

Vitale, Paolo

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## Optimal monetary policy for a pessimistic central bank

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## **CASMEF Working Paper Series**

OPTIMAL MONETARY POLICY FOR A PESSIMISTIC CENTRAL BANK

*Paolo Vitale*

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Department of Economics and Business

LUISS Guido Carli

Viale Romania 32, 00197, Rome -- Italy

<http://casmef.luiss.edu>

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# Optimal Monetary Policy for a Pessimistic Central Bank

**Paolo Vitale**

*University of Pescara\**

**August 2014**

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\*Department of Economics, Università Gabriele d'Annunzio, Viale Pindaro 42, 65127 Pescara (Italy); telephone: ++39-085-453-7647; fax: ++39-085-453-7565; webpage: <http://www.unich.it/~vitale>; e-mail: [p.vitale@unich.it](mailto:p.vitale@unich.it)

# Optimal Monetary Policy for a Pessimistic Central Bank

## ABSTRACT

We extend Svensson's (Svensson, 1997) model of optimal monetary policy to the case in which the monetary authorities are pessimistic. With respect to his formulation we show that: i) the inflation forecast is no longer an explicit intermediate target; ii) the monetary authorities move their instruments to hedge against the worst economic shocks, do not expect the inflation rate to mean revert to its first-best level and apply a more aggressive Taylor rule; and iii) the inflation rate is less volatile. Our conclusions also hold when the monetary authorities observe inflation and output gap with a time lag. Our analysis extends the analysis of van der Ploeg (van der Ploeg, 2009), as we allow for time-discounting of future social welfare losses due to deviations of output and inflation from first-best values.

**JEL Classification Numbers:** C61, E52, E58.

**Keywords:** Monetary Policy, Pessimism, Discounted Linear Exponential Quadratic Gaussian.

[..] a slide down a debt-deflation spiral could [..] create an existential crisis. In these circumstances patience is imprudent: the ECB should get a move on  
*The Economist*, August 2<sup>nd</sup> 2014

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## Introduction

A common feature of models of monetary policy (Galí (2008)) is that social preferences are represented by time-separable quadratic loss functions (see Rotemberg and Woodford (1999) and Woodford (2001)). As the economic environment is described by Markovian linear laws of motion, the optimal monetary policy is obtained exploiting standard results which apply to the linear-quadratic regulator (LQR) framework (Whittle (1982)). Specifically, it is possible to rely on the certainty equivalence principle (CEP) and replace unknown values with their maximum likelihood (ML) estimates. While convenient such a property is also problematic as it entails that uncertainty does not play a significant role in determining the optimal policy and that the quadratic loss function of models of monetary policy represents risk-aversion in an unsatisfactory manner. In fact, the prescribed optimal policy does not change when the environmental uncertainty varies, while risk-averse agents ought to care for the degree of uncertainty they face.

van der Ploeg (van der Pleog, 2009) corrects for these shortcomings by introducing a risk-adjustment in the loss function of the monetary authorities. He modifies the standard formulation of models of monetary policy (see Svensson, 1997), moving from a linear-quadratic framework to the linear-exponential-quadratic framework proposed by Whittle (Whittle, 1990). In doing so, he increases the convexity of the monetary authorities' loss function and derives an optimal policy which is influenced by the environmental uncertainty. In addition, he shows that the optimal policy is identified via a min-max choice mechanism, according to which the monetary authorities set their optimal policy in order to hedge against the worst economic conditions. This implies that first the worst economic outcomes are identified and then the optimal policy is chosen in order to minimize the social welfare loss such outcomes entail. In other words, according to the min-max mechanism described by van der Ploeg the monetary policy is set in order to hedge against the worst shocks to the economy.

A problem with Whittle's linear exponential quadratic Gaussian (LEQG) framework is that

it is not best suited to accommodate time-discounting. In fact, when a time-discounting factor is introduced into his LEQG framework, no stationary solutions exist, the impact of the risk-adjustment dissipates overtime and the optimal policy in the limit converges to the linear-quadratic Gaussian (LQG) counter-part.<sup>1</sup> van der Ploeg overcomes such problem by getting rid of time-discounting, or equivalently assuming a null discount rate. While this allows to obtain stationary solutions it poses two difficulties: the assumption that no discount factor is applied to losses to social welfare which accrue in the future is not economically sound and it is therefore not typically shared by models of monetary policy, such as Svensson's. This makes comparison with the existing literature on monetary policy arduous.

We extend van der Ploeg's analysis by introducing a different risk-adjustment to the loss function of the monetary authorities. We employ the recursive optimization criterion of Hansen and Sargent (Hansen and Sargent, 1994, 1995, 2013), which combines time-discounting with Whittle's risk-adjustment to the LQG formulation. Adopting Hansen and Sargent's discounted LEQG framework (DLEQG) we achieve several goals. Firstly, we are able to introduce a risk-adjustment into the loss function of the monetary authorities without the special assumption on the aggregation of social welfare losses across periods van der Ploeg employs. Secondly, we are able to exploit, with minor adjustments, a number of results proposed Whittle for the LEQG framework. Thirdly, we characterize the optimal monetary policy via a choice mechanism similar to that described by van der Ploeg. Fourthly, since we do not do away with time-discounting, our analysis of monetary policy is directly comparable with that of others and in particular with that of Svensson.

This manuscript is organized as follows. In Section 1 we extend Svensson's (Svensson, 1997) analysis of optimal monetary policy to the case in which the central bank is endowed with recursive preferences as in Hansen and Sargent's DLEQG framework. Because of the risk-adjustment introduced to the loss function of the discounted LQG formulation employed by Svensson, the central bank selects its monetary policy via a revised version of Whittle's pessimistic choice mechanism employed by van der Ploeg. As the CEP cannot be applied, in Section 2 we establish that: the inflation forecast is no longer an explicit intermediate target when inflation targeting is the exclusive mission of the central bank; and the monetary authorities do not necessarily expect the inflation rate to mean revert to its first-best level when the monetary policy is also aimed at output stabilization. We actually see that if the central bank is pessimistic it may expect the inflation rate to wander away from the first-best level even when it does not care for output stabilization. In comparison with Svensson's analysis,

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<sup>1</sup>See Bouakiz and Sobel (1984) and Whittle (1990).

we also find that the central bank follows a more aggressive Taylor rule when it is pessimistic. This results in a smaller volatility for the inflation rate, while the volatility of the output gap and the short-term interest rate is unaffected by the risk-adjustment we have introduced. This is interesting, because it means that empirically the impact of pessimism only appears in a reduced variability for the inflation rate.

In Section 3, we investigate the possibility that the central bank observes the state variables with a time lag and employ within this context the modified version of Whittle’s risk-sensitive separation principle (SP), confirming the empirical implications derived under perfect state observation. In Section 4, we show that the normalization for the first-best level of the inflation rate employed in the previous Sections is inconsequential for the economic implications of our analysis. A final Section offers some concluding remarks, while a separate Appendix contains proofs all the results proposed in the main body of the manuscript.

## 1 A Post-Keynesian Model of Monetary Policy

In our analysis of optimal monetary policy we refer to the analytical model developed by Svensson (Svensson, 1997), which describes the optimal monetary policy of a central bank with an infinite-horizon, time-separable quadratic loss function of inflation and output gap. Svensson considers a Post-Keynesian formulation with backward looking expectations and persistence in the dynamics of output and inflation. We employ it for several reasons. Firstly, it is genuinely dynamic, in that the monetary instrument and the economic shocks influence output and inflation across several periods. Its analysis is particularly interesting as it clearly reveals the impact of pessimism on monetary policy. This is not the case for the New-Keynesian formulation with forward looking expectations, in that, as expectations of future variables are supposed given when the monetary instrument is set, the corresponding analysis is static.<sup>2</sup> Secondly, Svensson’s formulation makes full use of the CEP, which breaks down in the presence of pessimistic monetary authorities. Thirdly, his formulation fits relatively well data from the US economy.

In Svensson’s formulation, the central bank controls the short-term (real) interest rate to minimize the expected value of the loss function  $\mathcal{L}_t \equiv \sum_{i=0}^{\infty} \delta^i c_{t+i}$ , where  $c_t$  is a quadratic cost function in the inflation rate,  $\pi_t$ , and the output gap  $y_t$ ,  $c_t \equiv \pi_t^2 + \lambda y_t^2$ , with  $\lambda \geq 0$ . The cost  $c_t$  captures the loss in welfare the economy incurs at time  $t$  when inflation and output deviates

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<sup>2</sup>Van der Ploeg’s (van der Ploeg, 2009) analysis of a New-Keynesian formulation with a forward-looking Phillips curve and a dynamic IS curve bears this out.

from first-best values.<sup>3</sup> As shown by Rotemberg and Woodford (1999) and Woodford (2001),  $\mathcal{L}_t$  can be derived as a second-order Taylor approximation of a representative agent's utility function and hence it represents social welfare loss.<sup>4</sup>

The dynamics of the state variables,  $\pi_t$  and  $y_t$ , is given by the following system of equations

$$\pi_t = \pi_{t-1} + \alpha y_{t-1} + \epsilon_t^\pi, \quad (1.1)$$

$$y_t = \beta y_{t-1} - \gamma r_{t-1} + \epsilon_t^y, \quad (1.2)$$

where  $r_t$  is a short-term (real) interest rate and the coefficients  $\alpha$ ,  $\beta$  and  $\gamma$  are non-negative constants. The variation in the inflation rate is increasing in lagged output, while the latter is serially correlated and decreasing in the lagged (real) interest rate. As noted by Svensson the short-term interest rate affects output with one lag and the inflation rate with two lags, this discrepancy being an important feature of this model which is however consistent with ample empirical evidence. Since in the plant equation the innovation terms  $\epsilon_t^\pi$  and  $\epsilon_t^y$  follow independent white noise processes, Svensson investigates a standard Markovian discounted linear quadratic Gaussian (DLQG) problem.

An unpleasant aspect of the quadratic loss function considered by Svensson, as stressed *inter alia* by Clarida, Galí, and Gertler (1999), is that it does not capture the impact of uncertainty on monetary policy. In fact, as mentioned before, in the LQR framework the optimal policy is independent of the variance of the shocks to the state variables. This implies that the convexity of the loss function in the DLQG problem studied by Svensson cannot represent crucial facets of monetary policy and that Svensson's analysis cannot shed light on how a central bank reacts to uncertainty and unpredictable shocks.

It can be argued that uncertainty heavily affects monetary policy and that monetary authorities are mostly concerned with adverse shocks, such as those associated with a strong deflation, which possess a large negative impact on social welfare. Therefore, it is important to determine within Svensson's formulation the optimal monetary policy of a central bank which seeks to hedge against the most adverse economic conditions.

To achieve this goal, van der Ploeg (2009) introduces a risk-adjustment into the central bank's loss function, by assuming that in  $t$  it minimizes the function  $\ln(E_t[\exp(\frac{\rho}{2} \sum_{i=0}^{\infty} c_{t+i})])$ ,

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<sup>3</sup>The long-run natural output level is normalized to zero so that  $y_t$  corresponds to output gap.

<sup>4</sup>The cost function,  $c_t$ , should depend on the deviation of the inflation rate from a positive first-best level  $\pi^*$ . We postpone the discussion of this more involving formulation to Section 4, where we show how the normalization introduced ( $\pi^* = 0$ ) here is inconsequential.

where  $\rho$  (with  $\rho > 0$ ) is a risk-enhancement coefficient. In this way he recasts Svensson's model within Whittle's LEQG framework. Importantly, no time-discounting is applied when social welfare losses are aggregated across periods in the definition of this loss function. This is because, as shown by Bouakiz and Sobel (1984), employing the time-discounting factor in the time-separable argument of the exponential function is problematic, as it leads to non-stationary solutions, where the impact of the risk-enhancement coefficient  $\rho$  dissipates over-time and in the limit (for  $t \rightarrow \infty$ ) the optimal policy converges to that of the DLQG framework.

To ensure that the optimal monetary policy is described by a stationary rule which depends on the variance of the shocks to the inflation rate and the output gap van der Ploeg simply sets the discount factor,  $\delta$ , equal to 1. As already mentioned this assumption contradicts economic reasoning and is inconsistent with standard models of monetary policy.<sup>5</sup>

While van der Ploeg's assumption on the discount factor is problematic, it is also unnecessary. This is because Svensson's model can be recast into the DLEQG framework proposed by Hansen and Sargent (Hansen and Sargent, 1994, 1995, 2013), which allows to combine properly Whittle's risk-adjustment and time-discounting.

We now present the general framework for the class of DLEQG problems, discussing some key results which illustrate its nexus with those of the LEQG and DLQG problems. Thus, Theorem 1 revises the min-max choice mechanism derived by Whittle to determine the optimal policy for the class of LEQG problems and employed by van der Ploeg in his analysis of monetary policy, while Theorem 2 shows the recursive equations which describe the optimal policy within the DLEQG framework and their close link with those which apply to the DLQG framework. In Section 2 we apply this framework and its associated Theorems to the analysis of optimal monetary policy.

## 1.1 A Discounted Linear Exponential Quadratic Gaussian Problem

DLEQG problems are characterized by: i) a Markovian linear dynamic structure for a vector of state variables,  $\mathbf{z}_t$ ; ii) a multi-normal distribution for an innovation vector  $\epsilon_t$ ; and iii) a

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<sup>5</sup>An alternative solution van der Ploeg pursues is to assume that the shocks to inflation and output are non-stationary and possess variances which augment geometrically overtime ( $\text{Var}[\epsilon_t^x] = (1/\delta)\text{Var}[\epsilon_{t-1}^x]$  and  $\text{Var}[\epsilon_t^y] = (1/\delta)\text{Var}[\epsilon_{t-1}^y]$ , where  $(1/\delta) > 1$ ). This is an ingenious way to get rid of the impact of time-discounting and obtain stationary solutions, but it is also unsatisfactory because as time elapses the volatility of shocks explode. In addition, while van der Ploeg defends this assumption on the ground that monetary authorities will typically be more uncertain about future shocks to the economic environment, it can be argued that even if the volatility of these shocks were constant overtime the central bank would be more uncertain about the more distant values of inflation and output. It is then unsurprising that van der Ploeg's assumption is not common in the literature.

recursive optimization criterion à la Epstein and Zin. The following Definition applies:

**Definition 1** An optimal control problem is said to be Markovian linear exponential quadratic Gaussian with time-discounting if the following recursive optimization

$$\mathbf{V}_t = \min_{\mathbf{u}_t} \left\{ c_t + \frac{2}{\rho} \ln \left( E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{V}_{t+1} \right) \right] \right) \right\}, \quad (1.3)$$

where  $\rho$  (with  $\rho > 0$ ) is the risk-enhancement coefficient,  $\delta$  (with  $0 < \delta < 1$ ) is the time-discounting factor,  $c_t$  is the (per-period) scalar-valued cost function and  $\mathbf{V}_t$  is the value function (with terminal condition  $\mathbf{V}_{T+1} = 0$ ), is solved at times  $t = 1, 2, \dots, T$  with respect to the free-valued control vector  $\mathbf{u}_t$  under the conditions that:

(i) the cost function,  $c_t$ , is a quadratic form in the control vector,  $\mathbf{u}_t$ , and the state vector,  $\mathbf{z}_t$ ,

$$c_t = \mathbf{u}_t' \mathbf{Q} \mathbf{u}_t + \mathbf{z}_t' \mathbf{R} \mathbf{z}_t + 2 \mathbf{u}_t' \mathbf{S} \mathbf{z}_t,$$

(ii) the vector of state variables,  $\mathbf{z}_t$ , is governed by the following linear plant equation

$$\mathbf{z}_t = \mathbf{A} \mathbf{z}_{t-1} + \mathbf{B} \mathbf{u}_{t-1} + \boldsymbol{\epsilon}_t, \text{ where } \boldsymbol{\epsilon}_t \sim \mathcal{N}[\mathbf{0}, \mathbf{N}] \text{ and } \boldsymbol{\epsilon}_t \perp \boldsymbol{\epsilon}_{t'}.$$

Imposing the condition that the recursive optimization is solved over a finite horizon  $T$  ensures that the value function  $\mathbf{V}_t$  is well defined. However, thanks to time-discounting an infinite horizon can be accommodated. That is what we will achieve in the next Section.

The parameter  $\rho$  represents a *risk-enhancement* coefficient, in that it introduces extra convexity *vis-a-vis* that of the loss function of the Markovian DLQG problem. Indeed, the convexity of  $c_t + \frac{\rho}{2} \ln \left( E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{V}_{t+1} \right) \right] \right)$  is increasing in  $\rho$ , while  $\lim_{\rho \downarrow 0} \frac{2}{\rho} \ln \left( E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{V}_{t+1} \right) \right] \right) = \frac{1}{2} \delta E_t[\mathbf{V}_{t+1}]$ . This indicates that in the limit, for  $\rho \downarrow 0$ , the solution of the recursive optimization in Definition 1 converges to that of the standard dynamic programming recursion of a discounted Markovian optimal control problem, i.e.  $\mathbf{V}_t = \min_{\mathbf{u}_t} \{c_t + \delta E_t[\mathbf{V}_{t+1}]\}$ , such as the one which applies to Svensson's version of the DLQG framework for his monetary policy model.<sup>6</sup>

By increasing the degree of risk-aversion of the optimizing agent, the DLEQG problem is better suited to capture the impact of risk-aversion on agents' decisions than the standard DLQG framework. In particular, the optimal policy is no longer independent of the agents' uncertainty (i.e. differently from what happens in the DLQG framework, the optimal control

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<sup>6</sup>The proof of these and other results are available on request. See also Vitale (2013).

will depend on the covariance matrix of the innovation vector,  $\mathbf{N}$ ).

The optimization criterion (1.3) is not a special characterization of preferences, as it is employed in the analysis of several economic issues (Hansen and Sargent, 2005; Hansen, Sargent, and Tallarini, 1999; Luo, 2004; Luo and Young, 2010). Moreover, as shown by Tallarini (Tallarini, 2000), it corresponds to the recursive preferences of Epstein and Zin (Epstein and Zin, 1991) when the elasticity of inter-temporal substitution is 1,  $\rho$  is the coefficient of relative risk-aversion and log consumption is approximated by a quadratic form of the state and control vectors.

The DLEQG framework subsumes Whittle's LEQG framework. In fact, for  $\delta \uparrow 1$  the optimal policy for the optimization criterion (1.3) converges to that for Whittle's Markovian LEQG problem.<sup>7</sup> This is important because it entails that our formulation of the monetary policy's model encompasses that of van der Ploeg (for  $\delta \uparrow 1$ ), alongside that of Svensson (for  $\rho \downarrow 0$ ). Furthermore, the DLEQG framework allows to introduce time-discounting in a satisfactory way while preserving most of Whittle's insights and results with some minor adjustments.

Following Whittle's lead we then introduce the concept of (*discounted*) *stress*:

**Definition 2** In  $t$  the discounted stress is  $\mathcal{S}_t \equiv c_t - \frac{1}{\rho} d_{t+1} + \delta \mathcal{V}_{t+1}$ , where  $d_t$  is a per-period discrepancy function equal to  $\epsilon_t' \mathbf{N}^{-1} \epsilon_t$  for  $t = 1, 2, \dots, T$  and 0 for  $t = T + 1$ .

The (discounted) stress in  $t$ ,  $\mathcal{S}_t$ , is said to respect a *saddle-point* condition if it admits the following min-max, or *extremized*, value  $\min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathcal{S}_t$ . The (discounted) stress is useful in that we can rely on the following Theorem, which adapts a result firstly outlined by Whittle for the LEQG problems:

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<sup>7</sup>In Whittle's LEQG framework the function  $\ln(E_t[\exp(\frac{\rho}{2} \sum_{t=1}^T c_t)])$  is minimized in  $t$  with respect to the control  $\mathbf{u}_t$ . For  $\delta \uparrow 1$  in the optimization criterion (1.3)  $\mathcal{V}_t$  does not converge to Whittle's function, since the minimization argument does not contain the past cost components,  $c_h$  with  $h < t$ . However, the optimal policy for the DLEQG framework converges to that for LEQG framework in that in  $t$   $c_h$ , with  $h < t$ , is deterministic and constant with respect to  $\mathbf{u}_t$ .

**Theorem 1** - (*Risk-sensitive Certainty Equivalence Principle*). In a Markovian DLEQG problem, if the saddle point condition for the stress is respected at all future dates, i.e.  $\min_{\mathbf{u}_{t+j}} \max_{\epsilon_{t+j+1}} \mathcal{S}_{t+j}$  exists for  $j = 0, 1, \dots, T-t$ , the optimal value of the vector  $\mathbf{u}_t$  is determined at time  $t$  by maximizing  $\mathcal{S}_t$  with respect to  $\epsilon_{t+1}$  and minimizing it with respect to  $\mathbf{u}_t$ . The value function is proportional to the extremized stress,  $\mathcal{V}_t \propto \min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathcal{S}_t$ .

**Proof.** See the Appendix.

Theorem 1 is particularly useful in that it suggests that, when the stress is well-behaved so that the saddle points exist and the DLEQG problem admits a meaningful solution, to pin down the optimal policy it is sufficient to *extremize* recursively the stress. The recursion starts at time  $T$  and proceeds backward. Therefore, if the saddle point condition is met at times  $T, T-1, \dots, t+1$ , in  $t$  the optimal policy is derived by first maximizing  $\mathcal{S}_t$  with respect to the innovation vector  $\epsilon_{t+1}$  and then by minimizing the resulting expression with respect to the control vector  $\mathbf{u}_t$ .

An economic interpretation of such extremization is that a risk-averse agent whose preferences are represented by the optimization criterion (1.3) attempts to hedge against the worst possible values for the vector  $\epsilon_{t+1}$ , by following a *min-max* strategy according to which she selects  $\mathbf{u}_t$  to minimize her welfare loss (i.e. the stress) against the most unfavorable innovation vector  $\epsilon_{t+1}$ . Such an agent acts *as if* she were pessimistic, considering these *worst-case* realizations very likely. Consequently she tunes her actions on their impact on her welfare, applying what we term, borrowing Whittle's terminology, a *pessimistic* choice mechanism.

