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Constant relative risk aversion utility and consumption CAPM: discount factors and risk aversions for Norway, Sweden, and the UK

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ABSTRACT

This paper applies the newly suggested Markov chained Monte Carlo Surface Sampling Algorithm of Zappa estimating European discount factors and relative risk aversions for the CRRA utility functions based on the consumption capital asset pricing model (CCAPM). The relatively challenging estimation focuses on parameter equalities and their interpretations for three European countries, Norway, Sweden, and the UK. Moreover, European consumer behavior regarding risk and inter-temporal consumption substitution (including asset markets risk pricing) are of general interest applying robust estimation techniques. The empirical results from the non-parametric Bayesian estimation technique with reported parameter paths, densities and diagnostics are compared. The UK (Swedish) investors show the highest (lowest) risk aversion (and lowest (highest) inter-temporal substitution in consumption). However, the risk aversion coefficient distributions are relatively wide, with an insignificant parameter for Sweden and Norway, and a significant parameter for the UK. The stochastic discount factor is approximately equal for all three countries reporting a yearly discount factor of 0.99. The quarterly discount factor is lowest for Sweden (0.997) followed by Norway and the UK (0.998), indicating that Sweden has the most impatient consumers. Bivariate conditional densities and simulation paths are extracted for interpretative purposes, specifically excessive shocks.

PUBLIC INTEREST STATEMENT

The paper uses the national account data for inflation adjusted household consumption (non-durables and services) together with inflation adjusted stock (index) with dividends on all stock series for Norway, Sweden, and the UK. The most interesting aspect of Asset Pricing considers how securities markets price risk (the time dimension alone is largely mechanical). For this question to be interesting, it must be that there is a positive price for risk – i.e. investors require some compensation for exposing their portfolios to risk. This in turn requires that investors dislike risk or that they are risk averse. This paper's main interest is therefore the stochastic discount factor for the constant relative risk aversion utility as described in the marginal rate of substitution.

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1. Introduction

The main purpose of this paper is to analyze the behavior of European consumers regarding risk and inter-temporal consumption. The relationship between European asset markets prices and consumer compensations of risk (inducing risk aversion) is not a closed subject in 2022. Applying the newly suggested nonparametric Bayesian estimation (NPB^1) subject to overidentified moment equations should extend our understanding of both market (risk) and consumption (inter-temporal substitution). However, the methodology is a challenge because the support of the posterior is a manifold of lower dimension than the number of model parameters. The manifold therefore has Lebesgue measure zero thus

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¹The C/C++ code with user guide is available at <http://aronaldg.org/webfile/npb/>.

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inhibiting the use of the most used Bayesian estimation method: Markov chain Monte Carlo (MCMC). However, using the C/C++ implementation Nonparametric Bayesian (NPB) methodology (Gallant & Tauchen, 2022c), solves this problem by adapting the innovative MCMC Surface Sampling Algorithm of Zappa et al. (2018)². The paper uses the MCMC Surface Sampling Algorithm estimating discount factor and relative risk aversion for the Norwegian, Swedish and UK exchange economies under constant relative risk aversion (CRRA) utility. The main purpose is discount factor and risk aversion comparison. For general moment densities and NPB starting values, the methodology uses the Semi-Nonparametric Time-series (SNP) (Gallant & McCulloch, 2022a) and maximum Likelihood Estimation (MLE) (Gallant & McCulloch, 2022b) estimation techniques.

The CRRA model is based on the Consumption Capital Asset Pricing Model (CCAPM) (Lucas, 1978). According to the CCAPM hypothesis, agents save wealth in the form of assets to carry out their future consumption intentions. The characteristics of consumption, the agent's risk aversion, the discount rate, and the intertemporal elasticity of substitution in consumption must therefore be connected to the characteristics of asset returns. Furthermore, the return premium of an asset will be inversely correlated with its consumption beta (systematic risk).

The paper implements the CRRA model for the period 1981 – 2022(Q2)³ for Norway, Sweden, and the UK. All three countries are well established European economies with developed financial markets and of importance for the CCAPM model, offer reliable and well-established household consumption of goods⁴. In addition, the three countries have a well-functioning trade pattern between them.

For an exchange economy under CRRA utility the algorithm estimates the parameters $(\theta = (\beta, \gamma))$ - the discount factor (β) and the relative risk aversion (γ). The Bayesian computer implementation consist of three steps. The first step determines the SNP specification (likelihood) (Gallant & McCulloch, 2022a). The SNP-model settings are lags in the location function (L_u), GARCH lags (L_g, Q_{type}), ARCH lags (L_r, P_{type}), and the density for $[P(z)]^2$ (K_x). Hermite functions for the density $([P(z)] \cdot \sqrt{\phi(z)})$ are added for deviations from a normal distribution of the series. The settings define an AR model with GARCH conditional scale function and a time homogenous nonparametric innovation density with skewness, fat tails, etc. The dependence on the past is through the linear location function (AR) and GARCH scale function⁵. The estimated parameters are used as start values for step number 2, the λ -prior method (MLE) (Gallant & McCulloch, 2022b). This step uses maximum likelihood estimation obtaining starting values for the third step requiring coding the moment equations, support conditions, and prior (NPB) (Gallant & Tauchen, 2022c). The surface sampling algorithm therefore fits an SNP likelihood

$$f(y|x, \rho) = \prod_{t=1}^n f(y_t|x_{t-1}, \rho), \quad (1.1.1)$$

subject to conditions

$$0 = q(\rho, \theta) = \frac{1}{n} \sum_{t=1}^n \int m(y, x_{t-1}, \rho, \theta) f(y|x_{t-1}, \rho) dy, q \in \mathbb{R}^m \quad (1.1.2)$$

and support conditions $h(\rho, \theta) > 0, h \in \mathbb{R}^l$, and a prior $\pi = (\rho, \theta)$ using MCMC.

Letting sans serif $x = (\rho, \theta)$, the support of the posterior is the manifold

$$M = \left\{ x \in \mathbb{R}^{da} : q_i(x) = 0, i = 1, \dots, m, h_j(x) > 0, j = 1, \dots, l \right\} \quad (1.1.3)$$

The Bayesian methodology gives access to model parameters with associated standard errors, diagnostics such as parameter traces/densities and autocorrelation functions for proposal scaling (parameter variance) and quadrature rules for coverage of the observed data⁶.

²The economic relevance of the methodology that is used in this paper is primarily methodological. The obvious approach is to emulate GMM directly and estimate θ and those elements of ρ upon which $m(y_t, x_{t-1}, \rho, \theta)$ depends using the MCMC-GMM method proposed by Chernozhukov and Hong (2003).

³The identical period in Norway, Sweden and the UK is related to availability of index and consumption data. All consumption data series are collected from SSB, SCB and ONS national statistics (www.ssb.no; www.scb.se, and www.ons.gov.uk).

⁴For example, Statistics Norway (www.ssb.no) has reported official statistics since 1876.

⁵Note that the methodology in these settings use quarterly data important for the SNP setting interpretations of $L_u, L_g, Q_{type}, L_r, P_{type}, K_x$.

⁶The Z-abscissae range (quadrature rule) must come close to covering the 5% through 95% quantiles of the transformed data.

The macroeconomic model is detailed in [Section 2](#). A description of the MCMC Surface Sampling Algorithm (SSA) is provided in [Section 3](#). The statistics are described in [Section 4](#) along with a brief overview of the computer's C/C++ implementation. [Section 5](#) presents the results, and [Section 5](#) also includes an extended interpretation of the results using simulations and conditional expectations from a post-estimation study. [Section 6](#) provides conclusions.

2. Literature review and the CCAPM theoretical model

2.1. Literature review

Formal econometric tests for the CCAPM were conducted by Hansen and Singleton (1982) with CRRA utility with Kreps-Porteus utility (formal econometric tests for the equity premium puzzle). Mehra and Prescott (1985) presented for US data, the basic CCAPM with time-additive CRRA (constant relative risk aversion) utility function did not perform well vis-a-vis the data, resulting in an important macroeconomic puzzle – the equity premium puzzle. The average equity premium vis-a-vis government bonds was 6.18% per year. Simulating the model for parameter values considered reasonable (a relative risk aversion coefficient between zero and ten), Mehra and Prescott concluded that the observed equity premium was inconsistent with the model, unless a risk aversion coefficient between 30 and 40 was considered. For the U.S. economy, these restrictions were consistently rejected either by Hansen and Singleton (1983) or by Epstein and Zin (1989, 1991), confirming the puzzle indicated by Mehra and Prescott.

The Issler and Piqueira (2000) paper⁷ presented estimates for the risk aversion coefficient, the discount rate of the future utility, and the inter-temporal elasticity of substitution in consumption for Brazil. Their work showed a discount factor in Brazil of 0.90. Moreover, they found that the Brazilian consumer was more impatient than its U.S. counterpart due to a discount factor close to one. Furthermore, the inter-temporal elasticity of substitution in consumption is not very high in Brazil, with values predominately close to 0.25 (evidence of consumption smoothing, or, as evidence that the Brazilian consumer is liquidity constrained). Finally, the paper found ambiguous results for relative risk aversion. However, in two out of three models, they found that Brazilians were more risk averse than U.S. citizens.

As a response to the equity-premium puzzle, the basic CCAPM with time-additive CRRA utility was re-formulated in two directions. The first tried to explain the puzzle from the existence of market imperfections – incomplete markets (Mankiw, 1986), liquidity constraints (Scheinkman & Weiss, 1986) and transaction costs (Grossman & Laroque, 1989) – which implied the non-validity of a representative consumer model due to the agent's restricted access to the asset markets. In this case, the model was re-formulated to incorporate heterogeneous agents. The second focused on incorporating a more general class of preferences for the representative consumer allowing for non-separability over time, for alternative ways of formulating preferences over risk, and by using non-expected utility models. Finally, the paper of Breeden et al. (2015) also found the CCAPM performs better than the original CAPM and nearly as well as the Fama and French (1992) three-factor model.

2.2. The CCAPM theoretical model

How risk is priced in the securities markets is one of the most intriguing aspects of asset pricing (the time dimension alone is largely mechanical although there are interesting interactions between the two). Risk must have a positive price for this question to be intriguing; investors must receive compensation for exposing their portfolios to risk. Theoretically, this calls for investors to be risk averse. A fundamental understanding of the connection between consumption, wealth, and risk aversion is provided by the CCAPM. The CCAPM serves as an asset valuation model that may be used to determine the expected premium that investors will pay for a particular stock and how the risk associated with consumption-driven stock price volatility would affect that return. The variations in the risk premium (return on asset and risk-free rate) with increasing consumption serve as a proxy for the amount of risk associated with the

⁷For Brazilian data, estimates of the relative risk aversion coefficient for the basic linearized model (under log-normality) have already been performed by Reis et al. (1998).

consumption beta. The CCAPM is helpful in determining how much stock market returns fluctuate in relation to consumer growth. An increased expected return on riskier assets is implied by a greater consumption beta. For instance, a consumption beta of 2.0 would indicate a 2% rise in asset return requirements if the market rose by 1%. The CCAPM covers a variety of wealth types outside of stock market wealth and offers a framework for comprehending variation in financial asset returns over a variety of time periods. Finally, the validity of the mean-variance theory is predicated on risk aversion (portfolios).

To analyze these features, consider a single-good general equilibrium model with infinite horizon and identical agents. The representative agent's general problem is:

$$\max_{\{C_{t+j}\}_{j=0}^{\infty}, \{\theta_{t+j+1}\}_{j=0}^{\infty}} U_t(*) \quad (2.2.1)$$

subject to the sequence of $C_{t+j} + q_{t+j}\theta_{t+j+1} \leq [q_{t+j} + q_{t+j}^*]\theta_{t+j} + y_{t+j}, \forall j = 0, 1, 2, \dots$

Moreover, there are N assets in the economy, where $\theta_{t+j(N \times 1)}, q_{t+j(N \times 1)}$ is a vector with their respective quantities and prices and $q_{t+j(N \times 1)}^*$ are vectors with their respective dividends. In each period, the agent receives an exogenous income y_{t+j} which is a state variable in the consumer problem.

2.3. The CRRA time-additive utility model

Following Hansen and Singleton (1982), the inter-temporal utility function in the benchmark model is a von Neumann-Morgenstern utility function (time-additive):

$$U_t = E_t \left[\sum_{j=0}^{\infty} \beta^j u[C_{t+j}] \right] \quad (2.3.1)$$

where C_t is real aggregate consumption per capita, $E_t(\cdot)$ is the conditional expectation on the information available to the agent at period t and β is the intertemporal discount rate, $0 < \beta < 1$. The maximization of this function subject to sequence of constraints results in the following N first-order conditions (Euler Equations):

$$u'[C_t] = E_t \{ \beta u'[C_{t+1}] (1 + R_{i,t+1}) \}, \forall i = 1, \dots, N \quad (2.3.2)$$

where the real rate of return for asset i at period $t+1$ is defined as $R_{i,t+1} := \frac{q_{i,t+1} + q_{i,t+1}^*}{q_{i,t}} - 1$, where $q_{i,t+1}$ and $q_{i,t+1}^*$ are respectively the price and return of asset i , $i = 1, 2, \dots, N$, in period $t+1$, with a similar notation applying to $q_{i,t}$. Dividing both sides by $u'[C_t]$ we obtain

$$1 = E_t [(1 + R_{i,t+1}) M_{t+1}], \forall i = 1, \dots, N \quad (2.3.3)$$

where M_{t+1} is pricing kernel (or stochastic discount factor). In this model, the instantaneous utility function is parameterized as Constant Relative Risk Aversion (CRRA),

$$u[C_t] = \frac{C_t^{1-\gamma} - 1}{1-\gamma} \quad (2.3.4)$$

where γ is the relative risk aversion coefficient. The main objection to this utility function specification is that only one parameter (γ) governs both the relative risk aversion coefficient and the inter-temporal elasticity of substitution in consumption ψ , since: $\psi \equiv \frac{\partial \ln \left(\frac{C_{t+1}}{C_t} \right)}{\partial \ln R_{t+1}} = \frac{1}{\gamma}$. For example, a risk averse agent (γ high) will not alter his consumption pattern significantly in response to interest rate variations ($1/\gamma$ low), given that risk aversion is the reciprocal of the inter-temporal elasticity of substitution.

