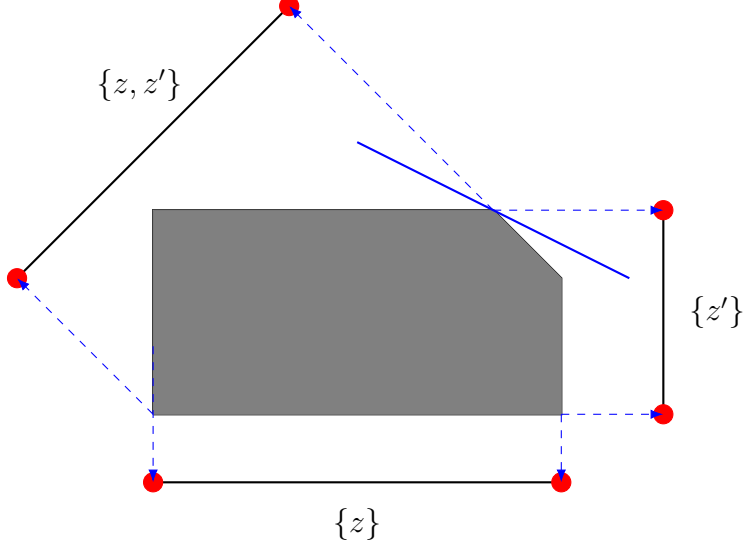


Figure 1: The projections with respect to test sets $\{z\}$, $\{z'\}$ and $\{z, z'\}$

The intuition is roughly conveyed by Figure 1, where the beings confined to the flatland see the shaded region as merely various segments from various perspectives. Their perspectives correspond to our test sets. And their attempt to characterize the shaded region is to try defining the region by the red dots that define the various segments. Each segment corresponds to an instance of (6), based on a particular test set S , with the lower and upper bounds of the inequality being the two endpoints of the segment.

The intuition explained above is substantially distinct from a perhaps better-known idea of support functions, which is to characterize the shaded region by its supporting hyperplanes such as the blue solid line in Figure 1. The problem of the latter idea, albeit natural, is that the family of supporting hyperplanes is often too large, and almost all its members may turn out to be redundant. For example, the said blue line is a convex combination between, and hence can be replaced by, the two edges of the region that the blue line intersects. According to the support function idea, every instance of (6) would be based on a linear combination of test sets rather than a single test set. Even if we restrict attention to a fixed collection of test sets, there is still a continuum of linear combinations among them. In fact, in almost every endeavor to characterize feasible interim allocations, the support function idea is the starting point, while the characterization by (6) is the final outcome, which corresponds to only a basis of the supporting hyperplanes.

We have thus come across a necessary condition for any interim allocation $((Q_i^j)_{i \in I})_{j \in J}$

to be feasible: that Ineq. (6) holds for all test sets $S \subseteq \mathcal{X}$. This condition specializes to Border's [3] condition in the single-unit symmetric model, where $J = \{1\}$, $T_i = T_1$, $\mu_i = \mu_1$ and $Q_i^j = Q_1^1$ for all $i \in I$, and $X = \{(x_i)_{i \in I} \in [0, 1]^I \mid \sum_{i \in I} x_i \leq 1\}$ for all $t \in T$: Apply (4) and (5) to this X to see that for any type profile $t \in T$ and any test set $S \subseteq \mathcal{X}$, or equivalently $(S_i)_{i \in I}$ with $S_i \subseteq T_i$ ($\forall i$), that $g(S, t) = 0$ and

$$f(S, t) = \chi_{I(S, t) \neq \emptyset} = 1 - \prod_{i \in I} \chi_{\neg S_i}(t_i).$$

Thus the right-hand side of (6) is equal to $1 - \prod_{i \in I} (1 - \mu_i(S_i))$, and the left-hand side, zero. This coupled with the symmetry assumption reduces (6) to the well-known condition

$$0 \leq \sum_{t_1 \in T_1} Q_1^1(t_1) \chi_{S_1}(t_1) \mu_1(t_1) \leq \frac{1}{|I|} \left(1 - (1 - \mu_1(S_1))^{|I|}\right)$$

for all $S_1 \subseteq T_1$ ($|A|$ denotes the cardinality of set A). This condition in turn is known to be equivalent to Kleiner et al.'s [18] majorization condition in the symmetric unidimensional-type model ($T_i = T_1 \subseteq \mathbb{R}$ and $Q_i^j = Q_1^j$ is monotone for all $(i, j) \in I \times J$). Condition (6) also specializes to Che et al.'s [7] condition in the multiunit homogeneous object model ($J = \{1\}$).

Let us denote the set of interim allocations $((Q_i^j)_{i \in I})_{j \in J}$ that satisfy (6) for all $S \in \mathcal{X}$ by \mathcal{Q}_B , B for Border. As is apparent from the above, it is easy to prove that the set \mathcal{Q} of feasible interim allocations is a subset of \mathcal{Q}_B , namely, that satisfaction of (6) is necessary for $((Q_i^j)_{i \in I})_{j \in J} \in \mathcal{Q}$ (Appendix B.1). The challenge is to figure out when $\mathcal{Q} \supseteq \mathcal{Q}_B$ is also true: When is (6) also sufficient for $((Q_i^j)_{i \in I})_{j \in J} \in \mathcal{Q}$?

3 The Locally Greedy Characterization

The following theorem provides a necessity and sufficiency test for $\mathcal{Q} \supseteq \mathcal{Q}_B$.

Theorem 1 *The following three statements are equivalent:*

- a. $\mathcal{Q} \supseteq \mathcal{Q}_B$.
- b. For any $w := (w(j, i, t_i))_{(j, i, t_i) \in \mathcal{X}} \in \mathbb{R}^{\mathcal{X}}$ there exist a function $(p_+, p_-) : 2^{\mathcal{X}} \rightarrow \mathbb{R}_+^2$ and an ex post allocation $((\bar{q}_i^j)_{i \in I})_{j \in J}$ such that $((\bar{q}_i^j(t))_{i \in I})_{j \in J}$ is an optimum of

$$\max_{(x_i^j)_{(i, j) \in I \times J} \in \text{cv}(X)} \sum_{(i, j) \in I \times J} x_i^j w(j, i, t_i) \quad (7)$$

for all $t := (t_i)_{i \in I} \in T$,

$$w(z) = \sum_{S \subseteq \mathcal{Z}} p_+(S) \chi_S(z) - \sum_{S \subseteq \mathcal{Z}} p_-(S) \chi_S(z) \quad (8)$$

for all $z \in \mathcal{Z}$, and

$$\begin{aligned} p_+(S) > 0 &\Rightarrow \forall t \in T \left[f(S, t) = \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i) \right] \\ p_-(S) > 0 &\Rightarrow \forall t \in T \left[g(S, t) = \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i) \right] \end{aligned} \quad (9)$$

for all $S \subseteq \mathcal{Z}$.

c. For any $w \in \mathbb{R}^{\mathcal{Z}}$ and any \bar{q} such that $\bar{q}(t)$ solves (7) for all $t \in T$ there exists $(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2$ that satisfies (8) for all $z \in \mathcal{Z}$ and (9) for all $S \subseteq \mathcal{Z}$.

To understand what the theorem says, let us recall from Section 2.3 that characterizing the set \mathcal{Q} of feasible interim allocations amounts to finding a family of unidimensional subspaces—each corresponding to a test set—and then characterizing the set \mathcal{Q} by the endpoints of its projections onto these subspaces. As illustrated by the Figure 1 there, the task is analogous to a flatlander’s attempt to characterize a higher-dimension shaded region from their unidimensional perspectives: to try defining the region by the small red dots that define the various segments, each being a projection of the shaded region.

If the task is achievable, it must be true that for any boundary point of the shaded region such as the big red dot in Figure 2, its projections onto some unidimensional subspaces are also boundary points of the corresponding segments, such as the small red dots in Figure 2. That is, the binding constraints at any original boundary point (the two thick blue lines in Figure 2) can be replaced by the binding constraints conditional on some subspaces (the small red dots in Figure 2).

That is the first layer of meaning in the theorem: For any interim allocation on the boundary of the feasibility set \mathcal{Q} , there is a test set S with respect to which the shadow price for either the ceiling or the floor in (6) is positive, with the shadow price for the ceiling constraint denoted by $p_+(S)$, and that for the floor constraint, $p_-(S)$. These binding constraints correspond to the small red dots in Figure 2. The vector w in the theorem is the gradient of a supporting hyperplane for the feasibility set at the said boundary interim allocation. Eq. (8) is the relation between the gradient w and the shadow prices.

However, there is a second, deeper layer of meaning in the theorem. From Figure 2 it might appear as though the existence of the small red dots were guaranteed. What the figure

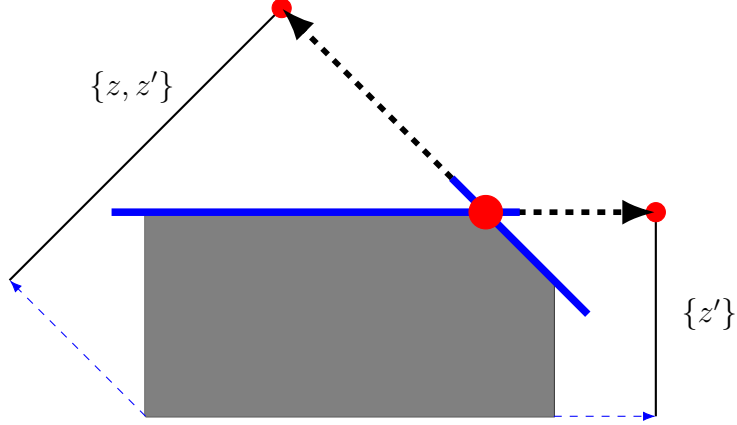


Figure 2: An extreme point of the shaded region and its projections

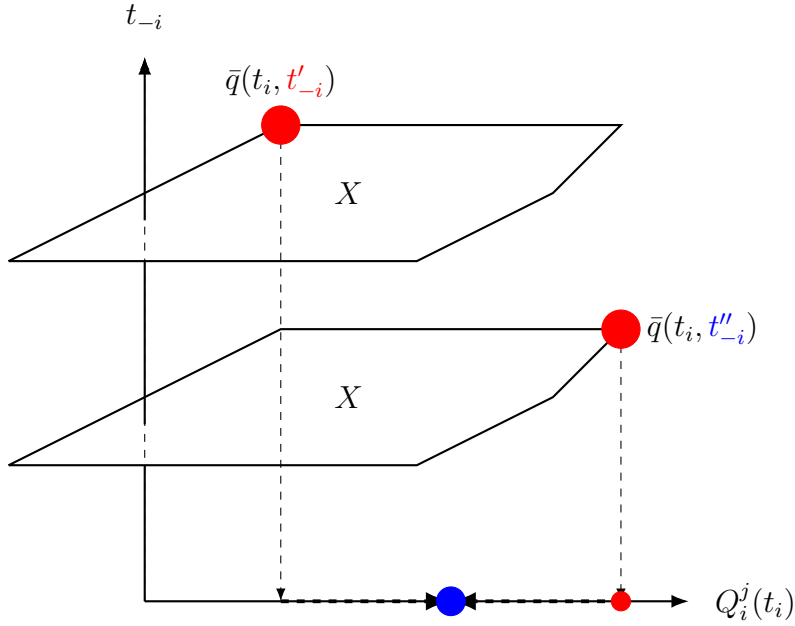


Figure 3: The projection fails to preserve the extremity of $((\bar{q}_i^j)_{i \in I})_{j \in J}$

misses to convey is the fact that an ex post allocation is stochastic, varying with the random type profile. That is, the location of the underlying big red dot is stochastic, as illustrated in the next Figure 3 with two type profiles, (t_i, t'_{-i}) and (t_i, t''_{-i}) . Then the expected value of its location projected onto a subspace may be an interior point such as the blue dot in Figure 3, rather than a boundary point of the projection such as the small red dot there. That is, this subspace fails to provide a binding constraint when a constraint in the higher

dimension is binding. Thus, to guarantee that the projection of the big red dot is a small red dot, we need to choose a certain kind of subspaces onto which the projection preserves the extremity of the big red dot. For a subspace to qualify this, the location of the big red dot needs to either always maximize, or always minimize, the quantity in the dimension that represents the subspace. Then the expected value of the projection of the big red dot onto that dimension is a boundary point of the feasible set within the subspace.

That is essentially what condition (9) in the theorem is saying. For a test set S to support the shadow price of the ceiling constraint in (6), the underlying ex post allocation \bar{q} for any boundary point of \mathcal{Q} needs to maximize the total quantity that can be feasibly assigned to the elements of S for every type profile that is consistent with some element of S (e.g., every type profile (t_i, t_{-i}) for all $t_{-i} \in T_{-i}$ if $S = \{(j, i, t_i)\}$). That is, $p_+(S) > 0$ only if S is *upward binding* for \bar{q} : $f(S, t) = \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i)$ for all type profiles $t \in T$. Symmetrically, for S to support the shadow price of the floor constraint in (6), \bar{q} needs to minimize the total quantity that can be feasibly assigned to the elements of S for every type profile that is consistent with some member of S . That is, $p_-(S) > 0$ only if the test set S is *downward binding* for \bar{q} : $g(S, t) = \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i)$ for all type profiles $t \in T$.

Thus, the theorem says that the characterization (6) of feasible interim allocations is valid if and only if for any interim allocation that is about to violate some feasibility constraint, the gradient w of its supporting hyperplane can be decomposed into a set of shadow prices $p_+(S)$ and $p_-(S)$ of (6) conditional on some test sets S each of which is either upward or downward binding for the underlying ex post allocation.

3.1 Greedy Solutions: The Easy Case

For any nonempty test set $S \subseteq \mathcal{Z}$ and any preference relation \succeq on S , arrange all the upper contour sets wrt (with respect to) \succeq within S into a nested sequence $(U^k)_{k=1}^m$ for some natural number m : $U^k = \{z \in S \mid z \succeq z'\}$ for some $z' \in S$ and

$$\emptyset =: U^0 \subsetneq U^1 \subsetneq U^2 \subsetneq \dots \subsetneq U^m := S. \quad (10)$$

Symmetrically, arrange all the lower contour sets wrt \succeq within S into a nested sequence $(L^k)_{k=1}^m$ for some m so that $L^k = \{z \in S \mid z' \succeq z\}$ for some $z' \in S$, and

$$\emptyset =: L^0 \subsetneq L^1 \subsetneq L^2 \subsetneq \dots \subsetneq L^m := S. \quad (11)$$

For any type profile $t \in T$, a vector $y := (y_z)_{z \in \mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}}$ is said to be a *greedy solution wrt \succeq within S given t* iff

$$\sum_{z \in U^n \setminus U^{n-1}} y_z = f(U^n, t) - f(U^{n-1}, t) \quad (12)$$

for all $n \in \{1, \dots, m\}$, and said to be a *generous solution wrt \succeq within S given t* iff

$$\sum_{z \in L^n \setminus L^{n-1}} y_z = g(L^n, t) - g(L^{n-1}, t) \quad (13)$$

for all $n \in \{1, \dots, m\}$.

I formulate the above definitions based on the fact that (12) is satisfied by any solution produced by the greedy algorithm applied to the problem

$$\max_{(y_z)_{z \in S} \in \mathbb{R}^S} \sum_{z \in S} y_z \alpha(z) \quad \text{s.t.} \quad \forall S' \subseteq S : \sum_{z \in S'} y_z \leq f(S', t)$$

for any nonnegative utility function $\alpha : S \rightarrow \mathbb{R}_+$ of \succeq , and (12) defines the unique greedy solution if \succeq is strict (no indifference between distinct members) on S , though whether the solution is optimal (or even feasible) for the above problem depends on the property of f (as well as the property of the choice set if it is only a proper subset of \mathbb{R}^S). By the same token as well as the same caveat, (13) is satisfied by any solution produced by the generous algorithm applied to

$$\max_{(y_z)_{z \in S} \in \mathbb{R}^S} \sum_{z \in S} y_z \beta(z) \quad \text{s.t.} \quad \forall S' \subseteq S : \sum_{z \in S'} y_z \geq g(S', t)$$

for any *nonpositive* utility function $\beta : S \rightarrow -\mathbb{R}_+$ of \succeq , and (13) defines the unique generous solution if \succeq is strict on S .⁶

The (Border-like) characterization $\mathcal{Q} \supseteq \mathcal{Q}_B$ can be proved easily if the underlying ex post allocation for every boundary point of \mathcal{Q} is a greedy or generous solution on the entire domain \mathcal{Z} of interim allocations. More precisely, for any ex post allocation $q := ((q_i^j)_{i \in I})_{j \in J}$, any type profile $t := (t_i)_{i \in I} \in T$ and any $(j, k, t'_k) \in \mathcal{Z}$, define

$$q(j, k, t'_k \mid t) := \begin{cases} q_k^j(t_k, t_{-k}) & \text{if } t'_k = t_k \\ 0 & \text{else.} \end{cases} \quad (14)$$

⁶Schrijver [26, Ch. 40] has extensive coverage of greedy algorithms. The hierarchical allocation familiar in the optimal auction literature is a special case thereof, with the said preference relation being the ranking of virtual utilities. See Hassin [15] for the term generous solutions, the mirror image of greedy solutions.

Then for any type profile $t \in T$ an ex post allocation q induces an element $q(\cdot|t)$ of $\mathbb{R}^{\mathcal{Z}}$. Through this definition, the ex post allocation \bar{q} that solves (7) in Theorem 1 becomes a stochastic choice among object-player-type triples (elements of \mathcal{Z}), stochastic in the sense that the choice may depend on the random type profile t . In the special case where this dependency is null, the choice $\bar{q}(\cdot|t)$ coincides with the solution produced by the greedy-generous algorithm that follows a ranking on \mathcal{Z} independent of t . The next lemma highlights this easy case.

Lemma 1 $\mathcal{Q} \supseteq \mathcal{Q}_B$ if for any $w \in \mathbb{R}^{\mathcal{Z}}$ there exists an ex post allocation \bar{q} such that $\bar{q}(t)$ solves (7) for all $t \in T$ and, with \succeq being the preference relation on \mathcal{Z} represented by $z \mapsto w(z)$, $\bar{q}(\cdot|t)$ is a greedy solution wrt \succeq within $U := \{z \in \mathcal{Z} \mid w(z) \geq 0\}$, and a generous solution wrt \succeq within $\mathcal{Z} \setminus U$, for each type profile $t \in T$.

The hypothesis in this lemma implies existence of the shadow price function (p_+, p_-) that Theorem 1 requires for $\mathcal{Q} \supseteq \mathcal{Q}_B$. First, every upper contour set in U is upward binding for \bar{q} . That is because, according to the greedy algorithm, \bar{q} allocates to the top indifference class $U^1 (= U^1 \setminus U^0)$ in U the largest feasible quantity $f(U^1, t) (= f(U^1, t) - f(U^0, t)$ as $U^0 = \emptyset$) and then allocates to the second highest indifference class $U^2 \setminus U^1$ the largest quantity $f(U^2, t) - f(U^1, t)$ among what remains, and so forth. Within any upper contour set U^m , these quantities add up to be equal to the maximum quantity $f(U^m, t)$. By the same token, every lower contour set in $\mathcal{Z} \setminus U$ is downward binding for \bar{q} . Second, plug the upper contour sets in U into (8) for $p_+(S)$, and the lower ones in $\mathcal{Z} \setminus U$ into (8) for $p_-(S)$. The equation (8) for each $z \in \mathcal{Z}$ is either $w(z) = \sum_k p_+(U^k) \chi_{U^k}(z)$ or $w(z) = -\sum_k p_-(L^k) \chi_{L^k}(z)$. Since these contour sets are nested, a nonnegative solution for (p_+, p_-) can be easily obtained.

The major result in the mainstream framework becomes a corollary of Lemma 1. The framework is based on an assumption that the set X of feasible allocation outcomes is determined by a paramodular pair of ceiling and floor functions.

Recall that \mathcal{R} denotes either \mathbb{R} or a discrete subset of \mathbb{R} . A subset Y of $\mathcal{R}^{I \times J}$ is said to be *determined* by a pair (φ, γ) of functions $\varphi : \mathcal{F} \rightarrow \mathcal{R}_+$ and $\gamma : \mathcal{G} \rightarrow \mathcal{R}_+$ ($\mathcal{F}, \mathcal{G} \subseteq 2^{I \times J}$) iff Y is equal to the set of $((y_i^j)_{i \in I})_{j \in J} \in \mathbb{R}^{I \times J}$ (i.e., *any* vector in the euclidean space $\mathbb{R}^{I \times J}$) such that $\sum_{(i,j) \in E} y_i^j \leq \varphi(E)$ for all $E \in \mathcal{F}$, and $\sum_{(i,j) \in E} y_i^j \geq \gamma(E)$ for all $E \in \mathcal{G}$.

A pair (φ, γ) of nonnegative functions defined on $2^{I \times J}$ is said to be *paramodular* iff

(i) φ and $-\gamma$ are each *submodular* on $2^{I \times J}$, namely,

$$\begin{aligned}\varphi(E) - \varphi(E \cap E') &\geq \varphi(E \cup E') - \varphi(E') \\ \gamma(E) - \gamma(E \cap E') &\leq \gamma(E \cup E') - \gamma(E')\end{aligned}$$

for all $E, E' \subseteq I \times J$; and (ii) (φ, γ) is *compliant* on $2^{I \times J}$, namely, for all $E, E' \subseteq I \times J$,

$$\varphi(E') - \varphi(E' \setminus E) \geq \gamma(E) - \gamma(E \setminus E').$$

Paramodularity regulates the total quantities that can be allocated to various sets of slots in a way that causes no complication from various combinations among the slots. Under the assumption, when we include a new slot to any set of slots, or remove an element from the set, the only binding feasibility constraint is the ceiling or floor of the marginal quantity of this element with respect to the set. That validates the greedy-generous algorithm because the algorithm is precisely to fill each positively valued slot up to its ceiling marginal quantity with respect to the set of higher-valued slots, and unload each negatively valued slot down to its floor marginal quantity with respect to the set of lower-valued slots. Consequently, for any $w \in \mathbb{R}^{\mathcal{Z}}$, the greedy-generous algorithm that follows the ordinal ranking of the w -values maximizes the total w -value among all feasible allocation outcomes, namely, solves problem (7) for every type profile $t \in T$. Then the Border-like characterization in the mainstream framework follows from Lemma 1.⁷

Corollary 1 *If the set X of feasible allocation outcomes is determined by some paramodular pair (φ, γ) such that $\varphi, \gamma : 2^{I \times J} \rightarrow \mathcal{R}_+$, then $\mathcal{Q} \supseteq \mathcal{Q}_B$.*

In the assumption of this corollary, the part that requires X be determined by (φ, γ) , albeit not explicit in the literature, is indispensable. Without it, the greedy algorithm need not be optimal despite (φ, γ) being paramodular. Section 4.1 has an example.

3.2 Local Greediness: The General Case

Without the paramodularity assumption, it is known that neither greedy nor generous solutions are necessarily optimal solutions to their corresponding problems. For example, Table 2 lists

⁷Corollary 1 is a trivial extension of Che et al. [7, Theorem 3] by allowing for multiple heterogeneous objects. The corollary can be further extended (trivially) to the case where X , as well as the pair (φ, γ) that determines X , depends on the type profile t .

	$(1, t_1)$	$(2, t_2)$
a	1.05	1
b	0.8	0.7
c	0.5	0

Table 2: Suboptimality of the greedy solution

the values of w on a test set $\{(j, i, t_i) \mid i = 1, 2; j \in \{a, b, c\}\}$ in a matching model between three objects (a , b and c) and two players (1 and 2). The feasibility constraint in this model is that matchings between players and objects be one-to-one. The greedy algorithm starts with the triple $(a, 1, t_1)$ since its w -value 1.05 is the largest. Then it goes to the triple whose w -value is the largest among the rest, which is $(b, 1, t_1)$ with w -value 0.8. But it is infeasible to allocate any quantity to $(b, 1, t_1)$ because it demands object b for player 1 while player 1 has been matched with object a entirely at the first step. Thus the algorithm moves on to $(b, 2, t_2)$, whose w -value 0.7 is the next largest, and raises its quantity up to the full, feasible amount. The matching so produced is $(a \mapsto 1, b \mapsto 2)$, with total w -value 1.75.

By contrast, the optimal matching is to assign object a to player 2 and object b to player 1, producing a total w -value 1.8. The reason why the greedy algorithm fails is known in combinatorial optimization: The set of feasible matchings is not a matroid, while the validity of greedy algorithms relies on matroid structures.⁸ Put intuitively, among the three triples $(a, 1, t_1)$, $(a, 2, t_2)$ and $(b, 1, t_1)$, once the matching $(a \mapsto 1)$ is made then the other two triples $(a, 2, t_2)$ and $(b, 1, t_1)$ become infeasible and so we can extract only one match from the three. Whereas, if the matching $(b \mapsto 1)$ is made first, one of the other triples, $(a, 2, t_2)$, is still feasible, and so we can extract out of the three two matches, which combine to produce a larger total value. In other words, the greedy algorithm misses some combinations.

Thus, the underlying ex post allocation \bar{q} for any boundary point of the feasible set \mathcal{Q} in Theorem 1 is not necessarily a greedy or generous solution on the domain \mathcal{Z} of interim allocations. However, according to our theorem, if the characterization (6) is valid then \bar{q} does have a greedy or generous property locally, restricted to sufficiently small subsets of \mathcal{Z} : The validity of the characterization implies that there are test sets that are upward (resp. downward) binding for \bar{q} , namely, always getting a maximal (resp. minimal) quantity from \bar{q}

⁸See Schrijver [26, Vol. B] or Vohra [29] on matroid.

regardless of the random type profile t . Thus, the revealed preference of \bar{q} should favor the elements in an upward binding set against those outside if it has to choose between them. Symmetrically, between elements inside a downward binding set and those outside, \bar{q} should choose the latter over the former. While such revealed preferences of \bar{q} need not constitute a single preference relation on the entire domain \mathcal{Z} (otherwise we are back to the easy case), among those elements of \mathcal{Z} that are mutually exclusive (such as the $(a, 1, t_1)$ and $(b, 1, t_1)$ in Table 2) there should be a consistent preference relation for \bar{q} .