Theorem 1 revises the certainty equivalence principle (CEP) of the Markovian DLQG problem: the normally distributed unobservable variables are no longer replaced by their maximum likelihood (ML) estimates, but by those that maximize the stress in order to compensate for risk-aversion. Therefore, while it is well known that in the Markovian DLQG problem the separation principle (SP) between optimization of the control vector and estimation of the unknown values applies, in that the control vector is chosen as it would be in the perfect information case with the unobservable values replaced by their ML estimates, in the DLEQG problem the derivation of the optimal control and the optimal estimation of the unknown values are intertwined, as the optimal control and optimal estimates are chosen in order to extremize the stress. Indeed, differently from the Markovian DLQG problem, uncertainty over the innovation vector  $\epsilon_{t+1}$  conditions the optimal choice of the control vector  $\mathbf{u}_t$ . Specifically, the statistical characteristics of  $\epsilon_{t+1}$ , and hence its covariance matrix  $\mathbf{N}$ , influence the optimal

value of the vector  $\mathbf{u}_t$ . *Viceversa*, the cost function and the degree of risk-aversion affect the optimal estimate of  $\epsilon_{t+1}$ , which no longer corresponds to the ML estimate.

The following Theorem describes the optimal policy for the DLEQG problem and the nexus with the common recursive solution which applies to the Markovian DLEQG problems:

**Theorem 2** *If the matrix  $(\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}$  is positive definite, at time  $t$  the optimal control is*

$$\mathbf{u}_t = \mathbf{K}_t \mathbf{z}_t, \text{ where} \quad (1.4)$$

$$\mathbf{K}_t = -(\mathbf{Q} + \mathbf{B}'\tilde{\mathbf{\Pi}}_{t+1}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}'\tilde{\mathbf{\Pi}}_{t+1}\mathbf{A}), \quad (1.5)$$

$$\tilde{\mathbf{\Pi}}_{t+1} = ((\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N})^{-1} \text{ and} \quad (1.6)$$

$$\mathbf{\Pi}_t = \mathbf{R} + \mathbf{A}'\tilde{\mathbf{\Pi}}_{t+1}\mathbf{A} - (\mathbf{S}' + \mathbf{A}'\tilde{\mathbf{\Pi}}_{t+1}\mathbf{B})(\mathbf{Q} + \mathbf{B}'\tilde{\mathbf{\Pi}}_{t+1}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}'\tilde{\mathbf{\Pi}}_{t+1}\mathbf{A}) \quad (1.7)$$

**Proof.** See the Appendix.

It is worth noticing that (1.7) represents a modified (risk-sensitive) version of the standard Riccati equation which applies to the Markovian DLQG problem. Indeed, the matrix  $\delta\mathbf{\Pi}_{t+1}$  is now replaced by the modified matrix  $\tilde{\mathbf{\Pi}}_{t+1}$ . This shows that in the DLEQG framework the optimal policy retains a specification which is similar to the one that would prevail in the DLQG one. Indeed, as for  $\rho = 0$  we obtain the standard Riccati equation of the DLQG problem, it is confirmed that the DLEQG problem encompasses the DLQG one. For  $\rho > 0$  a straightforward correction for the impact of uncertainty and risk-aversion must however be inserted in the expressions for the recursions of  $\mathbf{\Pi}_t$  and  $\mathbf{K}_t$ , as the optimal policy depends on the agent's risk-enhancement coefficient,  $\rho$ , and her uncertainty over future shocks,  $\mathbf{N}$ .

The requirement that the matrix  $(\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}$  being positive definite derives from a second order condition which must hold in  $t$  for the stress,  $\mathcal{S}_t$ , to satisfy the saddle point condition imposed by Theorem 1. As noted by Whittle, whenever the cost function  $c_t$  is non-negative such condition fails for  $\rho$  large enough, indicating that the value function  $\mathcal{V}_t$  is infinite. This means that for a sufficiently large degree of risk-aversion the DLEQG problem is not well-behaved and does not admit an optimizing solution. An economic interpretation of the failure of the optimization problem is that in these extreme circumstances the optimizing agent becomes so *pessimistic* as to consider her control ineffective and hence useless.

Because of time-discounting it is possible to consider the limit case for  $T \uparrow \infty$ , i.e. a DLEQG problem with infinite horizon. As indicated by Hansen and Sargent (Hansen and Sargent, 2004) there is no certainty that for  $T \uparrow \infty$  the value function  $\mathcal{V}_t$  is finite and as a consequence the DLEQG problem may be not well-defined. However, when a minimum is reached we can identify a stationary solution, in that in the limit  $\mathbf{\Pi}_t \rightarrow \mathbf{\Pi}$  and  $\mathbf{K}_t \rightarrow \mathbf{K}$ , where the limit matrices are determined by the fixed point in the risk-sensitive Riccati equation,

$$\mathbf{\Pi} = \mathbf{R} + \mathbf{A}'\tilde{\mathbf{\Pi}}\mathbf{A} - (\mathbf{S}' + \mathbf{A}'\tilde{\mathbf{\Pi}}\mathbf{B})(\mathbf{Q} + \mathbf{B}'\tilde{\mathbf{\Pi}}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}'\tilde{\mathbf{\Pi}}\mathbf{A}), \quad (1.8)$$

$$\text{with } \tilde{\mathbf{\Pi}} \equiv ((\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N})^{-1}. \quad (1.9)$$

This clearly confirms our claim that the optimization criterion (1.3) accommodates time-discounting in a satisfactory manner. In fact, the steady-state identified by the fixed point in (1.8) shows that stationary solutions where the impact of the risk-enhancement coefficient does not dissipate overtime are possible. This feature indicates that Hansen and Sargent's DLEQG framework is better suited than Whittle's framework employed by van der Ploeg to analyze the impact of risk-aversion on monetary policy.

## 2 Optimal Monetary Policy

Suppose that the function  $c_t$  introduced in Section 1 genuinely represents the welfare loss brought about by deviations of inflation and output from socially optimal levels, but that supplementary curvature must be imposed to capture the monetary authorities' attitude towards risk. We can achieve this by recasting Svensson's formulation into the DLEQG framework presented in Section 1.1. In this way the exponential transformation corresponds to a risk-adjustment imposed by the monetary authorities, while the risk-enhancement coefficient  $\rho$  measures their degree of risk-aversion.

According to Theorem 1, given the recursive optimization criterion (1.3), the monetary authorities will act as though they were *pessimistic*, choosing their monetary policy in order to minimize the social welfare loss against the worst possible economic shocks.<sup>8</sup> To see how this new attitude affects the monetary policy and the dynamics of inflation and output gap, let us introduce the value function  $\mathcal{V}_t$ , the vector of state variables  $\mathbf{z}_t \equiv (\pi_t \ y_t)'$ , the vector of innovation terms  $\boldsymbol{\epsilon}_t \equiv (\epsilon_t^\pi \ \epsilon_t^y)'$  and the scalar control variable  $u_t \equiv r_t$ . Given the dynamics of

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<sup>8</sup>Because of this result we henceforth use the terms pessimistic and risk-averse interchangeably.

inflation and output in equations (1.1) and (1.2), and the determinants of the welfare loss in Svensson's formulation,  $c_t = \pi_t^2 + \lambda y_t^2$ , we have that in our version of the DLEQG problem

$$\mathbf{A} \equiv \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, \mathbf{B} \equiv \begin{pmatrix} 0 \\ -\gamma \end{pmatrix}, \mathbf{R} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \mathbf{Q} \equiv \mathbf{S} \equiv \mathbf{0}, \mathbf{N} \equiv \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}.$$

As the optimization horizon of the monetary authorities is infinite we concentrate on the steady-state solution by solving the fixed point for the modified Riccati equation (1.8). Importantly, to identify such steady-state solution we do not need to assume that the shocks to output and inflation are non-stationary. This means that our steady-state solution is directly comparable with that of Svensson, as our formulation is a straightforward extension of his model. Applying Theorem 2 we can establish the following result, which posits that a unique optimal policy exists in steady state:

**Proposition 1** *With a pessimistic central bank the optimal monetary policy is*

$$r_t = \kappa_\pi \pi_t + \kappa_y y_t, \quad (2.1)$$

where

$$\kappa_\pi = \frac{1}{\gamma} \frac{\alpha \delta W}{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2}, \quad \kappa_y = \frac{1}{\gamma} \left( \beta + \frac{\alpha^2 \delta W}{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2} \right) y_t,$$

with  $\theta = \delta(\lambda + \delta(\alpha^2 + \lambda)W)$  and  $W$  the positive root of the quadratic equation

$$\left( 1 - \left( 1 + \frac{\lambda}{\alpha^2} \right) \sigma_\pi^2 \rho \right) W^2 - \left[ \left( 1 - \frac{(1-\delta)\lambda}{\alpha^2 \delta} \right) + \frac{\lambda}{\alpha^2} \sigma_\pi^2 \rho \right] W - \frac{\lambda}{\alpha^2 \delta} = 0. \quad (2.2)$$

**Proof.** See the Appendix.

Unsurprisingly, the Taylor rule in Proposition 1 subsumes that derived by Svensson for  $\rho = 0$ . In fact, in his formulation the corresponding Taylor rule's coefficients are

$$\kappa_\pi = \frac{1}{\gamma} \frac{\alpha \delta W}{\alpha^2 \delta W + \lambda} \quad \text{and} \quad \kappa_y = \frac{1}{\gamma} \left( \beta + \frac{\alpha^2 \delta W}{\alpha^2 \delta W + \lambda} \right) y_t,$$

where

$$W = \frac{1}{2} \left( 1 - \frac{(1-\delta)\lambda}{\alpha^2 \delta} + \sqrt{\left( 1 + \frac{(1-\delta)\lambda}{\alpha^2 \delta} \right)^2 + \frac{4\lambda}{\alpha^2}} \right),$$

which corresponds to the positive root of equation (2.2) for  $\rho = 0$ .

It is interesting to emphasize that similarities with Svensson's solution appear. In particular, denoting with  $\pi_{t+1|t}$  time  $t$  expectation of inflation rate in  $t + 1$ , we have that  $\pi_{t+1|t} = \pi_t + \alpha y_t$ . It is immediate to verify that

$$r_t = \frac{1}{\gamma} \left( \beta y_t + \frac{\alpha \delta W}{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2} \pi_{t+1|t} \right)$$

and that

$$\mathcal{V}_t = \nu + \pi_t^2 + \lambda y_t^2 + \delta W \pi_{t+1|t}^2,$$

so that the control path and the value function can be defined in terms of the inflation forecast.<sup>9</sup> In addition, denoting with  $\pi_{t+2|t}$  time  $t$  expectation of inflation rate in  $t + 2$ , we find that at the optimum

$$\pi_{t+2|t} = -\frac{1}{\alpha \delta W} (\lambda - \theta \rho \sigma_\pi^2) y_{t+1|t},$$

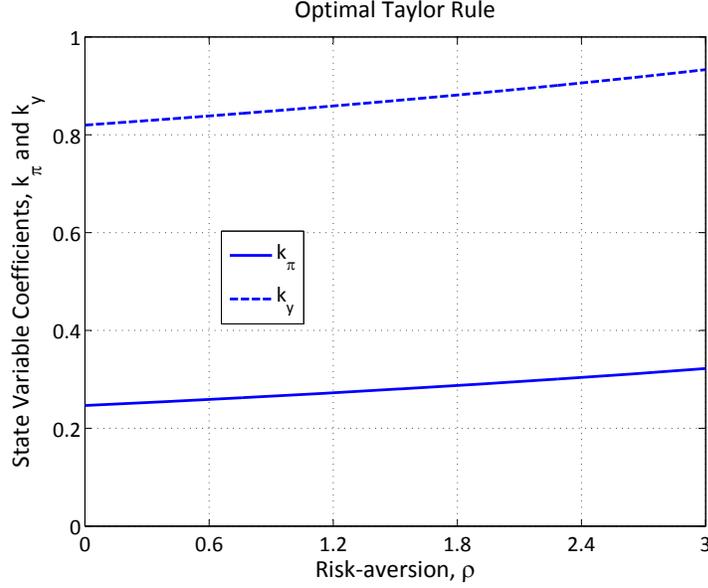
where  $y_{t+1|t}$  denotes time  $t$  expectation of output gap in  $t + 1$ . This condition implies that the two-period ahead inflation forecast is equal to its first-best level ( $\pi_{t+2|t} = 0$ ) insofar the one-period ahead expected output gap is null.

However, significant differences also emerge between Svensson's analysis and ours. When  $\lambda = 0$ , and hence only inflation targeting motivates the monetary authorities, time  $t$  expectation of the inflation rate  $\pi_{t+2}$  is always null for  $\rho = 0$ . In other words, in Svensson's formulation, for  $\lambda = 0$  the inflation forecast becomes an explicit intermediate target, in that exploiting the CEP one can see that the monetary policy is optimal insofar  $\pi_{t+2|t} = 0$ . On the other hand, for  $\rho > 0$ , that is in the DLEQG formulation we consider, when  $\lambda = 0$  at the optimum  $\pi_{t+2|t} = \alpha \delta \rho \sigma_\pi^2 y_{t+1|t} \neq 0$  (since  $\theta = \alpha^2 \delta^2 W$  for  $\lambda = 0$ ). This is because within our formulation the CEP cannot be applied and consequently the inflation forecast is not longer an explicit intermediate target when inflation targeting is the only mission of the central bank.

In addition, even when the monetary policy is also aimed at output stabilization ( $\lambda > 0$ ) with a risk-neutral central bank the inflation forecasts dampen out, in that for  $\rho = 0$   $\pi_{t+2|t} = \frac{\lambda}{\alpha^2 \delta W + \lambda} \pi_{t+1|t}$ . This indicates that within Svensson's formulation with output stabilization, as the inflation forecasts slowly converge to zero, the central bank expects the inflation rate to reach the first-best level in the long-run. This does not necessarily hold with a pessimistic

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<sup>9</sup>Details of this and other derivations are available on request.



**Figure 1:** The values of the state variable coefficients  $k_\pi$  and  $k_y$  for  $\alpha = 1.5$ ,  $\beta = 0.9$ ,  $\delta = 0.95$ ,  $\gamma = 2$ ,  $\lambda = 1$  and  $\sigma_\pi^2 = \sigma_y^2 = 0.05$  are plotted against the coefficient  $\rho$ .

central bank, as for  $\rho > 0$   $\pi_{t+2|t} = \left(\frac{\lambda - \theta \rho \sigma_\pi^2}{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2}\right) \pi_{t+1|t}$ . Strikingly, the central bank may actually expect the inflation rate to wander away from the first-best level even if  $\lambda$  is small or null, that is even when output stabilization is not a goal of its monetary policy. In fact, for  $\lambda = 0$  and  $\rho > 0$   $\pi_{t+2|t} = \frac{-\delta \rho \sigma_\pi^2}{1 - \delta \rho \sigma_\pi^2} \pi_{t+1|t}$  and hence for  $1/2 < \delta \rho \sigma_\pi^2 < 1$  we see that  $\text{abs}(\pi_{t+2|t}) > \text{abs}(\pi_{t+1|t})$ . This implies that even for  $\lambda = 0$ , a situation in which a risk-neutral central bank would employ  $\pi_{t+2|t}$  as an intermediate target and set its value equal to the optimal level zero, a pessimistic central bank may expect the inflation forecast to wander away from zero.

Finally, the curvature of the optimization criterion (1.3) conditions heavily the Taylor rule selected by the monetary authorities. Figure 1 plots the inflation,  $k_\pi$ , and output gap,  $k_y$ , coefficients in the optimal Taylor rule described in equation (2.1) against the risk-enhancement coefficient,  $\rho$ . Figure 1 shows that the larger  $\rho$ , the more aggressive is the Taylor rule followed by the central bank, in that the short-term (real) interest rate is more sensitive to departures: i) of the inflation rate from its optimal level ( $k_\pi$  is larger); and ii) of output from full employment ( $k_y$  is larger). While Figure 1 is obtained for a specific choice of parameters, the same conclusion is reached for all parametric constellations for which an optimal monetary policy exists. This result is established in the following Proposition:

**Proposition 2** For  $\sigma_\pi^2$  small enough, the coefficients on inflation,  $\kappa_\pi$ , and output gap,  $\kappa_y$ , in the optimal Taylor rule are increasing in the risk-enhancement coefficient,  $\rho$ .

**Proof.** See the Appendix.

This result may appear counter-intuitive, in that one may conjecture that a pessimistic agent will necessarily act more cautiously, selecting a more conservative policy rule (ie. smaller values for the Taylor's coefficients  $\kappa_\pi$  and  $\kappa_y$ ). However, a pessimistic central bank cares for the uncertainty over the inflation rate and the output gap and attempts to reduce it by reacting more aggressively to monetary and real shocks. With respect to Svensson's formulation, the risk-adjustment introduced in the optimization criterion (1.3) favors early resolution of uncertainty and leads to optimal control rules which reduce the volatility of state variables and agents' uncertainty.<sup>10</sup> This is a facet of risk-aversion which is shared by other multi-period models.<sup>11</sup> In addition, as our formulation subsumes that of van der Ploeg for  $\delta \uparrow 1$  it is unsurprising that an analogous conclusion is drawn in his analysis of monetary policy within his formulation.

Despite the monetary authorities select a more aggressive Taylor rule for  $\rho > 0$ , it may be difficult to detect empirically the impact of risk-aversion on the optimal monetary policy. In fact, if the monetary policy is analyzed on the basis of the moments of the short-term interest rate, that of a pessimistic central bank ( $\rho > 0$ ) is observationally equivalent to that of a risk-neutral central bank ( $\rho = 0$ ). To see this result consider that in steady state for  $\mathbf{K} \equiv (\kappa_\pi \ \kappa_y)$   $\mathbf{z}_t = \mathbf{\Gamma}\mathbf{z}_{t-1} + \epsilon_t$ , where  $\mathbf{\Gamma} = \mathbf{A} + \mathbf{BK}$ , so that  $\mathbf{z}_t = (\mathbf{I}_2 - \mathbf{\Gamma}L)^{-1}\epsilon_t$ . Then, we find that  $\text{Var}[\mathbf{z}_t] = \mathbf{\Lambda}\mathbf{N}\mathbf{\Lambda}'$ , where  $\mathbf{\Lambda} = (\mathbf{I}_2 - \mathbf{\Gamma})^{-1}$ . In addition, since  $r_t = \mathbf{K}\mathbf{z}_t$  we see that  $\text{Var}[r_t] = \mathbf{K}\mathbf{\Lambda}\mathbf{N}\mathbf{\Lambda}'\mathbf{K}'$ . Some tedious but straightforward algebra then shows that

$$\text{Var}[\mathbf{z}_t] = \begin{pmatrix} \frac{(1+\alpha\gamma\kappa_\pi)^2}{\alpha^2\gamma^2\kappa_\pi^2}\sigma_\pi^2 + \frac{1}{\gamma^2\kappa_\pi^2}\sigma_y^2 & -\frac{\gamma(1+\alpha\gamma\kappa_\pi)\kappa_\pi}{\alpha^2\gamma^2\kappa_\pi^2}\sigma_\pi^2 \\ -\frac{\gamma(1+\alpha\gamma\kappa_\pi)\kappa_\pi}{\alpha^2\gamma^2\kappa_\pi^2}\sigma_\pi^2 & \frac{1}{\alpha^2}\sigma_\pi^2 \end{pmatrix} \quad (2.3)$$

<sup>10</sup>As shown by Tallarini (Tallarini, 2000), in the recursive optimization (1.3) the elasticity of inter-temporal substitution is one. For  $\rho > 0$  the objective function of the recursive optimization (1.3) presents a coefficient of relative risk-aversion that is larger than one and hence it is greater than the inverse of the inter-temporal elasticity of substitution, the condition under which, according to Kreps and Porteus (Kreps and Porteus, 1978), the monetary authorities will prefer early resolution of uncertainty. See also Epstein and Zin (1991).

<sup>11</sup>For instance, in Holden and Subrahmanyam (1994) and Vitale (2012) risk-aversion induces agents to act more aggressively to reduce volatility and uncertainty.

and that

$$\text{Var}[r_t] = \frac{1}{\gamma^2} \left[ \left( \frac{1-\beta}{\alpha} \right)^2 \sigma_\pi^2 + \sigma_y^2 \right]. \quad (2.4)$$

Considering that  $\kappa_\pi$  is increasing in  $\rho$  and that  $\frac{(1+\alpha\gamma\kappa_\pi)^2}{\alpha^2\gamma^2\kappa_\pi^2}$  and  $\frac{1}{\gamma^2\kappa_\pi^2}$  are both decreasing in  $\kappa_\pi$ , the following result is established:

**Proposition 3** *The unconditional variances of the output gap,  $\text{Var}[y_t]$ , and the short-term (real) interest rate,  $\text{Var}[r_t]$ , are unaffected by the risk-enhancement coefficient,  $\rho$ , and will coincide with the values which prevail under risk-neutrality. The unconditional variance of the inflation rate,  $\text{Var}[\pi_t]$ , is decreasing in  $\rho$ .*

Proposition 3 indicates that the unconditional variance of the short-term (real) interest rate is independent of  $\rho$  and that the monetary policy of a pessimistic central bank shows the same level of volatility which occurs within Svensson's formulation. The unconditional variances for the inflation rate,  $\pi_t$ , and the output gap,  $y_t$ , explains how this is possible. The variance of the latter,  $\text{Var}[y_t]$ , is also independent of the risk-enhancement coefficient, while that of the former,  $\text{Var}[\pi_t]$ , is decreasing in  $\rho$ . One can see that for  $\rho > 0$  the reduced variability of the inflation rate exactly compensates the augmented aggressiveness of the monetary authorities, so that, even if  $\kappa_\pi$  and  $\kappa_y$  are larger than for  $\rho = 0$ , the unconditional variance of the short-term (real) interest rate remains the same.

The values of the unconditional variances  $\text{Var}[\pi_t]$  and  $\text{Var}[y_t]$  indicate that empirically Svensson's formulation and ours only differ in the variability of the inflation rate. While the unconditional variance of the output gap is unaffected by risk-aversion, that of the inflation rate is smaller for  $\rho > 0$ , suggesting that a pessimistic central bank will appear to be particularly concerned with the inflation rate volatility. This is because, given the specific lag structure in the law of motion for the state variables,  $\mathbf{z}_t$ , the variability of the inflation rate represents the key factor in determining the loss of social welfare and it is therefore the main driver of the pessimistic central bank's monetary policy.