Applying the utility function parameterization, we obtain the Euler equation

$$1 = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} (1 + R_{i,t+1}) \right], \forall i = 1, \dots, N \quad (2.3.5)$$

and, in this case, the pricing kernel is a function of consumption growth

$$M_{t+1}^{CRRA} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \quad (2.3.6)$$

Therefore, the SDF discounts gross return to unity and we use

$$0 = E_{t-1} \left[1 - \exp \left(\log \beta - \gamma \log \frac{C_t}{C_{t-1}} + \log \frac{y_t}{y_{t-1}} \right) \right], \quad (2.3.7)$$

where E_{t-1} is the conditional expectation given C_{t-1}, C_{t-2}, \dots

2.4. The equity premium puzzle⁸

The puzzle behind the equity premium is still open in the field of economics. In this work, assets have been valued using the Euler equation $E_t(m_{t+1}R_{t+1}) = 1$, where $R_{t+1} = 1 + r_{t+1}$ represents the gross rate of return on an asset, and m_{t+1} is referred to as the stochastic discount factor or occasionally the pricing kernel. The key to asset pricing is determining m_{t+1} since we can infer information about predicted interest rates from data and resolving the stock premium conundrum requires producing enough dispersion in the stochastic discount factor.

The puzzle's central thesis is that frequently employed economic models are unable to foresee the empirical premium of stock market returns over short-term, comparatively risk-free debt. The intriguing query at this point is if the CRRA utility models for the three nations of Norway, Sweden, and the UK have any potential relationship to the equity premium puzzle.

Since the representative agent has power utility, we can rewrite the stochastic discount factor with a concrete utility function:

$$m = \beta \frac{u'(c_{t+1}(\omega))}{u'(c_t)} = \beta \left(\frac{c_{t+1}(\omega)}{c_t} \right)^{-\gamma} \quad (2.4.1)$$

and the Euler equation. $E(mr) = 1$, can be written for the international all-share indices

$$1 = \beta E_t \left(\left(1 + r_{t+1}^{index} \right) \left(\frac{c_{t+1}(\omega)}{c_t} \right)^{-\gamma} \right) \quad (2.4.2)$$

The price for equity, p_{t+1}^e , can be calculated in the same fashion, but the dividend is uncertain and related to consumption growth:

$$p_t^e = E_t(m_{t+1}g_{t+1}(\omega)) = E_t(\beta g_{t+1}(\omega)^{1-\gamma}) \quad (2.4.3)$$

The last line follows from the moment generating function of the log normal random variable $g_{t+1}(\omega)$. We can also calculate our agent's required return on equity by

$$E(\log(1 + r_{t+1}^e)) = E(\log g_{t+1}(\omega)) - \log(p_t^e) \quad (2.4.4)$$

Now the equity premium is calculated by taking the difference of the returns.

3. The MCMC surface sampling algorithm and the normalization factor⁹

3.1. The MCMC surface sampling algorithm

Define

$$Q_x = \left[\frac{\partial}{\partial x} q_1(x), \dots, \frac{\partial}{\partial x} q_m(x) \right] \quad (3.1.1)$$

which is the transpose of the Jacobian of $q(x)$ and has dimension d_a by m . Place $A = [Q_x | 0]$, which is a square matrix of dimension d_a by d_a whose last $d = d_a - m$ columns are filled with zeros. Apply the singular value decomposition algorithm and obtain $A = USV^T$; U will be orthogonal and S diagonal with the first m diagonal entries positive and the remainder zero. If S is not so, Q_x does not have full rank, which violates a regularity condition of the Surface Sampling Algorithm (SSA). Partition U as $[T_x^\perp | T_x]$,

⁸See Mehra and Prescott (1985), Myatt, J, 2012 and Issler & Piqueira 2000.

⁹The algorithm and factor constant are in this section, based on the NPB user guide (Gallant & Tauchen, 2022c).

where T_x^\perp has m columns and T_x have d columns. Solving¹⁰ $q(x + v + Q_x a) = 0$ for a , using the Newton's method with Fletcher's (1987) line search from a start of $a = 0$. Tuning parameters are the solution tolerance and iteration limit. An iteration of the SSA has the property that if X_k is a draw from the posterior $p(x|y) \propto f(y|x)\pi(x)$, then so is X_{k+1} .

An iteration of the SSA algorithm is therefore:

1. Begin: $x = X_k$ (X_k must be in M).
2. Proposal:
 - a. Calculate Qx .
 - b. Compute T_x^\perp and T_x using the SVD as described.
 - c. Draw $v \sim p(v)$ as described above.
3. Projection to M :
 - a. Solve $q(x + v + Q_x a) = 0$ for a using Newton's method.
 - b. If Newton's method fails, put $X_{k+1} = x$. Done.
 - c. Else $y = (x + v + Q_x a)$. Continue.
4. Inequality check:
 - a. If $h_i(y) < 0$ for some i , put $X_{k+1} = x$. Done.
 - b. Else y satisfies (3). Continue.
5. Metropolis-Hastings acceptance/rejection step:
 - a. Calculate Qy as shown above.
 - b. Compute T_y^\perp and T_y using the SVD.
 - c. Find $v' \in T_y$ and $\omega' \in T_y^\perp$ $v' \in T_y$ and $\omega' \in T_y^\perp$ so that $x = y + v' + \omega'$.¹¹
 - d. $P_a = P_a = \min(1, \frac{f(y|y)\pi(y)p(v')}{f(y|x)\pi(x)p(v)})$
 - e. Generate $U \sim \text{Uniform}(0,1)$.
 - f. If $U > P_a$, put $X_{k+1} = x$. Done.
 - g. Else Continue.
6. Reverse Projection:
 - a. Solve $q(x + v + Q_x a) = 0$ for a using Newton's method.
 - b. If Newton's method fails, put $X_{k+1} = x$. Done.
 - c. Else accept move, $X_{k+1} = y$. Done.

In practice, identifying chain failure occurrences is important for chain tuning. The code gives rejection rates and indications as to which of the steps in the Surface Sampling Algorithm have the most influence on the accept/reject decision.

3.2. The normalization constant

The normalization constant is

$$Z = \int_M f(y|x)\pi(x)d\sigma(x) \quad (3.2.1)$$

where $\sigma(x)$ is d dimensional Hausdorff measure on \mathbb{R}^{d_a} . If a mapping from R_d to M can be found, then computing Z can be accomplished by Riemann integration after multiplication by a Jacobian term (Morgan, 2016, p. 29). The strategy proposed by Zappa et al. (2018) consists of successively reducing the domain of integration until such a mapping can be found. The remaining part of the integral can be computed from Surface Sampling draws. Therefore, computing Z_k by Monte Carlo integration using

$$S = \frac{1}{n_k} \sum_{i=1}^{n_k} I_{B_k}(y_i) (J_i)^{-1} \cdot \exp [\log f(y|y_i) + \log \pi(y_i) - \log f(y|x_0) - \log \pi(x_0)] \quad (3.2.2)$$

¹⁰Notational clash: x and y represent values for (ρ, θ) . X_k and Y_k represent either (ρ, θ) regarded as random variables or as their *ex-post* values as elements of an MCMC chain.

¹¹That is, put $z = \begin{bmatrix} T_y^\perp \\ T_y \end{bmatrix} \cdot (x - y)$, then $\omega' = T_y^\perp z$ and $v' = T_y z$.

and

$$\log Z_k = \left(\frac{d}{2}\right) \log \pi - \log \Gamma\left(\frac{d}{2} + 1\right) + d \log(r_k) + \log(S) + \log(y|x_0) + \log \pi(x_0) \quad (3.2.3)$$

where $\log Z = \log Z_k + \sum_{i=0}^{k-1} \log \hat{R}_i$.

4. Share indices and household consumption for Norway, Sweden, and the UK

The above formed equations are estimated for European data using the three well-known utility models. Previous empirical work in the international literature (Hansen & Singleton, 1982, 1983; Epstein & Zin, 1991) used monthly data, arguing that they are more likely to capture the agent's timing decision. Mainly due to data availability, this paper uses quarterly frequencies adjusted for dividends and inflation using quarterly consumer price index data (no seasonal adjustment). We distinguish between durable and non-durable goods (See Bernanke (1985) for an explanation of the durable goods problem). Let lsr_t denote gross geometric stock returns observed at time t ; i.e. $lsr_t = \log(P_t/P_{t-1})$, where P_t is the stock price (with dividends). The log consumption growth observed at time t is $lcg_t = \log(C_t/C_{t-1})$. Both series are adjusted for inflation. Lucas (1978) exchange economy set dividends equal to consumption at time t .

The data is therefore a matrix y with columns $y_t = \begin{pmatrix} lsr_t \\ lcg_t \end{pmatrix}$. The variable x_{t-1} denotes the information set and has as many lags in it as are needed to compute $f(y_t|x_{t-1}, \rho)$ of the equation $f(y|x, \rho) = n \cdot Y \cdot t = 1 \cdot f(y_t|x_{t-1}, \rho)$. The y_t are quarterly observations from 1980 to 2022 (q2) (166 observations) for Norway, Sweden, and the UK. The stochastic discount factor (SDF) for CRRA utility is the marginal rate of substitution $M_{t,t-1} = \beta(\frac{C_t}{C_{t-1}})^{-\gamma}$, where β is the representative agent's discount factor, γ is a risk aversion parameter, and the elasticity of intertemporal substitution (EIS) is $1/\gamma$. The moment conditions we use are

$$0 = q(\rho, \theta) = \frac{1}{n} \sum_{t=T_0}^n \int m(y, x_{t-1}, \rho, \theta) f(y|x_{t-1}, \rho) dy, q \in \mathbb{R}^m \quad (4.1.1)$$

where ρ contains the *SNP* parameters and *MLE/NPB* extends the $\theta = (\beta, \gamma)$. Finally, the implementation includes the support conditions $h(\rho, \theta) > 0, h \in \mathbb{R}^l$ and the prior $\pi(\rho, \theta)$. The computations suggest a three-step procedure. The first step is the *SNP* specification determining the ρ -parameters (Gallant and McCulloch, 2022a). The x_{t-1} allows for the information set. The second step establishes starting values applying the λ -prior method (*MLE*) (Gallant and McCulloch, 2022b) encompassing coding the moment equations, support conditions and the prior (mean and standard deviation of a normal)¹². The last step is to tune the nonparametric Bayes methodology (*NPB*) (Gallant & Tauchen, 2022c), which also requires coding the moment equations, support conditions and the prior. The implementation is sensitive to viable starting values and to choosing the correct scale.

Table 1 characterizes the index and consumption for all three countries. The table reports the characteristics of the Norwegian (panel A) index (OSEAX) and the consumption movements; the Swedish (panel B) index (OMXSPI) and consumption movements; the UK (panel C) index (FTSE All Shares) and consumption movements. The index means all are positive indicating positive drifts. Norway and Sweden report a drift almost three times higher than the UK, 0.009 versus 0.003. However, the standard deviation is relatively high not excluding a zero drift for all three countries. The UK FTSE index therefore also report the smallest standard deviation. The Norwegian OSEAX index report a drift of 0.00858 with an associated standard deviation of 0.05174. A similar index drift for Sweden (the UK) is 0.00882 (0.00337) with associated standard deviation of 0.05143 (0.03832). The three indices report a maximum (minimum) mean for Norway, Sweden and the UK of 0.11014 (-0.23248), 0.14814 (-0.16569), 0.08564 (-0.14592), respectively. The kurtosis is positive (tails) and the skew is negative (left skewed) for all three countries. Norway,

¹²The λ -prior and *NPB* methods normally distributed prior values for the OSEAX index and Consumption growth mean and standard deviation are generated with associated mean and standard deviations values and correct BIC decided model dimensions. A non-normal prior can be applied for the prior values (not implemented).

Table 1. All Share Indices and Household Consumptions Characteristics for Norway, Sweden, and the UK, 1980–2022(q2).