More precisely, for any nested sequence $\mathbb{S} := (S^k)_{k=1}^m$ of test sets (for some $m = 1, 2, \dots$)

$$\emptyset =: S^0 \subsetneq S^1 \subsetneq S^2 \subsetneq \dots \subsetneq S^m = \bigcup_{k=1}^m S^k =: \bigcup \mathbb{S}, \quad (15)$$

define a preference relation $\succeq_{\mathbb{S}}$ on $\bigcup \mathbb{S}$ by its strict part $\succ_{\mathbb{S}}$ and symmetric part $\sim_{\mathbb{S}}$:

$$\begin{aligned} z \succ_{\mathbb{S}} z' &\iff \left[k < k', z \in S^k, z' \in S^{k'} \setminus S^k \right] \\ z \sim_{\mathbb{S}} z' &\iff \exists k \left[z, z' \in S^k \setminus S^{k-1} \right]. \end{aligned}$$

Thus, every S^k is an upper contour set wrt $\succeq_{\mathbb{S}}$ within $\bigcup \mathbb{S}$. Analogously, define a preference relation $\succeq_{\mathbb{S}}$ on $\bigcup \mathbb{S}$ by the same symmetric part as the above $\sim_{\mathbb{S}}$ and a strict part $\triangleright_{\mathbb{S}}$ opposite to the above:

$$z \triangleright_{\mathbb{S}} z' \iff z' \succ_{\mathbb{S}} z.$$

Thus, every S^k is a *lower* contour set wrt $\succeq_{\mathbb{S}}$ within $\bigcup \mathbb{S}$.

Lemma 2 *For any ex post allocation q and any nested sequence $\mathbb{S} := (S^k)_{k=1}^m$ of test sets (for some $m = 1, 2, \dots$) in the form of (15), S^k is upward (resp. downward) binding for q for all $k \in \{1, \dots, m\}$ if and only if $q(\cdot|t)$ is a greedy (resp. generous) solution wrt $\succeq_{\mathbb{S}}$ (resp. $\triangleright_{\mathbb{S}}$) within $\bigcup \mathbb{S}$ for every type profile $t \in T$.*

Thus, although paramodularity is unavailable to validate a single greedy-generous solution on the entire \mathcal{Z} , the ex post allocation \bar{q} that we need to verify the characterization $\mathcal{Q} \supseteq \mathcal{Q}_B$ is nonetheless greedy or generous locally, though according to a ranking that may be different from that of the w -values aimed by the boundary point. Every member of \mathcal{Z} with a nonzero w -value is contained in some sufficiently small subset $\bigcup \mathbb{S}$ of \mathcal{Z} within which \bar{q} acts as if it were following a greedy or generous algorithm according to some ranking. Within every such a subset, the upper or lower contour sets with respect to this ranking are binding

test sets and hence are part of the support for the shadow price function. These upper or lower contour sets, across the various restricted domains within each of which \bar{q} behaves greedily or generously, constitute a covering of the entire domain \mathcal{Z} of interim allocations (except the elements that are assigned zero w -value). That is, the validity of the characterization requires that the domain \mathcal{Z} of interim allocations be covered by a family of subsets within each of which the boundary point is greedy or generous:⁹

Proposition 1 *If $\mathcal{Q} \supseteq \mathcal{Q}_B$ then, for any $w \in \mathbb{R}^{\mathcal{Z}}$ and any \bar{q} such that $\bar{q}(t)$ solves (7) for all $t \in T$, $\{z \in \mathcal{Z} \mid w(z) \neq 0\} \subseteq \bigcup_{S \in \mathcal{S}_+ \cup \mathcal{S}_-} S$ for some subsets \mathcal{S}_+ and \mathcal{S}_- of $2^{\mathcal{Z}}$ such that*

$$\begin{aligned}\mathcal{S}_+ &= \bigcup_{k \in \mathcal{K}_+} \{S^n \mid n = 1, \dots, m_k\} \\ \mathcal{S}_- &= \bigcup_{k \in \mathcal{K}_-} \{S^n \mid n = 1, \dots, m_k\}\end{aligned}$$

for some index sets \mathcal{K}_+ and \mathcal{K}_- such that for every $k \in \mathcal{K}_+$ (resp., every $k \in \mathcal{K}_-$), $m_k \in \{1, 2, \dots\}$, $(S^n)_{n=1}^{m_k}$ is an increasing nested sequence of subsets of \mathcal{Z} in the form of (15), and $\bar{q}(\cdot|t)$ is a greedy (resp., generous) solution wrt \succeq_S (resp., \supseteq_S) within $\bigcup S$.

3.3 How to Verify the Existence of the Shadow Prices

Part of the condition required in Theorem 1 to verify the characterization $\mathcal{Q} \supseteq \mathcal{Q}_B$ is that the linear system (8) has a nonnegative solution for the shadow price function (p_+, p_-) . In simple cases, this can be done through directly solving the equation system. For general cases, the following lemma provides a method based on the hyper-rectangle cover theory.

Let \mathcal{S}_+ be a collection of upward binding test sets, and \mathcal{S}_- a collection of downward binding test sets. Let \mathbf{M}_+ be a $|\mathcal{Z}|$ -by- $|\mathcal{S}_+|$ matrix whose rows are indexed by \mathcal{Z} and columns by \mathcal{S}_+ , and whose entry at the intersection between row z and column S is equal to $\chi_S(z)$ ($\forall z \in \mathcal{Z} \forall S \in \mathcal{S}_+$). Analogously, let \mathbf{M}_- be a $|\mathcal{Z}|$ -by- $|\mathcal{S}_-|$ matrix whose rows are

⁹If the sets $\bigcup S$ in the next Proposition 1 are all singleton, then its local greediness implication would have no bite. But that is a nonissue, because $\bigcup S$ by Lemma 2 is upward or downward binding for \bar{q} and one can easily prove that if a singleton $\{(j, i, t_i)\}$ is upward binding then $\bar{q}_i^j(t_i, t_{-i}) = \max_{x \in X} x_i^j$ for all type profiles $t_{-i} \in T_{-i}$ across the other players, and if $\{(j, i, t_i)\}$ is downward binding then $\bar{q}_i^j(t_i, t_{-i}) = \min_{x \in X} x_i^j$ for all $t_{-i} \in T_{-i}$. This occurs only for those $w \in \mathbb{R}^{\mathcal{Z}}$ given which the choice of \bar{q} is to either always favor the match $j \mapsto (i, t_i)$ between object j and player i of type t_i against all other matches, or always choose against the match. Thus, since the w in the proposition is arbitrary, the $\bigcup S$ cannot always be singleton.

indexed by \mathcal{Z} and columns by \mathcal{S}_- , and whose entry at the intersection between row z and column S is equal to $-\chi_S(z)$ ($\forall z \in \mathcal{Z} \forall S \in \mathcal{S}_-$). Thus $[\mathbf{M}_+, \mathbf{M}_-]$ is a matrix with $|\mathcal{Z}|$ rows and $|\mathcal{S}_+| + |\mathcal{S}_-|$ columns. Let \mathbf{p} denote the column vector

$$\mathbf{p} := [(p_+(S))_{S \in \mathcal{S}_+}, (p_-(S))_{S \in \mathcal{S}_-}]^\top,$$

and \mathbf{w} the column vector

$$\mathbf{w} := [(w(z))_{z \in \mathcal{Z}}]^\top.$$

Then (8) is equivalent to $[\mathbf{M}_+, \mathbf{M}_-] \mathbf{p} = \mathbf{w}$.

Lemma 3 *For any $w \in \mathbb{R}^{\mathcal{Z}}$, and any $\mathcal{S}_+, \mathcal{S}_- \subseteq 2^{\mathcal{Z}}$, let \mathbf{M}_+ and \mathbf{M}_- be the associated matrices defined above; if no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ can produce a nonnegative row whose entry at the $-\mathbf{w}$ position is (strictly) positive, then there exist $p_+ : \mathcal{S}_+ \rightarrow \mathbb{R}_+$ and $p_- : \mathcal{S}_- \rightarrow \mathbb{R}_+$ that satisfy (8).¹⁰*

To understand the intuition behind Lemma 3, consider two instances of (8) for some $z', z'' \in \mathcal{Z}$ such that $w(z') < w(z'')$. Subtract the instance of (8) when $z = z'$ by the instance of (8) when $z = z''$, so the left-hand side is a negative number. Then a contradiction occurs if the right-hand side is nonnegative, which occurs if $\chi_S(z') \geq \chi_S(z'')$ whenever $p_+(S) > 0$ ($S \in \mathcal{S}_+$), and $-\chi_S(z') \geq -\chi_S(z'')$ whenever $p_-(S) > 0$ ($S \in \mathcal{S}_-$). That is, the subtraction between the two instances of (8) produces a nonnegative row $[(\chi_S(z') - \chi_S(z''))_{S \in \mathcal{S}_+}, (-\chi_S(z') + \chi_S(z''))_{S \in \mathcal{S}_-}, -w(z') - (-w(z''))]$, with the last positive component signifying the contradictorily negative left-hand side $w(z') - w(z'')$. This contradictory case is ruled out by the hypothesis in the lemma, which also rules out any linear combination of similar subtractions that produces a contradiction to (8).

4 Applications

This section features three applications of Theorem 1. One is the impossibility of the Border-like characterization in the knapsack model, thanks to the necessity part of Theorem 1. The second is a positive result in a ranked-item auction model with a group-specific quota constraint. It still belongs to the easy case where a single greedy algorithm is valid globally,

¹⁰Extend (p_+, p_-) to $2^{\mathcal{Z}}$ trivially by setting $p_+(S) := 0$ and $p_-(S') := 0$ for all $S \notin \mathcal{S}_+$ and $S' \notin \mathcal{S}_-$.

although the paramodularity framework is not directly applicable. The third application is a matching model that departs from the paramodularity framework completely, with greedy algorithms valid only locally. The positive result has the novel feature that each player can have any number of types despite the lack of paramodularity. Appendix A applies Theorem 1 to another model outside the paramodularity framework, thanks to a main implication of total unimodularity. The appendix also extends part of the paper to infinite type spaces.

4.1 Impossibility in the Knapsack Model

Consider a knapsack model where a fixed capacity $C > 0$ is to be allocated among n players. For each $i \in I := \{1, \dots, n\}$, the capacity that player i occupies is equal to $c_i > 0$ if he gets to be included in the allocation. Players can share the total capacity c provided that there is room. More precisely, this is a multiunit homogeneous-good model (J being singleton and $I \times J$ identified with I) where the set X of feasible allocation outcomes is

$$X := \left\{ ((x_i)_{i \in I} \in \prod_{i \in I} \{0, c_i\} \mid \sum_{i \in I} x_i \leq C \right\}.$$

Assume that there is a nonempty subset I_* of $I \setminus \{1\}$ for which

$$c_1 < \sum_{i \in I_*} c_i \leq C - \sum_{i \in I \setminus (I_* \cup \{1\})} c_i < c_1 + c_k \quad \forall k \in I_*. \quad (16)$$

This assumption implies that the model is outside the paramodularity framework. To see that, consider the simplest case with only two players, $I = \{1, 2\}$, such that $I_* = \{2\}$. Then (16) becomes $c_1 < c_2 \leq C < c_1 + c_2$. With I_* singleton, one can show that the ceiling and floor functions φ and γ , defined by $\varphi(E) := \max_{x \in X} \sum_{i \in E} x_i$ and $\gamma(E) := \min_{x \in X} \sum_{i \in E} x_i$ for all $E \subseteq I$, constitute a paramodular pair.¹¹ However, the pair does not determine the set X of feasible allocation outcomes. For example, the allocation outcome $(x_1, x_2) = (c_1, C - c_1)$ satisfies all the bounds implied by (φ, γ) , yet it is infeasible because $0 < C - c_1 < c_2$ and hence $C - c_1 \notin \{0, c_2\}$. Thus, the requirement in Corollary 1 that X be determined by the pair is not satisfied, and hence the corollary does not apply. In fact, the greedy algorithm fails to be optimal despite the pair being paramodular: Since $c_2 > c_1$, there exist $w_1 > w_2 > 0$ for which w_1 is so close to w_2 that the optimum for

¹¹When the set I_* has at least two players, one can show that the ceiling function φ is not submodular, and so the paramodularity assumption does not hold.

$\max_{(x_1, x_2) \in X} (w_1 x_1 + w_2 x_2)$ is $(0, c_2)$. Whereas, the greedy algorithm produces the suboptimal $(c_1, 0)$ because $w_1 > w_2$.

Consequently, the main characterization result in the paramodularity framework is inapplicable. Furthermore, the literature provides no method to check whether the characterization $\mathcal{Q} \supseteq \mathcal{Q}_B$ is valid or not in knapsack models. With the necessity part of Theorem 1, by contrast, we can prove impossibility of the characterization:

Proposition 2 *In this knapsack model, if (16) holds and $|T_1| \geq 2$, then $\mathcal{Q} \not\supseteq \mathcal{Q}_B$.*

The impossibility result is driven by the integer constraint that one player needs a smaller quantity than some other players do and yet if the former's demand is satisfied then there is not enough room left to fit in the latter. In the two-player case described above, player 1 is the former, and player 2 the latter. Because $c_1 < c_2$, if the two players are equally weighed from the planner's viewpoint, player 2 gets the resource exclusively. If player 1 is weighed sufficiently more than player 2, whereas, player 1 gets the resource, leaving some capacity unused since $c_1 < C$, and yet not enough to fit in player 2, since $C < c_1 + c_2$.

Thus, let θ_1 be a type of player 1, and θ_2 a type of player 2. Pick any $w \in \mathbb{R}^{\mathcal{Z}}$ for which $w(1, \theta_1) = \bar{w}$, $w(2, \theta_2) = \underline{w}$, $\bar{w} > \underline{w} > 0$, and \bar{w} is so much larger than \underline{w} that $\bar{w}c_1 > \underline{w}c_2$. Consider any ex post allocation \bar{q} that maximizes the total w -value among all feasible allocation outcomes for every type profile. Since $w(2, \theta_2) > 0$, the player-type $(2, \theta_2)$ should be contained in some upward binding set. Otherwise, $p_+(S) = 0$ for any test set S that contains $(2, \theta_2)$, so (8) is impossible for the instance of $z = (2, \theta_2)$, and then the Border-like characterization cannot hold. Thus, suppose that $(2, \theta_2)$ is contained in some upward binding set S and consider the type profile $t := (\theta_1, \theta_2)$. Then $I(S, t)$ contains player 2 (by the definition (3) of $I(S, t)$) and so the maximum total quantity $f(S, t)$ that S can have is equal to c_2 , as $c_1 < c_2 \leq C < c_1 + c_2$. Meanwhile, under the type profile $t = (\theta_1, \theta_2)$, player 1 has the much larger weight \bar{w} than player 2 does and hence $\bar{q}(t)$ allocates c_1 to player 1 and zero to player 2. It follows that $\sum_{i \in I(S, t)} \bar{q}_i(t) < f(S, t)$, contradicting the premise that S is upward binding. Consequently, the Border-like characterization cannot hold. The proof is just slightly more complicated than the above to handle the general case where there may be more than two players and multiple individuals may be in the role of player 2.

4.2 Ranked-Item Matchings with Group-Specific Quotas

Consider a one-to-one matching model between n objects and n players such that all players have the same ordinal ranking on the objects and there is a group-specific quota constraint. The n objects are listed as $1, \dots, n$. Object j has quality $\rho_j \in \mathbb{R}_+$ so that

$$1 = \rho_1 \geq \rho_2 \geq \dots \geq \rho_n > 0.$$

The payoff for any player $i \in I$ of any type $t_i \in T_i$ is equal to $v(t_i, a) \sum_{j \in J} x_i^j \rho_j - c(a) - p_i$ if x_i^j is the probability with which i gets object j , a is his value-enhancing action before the allocation, $v(t_i, a)$ his payoff per unit of quality he receives, $c(a)$ his cost from the action a , and p_i his money payment. As explained in Gershkov et al. [10], the constructs v and c together imply that characterizing the feasible interim allocations is a necessary step in designing the optimal mechanism in any of such endogenous valuation models. As the former step is our focus, v and c will not concern us in the rest of this paper.

The set I of players is partitioned into two subsets of equal size, I_1 and I_2 with $|I_1| = |I_2| = n/2$ (n an even number). There are parametric integers ξ and ζ for which

$$\frac{n}{2} \geq \xi \geq \zeta$$

such that the objects are allocated to the players through one-to-one matchings subject to a *quota constraint* that at least ζ items among the objects of the top- ξ qualities (objects $1, \dots, \xi$) are allocated to the members of I_1 .

By the assumption of a player's payoff, we see that any allocation outcome $((x_i^j)_{i \in I})_{j=1}^n \in (\{0, 1\}^I)^n$ is equivalent to an allocation of qualities $(y_i)_{i \in I} \in [0, 1]^I$ such that $y_i = \rho_j$ if $x_i^j = 1$, namely, it allocates to each player i an amount ρ_j of quality. Thus, this model is equivalent to a multiunit homogeneous-good model where the total amount of quality available is equal to $\sum_{j \in J} \rho_j$, to which object j contributes an amount equal to ρ_j . That is, the set X of feasible allocation outcomes is equal to the set of $(y_i)_{i \in I} \in [0, 1]^n$ for each of which there exists a bijection $i \mapsto j(i)$ such that $y_i = \rho_{j(i)}$ for each $i \in I$ and

$$|\{i \in I_1 \mid j(i) \leq \xi\}| \geq \zeta. \tag{17}$$

Correspondingly, an interim allocation is a profile $(Q_i)_{i \in I}$ of functions such that each Q_i maps i 's type t_i to an interim expected quality $Q_i(t_i)$ that player i receives.

Due to the group-specific quota constraint (17), it is not suitable to assume that an allocation is symmetric across all players regardless which group (I_1 or I_2) they belong to. That means the majorization approach in Kleiner et al. [18] is not applicable.¹²

The paramodularity framework is not applicable either. That is because it assumes that the set X is determined by the ceiling and floor functions that regulate only the total quantities of various sets of players (cf. Che et al. [7], or Corollary 1 here). The problem is that the same total quantity may correspond to multiple allocation outcomes, and so the ceilings and floors of total quantities are insufficient to determine whether an allocation outcome is feasible or not.

For example, consider the case where $n = 4$, $\rho_1 > \rho_2 > \rho_3 > \rho_4$, $\rho_1 + \rho_3 = 2\rho_2$, I_1 contains player a but not player b , and $\xi = \zeta = 1$. Thus, the quota constraint is that the top-quality object 1 is set aside for I_1 -members. The feasible range of the total quantities is $[\rho_4, \rho_1]$ for player a alone, $[\rho_4, \rho_2]$ for player b alone, and $[\rho_3 + \rho_4, \rho_1 + \rho_2]$ for both players a and b : Since player a belongs to I_1 , she can get object 1 thereby receiving the highest quality ρ_1 ; but since the object can be allocated to another member of I_1 , player a can also be matched with the worst object thereby receiving quality ρ_4 . The case for player b is similar except that he does not belong to I_1 and hence the best quality he can get is ρ_2 (object 2). Together, the two players can get the best object ρ_1 (as one of them is in I_1) and the second best ρ_2 ; alternatively they can get the worst and second worst objects ($\rho_4 + \rho_3$). Now consider an allocation outcome $(y_i)_{i \in I} \in [0, 1]^4$ such that $y_a = y_b = \rho_2$. It satisfies all the bounds on total quantities mentioned before, because $\rho_1 > \rho_2 > \rho_4$ and $\rho_1 + \rho_3 = 2\rho_2$. However, this $(y_i)_{i \in I}$ is infeasible because $y_a = y_b$ whereas the qualities allocated to different players have to be distinct because different objects have distinct qualities in this example.

Theorem 1, by contrast, does not rely on X being determined by any pair of functions. Our analog to such a pair, $(f(\cdot, t), g(\cdot, t))$, is defined in (4) and (5) merely due to the compactness of X and is not assumed to determine X . To establish the Border-like characterization for this model, pick any $w := (w(i, t_i))_{(i, t_i) \in \mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}}$, where the domain \mathcal{Z} of interim allocations in this model consists of all player-types (i, t_i) for which $i \in I$ and

¹²Gershkov et al. [10, Thrm. 2] analyze a possibly asymmetric mechanism of two symmetric players in terms of majorization. But their majorization observations in that particular analysis is derived from a Border-like characterization (Maskin and Riley [22]) that belongs to the paramodularity framework rather than from the majorization characterization in Kleiner et al. [18].

$t_i \in T_i$. For any type profile $t := (t_i)_{i \in I} \in T$, consider a solution to problem (7), namely,

$$\max_{(y_i)_{i \in I} \in \text{cv}(X)} \sum_{i \in I} w(i, t_i) y_i. \quad (18)$$

The only complication in this otherwise simple linear programming is that a player i with a high w -value $w(i, t_i)$ may get a low-quality object just because he does not belong to I_1 and the high-quality object is reserved for members of I_1 . Nonetheless, problem (18) can be solved by a greedy algorithm that follows the ordinal ranking of the w -values except when the quota constraint is binding. Moreover, regardless of the particular type profile t that problem (18) takes parametrically, the order in which the greedy algorithm follows is derived from a common ranking on the entire domain \mathcal{Z} of interim allocations. Thus, this model still belongs to the easy case, where Lemma 1 delivers the Border-like characterization:

Proposition 3 *In the above-defined model of ranked items with quota constraints, $\mathcal{Q} \supseteq \mathcal{Q}_B$.*

The aforementioned greedy algorithm that solves problem (18) is:

1. Initiate $\zeta' := \zeta$, $j_1 := 1$, $j_2 := \xi + 1$ and $I' := I$.
2. Do while $I' \neq \emptyset$:
 - a. Let $i_* := \min(\arg \max_{i \in I'} w(i, t_i))$.
 - b. If $j_1 + \zeta' \leq \xi$, then let $\bar{q}_{i_*}(t) := \rho_{j_1}$; then update $I' := I' \setminus \{i_*\}$ and $j_1 := j_1 + 1$; if $i_* \in I_1$ then update $\zeta' := \max\{0, \zeta' - 1\}$; then go to Step 2a.
 - c. Else ($j_1 + \zeta' > \xi$):
 - i. If $i_* \in I_1$ and $\zeta' > 0$, then let $\bar{q}_{i_*}(t) := \rho_{j_1}$; then update $I' := I' \setminus \{i_*\}$, $j_1 := j_1 + 1$ and $\zeta' := \zeta' - 1$; then go to Step 2a.
 - ii. Else ($i_* \notin I_1$ or $\zeta' = 0$), let $\bar{q}_{i_*}(t) := \rho_{j_2}$; then update $I' := I' \setminus \{i_*\}$ and $j_2 := j_2 + 1$; then go to Step 2a.
3. Output $(\bar{q}_i(t))_{i \in I}$.

In other words, the algorithm makes the assignment player-by-player in descending order of their w -values under the type profile t . The variable j_1 keeps track of the best unassigned object in the top- ξ -quality category, and the variable ζ' keeps track of the unfulfilled quota, the difference between the minimally required number of top- ξ objects assigned

to I_1 -members and the actual number of top- ξ objects assigned to I_1 so far. If there is still an unassigned object available in the top- ξ category after ζ' of them are reserved for future I_1 -members ($j_1 + \zeta' \leq \xi$), assign the best available top- ξ object to the current player (Step 2b.). Otherwise, if the current player belongs to I_1 and if the unfulfilled quota ζ' is not yet zero, then there are still top- ξ objects available to him, as ζ' of them are reserved for I_1 members. The algorithm thus assigns to him the best unassigned object in the top- ξ category and reduces the unfulfilled quota by one (Step 2(c.i)). In any other case, either the current player belongs to I_1 while all the top- ξ objects have been assigned ($i_* \in I_1$, $\zeta' = 0$ and $j_1 + \zeta' > \xi$), or the current player belongs to the other group, and in the top- ξ category there is no unassigned object except those reserved for I_1 -members ($i_* \notin I_1$ and $j_1 + \zeta' > \xi$). Thus, in either subcase, the best object available to the current player is the best unassigned one among those below the top- ξ category, which is what the algorithm assigns to him (Step 2(c.ii)). Then the algorithm moves on to the next player in the list and repeats until all players are matched with objects.

The order in which the algorithm follows is derived from the ordinal ranking of the w -values, with ties broken in favor of the lower-indexed player:

$$(i, t_i) \succ (k, t_k) \iff [w(i, t_i) > w(k, t_k) \text{ or } [w(i, t_i) = w(k, t_k) \text{ and } i < k]].$$

For any $m \in \{1, \dots, |\mathcal{Z}|\}$, let U^m denote the upper contour set that consists of the top- m elements (player-types) of \mathcal{Z} according to \succ . For each type profile $t := (t_i)_{i \in I}$, the strict order \succ generates a listing of the players in I , (i_1, i_2, \dots, i_n) , so that $h < k \iff (i_h, t_{i_h}) \succ (i_k, t_{i_k})$. The algorithm makes assignments player-by-player in the order of this list. Importantly, $I(U^m, t) = \{i_1, \dots, i_{|I(U^m, t)|}\}$ for every U^m : If $i_k \in I(U^m, t)$ and $h < k$ then $(i_k, t_{i_k}) \in U^m$ (by (3)) and $(i_h, t_{i_h}) \succ (i_k, t_{i_k})$, and then by the definition of upper contour set we have $(i_h, t_{i_h}) \in U^m$ and hence $i_h \in I(U^m, t)$. Consequently, in the first $|I(U^m, t)|$ iterations of the algorithm, only the player-types that belong to U^m may possibly get assigned objects, and these objects are the best ones unless the quota constraint is binding. That is the main reason why the algorithm is greedy on the entire domain \mathcal{Z} so that Lemma 1 applies.