Finally, we should recall that for  $\rho$  large enough the condition, reported in Theorem 2, that  $(\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N}$  being positive definite is violated, indicating that no optimal monetary policy exists for an extremely risk-averse central bank. In other words, an important non-linearity emerges in the relation between the central bank's pessimism and monetary policy: as  $\rho$  augments the monetary authorities become more aggressive, but eventually their attempt

to minimize the loss of social welfare completely fails and no optimal monetary policy can be undertaken.

In Section 1 we explained why we prefer to analyze the monetary policy of a pessimistic central bank relying on a Post-Keynesian formulation rather than a New-Keynesian one. However, it is interesting to discuss briefly what changes would be required to consider a formulation with forward looking expectations. In this case the past inflation rate,  $\pi_{t-1}$ , in the supply schedule (1.1) is replaced by the expectation in  $t$  of the next period inflation rate,  $\pi_{t+1|t}$ .

Since the dynamic system in any period is described by the inflation rate and the output gap, it is safe to conjecture that the expected inflation is a linear function of these two state variables. Hence, applying the method of undetermined coefficients, firstly a fixed point should be found in equation (1.8) to determine an optimal policy for a given conjecture on the inflation expectations; secondly, the rationality condition on the inflation expectations should be imposed. This leads to two different fixed points that should hold to determine the optimal policy and a set of consistent conjectures on the inflation forecasts.

Analyzing this alternative formulation should give qualitatively similar results to those derived within the Post-Keynesian formulation we employ. Indeed, results derived by van der Ploeg within the LEQG framework indicate that the optimal monetary present similar characteristics in the Post-Keynesian and New-Keynesian formulations, so that conclusions drawn from the analysis of the former should carry forward in that of the latter.

We now wonder what happens when the monetary authorities observe only noisy signals of the output gap and the inflation rate or observe these state variables with a time lag. This is important in that in the DLEQG framework, the SP between estimation and control does not hold and hence it is not possible, as in the standard LQR framework, to replace unknown values with the ML estimates. In addition, it would be interesting to see whether pessimistic monetary authorities would be led to act more or less aggressively when observing imperfectly the economic environment. To investigate this scenario we need to discuss the properties of the class of DLEQG problems when state variables are imperfectly observed.

### 3 DLEQG Problems Under Imperfect State Observation

To allow for imperfect state observation we introduce the following modified Definition for the class of Markovian DLEQG problems, where the optimization criterion (1.3) must be amended in that the cost function  $c_t$  is no longer deterministic:<sup>12</sup>

**Definition 3** *An optimal control problem is said to be Markovian linear exponential quadratic Gaussian with time-discounting and imperfect state observation if the following recursive optimization*

$$E_t \left[ \exp \left( \frac{\rho}{2} \mathbf{V}_t \right) \right] = \min_{\mathbf{u}_t} E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) \right) \right] \quad (3.1)$$

is solved at times  $t = 1, 2, \dots, T$  with respect to the free-valued control vector  $\mathbf{u}_t$  under the conditions that:

- (i) for  $t = 1, 2, \dots, T$ , the cost function,  $c_t$ , is a positive-definite quadratic form in the control vector,  $\mathbf{u}_t$ , and the state vector,  $\mathbf{z}_t$ ,  $c_t = \mathbf{u}_t' \mathbf{Q} \mathbf{u}_t + \mathbf{z}_t' \mathbf{R} \mathbf{z}_t + 2\mathbf{u}_t' \mathbf{S} \mathbf{z}_t$ ;
- (ii) the vector of state variables,  $\mathbf{z}_t$ , follows a linear plant equation  $\mathbf{z}_t = \mathbf{A} \mathbf{z}_{t-1} + \mathbf{B} \mathbf{u}_{t-1} + \boldsymbol{\epsilon}_t$ ;
- (iii) in  $t$  the vector of observable variables is given by

$$\mathbf{w}_t = \mathbf{C} \mathbf{z}_{t-1} + \boldsymbol{\eta}_t,$$

with  $\boldsymbol{\psi}_t \equiv \begin{pmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{\eta}_t \end{pmatrix} \sim \mathcal{N} \left[ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{N} & \mathbf{L} \\ \mathbf{L}' & \mathbf{M} \end{pmatrix} \right]$  and  $\boldsymbol{\psi}_t \perp \boldsymbol{\psi}_{t'}$ .

As  $\mathbf{z}_t$  is now unobservable the (discounted) stress takes a new formulation. In particular, let  $\hat{\mathbf{z}}_{t-1}$  denote the expectation of the state vector  $\mathbf{z}_{t-1}$  conditional on the information contained in observation history, with  $\boldsymbol{\Omega}_{t-1}$  the corresponding conditional covariance matrix and

$$\mathbf{P} = \begin{pmatrix} \mathbf{N} & \mathbf{L}' \\ \mathbf{L} & \mathbf{M} \end{pmatrix}.$$

**Definition 4** *Under imperfect state observation, in  $t$  the discounted stress is  $\mathcal{S}_t \equiv c_t - \frac{1}{\rho} (\mathcal{D}_{t-1} + d_t + d_{t+1}) + \delta \mathbf{V}_{t+1}$ , where  $d_t$  is equal to  $\boldsymbol{\psi}_t' \mathbf{P}^{-1} \boldsymbol{\psi}_t$  for  $t = 1, 2, \dots, T$  and 0 for  $t = T + 1$ , while  $\mathcal{D}_{t-1} = (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \boldsymbol{\Omega}_{t-1}^{-1} (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})$ .*

<sup>12</sup>It can be shown that under perfect state observation the optimization criterion (3.1) is equivalent to the one employed in Definition 1.

Let  $\xi_t \equiv (\mathbf{z}'_{t-1} - \hat{\mathbf{z}}'_{t-1} \psi'_t \psi'_{t+1})$ . A revised version of Theorem 1 holds:

**Theorem 3** - (*Risk-sensitive Certainty Equivalence Principle*). In a Markovian DLEQG problem, under imperfect state observation, if the discounted stress,  $\mathcal{S}_{t+j}$ , respects the saddle point condition in all future dates, so that  $\min_{\mathbf{u}_{t+j}} \max_{\xi_{t+j}} \mathcal{S}_{t+j}$  exists for  $j = 0, 1, \dots, T-t$ , the optimal value of the vector  $\mathbf{u}_t$  is determined at time  $t$  by maximizing  $\mathcal{S}_t$  with respect to  $\xi_t$  and minimizing it with respect to  $\mathbf{u}_t$ . The value function is proportional to the extremized stress,  $\mathcal{V}_t \propto \min_{\mathbf{u}_t} \max_{\xi_t} \mathcal{S}_t$ .

**Proof.** See the Appendix.

The saddle point condition  $\min_{\mathbf{u}_t} \max_{\xi_t} \mathcal{S}_t$  can be satisfied proceeding in two stages: in stage i), conditionally on the current  $\mathbf{z}_t$ , the stress,  $\mathcal{S}_t$ , is extremized with respect to the other elements of the vector  $\xi_t$  and  $\mathbf{u}_t$ ; in stage ii) the resulting function is extremized with respect to  $\mathbf{z}_t$ . This allows a partial separation between estimation and control as illustrated by the following Theorem:

**Theorem 4** - (*Risk-sensitive Separation Principle*). Under imperfect state observation, conditionally on  $\mathbf{z}_t$  the stress in  $t$  is extremized for the optimal control,  $\mathbf{u}_t(\mathbf{z}_t) = \mathbf{K}_t \mathbf{z}_t$ , in Theorem 2. (*Risk-sensitive Certainty Equivalence Principle*). Under imperfect state observation, the optimal policy in  $t$  is recouped by replacing  $\mathbf{z}_t$  in  $\mathbf{u}_t(\mathbf{z}_t)$  with the maximum stress estimate (MSE),

$$\check{\mathbf{z}}_t = (\mathbf{I} - \rho \mathbf{\Omega}_t \mathbf{\Pi}_t)^{-1} \hat{\mathbf{z}}_t, \quad (3.2)$$

where  $\mathbf{\Pi}_t$  is the modified Riccati matrix defined in Theorem 2.

**Proof.** See the Appendix.

Interestingly, as the matrix  $\mathbf{\Pi}_t$  depends on the components of  $c_t$ , the MSE  $\check{\mathbf{z}}_t$  is affected by the shape of the cost function alongside the risk-enhancement coefficient  $\rho$  confirming the close nexus between control and estimation for the class of DLEQG problems.

### 3.1 Optimal Monetary Policy with Imperfect Observation of Inflation and Output

Reliable data on inflation and output are typically available with some delay. Consequently, within our analysis of monetary policy it is interesting to see what happens in the case the

central bank observes the output gap and the inflation rate with one time lag.<sup>13</sup>

Exploiting Theorem 4, it is possible to prove the following Proposition which identifies the unique optimal policy in steady state:

**Proposition 4** *When output and inflation are observed with one time lag, the optimal policy of a pessimistic central bank is*

$$r_t = \kappa_\pi \check{\pi}_t + \kappa_y \check{y}_t, \quad (3.3)$$

where the Taylor rule's coefficients,  $\kappa_\pi$  and  $\kappa_y$ , are as in Proposition 1, while the maximum stress estimates for the inflation rate and the output gap are

$$\begin{pmatrix} \check{\pi}_t \\ \check{y}_t \end{pmatrix} = \begin{pmatrix} \hat{\pi}_t \\ \hat{y}_t \end{pmatrix} + \rho \mathbf{G} \begin{pmatrix} \hat{\pi}_t \\ \hat{y}_t \end{pmatrix}, \quad \text{with } \mathbf{G} = \begin{pmatrix} \frac{\pi_1 - \det(\mathbf{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \sigma_\pi^2 & \frac{\pi_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \sigma_\pi^2 \\ \frac{\pi_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \sigma_y^2 & \frac{\pi_2 - \det(\mathbf{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \sigma_y^2 \end{pmatrix},$$

$\pi_1$ ,  $\pi_{1,2}$  and  $\pi_2$  the elements of modified Riccati matrix  $\mathbf{\Pi}$  in Proposition 1, and  $\hat{\pi}_t$  and  $\hat{y}_t$  the maximum likelihood estimates of  $\pi_t$  and  $y_t$  in  $t$ ,  $\hat{\pi}_t = \pi_{t-1} + \alpha y_{t-1}$  and  $\hat{y}_t = \beta y_{t-1} - \gamma r_{t-1}$ .

**Proof** See the Appendix.

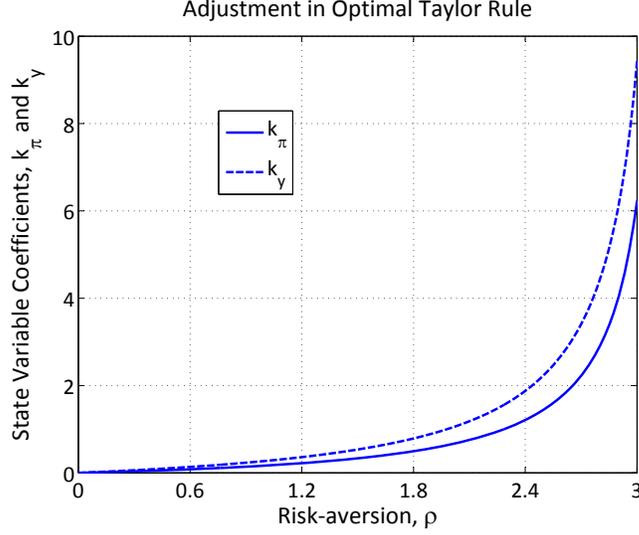
Proposition 4 implies that the optimal Taylor rule is given by a modified expression,

$$r_t = \kappa_\pi^I \hat{\pi}_t + \kappa_y^I \hat{y}_t, \quad (3.4)$$

where  $\mathbf{K}^I \equiv (\kappa_\pi^I \ \kappa_y^I) = \mathbf{K}(\mathbf{I} + \rho\mathbf{G})$ . The vector  $\rho\mathbf{K}\mathbf{G}$  contains adjustments to the Taylor rule's coefficients induced by the correction for risk-aversion to the ML estimate of  $\mathbf{z}_t$ . Imperfect state observation may entail a more (or less) aggressive Taylor rule, in so far the adjusted coefficients for inflation and output gap,  $\kappa_\pi^I$  and  $\kappa_y^I$ , are larger (smaller) than those which prevail under perfect state observation,  $\kappa_\pi$  and  $\kappa_y$ .

In Figure 2 we plot the differences between the adjusted coefficients,  $\kappa_\pi^I$  and  $\kappa_y^I$ , and the unadjusted ones,  $\kappa_\pi$  and  $\kappa_y$ , against  $\rho$ , using the same choice of parameters as in Figure 1. Once again, this plot proposes an apparently counter-intuitive result. In fact, we see that, as the difference is positive for both coefficients, the monetary authorities become even more aggressive when they observe with a time lag inflation and output. That is, when facing a more

<sup>13</sup>van der Ploeg considers instead the case in which inflation is perfectly observed, while only a noisy signal on the output gap is observed by the monetary authorities. While this means that these two scenarios are not directly comparable.

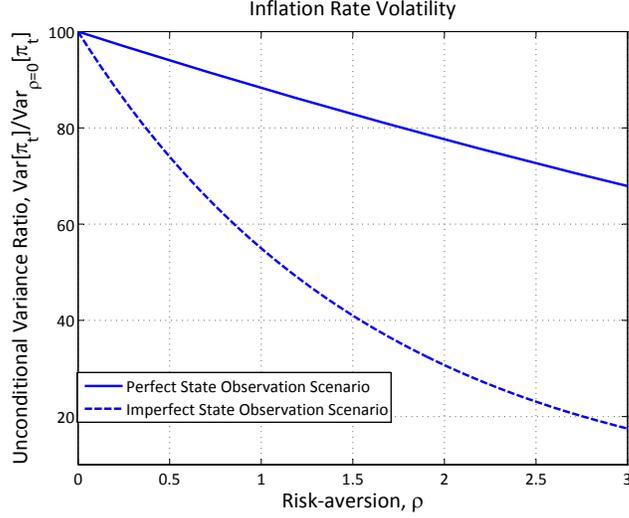


**Figure 2:** The adjustments to the Taylor rule coefficients  $k_\pi$  and  $k_y$  (ie. the differences  $\kappa_\pi^I - \kappa_\pi$  and  $\kappa_y^I - \kappa_y$ ) are plotted against  $\rho$  for  $\alpha = 1.5$ ,  $\beta = 0.9$ ,  $\delta = 0.95$ ,  $\gamma = 2$ ,  $\lambda = 1$  and  $\sigma_\pi^2 = \sigma_y^2 = 0.05$ .

uncertain environment the activism of the monetary authorities increases. As the adjustment to the Taylor rule's coefficients increases with  $\rho$  we also observe that such activism augments with the central bank's degree of risk-aversion.

While we do not have a result equivalent to Proposition 2, our numerical analysis shows that similar conclusions are drawn by other parametric choices. However, the increased activism manifest in Figure 2 is difficult to detect as the analysis of the unconditional variance of the inflation rate,  $\pi_t$ , the output gap,  $y_t$ , and the short-term (real) interest rate,  $r_t$ , reveals.

In fact, even under imperfect state observation the unconditional variance of the short-term interest rate is independent of the coefficient  $\rho$ , confirming that empirically it may be hard to appreciate the impact of pessimism on the monetary policy. To show this result consider that under imperfect state observation  $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \Psi\hat{\mathbf{z}}_{t-1} + \epsilon_t$ , where  $\Psi = \mathbf{B}\mathbf{K}_I$  and, as the state vector is observed with a lag,  $\hat{\mathbf{z}}_t = \mathbf{A}\mathbf{z}_{t-1} + \Psi\hat{\mathbf{z}}_{t-1}$ . This implies that  $\hat{\mathbf{z}}_t = \Phi\mathbf{z}_{t-1}$ , where  $\Phi = (\mathbf{I}_2 - \Psi)^{-1}\mathbf{A}$ . Replacing this expression in that for  $\mathbf{z}_t$  we find that  $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \Psi\Phi\mathbf{z}_{t-2} + \epsilon_t$ , which we can also write as  $\mathbf{z}_t = (\mathbf{I}_2 - \mathbf{A}\mathbf{L} - \Psi\Phi\mathbf{L}^2)^{-1}\epsilon_t$ . It follows that  $\text{Var}[\mathbf{z}_t] = \Lambda_I\mathbf{N}\Lambda_I'$ , where  $\Lambda_I = (\mathbf{I}_2 - \mathbf{A} - \Psi\Phi)^{-1}$ , while  $\text{Var}[\hat{\mathbf{z}}_t] = \Phi\Lambda_I\mathbf{N}\Lambda_I'\Phi'$ . Finally, since under imperfect state observation  $u_t = \mathbf{K}_I\hat{\mathbf{z}}_t$ , we have that  $\text{Var}[r_t] = \mathbf{K}_I\Phi\Lambda_I\mathbf{N}\Lambda_I'\Phi'\mathbf{K}_I'$ .



**Figure 3:** The ratio (in percentage terms) between the unconditional variance of the inflation rate  $\text{Var}[\pi_t]$  and its base value for  $\rho = 0$  is plotted against  $\rho$ , under perfect and imperfect state observation, for  $\alpha = 1.5$ ,  $\beta = 0.9$ ,  $\delta = 0.95$ ,  $\gamma = 2$ ,  $\lambda = 1$  and  $\sigma_\pi^2 = \sigma_y^2 = 0.05$ .

Once again some long but straightforward algebra shows that

$$\text{Var}^I[\mathbf{z}_t] = \begin{pmatrix} \left(1 + \frac{1 + \gamma\kappa_y^I - \beta}{\alpha\gamma\kappa_\pi^I}\right)^2 \sigma_\pi^2 + \left(\frac{1 - \gamma\kappa_y^I}{\gamma\kappa_\pi^I}\right)^2 \sigma_y^2 & -\frac{1}{\alpha} \left(1 + \frac{1 + \gamma\kappa_y^I - \beta}{\alpha\gamma\kappa_\pi^I}\right) \sigma_\pi^2 \\ -\frac{1}{\alpha} \left(1 + \frac{1 + \gamma\kappa_y^I - \beta}{\alpha\gamma\kappa_\pi^I}\right) \sigma_\pi^2 & \frac{1}{\alpha^2} \sigma_\pi^2 \end{pmatrix} \quad (3.5)$$

and that

$$\text{Var}^I[r_t] = \frac{1}{\gamma^2} \left[ \left(\frac{1 - \beta}{\alpha}\right)^2 \sigma_\pi^2 + \sigma_y^2 \right]. \quad (3.6)$$

This proves a result analogous to Proposition 3:

**Proposition 5** *With lagged observation of inflation and output, the unconditional variances of the output gap,  $\text{Var}[y_t]$ , and the short-term (real) interest rate,  $\text{Var}[r_t]$ , are unaffected by the risk-enhancement coefficient,  $\rho$ , and coincide with the values which prevail under risk-neutrality. The unconditional variance of the inflation rate,  $\text{Var}[\pi_t]$ , is instead influenced by the risk-enhancement coefficient in so far this affects the central bank's optimal Taylor rule.*

Proposition 5 indicates that the unconditional variances of the short-term interest rate and the output gap are equal to those which prevail under perfect state observation for  $\rho = 0$

( $\text{Var}^I[r_t] = \text{Var}[r_t]$  and  $\text{Var}^I[y_t] = \text{Var}[y_t]$ ), while the unconditional variance of the inflation rate is function of the Taylor rule's coefficients  $\kappa_{\pi}^I$  and  $\kappa_y^I$ . As these coefficients depend on the central bank's risk-aversion,  $\text{Var}^I[\pi_t]$  varies with the risk-enhancement coefficient and indeed numerical analysis shows that such value decreases with  $\rho$ , explaining why it is possible that for a larger  $\rho$  (and hence with a more aggressive Taylor-rule) the variability of the short-term interest remains unchanged.

Figure 3 plots in both the perfect state and imperfect state scenarios the ratio (in percentage terms) between the inflation rate's unconditional variance,  $\text{Var}[\pi_t]$ , and its base value for  $\rho = 0$  (respectively  $\text{Var}_{\rho=0}[\pi_t]$  and  $\text{Var}_{\rho=0}^I[\pi_t]$ ) against the risk-enhancement coefficient.<sup>14</sup> The plot clearly illustrates the reduction in the volatility of the inflation rate in the presence of a pessimistic central bank in both scenarios. As the volatility of the inflation rate is smaller with pessimistic monetary authorities, and decreasing in  $\rho$ , a more aggressive Taylor rule will not result in a more volatile short-term interest rate. We therefore conclude that in both scenarios the impact of pessimism on the central bank's optimal monetary policy only manifests via a reduced volatility in the inflation rate, as the variability of both the output gap and short-term interest rate is unaffected by  $\rho$ .

## 4 A Positive First-best Inflation Rate

In Section 2 we have assumed that the first-best value for the inflation rate is zero. We wonder what happens when we introduce the realistic assumption that such value is some positive constant  $\pi^*$ . Assuming, as in Svensson's original formulation, that at time  $t$  the cost function is  $c_t = (\pi_t - \pi^*)^2 + \lambda y_t^2$  implies that we should modify the DLEQG problem we investigated in Section 2. In particular, define  $\varsigma_t = \pi_t - \pi^*$  and rewrite the linear equations (1.1) and (1.2) governing the dynamics of the inflation rate and the output gap as follows

$$\varsigma_t = \varsigma_{t-1} + \alpha y_{t-1} + \epsilon_t^{\pi}, \quad (4.1)$$

$$y_t = \beta y_{t-1} - \gamma(\iota_{t-1} - \pi^*) + \epsilon_t^y, \quad (4.2)$$

---

<sup>14</sup>It should be noted that the base values for this unconditional variance differ between the two scenarios. In correspondence with the parametric choice of Figure 1, the unconditional variance of the inflation rate for  $\rho = 0$  in the imperfect state scenario is about 4 times bigger than the corresponding value for the perfect state scenario,  $\text{Var}_{\rho=0}^I[\pi_t] \approx 4\text{Var}_{\rho=0}[\pi_t]$ .

where now the control variable is the adjusted short-term real interest rate  $\iota_t \equiv r_t + \pi^* = i_t - \varsigma_t$ . For  $\mathbf{z}'_t \equiv (\varsigma_t \ y_t)$  we can rewrite the plant equation as

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}u_{t-1} + \boldsymbol{\mu} + \boldsymbol{\epsilon}_t,$$

with  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\boldsymbol{\epsilon}_t$  as in Section 2 and  $\boldsymbol{\mu} \equiv \begin{pmatrix} 0 \\ \gamma\pi^* \end{pmatrix}$ .