Characteristics for All Share Indices and Consumption Growth 1981–2022(Q2)									
Norway OSEAX Index Returns									
Panel A	Median	Maximum /	Moment	Quantile	Quantile	Cramer-	Serial dependence		VaR
Mean (all)/	Std.dev.	Minimum	Kurt/Skew	Kurt/Skew	Normal	von-Mises	Q(12)	Q ² (12)	(1; 2.5%)
M (-drop)	0.01689	0.11014	3.48554	0.09856	0.08058	0.55098	3.34892	5.10290	–0.187%
0.00858	0.05174	–0.23248	–1.37740	–0.02200	{0.9605}	{0.0000}	{0.7639}	{0.5307}	–0.138%
0.00901					Phillips -	Augmented	ARCH	RESET	CVaR
BDS-Z-statistic ($\epsilon = 1$)					Perron test	DF-test	(12)	(12;6)	(1; 2.5%)
m = 2	m = 3	m = 4	m = 5	m = 6	–12.64848	–12.64610	4.33604	0.31011	–0.232%
1.40569	1.77751	2.18022	2.06464	1.97461	{0.0000}	{0.0000}	{0.6313}	{0.9581}	–0.176%
{0.1598}	{0.0755}	{0.0292}	{0.0390}	{0.0483}					
Norway Consumption Growth									
Mean (all)/	Median	Maximum /	Moment	Quantile	Quantile	Cramer-	Serial dependence		VaR
M (-drop)	Std.dev.	Minimum	Kurt/Skew	Kurt/Skew	Normal	von-Mises	Q(12)	Q ² (12)	(1; 2.5%)
0.00263	0.00899	0.04699	–0.67084	0.21112	3.21597	1.09000	269.148	160.678	–4.820%
0.00294	0.02355	–0.04821	–0.66382	–0.32419	{0.2003}	{0.0000}	{0.0000}	{0.0000}	–4.610%
BDS-Z-statistic ($\epsilon = 1$)					Phillips -	Augmented	ARCH	RESET	CVaR
m = 2	m = 3	m = 4	m = 5	m = 6	Perron test	DF-test	(12)	(12;6)	(1; 2.5%)
3.11988	7.01564	16.24813	26.31176	38.34020	–41.26220	–4.10618	103.77979	8.72718	–4.821%
{0.0018}	{0.0000}	{0.0000}	{0.0000}	{0.0000}	{0.0001}	{0.0076}	{0.0000}	{0.0331}	–4.746%
Sweden OMXSPI Index Returns									
Panel B.	Median	Maximum /	Moment	Quantile	Quantile	Cramer-	Serial dependence		VaR
Mean (all)/	Std.dev.	Minimum	Kurt/Skew	Kurt/Skew	Normal	von-Mises	Q(12)	Q ² (12)	(1; 2.5%)
M (-drop)	0.02018	0.14814	1.21766	0.39526	3.63593	0.42380	8.03725	4.06777	–0.159%
0.00882	0.05143	–0.16569	–0.75585	–0.30391	{0.1624}	{0.0000}	{0.2354}	{0.6675}	–0.116%
0.00840					Phillips -	Augmented	ARCH	RESET	CVaR
BDS-Z-statistic ($\epsilon = 1$)					Perron test	DF-test	(12)	(12;6)	(1; 2.5%)
m = 2	m = 3	m = 4	m = 5	m = 6	–11.82775	–11.81650	3.38698	1.07375	–0.166%
3.15999	3.68117	4.04041	4.04766	4.44021	{0.0000}	{0.0000}	{0.7589}	{0.7834}	–0.143%
{0.0016}	{0.0002}	{0.0001}	{0.0001}	{0.0000}					
Sweden Consumption Growth									
Mean (all)/	Median	Maximum /	Moment	Quantile	Quantile	Cramer-	Serial dependence		VaR
M (-drop)	Std.dev.	Minimum	Kurt/Skew	Kurt/Skew	Normal	von-Mises	Q(12)	Q ² (12)	(1; 2.5%)
0.00200	–0.00106	0.05743	–1.19116	–0.64676	3.35725	0.77638	603.616	190.199	–3.865%
0.00219	0.02539	–0.03883	0.19790	0.12951	{0.1866}	{0.0000}	{0.0000}	{0.0000}	–3.463%
BDS-Z-statistic ($\epsilon = 1$)					Phillips -	Augmented	ARCH	RESET	CVaR
m = 2	m = 3	m = 4	m = 5	m = 6	Perron test	DF-test	(12)	(12;6)	(1; 2.5%)
123.126	173.147	237.368	346.256	516.416	–60.62088	–5.90918	117.653	21.3105	–3.883%
{0.0000}	{0.0000}	{0.0000}	{0.0000}	{0.0000}	{0.0001}	{0.0000}	{0.0000}	{0.0001}	–3.752%
UK FTSE All Shares Index Returns									
Panel C.	Median	Maximum /	Moment	Quantile	Quantile	Cramer-	Serial dependence		VaR
Mean (all)/	Std.dev.	Minimum	Kurt/Skew	Kurt/Skew	Normal	von-Mises	Q(12)	Q ² (12)	(1; 2.5%)
M (-drop)	0.00654	0.08564	2.00107	0.43585	1.35975	0.36511	5.75461	5.93282	–0.136%
0.00337	0.03832	–0.14592	–0.78800	–0.04069	{0.5067}	{0.0001}	{0.4512}	{0.4308}	–0.094%
0.00367					Phillips -	Augmented	ARCH	RESET	CVaR
BDS-Z-statistic ($\epsilon = 1$)					Perron test	DF-test	(12)	(12;6)	(1; 2.5%)
m = 2	m = 3	m = 4	m = 5	m = 6	–14.15929	–13.98857	2.83387	2.34651	–0.146%
1.70685	2.76085	3.63873	4.12829	4.88753	{0.0000}	{0.0000}	{0.8294}	{0.5037}	–0.123%
{0.0878}	{0.0058}	{0.0003}	{0.0000}	{0.0000}					
UK Consumption Growth									
Mean (all)/	Median	Maximum /	Moment	Quantile	Quantile	Cramer-	Serial dependence		VaR
M (-drop)	Std.dev.	Minimum	Kurt/Skew	Kurt/Skew	Normal	von-Mises	Q(12)	Q ² (12)	(1; 2.5%)
0.00252	0.00994	0.07982	3.16068	0.09964	12.76860	2.09111	173.386	160.678	–7.438%
0.00261	0.02267	–0.10641	–1.06456	–0.67752	{0.0017}	{0.0000}	{0.0000}	{0.0000}	–4.937%
BDS-Z-statistic ($\epsilon = 1$)					Phillips -	Augmented	ARCH	RESET	CVaR
m = 2	m = 3	m = 4	m = 5	m = 6	Perron test	DF-test	(12)	(12;6)	(1; 2.5%)
–1.2390	1.0714	7.4850	13.2827	19.4417	–22.5349	–3.08722	37.4212	54.1686	–10.641%
{0.2153}	{0.2840}	{0.0000}	{0.0000}	{0.0000}	{0.0000}	{0.1131}	{0.0000}	{0.0000}	–6.643%

*For all three bivariate series a breakpoint unit root test rejects unit root.

Sweden and the UK report a kurtosis (skew) of 3.48554 (–1.37740), 1.21766 (–0.75585), 2.00107 (–0.78800), respectively. Cramer-von-Mises test statistic reports significant non-normality for all three country indices. The Q(12) and the Q²(12) correlogram statistics (serial correlation) show no dependencies for the index mean and volatility. The indices report insignificant Q(12) and the Q²(12) correlogram statistics for all three countries. Similarly, the 12th lag ARCH test statistic (Engle, 1982) suggests insignificant conditional heteroscedasticity. The RESET test (Ramsey, 1969) reports stability for the three indices. For all index series, the adjusted series the ADF (Dickey & Fuller, 1979) and the Phillips and Perron (1988) statistics

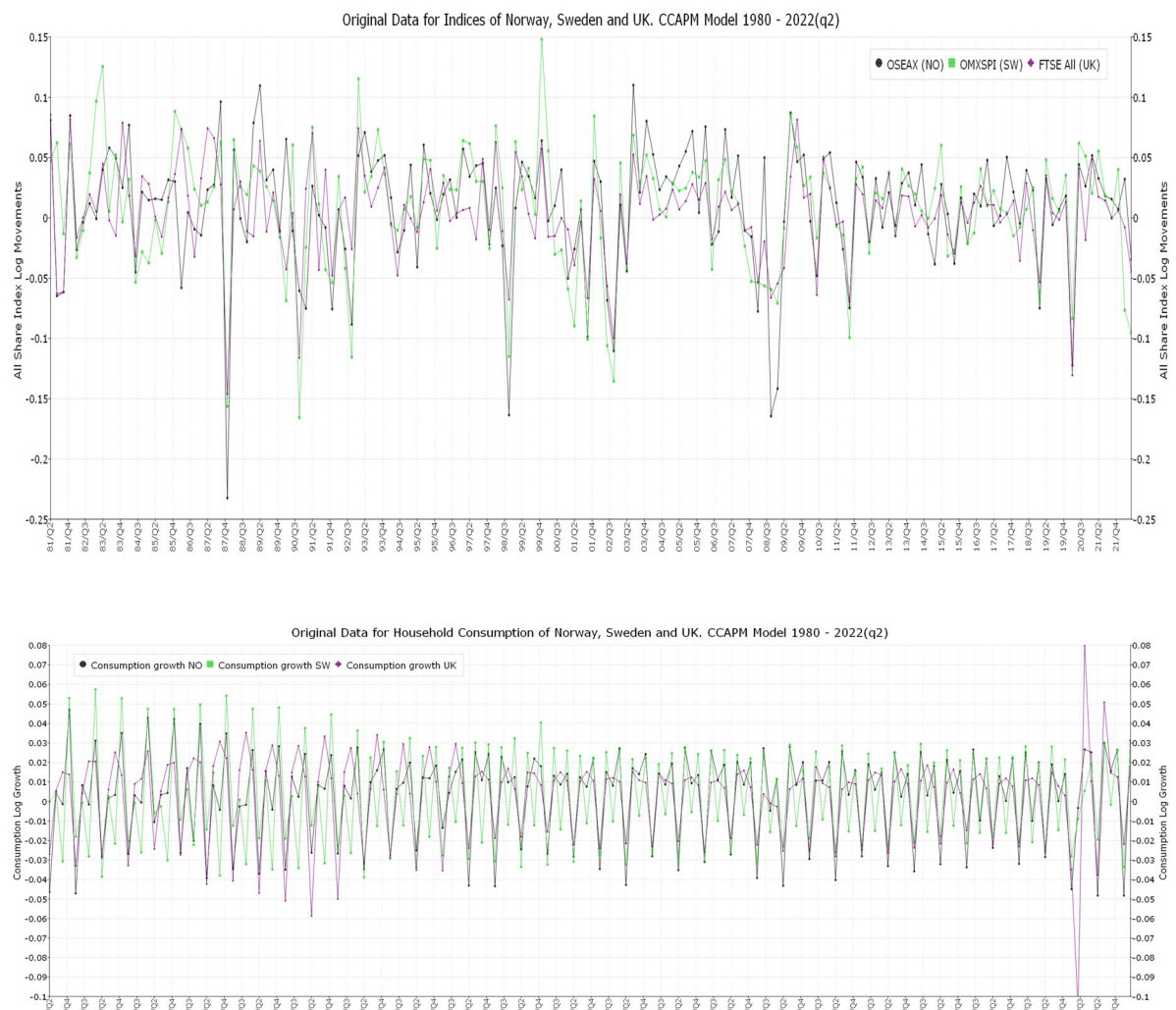


Figure 1. CCAPM data: Index log returns (top) and consumption growth (bottom) for Norway, Sweden, and the UK, 1980-2022(q2).

confirm stationarity. Finally, the BDS test statistic (Broock et al., 1996) reports general non-linear data dependence. We also experimented with breaking trends in the movement equations, but our results suggested little evidence for trend breaks. The Value at Risk (VaR) is a well-known concept for measures of risk and Table 1 includes the 2.5% and 1% VaR numbers for market participants. Figure 1 (top) reports the movement series for the indices of all three countries (OSEAX, OMXSPI, FTSE All Shares).

The consumption growth means are all positive indicating also here positive drift. Norway, Sweden and the UK report drifts of 0.00263, 0.00200 and 0.00252, respectively. The standard deviation is relatively high not excluding a zero drift, approximately 0.023 for the three countries. The Norwegian consumption growth reports a drift of 0.00263 with an associated standard deviation of 0.02355. A similar drift for Sweden (the UK) is 0.00200 (0.00252) with associated standard deviation of 0.02539 (0.02267). The consumption growth reports a maximum (minimum) mean for Norway, Sweden, and the UK of 0.04699 (-0.04821), 0.05743 (-0.03883), 0.07982 (-0.10641), respectively. The kurtosis is negative for Norway and Sweden, while the UK reports a positive kurtosis (tails). The skew is negative (left skewed) for Norway and the UK, while Sweden reports a positive skewness (right skewed). Norway, Sweden, and the UK report a kurtosis (skew) of -0.67083 (-0.66382), -1.19116 (0.19790), 3.16068 (-1.06456), respectively. Cramer-von-Mises test statistic reports significant non-normality for the consumption growth of all three countries. The $Q(12)$ and the $Q^2(12)$ correlogram statistics (serial correlation) show dependencies for the consumption growth mean and volatility. The consumption growth for Norway, Sweden and the UK all report significant $Q(12)$ and the $Q^2(12)$ correlogram statistics. Similarly, the 12th lag ARCH test statistic (Engle, 1982) suggests highly significant conditional heteroscedasticity for consumption growth.

The RESET test (Ramsey, 1969) reports stability for the three countries consumption growth. For all consumption series, the adjusted series the ADF (Dickey & Fuller, 1979) and the Phillips and Perron (1988) statistics confirm stationarity. Finally, the BDS test statistic (Broock et al., 1996) reports general non-linear data dependence. We also experimented with breaking trends in the movement equations, but our results suggested little evidence for trend breaks. The Value at Risk (VaR) is a well-known concept for measures of risk and Table 1 includes the 2.5% and 1% VaR numbers. Figure 1 (bottom) reports the movement series for and the consumption growth of all three countries.

5. Empirical results

The *SNP* model residuals for all three countries are more normal, serial correlation in mean and volatility are removed and the BDS statistic report no data dependence (not reported). The results for the structural parameter estimates are presented first for each model and country (MLE and NPB). All the results are summarized and the risk-aversion, the inter-temporal discount rate, the habit parameter, the uncertainty parameter, and the inter-temporal substitution of consumption are reported, analyzed, and compared.

The Norwegian, Swedish, and UK CRRA models are reported in the left, middle and right columns in Table 3. The bottom of Table 3 reports key statistics from the model estimations. The algorithm seems to work well. All the SSA algorithm steps from Section 3 above report acceptable values.