For example, consider four players such that $I_1 = \{1, 2\}$ and $I_2 = \{3, 4\}$ and each player has at least two types, θ and θ' . Let the quota constraint be $\xi = \zeta = 1$, namely, to set aside object 1 for player 1 or 2. Let the top four elements of \mathcal{Z} be

$$(2, \theta') \succ (3, \theta) \succ (1, \theta) \succ (4, \theta) \succ \dots,$$

so $U^1 = \{(2, \theta')\}$, $U^2 = \{(2, \theta'), (3, \theta)\}$ and so forth. Consider any type profile say $t := (\theta, \theta, \theta, \theta)$. Note from the definition (3) that $I(U^1, t) = \emptyset$ and so the maximum quantity $f(U^1, t)$ that U^1 can have is zero. Apply (3) to U^2 to see that $I(U^2, t) = \{3\}$ and so the maximum quantity $f(U^2, t)$ that U^2 can have is equal to ρ_2 , because player 3 can have up to the second-best object ρ_2 but not the best object ρ_1 , as $3 \notin I_1$. Note that the allocation outcome $\bar{q}(t)$ outputted by the algorithm does allocate ρ_2 to player 3 (in Step 2(c).ii of its first iteration), and hence

$$\sum_{i \in I(U^2, t)} \bar{q}_i(t) - \sum_{i \in I(U^1, t)} \bar{q}_i(t) = \bar{q}_3(t) = \rho_2 = f(U^2, t) - f(U^1, t).$$

Move on to U^3 . Note $I(U^3, t) = \{3, 1\}$. Since player 1 belongs to I_1 , players 1 and 3 together can get up to the total quantity $\rho_1 + \rho_2$; and the algorithm does allocate ρ_1 to player 1 because he ranks highest among all I_1 members at this t (while $(2, \theta') \succ (1, \theta)$ and player 2 belongs to I_1 as well, the type θ' is not consistent with the current type profile t). Thus

$$\sum_{i \in I(U^3, t)} \bar{q}_i(t) - \sum_{i \in I(U^2, t)} \bar{q}_i(t) = (\bar{q}_1(t) + \bar{q}_3(t)) - \bar{q}_3(t) = \bar{q}_1(t) = \rho_1 = (\rho_1 + \rho_2) - \rho_2 = f(U^3, t) - f(U^2, t).$$

For U^4 , analogously, $I(U^4, t) = \{3, 1, 4\}$ and hence $f(U^4, t) = \rho_1 + \rho_2 + \rho_3$ because among the three players someone is in I_1 . And the $\bar{q}(t)$ produced by the algorithm allocates the third-best object ρ_3 to player 4 because he ranks third given t , and the best object available now is object 3, with objects 1 and 2 already assigned. Thus,

$$\sum_{i \in I(U^4, t)} \bar{q}_i(t) - \sum_{i \in I(U^3, t)} \bar{q}_i(t) = \bar{q}_4(t) = \rho_3 = (\rho_1 + \rho_2 + \rho_3) - (\rho_1 + \rho_2) = f(U^4, t) - f(U^3, t).$$

Thus, $\bar{q}(\cdot | U^4, t)$ is a greedy solution wrt \succ within U^4 given t .

To compare Proposition 3 with the result in the majorization literature (e.g., Kleiner et al. [18, Theorem 3]), which considers symmetric mechanisms, let us restrict attention to mechanisms that are *group-wise symmetric* in the sense that all members within each group I_k are treated identically. Correspondingly, for each $k \in \{1, 2\}$ assume that all players in group I_k independently draw their types from an identical distribution. Relabel the players if necessary so that $1 \in I_1$ and $2 \in I_2$, and hence $T_i = T_k$ and $\mu_i = \mu_k$ for all $i \in I_k$ ($k = 1, 2$). Consider only interim allocations in the form of (Q_1, Q_2) such that for each $k \in \{1, 2\}$, $Q_k : T_k \rightarrow \mathbb{R}$ is the interim allocation for any member of I_k . Then the characterization condition (6) is required only for the test sets that treat members of the

same group identically. That is, we need only to consider test sets in the form of (S_1, S_2) such that $S_k \subseteq T_k$ for each $k = 1, 2$. Consequently, since $|I_1| = |I_2| = n/2$, Ineq. (6) becomes

$$\frac{2}{n} \sum_{t \in T} g(S, t) \mu(t) \leq \sum_{k=1}^2 \sum_{t_k \in T_k} Q_k(t_k) \chi_{S_k}(t_k) \mu_k(t_k) \leq \frac{2}{n} \sum_{t \in T} f(S, t) \mu(t) \quad (19)$$

for all such test sets (S_1, S_2) .

Furthermore, as in the majorization literature, let us assume that types are unidimensional so that there is no loss to consider only monotone interim allocations. Then we need only to consider test sets (S_1, S_2) such that for every $k \in \{1, 2\}$, $S_k = \{t_k \in T_k \mid F_k(t_k) > 1 - \alpha_k\}$ for some $\alpha_k \in (0, 1]$. For simplicity, consider only the special case of quota constraints that $\xi = \zeta = 1$, namely, that only the top-quality object 1 is set aside for I_1 -members.

Then $f(S, t)$ and $g(S, t)$ can be calculated easily for each type profile $t \in T$. Plug them into (19) to obtain a more explicit condition. The interim expected qualities defined below are just the shorthands for the long expressions in that condition.

$A_1(s_1 \mid s_2) :=$ The interim expected quality received by a member of I_1 whose type belongs to the s_1 upper quantile of F_1 (i.e., $F_1(t_i) > 1 - s_1$) given the following allocation:¹³ Object 1 goes to the member say i_* of I_1 whose type is the highest in I_1 ; the members of I_2 whose types belong to the s_2 upper quantile of F_2 get the highest-quality objects among objects $\{2, \dots, n\}$; the members of $I_1 \setminus \{i_*\}$ are matched with the objects among the remaining ones in the assortative manner (higher qualities with higher types).

$A_2(s_2 \mid s_1) :=$ The interim expected quality received by a member of I_2 whose type belongs to the s_2 upper quantile of F_2 given the following allocation: The highest-quality objects are assigned, in the assortative manner, to the members of I_1 whose types belong to the s_1 upper quantile of F_1 ; the members of I_2 are matched with the objects among the remaining ones in the assortative manner.

$Z_1(s_1 \mid s_2) :=$ The interim expected quality received by a member of I_1 whose type belongs to the s_1 upper quantile of F_1 given the following allocation: Object 1 goes to the member say i_0 of I_1 whose type is the *lowest* in I_1 ; the members of I_2 whose types belong to the s_2 upper quantile of F_2 get the *lowest*-quality objects among objects $\{2, \dots, n\}$; the

¹³This allocation, albeit well-defined, is not necessarily feasible, as not all objects are necessarily assigned. Same comment for the other allocations defined in this list.

remaining objects are assigned to $I_1 \setminus \{i_0\}$ in the *anti-assortative* manner: the higher the type, the lower the quality of the object assigned.

$Z_2(s_2 \mid s_1) :=$ The interim expected quality received by a member of I_2 whose type belongs to the s_2 upper quantile of F_2 given the following allocation: The highest-quality objects are assigned, in the *anti-assortative* manner, to the members of I_1 whose types belong to the s_1 upper quantile of F_1 ; the remaining objects are then assigned to members of I_2 in the *anti-assortative* manner.

Corollary 2 *In the model of ranked items with the set-aside rule $\xi = \zeta = 1$, a group-wise symmetric weakly increasing interim allocation (Q_1, Q_2) belongs to \mathcal{Q} if and only if*

$$\begin{aligned} \int_0^{\alpha_1} Z_1(s_1, 0) ds_1 + \int_0^{\alpha_2} Z_2(\alpha_1, s_2) ds_2 &\leq \int_0^{\alpha_1} Q_1(F_1^{-1}(1 - s_1)) ds_1 + \int_0^{\alpha_2} Q_2(F_2^{-1}(1 - s_2)) ds_2 \\ &\leq \int_0^{\alpha_1} A_1(s_1, 0) ds_1 + \int_0^{\alpha_2} A_2(\alpha_1, s_2) ds_2 \end{aligned} \quad (20)$$

for all $(\alpha_1, \alpha_2) \in [0, 1]^2$ (where F_k^{-1} denotes the generalized inverse of the cdf F_k for I_k).

Condition (20) differs from the majorization characterization in Kleiner et al. [18, Th. 3.1, Prop. 4] in that both the left- and right-hand sides involve the sum of two integrals rather than a single integral. The difference is mainly driven by the group-specific quota, without which their symmetry restriction could be applied to both groups (if players are iid), and then the upper bound inequality in (20) would reduce to that in Kleiner et al.

4.3 Matchings with Arbitrary Numbers of Types

Consider a matching model between $N \geq 2$ objects and two players such that the matching from players to objects has to be one-to-one, and each player can have any finite number of types. The set of players is $I := \{1, 2\}$, and the set of objects $J := \{1, \dots, N\}$, $N \geq 2$. An allocation outcome is in the form of $x := ((x_i^j)_{i=1}^2)_{j=1}^N \in \{0, 1\}^{2N}$, with $x_j^i = 1$ signifying player i getting object j . The set X of feasible allocation outcomes is defined to be the set of all $((x_i^j)_{i=1}^2)_{j=1}^N \in \{0, 1\}^{2N}$ subject to two constraints:

$$\begin{aligned} \forall i \in I : \sum_{j \in J} x_i^j &= 1 \\ \forall j \in J : \sum_{i \in I} x_i^j &\leq 1, \end{aligned}$$

	$(1, t_1)$	$(2, t_2)$	$(2, t'_2)$
a	-1	3	$1/2$
b	4	0	3
c	2	$1/2$	0

Table 3: The objects: a , b and c ; the player-types: $(1, t_1)$, $(2, t_2)$ and $(2, t'_2)$

the former being the unit-demand constraint,¹⁴ and the latter, resource-feasibility constraint.

It is known that paramodularity does not hold in such models. Given the set X defined above, paramodularity requires that the ceiling and floor functions (φ, γ) defined by $\varphi(E) := \max_{x \in X} \sum_{(i,j) \in E} x_i^j$ and $\gamma(E) := \min_{x \in X} \sum_{(i,j) \in E} x_i^j$ ($\forall E \subseteq I \times J$) satisfy the submodularity and compliance conditions (Section 3.1). It is easy to show that neither conditions are satisfied (Appendix B.9).

Thus, no method in the received, paramodularity framework applies. The new method introduced by Lang and Yang [20] does not apply either, because their applications so far are limited to at most two types per player. Theorem 1, by contrast, applies:

Proposition 4 *In the matching model defined above, $\mathcal{Q} \supseteq \mathcal{Q}_B$.*

The proof proceeds in three steps. First, for any $w := (w(j, i, t_i))_{(j,i,t_i) \in \mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}}$, consider a relatively simple (say integral) ex post allocation \bar{q} such that $\bar{q}(t)$ solves problem (7) for every type profile $t \in T$ given w . Second, find a family of subsets of \mathcal{Z} within each of which \bar{q} is greedy wrt some linear ordering such that the family covers \mathcal{Z} . Third, prove that this family is the support of a nonnegative shadow price function (p_+, p_-) that satisfies (8). Then the proposition follows from Theorem 1.

Let me illustrate these steps with an example of two players (1 and 2), three objects (a , b and c), one type t_1 for player 1, and two types (t_2 and t'_2) for player 2. The domain \mathcal{Z} of interim allocations thus consists of nine object-player-type triples such as $(a, 1, t_1)$, $(a, 2, t_2)$, $(b, 2, t'_2)$, etc. Table 3 displays one $w \in \mathbb{R}^{\mathcal{Z}}$, with the entry at row j and column (i, τ_i) being the w -value $w(j, i, \tau_i)$ of object j matched with player i of type τ_i (e.g., $w(a, 1, t_1) = -1$).

Step 1 is easy. When the type profile is (t_1, t_2) , the solution $\bar{q}(t_1, t_2)$ to problem (7) is to allocate object a to player 2, and object b to player 1. That is because, under the

¹⁴The Border-like characterization can be extended to the case where the unit-demand equation constraint is weakened to $\sum_{j \in J} x_i^j \leq 1$ for each $i \in I$.

	$(1, t_1)$	$(2, t_2)$
a	-1	3
b	4	0
c	2	$1/2$

Table 4: The optimal match, colored red, under type profile (t_1, t_2)

	$(1, t_1)$	$(2, t'_2)$
a	-1	$1/2$
b	4	3
c	2	0

Table 5: The optimal match, colored red, under type profile (t_1, t'_2)

type profile (t_1, t_2) , player 1 (of type t_1) gets larger w -value from object b than from any other object, and player 2 (of type t_2) gets larger w -value from object a than from any other object (Table 4). That is, in Table 4, the top contender $(b, 1, t_1)$ in column $(1, t_1)$ and the top contender $(a, 2, t_2)$ in column $(2, t_2)$ are compatible as they refer to different objects, and hence they constitute the maximum of the total w -value among all feasible allocation outcomes under type profile (t_1, t_2) . By contrast, consider the other type profile, (t_1, t'_2) . With player 2's realized type changed to t'_2 , the column $(2, t_2)$ in the previous table is replaced by the column $(2, t'_2)$ in Table 5. The top contender $(b, 2, t'_2)$ in the new column is incompatible with the top contender $(b, 1, t_1)$ in column $(1, t_1)$, as they are after the same object b . Thus, one of the top contenders needs to be replaced by the second-highest contender in the corresponding column, either the $(c, 1, t_1)$ in column $(1, t_1)$, or the $(a, 2, t'_2)$ in column $(2, t'_2)$. The former in pairing with the top contender $(b, 2, t'_2)$ yields $2 + 3 = 5$, whereas the latter in pairing with the top contender $(b, 1, t_1)$ yields $1/2 + 4 = 9/2$. Thus the solution $\bar{q}(t_1, t'_2)$ to problem (7) is to allocate object b to player 2, and object c to player 1.

Step 2: The idea is to find the revealed preference of \bar{q} among the object-player-type triples that are mutually exclusive (such as the $(b, 1, t_1)$ and $(b, 2, t'_2)$ noted before). There are two kinds of mutual exclusivity due to the feasibility constraint of one-to-one matching. One kind is that both triples refer to the same player-type, namely, both belong to the same column in Table 3. The other is that both triples refer to the same object, belonging to the

same row in Table 3. Let us inspect the behavior of \bar{q} within each kind.

Among the object-player-type triples that belong to the same column, the behavior of \bar{q} is largely consistent with the ordinal ranking of their w -values: The top or second-highest contender is chosen by \bar{q} , while any contender below them is never chosen. Specifically, the choice of \bar{q} is consistent with the following linear orderings in the three columns:

$$\text{column } (1, t_1) : (b, 1, t_1) \sim (c, 1, t_1) \succ (a, 1, t_1)$$

$$\text{column } (2, t_2) : (a, 2, t_2) \succ (c, 2, t_2) \succ (b, 2, t_2)$$

$$\text{column } (2, t'_2) : (b, 2, t'_2) \succ (a, 2, t'_2) \succ (c, 2, t'_2),$$

where $(b, 1, t_1) \sim (c, 1, t_1)$ in column $(1, t_1)$ because \bar{q} chooses $(b, 1, t_1)$ over $(c, 1, t_1)$ when the realized type τ_2 of player 2 is t_2 (so $(b, 1, t_1) \succeq (c, 1, t_1)$) as in Table 4, and chooses the reverse when $\tau_2 = t'_2$ (so $(c, 1, t_1) \succeq (b, 1, t_1)$) as in Table 5. It is easy to verify that within each column, \bar{q} is a greedy solution wrt the \succeq listed above for each type profile (τ_1, τ_2) .

Among the object-player-type triples that belong to the same row, the choice of \bar{q} is inconsistent with the ordinal ranking of the w -values. For example, in the row b of Table 3, $(b, 2, t'_2)$ is chosen over $(b, 1, t_1)$ given type profile (t_1, t'_2) , and given no type profile does the reverse hold. Thus, $(b, 2, t'_2) \succ (b, 1, t_1)$ despite $w(b, 2, t'_2) < w(b, 1, t_1)$. To derive from the w -values the revealed preference of \bar{q} within a row, define

$$\delta(j, i, \tau_i) := w(j, i, \tau_i) - \max_{j' \in J \setminus \{j\}} w(j', i, \tau_i)$$

for each object-player-type triple (j, i, τ_i) . For example, in row b of Table 3,

$$\delta(b, 1, t_1) = 4 - 2 = 2, \quad \delta(b, 2, t_2) = 0 - 3 = -3, \quad \delta(b, 2, t'_2) = 3 - 1/2 = 5/2.$$

The ordinal ranking according to the δ -values in this row is

$$\text{row } b : (b, 2, t'_2) \succ (b, 1, t_1) \succ (b, 2, t_2).$$

It is easy to verify that \bar{q} is greedy wrt this \succ within row b . Intuitively, \bar{q} has to choose between $(b, 1, \tau_1)$ and $(b, 2, \tau_2)$ only when both are the top contenders in their own columns. In that case, \bar{q} chooses $(b, 1, \tau_1)$ if the total w -value of $(b, 1, \tau_1)$ coupled with the second-highest contender in column $(2, \tau_2)$ is larger than the total w -value of $(b, 2, \tau_2)$ coupled with the second-highest contender in column $(1, \tau_1)$, and \bar{q} chooses $(b, 2, \tau_2)$ if the opposite is true. In other words, the choice of \bar{q} between $(b, 1, \tau_1)$ and $(b, 2, \tau_2)$ hinges on whether

$$w(b, 1, \tau_1) + \max_{j \neq b} w(j, 2, \tau_2) > w(b, 2, \tau_2) + \max_{j \neq b} w(j, 1, \tau_1)$$

holds or not, namely, whether $\delta(b, 1, \tau_1) > \delta(a, 2, \tau)$ holds or not.

The choice of \bar{q} within the other two rows in Table 3 is trivial. Neither row contains more than one top contenders, and hence \bar{q} never has to choose one over the other to resolve the conflict between two player-types going after the same object. Thus, we do not bother to find the revealed preference of \bar{q} within either row.

Within each column and row b in Table 3, the upper contour sets wrt the revealed preference of \bar{q} are then defined. And they constitute increasing nested sequences:

$$\begin{aligned} \text{row } b : \quad & U_b(2t'_2) = \{b2t'_2\} \subsetneq U_b(1t_1) = \{b2t'_2, b1t_1\} \subsetneq U_b(2t_2) = \{b2t'_2, b1t_1, b2t_2\} \\ \text{column } (1, t_1) : \quad & V_{1t_1}(b) = V_{1t_1}(c) = \{b1t_1, c1t_1\} \subsetneq V_{1t_1}(a) = \{b1t_1, c1t_1, a1t_1\} \\ \text{column } (2, t_2) : \quad & V_{2t_2}(a) = \{a2t_2\} \subsetneq V_{2t_2}(c) = \{a2t_2, c2t_2\} \subsetneq V_{2t_2}(b) = \{a2t_2, c2t_2, b2t_2\} \\ \text{column } (2, t'_2) : \quad & V_{2t'_2}(b) = \{b2t'_2\} \subsetneq V_{2t'_2}(a) = \{b2t'_2, a2t'_2\} \subsetneq V_{2t'_2}(c) = \{b2t'_2, a2t'_2, c2t'_2\}. \end{aligned}$$

In the above-displayed, an object-player-type triple (j, i, τ_i) is condensed to $j i \tau_i$, and the upper contour set in which (j, i, τ_i) is a minimum wrt the revealed preference is denoted by either $U_j(i \tau_i)$ if the set is within row j , or $V_{i \tau_i}(j)$ if the set is within column (i, τ_i) .

We have thus obtained a family of upper contour sets that covers the domain \mathcal{Z} :

$$\mathcal{S}_+ := \{U_b(2t'_2), U_b(1t_1), U_b(2t_2), V_{1t_1}(b), V_{1t_1}(a), V_{2t_2}(a), V_{2t_2}(c), V_{2t_2}(b), V_{2t'_2}(b), V_{2t'_2}(a), V_{2t'_2}(c)\}.$$

With an eye on the next step, to be sure that the negative w -values can be accommodated, let us append to the covering some lower contour sets:

$$\mathcal{S}_- := \{\{b1t_1, c1t_1, a1t_1\}, \{a2t_2, c2t_2, b2t_2\}, \{b2t'_2, a2t'_2, c2t'_2\}\}.$$

Denote the three sets in this \mathcal{S}_- by L_{1t_1} , L_{2t_2} and $L_{2t'_2}$, which are just the columns in Table 3.

Step 3 is to prove existence of a pair (p_+, p_-) of shadow price functions, $p_+ : \mathcal{S}_+ \rightarrow \mathbb{R}_+$ and $p_- : \mathcal{S}_- \rightarrow \mathbb{R}_+$, that satisfies both (7) and (9), namely,

$$w(j, i, \tau_i) = \sum_{S \subseteq \mathcal{S}_+} p_+(S) \chi_S(j, i, \tau_i) - \sum_{S \subseteq \mathcal{S}_-} p_-(S) \chi_S(j, i, \tau_i) \quad (21)$$

for all object-player-types (j, i, τ_i) . This equation system is the same as

$$[\mathbf{M}_+, \mathbf{M}_-] \mathbf{p} = \mathbf{w}$$

with the notations \mathbf{M}_+ , \mathbf{M}_- , \mathbf{p} and \mathbf{w} defined around Lemma 3. Here \mathbf{M}_+ is the 9-by-11 matrix ($|\mathcal{Z}| = 9$, $|\mathcal{S}_+| = 11$) whose entries are $\chi_S(j, i, \tau_i)$, \mathbf{M}_- the 9-by-3 matrix ($|\mathcal{S}_-| =$

3) whose entries are $-\chi_S(j, i, \tau_i)$, $\mathbf{p} := [\mathbf{p}_+, \mathbf{p}_-]^\top$ the 14-by-1 column vector with entries being $p_+(S)$ or $p_-(S)$, and \mathbf{w} the 9-by-1 column vector with entries $w(j, i, \tau_i)$.

By Lemma 3, it suffices Step 3 to prove that no sequence of row operations on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ can produce a nonnegative row whose entry at the $-\mathbf{w}$ position is positive, namely, to show that no linear combination among the instances of (21) can lead to a contradictory case where the left-hand side is negative and the right-hand side is nonnegative.

To see the intuition why such contradictory cases cannot occur, let us try to produce one. For every object-player-type triple (j, i, τ_i) , let $[ji\tau_i]$ denote the row of the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ corresponding to (j, i, τ_i) :

$$[ji\tau_i] := [(\chi_S(j, i, \tau_i))_{S \in \mathcal{S}_+}, (-\chi_S(j, i, \tau_i))_{S \in \mathcal{S}_-}, -w(j, i, \tau_i)].$$

Note that a contradictory case cannot be produced by an addition or subtraction between two object-player-types that refer to different player-types. That is, if $(i, \tau_i) \neq (k, \tau'_k)$ then neither $[ji\tau_i] + [j'k\tau'_k]$ nor $[ji\tau_i] - [j'k\tau'_k]$ can produce a nonnegative row, because at the position for L_{it_i} , $[ji\tau_i]$ has a negative entry -1 while $[j'k\tau'_k]$ has a zero entry, and at the position for $L_{k\tau'_k}$ the situation is symmetric. Thus, the operation leaves at least one of the two entries negative thereby failing to produce a nonnegative row.

It follows that a contradictory case has to result from an operation between two object-player-types that refer to the same player-type say (i, τ_i) , and the operation has to be a subtraction in order to eliminate the negative entry at the position for L_{it_i} . Furthermore, in order to produce a positive entry at the position for $-\mathbf{w}$, the subtraction has to be from a low- w -value object-player-type to a high- w -value one. That is, the operation has to be $[ji\tau_i] - [j'i\tau_i]$ for some player-type (i, τ_i) and some objects $j \neq j'$ such that $w(j, i, \tau_i) < w(j', i, \tau_i)$. Meanwhile, for $[ji\tau_i] - [j'i\tau_i]$ to be nonnegative in every entry, the -1 entry in $-[j'i\tau_i]$ at the position of $V_{i\tau_i}(j')$ needs to be canceled out by a positive 1 entry in $[ji\tau_i]$ at the same position. That requires, by the definition of the matrix \mathbf{M}_+ , $(j, i, \tau_i) \in V_{i\tau_i}(j')$, namely, $(j, i, \tau_i) \succeq (j', i, \tau_i)$. In sum, a contradictory case has to result from a subtraction $[ji\tau_i] - [j'i\tau_i]$ for which $w(j, i, \tau_i) < w(j', i, \tau_i)$ and $(j, i, \tau_i) \succeq (j', i, \tau_i)$. In our example, there is only one such possibility: $[c1t_1] - [b1t_1]$, namely, the second-highest contender in the column $(1, t_1)$ of Table 3 subtracted by the top contender in the same column of the table, as the two are indifferent according to the revealed preference of \bar{q} .

Table 6 displays $[c1t_1] - [b1t_1]$. The ellipses there stand for the omitted, less relevant

	$U_b(1t_1)$	$U_b(2t'_2)$	$U_b(2t_2)$	$V_{1t_1}(b)$	$V_{1t_1}(a)$	$V_{2t'_2}(b)$	L_{1t_1}	$L_{2t'_2}$	\dots	$-w$
$[c1t_1]$	0	0	0	1	1	0	-1	0	\dots	-2
$[b1t_1]$	1	0	1	1	1	0	-1	0	\dots	-4
$[c1t_1] - [b1t_1]$	-1	0	-1	0	0	0	0	0	\dots	2

Table 6: A row subtraction

entries, which are zero for both vectors. Right away we see that there are negative entries in $[c1t_1] - [b1t_1]$: At the position for $U_b(1t_1)$, for example, the entry is -1 , because $(b, 1, t_1) \in U_b(1t_1)$ while $(c, 1, t_1) \notin U_b(1t_1)$ (namely, $\chi_{U_b(1t_1)}(c, 1, t_1) - \chi_{U_b(1t_1)}(b, 1, t_1) = 0 - 1$). Thus, the operation has yet to produce a nonnegative vector.