This is a generalization of the DLEQG problem where the law of motion for the state vector is subject to predetermined disturbances. In fact, the vector  $\boldsymbol{\mu}$  contains deterministic values. The analysis of the DLEQG problem with pre-determined disturbances is involving. We deal with it in the next Section.

#### 4.1 DLEQG Problems with Pre-determined Disturbances

Let assume the state vector respects the following law of motion

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}u_{t-1} + \boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t,$$

where the vector  $\boldsymbol{\mu}_t$  contains pre-determined values. These values are known in advance and represent anticipated disturbances which modify the original plant equation introduced in Definition 1.

Under perfect state observation, with a pre-determined disturbance term  $\boldsymbol{\mu}_t$  in the law of motion for the state vector Theorem 1 holds, as the total stress in  $t$ ,  $\mathcal{S}_t$ , is still a quadratic form in  $\mathbf{u}_t$ ,  $\boldsymbol{\epsilon}_{t+1}$  and  $\mathbf{z}_t$ . However, because of the pre-determined disturbance term  $\boldsymbol{\mu}_t$  Theorem 2 must be extended as follows:

**Theorem 5** *Under perfect state observation, if the matrix  $(\delta\Pi_{t+1})^{-1} - \rho\mathbf{N}$  is positive definite and the state vector respects the linear plant equation with pre-determined disturbances, the optimal policy in  $t$  is*

$$\mathbf{u}_t = \mathbf{K}_t \mathbf{z}_t + (\mathbf{Q} + \mathbf{B}'\tilde{\Pi}_{t+1}\mathbf{B})^{-1} \mathbf{B}'\tilde{\Pi}_{t+1} (\Pi_{t+1}^{-1} \boldsymbol{\vartheta}_{t+1} - \boldsymbol{\mu}_{t+1}), \quad (4.3)$$

where  $\Pi_t$ ,  $\mathbf{K}_t$  and  $\tilde{\Pi}_{t+1}$  respect the recursive formulae presented in Theorem 2 and

$$\boldsymbol{\vartheta}_t = \Gamma_t' \tilde{\Pi}_{t+1} (\Pi_{t+1}^{-1} \boldsymbol{\vartheta}_{t+1} - \boldsymbol{\mu}_{t+1}), \quad \text{with } \Gamma_t = \mathbf{A} + \mathbf{B}\mathbf{K}_t. \quad (4.4)$$

**Proof.** See the Appendix.

Theorem 5 indicates that in the presence of pre-determined disturbances the optimal policy contains a risk-adjusted correction, the second component in the right hand side of equation (4.3), which takes into account their anticipated values.

A second adjustment must be introduced under imperfect state observation, when pre-determined disturbances enter into the plant equation for the state vector, to Theorem 4:

**Theorem 6** - (*Risk-sensitive Separation Principle*). Under imperfect state observation, conditionally on  $\mathbf{z}_t$  the stress in  $t$  is extremized for the optimal control,  $\mathbf{u}_t(\mathbf{z}_t) = \mathbf{K}_t \mathbf{z}_t + (\mathbf{Q} + \mathbf{B}' \tilde{\mathbf{\Pi}}_{t+1} \mathbf{B})^{-1} \mathbf{B}' \tilde{\mathbf{\Pi}}_{t+1} (\mathbf{\Pi}_{t+1}^{-1} \boldsymbol{\vartheta}_{t+1} - \boldsymbol{\mu}_{t+1})$ , in Theorem 5.

(*Risk-sensitive Certainty Equivalence Principle*). Under imperfect state observation, the optimal policy in  $t$  is recouped by replacing  $\mathbf{z}_t$  in  $\mathbf{u}_t(\mathbf{z}_t)$  with the maximum stress estimate (MSE),

$$\check{\mathbf{z}}_t = (\mathbf{I} - \rho \boldsymbol{\Omega}_t \mathbf{\Pi}_t)^{-1} (\hat{\mathbf{z}}_t - \rho \boldsymbol{\Omega}_t \boldsymbol{\vartheta}_t). \quad (4.5)$$

**Proof.** See the Appendix.

## 4.2 Optimal Monetary Policy with a Positive First-best Inflation Rate

Let us apply Theorem 5 to our analysis of monetary policy with a positive first-best inflation rate. We can still concentrate on a steady state solution because the pre-determined disturbance terms,  $\boldsymbol{\mu}$ , are time-invariant. To pin down the steady state solution we just notice that the recursive expression for the vector  $\boldsymbol{\vartheta}_t$  must yield a fixed point,  $\boldsymbol{\vartheta} = \mathbf{\Gamma}' \tilde{\mathbf{\Pi}} (\mathbf{\Pi}^{-1} \boldsymbol{\vartheta} - \boldsymbol{\mu})$ , which implies that

$$\boldsymbol{\vartheta} = -\mathbf{\Pi} (\mathbf{\Pi}^{-1} - \mathbf{\Gamma}' \tilde{\mathbf{\Pi}})^{-1} \mathbf{\Gamma}' \tilde{\mathbf{\Pi}} \boldsymbol{\mu}.$$

Given the expressions for  $\mathbf{\Gamma}' \tilde{\mathbf{\Pi}}$  and  $\boldsymbol{\mu}$  it can be checked that  $\boldsymbol{\vartheta} = \mathbf{0}$ . Inserting this vector in the expression for the optimal control in Theorem 5 we find after some manipulation that the optimal adjusted short-term interest rate is  $\iota_t = \kappa_\pi \varsigma_t + \kappa_y y_t + \pi^*$ , where  $\kappa_\pi$  and  $\kappa_y$  respect the expressions in Section 2. Given the definitions of  $\iota_t$  and  $\varsigma_t$  we conclude that the following Proposition holds:

**Proposition 6** With a positive first-best inflation rate,  $\pi^*$ , the optimal policy of a pessimistic central bank is

$$r_t = \kappa_\pi (\pi_t - \pi^*) + \kappa_y y_t, \quad (4.6)$$

where the coefficients  $\kappa_\pi$  and  $\kappa_y$  respect the formulae in Proposition 1.

In brief, the optimal short-term real interest rate corresponds to that obtained in Section 2 for the inflation rate,  $\pi_t$ , replaced by its deviation from the first-best level,  $\pi_t - \pi^*$ . Bar this adjustment, the optimal Taylor rule is identical to that derived for a normalized first-best inflation rate. This is also true under imperfect state observation. In fact, the following Proposition holds:

**Proposition 7** *With a positive first-best inflation rate, when output and inflation are observed with one time lag, the optimal policy of a pessimistic central bank is*

$$r_t = \kappa_{\pi}^I(\check{\pi}_t - \pi^*) + \kappa_y^I\check{y}_t, \quad (4.7)$$

where the coefficients  $\kappa_{\pi}^I$  and  $\kappa_y^I$  and the MSE for the inflation rate,  $\check{\pi}_t$ , and the output gap,  $\check{y}_t$ , respect the formulae in Proposition 4.

**Proof.** See the Appendix.

## 5 Concluding Remarks

Uncertainty plays a crucial role in determining the optimal actions of policymakers. The precautionary principle is often invoked in the conduct of monetary policy. It requires that monetary instruments should be set to counter-balance the effects of negative shocks to the economy. Surprisingly, such principle is usually ignored in the literature on optimal monetary policy. An exception is van der Ploeg (2009), who seeks to investigate the impact of the precautionary principle within standard models of optimal monetary policy introducing a risk-adjustment to the preferences of the monetary authorities.

A problem with his analysis is that for tractability issues, in aggregating across periods the loss in social welfare induced by deviations of output and inflation from first-best values, he does away with time-discounting. While special this assumption also limits the possibility to compare van der Ploeg's analysis with that conducted within traditional models of monetary policy.

We overcome this limitation by relying on a recursive representation of the monetary authorities's preferences, which allows to investigate their precautionary behavior under the assumption that a time-discounting factor is applied to future losses in social welfare. Exploiting recent advances within the optimal control literature (Whittle, 1990; Hansen and Sargent,

1994, 1995, 2013), we show that the optimal monetary policy is set applying a pessimistic choice mechanism, according to which monetary instruments are chosen in order to edge against the worst economic shocks. We also find, within Svensson's post-Keynesian monetary policy formulation, that such pessimism induces the monetary authorities to act more aggressively, contradicting the prevailing wisdom that amid an uncertain environment risk-aversion should lead policymakers to more conservative choices.

The policy implications of our analysis are relevant to the current debate on the conduct of monetary policy on the part of the ECB. In fact, given the large risks, i.e. the potential large economic costs, associated with a deflationary spiral in the euro area our analysis suggests that the ECB should not be hesitant and it should immediately embrace an aggressive expansionary monetary stance. This is because, as argued by *The Economist's* commentary cited in the preamble, any hesitance would contradict the precautionary principle which should guide monetary policy.

Importantly, we see that empirically it may be difficult to detect the impact of pessimism on monetary policy. In fact, a more aggressive Taylor rule is matched by reduced volatility in inflation, so that the volatility of the short-term monetary instrument is independent of the degree of pessimism on the part of the monetary authorities. Finally, our conclusions are valid both when the monetary authorities immediately observe inflation and output and when they do it with a time lag, a non-trivial result considering that with the recursive preferences we employ the certainty equivalence principle does not hold.

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## Appendix

### • Derivation of Theorem 1.

To prove this Theorem we first need to establish three preliminary Lemmas:

**Lemma 1** *The recursive optimization criterion (1.3) can be equivalently formulated as follows*

$$\mathbf{v}_t = \frac{2}{\rho} \ln \left( \min_{\mathbf{u}_t} \left\{ E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{v}_{t+1}) \right) \right] \right\} \right).$$

**Proof.** We can write

$$\begin{aligned} \exp \left( \frac{\rho}{2} \mathbf{v}_t \right) &= \exp \left( \min_{\mathbf{u}_t} \left\{ \frac{\rho}{2} c_t + \ln \left( E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{v}_{t+1} \right) \right] \right) \right\} \right) \\ &= \min_{\mathbf{u}_t} \left\{ \exp \left[ \frac{\rho}{2} c_t + \ln \left( E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{v}_{t+1} \right) \right] \right) \right] \right\} \\ &= \min_{\mathbf{u}_t} \left\{ \exp \left[ \ln \left( \exp \left( \frac{\rho}{2} c_t \right) \right) + \ln \left( E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{v}_{t+1} \right) \right] \right) \right] \right\} \\ &= \min_{\mathbf{u}_t} \left\{ \exp \left[ \ln \left( \exp \left( \frac{\rho}{2} c_t \right) E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{v}_{t+1} \right) \right] \right) \right] \right\} \\ &= \min_{\mathbf{u}_t} \left\{ \exp \left( \frac{\rho}{2} c_t \right) E_t \left[ \exp \left( \delta \frac{\rho}{2} \mathbf{v}_{t+1} \right) \right] \right\} \\ &= \min_{\mathbf{u}_t} \left\{ E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{v}_{t+1}) \right) \right] \right\}, \text{ so that} \\ \frac{\rho}{2} \mathbf{v}_t &= \ln \left( \min_{\mathbf{u}_t} \left\{ E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{v}_{t+1}) \right) \right] \right\} \right). \square \end{aligned}$$

**Lemma 2** *If  $Q(\mathbf{u}, \epsilon)$  is a quadratic form in the vectors  $\mathbf{u}$  and  $\epsilon$  which admits the saddle point  $\max_{\mathbf{u}} \min_{\epsilon} Q(\mathbf{u}, \epsilon)$ , then the following holds*

$$\min_{\mathbf{u}} \int \exp \left[ -\frac{1}{2} Q(\mathbf{u}, \epsilon) \right] d\epsilon \propto \exp \left[ -\frac{1}{2} \max_{\mathbf{u}} \min_{\epsilon} Q(\mathbf{u}, \epsilon) \right].$$

**Proof.** Consider the quadratic form  $Q(\mathbf{u}, \epsilon)$ , where  $Q(\mathbf{u}, \epsilon) = \mathbf{u}'\mathbf{Q}_{\mathbf{u}\mathbf{u}}\mathbf{u} + 2\mathbf{u}'\mathbf{Q}_{\mathbf{u}\epsilon}\epsilon + \epsilon'\mathbf{Q}_{\epsilon\epsilon}\epsilon$ . Assume  $Q$  admits a minimum in  $\epsilon$  in that  $\mathbf{Q}_{\epsilon\epsilon}$  is positive definite. The following holds

$$\int \exp\left[-\frac{1}{2}Q(\mathbf{u}, \epsilon)\right] d\epsilon \propto \exp\left[-\frac{1}{2}\min_{\epsilon} Q(\mathbf{u}, \epsilon)\right].$$

This is because, for  $\hat{\epsilon} = \operatorname{argmin} Q$ , we can write  $Q(\mathbf{u}, \epsilon) = Q(\mathbf{u}, \hat{\epsilon}) + (\epsilon - \hat{\epsilon})'\mathbf{Q}_{\epsilon\epsilon}(\epsilon - \hat{\epsilon})$ . In fact, as  $\mathbf{Q}_{\epsilon\epsilon}$  is positive definite and invertible, the minimum of  $Q$  with respect to  $\epsilon$  is obtained for  $\hat{\epsilon} = -\mathbf{Q}_{\epsilon\epsilon}^{-1}\mathbf{Q}_{\epsilon\mathbf{u}}\mathbf{u}$  and is equal to  $Q(\mathbf{u}, \hat{\epsilon}) = \mathbf{u}'[\mathbf{Q}_{\mathbf{u}\mathbf{u}} - \mathbf{Q}_{\mathbf{u}\epsilon}\mathbf{Q}_{\epsilon\epsilon}^{-1}\mathbf{Q}_{\epsilon\mathbf{u}}]\mathbf{u}$ . Then,

$$\begin{aligned} Q(\mathbf{u}, \epsilon) - Q(\mathbf{u}, \hat{\epsilon}) &= \epsilon'\mathbf{Q}_{\epsilon\epsilon}\epsilon + \epsilon'\mathbf{Q}_{\epsilon\mathbf{u}}\mathbf{u} + \mathbf{u}'\mathbf{Q}_{\mathbf{u}\epsilon}\epsilon + \mathbf{u}'\mathbf{Q}_{\mathbf{u}\epsilon}\mathbf{Q}_{\epsilon\epsilon}^{-1}\mathbf{Q}_{\epsilon\mathbf{u}}\mathbf{u} \\ &= \epsilon'\mathbf{Q}_{\epsilon\epsilon}\epsilon - \epsilon'\mathbf{Q}_{\epsilon\epsilon}\hat{\epsilon} - \hat{\epsilon}'\mathbf{Q}_{\epsilon\epsilon}\epsilon + \hat{\epsilon}'\mathbf{Q}_{\epsilon\epsilon}\hat{\epsilon} = (\epsilon - \hat{\epsilon})'\mathbf{Q}_{\epsilon\epsilon}(\epsilon - \hat{\epsilon}). \end{aligned}$$

As  $Q(\mathbf{u}, \hat{\epsilon}) = \min_{\epsilon} Q(\mathbf{u}, \epsilon)$  is a constant in the integration,

$$\int \exp\left[-\frac{1}{2}Q(\mathbf{u}, \epsilon)\right] d\epsilon = \exp\left[-\frac{1}{2}\min_{\epsilon} Q(\mathbf{u}, \epsilon)\right] \times \int \exp\left[-\frac{1}{2}(\epsilon - \hat{\epsilon})'\mathbf{Q}_{\epsilon\epsilon}(\epsilon - \hat{\epsilon})\right] d\epsilon.$$

Let  $\Delta$  denote  $(\epsilon - \hat{\epsilon})$ . Because  $\mathbf{Q}_{\epsilon\epsilon}$  is positive definite, for  $\Delta$  integrated over  $\mathbb{R}^n$  where  $n$  is the dimension of  $\epsilon$ , we find that  $\int \exp(-\frac{1}{2}\Delta'\mathbf{Q}_{\epsilon\epsilon}\Delta) d\Delta = (2\pi)^{n/2}\det(\mathbf{Q}_{\epsilon\epsilon})^{-1/2}$ . It follows

$$\int \exp\left[-\frac{1}{2}Q(\mathbf{u}, \epsilon)\right] d\epsilon = (2\pi)^{n/2}\det(\mathbf{Q}_{\epsilon\epsilon})^{-1/2} \times \exp\left[-\frac{1}{2}\min_{\epsilon} Q(\mathbf{u}, \epsilon)\right],$$

where the constant  $(2\pi)^{n/2}\det(\mathbf{Q}_{\epsilon\epsilon})^{-1/2}$  is independent of  $\mathbf{u}$ .

Suppose that we solve the program  $\min_{\mathbf{u}} \int \exp[-\frac{1}{2}Q(\mathbf{u}, \epsilon)]$ . Assume that  $Q$  admits a saddle point with respect to  $\epsilon$  and  $\mathbf{u}$ , so that  $\max_{\mathbf{u}} \min_{\epsilon} Q(\mathbf{u}, \epsilon)$  exists. This is the case if  $\mathbf{Q}_{\epsilon\epsilon} > 0$  and  $\mathbf{Q}_{\mathbf{u}\mathbf{u}} - \mathbf{Q}_{\mathbf{u}\epsilon}\mathbf{Q}_{\epsilon\epsilon}^{-1}\mathbf{Q}_{\epsilon\mathbf{u}} < 0$  hold. As a corollary of the former result we have

$$\min_{\mathbf{u}} \int \exp\left[-\frac{1}{2}Q(\mathbf{u}, \epsilon)\right] d\epsilon \propto \min_{\mathbf{u}} \exp\left[-\frac{1}{2}\min_{\epsilon} Q(\mathbf{u}, \epsilon)\right] = \exp\left[-\frac{1}{2}\max_{\mathbf{u}} \min_{\epsilon} Q(\mathbf{u}, \epsilon)\right]. \square$$

**Lemma 3** *In a Markovian DLEQG problem if the value function in  $t + 1$ ,  $\mathbf{V}_{t+1}$ , is a quadratic form in the state vector  $\mathbf{z}_{t+1}$  and the (discounted) stress,  $\mathbf{S}_t$ , satisfies a saddle point condition with respect to  $\epsilon_{t+1}$  and  $\mathbf{u}_t$ , so that  $\min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathbf{S}_t$  exists, the following proportionality condition holds*

$$\min_{\mathbf{u}_t} E_t \left[ \exp\left(\frac{\rho}{2}(c_t + \delta \mathbf{V}_{t+1})\right) \right] \propto \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathbf{S}_t\right),$$

where the proportionality constant is independent of the state vector  $\mathbf{z}_t$ , while the value function  $\mathbf{V}_t$  is a quadratic form in  $\mathbf{z}_t$  equal to the extremized stress,  $\min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathbf{S}_t$ , plus a constant independent of  $\mathbf{z}_t$ .

**Proof.** Consider that if  $\mathcal{V}_{t+1}$  is a quadratic form in  $\mathbf{z}_{t+1}$ , as the latter is linearly dependent on  $\epsilon_{t+1}$ ,

$$\begin{aligned} \min_{\mathbf{u}_t} E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathcal{V}_{t+1}) \right) \right] &\propto \min_{\mathbf{u}_t} \int \exp \left( \frac{\rho}{2} (c_t + \delta \mathcal{V}_{t+1}) - \frac{1}{2} \epsilon'_{t+1} \mathbf{N}^{-1} \epsilon_{t+1} \right) d\epsilon_{t+1} \\ &= \min_{\mathbf{u}_t} \int \exp \left( \rho \frac{\mathcal{S}_t}{2} \right) d\epsilon_{t+1}. \end{aligned}$$

Since  $\mathcal{V}_{t+1}$  is as a quadratic form in  $\epsilon_{t+1}$ ,  $\mathbf{u}_t$  and  $\mathbf{z}_t$ , so is  $\mathcal{S}_t$ . If the discounted stress in  $t$  admits the saddle point  $\min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathcal{S}_t$ , then  $-\mathcal{S}_t$  admits the saddle point  $\max_{\mathbf{u}_t} \min_{\epsilon_{t+1}} \{-\mathcal{S}_t\}$ . We can apply Lemma 2 for  $Q(\mathbf{u}_t, \epsilon_{t+1}) = -\rho \mathcal{S}_t$ , so that

$$\min_{\mathbf{u}_t} \int \exp \left( \rho \frac{\mathcal{S}_t}{2} \right) d\epsilon_{t+1} \propto \exp \left( -\frac{1}{2} \max_{\mathbf{u}_t} \min_{\epsilon_{t+1}} (-\rho \mathcal{S}_t) \right) = \exp \left( \frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathcal{S}_t \right),$$

where the constant of proportionality depends on the covariance matrix of the vector  $\epsilon_{t+1}$  and it is independent of both  $\mathbf{u}_t$  and  $\mathbf{z}_t$ . As  $\mathcal{S}_t$  is a quadratic form in  $\epsilon_{t+1}$ ,  $\mathbf{u}_t$  and  $\mathbf{z}_t$ , from the saddle point condition we conclude that the *extremized* value of the total stress ( $\min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathcal{S}_t$ ) is a quadratic form in  $\mathbf{z}_t$ , while  $\mathcal{V}_t$  in the statement of Lemma 1 is equal to the extremized value of  $\mathcal{S}_t$  plus a constant independent of  $\mathbf{z}_t$ .  $\square$

**Proof of Theorem 1.** From Lemma 1 we see that in any  $t$ ,  $\mathbf{u}_t$  is chosen minimizing  $E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathcal{V}_{t+1}) \right) \right]$ . Let us solve this minimization starting in  $T$ . We see that in  $T$   $c_T$  is a quadratic form in  $\mathbf{u}_T$ , while  $\mathcal{V}_{T+1} = d_{T+1} = 0$ . This implies that  $\mathcal{S}_T$  is a quadratic form in  $\mathbf{u}_T$  and  $\epsilon_{T+1}$  and hence that the conditions to apply Lemma 3 are met, so that the saddle point condition for  $\mathcal{S}_T$  yields the optimal control  $\mathbf{u}_T$ , with the extremized total stress,  $\min_{\mathbf{u}_T} \max_{\epsilon_{T+1}} \mathcal{S}_T$ , and the value function,  $\mathcal{V}_T$ , both quadratic forms in  $\mathbf{z}_T$ . Then, by backward induction the statement is established.  $\square$

- **Derivation of Theorem 2.**

First, we need to adapt to the Markovian DLEQG problem a result originally derived by Whittle.