5.1. CRRA utility functional model

The Bayesian results for the CRRA model are reported in Table 3¹³. The log-posterior traces in Figure 2 shows an acceptable path for a 50,000 long Bayes simulation. The z-abcissae range comes close to covering the 5% through 95% quantiles of the transformed data. Traces for beta (β) and gamma (γ) are shown in Figure 3. Distributions for the same parameters are reported in Figure 4.

The results suggest that beta (β) is close to 1. However, the quarterly β data of 0.99752, 0.99643, and 0.99769 for Norway, Sweden, and the UK, respectively, suggest an annual intertemporal discount rate of approximately 0.99 for all three countries. The β parameter is clearly significant. A discount rate of 0.99, suggests that consumers are quite patient (lowest for Sweden). These β results are also much closer to US CCAPM results than for example Brazilian data (Issler & Piqueira, 2000).

The results for the parameter γ of 3.27907, 0.19407, and 6.17745 for Norway, Sweden, and UK, respectively. Only the UK γ coefficient is significant (1%). Both Norway and Sweden report an insignificant γ coefficient. Moreover, note the rather wide density plots for the γ coefficients in Figure 4. For Norway and Sweden, the densities include zero and therefore risk-neutrality. For the CRRA utility model, the relative risk aversion is therefore high for UK consumers, and quite low and not different from zero for Norwegian and Swedish consumers indicating an 95% confidence interval that is close to zero (risk neutrality). Furthermore, the density plots are quite wide for all three countries but clearly significant for the UK. The density together with the reported means, shows that consumers are mildly risk averse, suggesting that the inter-temporal elasticity of substitution in consumption $\psi = \frac{1}{\gamma}$ is relatively low for the UK, higher for Norway and high for Sweden. That is, the aggregate substitution between investment and consumption is relatively high in the Swedish market (the agent will *not alter* consumption in response to Index/interest rate variations) while Norway and the UK report lower substitution.

¹³Extrinsic mean: $\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$ and $\hat{V}_{EC} = \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})(x_i - \bar{x})^T$

Alternative intrinsic mean (center): $\bar{x} : \hat{V}_{IC} = \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})(x_i - \bar{x})^T$

Adjust for geodesic distance:

1) Modified extrinsic var-covariance matrix: $\hat{V}_{ME} = \frac{1}{N} \sum_{i=1}^N D_{j(i)} D_{j(i)}^T$

2) Modified Riemann var-covariance matrix: $\hat{V}_{MR} = \frac{1}{N} \sum_{i=1}^N z_i z_i^T$

Credibility regions:

An estimated posterior probability can be estimated as: $P(R|x, y) = \frac{1}{N} \sum_{i=1}^N I_R(x_i)$, where .

the region: $R_t = X_{t-1}^{da} [\bar{x}_t - \tau sdev(x_t), \bar{x}_t + \tau sdev(x_t)]$ is of special interest. The.

$sdev(x_t)$ is the square root of the preferred diagonal elements of V_{EC} , V_{IC} , V_{ME} or V_{MR} .

Table 2. Gamma and certainty equivalent for the CRRA utility function.

Value of γ	1	2	5	10	30
Value of W_{CE}	70,711	66,667	58,566	53,991	51,209

Issler and Piqueira (2000), using general method of moments (GMM) methodologies¹⁴, report a discount factor (β) for the US market close to 1, while the Brazilian consumers are more impatient reporting a discount factor (β) of 0.9. Issler and Piqueira (2000), report results for the US market that appear similar to the UK market. However, their analysis reports more ambiguous results for risk aversion and inter-temporal elasticity of substitution in consumption for Brazil. The distributions reported in Figure 4 in this paper, the Issler and Piqueira (2000) ambiguity for risk aversion (using GMM) seems to re-emerge in the distributions reported in Figure 4.

To show an example, individuals with CRRA utility functions ($U(C) = \frac{C^{1-\gamma}}{1-\gamma}$) and risky projects equal to either 50,000 or 100,000 with a probability 0.5, giving an expected wealth $E[W] = 75,000$, report the following certainty equivalent W_{CE} in Table 2.

The first column of Table 2 indicates that an individual with a logarithmic utility function ($\gamma = 1$) would value the risky wealth as equivalent to 70,711 with certainty. This individual would be willing to pay a risk premium of 4,289 ($= E[W] - W_{CE}$) to exchange the risky wealth for its expected value of 75,000. The second column in Table 2 indicates that an individual with a value of $\gamma = 2$ values the risky wealth as equivalent to 66,667 with certainty. This individual would be willing to pay a risk premium of 8,333 to exchange the risky wealth for its expected value of 75,000. The risk premium is almost twice as large as that for an individual with $\gamma = 1$, confirming that the individual with the higher value for γ is, indeed, the more risk averse.

As we progress through Table 2, we can see that the certainty equivalent declines as the value of γ increases. Again, this indicates that risk aversion increases as γ increases. Take note of the last column in Table 2. A person with a score of $\gamma = 30$ is extremely risk averse. Such a person would be prepared to exchange the risky wealth for its anticipated worth of 75,000 by paying a risk premium of \$23,791. The person would have 51,209 with certainty after this trade, which is just a little bit more than the worst-case scenario of 50,000 for the hazardous wealth. This person is so fearful of taking risks that they would probably never consider leaving their house!

Other empirical studies based on the actual behavior of individuals have consistently yielded estimates in the range 1 to 4 for the coefficient of relative risk aversion, with a mean value for $\hat{\gamma} = 2$.

Bayesian parameter traces, densities (with means, modes and standard errors) and log-posterior trace is reported in Figures 2–4 for the CRRA utility model. The stochastic discount factor (MRS) for the period 1981 to 2022(q2) is shown in Figure 4. For the CRRA model, if we plug in for our results for β and γ , we achieve an equity premium close to zero¹⁵. Whether or not the CRRA implied above is plausible, the result has other problems as well. Notice that the expected value of log returns for both assets is quadratic in γ . This means that we can raise to allow for a large equity premium, but it comes at the cost of decreasing returns. At some point, we will raise γ and the return on an asset will decrease, an unrealistic consequence of the model.

The data are quarterly observations from 1981 to 2022(Q2) on inflation adjusted, value weighted returns with dividends on indices and inflation adjusted, per-capita nondurables and services from national statistics (SSB, 2022: (<https://www.ssb.no>); SCB, 2022: <http://www.scb.se>; ONS, 2022: <http://www.ons.gov.uk>).

Bayesian inference is based on the posterior: $Pr(\theta|x) = \frac{Pr(x|\theta) \times Pr(\theta)}{Pr(x)} = \frac{f(\theta|x)\pi(\theta)}{\int f(\theta|x)\pi(\theta)d\theta}$

In the lower part of the table, below the Bayesian parameters (θ) the performance of the SSA algorithm is reported. The normalization constant (marginal data density), that is $Pr(x)$, can be rewritten as: $Z = \int_M f(y|x)\pi(x)d\sigma(x)$ where $\sigma(x)$ is d -dimensional Hausdorff measure on $\mathbb{R}^{d\sigma}$. See Section 5.1 for the definitions of standard deviations ($V_{EC}, V_{IC}, V_{ME}, V_{MR}$). The moment conditions, likelihoods and support conditions are programmed in C/C++. See also Gallant and Tauchen (2022c).

¹⁴The weighting matrix ambiguity using classical GMM methodology seems not appropriately dealt with in Issler and Piqueira (2000).

¹⁵The equity premium can be calculated by taking the difference in returns between the risk free asset and the price for equity: $E_t(\log(1 + r_{t+1}^e) - \log(1 + r_{t+1}^f)) = (2 \cdot \beta - 1) \cdot \omega_2 / 2$, where ω_2 is the assumed $N(\omega_1, \omega_2)$ of the consumption growth.

Table 3. CRRA Utility Model Parameters for Norway, Sweden, and the UK, 1980–2022(q2).