Consequently, to produce a contradictory case the operation has to involve another object-player-type (j', k, τ'_k) such that the vector $[j'k\tau'_k]$ has a positive entry at the position for $U_b(1t_1)$ to cancel out the negative entry in $[c1t_1] - [b1t_1]$ at that position. This (j', k, τ'_k) is $(b, 2, t'_2)$, because $(b, 2, t'_2)$ is the only element of $U_b(1t_1)$ (other than $(b, 1, t_1)$), as $(b, 2, t'_2) \succ (b, 1, t_1)$ by the revealed preference of \bar{q} within row b of Table 3. Hence the operation requires an addition of $[b2t'_2]$. Since $[b2t'_2]$ has a -1 entry at the position for $L_{2t'_2}$, the addition of $[b2t'_2]$ in turn requires the subtraction of either $[a2t'_2]$ or $[c2t'_2]$ to cancel out the -1 . Say it is $[a2t'_2]$, so the operation aimed at producing a contradictory case entails a quadruple

$$[c1t_1] - [b1t_1] + [b2t'_2] - [a2t'_2].$$

Table 7 displays this quadruple operation. Note that as a result of the quadruple operation the entry at the $-w$ position becomes nonpositive, $-1/2$. In other words,

$$w(c, 1, t_1) - w(b, 1, t_1) + w(b, 2, t'_2) - w(a, 2, t'_2) \geq 0.$$

That is no coincidence, because $(b, 2, t'_2) \succ (b, 1, t_1)$ implies, by the revealed preference of \bar{q} within the row b in Table 3, that $\delta(b, 2, t'_2) \geq \delta(b, 1, t_1)$. This inequality in turn implies the inequality displayed above by the definition of δ . The displayed inequality remains intact if we replace the $[a2t'_2]$ in the quadruple by the other possible element, $[c2t'_2]$, because the w -value-added $w(b, 2, t'_2) - w(c, 2, t'_2) > \delta(b, 2, t'_2)$, with $w(c, 2, t'_2) < w(a, 2, t'_2)$.

Now that we have exhausted the last resort to produce a nonnegative row whose entry at the $-w$ position is positive, we see heuristically that the equation system (21) has no contradiction and hence admits a solution for the shadow price functions.

	$U_b(1t_1)$	$U_b(2t'_2)$	$U_b(2t_2)$	$V_{1t_1}(b)$	$V_{1t_1}(a)$	$V_{2t'_2}(b)$	$V_{2t'_2}(a)$	L_{1t_1}	$L_{2t'_2}$	\dots	$-w$
$[c1t_1]$	0	0	0	1	1	0	0	-1	0	\dots	-2
$[b1t_1]$	1	0	1	1	1	0	0	-1	0	\dots	-4
$[c1t_1] - [b1t_1]$	-1	0	-1	0	0	0	0	0	0	\dots	2
$[b2t'_2]$	1	1	1	0	0	1	1	0	-1	\dots	-3
$[a2t'_2]$	0	0	0	0	0	0	1	0	-1	\dots	-1/2
$[b2t'_2] - [a2t'_2]$	1	1	1	0	0	1	0	0	0	\dots	-5/2
net total	0	1	0	0	0	1	0	0	0	.	-1/2

Table 7: A quadruple operation $[c1t_1] - [b1t_1] + [b2t'_2] - [a2t'_2]$

The proof, deferred to Appendix B.10, is more involving than the above to handle two complications. One is that in an arbitrary sequence of row operations on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$, a row operation such as $[c1t_1] - [b1t_1]$ may be more generally $\alpha[c1t_1] - \beta[b1t_1]$ for some positive real numbers α and β . The other complication is that there may be multiple quadruples like the one shown above. The first complication is handled by dissecting the row with the larger coefficient, say $-\beta[b1t_1]$ with $\beta > \alpha$, into two portions, one being $-\alpha[b1t_1]$ to be paired with $\alpha[c1t_1]$ so that the pair becomes $[c1t_1] - [b1t_1]$ multiplied by α , and the other being $-(\beta - \alpha)[b1t_1]$, treated as a separate row in the sequence of operations (which in turn can be paired with another row due to the nonnegative-row condition). The second complication is handled by an induction argument that recursively removes such quadruples from the sequence of row operations until the sequence is reduced to null. At each step of the induction, a quadruple is selected so that both the nonnegative-row condition and the positivity of the $-\mathbf{w}$ entry are unchanged by its removal, and hence the inductive step on the remaining sequence follows. The sequence is eventually exhausted without occurrence of any contradictory cases. Thus by Lemma 3 a solution for the equation system (21) exists.

5 Conclusion

Characterizing feasible interim allocations is often an unavoidable step in the design of optimal mechanisms, and Border-like solutions have been the prototype of such characterizations. The literature has located only sufficient conditions to obtain Border-like solutions,

and those sufficient conditions are mostly confined to the paramodularity framework that rules out nontrivial combinatorial complications such as matchings. This paper adds to the literature a necessary and sufficient condition to obtain Border-like solutions without the confine of paramodularity. The main message is that the condition crucial to Border-like solutions is not paramodularity, but rather the local greediness of the interim allocations that are about to violate some feasibility constraints. Paramodularity merely guarantees the easy case where the greediness property is global. This connection between Border-like solutions and greedy algorithms is instrumental in the search for Border-like solutions such as those presented above.

The three applications presented above have various potentials for further development. The characterization result in the n -player ranked-item model with a group-specific quota constraint is comparable to its counterpart in the majorization literature. Thus an immediate question is how the implications of the characterization differ from those of the majorization characterization. The characterization result, coupled with the tractability due to the unidimensional-type assumption that the model allows, may facilitate an optimal mechanism study on a planner's group-specific preferential mandate versus the possible moral hazard problem of those who benefit from the preferential treatment.

The characterization result in the two-player matching model with arbitrary numbers of types is a step toward the direction of bringing the mechanism design approach into the study of matchings. Extension of the characterization beyond the two-player assumption has yet to be explored. The possibility that the ranking between two player-types may depend on the realized type of a third player brings out new challenges to the construction of the restricted domains for local greediness.

The impossibility observation is quite general in the knapsack model, which is relevant to sharing economies with integer constraints. Shattering the hope for an exact optimal mechanism study therein except the special case where maximization over interim allocations can reduce to pointwise maximization over ex post allocations, the impossibility observation makes approximate methods for the optimal mechanisms in such models compelling.

A Other Applications

This appendix applies Theorem 1 to another model outside the paramodularity framework and extends part of the paper to the case of infinite or continuum type spaces.

A.1 Total Unimodularity

Assume that the set X of feasible allocation outcomes is determined by a pair (\hat{f}, \hat{g}) of functions such that $\hat{f} : \mathcal{F} \rightarrow \mathbb{Z}_+$ and $\hat{g} : \mathcal{G} \rightarrow \mathbb{Z}_+$ for some $\mathcal{F}, \mathcal{G} \subseteq 2^{I \times J}$, with the properties $\hat{f} \geq \hat{g}$ on $\mathcal{F} \cap \mathcal{G}$, $\emptyset \in \mathcal{F} \Rightarrow \hat{f}(\emptyset) = 0$, $\emptyset \in \mathcal{G} \Rightarrow \hat{g}(\emptyset) = 0$ and X is nonempty and compact. Let \mathcal{P} be the set of vectors $(w, \varphi, \gamma) \in \mathbb{R}^{\mathcal{Z}} \times \mathbb{R}_+^{\mathcal{F} \times T} \times \mathbb{R}_+^{\mathcal{G} \times T}$ such that

$$\forall (i, j) \in I \times J \forall t \in T : w(j, i, t_i) - \sum_{F \in \mathcal{F}} \varphi(F, t) \chi_F(i, j) + \sum_{G \in \mathcal{G}} \gamma(G, t) \chi_G(i, j) = 0. \quad (22)$$

We can think of $\varphi(F, t)$ as the Lagrange multiplier for the ceiling constraint $\sum_{(i,j) \in F} x_i^j \leq \hat{f}(F)$, and $\gamma(G, t)$ the Lagrange multiplier for the floor constraint $\sum_{(i,j) \in G} x_i^j \geq \hat{g}(G)$, in the problem (7) given type profile $t \in T$. Note that these Lagrange multipliers are tailored individually for each specific type profile t . By contrast, the shadow prices $p_+(S)$ and $p_-(S)$ in Theorem 1 are each uniform across all type profiles $t \in T$.

Lang and Yang's [20] *total unimodular* assumption stipulates that the matrix $[\mathbf{M}_1, \mathbf{M}_2]^\top$ for which $\mathcal{P} = \{v \mid \mathbf{M}_1 v = 0; \mathbf{M}_2 v \geq 0\}$ is totally unimodular, namely, the determinant of every square submatrix of every order is 0, 1, or -1 . The main implication of this assumption is that every $(w, \varphi, \gamma) \in \mathcal{P}$ is a conic combination of some $(w_k, \varphi_k, \gamma_k)_{k=1}^K$ for which each w_k is $\{-1, 0, 1\}$ -valued, and all φ_k and γ_k are $\{0, 1\}$ -valued.

From the perspective of Theorem 1, the essence of total unimodularity is that for every supporting hyperplane of the set of feasible interim allocations, the gradient w can be decomposed into a conic combination of $\{-1, 0, 1\}$ -valued extreme rays such that the dual variable of problem (7), which determines the underlying ex post allocation outcome $\bar{q}(t)$ given type profile $t \in T$, can be decomposed by the same coefficients for all type profiles $t \in T$. This, coupled with two other conditions in Lang and Yang, turns out to imply that the ex post allocation \bar{q} follows the greedy-generous algorithm globally on the entire domain \mathcal{Z} of interim allocations according to any of the extreme rays that the gradient w is decomposed into. The corresponding upper or lower contour sets then constitute the support for the shadow prices sought after in Theorem 1.

I thus formulate the essential implication described above into a decomposability assumption. It requires only the elements of a subset of \mathcal{P} to be conic combinations of some $(w_k, \varphi_k, \gamma_k)_{k=1}^K$ for which only every w_k is required to be $\{-1, 0, 1\}$ -valued.

A constraint structure $(\mathcal{F}, \mathcal{G}, \hat{f}, \hat{g})$ is said to be *decomposable* iff for any $w \in \mathbb{R}^{\mathcal{Z}}$ there exists a solution (φ^*, γ^*) to the problem

$$\begin{aligned} \min_{(\varphi, \gamma) \in \mathbb{R}_+^{\mathcal{F} \times T} \times \mathbb{R}_+^{\mathcal{G} \times T}} \quad & \sum_{t \in T} \mu(t) \left(\sum_{F \in \mathcal{F}} \hat{f}(F) \varphi(F, t) - \sum_{G \in \mathcal{G}} \hat{g}(G) \gamma(G, t) \right) \\ & \text{subject to (22)} \end{aligned} \quad (23)$$

such that $(w_k, \varphi_k, \gamma_k)_{k=1}^K$ is a conic combination among some $(w_1, \varphi_1, \gamma_1), \dots, (w_K, \varphi_K, \gamma_K) \in \mathcal{P}$ (for some integer K) and

$$\forall k \in \{1, \dots, K\} \forall (i, j) \in I \times J \forall t_i \in T_i : w_k(j, i, t_i) \in \{0, 1, -1\}. \quad (24)$$

Remark 1 *If $(\mathcal{F}, \mathcal{G}, \hat{f}, \hat{g})$ satisfies total unimodularity then it is decomposable.*¹⁵

I also assume another implication of total unimodularity noted by Lang and Yang, that the convex hull $\text{cv}(X)$ of the set X of feasible allocation outcomes is determined by (\hat{f}, \hat{g}) . The other assumption I adopt from Lang and Yang is linearity of a function defined by

$$h(F, G) := \max_{x \in \text{cv} X} \left(\sum_{i \in F} x_i - \sum_{i \in G} x_i \right) \quad (25)$$

for any $F, G \subseteq I$ such that $F \cap G = \emptyset$. The function h is said to be *linear* iff

$$h(F, G) = h(F, \emptyset) + h(\emptyset, G)$$

for all $F, G \subseteq I$ such that $F \cap G = \emptyset$.¹⁶

¹⁵One can further prove that paramodularity implies decomposability. Thus, decomposability allows for arbitrary numbers of types per player, as paramodularity does. Different from decomposability, total unimodularity has not been known to include paramodularity, and it remains an open question whether total unimodularity in matching models allows a player to have more than two types.

¹⁶The linearity assumption is introduced by Lang and Yang [20]. Without the assumption, their main result is a necessity and sufficiency test against any pair $(S_+, S_-) \in (2^{\mathcal{Z}})^2$ of disjoint subsets. By contrast, the Border condition (6) in my paper is a necessity and sufficiency test against only any set $S \in 2^{\mathcal{Z}}$. As noted in their Corollary 1, their test reduces to mine if the linearity is assumed. Thus the next Proposition 5 corresponds to a slight generalization of their Corollary 1, which also assumes linearity.

Proposition 5 *If $(\mathcal{F}, \mathcal{G}, \hat{f}, \hat{g})$ is decomposable, (\hat{f}, \hat{g}) determines $\text{cv}(X)$, and h is linear, then $\mathcal{Q} \supseteq \mathcal{Q}_B$.*

To prove the proposition, pick any $w \in \mathbb{R}^{\mathcal{Z}}$. By the decomposability assumption, there exist $(\varphi^*, \gamma^*) \in \mathbb{R}_+^{\mathcal{F} \times T} \times \mathbb{R}_+^{\mathcal{G} \times T}$, $(w_k, \varphi_k, \gamma_k)_{k=1}^K$ and $(\beta_k)_{k=1}^K \in \mathbb{R}_{++}^K$ for some integer K such that (φ^*, γ^*) is a solution to problem (23), $(w_k, \varphi_k, \gamma_k) \in \mathcal{P}$ and satisfies (24) for any $k = 1, \dots, K$, and $w = \sum_{k=1}^K \beta_k w_k$, $\varphi^* = \sum_{k=1}^K \beta_k \varphi_k$, and $\gamma^* = \sum_{k=1}^K \beta_k \gamma_k$. Now that (23) has a solution, its dual has a solution $\bar{q} = (\bar{q}_i^j)_{(i,j) \in I \times J}$. Since $\text{cv}(X)$ is determined by (\hat{f}, \hat{g}) and $\mu(t) > 0$ for all $t \in T$ (the finite set T being the support of μ), the dual is equivalent to the problem (7) for each $t \in T$, and hence $\bar{q}(t)$ is a solution thereof.

For any $k = 1, \dots, K$, define

$$\begin{aligned} S^{k,+} &:= \{z \in \mathcal{Z} \mid w_k(z) = 1\}, \\ S^{k,-} &:= \{z \in \mathcal{Z} \mid w_k(z) = -1\}. \end{aligned}$$

Due to the decomposability and linearity assumptions, one can prove (Appendix B.11):

Lemma 4 *For every $k \in \{1, \dots, K\}$ and every $t := (t_i)_{i \in I} \in T$,*

$$\begin{aligned} \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_{S^{k,+}}(j, i, t_i) &= f(S^{k,+}, t), \\ \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_{S^{k,-}}(j, i, t_i) &= g(S^{k,-}, t). \end{aligned}$$

The lemma implies immediately that $S^{k,+}$ is upward binding for \bar{q} , and $S^{k,-}$ downward binding for \bar{q} , for all $k = 1, \dots, K$.

Finally, use the conic combination condition

$$\forall (i, j) \in I \times J \forall t_i \in T_i : w(j, i, t_i) = \sum_{k=1}^K \beta_k (\chi_{S^{k,+}}(j, i, t_i) - \chi_{S^{k,-}}(j, i, t_i))$$

to obtain a solution for (8): $p_+(S^{k,+}) = p_-(S^{k,-}) = \beta_k$ for all k and $p_+(S) = p_-(S) = 0$ for all other subsets S of \mathcal{Z} . Thus, Theorem 1 implies $\mathcal{Q} \supseteq \mathcal{Q}_B$ and hence the proposition.

A.2 Infinite Type Spaces

This section removes the assumption $|T| < \infty$ in the main text.

Lemma 5 *Let T be any (possibly infinite) type space and $X \subseteq \mathcal{R}_+^{I \times J}$ be any set of feasible allocation outcomes such that the pair (f, g) of functions is defined in (4) and (5). Suppose:*

a. there exists a sequence $(X^m)_{m=M}^\infty$ that satisfies the following for each m :

i. $X^m \subseteq \mathcal{R}_+^{I \times J}$ and the pair (f^m, g^m) defined in (4) and (5) given (T, X^m) satisfies $f^m = f$ and

$$\forall S \subseteq \mathcal{Z} \ \forall t \in T : g^m(E, t) = \max\{0, g(E, t) - 1/m\}; \quad (26)$$

ii. if $|T^m| < \infty$ then $\mathcal{Q}_B = \mathcal{Q}$ in the model (T^m, X^m) ;

b. X^m converges to X (in Hausdorff metric) as $m \rightarrow \infty$.

Then $\mathcal{Q}_B = \mathcal{Q}$ in the original model (T, X) .

The proof of Lemma 5 is an extension of Che et al.'s [7, Supplemental Appendix B.2] passing-to-limit argument. The assumption (26) is to ensure that, when any given interim allocation is being approximated from below by the nearest nonnegative grid points, the floor constraints in the Border condition within the discretized model is satisfied. This assumption rules out models where a floor constraint has to be positive-integer-valued. For example, in the matching model of Section 4.3 and Lang and Yang's [20] total unimodular model, the floor constraint involves positive integers and hence cannot be perturbed downward. By contrast, the assumption is satisfied in both the paramodularity model (Section 3.1) and the ranked-item model (Section 4.2), as the next two corollaries observe.

Corollary 3 *If the set $X \subseteq \mathbb{R}_+^{I \times J}$ of feasible allocation outcomes is determined by a paramodular pair (φ, γ) of functions $2^{I \times J} \rightarrow \mathbb{R}_+$, then $\mathcal{Q}_B = \mathcal{Q}$ whether $|T|$ is finite or not.*

Corollary 4 *In the ranked-item auction model with quota constraints (Section 4.2), $\mathcal{Q}_B = \mathcal{Q}$ whether $|T|$ is finite or not.*

B Proofs

B.1 Necessity of Condition (6) ($\mathcal{Q} \subseteq \mathcal{Q}_B$)

For any $Q := (Q_i^j(t_i))_{(j,i,t_i) \in \mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}}$ and $w := (w(j, i, t_i))_{(j,i,t_i) \in \mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}}$, denote

$$\langle Q, w \rangle := \sum_{(i,j) \in I \times J} \sum_{t_i \in T_i} Q_i^j(t_i) w(j, i, t_i) \mu_i(t_i). \quad (27)$$

Then $\langle Q, w \rangle \leq \sup_{Q' \in \mathcal{Q}} \langle Q', w \rangle$ for any $Q \in \mathcal{Q}$. To prove $Q \in \mathcal{Q}_B$, namely, that Q satisfies (6), pick any $S \subseteq \mathcal{Z}$ and apply $\langle Q, w \rangle \leq \sup_{Q' \in \mathcal{Q}} \langle Q', \varphi \rangle$ to the case where $w = \chi_S$. For any $Q' \in \mathcal{Q}$, Q' satisfies (1) with respect to some ex post allocation q . Thus

$$\begin{aligned}
\sup_{Q' \in \mathcal{Q}} \langle Q', \chi_S \rangle &= \sup_{(q(t))_{t \in T} \in \prod_{t \in T} \text{cv}(X)} \sum_{(i,j) \in I \times J} \sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} q_i^j(t_i, t_{-i}) \chi_S(j, i, t_i) \mu_{-i}(t_{-i}|t_i) \mu_i(t_i) \\
&= \sum_{t \in T} \sup_{q(t) \in \text{cv}(X)} \sum_{(i,j) \in I \times J} q_i^j(t) \chi_S(j, i, t_i) \mu(t) \\
&= \sum_{t \in T} \sup_{q(t) \in X} \sum_{(i,j) \in I \times J} q_i^j(t) \chi_S(j, i, t_i) \mu(t) \\
&\leq \sum_{t \in T} f(S, t) \mu(t),
\end{aligned}$$

with the third line due to the fact that X contains all extremal points of its convex hull, and the last line due to (4). Thus the second inequality in (6) follows. The first inequality in (6) is analogous via $w := -\chi_S$. Thus $Q \in \mathcal{Q}_B$. ■

B.2 Theorem 1

Since T is assumed finite, the domain \mathcal{Z} of any interim allocation (in the form of $(j, i, t_i) \mapsto Q_i^j(t_i)$) is a finite set. Thus, the set of interim allocations is $\mathbb{R}^{\mathcal{Z}}$, and hence any continuous linear operator on \mathcal{Q} corresponds to a vector $w := (w(z))_{z \in \mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}}$.

Lemma 6 $\mathcal{Q}_B \subseteq \mathcal{Q}$ if and only if, for any $w \in \mathbb{R}^{\mathcal{Z}}$,

$$\max_{Q \in \mathcal{Q}_B} \sum_{(i,j) \in I \times J} \sum_{t_i \in T_i} w(j, i, t_i) Q_i^j(t_i) \mu_i(t_i) \leq \sum_{t \in T} \mu(t) \max_{q(t) \in \text{cv}(X)} \sum_{(i,j) \in I \times J} q_i^j(t) w(j, i, t_i). \quad (28)$$

Proof By the definition of \mathcal{Q} and (1), the right-hand side of (28) is equal to

$$\max_{Q \in \mathcal{Q}} \sum_{(i,j) \in I \times J} \sum_{t_i \in T_i} w(j, i, t_i) Q_i^j(t_i) \mu_i(t_i).$$

Then, with the notation defined in (27), (28) is equivalent to

$$\max_{Q \in \mathcal{Q}_B} \langle Q, w \rangle \leq \max_{Q \in \mathcal{Q}} \langle Q, w \rangle.$$

Note that the set \mathcal{Q} of feasible interim allocations is convex and compact, since the set of ex post allocation is convex and compact (each assigning to every $t \in T$ an element of the convex hull of the compact X) and the mapping from ex post allocations to their reduced

forms, Eq. (1), is linear and continuous. It then follows from a finite-dimensional application of the separating hyperplane theorem (Theorem 7.51 of Aliprantis and Border [2, p288]) that

$$\mathcal{Q} = \left\{ \bar{Q} \in \mathbb{R}^{\mathcal{Z}} \mid \forall w \in \mathbb{R}^{\mathcal{Z}} \left[\langle \bar{Q}, w \rangle \leq \max_{Q \in \mathcal{Q}} \langle Q, w \rangle \right] \right\}.$$

Claim: $\mathcal{Q}_B \subseteq \mathcal{Q}$ if and only if (28) holds for all $w \in \mathbb{R}^{\mathcal{Z}}$. Clearly $\mathcal{Q}_B \subseteq \mathcal{Q}$ implies (28) for all $w \in \mathbb{R}^{\mathcal{Z}}$. To prove the converse, suppose that (28) is true for all $w \in \mathbb{R}^{\mathcal{Z}}$. If $\bar{Q} \notin \mathcal{Q}$ then, by the equation displayed above, there exists w given which $\langle \bar{Q}, w \rangle > \max_{Q \in \mathcal{Q}} \langle Q, w \rangle$. Then $\langle \bar{Q}, w \rangle > \max_{Q \in \mathcal{Q}_B} \langle Q, w \rangle$ by (28). Thus $\bar{Q} \notin \mathcal{Q}_B$. ■

Proof of Theorem 1 First, we prove (a) \iff (c). As \mathcal{Q}_B is defined by the system of inequalities (6), the left-hand side of (28) is equal to

$$\begin{aligned} & \max_{((Q_i^j)_{i \in I})_{j \in J} \in \mathbb{R}^{\mathcal{Z}}} \sum_{(i,j) \in I \times J} \sum_{t_i \in T_i} \mu_i(t_i) Q_i^j(t_i) w(j, i, t_i) \\ & \forall S \subseteq \mathcal{Z} : \quad \sum_{(i,j) \in I \times J} \sum_{t_i \in T_i} \mu_i(t_i) Q_i^j(t_i) \chi_S(j, i, t_i) \leq \sum_{t \in T} \mu(t) f(S, t) \\ & \quad \sum_{(i,j) \in I \times J} \sum_{t_i \in T_i} \mu_i(t_i) Q_i^j(t_i) \chi_S(j, i, t_i) \geq \sum_{t \in T} \mu(t) g(S, t). \end{aligned}$$

The dual of this problem is

$$\begin{aligned} & \min_{(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2} \sum_{t \in T} \mu(t) \sum_{S \subseteq \mathcal{Z}} (p_+(S) f(S, t) - p_-(S) g(S, t)) \\ & \forall z \in \mathcal{Z} : \quad w(z) = \sum_{S \subseteq \mathcal{Z}} (p_+(S) - p_-(S)) \chi_S(z). \end{aligned}$$

Thus, it follows from Lemma 6 that $\mathcal{Q}_B \subseteq \mathcal{Q}$ if and only if for any $w \in \mathbb{R}^{\mathcal{Z}}$ there exists $(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2$ that satisfies (8)—the constraint in the dual—for all $z \in \mathcal{Z}$ and

$$\sum_{t \in T} \mu(t) \sum_{S \subseteq \mathcal{Z}} (p_+(S) f(S, t) - p_-(S) g(S, t)) \leq \sum_{t \in T} \mu(t) \max_{q(t) \in \text{cv}(X)} \sum_{(i,j) \in I \times J} q_i^j(t) w(j, i, t_i). \quad (29)$$

For any \bar{q} such that $\bar{q}(t)$ solves (7) for all $t \in T$, the right-hand side of the above inequality is equal to $\sum_{t \in T} \mu(t) \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) w(j, i, t_i)$ and hence the inequality is the same as

$$\sum_{t \in T} \mu(t) \sum_{S \subseteq \mathcal{Z}} (p_+(S) f(S, t) - p_-(S) g(S, t)) \leq \sum_{t \in T} \mu(t) \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) w(j, i, t_i).$$

Plug (8) into the right-hand side to rewrite this inequality as

$$\begin{aligned} & \sum_{t \in T} \mu(t) \sum_{S \subseteq \mathcal{Z}} (p_+(S) f(S, t) - p_-(S) g(S, t)) \\ & \leq \sum_{t \in T} \mu(t) \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \left(\sum_{S \subseteq \mathcal{Z}} p_+(S) \chi_S(j, i, t_i) - \sum_{S \subseteq \mathcal{Z}} p_-(S) \chi_S(j, i, t_i) \right). \end{aligned}$$

Rearrange terms to rewrite this inequality as

$$\begin{aligned} & \sum_{S \subseteq \mathcal{Z}} p_+(S) \sum_{t \in T} \mu(t) \left(f(S, t) - \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i) \right) \\ & + \sum_{S \subseteq \mathcal{Z}} p_-(S) \sum_{t \in T} \mu(t) \left(-g(S, t) + \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i) \right) \leq 0. \end{aligned} \quad (30)$$

Thus, $\mathcal{Q}_B \subseteq \mathcal{Q}$ if and only if for any $w \in \mathbb{R}^{\mathcal{Z}}$ there exists $(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2$ that satisfies (8) and (30) for any \bar{q} such that $\bar{q}(t)$ solves (7) for all $t \in T$. For any $S \subseteq \mathcal{Z}$ and $t \in T$,

$$f(S, t) = \max_{x \in X} \sum_{(i,j) \in I(S,t)} x_i^j = \max_{x \in \text{cv}(X)} \sum_{(i,j) \in I(S,t)} x_i^j \geq \sum_{(i,j) \in IS,t} \bar{q}_i^j(t) = \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i),$$

with the first “=” due to (4), the second “=” due to X containing all extremal points of $\text{cv}(X)$, the inequality due to $\bar{q}_i^j(t) \in \text{cv}(X)$, and the last “=” due to the definition of $I(S, t)$. By the same token,

$$-g(S, t) + \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_S(j, i, t_i) \geq 0$$

for all $S \subseteq \mathcal{Z}$ and all $t \in T$. This, coupled with the fact that $\mu(t) > 0$ for all $t \in T$ (T being finite and the support of μ), implies that (30) holds if and only if (9) holds for all $S \subseteq \mathcal{Z}$. In sum, $\mathcal{Q}_B \subseteq \mathcal{Q}$ if and only if for any $w \in \mathbb{R}^{\mathcal{Z}}$ there exists $(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2$ that satisfies (8) and (9) for any \bar{q} such that $\bar{q}(t)$ solves (7). In other words, (a) \iff (c).