**Lemma 4** *The saddle point conditions for the discounted total stress,  $\mathcal{S}_t$ , with  $t = 1, 2, \dots, T$ , can be satisfied by solving the following discounted future stress backward recursion*

$$F_t(\mathbf{z}_t) = \min_{\mathbf{u}_t} \left\{ \max_{\epsilon_{t+1}} \left[ c_t - \frac{1}{\rho} d_{t+1} + \delta F_{t+1}(\mathbf{z}_{t+1}) \right] \right\},$$

with  $t = T, T-1, \dots, 1$ , where  $F_t(\mathbf{z}_t)$ , denoted as the *extremized discounted future stress*, is a quadratic form in the state vector  $\mathbf{z}_t$ ,  $F_t(\mathbf{z}_t) \equiv \mathbf{z}'_t \mathbf{\Pi}_t \mathbf{z}_t$  with  $\mathbf{\Pi}_{T+1} \equiv \mathbf{0}$ . The value function in  $t$  is  $\mathcal{V}_t = \nu_t + F_t(\mathbf{z}_t)$ , where  $\nu_t$  is independent of  $\mathbf{z}_t$ .

**Proof.** Let us start from  $t = T$ . By definition  $\mathcal{V}_{T+1} = 0$  and  $d_{T+1} = 0$ . Given that  $c_T$  is a quadratic form in  $\mathbf{u}_T$  and  $\mathbf{z}_T$ , we immediately see that: i) imposing the saddle point condition for the total stress in  $T$ ,  $\mathcal{S}_T$ , is equivalent to solving the discounted future stress backward recursion; and ii) there exist a

matrix  $\mathbf{\Pi}_T$  such that the extremized (discounted) future stress is  $F_T(\mathbf{z}_T) \equiv \mathbf{z}'_T \mathbf{\Pi}_T \mathbf{z}_T$  and a constant  $\nu_T$  independent of  $\mathbf{z}_T$  such that  $\exp(\rho \mathcal{V}_T/2) = \exp(\frac{1}{2} \rho [\nu_T + F_T(\mathbf{z}_T)])$ . Proceeding backward, the optimal control vector at time  $T - 1$  is obtained by imposing the following saddle point condition

$$\min_{\mathbf{u}_{T-1}} \max_{\boldsymbol{\epsilon}_T} \mathcal{S}_{T-1} = \min_{\mathbf{u}_{T-1}} \left\{ \max_{\boldsymbol{\epsilon}_T} \left[ c_{T-1} - \frac{1}{\rho} d_T + \delta \mathcal{V}_T \right] \right\}$$

Since  $\mathcal{V}_T = \nu_T + F_T(\mathbf{z}_T)$  and  $\nu_T$  is independent of  $\mathbf{z}_T$ , this is equivalent to the saddle point condition

$$\min_{\mathbf{u}_{T-1}} \left\{ \max_{\boldsymbol{\epsilon}_T} \left[ c_{T-1} - \frac{1}{\rho} d_T + \delta F_T(\mathbf{z}_T) \right] \right\}.$$

Given that  $c_{T-1}$  is a quadratic function in  $\mathbf{u}_{T-1}$  and  $\mathbf{z}_{T-1}$ ,  $d_T$  is a quadratic form in  $\boldsymbol{\epsilon}_T$  and  $F_T(\mathbf{z}_T)$  is a quadratic form in  $\mathbf{z}_T$  while this is linear in  $\mathbf{u}_{T-1}$ ,  $\mathbf{z}_{T-1}$  and  $\boldsymbol{\epsilon}_T$ , we find that the result of this extremization is given by a quadratic form of  $\mathbf{z}_{T-1}$ , so that there exists a matrix  $\mathbf{\Pi}_{T-1}$  such that  $F_{T-1}(\mathbf{z}_{T-1}) = \mathbf{z}'_{T-1} \mathbf{\Pi}_{T-1} \mathbf{z}_{T-1}$  and  $\exp(\rho \mathcal{V}_{T-1}/2) = \exp(\frac{1}{2} \rho [\nu_{T-1} + F_{T-1}(\mathbf{z}_{T-1})])$ . Since the same argument applies at any other date  $t$  as long as  $F_{t+1}$  is a quadratic form in  $\mathbf{z}_{t+1}$ , by backward induction the statement is proved.  $\square$

**Proof of Theorem 2.** In the Markovian DLEQG problem the extremized future stress  $F_t(\mathbf{z}_t)$  respects the double recursion  $F_t = \mathcal{L} \tilde{\mathcal{L}} F_{t+1}$ , based on the following two operators

$$\mathcal{L} \phi(\mathbf{z}) = \min_{\mathbf{u}} [c(\mathbf{z}, \mathbf{u}) + \phi(A\mathbf{z} + B\mathbf{u})] \quad \text{and} \quad \tilde{\mathcal{L}} \phi(\mathbf{z}) = \max_{\boldsymbol{\epsilon}} [\phi(\mathbf{z} + \boldsymbol{\epsilon}) - \frac{1}{\rho} \boldsymbol{\epsilon}' \mathbf{N}^{-1} \boldsymbol{\epsilon}],$$

where  $\phi(\mathbf{z}) = \delta \mathbf{z}' \mathbf{\Pi} \mathbf{z}$ , so that  $\tilde{\mathcal{L}} \phi(\mathbf{z}) = \max_{\boldsymbol{\epsilon}} [(\mathbf{z} + \boldsymbol{\epsilon})' \delta \mathbf{\Pi} (\mathbf{z} + \boldsymbol{\epsilon}) - \frac{1}{\rho} \boldsymbol{\epsilon}' \mathbf{N}^{-1} \boldsymbol{\epsilon}]$ . Taking first derivatives, we find that

$$\tilde{\boldsymbol{\epsilon}} = -(\delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1})^{-1} \delta \mathbf{\Pi} \mathbf{z} = -\check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} \mathbf{z},$$

which pins down a maximum if  $\check{\mathbf{\Pi}}$  is negative definite, or equivalently if  $(\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N}$  is positive definite. Replacing this expression we conclude that  $\tilde{\mathcal{L}} \phi(\mathbf{z}) = \mathbf{z}' ((\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N})^{-1} \mathbf{z} = \mathbf{z}' \tilde{\mathbf{\Pi}} \mathbf{z}$ . For  $\tilde{\mathcal{L}} \phi(\mathbf{z}) = \mathbf{z}' \tilde{\mathbf{\Pi}} \mathbf{z}$ , solution of the operator  $\mathcal{L}$  yields the standard recursive formulae for  $\mathbf{\Pi}$  and  $\mathbf{K}$  from the Markovian LQG problem where  $\tilde{\mathbf{\Pi}} = ((\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N})^{-1}$  replaces  $\mathbf{\Pi}$ . Applying the two operators at time  $t$  we obtain the recursive formulae for  $\mathbf{\Pi}_t$  and  $\mathbf{K}_t$ , with the terminal condition  $\mathbf{\Pi}_{T+1} = \mathbf{0}$ , presented in the statement. Importantly, as the cost function  $c_t$  is positive definite in  $\mathbf{u}_t$  and  $\mathbf{z}_t$ ,  $\mathbf{Q} + \mathbf{B}' \tilde{\mathbf{\Pi}}_{t+1} \mathbf{B}$  is positive definite, so that the second order condition for minimization in the operator  $\mathcal{L}$  holds and  $\mathbf{\Pi}_t$  is positive semidefinite.  $\square$

• **Proof of Proposition 1.**

Solving for the fixed point in the modified Riccati equation (1.8), we find after some long but straightforward algebra that  $\mathcal{V}_t = \nu + \mathbf{z}'_t \mathbf{\Pi} \mathbf{z}_t$ , where  $\nu$  is a constant independent of  $\mathbf{z}_t$  and

$$\mathbf{\Pi} = \begin{pmatrix} 1 + \delta W & \alpha \delta W \\ \alpha \delta W & \lambda + \alpha^2 \delta W \end{pmatrix},$$

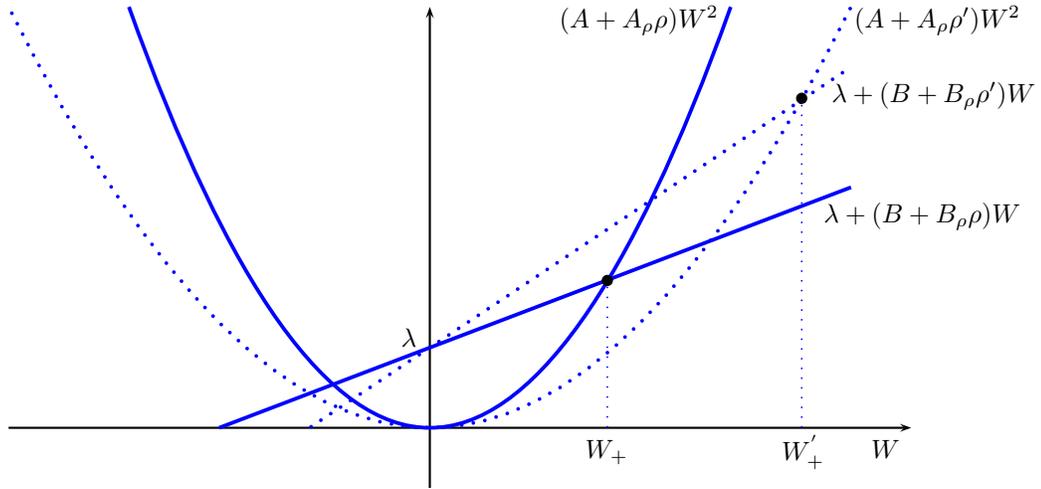
$W$  is a positive root of the equation  $\delta(\alpha^2 - \delta(\alpha^2 + \lambda)\rho\sigma_\pi^2)W^2 - (\delta(\alpha^2 + \lambda) - \lambda(1 - \delta\rho\sigma_\pi^2))W - \lambda = 0$ , which re-arranged can be written in the form employed in the statement of the Proposition, while  $u_t = \mathbf{Kz}_t$  with  $\mathbf{K} = \frac{1}{\gamma} \left( \frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2} \beta + \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2} \right)$  and  $\theta = \delta(\lambda + \delta(\alpha^2 + \lambda)W)$ .  $\square$

• **Proof of Proposition 2.**

We notice that  $\kappa_\pi$  and  $\kappa_y$  can be written as functions of  $\rho$  and  $W$ ,  $\kappa_\pi = K_\pi(W, \rho)$  and  $\kappa_y = K_y(W, \rho)$ . Then, we show that  $W$  is increasing in  $\rho$  and hence that  $K_\pi(\cdot, \cdot)$  and  $K_y(\cdot, \cdot)$  are increasing in both arguments,  $W$  and  $\rho$ . To see that  $W$  rises with  $\rho$ , notice that the equation solved by  $W$  can be rewritten as follows

$$(A + A_\rho\rho)W^2 = \lambda + (B + B_\rho\rho)W,$$

where  $A = \delta\alpha^2 > 0$ ,  $A_\rho = -\delta(\alpha^2 + \delta) < 0$ ,  $B = \delta\alpha^2 - (1 - \delta)\lambda$  and  $B_\rho = \delta\lambda\sigma_\pi^2 > 0$ . Graphical inspection shows that this equation admits only one positive solution  $W_+$ . As  $\rho$  increases,  $A + A_\rho\rho$  diminishes, so that in the Cartesian space the parabola on the left hand side moves downward, while  $(B + B_\rho\rho)$  increases, so that the straight line on the right hand side rotates counter-clockwise. Graphical inspection, as represented in the following diagram, shows that  $W_+$  augments.



The functions  $K_\pi(\cdot, \cdot)$  and  $K_y(\cdot, \cdot)$  depend on  $W$  and  $\rho$  through the ratio  $\frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}$ , with  $\theta = \delta\lambda + \delta^2(\alpha^2 + \lambda)W$ .  $K_\pi(\cdot, \cdot)$  and  $K_y(\cdot, \cdot)$  rise with this ratio, which is positive for  $\sigma_\pi^2$  small. For  $\sigma_\pi^2$  small, so that  $\lambda > \theta\sigma_\pi^2\rho$ , this ratio is increasing in  $W$ . In addition, this ratio is clearly increasing in  $\rho$ .  $\square$

• **Derivation of Theorem 3.**

First we establish a preliminary result, which reformulates Lemma 3 under imperfect state observation:

**Lemma 5** *In a Markovian DLEQG problem, under imperfect state observation, if the value function  $\mathcal{V}_{t+1}$  is a quadratic form in the state vector  $\mathbf{z}_{t+1}$  and the stress  $\mathcal{S}_t$  satisfies a saddle point condition with respect to  $\xi_t$  (with  $\xi'_t \equiv (\mathbf{z}'_{t-1} - \hat{\mathbf{z}}'_{t-1} \psi'_t \psi'_{t+1})$ ) and  $\mathbf{u}_t$ , so that  $\min_{\mathbf{u}_t} \max_{\xi_t} \mathcal{S}_t$  exists, the following proportionality*

condition holds

$$\min_{\mathbf{u}_t} E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) \right) \right] \propto \exp \left( \frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\xi_t} \mathcal{S}_t \right),$$

where the proportionality constant is independent of  $\mathbf{z}_t$ , while the value function  $\mathbf{V}_t$  is a quadratic form in  $\mathbf{z}_t$  equal to the extremized discounted stress  $\min_{\mathbf{u}_t} \max_{\xi_t} \mathcal{S}_t$  plus a constant independent of  $\mathbf{z}_t$ .

**Proof.** Under imperfect state observation  $\mathbf{V}_{t+1}$  is a function of  $\mathbf{z}_{t+1}$ , while  $c_t$  is function of  $\mathbf{z}_t$ . Since under imperfect state information  $\mathbf{z}_t$  and  $\mathbf{z}_{t+1}$  can be expressed in terms of the vector  $\xi_t$ , we have

$$\min_{\mathbf{u}_t} E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) \right) \right] \propto \min_{\mathbf{u}_t} \int \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) - \frac{1}{2} \xi_t' \Upsilon_{t-1}^{-1} \xi_t \right) d\xi_t,$$

where  $\Upsilon_{t-1}$  denotes the covariance matrix of  $\xi_t$  conditional on observation history. In addition, since  $(\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \perp \psi_t' \perp \psi_{t+1}'$ , we can write

$$\begin{aligned} \min_{\mathbf{u}_t} E_t \left[ \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) \right) \right] &\propto \min_{\mathbf{u}_t} \int \exp \left( \frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) - \frac{1}{2} \left( \psi_{t+1}' \mathbf{P}^{-1} \psi_{t+1} + \right. \right. \\ &\quad \left. \left. \psi_t' \mathbf{P}^{-1} \psi_t + (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \Omega_{t-1}^{-1} (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1}) \right) \right) d\xi_t \\ &= \min_{\mathbf{u}_t} \int \exp \left( \rho \frac{\mathcal{S}_t}{2} \right) d\xi_t. \end{aligned}$$

We proceed as in the Proof of Lemma 3. Since  $\mathbf{V}_{t+1}$  is a quadratic function of  $\mathbf{z}_{t+1}$  and the latter is linearly dependent on  $\xi_t$  and  $\mathbf{u}_t$ ,  $\mathcal{S}_t$  is a quadratic form in  $\xi_t$  and  $\mathbf{u}_t$ . If  $\mathcal{S}_t$  respects the aforementioned saddle point condition, exploiting Lemma 2, we conclude that

$$\begin{aligned} \min_{\mathbf{u}_t} \int \exp \left( \rho \frac{\mathcal{S}_t}{2} \right) d\xi_t &= \min_{\mathbf{u}_t} \int \exp \left( -\frac{1}{2} \underbrace{(-\rho \mathcal{S}_t)}_{Q(\mathbf{u}_t, \xi_t)} \right) d\xi_t \\ &\propto \exp \left( -\frac{1}{2} \max_{\mathbf{u}_t} \min_{\xi_t} (-\rho \mathcal{S}_t) \right) = \exp \left( \frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\xi_t} \mathcal{S}_t \right), \end{aligned}$$

where we have made use of the fact that  $-\mathcal{S}_t$  admits a saddle point in  $\mathbf{u}_t$  and in  $\xi_t$ . As  $\mathcal{S}_t$  respects the saddle point condition, its extremized value will be a quadratic form in  $\mathbf{z}_t$  and so will be  $\mathbf{V}_t$ .  $\square$

**Proof of Theorem 3.** It is as that of **Theorem 1** with  $\xi_t$  replacing  $\epsilon_{t+1}$  throughout.  $\square$

• **Derivation of Theorem 4.**

Let  $P_t(\mathbf{z}_t)$  denote the *extremized discounted past stress* defined as

$$P_t(\mathbf{z}_t) = \max_{\mathbf{z}_{t-1}} \left\{ -\frac{1}{\rho} \left( d_t + \mathcal{D}_{t-1} \right) \right\}.$$

The following Lemma adapts to the Markovian DLEQG problem a result originally derived by Whittle:

**Lemma 6** *Under imperfect state observation, the extremization of the discounted stress at time  $t$ , with  $t = 1, 2, \dots, T$ , is obtained operating into two stages. In the first stage, the extremized discounted past and future stresses,  $P_t(\mathbf{z}_t)$  and  $F_t(\mathbf{z}_t)$ , are calculated conditionally on  $\mathbf{z}_t$ .  $P_t$  and  $F_t$ , relate to estimation and control respectively: the former identifies the estimate for  $\mathbf{z}_t$  conditional on past observations; the latter pins down the control  $\mathbf{u}_t(\mathbf{z}_t)$  which would be optimal if  $\mathbf{z}_t$  were known. In the second stage, the saddle point for the discounted stress  $\mathcal{S}_t$  is achieved by maximizing  $P_t(\mathbf{z}_t) + F_t(\mathbf{z}_t)$  with respect to  $\mathbf{z}_t$ . This yields the maximum stress estimate (MSE),  $\check{\mathbf{z}}_t$ , for  $\mathbf{z}_t$  and the optimal control,  $\mathbf{u}_t(\check{\mathbf{z}}_t)$ .*

**Proof.** To establish this result notice that  $\xi_t$  contains  $\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1}$ ,  $\epsilon_t$ ,  $\eta_t$ ,  $\epsilon_{t+1}$  and  $\eta_{t+1}$ . They can be expressed as linear functions of the unobservable (at time  $t$ ) vectors  $\mathbf{z}_{t-1}$ ,  $\mathbf{z}_t$ ,  $\mathbf{z}_{t+1}$  and  $\mathbf{w}_{t+1}$ . The saddle-point condition for the stress in  $t$  in Theorem 3 can be equivalently written as

$$\min_{\mathbf{u}_t} \max_{\mathbf{z}_{t-1}, \mathbf{z}_t, \mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \mathcal{S}_t.$$

It can be satisfied proceeding in two stages: in stage i), conditionally on  $\mathbf{z}_t$ ,  $\mathcal{S}_t$  is extremized with respect to  $\mathbf{u}_t$ ,  $\mathbf{z}_{t-1}$ ,  $\mathbf{z}_{t+1}$  and  $\mathbf{w}_{t+1}$ ; in stage ii) the resulting function is extremized with respect to  $\mathbf{z}_t$ :

$$\min_{\mathbf{u}_t} \max_{\mathbf{z}_{t-1}, \mathbf{z}_t, \mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \mathcal{S}_t \Leftrightarrow \max_{\mathbf{z}_t} \left\{ \min_{\mathbf{u}_t} \max_{\mathbf{z}_{t-1}, \mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \mathcal{S}_t \right\}.$$

In stage i), conditionally on  $\mathbf{z}_t$ , the extremization of the stress is achieved by isolating terms in  $\mathcal{S}_t$  pertaining to *past* and *future*,  $-\frac{1}{\rho}(d_t + \mathcal{D}_{t-1})$  and  $c_t - \frac{1}{\rho}d_{t+1} + \delta\mathcal{V}_{t+1}$ , and solving the programs

$$\max_{\mathbf{z}_{t-1}} \left\{ -\frac{1}{\rho} \left( d_t + \mathcal{D}_{t-1} \right) \right\} \text{ and } \min_{\mathbf{u}_t} \max_{\mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \left\{ c_t - \frac{1}{\rho} d_{t+1} + \delta\mathcal{V}_{t+1} \right\}.$$

As  $\mathbf{z}_{t+1}$  and  $\mathbf{w}_{t+1}$  are linearly dependent on  $\epsilon_{t+1}$  and  $\eta_{t+1}$ , the latter program can be written as follows

$$\min_{\mathbf{u}_t} \max_{\epsilon_{t+1}, \eta_{t+1}} \left\{ c_t - \frac{1}{\rho} d_{t+1} + \delta\mathcal{V}_{t+1} \right\}.$$

The maximization of  $c_t - (1/\rho)d_{t+1} + \mathcal{V}_{t+1}$  with respect to  $\eta_{t+1}$  reduces to  $\max_{\eta_{t+1}} \{ -(1/\rho)d_{t+1} \} = -\frac{1}{\rho}\epsilon'_{t+1}\mathbf{N}^{-1}\epsilon_{t+1}$ . This means that under imperfect state information the extremization, conditionally on  $\mathbf{z}_t$ , of  $\mathcal{S}_t$  with respect to  $\mathbf{u}_t$ ,  $\mathbf{z}_{t+1}$  and  $\mathbf{w}_{t+1}$  is equivalent to the extremization of the stress under perfect state information with respect to  $\mathbf{u}_t$  and  $\epsilon_{t+1}$ . Lemma 4 shows that this corresponds to calculating the extremized future stress,  $F_t(\mathbf{z}_t)$ , which yields the optimal policy,  $\mathbf{u}(\mathbf{z}_t)$ , conditional on  $\mathbf{z}_t$ . The former program instead corresponds to calculating the extremized past stress in  $t$ ,  $P_t(\mathbf{z}_t)$ . As the maximand in the definition of the extremized past stress is proportional to the log of the conditional density function of  $\mathbf{z}_t$ , its arg max yields the ML estimate of  $\mathbf{z}_t$ . In stage ii) estimation and control are recoupled by maximizing the sum  $P_t(\mathbf{z}_t) + F_t(\mathbf{z}_t)$  with respect to the current state vector  $\mathbf{z}_t$ .  $\square$

**Proof of Theorem 4.** We can apply Theorem 3 and Lemma 6. Conditionally on  $\mathbf{z}_t$  the extremized future stress,  $F_t$ , respects the backward-looking recursion in Lemma 4 and it is therefore a quadratic form in  $\mathbf{z}_t$ ,  $F_t(\mathbf{z}_t) = \mathbf{z}_t' \mathbf{\Pi}_t \mathbf{z}_t$ . Theorem 2 provides the modified Riccati equation for the matrix  $\mathbf{\Pi}_t$  and the conditional optimal policy,  $\mathbf{u}_t(\mathbf{z}_t)$ . As for the extremized past stress consider that given the definition of  $\mathcal{D}_{t-1}$  we see that  $\max_{\mathbf{z}_{t-1}} -\frac{1}{\rho} \{d_t + \mathcal{D}_{t-1}\}$  is equivalent to  $\max_{\mathbf{z}_{t-1}} \frac{1}{\rho} 2 \ln f(\mathbf{z}_t)$  where  $f(\cdot)$  is the conditional density function of  $\mathbf{z}_t$ . The maximum corresponds to  $P_t(\mathbf{z}_t) = -\frac{1}{\rho} \mathcal{D}_t + \dots$ , where  $\dots$  indicates terms independent of  $\mathbf{z}_t$ .