Norway Utility Models Parameter Estimates (MCMC size 250 k)										Sweden Utility Models Parameter Estimates (MCMC size 250 k)										UK Utility Models Parameter Estimates (MCMC size 250 k)															
CRRA utility NPB										CRRA utility NPB										CRRA utility NPB															
Posterior Mode = −392.077										Posterior Mode = −404.549										Posterior Mode = −405.855															
Standard Deviations($V_{EC} V_{IC} V_{ME} V_{MR}$)										Standard Deviations($V_{EC} V_{IC} V_{ME} V_{MR}$)										Standard Deviations($V_{EC} V_{IC} V_{ME} V_{MR}$)															
Parameters		Mean θ		Extrinsic		Intrinsic		θ		Extrinsic		Intr Ctr		Modified Extrinsic		Riemann		Extrinsic		Intr Ctr		Modified Extrinsic		Riemann		Extrinsic		Intr Ctr		Modified Extrinsic		Riemann			
Param		Extrinsic	Intrinsic	θ		Extrinsic	Intrinsic	θ		Extrinsic	Intr Ctr	Modified Extrinsic	Riemann	Extrinsic	Intr Ctr	Modified Extrinsic	Riemann	Extrinsic	Intr Ctr	θ		Extrinsic	Intr Ctr	Modified Extrinsic	Riemann	Extrinsic	Intr Ctr	θ		Extrinsic	Intr Ctr	Modified Extrinsic	Riemann		
a0[1]		−0.30449	−0.33158	0.04263	0.05049	0.05048	0.05108	0.05048	0.05025	0.07759	0.09723	0.11803	0.12514	0.12111	0.11803	0.12514	0.12111	0.12111	0.11803	−0.86711	−0.87711	0.13393	0.13540	0.18744	0.13836	0.13393	0.13540	0.18744	0.13836	0.13393	0.13540	0.18744	0.13836		
a0[2]		0.46780	0.48436	0.08117	0.08283	0.08284	0.08285	0.08285	0.09389	0.09118	0.15629	0.14069	0.25176	0.14241	0.14069	0.25176	0.14241	0.14241	0.14069	−0.47725	−0.47725	0.07367	0.07961	0.08746	0.08054	0.07367	0.07961	0.08746	0.08054	0.07367	0.07961	0.08054	0.07367	0.07961	
a0[3]		−0.10037	−0.03477	0.09273	0.11359	0.11360	0.11310	0.11310	−0.05582	−0.00232	0.04293	0.06583	0.05533	0.06582	0.04293	0.06583	0.06582	0.06582	0.04293	−0.09026	−0.07506	0.04962	0.05190	0.05967	0.05246	0.04962	0.05190	0.05967	0.05246	0.04962	0.05190	0.05967	0.05246	0.04962	0.05190
a0[4]		−0.10723	−0.12684	0.04215	0.04649	0.04650	0.04610	0.04610	−0.00281	−0.03385	0.04297	0.05476	0.05665	0.05494	0.04297	0.05476	0.05665	0.05494	0.04297	0.05566	0.00637	0.06411	0.08087	0.07099	0.08087	0.06411	0.08087	0.07099	0.08087	0.06411	0.08087	0.07099	0.08087	0.06411	0.08087
a0[5]		0.17072	0.19888	0.05981	0.06611	0.06615	0.06573	0.06573	−0.09101	−0.07668	0.03462	0.04032	0.03894	0.04350	0.03462	0.04032	0.03894	0.04350	0.03462	−0.39987	−0.42476	0.10171	0.10470	0.18350	0.10608	0.10171	0.10470	0.18350	0.10608	0.10171	0.10470	0.18350	0.10608	0.10171	0.10470
a0[6]		0.04560	0.09462	0.05950	0.07709	0.07710	0.07700	0.07700	0.00653	−0.00016	0.04624	0.04575	0.05581	0.04656	0.04624	0.04575	0.05581	0.04656	0.04575	−0.14415	−0.06954	0.06504	0.06504	0.09897	0.10611	0.06504	0.06504	0.09897	0.10611	0.06504	0.06504	0.09897	0.10611	0.06504	0.06504
a0[7]		−0.05537	−0.01592	0.03181	0.05067	0.05068	0.05051	0.05051	−0.05801	−0.05451	0.02945	0.02654	0.03313	0.02756	0.02945	0.02654	0.03313	0.02756	0.02654	−0.02860	0.01293	0.03704	0.03704	0.05567	0.05602	0.03704	0.03704	0.05567	0.05602	0.03704	0.03704	0.05567	0.05602	0.03704	0.03704
a0[8]		0.05853	0.02230	0.03258	0.04872	0.04873	0.04848	0.04848	0.17233	0.19084	0.04301	0.04321	0.03192	0.04518	0.04301	0.04321	0.03192	0.04518	0.03192	0.09084	0.07692	0.05882	0.05882	0.06042	0.08415	0.05882	0.05882	0.06042	0.08415	0.05882	0.05882	0.06042	0.08415	0.05882	0.05882
b0[1]		−0.31019	−0.37098	0.12186	0.13618	0.13623	0.13670	0.13670	0.05408	0.06203	0.04346	0.04366	0.05618	0.04394	0.04346	0.04366	0.05618	0.04394	0.04366	0.05064	0.01983	0.04479	0.04479	0.05437	0.10784	0.04479	0.04479	0.05437	0.10784	0.04479	0.04479	0.05437	0.10784	0.04479	0.04479
b0[2]		0.53814	0.46387	0.09854	0.12326	0.12326	0.12533	0.12533	−0.03779	−0.03311	0.13378	0.14620	0.16643	0.15054	0.13378	0.14620	0.16643	0.15054	0.14620	−0.11234	−0.12948	0.08187	0.08366	0.09554	0.08505	0.08187	0.08366	0.09554	0.08505	0.08187	0.08366	0.09554	0.08505	0.08187	0.08366
B(1,1)		0.01637	−0.02238	0.02601	0.04666	0.04669	0.04448	0.04448	0.00166	0.00313	0.01331	0.03527	0.04601	0.01462	0.01331	0.03527	0.04601	0.01462	0.03527	−0.12171	−0.13863	0.05970	0.06207	0.07208	0.06273	0.05970	0.06207	0.07208	0.06273	0.05970	0.06207	0.07208	0.06273	0.05970	0.06207
B(2,1)		−0.00749	−0.02880	0.02376	0.03192	0.03193	0.04039	0.04039	−0.07041	−0.07377	0.05498	0.06876	0.08313	0.07208	0.05498	0.06876	0.08313	0.07208	0.06876	−0.02315	−0.01997	0.02042	0.02067	0.02693	0.01626	0.02042	0.02067	0.02693	0.01626	0.02042	0.02067	0.02693	0.01626	0.02042	0.02067
B(1,2)		−0.04397	−0.10560	0.02784	0.06763	0.06761	0.05916	0.05916	−0.02663	−0.05400	0.04820	0.05859	0.04283	0.06493	0.04820	0.05859	0.04283	0.06493	0.04283	−0.15228	−0.13955	0.05675	0.05814	0.08195	0.05844	0.05675	0.05814	0.08195	0.05844	0.05675	0.05814	0.08195	0.05844	0.05675	0.05814
B(2,2)		−0.06757	−0.07498	0.04362	0.04424	0.04423	0.05394	0.05394	−0.42947	−0.43576	0.05633	0.05359	0.06813	0.05462	0.05633	0.05359	0.06813	0.05462	0.05359	−0.01480	−0.00696	0.01466	0.01662	0.02051	0.01540	0.01466	0.01662	0.02051	0.01540	0.01466	0.01662	0.02051	0.01540	0.01466	0.01662
R0[1]		0.91946	0.84989	0.04349	0.08205	0.08205	0.08223	0.08223	0.28207	0.30575	0.04605	0.04236	0.07371	0.04417	0.04605	0.04236	0.07371	0.04417	0.04236	0.30442	0.32573	0.03743	0.04306	0.04795	0.04332	0.03743	0.04306	0.04795	0.04332	0.03743	0.04306	0.04795	0.04332	0.03743	0.04306
R0[2]		−0.04968	−0.05934	0.02439	0.02624	0.02624	0.02637	0.02637	−0.04762	−0.05642	0.02832	0.02699	0.03832	0.02763	0.02832	0.02699	0.03832	0.02763	0.02699	−0.11864	−0.12175	0.02134	0.02156	0.04844	0.02171	0.02134	0.02156	0.04844	0.02171	0.02134	0.02156	0.04844	0.02171	0.02134	0.02156
R0[3]		0.64990	0.62037	0.04289	0.05206	0.05204	0.05075	0.05075	0.37708	0.41703	0.05281	0.07087	0.06957	0.07164	0.05281	0.07087	0.06957	0.07164	0.06957	0.30362	0.29537	0.03442	0.03540	0.04123	0.03590	0.03442	0.03540	0.04123	0.03590	0.03442	0.03540	0.04123	0.03590	0.03442	0.03540
P(1,1)		0.17758	0.20300	0.06963	0.07412	0.07415	0.07401	0.07401	0.59268	0.66058	0.07781	0.08759	0.29275	0.08942	0.07781	0.08759	0.29275	0.08942	0.29275	0.51001	0.53036	0.04658	0.05081	0.07311	0.05124	0.04658	0.05081	0.07311	0.05124	0.04658	0.05081	0.07311	0.05124	0.04658	0.05081
P(2,1)		−0.13339	−0.21189	0.05299	0.09472	0.09472	0.09378	0.09378	−0.07804	−0.11742	0.06465	0.06906	0.08781	0.06908	0.06465	0.06906	0.08781	0.06908	0.06906	−0.21319	−0.23617	0.03675	0.04336	0.05033	0.04360	0.03675	0.04336	0.05033	0.04360	0.03675	0.04336	0.05033	0.04360	0.03675	0.04336
Q(1,1)		−	−	−	−	−	−	−	0.76894	0.71946	0.05556	0.06253	0.15522	0.06148	0.05556	0.06253	0.15522	0.06148	0.06253	0.83649	0.83620	0.02489	0.02491	0.03239	0.02513	0.02489	0.02491	0.03239	0.02513	0.02489	0.02491	0.03239	0.02513	0.02489	0.02491
beta (b)		0.99752	0.99827	0.00097	0.00123	0.00123	0.00671	0.00671	0.99693	0.99640	0.00095	0.00105	0.00091	0.00141	0.00095	0.00105	0.00091	0.00141	0.00091	0.99769	0.99868	0.00096	0.00138	0.00142	0.00243	0.00096	0.00138	0.00142	0.00243	0.00096	0.00138	0.00142	0.00243	0.00096	0.00138
gamma (γ)		3.27907	3.13085	2.49621	2.50065	2.50154	2.50050	2.50050	0.19407	0.25813	0.18681	0.21928	0.34966	0.21006	0.18681	0.21928	0.34966	0.21006	0.21928	6.17745	6.15927	1.55603	1.55592	2.64287	1.55846	1.55603	1.55592	2.64287	1.55846	1.55603	1.55592	2.64287	1.55846	1.55603	1.55592
β ⁴		0.99012	0.99310	−	−	−	−	−	0.98778	0.98568	−	−	−	−	0.98778	0.98568	−	−	−	0.99079	0.99473	0.99473	0.99473	−	−	−	−	−	−	−	−	−	−	−	−
1/(γ)		0.30496	0.31940	−	−	−	−	−	5.15278	3.87402	−	−	−	−	5.15278	3.87402	−	−	−	0.16188	0.16236	0.16236	0.16236	−	−	−	−	−	−	−	−	−	−	−	−
Norway CRRA										Sweden CRRA										Credibility regions CRRA Norway utility:															
Posterior Mode = −392.077										Posterior Mode = −404.549										Posterior Mode = −405.855															
Standard Deviations($V_{EC} V_{IC} V_{ME} V_{MR}$)										Standard Deviations($V_{EC} V_{IC} V_{ME} V_{MR}$)										Standard Deviations($V_{EC} V_{IC} V_{ME} V_{MR}$)															
Rejection rate		Norway CRRA		Sweden CRRA		Credibility regions CRRA		UK CRRA		Norway CRRA		Sweden CRRA		Credibility regions CRRA		UK CRRA		Norway CRRA		Sweden CRRA		Credibility regions CRRA		UK CRRA		Norway CRRA		Sweden CRRA		Credibility regions CRRA		UK CRRA			
		0.30094		0.45275		0.45275		0.63594		0.30094		0.45275		0.45275		0.63594		0.30094		0.45275		0.45275		0.63594		0.30094		0.45275		0.45275		0.63594			
Metropolis-Hastings reject average		0.25510		0.30516		0.30516		0.40987		0.25510		0.30516		0.30516		0.40987		0.25510		0.30516		0.30516		0.40987		0.25510		0.30516		0.30516		0.40987			
Inequality failure average		0.04584		0.14758		0.14758		0.22607		0.04584		0.14758		0.14758																					

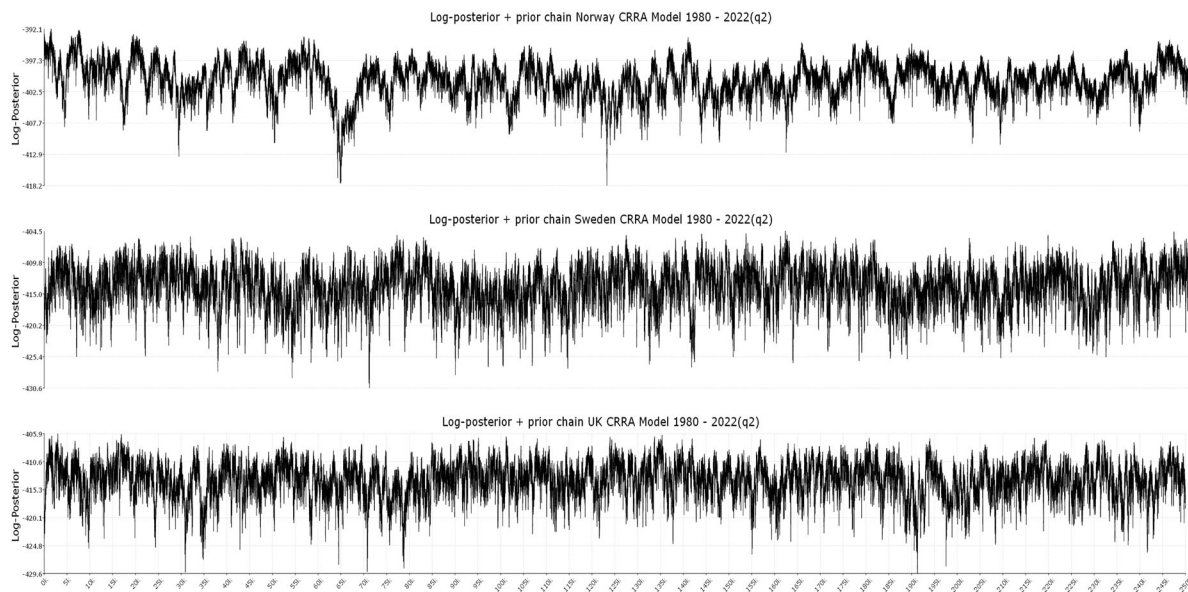


Figure 2. Trace plots (50k) of Norwegian (top), Swedish (middle) and UK (bottom) CRRA log posterior paths. (a) Norway traces and parameter densities. (b) Sweden traces and parameter densities. (c) The UK traces and parameter densities.

5.2. Model specifications and the marginal likelihood

Figure 2 reports log-likelihoods for the CRRA models of Norway, Sweden, and the UK, respectively. The plots report satisfactory variance and diversion. The log-likelihood reports the lowest optimal likelihood for Norway.

Tables 3 and 4 report the normalization constants. In Bayes rule, the denominator $Pr(x)$ (the marginal probability of the data) is the normalization constant (a constant that makes the posterior density integrate to one). The CRRA model reports a log normalization constant of -438.75 , 455.29 , 450.70 , with an associated standard error of 0.01854 , 0.0103 , 0.0177 , for Norway, Sweden, and the UK, respectively. The result is also the marginal data density $Z = \int_M f(y|x) \cdot \pi(x) \cdot d\sigma(x)$, where $\sigma(x)$ is d -dimensional Hausdorff measure on \mathbb{R}^{da} .

Furthermore, for model specifications, the probability of simultaneously valid credibility intervals using the center, standard deviations V_{IC} and V_{MR} from above and critical values of 1.96 (2.70), the CRRA model reports 0.5239 (0.9708), and 0.5490 (0.9703) for Norway, 0.5714 (0.8975), and 0.3366 (0.6703) for Sweden and 0.6108 (0.9286), and 0.3863 (0.8564) for the UK.

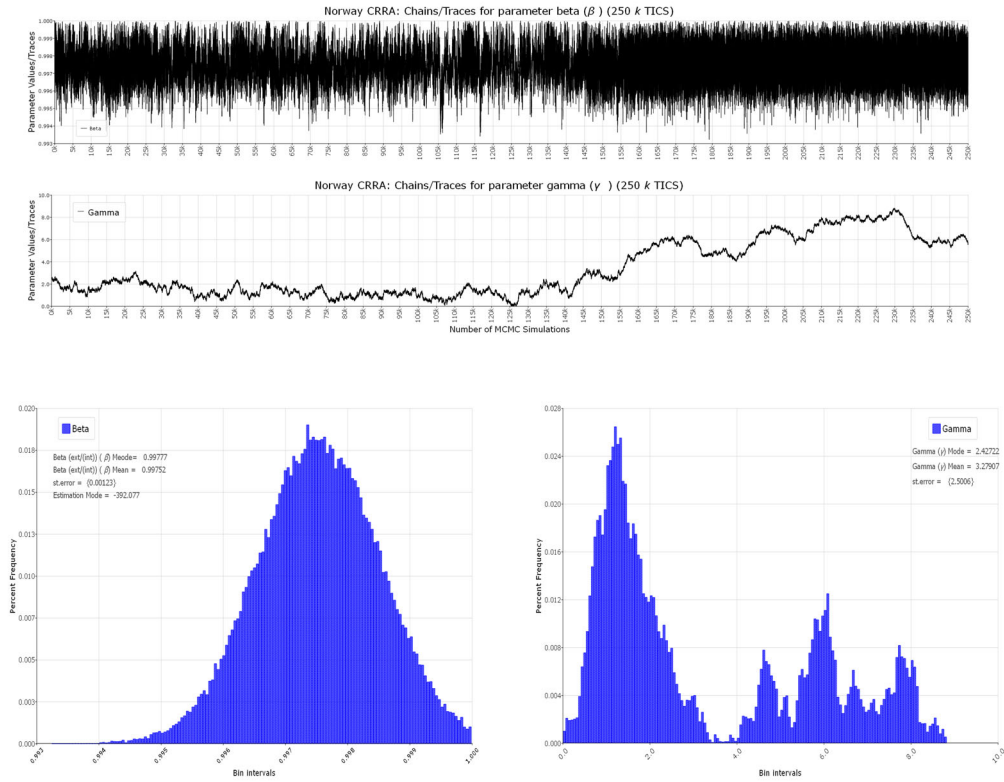
For all three countries, the z-abscissae range (quadrature rule) is well within the quadrature rule covering the 5% through 95% quantiles of the transformed data (Table 3). For Norway, $(-1.87822, 1.33526) \subset (-2.85697, 2.85697)$ and $(-1.18963, 1.92945) \subset (-2.85697, 2.85697)$. For Sweden, $(-2.11780, 1.28166) \subset (-2.85697, 2.85697)$ and $(-1.38673, 1.74788) \subset (-2.85697, 2.85697)$. For the UK, $(-1.79326, 1.70977) \subset (-2.85697, 2.85697)$ and $(-1.26017, 1.71638) \subset (-2.85697, 2.85697)$. Moreover, trace plots for the CRRA model (appendix) report satisfactory variance for all parameters in the NPB model (except for β , γ) already reported above for the same three countries).