To prove (b) \Rightarrow (a), pick any $w \in \mathbb{R}^{\mathcal{Z}}$. If there exists $(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2$ that satisfies (8) and (9) for *some* \bar{q} such that $\bar{q}(t)$ solves (7) for all t , Ineq. (29) holds by the previous reasoning below (29). Then the reasoning in the previous paragraph preceding (29) implies $\mathcal{Q} \supseteq \mathcal{Q}_B$. Thus, (b) \Rightarrow (a). Finally, note that (c) \Rightarrow (b) follows directly from the fact that a solution to (7) exists. Thus, (b) \iff (a) \iff (c), as asserted. ■

B.3 Lemma 1 and Corollary 1

Proof of Lemma 1 Let N be the cardinality of $\{w(z) \mid z \in \mathcal{Z}\}$ and list its elements in descending order so that

$$w^1 > \dots > w^m \geq 0 > w^{m+1} > \dots > w^N. \quad (31)$$

Define $U^n := \{z \in \mathcal{Z} \mid w(z) \geq w^n\}$ for each $n \in \{1, \dots, m\}$, and $L^n := \{z \in \mathcal{Z} \mid w(z) \leq w^{N-n+1}\}$ for each $n \in \{1, \dots, N-m\}$. Thus,

$$\begin{aligned} \emptyset &=: U^0 \subsetneq U^1 \subsetneq \dots \subsetneq U^m = U \\ \emptyset &=: L^0 \subsetneq L^1 \subsetneq \dots \subsetneq L^{N-m} = \mathcal{Z} \setminus U. \end{aligned}$$

Since \succeq is represented by w , $\{U^n\}_{n=1}^m$ is the family of all the upper contour sets wrt \succeq within U , and $\{L^n\}_{n=1}^{N-m}$ the family of all the lower contour sets wrt \succeq within $\mathcal{Z} \setminus U$.

By the hypothesis of the lemma, \bar{q} given any $t \in T$ is a greedy solution within U , and a generous solution within $\mathcal{Z} \setminus U$, with respect to the ordinal ranking \succeq of w . It then follows from Lemma 2 (proved in Appendix B.4) that U^n is upward binding for \bar{q} for all $n \in \{1, \dots, m\}$, and L^n downward binding for \bar{q} for all $n \in \{1, \dots, N-m\}$.

Thus, by Theorem 1, it suffices to show existence of a nonnegative solution for the p_+ and p_- in (8) that are defined respectively on $\{U^n\}_{n=1}^m$ and $\{L^n\}_{n=1}^{N-m}$. To show that, note from the definition of U^n and L^n that (8) is equivalent to

$$\begin{aligned} \forall n \leq m : \quad w^n &= p_+(U^n) + p_+(U^{n+1}) + \dots + p_+(U^m) \\ \forall n > m : \quad w^n &= -p_-(L^{N-n+1}) - p_-(L^{N-n+2}) - \dots - p_-(L^{N-m}) \end{aligned}$$

(Note on the second line that $N-n+1 \leq N-m$ because $n \geq m+1$). A solution is $p_+(U^n) = w^n - w^{n+1}$ for all $n < m$, $p_+(U^m) = w^m$, $p_-(L^{N-m}) = -w^{m+1}$, and $p_-(L^{N-n+1}) = w^{n-1} - w^n$ for all $n > m+1$ (namely, $p_-(L^k) = w^{N-k} - w^{N-k+1}$ for all $N-n+1 \leq k < N-m$). ■

Proof of Corollary 1 Pick any type profile $t := (t_i)_{i \in I} \in T$. There is a one-to-one function $\psi_t : \mathcal{R}^{I \times J} \rightarrow \mathcal{R}^{\mathcal{Z}}$ defined by $\psi_t(((x_i)_{i \in I})_{j \in J}) := (x(z))_{z \in \mathcal{Z}}$ such that for any $(j, i, t'_i) \in \mathcal{Z}$,

$$x(j, i, t'_i) = \begin{cases} x_i^j & \text{if } t'_i = t_i \\ 0 & \text{else.} \end{cases}$$

Consequently, for any $w \in \mathbb{R}^{\mathcal{Z}}$ and any $t := (t_i)_{i \in I} \in T$, problem (7) is equivalent to

$$\max_{(x(z))_{z \in \mathcal{Z}} \in \text{cv}(\psi_t(X))} \sum_{z \in \mathcal{Z}} x(z)w(z). \quad (32)$$

Claim: $\psi_t(X)$ ($= \{\psi_t(x) \mid x \in X\}$) is determined by the pair $(f(\cdot, t), g(\cdot, t))$ of functions defined on $2^{\mathcal{Z}}$ by (4) and (5). To prove that, note from the definition of ψ_t that $(x(z))_{z \in \mathcal{Z}} \in \psi_t(X)$ is equivalent to both

$$\exists ((x_i^j)_{i \in I})_{j \in J} \in X \forall (i, j) \in I \times J [x(j, i, t_i) = x_i^j] \quad (33)$$

and

$$t_i \neq t'_i \Rightarrow x(j, i, t'_i) = 0$$

hold. By (4) and (5), $f(S, t) = \varphi(I(S, t))$ and $g(S, t) = \gamma(I(S, t))$ for any $S \subseteq \mathcal{Z}$. Thus,

$$\begin{aligned} ((x_i^j)_{i \in I})_{j \in J} \in X &\iff \left[\forall S \subseteq \mathcal{Z} : \gamma(I(S, t)) \leq \sum_{(i, j) \in I(S, t)} x_i^j \leq \varphi(I(S, t)) \right] \\ &\iff \left[\forall S \subseteq \mathcal{Z} : g(S, t) \leq \sum_{(j, i, t_i) \in S} x_i^j \leq f(S, t) \right], \end{aligned}$$

with the first line due to the hypothesis of the corollary that X is determined by (φ, γ) and the fact that every subset E of I corresponds to some $S \subseteq \mathcal{Z}$ for which $I(S, t) = E$. It follows that (33) is equivalent to

$$\forall S \subseteq \mathcal{Z} : g(S, t) \leq \sum_{z \in S} x(z) \leq f(S, t). \quad (34)$$

This in turn implies $x(j, i, t'_i) = 0$ for all $t'_i \neq t_i$, because

$$t_i \neq t'_i \Rightarrow I(\{(j, i, t'_i)\}, t) = \emptyset \Rightarrow f(\{z\}, t) = g(\{z\}, t) = 0 \stackrel{(34)}{\Rightarrow} x(j, i, t'_i) = 0.$$

In sum, $(x(z))_{z \in \mathcal{Z}} \in \psi_t(X)$ if and only if (34) holds. This proves the claim.

By the definition of $I(S, t)$, it is easy to prove that $I(S \cup S', t) = I(S, t) \cup I(S', t)$ and $I(S \cap S', t) = I(S, t) \cap I(S', t)$ for all $S, S' \subseteq \mathcal{Z}$ and all $t \in T$. Thus, the pair $(f(\cdot, t), g(\cdot, t))$ is paramodular on $2^{\mathcal{Z}}$. This coupled with the claim above implies that problem (32) is solved by the greedy-generous solution according to the ordinal ranking of w on \mathcal{Z} (Hassin [15, Theorems 4 & 5]):¹⁷ List the elements of $\{w(z) \mid z \in \mathcal{Z}\}$ in descending order as in (31). Define $U^n := \{z \in \mathcal{Z} \mid w(z) \geq w^n\}$ for each $n \in \{1, \dots, m\}$, $L^n := \{z \in \mathcal{Z} \mid w(z) \leq w^{N-n+1}\}$ for each $n \in \{1, \dots, N-m\}$, and $U^0 := L^0 := \emptyset$. Then a solution to problem (32) is $\bar{x}(\cdot | t) \in \psi_t(X)$ defined by

$$\bar{x}(j, i, t'_i | t) := \begin{cases} \frac{1}{|U^n \setminus U^{n-1}|} (f(U^n, t) - f(U^{n-1}, t)) & \text{if } (j, i, t'_i) \in U^n \\ \frac{1}{|L^n \setminus L^{n-1}|} (g(L^n, t) - g(L^{n-1}, t)) & \text{if } (j, i, t'_i) \in L^n. \end{cases}$$

(This $\bar{x}(j, i, t'_i | t)$ is well-defined because if $z \in U^n$ then $w(z) \geq w^n$ for some $n \leq m$ and hence $z \notin L^k$ for all $k \leq N-m$. And $\bar{x}(\cdot | t) \in \psi_t(X)$ because if $t'_i \neq t_i$ then $f(U^n, t) - f(U^{n-1}, t) = g(L^n, t) - g(L^{n-1}, t) = 0$ and hence $\bar{x}(j, i, t'_i) = 0$.)

Apply the above reasoning to all $t \in T$ and let $\bar{q}(t) := \psi_t^{-1}(\bar{x}(\cdot | t))$. Then \bar{q} is a solution for (7) and is a greedy-generous solution that suffices $\mathcal{Q} \supseteq \mathcal{Q}_B$ according to Lemma 1. ■

¹⁷The compliance condition in Hassin's Theorem 4 is meant to be assumed for all subsets rather than only those related by set inclusion.

B.4 Lemma 2 and Proposition 1

Proof of Lemma 2 For any ex post allocation q , any type profile $t := (t_i)_{i \in I} \in T$ and any $S \subseteq \mathcal{Z}$, by (14) we have

$$\sum_{(i,j) \in I \times J} q_i^j(t) \chi_S(j, i, t_i) = \sum_{(j,i,t'_i) \in S} q_i^j(t) \chi_{\{t'_i\}}(t_i) \stackrel{(14)}{=} \sum_{(j,i,t'_i) \in S} q(j, i, t'_i | t) = \sum_{z \in S} q(z | t).$$

Thus, for any nested sequence $(S^n)_{n=0}^m$ in the form of (15), every S^n ($\forall n$) is upward binding for q if and only if

$$\forall n = 1, \dots, m : \sum_{z \in S^n} q(z | t) = f(S^n, t). \quad (35)$$

By our definition of greedy solutions within $\bigcup \mathbb{S}$ given t , to prove the equivalence between upward bindingness and greediness, it suffices to prove that (35) is equivalent to

$$\forall k = 1, \dots, m : \sum_{z \in S^k \setminus S^{k-1}} q(z | t) = f(S^k, t) - f(S^{k-1}, t). \quad (36)$$

(35) \Rightarrow (36): Just apply (35) to the cases of $n = k$ and $n = k - 1$ to obtain

$$\sum_{z \in S^k \setminus S^{k-1}} q(z | t) = \sum_{z \in S^k} q(z | t) - \sum_{z \in S^{k-1}} q(z | t) = f(S^k, t) - f(S^{k-1}, t).$$

(36) \Rightarrow (35): For any $n \in \{1, \dots, m\}$, by (15) we have $S^n = \bigsqcup_{k=1}^n (S^k \setminus S^{k-1})$ and hence

$$\sum_{z \in S^n} q(z | t) = \sum_{k=1}^n \sum_{z \in S^k \setminus S^{k-1}} q(z | t) \stackrel{(36)}{=} \sum_{k=1}^n (f(S^k, t) - f(S^{k-1}, t)) = f(S^n, t),$$

with the last “=” due to telescoping and $f(\emptyset, t) = 0$. Thus, upward bindingness is equivalent to greediness. The part for downward bindingness and generosity is symmetric. ■

Proof of Proposition 1 Let $\mathcal{Q} \supseteq \mathcal{Q}_B$ and pick any $w \in \mathbb{R}^{\mathcal{Z}}$ and any \bar{q} that solves (7). By Statement (c) of Theorem 1, let $\mathcal{S}_+ := \{S \subseteq \mathcal{Z} \mid p_+(S) > 0\}$, and $\mathcal{S}_- := \{S \subseteq \mathcal{Z} \mid p_-(S) > 0\}$. For any $z \in \mathcal{Z}$ such that $w(z) \neq 0$, Eq. (8) requires that $z \in S$ for some $S \in \mathcal{S}_+ \cup \mathcal{S}_-$. Thus $\{z \in \mathcal{Z} \mid w(z) \neq 0\} \subseteq \bigcup_{S \in \mathcal{S}_+ \cup \mathcal{S}_-} S$. Every $S \in \mathcal{S}_+$ belongs to a collection of elements of \mathcal{S}_+ such that the set inclusion relation \subseteq is total on the collection. Extend the collection to maximal to obtain a maximal nested sequence $\mathbb{S} := (S^k)_{k=1}^m$ for some $m \in \{1, 2, \dots, |\mathcal{S}_+|\}$ within \mathcal{S}_+ . By (9), every S^k ($\forall k$) is upward binding for \bar{q} , then it follows from Lemma 2

that $\bar{q}(\cdot|t)$ is a greedy solution within $\bigcup \mathbb{S}$ wrt $\succeq_{\mathbb{S}}$ for every type profile $t \in T$. Obviously \mathcal{S}_+ is the union of all these maximal nested sequences. By the same token, \mathcal{S}_- is the union of all the maximal nested sequences among its elements, and $\bar{q}(\cdot|t)$ is generous within $\bigcup \mathbb{S}$ wrt $\succeq_{\mathbb{S}}$ for every such nested sequence \mathbb{S} . ■

B.5 Lemma 3

By the definitions of \mathbf{M}_+ , \mathbf{M}_- , \mathbf{w} and \mathbf{p} in the text preceding the statement of the lemma, the equation system (8) is equivalent to $[\mathbf{M}_+, \mathbf{M}_-] \mathbf{p} = \mathbf{w}$. By Theorem 2 of Chu et al. [8], a nonnegative solution for \mathbf{p} exists if the cover order of $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ is less than or equal to the cover order of $[\mathbf{M}_+, \mathbf{M}_-]$. According to the procedure in their Section 4, the cover order of any matrix say \mathbf{A} is equal to the maximum number of (strictly) positive entries among all the nonnegative rows that any Gaussian elimination on \mathbf{A} can produce. Since the only difference between $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ and $[\mathbf{M}_+, \mathbf{M}_-]$ is the $-\mathbf{w}$ column, any nonnegative row produced by a Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ that has zero at the entry for $-\mathbf{w}$ can be produced by the same Gaussian elimination on $[\mathbf{M}_+, \mathbf{M}_-]$. Thus, the desired inequality between their cover orders follows from the hypothesis in the lemma. ■

B.6 Proposition 2

Since $|T_1| \geq 2$ by assumption, pick any $\underline{\theta}, \bar{\theta} \in T_1$ such that $\underline{\theta} \neq \bar{\theta}$. Let me start by assuming $\{1\} \cup I_* = I$ and hence (16) becomes

$$c_1 < \sum_{i \in I_*} c_i \leq C < c_1 + c_k \quad \forall k \in I_*. \quad (37)$$

When this case is done, I will extend the proof to the other case.

By the necessity assertion in Theorem 1, it suffices $\mathcal{Q} \not\supseteq \mathcal{Q}_B$ to find a $w \in \mathbb{R}^{\mathcal{Z}}$ for which there exists no $(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2$ that satisfies both (8) and (9) given any \bar{q} that solves (7). To that end, pick any $\bar{w} > \underline{w} > 0$ for which

$$\bar{w}c_1 > \underline{w} \sum_{i \in I_*} c_i. \quad (38)$$

For each $i \in I$ ($= I_* \cup \{1\}$) and any $t_i \in T_i$, let

$$w(i, t_i) := \begin{cases} \bar{w} & \text{if } (i, t_i) = (1, \bar{\theta}) \\ \underline{w} & \text{else.} \end{cases}$$

Then, for every $t := (t_i)_{i \in I} \in T$ the solution to problem (7) is

$$\bar{q}(t) = \begin{cases} (c_1, \underbrace{0, \dots, 0}_{I_*}) & \text{if } t_1 = \bar{\theta} \\ (0, (c_i)_{i \in I_*}) & \text{else.} \end{cases} \quad (39)$$

To verify that, note from the positivity of w that any solution is to max out the total usage of the capacity as long as there is still room to fit in one more player. Thus, (37) implies that there are only two contenders for the allocation outcome, either $(0, (c_i)_{i \in I_*})$ or $(c_1, 0, \dots, 0)$. When $t_1 = \bar{\theta}$, all the players in I_* gets the low weight \underline{w} while player 1 gets the high weight \bar{w} ; then the total w -value of $(0, (c_i)_{i \in I_*})$ is equal to $\underline{w} \sum_{i \in I_*} c_i$, while the total w -value of $(c_1, 0, \dots, 0)$ is $\bar{w} c_1$. By (38), $\underline{w} \sum_{i \in I_*} c_i < \bar{w} c_1$ and hence $\bar{q}(t) = (c_1, 0, \dots, 0)$. When $t_1 \neq \bar{\theta}$, every player gets the same weight \underline{w} . Then the w -value of $(c_1, 0, \dots, 0)$ is equal to $\underline{w} c_1$, while that of $(0, (c_i)_{i \in I_*})$ is $\underline{w} \sum_{i \in I_*} c_i$. Since $c_1 < \sum_{i \in I_*} c_i$ by (16), $(c_1, 0, \dots, 0)$ has less w -value than $(0, (c_i)_{i \in I_*})$ does. Thus $\bar{q}(t) = (0, (c_i)_{i \in I_*})$, and hence (39) holds.

Next, let us observe that no $S \subseteq \mathcal{S}$ that contains $(1, \underline{\theta})$ can be upward binding for \bar{q} . First, apply (3), (4) and (5) to the definition of X to obtain

$$f(S, t) = \begin{cases} \sum_{i \in I_*} c_i & \text{if } I_* \subseteq I(S, t) \\ c_1 & \text{if } 1 \in I(S, t) \neq \{1\} \cup I_* \\ \sum_{i \in I(S, t)} c_i & \text{if } 1 \notin I(S, t) \end{cases} \quad (40)$$

because $\sum_{i \in K} c_i < c_1 < \sum_{i \in I_*} c_i \leq C$ for all $K \subsetneq I_*$ by (37). Second, note that

$$[(1, \underline{\theta}) \in S, t_1 = \underline{\theta}, S \text{ is upward binding for } \bar{q}] \Rightarrow I(S, t) = \{1\} \cup I_*. \quad (41)$$

That is because $t_1 = \underline{\theta}$ implies $\bar{q}(t) = (0, (c_i)_{i \in I_*})$ by (39). Meanwhile, $(1, \underline{\theta}) \in S$ implies $1 \in I(S, t)$. If $I(S, t) \neq \{1\} \cup I_*$ then $f(S, t) = c_1$ by (40). But then S is not upward binding, because $\sum_{i \in I(S, t)} \bar{q}_i(t)$ is equal to either 0 if $I(S, t) = \{1\}$, or $\sum_{i \in K} \bar{q}_i(t) = \sum_{i \in K} c_i$ if $I(S, t) = \{1\} \cup K$ for some $K \subsetneq I_*$. In the former case, clearly $\sum_{i \in I(S, t)} \bar{q}_i(t) < f(S, t)$. In the latter case, since $\sum_{i \in I(S, t)} \bar{q}_i(t) = \sum_{i \in K} c_i < c_1 = f(S, t)$ due to (37). Thus (41) is true.

Third, note

$$[(1, \underline{\theta}) \in S, S \text{ is upward binding for } \bar{q}] \Rightarrow \forall i \in I_* \forall t_i \in T_i [(i, t_i) \in S]. \quad (42)$$

Otherwise, say $(i', s_{i'})$ is missing for some $i' \in I_*$ and $s_{i'} \in T_{i'}$, then take t such that $t_1 = \underline{\theta}$ and $t_{i'} = s_{i'}$ to see that $I(S, t) \neq \{1\} \cup I_*$, which contradicts (41), as $t_1 = \underline{\theta}$.

Fourth, consider any type profile t for which $t_1 = \bar{\theta}$. Then $\bar{q}(t) = (c_1, 0, \dots, 0)$ by (39). Given this t , if S is upward binding then $I(S, t) \supseteq I_*$ by (42); consequently, $f(S, t) = \sum_{i \in I_*} c_i$ (by (40)) on one hand, and $\sum_{i \in I(S, t)} \bar{q}_i(t) \leq \sum_{i \in \{1\} \cup I_*} \bar{q}_i(t) = c_1$ on the other hand. Thus $\sum_{i \in I(S, t)} \bar{q}_i(t) < f(S, t)$, which contradicts the premise that S is upward binding for \bar{q} .

In sum, any $S \subseteq \mathcal{Z}$ that contains $(1, \underline{\theta})$ is not upward binding for \bar{q} . Then (8) in the case of $z = (1, \underline{\theta})$ becomes

$$w(1, \underline{\theta}) = - \sum_{S \subseteq \mathcal{Z}} p_-(S) \chi_S(1, \underline{\theta}) \leq 0,$$

contradicting the fact that $w(1, \underline{\theta}) = \underline{w} > 0$. Thus, there exists no $(p_+, p_-) : 2^{\mathcal{Z}} \rightarrow \mathbb{R}_+^2$ that satisfies both (8) and (9) given any \bar{q} , as asserted.

Finally, consider the other case, where $I_0 := I \setminus (\{1\} \cup I_*) \neq \emptyset$. Since $c_i > 0$ for all $i \in I$, we can pick any $\bar{\bar{w}} > 0$ for which

$$\forall i \in I_0 : \bar{\bar{w}} c_i > \bar{w} \sum_{i \in \{1\} \cup I_*} c_i.$$

Extend the previously chosen w to the player-types in $\bigcup_{i \in I_0} (\{i\} \times T_i)$ by

$$\forall i \in I_0 \forall t_i \in T_i : w(i, t_i) := \bar{\bar{w}}.$$

Given the extended w , for every type profile t the solution $\bar{q}(t)$ to problem (7) satisfies $\bar{q}_i(t) = c_i$ for all $i \in I_0$ since every such i weighs more than the highest possible total weight of $I \setminus I_0 (= \{1\} \cup I_*)$. Let $C' = C - \sum_{i \in I_0} c_i$. By (16), $C' > 0$ and C' plays the role of the C in the proof of the previous case. Thus, for all $i \in \{1\} \cup I_*$, the solution $\bar{q}_i(t)$ is the same as what is given in (39). Consequently, the proof follows as in the previous case. ■

B.7 Proposition 3

Given any $w := (w(i, t_i))_{(i, t_i) \in \mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}}$, define a binary relation \succ on \mathcal{Z} by

$$(i, t_i) \succ (k, t_k) \iff [w(i, t_i) > w(k, t_k) \text{ or } [w(i, t_i) = w(k, t_k) \text{ and } i < k]].$$

List the elements of \mathcal{Z} as $(z^k)_{k=1}^{|\mathcal{Z}|}$ in descending order of \succ so that

$$z^1 \succ z^2 \succ \dots \succ z^{n_*} \succ z^{n_*+1} \succ \dots \succ z^{|\mathcal{Z}|},$$

with $w(z^{n*}) \geq 0 > w(z^{n*+1})$. For any $m \in \{1, \dots, |\mathcal{Z}|\}$, let

$$\begin{aligned} U^m &:= \{z^k \mid 1 \leq k \leq m\} \\ L^{|\mathcal{Z}|-m+1} &:= \{z^k \mid m \leq k \leq |\mathcal{Z}|\}. \end{aligned}$$

Thus, each U^m is an upper contour set, and each $L^{|\mathcal{Z}|-m+1}$ a lower contour set, wrt \succ .