Re-coupling the extremization of the past and future stresses requires maximizing the sum  $P_t(\mathbf{z}_t) + F_t(\mathbf{z}_t)$  with respect to  $\mathbf{z}_t$  to obtain the MSE,  $\check{\mathbf{z}}_t$ . Given that  $P_t(\mathbf{z}_t) + F_t(\mathbf{z}_t) = -(1/\rho)(\mathbf{z}_t - \hat{\mathbf{z}}_t)' \mathbf{\Omega}_t^{-1} (\mathbf{z}_t - \hat{\mathbf{z}}_t) + \mathbf{z}_t' \mathbf{\Pi}_t \mathbf{z}_t$  plus terms independent of  $\mathbf{z}_t$ , from the first derivative of this sum with respect to  $\mathbf{z}_t$  it is immediate to see that, for  $\mathbf{\Omega}_t^{-1} - \rho \mathbf{\Pi}_t$  positive definite,  $\check{\mathbf{z}}_t$  is given by the following expression

$$\check{\mathbf{z}}_t = (\mathbf{I} - \rho \mathbf{\Omega}_t \mathbf{\Pi}_t)^{-1} \hat{\mathbf{z}}_t.$$

Finally, the optimal control vector under imperfect state observation is given by Theorem 2 where  $\check{\mathbf{z}}_t$  replaces  $\mathbf{z}_t$ , i.e.  $\mathbf{u}_t = \mathbf{K}_t \check{\mathbf{z}}_t$ , and  $\mathbf{K}_t$  is the matrix of optimal coefficients presented in Theorem 2.  $\square$

• **Proof of Proposition 4.**

When the lag of the state vector is observed in  $t$  the stress is simplified, in that  $d_t = \epsilon_t' \mathbf{N}^{-1} \epsilon_t$  for  $t = 1, 2, \dots, T$ . It follows that the extremization of the past stress is reached for  $\mathbf{z}_{t-1} = \hat{\mathbf{z}}_{t-1}$  and is given by  $P_t(\mathbf{z}_t) = -\frac{1}{\rho} \epsilon_t' \mathbf{N}^{-1} \epsilon_t + \dots$ , where once again  $\dots$  denotes terms independent of  $\mathbf{z}_t$ . Since in steady state  $F_t(\mathbf{z}_t) = \mathbf{z}_t' \mathbf{\Pi} \mathbf{z}_t$ , in re-coupling past and future extremization we solve

$$\max_{\mathbf{z}_t} \left\{ -\frac{1}{\rho} \epsilon_t' \mathbf{N}^{-1} \epsilon_t + \mathbf{z}_t' \mathbf{\Pi} \mathbf{z}_t \right\}.$$

Given that at time  $t$  the observable vector is  $\mathbf{w}_t = \mathbf{z}_{t-1}$ , the conditional expectation of  $\mathbf{z}_t$  is  $\hat{\mathbf{z}}_t = \mathbf{A} \mathbf{z}_{t-1} + \mathbf{B} \mathbf{u}_{t-1}$ . As we can write  $\epsilon_t = \mathbf{z}_t - \hat{\mathbf{z}}_t$ , we need to solve

$$\max_{\mathbf{z}_t} \left\{ -\frac{1}{\rho} (\mathbf{z}_t - \hat{\mathbf{z}}_t)' \mathbf{N}^{-1} (\mathbf{z}_t - \hat{\mathbf{z}}_t) + \mathbf{z}_t' \mathbf{\Pi} \mathbf{z}_t \right\}.$$

We immediately conclude that, for  $\mathbf{N}^{-1} - \rho \mathbf{\Pi}$  positive definite, the maximum stress estimate (MSE)  $\check{\mathbf{z}}_t$  is given by

$$\check{\mathbf{z}}_t = (\mathbf{I} - \rho \mathbf{N} \mathbf{\Pi})^{-1} \hat{\mathbf{z}}_t.$$

Simple algebraic transformations show that  $\check{\mathbf{z}}_t = (\mathbf{I} + \rho \mathbf{G}) \hat{\mathbf{z}}_t$ . As indicated in Theorem 4, the optimal control is then obtained by inserting the MSE,  $\check{\mathbf{z}}_t$ , in lieu of  $\mathbf{z}_t$  into the control rule which would prevail under perfect state observation.  $\square$

• **Proof of Theorem 5**

Lemma 3 and Theorem 1 hold, as the total stress in  $t$ ,  $\mathcal{S}_t$ , is still a quadratic form in  $\mathbf{u}_t$ ,  $\epsilon_{t+1}$  and

$\mathbf{z}_t$ . However, this quadratic form is no longer homogeneous in  $\mathbf{z}_t$ . This implies that Lemma 4 must be amended, in that the extremized future stress is now a non-homogenous quadratic form,  $F_t(\mathbf{z}_t) = \mathbf{z}'_t \mathbf{\Pi}_t \mathbf{z}_t - 2\mathbf{\vartheta}'_t \mathbf{z}_t \cdots$  (with  $\mathbf{\vartheta}_t$  a vector of coefficients for the components of  $\mathbf{z}_t$  and  $\cdots$  indicating terms independent of  $\mathbf{z}_t$ ). Given that in  $T+1$   $F_{T+1} \equiv 0$ , the terminal conditions  $\mathbf{\Pi}_{T+1} = \mathbf{0}$  and  $\mathbf{\vartheta}_{T+1} = \mathbf{0}$  apply. Therefore, we just repeat the steps followed in the proof of Theorem 2. Recall that the extremized future stress respects the double recursion  $F_t = \mathcal{L} \tilde{\mathcal{L}} F_{t+1}$  based on the two operators

$$\mathcal{L} \phi(\mathbf{z}) = \min_{\mathbf{u}} [c(\mathbf{z}, \mathbf{u}) + \phi(\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} + \boldsymbol{\mu})] \quad \text{and} \quad \tilde{\mathcal{L}} \phi(\mathbf{z}) = \max_{\boldsymbol{\epsilon}} [\phi(\mathbf{z} + \boldsymbol{\epsilon}) - \frac{1}{\rho} \boldsymbol{\epsilon}' \mathbf{N}^{-1} \boldsymbol{\epsilon}].$$

Assume that  $\phi(\mathbf{z}) = \delta \mathbf{z}' \mathbf{\Pi} \mathbf{z} - 2\delta \mathbf{\vartheta}' \mathbf{z} + \cdots$ , so that  $\tilde{\mathcal{L}} \phi(\mathbf{z}) = \max_{\boldsymbol{\epsilon}} [(\mathbf{z} + \boldsymbol{\epsilon})' \delta \mathbf{\Pi} (\mathbf{z} + \boldsymbol{\epsilon}) - 2\delta \mathbf{\vartheta}' (\mathbf{z} + \boldsymbol{\epsilon}) - \frac{1}{\rho} \boldsymbol{\epsilon}' \mathbf{N}^{-1} \boldsymbol{\epsilon} + \cdots]$ . Taking first derivatives, we find that

$$\tilde{\boldsymbol{\epsilon}} = -(\delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1})^{-1} \delta \mathbf{\Pi} \mathbf{z} + (\delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1})^{-1} \delta \mathbf{\vartheta},$$

which pins down a maximum if  $(\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N}$  is positive definite. Replacing this expression we conclude that  $\tilde{\mathcal{L}} \phi(\mathbf{z}) = \mathbf{z}' \tilde{\mathbf{\Pi}} \mathbf{z} - 2\tilde{\mathbf{\vartheta}}' \mathbf{z} + \cdots$ , where  $\tilde{\mathbf{\Pi}} = ((\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N})^{-1}$ ,  $\tilde{\mathbf{\vartheta}} = \tilde{\mathbf{\Pi}} \mathbf{\Pi}^{-1} \mathbf{\vartheta}$  and  $\cdots$  denotes terms independent of  $\mathbf{z}$ . For  $\tilde{\mathcal{L}} \phi(\mathbf{z}) = \mathbf{z}' \tilde{\mathbf{\Pi}} \mathbf{z} - 2\tilde{\mathbf{\vartheta}}' \mathbf{z} + \cdots$ , the solution of the operator  $\mathcal{L}$  yields the standard recursive formulae for  $\mathbf{\Pi}$ ,  $\mathbf{K}$  and  $\mathbf{\vartheta}$  from the Markovian LQG problem with pre-determined disturbances, where  $\tilde{\mathbf{\Pi}} = ((\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N})^{-1}$  and  $\tilde{\mathbf{\vartheta}}$  replace respectively  $\mathbf{\Pi}$  and  $\mathbf{\vartheta}$ . Specifically applying the double recursion  $F_t = \mathcal{L} \tilde{\mathcal{L}} F_{t+1}$ , we find that  $F_t(\mathbf{z}_t) = \mathbf{z}'_t \mathbf{\Pi}_t \mathbf{z}_t - 2\tilde{\mathbf{\vartheta}}'_{t+1} \mathbf{z}_t + \cdots$  for  $\mathbf{u}_t = \mathbf{K}_t \mathbf{z}_t + (\mathbf{Q} + \mathbf{B}' \tilde{\mathbf{\Pi}}_{t+1} \mathbf{B})^{-1} \mathbf{B}' (\tilde{\mathbf{\vartheta}}_{t+1} - \tilde{\mathbf{\Pi}}_{t+1} \boldsymbol{\mu}_{t+1})$ , where  $\mathbf{K}_t = -(\mathbf{Q} + \mathbf{B}' \tilde{\mathbf{\Pi}}_{t+1} \mathbf{B})^{-1} (\mathbf{S} + \mathbf{B}' \tilde{\mathbf{\Pi}}_{t+1} \mathbf{A})$ . Replacing  $\tilde{\mathbf{\vartheta}}_{t+1}$  with  $\tilde{\mathbf{\Pi}}_{t+1} \mathbf{\Pi}_{t+1}^{-1} \mathbf{\vartheta}_{t+1}$  we find the recursive formulae presented in the statement.  $\square$

#### • Proof of Theorem 6

Lemma 6 still applies. In recoupling the extremized past and future stresses, the sum  $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$  is maximized with respect to  $\mathbf{z}_t$  to obtain the MSE,  $\check{\mathbf{z}}_t$ . Given that  $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t) = -(1/\rho)(\mathbf{z}_t - \hat{\mathbf{z}}_t)' \boldsymbol{\Omega}_t^{-1} (\mathbf{z}_t - \hat{\mathbf{z}}_t) + \mathbf{z}'_t \mathbf{\Pi}_t \mathbf{z}_t - 2\mathbf{\vartheta}'_t \mathbf{z}_t$  plus terms independent of  $\mathbf{z}_t$ , taking the first derivative of this sum with respect to  $\mathbf{z}_t$  we see that, for  $\boldsymbol{\Omega}_t^{-1} - \rho \mathbf{\Pi}_t$  positive definite,

$$\check{\mathbf{z}}_t = (\mathbf{I} - \rho \boldsymbol{\Omega}_t \mathbf{\Pi}_t)^{-1} (\hat{\mathbf{z}}_t - \rho \boldsymbol{\Omega}_t \mathbf{\vartheta}_t),$$

where  $\hat{\mathbf{z}}_t$  is still the ML estimate of  $\mathbf{z}_t$ .  $\square$

#### • Proof of Proposition 6.

Since  $\mathbf{\vartheta} = \mathbf{0}$ ,  $F_t(\mathbf{z}_t) = \mathbf{z}'_t \mathbf{\Pi}_t \mathbf{z}_t$ . In addition in the proof of Proposition 4 we have seen that  $P_t(\mathbf{z}_t) = -\frac{1}{\rho} \boldsymbol{\epsilon}'_t \mathbf{N}^{-1} \boldsymbol{\epsilon}_t + \cdots$ . Then, applying Lemma 6 we find that

$$\check{\mathbf{z}}_t = (\mathbf{I} - \rho \mathbf{N} \mathbf{\Pi})^{-1} \hat{\mathbf{z}}_t,$$

where now the ML estimate for  $\mathbf{z}_t$  is  $\hat{\mathbf{z}}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}u_{t-1} + \boldsymbol{\mu}$ . Given the expressions for  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\boldsymbol{\mu}$ , and the definitions of  $\mathbf{z}_t$  and  $\iota_t$ , we have that this ML estimate can be written as in Section 2,

$$\begin{aligned}\hat{\pi}_t &= \pi_{t-1} + \alpha y_t, \\ \hat{y}_t &= \beta y_{t-1} - \gamma r_{t-1}. \square\end{aligned}$$

## Technical Appendix: Detailed Calculations

### A.1. Monotonicity and Convexity of the Optimization Criterion (1.3).

Let  $\mathcal{R}(\mathbf{v}) \equiv \ln(E[\exp(\delta \frac{\rho}{2} \mathbf{v})])$ . Then assume  $\mathbf{v}_1 \geq \mathbf{v}_2 \geq 0$ . Consider that

$$\mathcal{R}(\mathbf{v}_1) - \mathcal{R}(\mathbf{v}_2) = \ln(E[\exp(\delta \frac{\rho}{2} \mathbf{v}_1)]) - \ln(E[\exp(\delta \frac{\rho}{2} \mathbf{v}_2)]) = \ln\left(\frac{E[\exp(\delta \frac{\rho}{2} \mathbf{v}_1)]}{E[\exp(\delta \frac{\rho}{2} \mathbf{v}_2)]}\right) \geq 0.$$

This means that  $\mathcal{R}(\mathbf{v})$  is monotone increasing in  $\mathbf{v}$ . So, consider the convex combination  $\theta \mathbf{v}_1 + (1 - \theta) \mathbf{v}_2$ , with  $0 < \theta < 1$ .

$$\begin{aligned} \mathcal{R}(\theta \mathbf{v}_1 + (1 - \theta) \mathbf{v}_2) &= \ln(E[\exp(\delta \frac{\rho}{2} [\theta \mathbf{v}_1 + (1 - \theta) \mathbf{v}_2])]) = \ln(E[\exp(\delta \frac{\rho}{2} \mathbf{v}_1)^\theta \exp(\delta \frac{\rho}{2} \mathbf{v}_2)^{1-\theta}]) \\ &\leq \ln\left(\left\{E[\exp(\delta \frac{\rho}{2} \mathbf{v}_1)]\right\}^\theta \cdot \left\{E[\exp(\delta \frac{\rho}{2} \mathbf{v}_2)]\right\}^{1-\theta}\right) \\ &= \ln(\theta) \ln(E[\exp(\delta \frac{\rho}{2} \mathbf{v}_1)]) + \ln(1 - \theta) \ln(E[\exp(\delta \frac{\rho}{2} \mathbf{v}_2)]) \\ &= \ln(\theta) \mathcal{R}(\mathbf{v}_1) + \ln(1 - \theta) \mathcal{R}(\mathbf{v}_2), \end{aligned}$$

where the inequality follows from Hölder's inequality. In fact, Hölder's inequality states that for  $p$  and  $q$  such that  $1 < p, q$ , with  $1/p + 1/q = 1$ ,  $E[|X \cdot Y|] \leq (E[|X|^p])^{1/p} \cdot (E[|Y|^q])^{1/q}$ . We can apply Hölder's inequality setting  $X \equiv \exp(\delta \frac{\rho}{2} \mathbf{v}_1)^\theta$  and  $Y \equiv \exp(\delta \frac{\rho}{2} \mathbf{v}_2)^{1-\theta}$ , choosing  $p = 1/\theta$  and  $q = 1/(1 - \theta)$  and noticing that the exponential function is non-negative. This means that  $\mathcal{R}$  is convex in  $\mathbf{v}$ . Define  $\Gamma(\mathbf{u}, \mathbf{z}, \mathbf{v}) \equiv Q(\mathbf{u}, \mathbf{z}) + \mathcal{R}(\mathbf{v})$ , where  $Q$  is positive definite in  $\mathbf{u}$  and  $\mathbf{z}$ . From the properties of the function  $\mathcal{R}$ , it follows that  $\Gamma(\mathbf{u}, \mathbf{z}, \mathbf{v})$  is monotone increasing in  $\mathbf{v}$  and convex in  $\mathbf{u}$ ,  $\mathbf{z}$  and  $\mathbf{v}$ , so that the recursive optimization criterion captures risk-aversion. In fact, the larger  $\rho$  the larger the convexity of the function  $\Gamma$ .

### A.2. Limit Properties of the Optimization Criterion (1.3).

For  $\rho > 0$  we have that

$$\rho \mathbf{v}_t = \min_{\mathbf{u}_t} \left\{ \rho c_t + \ln(E_t[\exp(\delta \frac{\rho}{2} \mathbf{v}_{t+1})]) \right\} = \rho \min_{\mathbf{u}_t} \left\{ c_t + \frac{2}{\rho} \ln(E_t[\exp(\delta \frac{\rho}{2} \mathbf{v}_{t+1})]) \right\},$$

$$\text{i.e. } \mathbf{v}_t = \min_{\mathbf{u}_t} \left\{ c_t + \frac{2}{\rho} \ln(E_t[\exp(\delta \frac{\rho}{2} \mathbf{v}_{t+1})]) \right\}.$$

We proceed by backward induction. Assume that  $\mathbf{v}_{t+1}$  is independent of  $\rho$  (this is certainly true for in  $N + 1$ ). Then,

$$\lim_{\rho \downarrow 0} \frac{1}{\rho} \ln(E_t[\exp(\delta \frac{\rho}{2} \mathbf{v}_{t+1})]) = \lim_{\rho \downarrow 0} \delta \frac{1}{2} \frac{E_t[\exp(\delta \frac{\rho}{2} \mathbf{v}_{t+1}) \cdot \mathbf{v}_{t+1}]}{E_t[\exp(\delta \frac{\rho}{2} \mathbf{v}_{t+1})]} = \frac{1}{2} \delta E_t[\mathbf{v}_{t+1}],$$

where we have used the Hôpital's rule and moved the derivative operator inside the expectation operator. This implies that for  $\rho \downarrow 0$  we have  $\mathbf{v}_t = \min_{\mathbf{u}_t} \{c_t + \delta E_t [\mathbf{v}_{t+1}]\}$ , with  $\mathbf{v}_t$  independent of  $\rho$ . Because by definition  $\mathbf{v}_{T+1}$  is independent of  $\rho$ , by backward induction the argmin of the Markovian DLEQG problem converges to that of the corresponding Markovian DLQG problem for  $\rho \downarrow 0$ . This implies that the Markovian DLEQG problem encompasses the Markovian DLQG problem and it can be considered an extension of the latter.

### A.3. The Optimization Criterion (1.3) and Epstein and Zin's Preferences.

Suppose  $\mathbf{u}_t$  solves Epstein and Zin's recursion

$$\mathbf{u}_t = \max \left\{ (1 - \delta) C_t^{1-\eta} + \delta E_t \left[ \mathbf{u}_{t+1}^{1-\gamma} \right]^{\left(\frac{1-\eta}{1-\gamma}\right)} \right\}^{\left(\frac{1}{1-\eta}\right)},$$

where  $1/\eta$  is the elasticity of inter-temporal substitution. Let  $\eta = 1$ . Tallarini (2000) shows that

$$\mathbf{u}_t = \max \left\{ C_t^{1-\delta} \left( E_t \left[ \mathbf{u}_{t+1}^{1-\gamma} \right] \right)^{\left(\frac{\delta}{1-\gamma}\right)} \right\}.$$

Taking logs,

$$\ln \mathbf{u}_t = \max \left\{ (1 - \delta) \ln C_t + \frac{\delta}{1 - \gamma} \ln E_t \left[ \mathbf{u}_{t+1}^{1-\gamma} \right] \right\},$$

or equivalently

$$\frac{\ln \mathbf{u}_t}{1 - \delta} = \max \left\{ \ln C_t + \frac{\delta}{(1 - \delta)(1 - \gamma)} \ln E_t \left[ \mathbf{u}_{t+1}^{1-\gamma} \right] \right\}.$$

We can re-write this as

$$-\frac{\ln \mathbf{u}_t}{1 - \delta} = \min \left\{ -\ln C_t - \frac{\delta}{(1 - \delta)(1 - \gamma)} \ln E_t \left[ \mathbf{u}_{t+1}^{1-\gamma} \right] \right\}.$$

For  $\mathbf{v}_t = -\frac{\ln \mathbf{u}_t}{1 - \delta}$ , we have that  $-(1 - \delta)\mathbf{v}_t = \ln \mathbf{u}_t$ , so that  $\mathbf{u}_{t+1} = \exp(-(1 - \delta)\mathbf{v}_{t+1})$  and

$$\mathbf{u}_{t+1}^{1-\gamma} = (\exp(-(1 - \delta)\mathbf{v}_{t+1}))^{1-\gamma} = \exp(-(1 - \delta)(1 - \gamma)\mathbf{v}_{t+1}).$$

Setting  $\rho' = -2(1 - \delta)(1 - \gamma)$ , we can write

$$\mathbf{v}_t = \min \left\{ -\ln C_t + \delta \frac{2}{\rho'} \ln E_t \left[ \exp \left( \frac{\rho'}{2} \mathbf{v}_{t+1} \right) \right] \right\},$$

which corresponds to the optimization criterion (1.3) for  $\rho = \frac{\rho'}{\delta}$  and  $-\ln C_t$  equal to a quadratic form in the control and state vectors,  $\mathbf{u}_t$  and  $\mathbf{z}_t$ .