5.3. CRRA comparison of the European countries

Table 4 reports the median European results using the CRRA utility function. The results for beta (β) the discount rate is reported in terms of quarter equivalent rate. The results show the following immediate results:

1. The estimates for the discount rate (β) are approximately 0.99 on an annual basis for all three models $0.99 (= 0.9975^4)$. The result is close to US consumer data. The Brazilian discount factor is lower (Issler & Piqueira, 2000).

Panel A. Norway traces and parameter densities



Panel B. Sweden traces and parameter densities

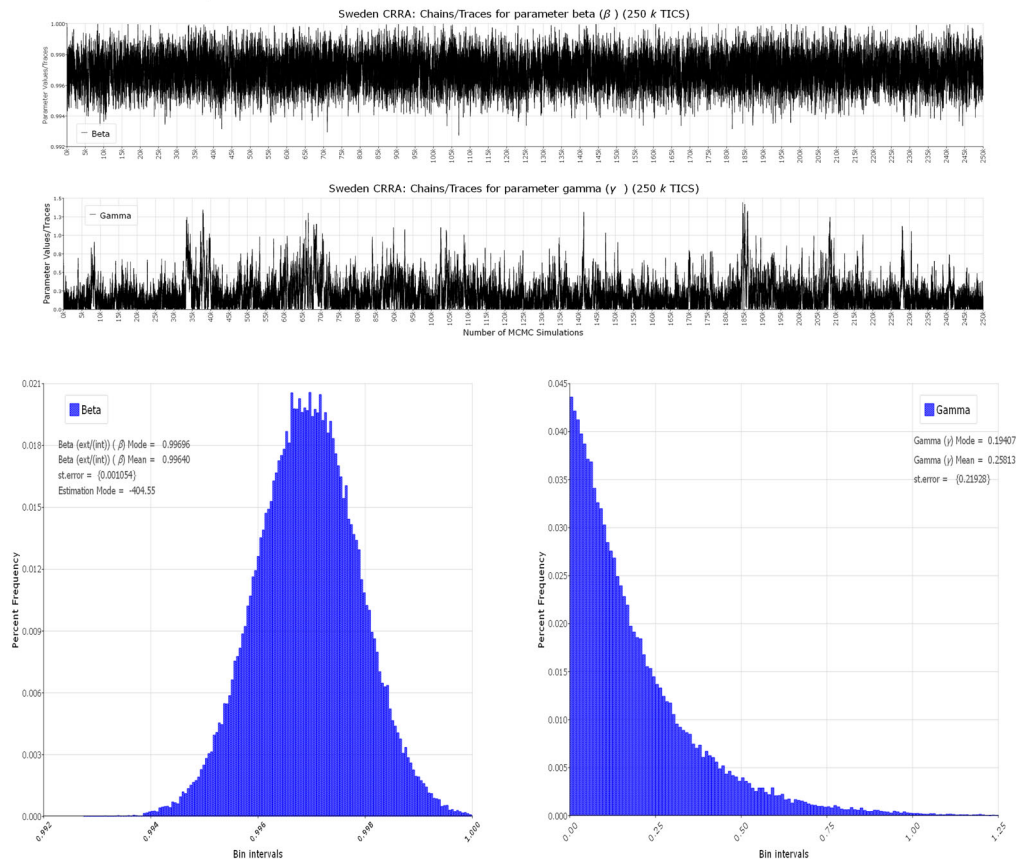


Figure 3. Trace and Density plots (50k) of the Norwegian (top), Swedish (middle) and UK (bottom) CRRA utility function's subjective discount factor β and the risk aversion parameter γ . Mean and standard errors of β and γ are reported in the plots.

Panel C. The UK traces and parameter densities

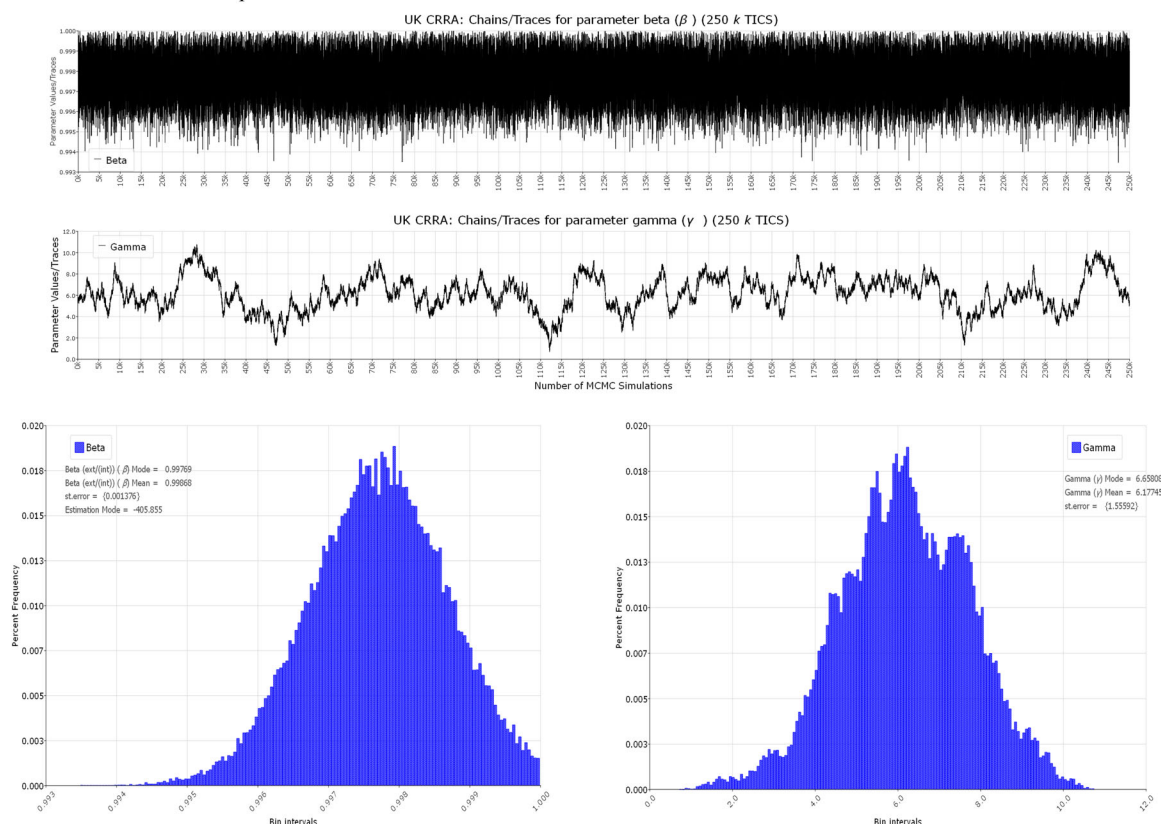


Figure 3. Continued.

2. The relative risk aversion coefficients (γ) varies between the European countries. For the CRRA utility model the γ is approximately between 0 and 6.5. For Norway the γ risk aversion coefficient is 3.38 and insignificant ($t < 1.6$). Furthermore, the Norwegian distribution is both wide and divided into several parts. The distribution seems therefore to include zero (risk neutrality). For Sweden, the γ coefficient is low 0.194 and insignificant ($t < 1.25$). The distribution is toward zero with a right tail. The UK seems to be the only country that report a positive coefficient of 6.18 that is significant ($t > 3.95$). The distribution is not split into several parts and lies between 3.1 and 9.3 (95% confidence interval).
3. The Swedish consumers seem therefore risk neutral. The Norwegian consumers are less risk-averse than UK consumers but seem more risk averse than Swedish consumers. The UK reports risk aversion that seems to exceed US consumers.
4. The equity premium seems quite low for all three countries and is comparable to the US and the Brazilian market. Issler and Piqueira (2000) find no equity premium in the Brazilian market.
5. The inter-temporal substitution in consumption ($1/\gamma$) does therefore vary between the countries. For the UK, the inter-temporal substitution is relatively low, higher for Norway, while the inter-temporal substitution for Sweden is relatively high.

The model strengthens the argument of risk aversion among consumers/investors in the Norwegian and UK markets. In contrast, Sweden report risk neutrality. However, for Norway, the risk aversion coefficient is not significant, and the distribution includes zero signaling risk neutrality. Finally, the yearly discount factor of 0.99 ($0.99 = 0.9975^4$) is quite stable over all three countries, indicating relatively patient market participants. The stochastic discount factors from Figure 4 have a quarterly average of 0.99752, 0.99693, and 0.99769, for Norway, Sweden, and the UK, respectively. Hence, the most impatient (β) and risk neutral (γ) customers/investors are found in Sweden.

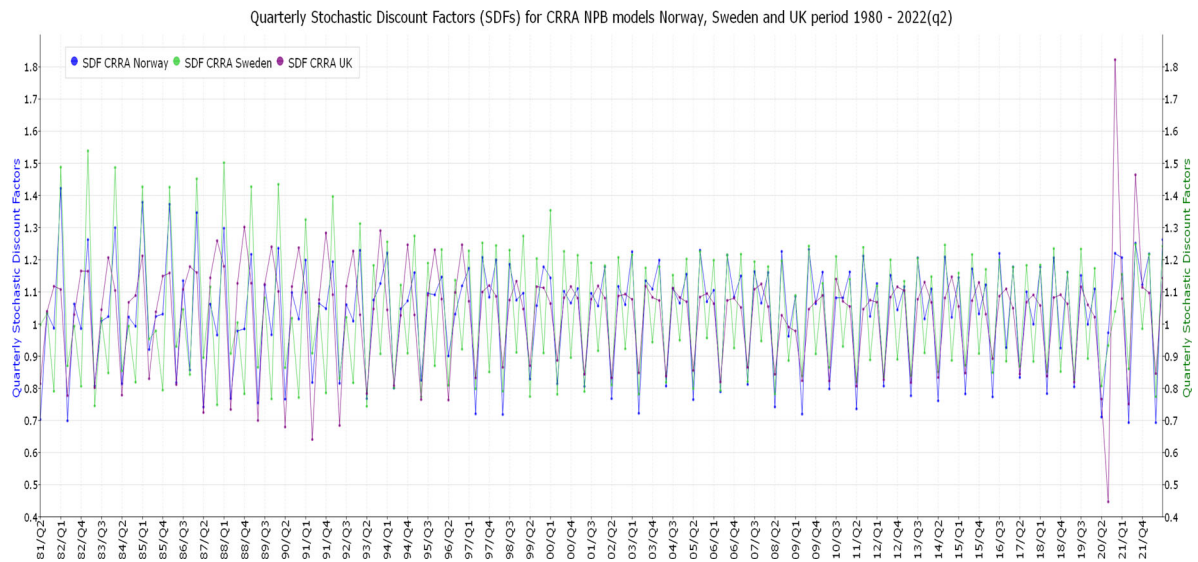


Figure 4. Stochastic discount factor for a CRRA utility model Norway, Sweden, and the UK, 1980-2022(q2).

Table 4. The mode, parameters for beta (β) and gamma (γ) and the normalization constant. Comparison Norway, Sweden, and the UK.

Comparing Mean Results for Norway, Sweden and the UK

		Period – 1980–2022 (q2)		
		Norway	Sweden	the UK
Posterior Mode		–392.07741	–404.54867	–405.85519
Mean	Beta (β)	0.99752	0.99693	0.99769
discounting		{0.0010}	{0.0009}	{0.0010}
Relative	Gamma (γ)	3.27907	0.19407	6.17745
risk aversion		{2.4962}	{0.1868}	{1.5560}
Normalisation				
constant		–438.7531	–455.2890	–450.7014
Standard dev		{0.01854}	{0.0103}	{0.0177}

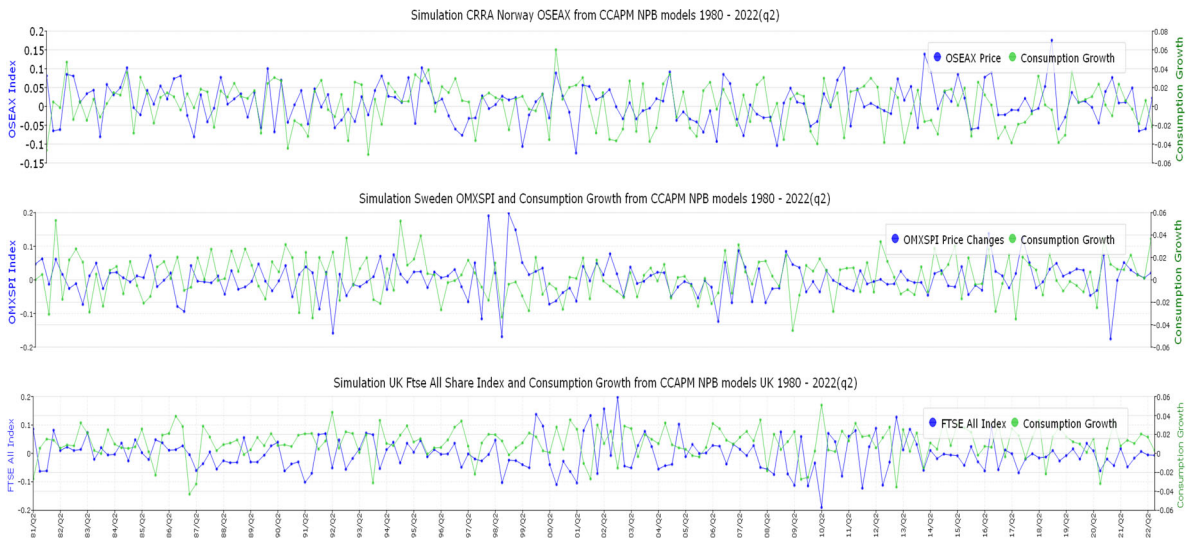
Another important conclusion from our NPB estimation results is that there is almost no equity premium for all three countries. Plugging in for the CRRA model reports a difference of $(2\beta - 1) \cdot (\omega_2/2)$, where ω_2 is the consumption sample variance of 0.0558% given equity premiums of less than 1%.