Denote the output of the algorithm defined in Section 4.2 given type profile $t \in T$ by $\bar{q}(t)$. The proof of the proposition proceeds with the following lemmas.

Lemma 7 *For any $t \in T$ and any $m \in \{1, \dots, |\mathcal{Z}|\}$, $\sum_{i \in I(U^m, t)} \bar{q}_i(t) = f(U^m, t)$.*

Proof By the definition (4) of f , the claim is the same as $\bar{q}(t)$ maximizing the total quality $\sum_{i \in I(U^m, t)} y_i$ received by the members of $I(U^m, t)$ among all feasible allocation outcomes $(y_i)_{i \in I} \in X$. Given type profile $t := (t_i)_{i \in I}$, list all the players in I as

$$i_1, i_2, \dots, i_{|I(U^m, t)|}, \dots, i_{n-1}, i_n,$$

so that $h < k \iff (i_h, t_{i_h}) \succ (i_k, t_{i_k})$. The algorithm proceeds player-by-player in the order of this list. Note $I(U^m, t) = \{i_1, \dots, i_{|I(U^m, t)|}\}$: If $i_k \in I(U^m, t)$ and $h < k$, then $(i_k, t_{i_k}) \in U^m$ and $(i_h, t_{i_h}) \succ (i_k, t_{i_k})$; since U^m is an upper contour set wrt \succ , $(i_h, t_{i_h}) \in U^m$ and hence $i_h \in I(U^m, t)$. Thus, objects are allocated only to the members of $I(U^m, t)$ in the first $|I(U^m, t)|$ iterations. Since objects are allocated in descending order of their qualities (from low indices j_1 and j_2 to high ones), the players in $I(U^m, t)$ get higher-quality objects than those outside $I(U^m, t)$ do except when there are not enough I_1 -members in $I(U^m, t)$ to fulfill the quota ($|I(U^m, t) \cap I_1| < \zeta$) and so exactly $\zeta - |I(U^m, t) \cap I_1|$ of the top- ξ quality objects are reserved for the I_1 -members outside $I(U^m, t)$. In that case, the algorithm reserves only the lowest-quality ones in the top- ξ category, and assigns to the non- I_1 -members in $I(U^m, t)$ the highest-quality objects below the top- ξ category only if no more top- ξ unreserved object is available. Thus, the quota constraint is binding in this case, and hence these non- I_1 members cannot switch their below-top- ξ objects to the reserved top- ξ one. Thus, the total quality assigned to $I(U^m, t)$ is maximized among all feasible allocation outcomes. ■

Lemma 8 *For any $t \in T$ and any $m \in \{1, \dots, |\mathcal{Z}|\}$, $\sum_{i \in I(L^{|\mathcal{Z}|-m+1}, t)} \bar{q}_i(t) = g(L^{|\mathcal{Z}|-m+1}, t)$.*

Proof First observe, for any test set $S \subseteq \mathcal{Z}$ and any type profile $t \in T$, that

$$g(S, t) = \sum_{j=1}^n \rho_j - f(\mathcal{Z} \setminus S, t). \quad (43)$$

This follows from the feasibility requirement in this model that all objects be assigned, which implies that minimizing the total quality assigned to $I(S, t)$ is the same as maximizing the total quality assigned to $I \setminus I(S, t)$ ($= I(\mathcal{Z} \setminus S, t)$).¹⁸ Then (43) follows from the definition (5) of g . Second, note $L^{|\mathcal{Z}|-m+1} = \mathcal{Z} \setminus U^{m-1}$ by the definition of U^m and $L^{|\mathcal{Z}|-m+1}$. Thus,

$$\begin{aligned} \sum_{i \in I(L^{|\mathcal{Z}|-m+1}, t)} \bar{q}_i^j(t) &= \sum_{j=1}^n \rho_j - \sum_{i \in I(U^{m-1}, t)} \bar{q}_i^j(t) \\ &= \sum_{j=1}^n \rho_j - f(U^{m-1}, t) \\ &= g(L^{|\mathcal{Z}|-m+1}, t), \end{aligned}$$

where the first line is due to the feasibility of \bar{q} , the second line due to the Lemma 7, and the third line due to (43). ■

Lemma 9 *For any type profile $t \in T$, $\bar{q}(\cdot | t)$ is a greedy solution wrt \succ within U^{n*} , and a generous solution wrt \succ within $\mathcal{Z} \setminus U^{n*}$.*

Proof By the definitions of U^m and $L^{|\mathcal{Z}|-m+1}$, we have the nested sequence $\emptyset =: U^0 \subsetneq U^1 \subsetneq \dots \subsetneq U^{n*}$ of the upper contour sets wrt \succ in the subset of \mathcal{Z} that are assigned nonnegative w values, and the nested sequence $\emptyset =: L^0 \subsetneq L^1 \subsetneq \dots \subsetneq L^{|\mathcal{Z}|-n*} = \mathcal{Z} \setminus U^{n*}$ of the lower contour sets wrt \succ in the rest of \mathcal{Z} . Then the previous two lemmas together imply the greediness and generosity of \bar{q} due to Lemmas 2. ■

Lemma 10 *For any $t \in T$, the output $\bar{q}(t)$ of the algorithm solves problem (7), namely, maximizes $\sum_{i \in I} y_i w(i, t_i)$ among all $(y_i)_{i \in I} \in \text{cv}(X)$.*

proof Let $t \in T$. The algorithm given t makes the allocation player-by-player in the order of i_1, i_2, \dots, i_n , such that $h < k \iff (i_h, t_{i_h}) \succ (i_k, t_{i_k})$. For the maximization of $\sum_{i \in I} y_i w(i, t_i)$, there is no loss to restrict the choice set $\text{cv}(X)$ to X , because X is finite and the objective linear. Thus, pick any $(y_i)_{i \in I} \in X$ as an alternative to $\bar{q}(t)$. For any $m \in \{1, \dots, n\}$, there exists an upper contour set U wrt \succ for which $I(U, t) = \{i_1, \dots, i_m\}$ (shown in the proof of Lemma 7). Lemma 7 applied to U implies that

$$\sum_{k=1}^m y_{i_k} \leq \sum_{k=1}^m \bar{q}_{i_k}(t).$$

¹⁸I thank Yijia Ding for suggesting this symmetry to me.

Thus, $(\bar{q}_{i_k}(t))_{k=1}^n$ weakly majorizes $(y_{i_k})_{k=1}^n$. This, coupled with the fact that $(w(i_k, t_{i_k}))_{k=1}^n$ is a weakly decreasing sequence (by the definition of \succ), implies the desired inequality

$$\sum_{k=1}^n w(i_k, t_{i_k}) y_{i_k} \leq \sum_{k=1}^n w(i_k, t_{i_k}) \bar{q}_{i_k}(t). \quad \blacksquare$$

Due to the above lemmas, Proposition 3 follows from Lemma 1.

B.8 Corollary 2

Pick any $(s_1, s_2) \in (0, 1)^2$ and consider the test set (S_1, S_2) such that $S_k = \{t_k \in T_k \mid F_k(t_k) > 1 - s\}$ for each $k \in \{1, 2\}$. Thus, both the left- and right-hand sides of (19) are functions of (s_1, s_2) . Denote the left-hand side by $\Gamma(s_1, s_2)$, and right-hand side by $\Phi(s_1, s_2)$. Note from (43) that

$$\Gamma(s_1, s_2) = \frac{2}{n} \sum_{j=1}^n \rho_j - \Phi(1 - s_1, 1 - s_2). \quad (44)$$

Since the corollary assumes the quota constraint to be $\xi = \zeta = 1$, any allocation outcome that maximizes the total quality for any set $I(S, t)$ of players assigns object 1 to any I_1 -member in $I(S, t)$ and, for the other members of $I(S, t)$, assign them the top-quality objects below object 1. To ease the inspection on which group (I_1 or I_2) of players that an expression refers to, let $n_k := |I_k|$ ($= n/2$) for each $k \in \{1, 2\}$. Since types are iid within each I_1 and I_2 ,

$$\begin{aligned} \Phi(s_1, s_2) &= \sum_{k_2=0}^{n_2} \sum_{j=2}^{k_2+1} \rho_j \binom{n_1}{0} \binom{n_2}{k_2} (1 - s_1)^{n_1} s_2^{k_2} (1 - s_2)^{n_2 - k_2} \\ &\quad + \sum_{k_1=1}^{n_1} \sum_{k_2=0}^{n_2} \sum_{j=1}^{k_1+k_2} \rho_j \binom{n_1}{k_1} \binom{n_2}{k_2} s_1^{k_1} (1 - s_1)^{n_1 - k_1} s_2^{k_2} (1 - s_2)^{n_2 - k_2}. \end{aligned}$$

After a lengthy calculation (similar to Gershkov et al. [10, Appendix C]), one can obtain

$$\begin{aligned} \frac{\partial}{\partial s_1} \Phi(s_1, s_2) &= n_1 \rho_1 (1 - s_1)^{n_1 - 1} \\ &\quad + n_1 \sum_{k_1=2}^{n_1} \sum_{k_2=0}^{n_2} \rho_{k_1+k_2} \binom{n_1 - 1}{k_1 - 1} \binom{n_2}{k_2} s_1^{k_1 - 1} (1 - s_1)^{n_1 - k_1} s_2^{k_2} (1 - s_2)^{n_2 - k_2} \\ \frac{\partial}{\partial s_2} \Phi(s_1, s_2) &= n_2 \sum_{k_1=0}^{n_1} \sum_{k_2=1}^{n_2} \rho_{\max\{k_1, 1\} + k_2} \binom{n_1}{k_1} \binom{n_2 - 1}{k_2 - 1} s_1^{k_1} (1 - s_1)^{n_1 - k_1} s_2^{k_2 - 1} (1 - s_2)^{n_2 - k_2}. \end{aligned}$$

Combining these two equation with (44), one can obtain the partial derivatives of Γ :

$$\begin{aligned}\frac{\partial}{\partial s_1}\Gamma(s_1, s_2) &= n_1 \rho_1 s_1^{n_1-1} \\ &\quad + n_1 \sum_{k_1=1}^{n_1-1} \sum_{k_2=0}^{n_2} \rho_{n-k_1-k_2+1} \binom{n_1-1}{k_1-1} \binom{n_2}{k_2} s_1^{k_1-1} (1-s_1)^{n_1-k_1} s_2^{k_2} (1-s_2)^{n_2-k_2} \\ \frac{\partial}{\partial s_2}\Gamma(s_1, s_2) &= n_2 \sum_{k_1=0}^{n_1} \sum_{k_2=1}^{n_2} \rho_{n-\min\{n_1-1, k_1\}-k_2+1} \binom{n_2-1}{k_2-1} \binom{n_1}{k_1} s_1^{k_1} (1-s_1)^{n_1-k_1} s_2^{k_2-1} (1-s_2)^{n_2-k_2}.\end{aligned}$$

These expressions of the partial derivatives are equal to the interim expected qualities (as functions of upper quantiles) defined in the text preceding the statement of the corollary:

$$\begin{aligned}\frac{\partial}{\partial s_1}\Phi(s_1, s_2) &= n_1 A_1(s_1 | s_2) \\ \frac{\partial}{\partial s_2}\Phi(s_1, s_2) &= n_2 A_2(s_2 | s_1) \\ \frac{\partial}{\partial s_1}\Gamma(s_1, s_2) &= n_1 Z_1(s_1 | s_2) \\ \frac{\partial}{\partial s_2}\Gamma(s_1, s_2) &= n_2 Z_2(s_2 | s_1).\end{aligned}$$

Thus, for any $(\alpha_1, \alpha_2) \in (0, 1)^2$, $\Gamma(\alpha_1, \alpha_2)$ and $\Phi(\alpha_1, \alpha_2)$ are equal to the left- and right-hand sides of (20). Then the corollary follows from Proposition 3. ■

B.9 Non-Paramodularity of the Matching Model (Section 4.3)

By the definitions of X in Section 4.3 and the functions φ and γ defined by $\varphi(E) := \max_{x \in X} \sum_{(i,j) \in E} x_i^j$ and $\gamma(E) := \min_{x \in X} \sum_{(i,j) \in E} x_i^j$ ($\forall E \subseteq I \times J$), we have:

$$\begin{aligned}\varphi\{(1, j), (2, j)\} &= \varphi\{(i, j') \mid j' \in J'\} = \gamma\{(i, j') \mid j' \in J\} = 1 \\ [M \supseteq \{(1, j), (2, j')\}, j' \neq j] &\Rightarrow \varphi(M) = 2 \\ J'' \subsetneq J &\Rightarrow \gamma(\{i\} \times J'') = 0\end{aligned}$$

for any player $i \in I$, any object $j \in J$, and any nonempty subset $J' \subseteq J$.

A counterexample to the submodularity of φ is: $\varphi\{(1, 1), (2, 1)\} + \varphi\{(2, 1), (2, 2)\} = 1 + 1 = 2$, whereas

$$\begin{aligned}&\varphi(\{(1, 1), (2, 1)\} \cup \{(2, 1), (2, 2)\}) + \varphi(\{(1, 1), (2, 1)\} \cap \{(2, 1), (2, 2)\}) \\ &= \varphi\{(1, 1), (2, 1), (2, 2)\} + \varphi\{(2, 1)\} = 2 + 1 = 3.\end{aligned}$$

A counterexample to compliance is:

$$\varphi\{(1, 1), (2, 1)\} - \varphi(\{(1, 1), (2, 1)\} \setminus (\{1\} \times J)) = 1 - \varphi\{(2, 1)\} = 1 - 1 = 0;$$

whereas

$$\gamma(\{1\} \times J) - \gamma((\{1\} \times J) \setminus \{(1, 1), (2, 1)\}) = 1 - \gamma(\{1\} \times (J \setminus \{1\})) = 1 - 0 = 1. \quad \blacksquare$$

B.10 Proposition 4

B.10.1 The Revealed Preferences of \bar{q}

Pick any $w \in \mathbb{R}^{\mathcal{X}}$. For any type profile $t := (t_1, t_2) \in T$, the problem (7) is

$$\max_{x \in \text{cv} X} \sum_{(i,j) \in I \times J} x_i^j w(j, i, t_i) = \max_{x \in X} \sum_{(i,j) \in I \times J} x_i^j w(j, i, t_i),$$

with the equality due to linearity of the objective and finite cardinality of X . We can thus choose \bar{q} such that $\bar{q}(t)$ is the (integral) solution to the problem on the right-hand side that breaks ties by picking the player-object pair with the (lexicographically) lower index.

To express \bar{q} explicitly, define, for any $i \in \{1, 2\}$ and any $t_i \in T_i$, the top and second-highest contenders for player-type (i, t_i) by

$$\begin{aligned} j^1(i, t_i) &:= \min \left(\arg \max_{j \in J} w(j, i, t_i) \right) \\ j^2(i, t_i) &:= \min \left(\arg \max_{j \in J \setminus \{j^1(i, t_i)\}} w(j, i, t_i) \right). \end{aligned}$$

For every object j and every player-type (i, t_i) , define

$$\delta(j, i, t_i) := w(j, i, t_i) - \max_{j' \in J \setminus \{j\}} w(j', i, t_i). \quad (45)$$

Claim: For every type profile $t := (t_1, t_2) \in T$, $\bar{q}(t)$ chooses the allocation outcome $(1 \mapsto j^1(1, t_1), 2 \mapsto j^1(2, t_2))$ (matching player 1 with object $j^1(1, t_1)$, and player 2 with object $j^1(2, t_2)$) if $j^1(1, t_1) \neq j^1(2, t_2)$; if $j^1(1, t_1) = j^1(2, t_2)$ and $\delta(j^1(1, t_1), 1, t_1) \geq \delta(j^1(2, t_2), 2, t_2)$, $\bar{q}(t)$ chooses the outcome $(1 \mapsto j^1(1, t_1), 2 \mapsto j^2(2, t_2))$ (matching player 1 with the top contender for player-type $(1, t_1)$, and player 2 with the second-highest contender for player-type $(2, t_2)$); else it chooses $(1 \mapsto j^2(1, t_1), 2 \mapsto j^1(2, t_2))$.

The claim is obvious when $j^1(1, t_1) \neq j^1(2, t_2)$. When $j^1(1, t_1) = j^1(2, t_2) =: j_*$, both players 1 and 2 rank object j_* the highest in terms of w -values, while \bar{q} can match only

one of them with j_* due to the one-to-one matching constraint. That is, $\bar{q}(t)$ either couples $(j_*, 1, t_1)$ with the second-highest contender in the column for $(2, t_2)$, or couples $(j_*, 2, t_2)$ with the second-highest contender in the column for $(1, t_1)$, whichever generates the larger total w -value (with ties broken in favor of player 1). In other words, if

$$w(j_*, 1, t_1) + w(j^2(2, t_2), 2, t_2) \geq w(j_*, 2, t_2) + w(j^2(1, t_1), 1, t_1),$$

or equivalently, $\delta(j_*, 1, t_1) \geq \delta(j_*, 2, t_2)$, then $\bar{q}(t_1, t_2)$ chooses $(j_*, 1, t_1)$ over $(j_*, 2, t_2)$; else $\bar{q}(t_1, t_2)$ chooses $(j_*, 2, t_2)$ over $(j_*, 1, t_1)$. Thus, the claim follows. ■

Competition for the Same Player-Type For any player-type (i, t_i) ($i \in I, t_i \in T_i$), the behavior of \bar{q} within the set $J \times \{(i, t_i)\}$ is consistent with the ordinal ranking of the w -values of its elements, as it chooses either the top or second-highest contender and never chooses any other element in the set. More precisely, arrange the elements of $J \times \{(i, t_i)\}$ into a list

$$(j^1, i, t_i) \sim_{it_i} (j^2, i, t_i) \succ_{it_i} (j^3, i, t_i) \succ_{it_i} \cdots \succ_{it_i} (j^{N-1}, i, t_i) \quad (46)$$

($N = |J|$) such that $j^1 = j^1(i, t_i)$, $j^2 = j^2(i, t_i)$, and $n < m \Rightarrow w(j^n, i, t_i) \geq w(j^m, i, t_i)$. Define

$$\begin{aligned} V_{it_i}^n &:= \{j^k \mid 1 \leq k \leq n+1\} \quad \forall n = 1, \dots, N-1, \\ L_{it_i} &:= J \times \{(i, t_i)\}. \end{aligned}$$

Let \succeq_{it_i} denote “ \succ_{it_i} or \sim_{it_i} .” Then the $(V_{it_i}^n)_{n=1}^{N-1}$ is the set of the upper contour sets wrt \succeq_{it_i} within $J \times \{(i, t_i)\}$ and constitutes a nested sequence, and L_{it_i} is a lower contour set therein.

Competition for the Same Object For any $j \in J$, let us inspect the behavior of \bar{q} within $\{(j, i, t_i) \mid i \in I; t_i \in T_i\}$, temporarily called “row” due to its position in Table 3. First, as noted before, \bar{q} never picks a member (j, i, t_i) in the row that is neither the top nor the second-highest contender in its “column” $J \times \{(i, t_i)\}$. Second, a member (j, i, t_i) of the row that is not the top contender in its column is never chosen by \bar{q} over a member (j, k, t'_k) that is the top contender in its column $J \times \{(k, t'_k)\}$: The two members are both relevant to \bar{q} only when the type profile is (t_i, t'_k) . Now that (j, i, t_i) is not the top contender in its column, the top contender in that column gets to be chosen together with (j, k, t'_k) , and so

(j, k, t'_k) is not crowded out. Thus, the competition within a row is only among its members that are the top contenders in their own columns, namely, the members of the set

$$\mathcal{Z}_j := \{(j, i, t_i) \mid i \in I; t_i \in T_i; j = j^1(i, t_i)\}.$$

When the type profile is (t_1, t_2) and both $(j, 1, t_1)$ and $(j, 2, t_2)$ belong to \mathcal{Z}_j , the claim just proved above says that $\bar{q}(t_1, t_2)$ chooses between $(1, t_1)$ and $(2, t_2)$ according to the ordinal ranking between $\delta(j, 1, t_1)$ and $\delta(j, 2, t_2)$, with ties broken in favor of the lower-indexed one. Thus, let \succ_j denote this ranking on \mathcal{Z}_j , so

$$\delta(j, i, t_i) > \delta(j, k, t'_k) \implies (j, i, t_i) \succ_j (j, k, t'_k)$$

and, in the case of $\delta(j, i, t_i) = \delta(j, k, t'_k)$, $(j, i, t_i) \succ_j (j, k, t'_k)$ if $i < k$ or “ $i = k$ and $t_i \triangleleft t'_k$ ” (\triangleleft a strict total order on T_i). Label the elements of \mathcal{Z}_j so that $\mathcal{Z}_j = \{z^n \mid n = 1, \dots, |\mathcal{Z}_j|\}$ and

$$z^1 \succ_j z^2 \succ_j \dots \succ_j z^{|\mathcal{Z}_j|}. \quad (47)$$

For any $n = 1, \dots, |\mathcal{Z}_j|$, let

$$U_j^n := \{z^k \mid 1 \leq k \leq n\}.$$

Then the nested sequence $(U_j^n)_{n=1}^{|\mathcal{Z}_j|}$ consists of all the upper contour sets wrt \succ_j within \mathcal{Z}_j .

B.10.2 The Bindingness of Test Sets U_j^n , $V_{it_i}^n$ and L_{it_i}

By Lemma 2, to prove that U_j^n and $V_{it_i}^n$ are upward binding, it suffices to prove that \bar{q} is greedy within the corresponding restricted domains. The case for L_{it_i} is similar and simpler.

First, pick any $j \in J$ to show that \bar{q} is greedy wrt \succ_j within \mathcal{Z}_j for all type profiles. Let $n \in \{1, \dots, |\mathcal{Z}_j|\}$ and $z^n = (j, k, t_k)$ for some player-type (k, t_k) . We just need

$$\sum_{(i,j) \in I(U_j^n \setminus U_j^{n-1}, t')} \bar{q}_i^j(t') = f(U_j^n, t') - f(U_j^{n-1}, t') \quad (48)$$

for any type profile $t' := (t'_i)_{i \in I} \in T$ (let $U_j^0 := \emptyset$). If $t'_k \neq t_k$ then $I(U_j^n, t') = I(U_j^{n-1}, t')$ and $I(U_j^n \setminus U_j^{n-1}, t') = \emptyset$, and hence (48) holds, as both sides of are zero. Thus, suppose $t'_k = t_k$. Then $I(U_j^n \setminus U_j^{n-1}, t') = \{k\}$ and hence the left-hand side of (48) is equal to $\bar{q}_k(t')$. If $I(U_j^{n-1}, t') = \emptyset$, then the right-hand side is equal to $f(U_j^n, t') = 1$. Meanwhile, $I(U_j^{n-1}, t') = \emptyset$ implies that $(j, -k, t'_{-k}) \notin U_j^{n-1}$; thus, either $(j, -k, t'_{-k}) \in \mathcal{Z}_j$ and hence $(j, k, t'_k) = (j, k, t_k) \succ_j (j, -k, t'_{-k})$, or $(j, -k, t'_{-k}) \notin \mathcal{Z}_j$. In either case, (j, k, t'_k) gets chosen, namely,

$\bar{q}_k(t') = 1$. Thus, (48) holds when $I(U_j^{n-1}, t') = \emptyset$. When $I(U_j^{n-1}, t') \neq \emptyset$, $(j, i, t'_i) \in U_j^{n-1}$ for some $i \in I$, which cannot be k , because $t'_k = t_k$. That is, $i = -k$. Then $(j, i, t'_i) \succ_j (j, k, t'_k)$ and hence $\bar{q}_k(t') = 0$, so the left-hand side of (48) is zero. Meanwhile, $I(U_j^{n-1}, t') \neq \emptyset$ implies $f(U_j^{n-1}, t') = 1$ and hence the right-hand side of (48) is equal to zero. Thus (48) holds again. All cases considered, we have proved the greediness of \bar{q} wrt \succ_j .

Second, pick any $i \in I$ and any $t_i \in T_i$ to show that \bar{q} is greedy wrt \succeq_{it_i} within $J \times \{(i, t_i)\}$ for all type profiles. For any $n = 1, \dots, N-1$ with $(j^n, i, t_i) \in V_{it_i}^n$, we just need

$$\sum_{(i,j) \in I(V_{it_i}^n \setminus V_{it_i}^{n-1}, t')} \bar{q}_i^j(t') = f(V_{it_i}^n, t') - f(V_{it_i}^{n-1}, t')$$

for any type profile $t' := (t'_k)_{k \in I} \in T$ (let $V_{it_i}^0 := \emptyset$). This is trivial if $t'_i \neq t_i$. Suppose $t'_i = t_i$. Then $I(V_{it_i}^{n-1}, t')$ contains $\{(i, j^1), (i, j^2)\}$ for all $n = 2, \dots, N-1$. For all such n , $f(V_{it_i}^n, t') - f(V_{it_i}^{n-1}, t') = 1 - 1 = 0$, and (j^n, i, t_i) is not a top-two contender and hence $\bar{q}_i^{j^n}(t') = 0$; thus, the displayed equation holds for all such $n > 1$. When $n = 1$, $I(V_{it_i}^{n-1}, t') = \emptyset$ and $V_{it_i}^n \setminus V_{it_i}^{n-1} = V_{it_i}^n$; hence $f(V_{it_i}^n, t') - f(V_{it_i}^{n-1}, t') = 1 - 0 = 1$ and the left-hand side of the displayed equation is equal to $\bar{q}_i^{j^1}(t') + \bar{q}_i^{j^2}(t') = 1$. Again the equation holds in all cases.