#### A.4. The Discounted Future Stress Recursion.

From Lemma 3 we know that if  $\mathbf{V}_{t+1}$  is a quadratic form in  $\mathbf{z}_{t+1}$

$$\exp\left(\frac{\rho}{2} \mathbf{V}_t\right) = \text{constant} \times \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \mathcal{S}_t\right) = \exp\left(\frac{\rho}{2} \left[\gamma_t + \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \mathcal{S}_t\right]\right),$$

for  $\gamma_t$  a constant independent of  $\mathbf{z}_t$ . This implies that  $\mathbf{V}_t = \gamma_t + \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \mathcal{S}_t$ . Then, assume that  $\mathbf{V}_{t+1} = \nu_{t+1} + \mathbf{z}'_{t+1} \boldsymbol{\Pi}_{t+1} \mathbf{z}_{t+1}$ ,  $\mathcal{S}_t = c_t - \frac{1}{\rho} d_{t+1} + \delta \mathbf{V}_{t+1}$ , it follows that

$$\begin{aligned} \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \mathcal{S}_t &= \min_{\mathbf{u}_t} \left\{ \max_{\boldsymbol{\epsilon}_{t+1}} \left[ c_t - \frac{1}{\rho} d_{t+1} + \delta \nu_{t+1} + \delta \mathbf{z}'_{t+1} \boldsymbol{\Pi}_{t+1} \mathbf{z}_{t+1} \right] \right\} \\ &= \delta \nu_{t+1} + \min_{\mathbf{u}_t} \left\{ \max_{\boldsymbol{\epsilon}_{t+1}} \left[ c_t - \frac{1}{\rho} d_{t+1} + \delta \mathbf{z}'_{t+1} \boldsymbol{\Pi}_{t+1} \mathbf{z}_{t+1} \right] \right\} \\ &= \delta \nu_{t+1} + \mathbf{z}'_t \boldsymbol{\Pi}_t \mathbf{z}_t = \delta \nu_{t+1} + F_t(\mathbf{z}_t), \end{aligned}$$

so that  $\mathbf{V}_t = \gamma_t + \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \mathcal{S}_t = \nu_t + F_t(\mathbf{z}_t)$ , with  $\nu_t = \gamma_t + \delta \nu_{t+1}$  and  $F_t(\mathbf{z}_t) = \mathbf{z}'_t \boldsymbol{\Pi}_t \mathbf{z}_t$ .

#### A.5. The Value Function.

Consider that (for  $n$  the dimension of the vector  $\boldsymbol{\epsilon}_{t+1}$ )

$$\begin{aligned} E_t \left[ \exp\left(\frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1})\right) \right] &= (2\pi)^{-n/2} \det(\mathbf{N})^{-1/2} \int \exp\left(\frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) - \frac{1}{2} \boldsymbol{\epsilon}'_{t+1} \mathbf{N}^{-1} \boldsymbol{\epsilon}_{t+1}\right) d\boldsymbol{\epsilon}_{t+1} \\ &= (2\pi)^{-n/2} \det(\mathbf{N})^{-1/2} \int \exp\left(\rho \frac{\mathcal{S}_t}{2}\right) d\boldsymbol{\epsilon}_{t+1}, \text{ so that} \\ \exp\left(\frac{\rho}{2} \mathbf{V}_t\right) &= (2\pi)^{-n/2} \det(\mathbf{N})^{-1/2} \min_{\mathbf{u}_t} \int \exp\left(\rho \frac{\mathcal{S}_t}{2}\right) d\boldsymbol{\epsilon}_{t+1}. \end{aligned}$$

The function  $-\rho \mathcal{S}_t$  is a quadratic form in  $\mathbf{u}_t$  and  $\boldsymbol{\epsilon}_{t+1}$ , which we can write as  $\mathbf{u}'_t \mathbf{S}_{\mathbf{u}\mathbf{u}} \mathbf{u}_t + 2\mathbf{u}'_t \mathbf{S}_{\mathbf{u}\boldsymbol{\epsilon}} \boldsymbol{\epsilon}_{t+1} + \boldsymbol{\epsilon}'_{t+1} \mathbf{S}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}} \boldsymbol{\epsilon}_{t+1}$ , with  $\mathbf{S}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}} = \mathbf{N}^{-1} - \delta \rho \boldsymbol{\Pi}_{t+1}$ . We can apply Lemma 2. From its proof we know that

$$\min_{\mathbf{u}_t} \int \exp\left(\rho \frac{\mathcal{S}_t}{2}\right) d\boldsymbol{\epsilon}_{t+1} = (2\pi)^{n/2} \det(\mathbf{N}^{-1} - \delta \rho \boldsymbol{\Pi}_{t+1})^{-1/2} \times \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \mathcal{S}_t\right).$$

Notice that  $\mathbf{N}^{-1} - \delta \rho \boldsymbol{\Pi}_{t+1} = \mathbf{N}^{-1} (\mathbf{I} - \delta \rho \mathbf{N} \boldsymbol{\Pi}_{t+1})$ , so that  $\det(\mathbf{N}^{-1} - \delta \rho \boldsymbol{\Pi}_{t+1}) = \det(\mathbf{I} - \delta \rho \mathbf{N} \boldsymbol{\Pi}_{t+1}) / \det(\mathbf{N})$ . Therefore,

$$\exp\left(\frac{\rho}{2} \mathbf{V}_t\right) = \det(\mathbf{I} - \delta \rho \mathbf{N} \boldsymbol{\Pi}_{t+1})^{-1/2} \min_{\mathbf{u}_t} \int \exp\left(\rho \frac{\mathcal{S}_t}{2}\right) d\boldsymbol{\epsilon}_{t+1}.$$

This implies that in **A.4.**  $\gamma_t = \frac{2}{\rho} \ln(\det(\mathbf{I} - \delta \rho \mathbf{N} \boldsymbol{\Pi}_{t+1})^{-1/2}) = -\frac{1}{\rho} \ln(\det(\mathbf{I} - \delta \rho \mathbf{N} \boldsymbol{\Pi}_{t+1}))$ , while  $\nu_t = \gamma_t + \delta \nu_{t+1}$ . In steady state  $\mathbf{V}_t = F(\mathbf{z}_t) + \nu$ , where  $F(\mathbf{z}_t) = \mathbf{z}'_t \boldsymbol{\Pi} \mathbf{z}_t$ , while

$$\nu = -\frac{1}{1-\delta} \left( \frac{1}{\rho} \ln(\det(\mathbf{I} - \delta \rho \mathbf{N} \boldsymbol{\Pi})) \right).$$

For  $\rho \downarrow 0$  we can show that  $\gamma_t \rightarrow \delta \text{Tr}(\mathbf{N}\mathbf{\Pi}_{t+1})$ . This also implies that in steady state  $\rho \downarrow 0$   $\nu \rightarrow \frac{\delta}{1-\delta} \text{Tr}(\mathbf{N}\mathbf{\Pi})$ . To prove this result consider that to calculate the limit for  $\rho \downarrow 0$  of  $\gamma_t$  we need to apply Hôpital's rule. Then, we should use the following results which apply to any invertible matrix  $\Xi$

$$\frac{d \ln(\det(\Xi))}{dy} = \frac{1}{\det(\Xi)} \frac{d \det(\Xi)}{dy} \quad \text{and} \quad \frac{d \det(\Xi)}{dy} = \det(\Xi) \text{Tr} \left( \Xi^{-1} \frac{d \Xi}{dy} \right).$$

Then,

$$\frac{d \ln(\det(\mathbf{I} - \delta \rho \mathbf{N}\mathbf{\Pi}_{t+1}))}{d \rho} = -\delta \text{Tr} \left( (\mathbf{I} - \delta \rho \mathbf{N}\mathbf{\Pi}_{t+1})^{-1} \mathbf{N}\mathbf{\Pi}_{t+1} \right).$$

For  $\rho \downarrow 0$  this converges to  $-\delta \text{Tr}(\mathbf{N}\mathbf{\Pi}_{t+1})$ . Applying Hôpital's rule we conclude that  $\gamma_t \rightarrow \delta \text{Tr}(\mathbf{N}\mathbf{\Pi}_{t+1})$ , so that in steady state  $\nu \rightarrow \frac{\delta}{1-\delta} \text{Tr}(\mathbf{N}\mathbf{\Pi})$ .

#### A.6. The $\tilde{\mathcal{L}}$ -Recursion.

Suppose  $\phi(\mathbf{z}) = \delta \mathbf{z}' \mathbf{\Pi} \mathbf{z}$ , so that  $\tilde{\mathcal{L}} \phi(\mathbf{z}) = \max_{\epsilon} [(\mathbf{z} + \epsilon)' \delta \mathbf{\Pi} (\mathbf{z} + \epsilon) - \frac{1}{\rho} \epsilon' \mathbf{N}^{-1} \epsilon]$ . Taking first derivatives, we find that

$$2 \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right) \epsilon + 2 \delta \mathbf{\Pi} \mathbf{z} = 0 \quad \Leftrightarrow \quad \tilde{\epsilon} = - \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} \delta \mathbf{\Pi} \mathbf{z} = -\check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} \mathbf{z},$$

which identifies a maximum for  $\check{\mathbf{\Pi}}$  negative definite. Plugging this formula in the expression for  $\tilde{\mathcal{L}} \phi(\mathbf{z})$ , we find that

$$\begin{aligned} \tilde{\mathcal{L}} \phi(\mathbf{z}) &= -\frac{1}{\rho} \tilde{\epsilon}' \mathbf{N}^{-1} \tilde{\epsilon} + (\mathbf{z} + \tilde{\epsilon})' \delta \mathbf{\Pi} (\mathbf{z} + \tilde{\epsilon}) \\ &= -\frac{1}{\rho} \mathbf{z}' \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \mathbf{N}^{-1} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} \mathbf{z} + \mathbf{z}' (\mathbf{I} - \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi})' \delta \mathbf{\Pi} (\mathbf{I} - \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi}) \mathbf{z}. \end{aligned}$$

Now,

$$\begin{aligned} &-\frac{1}{\rho} \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \mathbf{N}^{-1} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} + (\mathbf{I} - \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi})' \delta \mathbf{\Pi} (\mathbf{I} - \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi}) = \\ &-\frac{1}{\rho} \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \mathbf{N}^{-1} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} + \delta \mathbf{\Pi} - 2 \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} + \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} = \\ &\delta \mathbf{\Pi} - 2 \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} + \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \left[ \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right] \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} = \\ &\delta \mathbf{\Pi} - 2 \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} + \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} = \delta \mathbf{\Pi} - \delta \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} = \delta \mathbf{\Pi} [\mathbf{I} - \check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi}]. \end{aligned}$$

Consider that

$$\begin{aligned}\delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} &= \delta \frac{1}{\rho} \mathbf{N}^{-1} \rho \mathbf{N} \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1}, \text{ so that} \\ \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} &= (\delta \rho \mathbf{N} \mathbf{\Pi} - \mathbf{I})^{-1} \rho \mathbf{N}. \text{ Hence,}\end{aligned}$$

$$\mathbf{I} - \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} \delta \mathbf{\Pi} = \mathbf{I} + (\mathbf{I} - \delta \rho \mathbf{N} \mathbf{\Pi})^{-1} \delta \rho \mathbf{N} \mathbf{\Pi}.$$

Since for  $\mathbf{I} + \mathbf{A}$  invertible  $(\mathbf{I} + \mathbf{A})^{-1} = \mathbf{I} - (\mathbf{I} + \mathbf{A})^{-1} \mathbf{A}$ , we find that

$$\begin{aligned}\mathbf{I} - \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} \delta \mathbf{\Pi} &= (\mathbf{I} - \delta \rho \mathbf{N} \mathbf{\Pi})^{-1} \text{ and hence} \\ \delta \mathbf{\Pi} \left[ \mathbf{I} - \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} \delta \mathbf{\Pi} \right] &= \delta \mathbf{\Pi} (\mathbf{I} - \delta \rho \mathbf{N} \mathbf{\Pi})^{-1}.\end{aligned}$$

Then,

$$\tilde{\mathcal{L}}\phi(\mathbf{z}) = \mathbf{z}' \delta \mathbf{\Pi} [\mathbf{I} - (\mathbf{I} - (\rho \mathbf{N} \delta \mathbf{\Pi})^{-1})^{-1}] \mathbf{z} = \mathbf{z}' \delta \mathbf{\Pi} (\mathbf{I} - \delta \rho \mathbf{N} \mathbf{\Pi})^{-1} \mathbf{z}.$$

For  $\mathbf{\Pi}$  invertible,

$$\delta \mathbf{\Pi} (\mathbf{I} - \rho \mathbf{N} \delta \mathbf{\Pi})^{-1} = \delta \mathbf{\Pi} [(\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N}]^{-1} = ((\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N})^{-1}.$$

We conclude that  $\tilde{\mathcal{L}}\phi(\mathbf{z}) = \mathbf{z}' \tilde{\mathbf{\Pi}} \mathbf{z}$ , where  $\tilde{\mathbf{\Pi}} = ((\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N})^{-1}$  if  $\mathbf{\Pi}$  is invertible and  $\delta \mathbf{\Pi} (\mathbf{I} - \delta \rho \mathbf{N} \mathbf{\Pi})^{-1}$  otherwise.

#### A.7. Second Order Conditions for the $\tilde{\mathcal{L}}$ -Recursion.

Consider that the second order condition for the maximization in the  $\tilde{\mathcal{L}}$ -recursion is that  $\delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1}$  being negative definite. Now, as this is a symmetric matrix, there exists a coordinate transformation which diagonalizes it. This matrix will be negative definite iff all its eigenvalues are negative, or equivalently iff its elements on the main diagonal are negative, suggesting that is possible to proceed as in the scalar case. Hence, suppose that  $\mathbf{\Pi}$  is invertible. The condition  $\delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} < 0$  is equivalent to  $(\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N} > 0$ , as the elements on the main diagonal of the former matrix will be negative iff those on the latter are positive, or equivalently the former matrix is negative definite iff the latter is positive definite. We therefore establish that a solution to the  $\tilde{\mathcal{L}}$ -recursion exists if and only if  $\tilde{\mathbf{\Pi}}$ , the inverse of  $(\delta \mathbf{\Pi})^{-1} - \rho \mathbf{N}$ , is positive definite.

#### A.8. Second Order Conditions for the $\mathcal{L}$ -Recursion.

Suppose that  $\mathbf{\Pi}$  is positive semi-definite and  $\mathbf{Q}$  and  $\mathbf{R}$  are positive definite. This will be true if the cost function  $c$  is positive definite in  $\mathbf{u}$  and  $\mathbf{z}$  (that  $\mathbf{Q}$  and  $\mathbf{R}$  are positive definite when the cost function is a positive definite quadratic form is obvious; that in this case  $\mathbf{\Pi}$  is also positive semi-definite will be

shown below). Assume also that the condition in Theorem 2 for the DLEQG problem to have a proper solution holds, so that  $\tilde{\Pi}$  is positive definite.

In solving the  $\mathcal{L}$ -recursion, standard result shows that the control is  $\mathbf{u} = (\mathbf{Q} + \mathbf{B}'\tilde{\Pi}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}'\tilde{\Pi}\mathbf{A})$ . As  $\mathbf{Q}$  and  $\tilde{\Pi}$  are positive definite a minimum is certainly reached since the denominator in the expression for the optimal control is also positive definite.

Suppose that in  $t+1$   $\Pi_{t+1}$  is positive semi-definite, while  $\tilde{\Pi}_{t+1}$  is positive definite. At time  $t$ , plugging the optimal control vector into the  $\mathcal{L}$ -recursion, standard algebra shows that  $\mathcal{L}\phi(\mathbf{z}_t) = \mathbf{z}_t'\Pi_t\mathbf{z}_t$ , where

$$\Pi_t = \mathbf{R} + \mathbf{A}'\tilde{\Pi}_{t+1}\mathbf{A} - \left(\mathbf{S}' + \mathbf{A}'\tilde{\Pi}_{t+1}\mathbf{B}\right) \left(\mathbf{Q} + \mathbf{B}'\tilde{\Pi}_{t+1}\mathbf{B}\right)^{-1} \left(\mathbf{S} + \mathbf{B}'\tilde{\Pi}_{t+1}\mathbf{A}\right).$$

Now, consider that

$$\mathcal{L}\phi(\mathbf{z}_t) = \min_{\mathbf{u}_t} [c(\mathbf{z}_t, \mathbf{u}_t) + (\mathbf{A}\mathbf{z}_t + \mathbf{B}\mathbf{u}_t)'\tilde{\Pi}_{t+1}(\mathbf{A}\mathbf{z}_t + \mathbf{B}\mathbf{u}_t)].$$

Since the cost function  $c$  is a positive definite quadratic form in  $\mathbf{u}_t$  and  $\mathbf{z}_t$  and  $\tilde{\Pi}_{t+1}$  is positive definite,  $\mathcal{L}\phi(\mathbf{z}_t)$  is non-negative and therefore  $\Pi_t$  must be positive semi-definite. As in  $T$   $\Pi_T$  is equal to  $\mathbf{0}$ , by induction we prove that at any time  $t$  the  $\mathcal{L}$ -recursion has the solution discussed in the proof of Theorem 2, as the second order condition of the minimization is always respected, while the matrix  $\Pi_t$  is positive semi-definite.

That the cost function  $c$  is a positive definite quadratic form in  $\mathbf{u}_t$  and  $\mathbf{z}_t$  is a *sufficient* condition for the DLEQG problem to have the recursive solution presented in Theorem 2, but it is not *necessary*. If this assumption is abandoned, it will be necessary to verify that the matrix  $\mathbf{Q} + \mathbf{B}'\tilde{\Pi}_{t+1}\mathbf{B}$  is positive definite at any  $t$ .

#### A.9. The Coefficient $\rho$ and the Relative Risk-aversion.

Using results from Tallarini (Tallarini, 2000), we have seen (A.3) that the risk-enhancement coefficient is

$$\rho = 2 \left( \frac{1}{\delta} - 1 \right) (\gamma - 1).$$

This value is larger than zero if  $\gamma > 1 = 1/\eta$ , i.e. if the coefficient of relative risk-aversion is larger than the inverse of the inter-temporal elasticity of substitution in Epstein and Zin's recursive preferences. In other words, a positive risk-enhancement coefficient is equivalent to the condition that the coefficient of relative risk-aversion is larger than the inverse of the inter-temporal elasticity of substitution.

#### A.10. The Coefficient $\rho$ and Early Resolution of Uncertainty.

Kreps and Porteus (Kreps and Porteus, 1978) note that when the relative-risk aversion is greater than the inverse of the inter-temporal elasticity of substitution, i.e. for  $\gamma > 1/\eta$ , preferences favor early resolution of uncertainty. In fact, for  $\gamma = 1/\eta$  (or equivalently  $\rho = 0$ ) Epstein and Zin's recursive preferences become linear, so that the utility function assumes the familiar time-separable form, while the value func-

tion solves the standard Bellman's equation from dynamic programming,  $\mathbf{V}_t = \min_{\mathbf{u}_t} \{c_t + \delta E_t [\mathbf{V}_{t+1}]\}$ . We have seen (A.5) that in our specification for  $\rho > 0$   $\gamma > 1$ , while  $1 = 1/\eta$ . This means that, applying Kreps and Porteus's argument, our recursive optimization induces earlier resolution of uncertainty *vis-a-vis* the case of the time-separable quadratic form.