5.4. Simulations (100 k) and step ahead mean and volatility densities

The MCMC SSA algorithm together with the SNP methodology obtain a convenient representation of one-step ahead conditional mean \hat{y}_{t+1} given $\{\hat{y}_\tau\}_{\tau=1}^t$ and variance $\hat{\sigma}_t^2$ of \hat{y}_{t+1} given $\{\hat{y}_\tau\}_{\tau=1}^t$. In Figure 5(a), the paper therefore reports a simulation for both indices and consumptions growth for Norway (top), Sweden (middle) and the UK (bottom). Furthermore, in Figure 5(b) the simulations are extended to 100k and index and consumption growth are split into sub-plots. The index paths are plotted on the left and consumption growth on the right. Figure 6 report 100k densities for bivariate simulations. The figures report 100k densities all-share indices (left) and the consumption growth (right) for Norway (top), Sweden (middle) and the UK (bottom). The simulations show that the consumption densities are skewed to the left for all three countries and the country's indices have similar mean movements but quite different maximum and minimum.

The conditional mean, volatility and correlation is reported in Figures 7–10. The movements report co-movements clearly indicated by the correlation numbers in Figure 9. Only the habit indicates some form of negative correlation between the index and the consumption growth. Finally, Figure 10 reports plots for the three countries bivariate one-step-ahead index and consumption movements. The 3D-plots show similar high surface densities when the indices are around 0.1, indicating that appositive

Panel A. Simulation 166 Index Returns and Consumption Growth observations from NPB model



Panel B. Simulation 100k Index Returns and Consumption Growth observations from NPB model

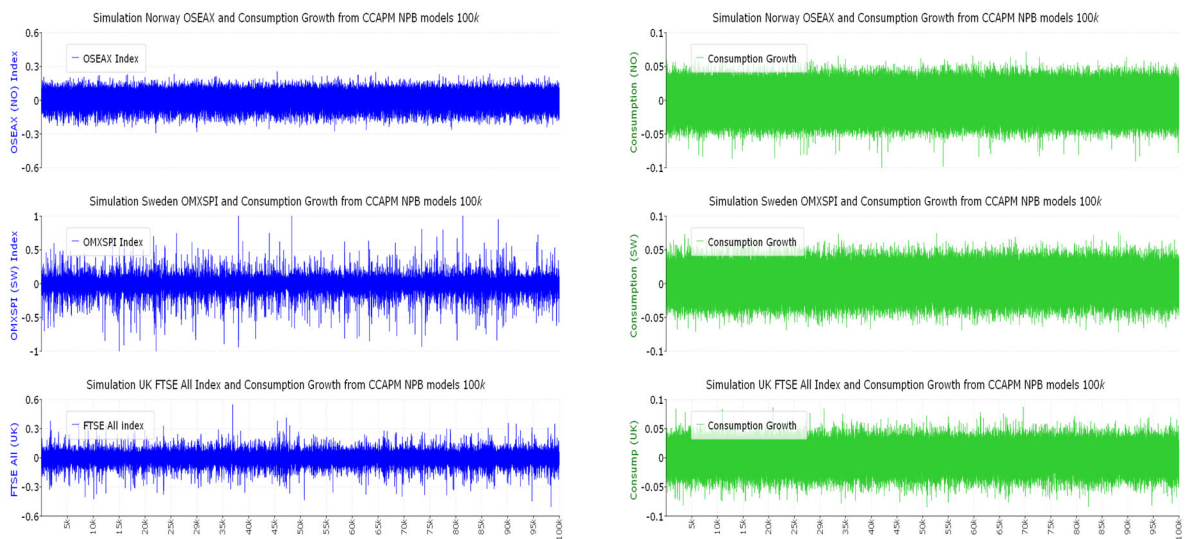


Figure 5. NPB Simulation of 166 and 100k Index Returns and Consumption Growth. (a) Simulation 166 Index Returns and Consumption Growth observations from NPB model. (b) Simulation 100k Index Returns and Consumption Growth observations from NPB model.

consumption movements are dependent on a positive index market. Furthermore, when consumption growth is strongly negative there are some spectacular movements of the OSEAX index (or vice versa).

6. Summary and conclusions

The main objective of this paper has been to present estimates for the risk aversion, coefficient, the discount rate of the future utility, and the inter-temporal elasticity of consumption for Norwegian, Swedish, and UK consumers adapting the highly innovative MCMC Surface Sampling Algorithm of Zappa et al. (2018). Both GMM ambiguity and inhibited use of MCMC models (overidentified moment conditions) are circumvented. Moreover, the algorithm extracts the posterior, estimates scale, and calculate the normalization constant of the posterior density. For implementation, the manuscript uses the CRRA utility specification CCAPM model for three European countries.

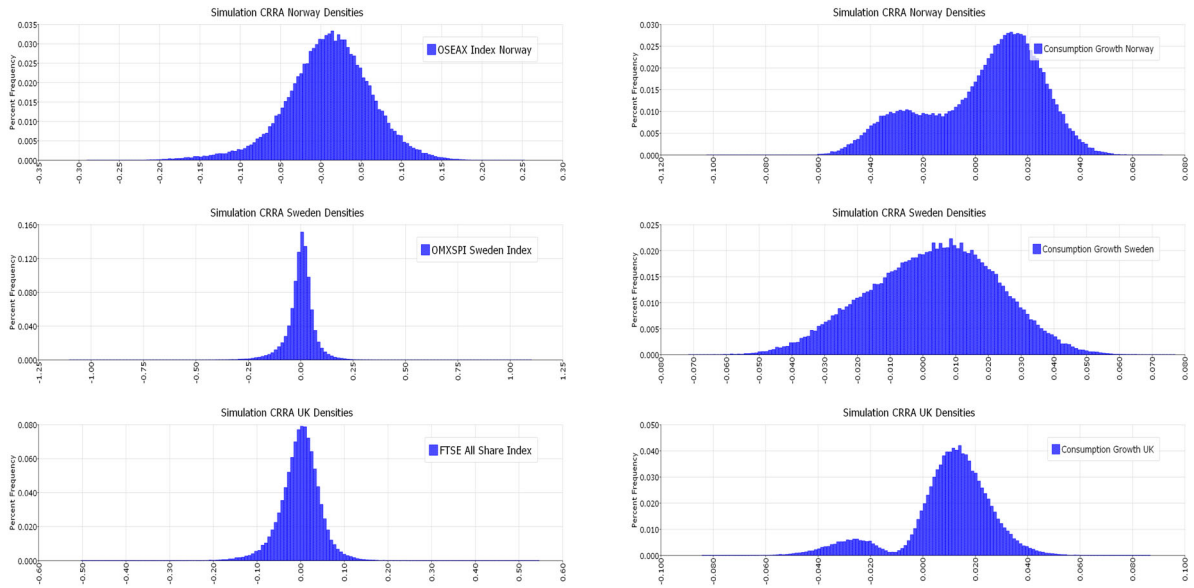


Figure 6. Simulation 100k OSEAX returns and Consumption Movements Distributions.



Figure 7. Conditional Mean applying the original data-points.

The paper's empirical results allow us to make some inference about these three countries market characteristics extracting discount factors and risk aversion/elasticity of consumption coefficients. The main results can be summarized in the following way:

1. An estimate of the yearly discount factor β in Norway, Sweden, and the UK is close to 0.99 for all three countries. The most impatient consumers are located in Norway. The result is close to the US market.
2. The relative risk aversion γ is highest in the UK (6.18) significant at 1%, followed by Norway with a coefficient of 3.28 and a standard error of 2.5 (insignificant). In contrast, Sweden reports a risk aversion coefficient of 0.2. The Swedish result is insignificant and clearly reports a risk aversion (γ) parameter indicating risk neutrality (the possibility that γ is zero). Furthermore, the gamma density of Sweden extends risk neutrality with a right tail toward 1. Norway's distribution is both wide and split over several areas including zeroes (risk neutrality). The UK reports a continuous clearly positive



Figure 8. Conditional Volatility using original data-points.

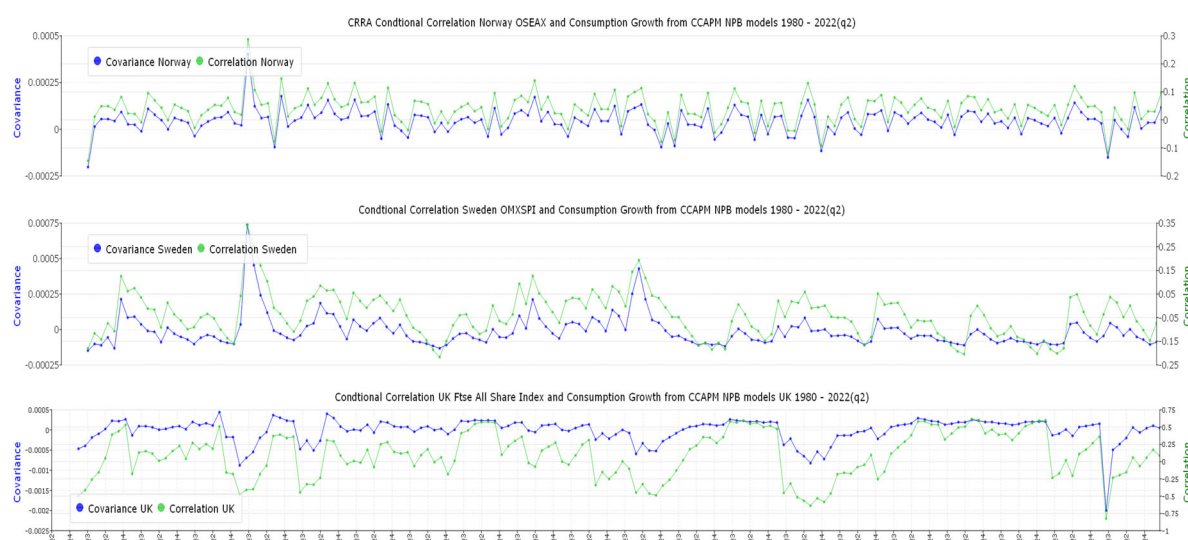


Figure 9. Conditional Covariance/Correlation using original data-points. (a) Norway's CRRA Mean One-Step-Ahead Density 1980-2022(q2). (b) Sweden's CRRA Mean One-Step-Ahead Density 1980-2022(q2). (c) UK's CRRA Mean One-Step-Ahead Density 1980-2022(q2).

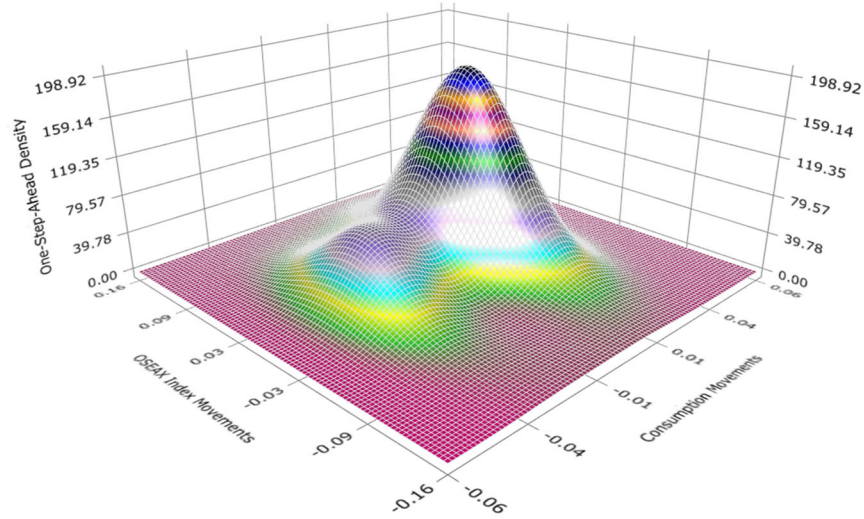
coefficient between 3 and 9. The ambiguity found in other international studies using GMM, gives a better insight in the importance of methodology.

3. The equity premium seems low for all three European countries ($<1\%$).
4. The inter-temporal elasticity of substitution $1/7$ in consumption is low for the UK (0.16), higher for Norway (0.3) and high for Sweden (5).

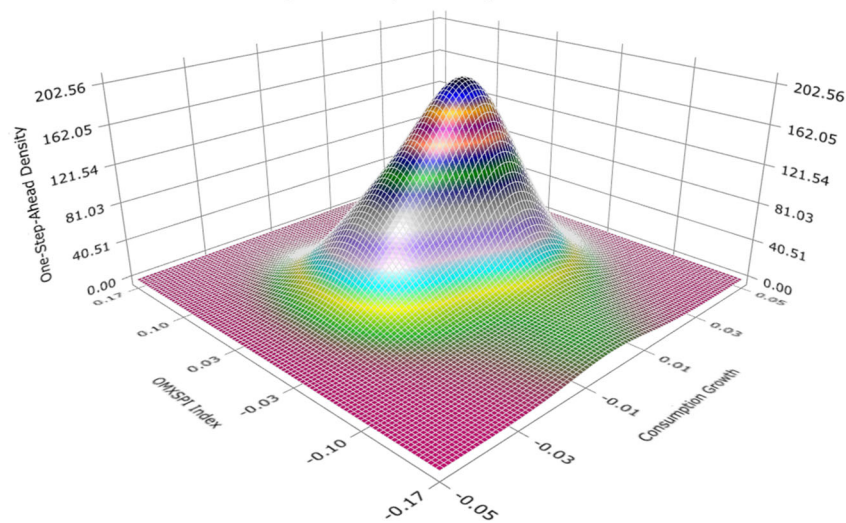
This article therefore reports a successful implementation of nonparametric Bayesian estimation, giving estimated and thoroughly tested CCAPM model coefficients for discount factors and risk aversion/elasticity of consumption (with associated scale and normalization constant). The results show that European asset markets price positive risk and investors require some compensation for exposing their portfolios to risk (risk aversion).