Third, by a symmetric argument to the previous paragraph, \bar{q} is generous wrt \succeq_{it_i} within $J \times \{(i, t_i)\}$ for all type profiles $t' \in T$. In particular, if $t'_i = t_i$ then

$$\sum_{(i,j) \in I(L_{it_i}, t')} \bar{q}_i^j(t') = \sum_{j \in J} \bar{q}_i^j(t') = 1 = g(L_{it_i}, t');$$

else $I(L_{it_i}, t') = \emptyset$ and hence $\sum_{(i,j) \in I(L_{it_i}, t')} \bar{q}_i^j(t') = 0 = g(L_{it_i}, t')$. ■

B.10.3 Existence of the Shadow Price Function

We have therefore obtained collections of binding test sets:

$$\begin{aligned} \mathcal{S}_+ &:= \{U_j^n \mid j \in J; n = 1, \dots, |\mathcal{Z}_j|\} \cup \{V_{it_i}^n \mid i \in I; t_i \in T_i; n = 1, \dots, N-1\} \\ \mathcal{S}_- &:= \{L_{it_i} \mid i \in I; t_i \in T_i\}. \end{aligned}$$

Note that $\mathcal{S}_+ \cup \mathcal{S}_-$ is a covering of the domain \mathcal{Z} of interim allocations, as every $(j, i, t_i) \in \mathcal{Z}$ belongs to L_{it_i} ($= J \times \{(i, t_i)\}$). Note also that \mathcal{S}_- does not include all lower contour sets, as there is no need to do so.

To apply Theorem 1, the final step is to prove existence of the shadow price functions $p_+ : \mathcal{S}_+ \rightarrow \mathbb{R}_+$ and $p_- : \mathcal{S}_- \rightarrow \mathbb{R}_+$. According to Lemma 3, let $[\mathbf{M}_+, \mathbf{M}_-]$ be the matrix

defined there with respect to the \mathcal{S}_+ and \mathcal{S}_- here. That is, for any $(j, i, t_i) \in \mathcal{Z}$, let $[jit_i]$ denote the row of $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ corresponding to (j, i, t_i) , so that

$$\begin{aligned} [jit_i] &:= \left(([jit_i](S))_{S \in \mathcal{S}_+ \sqcup \mathcal{S}_-}, [jit_i](-\mathbf{w}) \right) \\ &:= \left((\chi_S(j, i, t_i))_{S \in \mathcal{S}_+}, (-\chi_S(j, i, t_i))_{S \in \mathcal{S}_-}, -w(j, i, t_i) \right). \end{aligned}$$

By Lemma 3, to verify the condition required in Theorem 1, it suffices to prove that no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$ can produce a nonnegative row whose entry at the $-\mathbf{w}$ position is positive. Since any Gaussian elimination consists of a sequence of row-wise operations, through addition or subtraction with each row multiplied by a coefficient, the row that the elimination procedure produces is equal to $\sum_{z \in Z} \beta_z[z]$ for some $Z \subseteq \mathcal{Z}$ such that $\beta_z \in \mathbb{R} \setminus \{0\}$ is the coefficient for the row $[z]$ in the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\mathbf{w}]$. The nonnegative-row condition in Lemma 3 means $\sum_{z \in Z} \beta_z[z] \geq \mathbf{0}$.

Thus, pick any $Z \subseteq \mathcal{Z}$ and $(\beta_z)_{z \in Z} \in (\mathbb{R} \setminus \{0\})^Z$ for which $\sum_{z \in Z} \beta_z[z] \geq \mathbf{0}$. For any $i \in I$ and any $t_i \in T_i$, define

$$Z_{it_i} := (J \times \{(i, t_i)\}) \cap Z.$$

Note that $Z = \sqcup_{i \in I} \sqcup_{t_i \in T_i} Z_{it_i}$. Thus,

$$\sum_{z \in Z} \beta_z[z] = \sum_{i \in I} \sum_{t_i \in T_i} \sum_{z \in Z_{it_i}} \beta_z[z].$$

By the definition of $L_{kt'_k}$,

$$[jit_i](L_{kt'_k}) = \begin{cases} -1 & \text{if } (i, t_i) = (k, t'_k) \\ 0 & \text{else.} \end{cases} \quad (49)$$

Similarly, $[jit_i](V_{kt'_k}^n) = 0$ for all n whenever $(k, t'_k) \neq (i, t_i)$. Thus,

$$[S = V_{it_i}^n \text{ or } S = L_{it_i}] \implies \sum_{z \in Z} \beta_z[z](S) = \sum_{z \in Z_{it_i}} \beta_z[z](S).$$

By $\sum_{z \in Z} \beta_z[z] \geq \mathbf{0}$,

$$[S = V_{it_i}^n \text{ or } S = L_{it_i}] \implies \sum_{z \in Z_{it_i}} \beta_z[z](S) \geq 0. \quad (50)$$

Rearrangement for the Same Player-Type Let $i \in I$ and $t_i \in T_i$. Define

$$\begin{aligned} Z_{it_i}^- &:= \{z \in Z_{it_i} \mid \beta_z < 0\} \\ Z_{it_i}^+ &:= Z_{it_i} \setminus Z_{it_i}^- (= \{z \in Z_{k,t_k} \mid \beta_z > 0\}). \end{aligned}$$

Thus,

$$\sum_{z \in Z_{it_i}} \beta_z[z] = \sum_{z \in Z_{it_i}^+} |\beta_z|[z] - \sum_{z \in Z_{it_i}^-} |\beta_z|[z].$$

Lemma 11 For each (i, t_i) there exist a finite set K , functions $\alpha : K \rightarrow Z_{it_i}^-$ and $\zeta : K \rightarrow Z_{it_i}^+$, and a positive vector $(\tilde{\beta}_k)_{k \in K} \in \mathbb{R}_{++}^K$ such that $\zeta(k) \succeq_{it_i} \alpha(k)$ for each $k \in K$ and

$$\sum_{z \in Z_{it_i}} \beta_z[z] = \sum_{k \in K} \tilde{\beta}_k ([\zeta(k)] - [\alpha(k)]).$$

Proof Let $u : J \times \{(i, t_i)\} \rightarrow \{1, \dots, N-1\}$ be a utility function representing the preference relation \succeq_{it_i} on $J \times \{(i, t_i)\}$ ($N-1 = |J| - 1$ is the number of \sim_{it_i} -indifference sets in $J \times \{(i, t_i)\}$). By the definition of $(V_{it_i}^n)_{n=1}^{N-1}$, for any $m \in \{1, \dots, N-1\}$,

$$\{z \in J \times \{(i, t_i)\} \mid u(z) \geq m\} = V_{it_i}^{N-m}. \quad (51)$$

Let us construct the set K recursively. Start with any element of $Z_{it_i}^-$ that minimizes u on $Z_{it_i}^-$, and denote this element by $\alpha(1)$. Initiate K by $K := \{1\}$. There exists $z' \in Z_{it_i}^+$ for which $z' \succeq_{it_i} \alpha(1)$. That is because $[\alpha(1)](V_{it_i}^{N-u(\alpha(1))}) = 1$ and hence $\alpha(1)$ contributes a negative entry to $\sum_{z \in Z_{it_i}} \beta_z[z](V_{it_i}^{N-u(\alpha(1))})$ since $\alpha(1) \in Z_{it_i}^-$; the nonnegativity of this sum then requires that there be $z' \in Z_{it_i}^+$ for which $[z'](V_{it_i}^{N-u(\alpha(1))}) = 1$ to cancel out the negative entry. That means $z' \in V_{it_i}^{N-u(\alpha(1))}$ and hence $u(z') \geq u(\alpha(1))$. Let $\zeta(1)$ denote the $u(z')$ -minimum among all the $z' \succeq_{it_i} \alpha(1)$ in $Z_{it_i}^+$. Let

$$\tilde{\beta}_1 := \min \{|\beta_{\alpha(1)}|, |\beta_{\zeta(1)}|\} \quad (= \min \{-\beta_{\alpha(1)}, \beta_{\zeta(1)}\}). \quad (52)$$

For any $z \in Z_{it_i}$, let

$$\begin{aligned} \beta'_z &:= \begin{cases} \beta_{\alpha(1)} + \tilde{\beta}_1 & \text{if } z = \alpha(1) \\ \beta_{\zeta(1)} - \tilde{\beta}_1 & \text{if } z = \zeta(1) \\ \beta_z & \text{else} \end{cases} \\ \tilde{Z}_{it_i}^- &:= \begin{cases} Z_{it_i}^- \setminus \{\alpha(1)\} & \text{if } \beta'_{\alpha(1)} = 0 \\ Z_{it_i}^- & \text{else} \end{cases} \\ \tilde{Z}_{it_i}^+ &:= \begin{cases} Z_{it_i}^+ \setminus \{\zeta(1)\} & \text{if } \beta'_{\zeta(1)} = 0 \\ Z_{it_i}^+ & \text{else.} \end{cases} \end{aligned} \quad (53)$$

Then

$$\sum_{z \in Z_{it_i}} \beta_z[z] = \sum_{k \in K} \tilde{\beta}_k([\zeta(k)] - [\alpha(k)]) + \sum_{z \in \tilde{Z}_{it_i}^- \sqcup \tilde{Z}_{it_i}^+} \beta'_z[z]. \quad (54)$$

Claim: (50) is preserved when Z_{it_i} is replaced by $\tilde{Z}_{it_i}^+ \sqcup \tilde{Z}_{it_i}^-$, and β_z replaced by β'_z . The case for $S = L_{it_i}$ follows directly from (49). For the other case, we need to show

$$\sum_{z \in \tilde{Z}_{it_i}^- \sqcup \tilde{Z}_{it_i}^+} \beta'_z[z](V_{it_i}^{N-m}) \geq 0 \quad (55)$$

for all $m = 1, \dots, N-1$. To that end, plug (53) into the left-hand side of (55) to have

$$\begin{aligned} \sum_{z \in \tilde{Z}_{it_i}^- \sqcup \tilde{Z}_{it_i}^+} \beta'_z[z](V_{it_i}^{N-m}) &= \sum_{z \in \tilde{Z}_{it_i}^+} \beta'_z[z](V_{it_i}^{N-m}) - \sum_{z \in \tilde{Z}_{it_i}^-} |\beta'_z[z]| (V_{it_i}^{N-m}) \\ &= \sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-m}) - \tilde{\beta}_1[\zeta(1)](V_{it_i}^{N-m}) \\ &\quad - \sum_{z \in Z_{it_i}^-} |\beta_z[z]| (V_{it_i}^{N-m}) + \tilde{\beta}_1[\alpha(1)](V_{it_i}^{N-m}). \end{aligned}$$

The right-hand side of the equation displayed above reduces to

$$\sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-m}) - \sum_{z \in Z_{it_i}^-} |\beta_z[z]| (V_{it_i}^{N-m})$$

when $m \leq u(\alpha(1))$ or $m > u(\zeta(1))$. That is because $m \leq u(\alpha(1))$ implies that both $\zeta(1)$ and $\alpha(1)$ belong to $V_{it_i}^{N-m}$ by (51) and so $[\zeta(k)](V_{it_i}^{N-m}) = [\alpha(k)](V_{it_i}^{N-m})$, and $m > u(\zeta(1))$ implies that neither $\zeta(1)$ nor $\alpha(1)$ belong to such $V_{it_i}^{N-m}$ and hence again $[\zeta(k)](V_{it_i}^{N-m}) = [\alpha(k)](V_{it_i}^{N-m})$. Thus, by the (50) for the original Z_{it_i} , (55) holds for all such m . For any other m , namely, $u(\zeta(1)) \geq m > u(\alpha(1))$, recall that $\zeta(1)$ is a minimum of $u(z')$ among all $z' \in Z_{it_i}^+$ such that $z' \succeq_{it_i} \alpha(1)$. Thus, decreasing m from $u(\zeta(1))$ to $u(\alpha(1))$ cannot enlarge $\sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-m})$. Hence

$$\begin{aligned} &\sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-m}) - \tilde{\beta}_1[\zeta(1)](V_{it_i}^{N-m}) - \sum_{z \in Z_{it_i}^-} |\beta_z[z]| (V_{it_i}^{N-m}) + \tilde{\beta}_1[\alpha(1)](V_{it_i}^{N-m}) \\ &\geq \sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}) - \tilde{\beta}_1[\zeta(1)](V_{it_i}^{N-m}) - \sum_{z \in Z_{it_i}^-} |\beta_z[z]| (V_{it_i}^{N-m}) + \tilde{\beta}_1[\alpha(1)](V_{it_i}^{N-m}) \\ &= \sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}) - \tilde{\beta}_1 - \sum_{z \in Z_{it_i}^- \setminus \{\alpha(1)\}} |\beta_z[z]| (V_{it_i}^{N-m}), \end{aligned}$$

with the last line due to $u(\zeta(1)) \geq m > u(\alpha(1))$ and hence $\alpha(1) \notin V_{it_i}^{N-m} \ni \zeta(1)$, namely, $[\alpha(1)](V_{it_i}^{N-m}) = 0$ and $\zeta(1)[(V_{it_i}^{N-m})] = 1$. Since $(V_{it_i}^n)_{n=1}^{N-1}$ is a nested increasing sequence and $m > u(\alpha(1))$, $z \in V_{it_i}^{N-m} \Rightarrow z \in V_{it_i}^{N-u(\alpha(1))}$. Thus,

$$\sum_{z \in Z_{it_i}^- \setminus \{\alpha(1)\}} |\beta_z[z]| (V_{it_i}^{N-m}) \leq \sum_{z \in Z_{it_i}^- \setminus \{\alpha(1)\}} |\beta_z[z]| (V_{it_i}^{N-u(\alpha(1))}).$$

Plug this into the multiline formulas displayed above to obtain

$$\sum_{z \in \tilde{Z}_{it_i}^- \sqcup \tilde{Z}_{it_i}^+} \beta'_z[z](V_{it_i}^{N-m}) \geq \sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}) - \tilde{\beta}_1 - \sum_{z \in Z_{it_i}^- \setminus \{\alpha(1)\}} |\beta_z[z]| (V_{it_i}^{N-u(\alpha(1))}).$$

By the definition (Eq. (52)) of $\tilde{\beta}_1$, $\tilde{\beta}_1 \leq |\beta_{\alpha(1)}| = |\beta_{\alpha(1)}| [\alpha(1)](V_{it_i}^{N-u(\alpha(1))})$ and hence

$$-\tilde{\beta}_1 - \sum_{z \in Z_{it_i}^- \setminus \{\alpha(1)\}} |\beta_z[z]| (V_{it_i}^{N-u(\alpha(1))}) \geq - \sum_{z \in Z_{it_i}^-} |\beta_z[z]| (V_{it_i}^{N-u(\alpha(1))}).$$

Plug this back into the previous inequality to obtain

$$\sum_{z \in \tilde{Z}_{it_i}^- \sqcup \tilde{Z}_{it_i}^+} \beta'_z[z](V_{it_i}^{N-m}) \geq \sum_{z \in Z_{it_i}^+} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}) - \sum_{z \in Z_{it_i}^-} |\beta_z[z]| (V_{it_i}^{N-u(\alpha(1))}) = \sum_{z \in Z_{it_i}} \beta_z[z](V_{it_i}^{N-u(\alpha(1))}),$$

which is nonnegative by the (50) for the original Z_{it_i} . Thus (55) follows.

Now let $\tilde{Z}_{it_i}^+$, $\tilde{Z}_{it_i}^-$ and β'_z take the roles of $Z_{it_i}^+$, $Z_{it_i}^-$ and β_z . This preserves (50) by the above reasoning and reduces the cardinality of $Z_{it_i}^+$ or $Z_{it_i}^-$ by one.

Repeat the above procedure on the updated $Z_{it_i}^+$, $Z_{it_i}^-$ and β_z . Let $\alpha(2)$ be an element of $Z_{it_i}^-$ that minimizes $u(\cdot)$ on $Z_{it_i}^-$, and $\zeta(2)$ an element of $Z_{it_i}^+$ that minimizes $u(z')$ among all $z' \in Z_{it_i}^+$ for which $z' \succeq_{it_i} \alpha(2)$. Update $K := K \sqcup \{2\}$ and define $\tilde{\beta}_2$ as in (52). Then update $Z_{it_i}^+$, $Z_{it_i}^-$ and β_z again accordingly. As before, this update preserves (50) and further reduces the cardinality of $Z_{it_i}^+$ or $Z_{it_i}^-$ by one.

Continue this procedure until $Z_{it_i}^+$ or $Z_{it_i}^-$ becomes empty. In the end, $Z_{it_i}^- \neq \emptyset = Z_{it_i}^+$ is impossible, otherwise the nonnegativity condition in (50) is violated. Neither is $Z_{it_i}^- = \emptyset \neq Z_{it_i}^+$ possible, otherwise the elements of $Z_{it_i}^+$, by (49), have negative entries at the position for L_{it_i} that are not canceled out, again violating the nonnegativity condition. Both sets empty, (54) turns into the equation that the lemma claims. ■

Apply the lemma to all player-types to obtain

$$\sum_{z \in Z} \beta_z[z] = \sum_{i \in I} \sum_{t_i \in T_i} \sum_{k \in K_{it_i}} \tilde{\beta}_k([\zeta_{it_k}(k)] - [\alpha_{it_k}(k)])$$

such that for every player-type (i, t_i) , α_{it_k} and ζ_{it_k} are functions $K_{it_i} \rightarrow Z_{it_i}^-$ and $K_{it_i} \rightarrow Z_{it_i}^+$, and for each $k \in K_{it_i}$, $\zeta_{it_i}(k)$ is a \succeq_{it_i} -minimum among $z \in Z_{it_i}^+$ for which $z \succeq_{it_i} \alpha(k)$. Let

$$\mathbb{K} := \sqcup_{i \in I} \sqcup_{t_i \in T_i} K_{it_i}$$

and extend α and ζ to \mathbb{K} by $k \in K_{it_i} \Rightarrow [\alpha(k) := \alpha_{it_i}(k) \text{ and } \zeta(k) := \zeta_{it_i}(k)]$. Then the above equation is more succinctly rewritten as

$$\sum_{z \in Z} \beta_z[z] = \sum_{k \in \mathbb{K}} \tilde{\beta}_k([\zeta(k)] - [\alpha(k)]). \quad (56)$$

Rearrangement for the Same Object Recall that for every $j \in J$, \mathcal{Z}_j is the set of (j, i, t_i) across player-types (i, t_i) such that (j, i, t_i) is the top contender in $J \times \{(i, t_i)\}$. Let

$$\mathbb{K}_j := \{k \in \mathbb{K} \mid \exists i \in I \exists t_i \in T_i [(j, i, t_i) \in \mathcal{Z}_j \cap \{\alpha(k), \zeta(k)\}]\}$$

for each $j \in J$. Note that $j \neq j' \Rightarrow \mathbb{K}_j \cap \mathbb{K}_{j'} = \emptyset$. That is because for each player-type (i, t_i) there is a unique top contender $j^1(i, t_i)$ in $J \times \{(i, t_i)\}$ by the definition of $j^1(i, t_i)$ in Section 4.3. Thus, if one of $\alpha(k)$ and $\zeta(k)$ is a top contender then the other is not.

By the definition of $[\mathbf{M}_+, \mathbf{M}_-]$ and that of the sets $(U_j^n)_{n=1}^{|\mathcal{Z}_j|}$, $[j'it_i](U_j^n) = 0$ for all n unless $j' = j$. Thus, for all $j \in J$ and all $n \in \{1, \dots, |\mathcal{Z}_j|\}$,

$$\sum_{z \in Z} \beta_z[z](U_j^n) = \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k([\zeta(k)](U_j^n) - [\alpha(k)](U_j^n)).$$

For all such j and n , by the nonnegative-row condition,

$$\sum_{k \in \mathbb{K}_j} \tilde{\beta}_k([\zeta(k)](U_j^n) - [\alpha(k)](U_j^n)) \geq 0. \quad (57)$$

For any $j \in J$ such that $\mathbb{K}_j \neq \emptyset$, define

$$\begin{aligned} \mathbb{K}_j^- &:= \{k \in \mathbb{K}_j \mid \alpha(k) \in \mathcal{Z}_j\} \\ \mathbb{K}_j^+ &:= \{k \in \mathbb{K}_j \mid \zeta(k) \in \mathcal{Z}_j\}. \end{aligned}$$

Thus, if $k \in \mathbb{K}_j^-$ then $[\alpha(k)]$ enters the Gaussian elimination through subtraction (negative $\beta_{\alpha(k)}$) and, since $\alpha(k) \in \mathcal{Z}_j$, the subtraction $-[\alpha(k)]$ contributes a negative entry at the position for some U_j^n . Analogously, if $k \in \mathbb{K}_j^+$ then $[\zeta(k)]$ enters the Gaussian elimination through addition and contributes a positive entry at the position for some U_j^n .

Since $\alpha(k)$ and $\zeta(k)$ cannot be both top contenders, $\mathbb{K}_j^- \cap \mathbb{K}_j^+ = \emptyset$. Thus, (57) becomes

$$\sum_{k \in \mathbb{K}_j^+} \tilde{\beta}_k ([\zeta(k)](U_j^n) - [\alpha(k)](U_j^n)) \geq \sum_{k \in \mathbb{K}_j^-} \tilde{\beta}_k ([\alpha(k)](U_j^n) - [\zeta(k)](U_j^n))$$

for all n . Since neither the $\alpha(k)$ on the left-hand side nor the $\zeta(k)$ on the right-hand side are top contenders, their values at U_j^n are both zero. Thus, for any $n = 1, \dots, |\mathcal{Z}_j|$,

$$\sum_{k \in \mathbb{K}_j^+} \tilde{\beta}_k [\zeta(k)](U_j^n) \geq \sum_{k \in \mathbb{K}_j^-} \tilde{\beta}_k [\alpha(k)](U_j^n). \quad (58)$$

Lemma 12 *For each $j \in J$ such that $\mathbb{K}_j \neq \emptyset$, there exist a finite set H , functions $\theta : H \rightarrow \mathbb{K}_j^-$ and $\phi : H \rightarrow \mathbb{K}_j^+$, a positive vector $(\beta_h^*)_{h \in H} \in \mathbb{R}_{++}^H$ and a $\mathbb{K}_j^* \subseteq \mathbb{K}_j^+$ such that $\zeta(\phi(h)) \succeq_j \alpha(\theta(h))$ for each $h \in H$ and*

$$\begin{aligned} \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k ([\zeta(k)] - [\alpha(k)]) &= \sum_{h \in H} \beta_h^* ([\zeta(\phi(h))] - [\alpha(\phi(h))] + [\zeta(\theta(h))] - [\alpha(\theta(h))]) \\ &\quad + \sum_{k \in \mathbb{K}_j^*} \tilde{\beta}_k ([\zeta(k)] - [\alpha(k)]). \end{aligned}$$

Proof Mimic the proof of Lemma 11. Let $v : \mathcal{Z} \rightarrow \{1, \dots, |\mathcal{Z}_j|\}$ be a utility function representing the preference relation \succeq_j on \mathcal{Z}_j . By the definition of $(U_j^n)_{n=1}^{|\mathcal{Z}_j|}$, for any $m \in \{1, \dots, |\mathcal{Z}_j|\}$,

$$\{z \in \mathcal{Z}_j \mid v(z) \geq m\} = U_j^{|\mathcal{Z}_j| - m + 1}.$$

To construct the set H , start with any element of \mathbb{K}_j^- that minimizes $v(\alpha(k))$ among all $k \in \mathbb{K}_j^-$ and denote it by $\theta(1)$. Initiate H by $H := \{1\}$. By (58), let $\phi(1)$ be any $k' \in \mathbb{K}_j^+$ that minimizes $v(\zeta(k'))$ among all $k' \in \mathbb{K}_j^+$ for which $\zeta(k') \succeq_j \alpha(\theta(1))$. Let

$$\beta_1^* := \min \left\{ \tilde{\beta}_{\theta(1)}, \tilde{\beta}_{\phi(1)} \right\}.$$

Just as in (53) for Lemma 11, dissect the larger one between $\tilde{\beta}_{\theta(1)}$ and $\tilde{\beta}_{\phi(1)}$ into two portions, one equal to β_1^* , and the other equal to the remaining portion. Extract the β_1^* portions of $\theta(1)$ and $\phi(1)$ from \mathbb{K}_j^- and \mathbb{K}_j^+ and put them into the set indexed by H . Then, analogous to (54),

$$\begin{aligned} \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k ([\zeta(k)] - [\alpha(k)]) &= \sum_{h \in H} \beta_h^* ([\zeta(\phi(h))] - [\alpha(\phi(h))] + [\zeta(\theta(h))] - [\alpha(\theta(h))]) \quad (59) \\ &\quad + \sum_{k \in \tilde{\mathbb{K}}_j^- \sqcup \tilde{\mathbb{K}}_j^+} \tilde{\beta}'_k ([\zeta(k)] - [\alpha(k)]), \end{aligned}$$

with $\tilde{\mathbb{K}}_j^-$, $\tilde{\mathbb{K}}_j^+$ and $\tilde{\beta}'_k$ analogously defined. Just like (55), for all $n \in \{1, \dots, |\mathcal{Z}_j|\}$,

$$\sum_{k \in \tilde{\mathbb{K}}_j^- \sqcup \tilde{\mathbb{K}}_j^+} \tilde{\beta}'_k ([\zeta(k)](U_j^n) - [\alpha(k)](U_j^n)) \geq 0$$

due to (58), $(U_j^n)_{n=1}^{|\mathcal{Z}_j|}$ being an increasing nested sequence, and $\phi(1)$ minimizes $v(\zeta(k))$ among the $k \in \mathbb{K}_j^+$ such that $\zeta(k) \succeq_j \alpha(\theta(1))$.