The empirical results from the optimal nonparametric Bayesian estimation's simulation plots, and the bivariate step ahead index and consumption plots, indicate a well-functioning algorithm. Furthermore, the estimation challenge because the support of the posterior is a manifold of lower dimension than

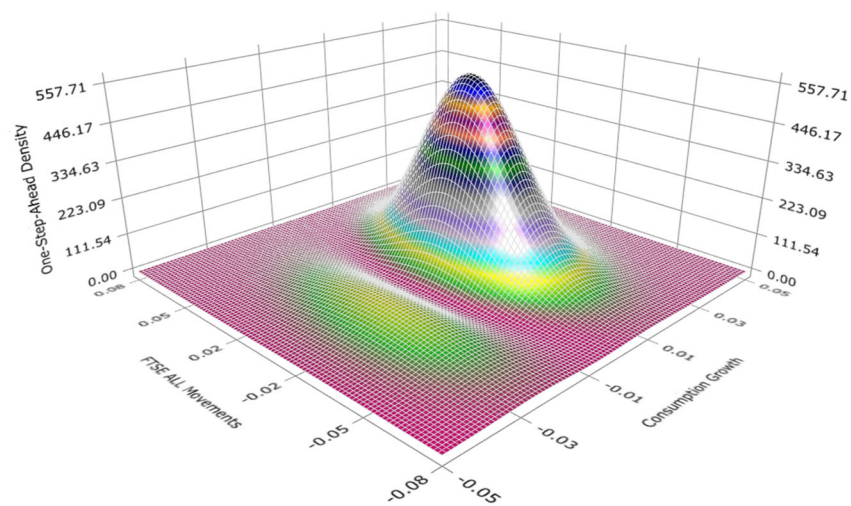
Panel A: Norway's CRRA Mean One-Step-Ahead Density 1980-2022(q2)



Panel B: Sweden's CRRA Mean One-Step-Ahead Density 1980-2022(q2)



Panel C: UK's CRRA Mean One-Step-Ahead Density 1980-2022(q2)

**Figure 10.** One-Step Ahead Mean Surface Densities NPB estimations 1980-2022(Q2) for Norway, Sweden, and UK.

the number of model parameters (Lebesgue measure of zero), seems overcome and solved methodologically.

Finally, some words of caution regarding the main results. Notice that the expected value of log returns for both assets is quadratic in γ . This means that we can raise γ to allow for large equity premiums, but it comes at the cost of decreasing returns. Furthermore, quarterly dividends are from statistics Norway/Sweden and the UK. The calculated index accuracy may be periodized somewhat inaccurately giving small index return mistakes (see Statistics Norway).

Disclosure statement

No potential conflict of interest was reported by the author(s).

Author contributions

The author contributed to the study conception and design. Material preparation, data collection and analysis were performed by Per Bjarte Solibakke. Data for indices are available from financial markets and associated research databases. Consumption growth and CPI are available from national statistics. The first draft of the manuscript was written by Per Bjarte Solibakke. Software for NPB estimation from Gallant & Tauchen, 2022c. Presentation software (graphics) available from Aptech Systems, GAUSS 23 (<seurl>www.aptech.com</seurl>, 2023).

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Appendix A.

Trace Plots for Norway¹⁶

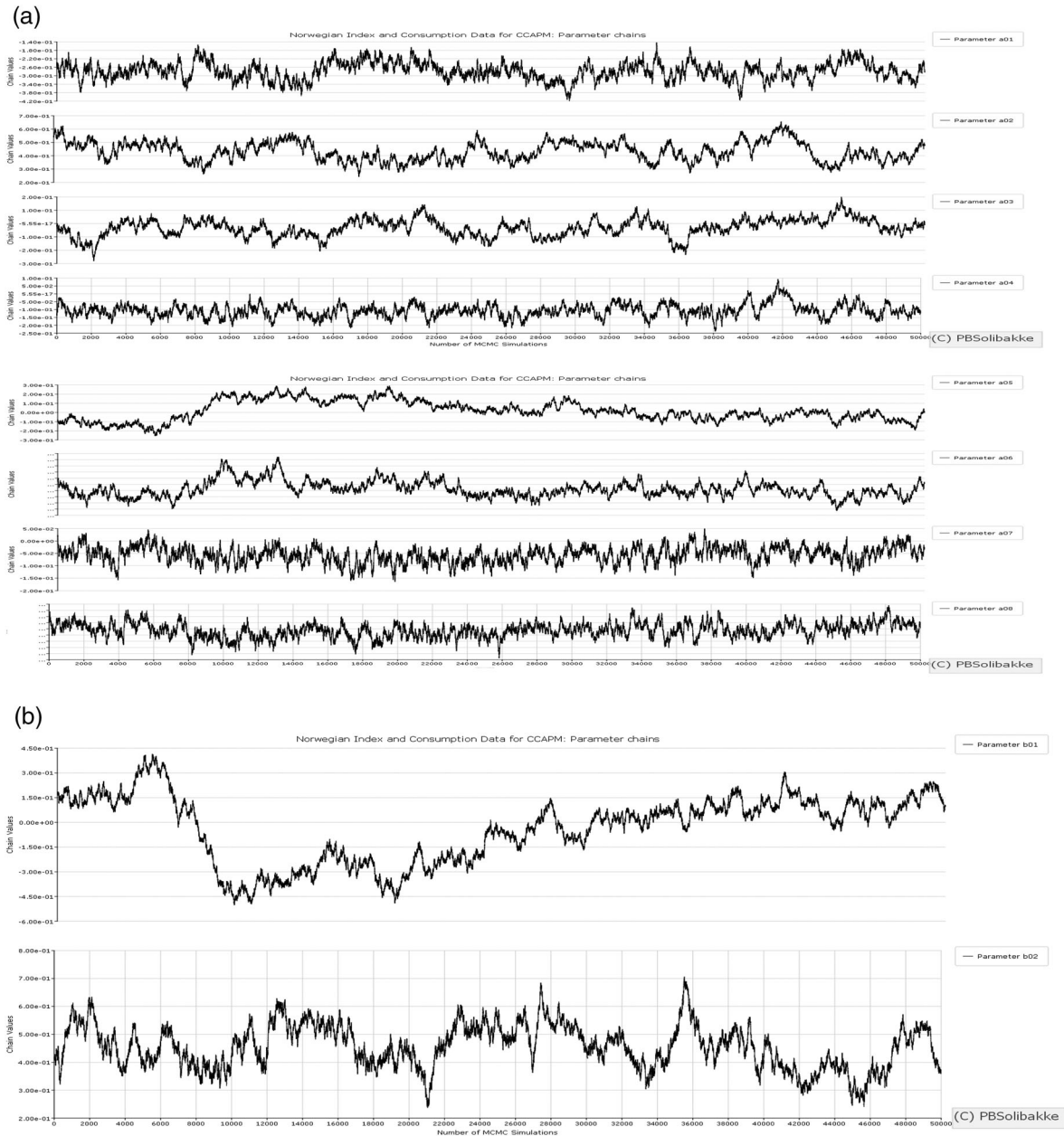


Figure A1. Trace Plots polynomial terms (a), location function terms (b and c), variance function terms (d and e). (a) Trace plots of SNP-GARCH Polynomial Terms. The bivariate SNP_GARCH density is proportional to $P^2(y)n_2(y|\mu_{t-1}, \sum_{t-1})$ where $P(y) = 1 + a0[1]y_2 + a0[2]y_2^2 + a0[3]y_2^3 + a0[4]y_2^4 + a0[5]y_1 + a0[6]y_1^2 + a0[7]y_1^3 + a0[8]y_1^4$. (b) Trace plot for SNP-GARCH Location Function Terms. The bivariate SNP_GARCH density is proportional to $P^2(y)n_2(y|\mu_{t-1}, \sum_{t-1})$ where $\mu_{t-1} = b_0 + By_{t-1}$, $b_0 = (b0[1], b0[2])'$, and B has element $b_{i,j} = B(i,j)$. (c) Trace plot for SNP-GARCH Location Function Terms. The bivariate SNP_GARCH density is proportional to $P^2(y)n_2(y|\mu_{t-1}, \sum_{t-1})$ where $\mu_{t-1} = b_0 + By_{t-1}$, $b_0 = (b0[1], b0[2])'$, and B has element $b_{i,j} = B(i,j)$. (d) Trace plot for SNP-GARCH Variance Function Terms. The bivariate SNP_GARCH density is proportional to $P^2(y)n_2(y|\mu_{t-1}, \sum_{t-1})$ where $\mu_{t-1} = b_0 + By_{t-1}$, $b_0 = (b0[1], b0[2])'$, and $\sum_{t-1} = R_0R'_0 + Q\sum_{y_{t-1}}Q' + P(y_{t-1} - \mu_{t-2})(y_{t-1} - \mu_{t-2})'P'$. R_0 has element $r_{1,1} = R0[1]$, $r_{2,1} = 0$, $r_{1,2} = R0[4]$, and $r_{2,2} = R0[3]$. P has diagonal with elements $p_{1,1} = P[1,1]$, and $p_{2,2} = P[2,1]$. Q is zero for Norway.

¹⁶Trace plots for all parameters for the Swedish and UK estimations are available from author upon request.

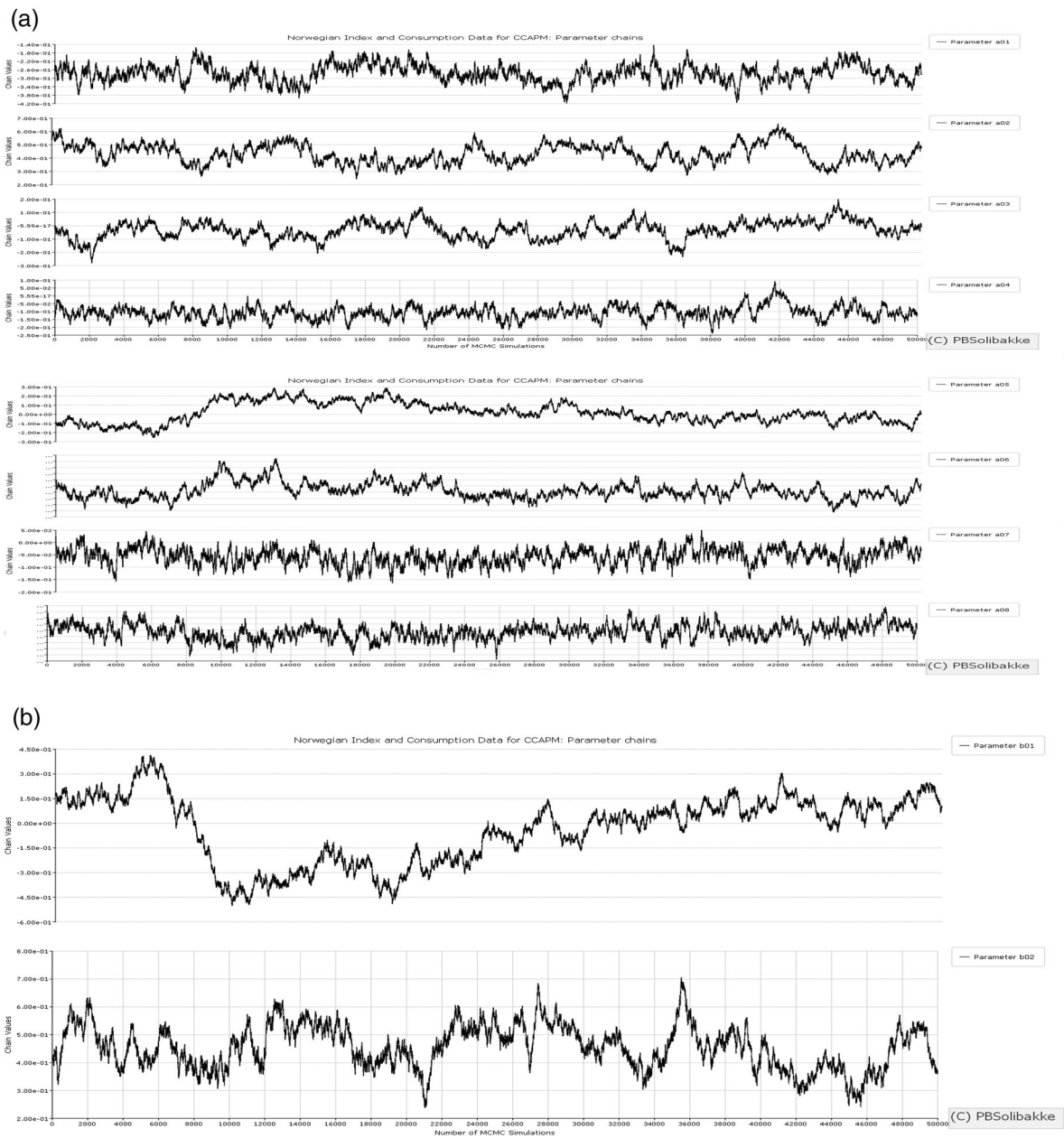


Figure A1. Continued.