Now let $\tilde{\mathbb{K}}_j^-$, $\tilde{\mathbb{K}}_j^+$ and $\tilde{\beta}'_k$ take the roles of \mathbb{K}_j^- , \mathbb{K}_j^+ and β_k . This preserves (58) and reduces the cardinality of \mathbb{K}_j^- or \mathbb{K}_j^+ by one. Repeat until at least one of the two sets becomes empty. By (58), $\mathbb{K}_j^- \neq \emptyset = \mathbb{K}_j^+$ is impossible. Thus, in the end, $\mathbb{K}_j^- = \emptyset$ and thus the last sum in (59) becomes

$$\sum_{k \in \tilde{\mathbb{K}}_j^+} \tilde{\beta}'_k ([\zeta(k)] - [\alpha(k)])$$

with the updated $\tilde{\mathbb{K}}_j^+$ in the end. Assign this $\tilde{\mathbb{K}}_j^+$ to \mathbb{K}_j^* and the desired equation obtains. ■

B.10.4 Proof of Proposition 4

By Lemma 3, the proposition follows from Theorem 1 if $\sum_{z \in Z} \beta_z[z](-\mathbf{w}) \leq 0$. Suppose, to the contrary, that the inequality does not hold. Then by (56),

$$\sum_{k \in \mathbb{K}} \tilde{\beta}_k (w(\zeta(k)) - w(\alpha(k))) < 0.$$

This inequality implies that $w(\zeta(k)) - w(\alpha(k)) < 0$ for some $k \in \mathbb{K}$, because $\tilde{\beta}_k > 0$ for all k (Lemma 11). Recall that for every $k \in K$, both $\zeta(k)$ and $\alpha(k)$ belong to some $J \times \{(i, t_i)\}$ and $\zeta(k) \succeq_{it_i} \alpha(k)$. Thus, if $w(\zeta(k)) - w(\alpha(k)) < 0$ then $\alpha(k) \succeq_{it_i} \zeta(k)$ and so $\alpha(k) \sim_{it_i} \zeta(k)$, which coupled with $w(\zeta(k)) - w(\alpha(k)) < 0$ implies that $\alpha(k)$ is the top contender in $J \times \{(i, t_i)\}$. Consequently, $\alpha(k) \in \mathcal{Z}_j$ for some $j \in J$. In other words, $w(\zeta(k)) - w(\alpha(k)) < 0 \Rightarrow k \in \mathbb{K}_j$ for some $j \in J$, and hence the strict inequality displayed above implies

$$\sum_{j \in J} \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k (w(\zeta(k)) - w(\alpha(k))) < 0.$$

Since $\tilde{\beta}_k > 0$ for all $k \in \mathbb{K}$, the above inequality in turn implies that for some $j \in J$,

$$\sum_{k \in \mathbb{K}_j} \tilde{\beta}_k (w(\zeta(k)) - w(\alpha(k))) < 0.$$

Meanwhile, apply the equation asserted by Lemma 12 to the entry for $-\mathbf{w}$ and multiply both sides of the equation by -1 to obtain

$$\begin{aligned} \sum_{k \in \mathbb{K}_j} \tilde{\beta}_k (w(\zeta(k)) - w(\alpha(k))) &= \sum_{h \in H} \beta_h^* (w(\zeta(\phi(h))) - w(\alpha(\phi(h))) + w(\zeta(\theta(h))) - w(\alpha(\theta(h)))) \\ &\quad + \sum_{k \in \mathbb{K}_j^*} \tilde{\beta}_k (w(\zeta(k)) - w(\alpha(k))). \end{aligned}$$

To derive a contradiction, I claim that the right-hand side is nonnegative. First, $\sum_{k \in \mathbb{K}_j^*} \tilde{\beta}_k (w(\zeta(k)) - w(\alpha(k))) \geq 0$. That follows from the definition of \mathbb{K}_j^* in Lemma 12: For any $k \in \mathbb{K}_j^*$, with $\mathbb{K}_j^* \subseteq \mathbb{K}_j^+$, $\alpha(k)$ is not a top contender and hence, by the construction of ζ , $w(\zeta(k)) \geq w(\alpha(k))$. Thus, the sum is nonnegative ($\tilde{\beta}_k \geq 0$ for all k). Second,

$$w(\zeta(\phi(h))) - w(\alpha(\phi(h))) + w(\zeta(\theta(h))) - w(\alpha(\theta(h))) \geq 0 \quad (60)$$

for all $h \in H$ due to the property of (θ, ϕ) : By Lemma 12, $\zeta(\phi(h)) \succeq_j \alpha(\theta(h))$. Thus, $\delta(\zeta(\phi(h))) \geq \delta(\alpha(\theta(h)))$ by the definition of \succeq_j . Consequently, by the definition of δ ,

$$w(\zeta(\phi(h))) - w(\alpha(\phi(h))) \geq \delta(\zeta(\phi(h))) \geq \delta(\alpha(\theta(h))),$$

with the first inequality due to the fact that $\zeta(\phi(h))$ is a top contender (the fact that $\zeta(\phi(h)) \succeq_j \alpha(\theta(h))$ requires $\zeta(\phi(h)) \in \mathcal{Z}_j$). Furthermore, by Lemma 12, $\theta(h) \in \mathbb{K}_j^-$ and hence $\alpha(\theta(h)) \in \mathcal{Z}_j$, namely, $\alpha(\theta(h))$ is a top contender. This coupled with the fact $\zeta(\theta(h)) \succeq_{it_i} \alpha(\theta(h))$ (Lemma 11) implies that $\zeta(\theta(h))$ is the second-highest contender in the $J \times \{(i, t_i)\}$ where $\alpha(\theta(h))$ is the top contender. Thus, by the definition of δ ,

$$\delta(\alpha(\theta(h))) = w(\alpha(\theta(h))) - w(\zeta(\theta(h))).$$

This combined with the inequality displayed above gives (60), as desired. ■

B.11 Lemma 4 and Remark 1

Proof of Lemma 4 Since (φ^*, γ^*) is a solution to (23) and \bar{q} a solution to the dual thereof, $(\bar{q}, \varphi^*, \gamma^*)$ is a saddle point of the Lagrangian

$$\begin{aligned} L(q, \varphi, \gamma) &:= \sum_{(i,j) \in I \times J} \sum_{t \in T} w(j, i, t_i) q_i^j(t) \mu(t) + \sum_{F \in \mathcal{F}} \sum_{t \in T} \varphi(F, t) \left(\hat{f}(F) - \sum_{i \in F} q_i^j(t) \right) \mu(t) \\ &\quad + \sum_{G \in \mathcal{G}} \sum_{t \in T} \gamma(G, t) \left(\sum_{i \in F} q_i^j(t) - \hat{g}(G) \right) \mu(t). \end{aligned}$$

For any $t \in T$, $k \in \{1, \dots, K\}$, $q(t) \in \text{cv}X$, $\varphi_k(\cdot, t) \in \mathbb{R}_+^{\mathcal{F}}$ and $\gamma(\cdot, t) \in \mathbb{R}_+^{\mathcal{G}}$, define

$$\begin{aligned} L_k^t(q(t), \varphi(\cdot, t), \gamma(\cdot, t)) &:= \sum_{(i,j) \in I \times J} w_k(j, i, t_i) q_i^j(t) \mu(t) + \sum_{F \in \mathcal{F}} \varphi_k(F, t) \left(\hat{f}(F) - \sum_{i \in F} q_i^j(t) \right) \mu(t) \\ &\quad + \sum_{G \in \mathcal{G}} \gamma_k(G, t) \left(\sum_{i \in G} q_i^j(t) - \hat{g}(G) \right) \mu(t). \end{aligned}$$

Claim: For any $k = 1, \dots, K$ and any $t \in T$, $(\bar{q}(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ is a saddle point of $L_k^t(q(t), \varphi(\cdot, t), \gamma(\cdot, t))$. First, since $(w_k, \varphi_k^*, \gamma_k^*)$ belongs to \mathcal{P} and hence satisfies (22), $\bar{q}(t)$ maximizes $L_k(q(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ over all $q(t) \in \mathbb{R}^I$. Second, note that

$$L\left(q, \sum_k \beta_k \varphi_k, \sum_k \beta_k \gamma_k\right) = \sum_{k=1}^K \beta_k \sum_{t \in T} L_k^t(q(t), \varphi_k(\cdot, t), \gamma_k(\cdot, t)).$$

Since $(\bar{q}, \varphi^*, \gamma^*)$ is a saddle point of L , L is minimized by $(\sum_k \beta_k \varphi_k^*, \sum_k \beta_k \gamma_k^*)$ given $q = \bar{q}$, as $\varphi^* = \sum_k \beta_k \varphi_k^*$ and $\gamma^* = \sum_k \beta_k \gamma_k^*$. Consequently, since $\beta_k > 0$ for all k and $(\varphi_k(\cdot, t), \gamma_k(\cdot, t))$ does not enter $L_{k'}^t$ for any $(k', t') \neq (k, t)$, $(\varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ minimizes L_k^t given $q(t) = \bar{q}(t)$, for any k and any t . Thus, $(\bar{q}(t), \varphi_k^*(\cdot, t), \gamma_k^*(\cdot, t))$ is a saddle point of L_k^t , as claimed.

By the claim proved above, for any $k = 1, \dots, K$ and any $t \in T$, $\bar{q}(t)$ solves

$$\begin{aligned} &\max_{(q_i^j(t))_{(i,j) \in I \times J} \in \mathbb{R}^I} \sum_{(i,j) \in I \times J} w_k(j, i, t_i) q_i^j(t) \mu(t) \\ &\text{s.t.} \quad \left(\hat{f}(F) - \sum_{i \in F} q_i^j(t) \right) \mu(t) \geq 0 \quad (\forall F \in \mathcal{F}) \\ &\quad \left(\sum_{i \in G} q_i^j(t) - \hat{g}(G) \right) \mu(t) \geq 0 \quad (\forall G \in \mathcal{G}), \end{aligned}$$

because L_k^t is the Lagrangian associated to this problem. This, combined with $\mu(t) > 0$ and the assumption that (\hat{f}, \hat{g}) determines $\text{cv}(X)$, implies that for any $k = 1, \dots, K$ and any $t \in T$, $\bar{q}(t)$ solves $\max_{x \in \text{cv}X} \sum_{(i,j) \in I \times J} w_k(j, i, t_i) x_i$.

For any $k = 1, \dots, K$ and any $t := (t_i)_{i \in I} \in T$, since w_k is $\{0, 1, -1\}$ -valued by (24),

$$\max_{x \in \text{cv}(X)} \sum_{(i,j) \in I \times J} w_k(j, i, t_i) x_i = \max_{x \in \text{cv}(X)} \left(\sum_{i: w_k(j, i, t_i) = 1} x_i - \sum_{i: w_k(j, i, t_i) = -1} x_i \right) = h(I(S^{k,+}, t), I(S^{k,-}, t)),$$

where the last “=” uses (25), which applies because $I(S^{k,+}, t) \cap I(S^{k,-}, t) = \emptyset$ by the

definitions of $S^{k,+}$ and $S^{k,-}$. Since h is assumed linear,

$$\begin{aligned} h(I(S^{k,+}, t), I(S^{k,-}, t)) &= \max_{x \in \text{cv} X} \sum_{(i,j) \in I(S^{k,+}, t)} x_i^j - \min_{x \in \text{cv} X} \sum_{(i,j) \in I(S^{k,-}, t)} x_i^j \\ &= \max_{x \in X} \sum_{(i,j) \in I(S^{k,+}, t)} x_i^j - \min_{x \in X} \sum_{(i,j) \in I(S^{k,-}, t)} x_i^j, \end{aligned}$$

where the first line is due to linearity of h and (25), and the second line due to the fact that X contains all extremal points of its convex hull. Let $(b_i^j)_{(i,j) \in I \times J}$ be a solution to $\max_{x \in X} \sum_{(i,j) \in I(S^{k,+}, t)} x_i^j$, and $(a_i^j)_{(i,j) \in I \times J}$ a solution to $\min_{x \in X} \sum_{(i,j) \in I(S^{k,-}, t)} x_i^j$. Since $(\bar{q}_i^j(t))_{(i,j) \in I \times J}$ is a solution to $\max_{x \in \text{cv}(X)} \sum_{(i,j) \in I \times J} w_k(j, i, t_i) x_i^j$, the two formulas displayed above together yield

$$\sum_{(i,j) \in I(S^{k,+}, t)} \bar{q}_i^j(t) - \sum_{(i,j) \in I(S^{k,-}, t)} \bar{q}_i^j(t) = \sum_{(i,j) \in I(S^{k,+}, t)} b_i^j - \sum_{(i,j) \in I(S^{k,-}, t)} a_i^j.$$

Since $I(S^{k,+}, t) \cap I(S^{k,-}, t) = \emptyset$, the above equation implies that $\bar{q}_i^j(t) = b_i^j$ for all $(i, j) \in I(S^{k,+}, t)$, and $\bar{q}_i^j(t) = a_i^j$ for all $(i, j) \in I(S^{k,-}, t)$. Thus,

$$\begin{aligned} \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_{S^{k,+}}(j, i, t_i) &= \max_{x \in X} \sum_{(i,j) \in I(S^{k,+}, t)} x_i^j = f(S^{k,+}, t), \\ \sum_{(i,j) \in I \times J} \bar{q}_i^j(t) \chi_{S^{k,-}}(j, i, t_i) &= \min_{x \in X} \sum_{(i,j) \in I(S^{k,-}, t)} x_i^j = g(S^{k,-}, t), \end{aligned}$$

where the last “=” on each line uses the notation defined in (4) and (5). ■

Proof of Remark 1 Let $w \in \mathbb{R}^{\mathcal{Z}}$. Since X is assumed nonempty and compact, so is $\text{cv} X$. Thus, problem (7) has a finite optimum, and hence its dual, problem (23), has a solution (φ^*, γ^*) . It follows that $(w, \varphi^*, \gamma^*) \in \mathcal{P}$. By total unimodularity, every element of \mathcal{P} is a conic combination of some $\{(w_k, \varphi_k, \gamma_k) \mid k = 1, \dots, K\} \subset \mathcal{P}$ for some integer K such that (24) is true and the ranges of both φ_k and γ_k are $\{0, 1\}$ (Lang and Yang’s [20] Lemma 13, or Hoffman’s [17] Lemma 3.1). Thus, (w, φ^*, γ^*) is the conic combination of $\{(w_k, \varphi_k, \gamma_k) \mid k = 1, \dots, K\} \subset \mathcal{P}$ satisfying (24), and hence decomposability is satisfied. ■

B.12 Lemma 5 and Corollaries 3 and 4

Proof of Lemma 5 It is easy to adapt Appendix B.1 to obtain $\mathcal{Q} \subseteq \mathcal{Q}_B$ for infinite type spaces.¹⁹ Following is the proof of $\mathcal{Q}_B \subseteq \mathcal{Q}$ by a passing-to-limit argument.

¹⁹Since X is assumed compact, every element of \mathcal{Q} is μ -essentially bounded due to (1). Thus, following Border [3], treat both \mathcal{Q} and \mathcal{Q}_B as subsets of the $L_\infty(\mu)$ -space of functions $\mathcal{Z} \rightarrow \mathbb{R}$. For any L_∞ -function $Q :=$

Let $Q := (Q_i^j)_{(i,j) \in I \times J} \in \mathcal{Q}_B$ in the original model (T, X) such that (f, g) is defined in (4) and (5). Write Q as $(Q_i)_{i \in I}$ such that $Q_i := (Q_i^j)_{j \in J}$ for each $i \in I$. For any $m = 1, 2, 3, \dots$, partition \mathbb{R}^J into a collection \mathcal{C}_m of cells each of which has diameter at most $1/m$.²⁰ For each player $i \in I$ and each cell $C \in \mathcal{C}_m$ that has nonempty intersection with the range of Q_i , denote $([\min C]_j)_{j \in J}$ for the coordinate-wise minimum among all elements of C , with $[\min C]_j$ being its coordinate in the j th dimension, and define

$$(Q_i^j)^m(t_i) := \max \{0, [\min C]_j\}$$

for all $j \in J$ and all t_i in the inverse image $Q_i^{-1}(C)$ of C . Thus,

$$\max \{0, Q_i^j(t_i) - 1/m\} \leq (Q_i^j)^m(t_i)$$

for each m , each $(i, j) \times J$ and each $t_i \in T_i$. Since $Q \in \mathcal{Q}_B$ in the model (T, X) , Q satisfies (6) with respect to the (f, g) defined in (4) and (5) with respect to (T, X) . Since all elements of X are assumed nonnegative in the lemma, both f and g are nonnegative-valued. Then (6) implies that the range of Q_i is contained in \mathbb{R}_+^J . This coupled with the definition of $(Q_i^j)^m(t_i)$ implies that $(Q_i^j)^m(t_i) \leq Q_i^j(t_i)$. Thus we have

$$\max \{0, Q_i^j(t_i) - 1/m\} \leq (Q_i^j)^m(t_i) \leq Q_i^j(t_i).$$

Combine this inequality with the fact that Q satisfies (6) with respect to (f, g) to see that $Q^m := ((Q_i^j)^m)_{(i,j) \in I \times J}$ satisfies the (6) with respect to (f, g^m) because g^m satisfies (26) by the hypothesis of the lemma.

Since there is no loss to restrict the range of Q_i to a bounded set (as f and g are each finite-valued), for each m there are only finitely many cells in \mathcal{C}_m that intersect with the range of Q_i . Thus, $(Q_i)^m$ is equivalent to a function defined on the finite type space

$$T_i^m := \{Q_i^{-1}(C) \mid Q_i^{-1}(C) \neq \emptyset; C \in \mathcal{C}_m\}.$$

This, coupled with the fact that Q^m satisfies (6) with respect to (f, g^m) , means that $Q^m \in \mathcal{Q}_B$ with respect to (f, g^m) given type space $T^m := \prod_{i \in I} T_i^m$. Since T^m is finite, Assumption (a.ii) in the lemma implies that $Q^m \in \mathcal{Q}$ in the model (T^m, X^m) . Consequently, in the model

$(Q_i^j)_{(i,j) \in I \times J} \in \mathbb{R}^{\mathcal{X}}$ and any $L_1(\mu)$ -function $w \in \mathbb{R}^{\mathcal{X}}$, define $\langle Q, w \rangle := \int_T \sum_{(i,j) \in I \times J} Q_i^j(t_i) w(j, i, t_i) d\mu(t)$. Then $\langle \cdot, w \rangle$ is a continuous linear functional on the L_∞ -space of interim allocations.

²⁰A cell in \mathbb{R}^J is any set $\prod_{j \in J} [y_j, y'_j)$ for some real numbers $y_j < y'_j$ ($\forall j \in J$).

(T^m, X^m) there exists an ex post allocation q^m of which Q^m is the reduced form. Since T^m by definition is the collection of partition cells of the original type space T , we can extend q^m and Q^m to the entire T so that q^m is an ex post allocation in the model (T, X^m) , and Q^m its reduced form in the same model. This being true for all m , we can extract a subsequence $(q^{m_k})_{k=1}^\infty$ converging to some ex post allocation q . Since X^m converges to X by Assumption (b) in the lemma, q is an ex post allocation in the original model (T, X) . Then follow the reasoning in Border [3] to see that $\lim_{k \rightarrow \infty} Q^{m_k}$ is the reduced form of q , and $Q = \lim_{k \rightarrow \infty} Q^{m_k}$. Thus, $Q \in \mathcal{Q}$ in the original model (T, X) , as desired. ■

Proof of Corollary 3 Since X is determined by a paramodular pair (φ, γ) , each being defined on $2^{I \times J}$, we can identify X^m with γ^m such that

$$\gamma^m(E) := \max\{0, g(E) - 1/m\}$$

for every $E \subseteq I \times J$ and every $m \in \{1, 2, \dots\}$. Thus, by Eqs. (4) and (5), the pair (f, g) is defined for the original (T, X) , (f, g^m) is defined for each (T, X^m) , and g^m satisfies (26). By the definition of γ^m and the assumption that $X \subseteq \mathbb{R}_+$, γ^m converges to γ in sup-norm as $m \rightarrow \infty$. Consequently, $X^m \rightarrow_m X$ in Hausdorff metric and $g^m \rightarrow_m g$ in sup-norm.

Thus, by Lemma 5 and Corollary 1, it suffices to prove that, for any sufficiently large integer m , $-\gamma^m$ is submodular and (φ, γ^m) is compliant. Submodularity of $-\gamma^m$ means

$$\gamma^m(E) + \gamma^m(E') \leq \gamma^m(E \cup E') + \gamma^m(E \cap E') \quad (61)$$

for all $E, E' \subseteq I$. Since 2^I is finite, it suffices to show, given any $E, E' \subseteq I$, that (61) holds for all sufficiently large m . If $\gamma(E) > 0$ and $\gamma(E') > 0$, then (26) implies that, for any large enough m , $\gamma^m(E) = \gamma(E) - 1/m$ and $\gamma^m(E') = \gamma(E') - 1/m$; meanwhile, the right-hand side of (61) is never less than $\gamma(E \cup E') + \gamma(E \cap E') - 2/m$ (by (26)). Thus (61) follows from $\gamma(E) + \gamma(E') \leq \gamma(E \cup E') + \gamma(E \cap E')$ (submodularity of $-\gamma$) for all large m . If $\gamma(E) = 0$ and $\gamma(E') = 0$, then $\gamma^m(E) = \gamma^m(E') = 0$ by the definition of γ^m , and (61) follows trivially because its right-hand side is always nonnegative (by the definition of γ^m). Else, one of $\gamma(E)$ and $\gamma(E')$ is zero, and the other positive. Then $\gamma(E \cup E') > 0$ and $\gamma(E \cap E') = 0$ (monotonicity of γ , due to submodularity of $-\gamma$). Without loss of generality, say $\gamma(E) > 0 = \gamma(E')$. Then for any m sufficiently large, (61) becomes $\gamma(E) - 1/m \leq \gamma(E \cup E') - 1/m$, which is true by $\gamma(E') \leq \gamma(E \cup E')$ (submodularity of $-\gamma$). Thus, (61) is true for any sufficiently large m .

Compliance of (φ, γ^m) means

$$\varphi(E') - \varphi(E' \setminus E) \geq \gamma^m(E) - \gamma^m(E \setminus E') \quad (62)$$

for all $E, E' \subseteq I$. Suppose that (62) does not hold no matter how large m is. Then, it follows from the fact $\varphi(E') - \varphi(E' \setminus E) \geq \gamma(E) - \gamma(E \setminus E')$ (compliance of (φ, γ)) that $\gamma^m(E \setminus E') = \gamma(E \setminus E') - 1/m$ and $\gamma^m(E) = 0$ for any m . Then by the definition of γ^m we have $\gamma(E) = 0 < \gamma(E \setminus E')$, contradicting the monotonicity of γ noted previously. Thus, (62) holds for all sufficiently large m . Since there are only finitely many subsets of I , (62) holds for all subsets of I when m is sufficiently large. Thus, for any sufficiently large m , (φ, γ^m) is paramodular. Then the conclusion follows from Corollary 1 and Lemma 5. ■

Proof of Corollary 4 Since $\rho_n > 0$ by assumption in the ranked-item model, there exists $\eta > 0$ for which $\rho_n - \eta > 0$. For any positive integer r such that $1/r \in (0, \eta]$, let X^r be the set of all $(y_i)_{i \in I} \in [0, 1]^n$ for which there exists a bijection $j : I \rightarrow J$ such that $j^{-1}(1) \in I_1$ and, for all $i \in I$,

$$\begin{aligned} j(i) \neq n &\Rightarrow y_i = \rho_{j(i)} \quad \text{and} \\ j(i) = n &\Rightarrow \rho_n - 1/r \leq y_i \leq \rho_n. \end{aligned} \quad (63)$$

That is, X^r is the same as X except that the condition $y_{j^{-1}(n)} = \rho_n$ in X is replaced by (63). Obviously, X^r converges to the original X when $r \rightarrow \infty$. It is also clear that, for any $S \subseteq \mathcal{Z}$ and any $t \in T$, the ceiling $f(S, t)$ defined by (4) remains the same when X is replaced by X^r , as there is no difference regarding upper bounds between the definitions of X and X^r . Since the only difference between the two sets is the lower bound change in (63), the $g(S, t)$ defined by (43) when X is replaced by X^r becomes

$$g^r(S, t) = \sum_{j=1}^n \rho_j - f(\mathcal{Z} \setminus S, t) - \frac{1}{r}. \quad (64)$$

It suffices to extend Proposition 3 to the case where X is replaced by X^r for any sufficiently large r , for then the extended Proposition 3 implies $\mathcal{Q} = \mathcal{Q}_B$ with respect to X^r and any finite type space, and then Lemma 5 implies the desired conclusion.

To that end, pick any $w \in \mathbb{R}^{\mathcal{Z}}$. For any type profile $t \in T$, modify the algorithm in Section 4.2 by replacing the assignment $\bar{q}_{i_*}(t) := \rho_{j_k}$ ($k = 1, 2$) therein with

$$\bar{q}_{i_*}(t) := \begin{cases} \rho_{j_k} & \text{if } j_k \neq n \text{ or } w(i_*, t_{i_*}) \geq 0 \\ \rho_n - 1/r & \text{if } j_k = n \text{ and } w(i_*, t_{i_*}) < 0. \end{cases}$$

Then Lemma 7 remains true when its statement is modified by replacing the phrase “any $m \in \{1, \dots, |\mathcal{Z}|\}$ ” with “any $m \in \{1, \dots, |\mathcal{Z}|\}$ for which $w(z^m) \geq 0$.” Lemma 8 remains true when its statement is modified into “for any $t \in T$ and any $m \in \{1, \dots, |\mathcal{Z}|\}$ for which $w(z^m) < 0$, $\sum_{I(L^{|\mathcal{Z}|-m+1}, t)} \bar{q}_i(t) = g^r(L^{|\mathcal{Z}|-m+1}, t)$.” Lemma 9 remains true without any change. Lemma 10 remains true with the X in its statement replaced by X^r . Thus, Proposition 3 is extended, as desired. ■

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