#### A.11. The Recursive Optimization with Deterministic Disturbances to the Plant Equation.

Assume that  $\mathbf{V}_{t+1} = \mathbf{z}'_{t+1} \mathbf{\Pi}_{t+1} \mathbf{z}'_{t+1} - 2\mathbf{v}'_{t+1} \mathbf{z}_{t+1} + \nu_{t+1}$ . Then, consider

$$\exp\left(\frac{\rho}{2} \mathbf{V}_t\right) = \min_{\mathbf{u}_t} E \left[ \exp\left(\frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1})\right) \right].$$

Given that  $\mathbf{z}_{t+1}$  is a linear function of  $\epsilon_{t+1}$ , this expression is equal to

$$\exp\left(\frac{\rho}{2} \mathbf{V}_t\right) = \min_{\mathbf{u}_t} \left\{ \frac{(2\pi)^{-n/2}}{\det(\mathbf{N})^{1/2}} \int \left[ \exp\left(\frac{\rho}{2} (c_t + \delta \mathbf{V}_{t+1}) - \frac{1}{2} \epsilon'_{t+1} \mathbf{N}^{-1} \epsilon_{t+1}\right) \right] d\epsilon_{t+1} \right\}.$$

Given the expression for  $\mathbf{V}_{t+1}$ , the argument inside the exponential can be written as  $\frac{\rho}{2} \kappa_t + \mathcal{Q}(\epsilon_{t+1})$ , where  $\kappa_t$  contains terms independent of  $\epsilon_{t+1}$ , while  $\mathcal{Q}(\epsilon_{t+1}) = -\frac{1}{2} \epsilon'_{t+1} \mathbf{Q}_{\epsilon, \epsilon} \epsilon_{t+1} + \mathbf{Q}'_{\epsilon} \epsilon_{t+1}$  with

$$\mathbf{Q}_{\epsilon, \epsilon} = (\mathbf{I} - \delta \rho \mathbf{\Pi}_{t+1} \mathbf{N}) \mathbf{N}^{-1} \quad \text{and} \quad \mathbf{Q}_{\epsilon} = \delta \rho \mathbf{\Pi}_{t+1} (\mathbf{A} \mathbf{z}_t + \mathbf{B} \mathbf{u}_t) - \delta \rho \mathbf{v}_{t+1}.$$

For  $\epsilon_{t+1} = \hat{\epsilon}_{t+1} = \mathbf{Q}_{\epsilon, \epsilon}^{-1} \mathbf{Q}_{\epsilon}$ ,  $\mathcal{Q}(\epsilon_{t+1})$  has a maximum equal to  $\mathcal{Q}(\hat{\epsilon}_{t+1}) = \frac{1}{2} \mathbf{Q}'_{\epsilon} \mathbf{Q}_{\epsilon, \epsilon}^{-1} \mathbf{Q}_{\epsilon}$ . Moreover, and more importantly, even if  $\mathcal{Q}$  is not an homogeneous quadratic form, it is immediate to see that

$$\mathcal{Q}(\epsilon_{t+1}) = \max_{\epsilon_{t+1}} \mathcal{Q}(\epsilon_{t+1}) - \frac{1}{2} (\epsilon_{t+1} - \hat{\epsilon}_{t+1})' \mathbf{Q}_{\epsilon, \epsilon} (\epsilon_{t+1} - \hat{\epsilon}_{t+1}).$$

In fact,  $-\frac{1}{2} (\epsilon_{t+1} - \hat{\epsilon}_{t+1})' \mathbf{Q}_{\epsilon, \epsilon} (\epsilon_{t+1} - \hat{\epsilon}_{t+1}) = -\frac{1}{2} \epsilon'_{t+1} \mathbf{Q}_{\epsilon, \epsilon} \epsilon_{t+1} - \frac{1}{2} \hat{\epsilon}'_{t+1} \mathbf{Q}_{\epsilon, \epsilon} \hat{\epsilon}_{t+1} + \hat{\epsilon}'_{t+1} \mathbf{Q}_{\epsilon, \epsilon} \epsilon_{t+1}$ . Substituting  $\hat{\epsilon}_{t+1}$  with  $\mathbf{Q}_{\epsilon, \epsilon}^{-1} \mathbf{Q}_{\epsilon}$ , we find  $-\frac{1}{2} \epsilon'_{t+1} \mathbf{Q}_{\epsilon, \epsilon} \epsilon_{t+1} + \mathbf{Q}'_{\epsilon} \epsilon_{t+1} - \frac{1}{2} \mathbf{Q}'_{\epsilon} \mathbf{Q}_{\epsilon, \epsilon}^{-1} \mathbf{Q}_{\epsilon} = \mathcal{Q}(\epsilon_{t+1}) - \max_{\epsilon_{t+1}} \mathcal{Q}(\epsilon_{t+1})$ . Then,

$$\begin{aligned} \exp\left(\frac{\rho}{2} \mathbf{V}_t\right) &= \frac{(2\pi)^{-n/2}}{\det(\mathbf{N})^{1/2}} \min_{\mathbf{u}_t} \left\{ \int \left[ \exp\left(\frac{\rho}{2} \kappa_t + \max_{\epsilon_{t+1}} \mathcal{Q}(\epsilon_{t+1}) - \frac{1}{2} (\epsilon_{t+1} - \hat{\epsilon}_{t+1})' \mathbf{Q}_{\epsilon, \epsilon} (\epsilon_{t+1} - \hat{\epsilon}_{t+1})\right) \right] d\epsilon_{t+1} \right\} \\ &= \frac{(2\pi)^{-n/2}}{\det(\mathbf{N})^{1/2}} \min_{\mathbf{u}_t} \left\{ \exp\left(\frac{\rho}{2} \kappa_t + \max_{\epsilon_{t+1}} \mathcal{Q}(\epsilon_{t+1})\right) \int \left[ \exp\left(-\frac{1}{2} \Delta' \mathbf{Q}_{\epsilon, \epsilon} \Delta\right) \right] d\Delta \right\}, \end{aligned}$$

where  $\Delta = \epsilon_{t+1} - \hat{\epsilon}_{t+1}$ . This implies that

$$\exp\left(\frac{\rho}{2} \mathbf{V}_t\right) = \frac{(2\pi)^{-n/2}}{\det(\mathbf{N})^{1/2}} \frac{(2\pi)^{n/2}}{\det(\mathbf{Q}_{\epsilon, \epsilon})^{1/2}} \min_{\mathbf{u}_t} \left\{ \exp\left(\frac{\rho}{2} \kappa_t + \max_{\epsilon_{t+1}} \mathcal{Q}(\epsilon_{t+1})\right) \right\}$$

Considering that  $\det(\mathbf{Q}_{\epsilon, \epsilon}) = \det(\mathbf{I} - \delta \rho \mathbf{\Pi}_{t+1} \mathbf{N}) / \det(\mathbf{N})$ , we find that

$$\begin{aligned} \exp\left(\frac{\rho}{2} \mathbf{V}_t\right) &= \det(\mathbf{I} - \delta \rho \mathbf{\Pi}_{t+1} \mathbf{N})^{-1/2} \exp\left(\min_{\mathbf{u}_t} \left(\frac{\rho}{2} \kappa_t + \max_{\epsilon_{t+1}} \mathcal{Q}(\epsilon_{t+1})\right)\right) \\ &= \det(\mathbf{I} - \delta \rho \mathbf{\Pi}_{t+1} \mathbf{N})^{-1/2} \exp\left(\min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \left(\frac{\rho}{2} \kappa_t + \mathcal{Q}(\epsilon_{t+1})\right)\right). \end{aligned}$$

It is immediate to see that, given the definitions of  $\mathcal{Q}(\epsilon_{t+1})$  and  $\mathcal{S}_t$ ,

$$\begin{aligned} \frac{\rho}{2}\kappa_t + \mathcal{Q}(\epsilon_{t+1}) &= \frac{\rho}{2} \left( c_t + \delta \mathbf{z}'_{t+1} \mathbf{\Pi}_{t+1} \mathbf{z}'_{t+1} - 2\delta \mathbf{v}'_{t+1} \mathbf{z}_{t+1} - \frac{1}{\rho} \epsilon'_{t+1} \mathbf{N}^{-1} \epsilon_{t+1} + \delta \nu_{t+1} \right) \\ &= \frac{\rho}{2} (\mathcal{S}_t + \delta \nu_{t+1}) \quad \text{and hence even in this case} \\ \exp\left(\frac{\rho}{2} \mathbf{V}_t\right) &= \det(\mathbf{I} - \delta \rho \mathbf{\Pi}_{t+1} \mathbf{N})^{-1/2} \exp\left(\frac{\rho}{2} \left( \min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathcal{S}_t + \delta \nu_{t+1} \right)\right). \end{aligned}$$

This implies that as in the homogenous case (see **A.4** and **A.5**)

$$\mathbf{V}_t = \gamma_t + \delta \nu_{t+1} + \min_{\mathbf{u}_t} \max_{\epsilon_{t+1}} \mathcal{S}_t, \quad \text{where } \gamma_t = -\frac{1}{\rho} \log(\det(\mathbf{I} - \delta \rho \mathbf{\Pi}_{t+1} \mathbf{N})).$$

### A.12. The $\tilde{\mathcal{L}}$ -Recursion with Deterministic Disturbances to the Plant Equation.

Suppose  $\phi(\mathbf{z}) = \delta \mathbf{z}' \mathbf{\Pi} \mathbf{z} - 2\delta \mathbf{v}' \mathbf{z} + \delta \nu$ , so that  $\tilde{\mathcal{L}}\phi(\mathbf{z}) = \max_{\epsilon} [(z + \epsilon)' \delta \mathbf{\Pi} (z + \epsilon) - 2\delta \mathbf{v}' (z + \epsilon) + \delta \nu - \frac{1}{\rho} \epsilon' \mathbf{N}^{-1} \epsilon]$ . Taking first derivatives, we find that

$$2 \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right) \epsilon + 2\delta (\mathbf{\Pi} \mathbf{z} - \mathbf{v}) = 0 \Leftrightarrow \tilde{\epsilon} = - \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} \delta \mathbf{\Pi} \mathbf{z} + \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} \delta \mathbf{v},$$

ie.  $\tilde{\epsilon} = \tilde{\epsilon}_o + \tilde{\epsilon}_e$ , with  $\tilde{\epsilon}_o = -\check{\mathbf{\Pi}}^{-1} \delta \mathbf{\Pi} \mathbf{z}$  and  $\tilde{\epsilon}_e = \check{\mathbf{\Pi}}^{-1} \delta \mathbf{v}$ . Plugging this formula in the expression for  $\tilde{\mathcal{L}}\phi(\mathbf{z})$ , we find that

$$\begin{aligned} \tilde{\mathcal{L}}\phi(\mathbf{z}) &= -\frac{1}{\rho} (\tilde{\epsilon}_o + \tilde{\epsilon}_e)' \mathbf{N}^{-1} (\tilde{\epsilon}_o + \tilde{\epsilon}_e) + (\mathbf{z} + \tilde{\epsilon}_o + \tilde{\epsilon}_e)' \delta \mathbf{\Pi} (\mathbf{z} + \tilde{\epsilon}_o + \tilde{\epsilon}_e) - 2\delta \mathbf{v}' (\mathbf{z} + \tilde{\epsilon}_o + \tilde{\epsilon}_e) + \delta \nu \\ &= \underbrace{-\frac{1}{\rho} \check{\epsilon}'_o \mathbf{N}^{-1} \check{\epsilon}_o + (\mathbf{z} + \check{\epsilon}_o)' \delta \mathbf{\Pi} (\mathbf{z} + \check{\epsilon}_o)}_{\mathbf{z}' \tilde{\mathbf{\Pi}} \mathbf{z}} + \underbrace{-\frac{1}{\rho} \check{\epsilon}'_e \mathbf{N}^{-1} \check{\epsilon}_e + \check{\epsilon}'_e \delta \mathbf{\Pi} \check{\epsilon}_e - 2\delta \mathbf{v}' \check{\epsilon}_e + \delta \nu}_{\text{independent of } \mathbf{z}} \\ &\quad \underbrace{-\frac{2}{\rho} \check{\epsilon}'_e \mathbf{N}^{-1} \check{\epsilon}_o + 2\check{\epsilon}'_e \delta \mathbf{\Pi} (\mathbf{z} + \check{\epsilon}_o) - 2\delta \mathbf{v}' (\mathbf{z} + \check{\epsilon}_o)}_{\text{linear in } \mathbf{z}}. \end{aligned}$$

The terms tagged as “ $\mathbf{z}' \tilde{\mathbf{\Pi}} \mathbf{z}$ ” correspond to the function  $\tilde{\mathcal{L}}\phi(\mathbf{z})$  of the homogenous formulation (See **A.6**), so that we concentrate on those tagged as “independent of  $\mathbf{z}$ ” and “linear  $\mathbf{z}$ ”. Thus, starting from the terms “independent of  $\mathbf{z}$ ”, consider that, as  $\tilde{\epsilon}_e = \check{\mathbf{\Pi}}^{-1} \delta \mathbf{v}$ , the sum of the first three can be written as

$$\begin{aligned} \delta^2 \mathbf{v}' \left( -\frac{1}{\rho} \check{\mathbf{\Pi}}^{-1} \mathbf{N}^{-1} \check{\mathbf{\Pi}}^{-1} + \delta \check{\mathbf{\Pi}}^{-1} \mathbf{\Pi} \check{\mathbf{\Pi}}^{-1} - 2\check{\mathbf{\Pi}}^{-1} \right) \mathbf{v} &= \\ \delta^2 \mathbf{v}' \left( \check{\mathbf{\Pi}}^{-1} \left( \delta \mathbf{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right) \check{\mathbf{\Pi}}^{-1} - 2\check{\mathbf{\Pi}}^{-1} \right) \mathbf{v} &= -\delta^2 \mathbf{v}' \check{\mathbf{\Pi}}^{-1} \mathbf{v}. \end{aligned}$$

This implies that sum of the terms “independent of  $\mathbf{z}$ ” is equal to a constant  $\tilde{\nu}$ , where

$$\tilde{\nu} = \delta \nu - \delta^2 \boldsymbol{\vartheta}' \left( \delta \boldsymbol{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)^{-1} \boldsymbol{\vartheta} = \delta \nu + \delta^2 \rho \boldsymbol{\vartheta}' \mathbf{N} (\mathbf{I} - \delta \rho \boldsymbol{\Pi} \mathbf{N})^{-1} \boldsymbol{\vartheta}.$$

The terms “linear in  $\mathbf{z}$ ” can be re-written as follows

$$-2 \left[ \frac{1}{\rho} \tilde{\boldsymbol{\epsilon}}_e' \mathbf{N}^{-1} - \tilde{\boldsymbol{\epsilon}}_e' \delta \boldsymbol{\Pi} + \delta \boldsymbol{\vartheta}' \right] (\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o) + \frac{2}{\rho} \tilde{\boldsymbol{\epsilon}}_e' \mathbf{N}^{-1} \mathbf{z}.$$

Given the expression for  $\tilde{\boldsymbol{\epsilon}}_e$  we find that the sum in the squared brackets is equal to

$$\begin{aligned} & -2 \left[ \frac{1}{\rho} \delta \boldsymbol{\vartheta}' \check{\boldsymbol{\Pi}}^{-1} \mathbf{N}^{-1} - \delta \boldsymbol{\vartheta}' \check{\boldsymbol{\Pi}}^{-1} \delta \boldsymbol{\Pi} + \delta \boldsymbol{\vartheta}' \right] (\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o) \\ &= -2 \delta \boldsymbol{\vartheta}' \left[ \mathbf{I} - \check{\boldsymbol{\Pi}}^{-1} \underbrace{\left( \delta \boldsymbol{\Pi} - \frac{1}{\rho} \mathbf{N}^{-1} \right)}_{\check{\boldsymbol{\Pi}}} \right] (\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o) = 0, \text{ while} \\ & \frac{2}{\rho} \tilde{\boldsymbol{\epsilon}}_e' \mathbf{N}^{-1} \mathbf{z} = 2 \delta \boldsymbol{\vartheta}' \check{\boldsymbol{\Pi}}^{-1} (\rho \mathbf{N})^{-1} \mathbf{z}. \end{aligned}$$

which we can write as  $-2\tilde{\boldsymbol{\vartheta}}' \mathbf{z}$ , with  $-\tilde{\boldsymbol{\vartheta}} = \delta (-\rho \mathbf{N})^{-1} \check{\boldsymbol{\Pi}}^{-1} (-\boldsymbol{\vartheta})$ . Now,

$$\begin{aligned} (-\rho \mathbf{N})^{-1} \check{\boldsymbol{\Pi}}^{-1} &= \delta (-\rho \mathbf{N})^{-1} (\delta \boldsymbol{\Pi} - (\rho \mathbf{N})^{-1})^{-1} \\ &= (-\rho \mathbf{N})^{-1} [(\mathbf{I} + \delta \boldsymbol{\Pi} (-\rho \mathbf{N})) (-\rho \mathbf{N})^{-1}]^{-1} \\ &= (\mathbf{I} - \delta \rho \boldsymbol{\Pi} \mathbf{N})^{-1}. \end{aligned}$$

Then, we conclude that  $\mathcal{L}\phi(\mathbf{z}) = \mathbf{z}' \tilde{\boldsymbol{\Pi}} \mathbf{z} - 2\tilde{\boldsymbol{\vartheta}}' \mathbf{z} + \tilde{\nu}$ , with  $\tilde{\boldsymbol{\vartheta}} = \delta (-\rho \mathbf{N})^{-1} \check{\boldsymbol{\Pi}}^{-1} \boldsymbol{\vartheta} = \delta (\mathbf{I} - \delta \rho \boldsymbol{\Pi} \mathbf{N})^{-1} \boldsymbol{\vartheta}$  and  $\tilde{\nu} = \delta \nu + \delta^2 \rho \boldsymbol{\vartheta}' \mathbf{N} (\mathbf{I} - \delta \rho \boldsymbol{\Pi} \mathbf{N})^{-1} \boldsymbol{\vartheta}$ . Notice that for  $\boldsymbol{\Pi}$  invertible we can also write

$$\begin{aligned} \tilde{\boldsymbol{\vartheta}} &= \delta ((\delta \boldsymbol{\Pi})^{-1} - \rho \mathbf{N})^{-1} (\delta \boldsymbol{\Pi})^{-1} \boldsymbol{\vartheta} \\ &= ((\delta \boldsymbol{\Pi})^{-1} - \rho \mathbf{N})^{-1} \boldsymbol{\Pi}^{-1} \boldsymbol{\vartheta} = \tilde{\boldsymbol{\Pi}} \boldsymbol{\Pi}^{-1} \boldsymbol{\vartheta}. \end{aligned}$$

### A.13. The $\mathcal{L}$ -Recursion with Deterministic Disturbances to the Plant Equation.

Suppose that  $\phi(\mathbf{z}) = \mathbf{z}' \boldsymbol{\Pi} \mathbf{z} - 2\boldsymbol{\vartheta}' \mathbf{z} + \nu$ . Then, consider that

$$\begin{aligned} \mathcal{L}\phi(\mathbf{z}) &= \min_{\mathbf{u}} [c(\mathbf{z}, \mathbf{u}) + (\mathbf{A} \mathbf{z} + \mathbf{B} \mathbf{u} + \boldsymbol{\mu})' \boldsymbol{\Pi} (\mathbf{A} \mathbf{z} + \mathbf{B} \mathbf{u} + \boldsymbol{\mu}) - 2\boldsymbol{\vartheta}' (\mathbf{A} \mathbf{z} + \mathbf{B} \mathbf{u} + \boldsymbol{\mu}) + \nu] \text{ where} \\ c(\mathbf{z}, \mathbf{u}) &= \mathbf{u}' \mathbf{Q} \mathbf{u} + 2\mathbf{u}' \mathbf{S} \mathbf{z} + \mathbf{z}' \mathbf{R} \mathbf{z}. \end{aligned}$$

The first order condition is

$$(\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})\mathbf{u} + (\mathbf{S} + \mathbf{B}\mathbf{\Pi}\mathbf{A})\mathbf{z} + \mathbf{B}'(\mathbf{\Pi}\boldsymbol{\mu} - \boldsymbol{\vartheta}) = \mathbf{0},$$

so that the optimal control is  $\mathbf{u}_o + \mathbf{u}_e$  where

$$\begin{aligned}\mathbf{u}_o &= \mathbf{K}_o\mathbf{z} = -(\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}\mathbf{\Pi}\mathbf{A})\mathbf{z}, \\ \mathbf{u}_e &= \mathbf{K}_e(\mathbf{\Pi}\boldsymbol{\mu} - \boldsymbol{\vartheta}) = -(\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})^{-1}\mathbf{B}'(\mathbf{\Pi}\boldsymbol{\mu} - \boldsymbol{\vartheta}).\end{aligned}$$

Plugging the expressions for  $\mathbf{u}_o$  and  $\mathbf{u}_e$  into  $\mathcal{L}\phi(\mathbf{z})$ , we find that

$$\begin{aligned}\mathbf{z}'\mathbf{\Pi}_{-1}\mathbf{z} - 2\boldsymbol{\vartheta}'_{-1}\mathbf{z} + \nu_{-1} &= \mathbf{z}'\mathbf{R}\mathbf{z} + (\mathbf{u}_o + \mathbf{u}_e)'\mathbf{Q}(\mathbf{u}_o + \mathbf{u}_e) + 2\mathbf{z}'\mathbf{S}'(\mathbf{u}_o + \mathbf{u}_e) + \\ &[\mathbf{A}\mathbf{z} + \mathbf{B}(\mathbf{u}_o + \mathbf{u}_e) + \boldsymbol{\mu}]'\mathbf{\Pi}[\mathbf{A}\mathbf{z} + \mathbf{B}(\mathbf{u}_o + \mathbf{u}_e) + \boldsymbol{\mu}] - 2\boldsymbol{\vartheta}'[\mathbf{A}\mathbf{z} + \mathbf{B}(\mathbf{u}_o + \mathbf{u}_e) + \boldsymbol{\mu}] + \nu = \\ &\underbrace{\mathbf{z}'\mathbf{R}\mathbf{z} + \mathbf{u}'_o\mathbf{Q}\mathbf{u}_o + 2\mathbf{z}'\mathbf{S}'\mathbf{u}_o + (\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}_o)'\mathbf{\Pi}(\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}_o)}_{\text{quadratic in } \mathbf{z}} + \\ &\underbrace{2\mathbf{u}'_e[(\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})\mathbf{u}_o + 2(\mathbf{S} + \mathbf{B}'\mathbf{\Pi}\mathbf{A})\mathbf{z}] + 2(\mathbf{\Pi}\boldsymbol{\mu} - \boldsymbol{\vartheta})'(\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}_o)}_{\text{linear in } \mathbf{z}} + \\ &\underbrace{\mathbf{u}'_e\mathbf{Q}\mathbf{u}_e + (\mathbf{B}\mathbf{u}_e + \boldsymbol{\mu})'\mathbf{\Pi}(\mathbf{B}\mathbf{u}_e + \boldsymbol{\mu}) - 2\boldsymbol{\vartheta}'(\mathbf{B}\mathbf{u}_e + \boldsymbol{\mu}) + \nu}_{\text{independent of } \mathbf{z}}.\end{aligned}$$

The terms in the right hand side of this equation tagged as “quadratic in  $\mathbf{z}$ ” will correspond to  $\mathbf{z}'\mathbf{\Pi}_{-1}\mathbf{z}$ , the terms tagged as “linear in  $\mathbf{z}$ ” will correspond to  $-2\boldsymbol{\vartheta}'_{-1}\mathbf{z}$ , while those tagged “independent of  $\mathbf{z}$ ” will correspond to  $\nu_{-1}$ . In particular, as

$$\mathbf{z}'\mathbf{\Pi}_{-1}\mathbf{z} = \mathbf{z}'\mathbf{R}\mathbf{z} + \mathbf{u}'_o\mathbf{Q}\mathbf{u}_o + 2\mathbf{z}'\mathbf{S}'\mathbf{u}_o + [\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}_o]'\mathbf{\Pi}[\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}_o]$$

corresponds to the Riccati recursion of the homogenous LQG framework, standard results indicate that

$$\mathbf{\Pi}_{-1} = \mathbf{R} + \mathbf{A}'\mathbf{\Pi}\mathbf{A} - (\mathbf{S}' + \mathbf{A}'\mathbf{\Pi}\mathbf{B})(\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}'\mathbf{\Pi}\mathbf{A}).$$

Consider hence the sum of the terms “linear in  $\mathbf{z}$ ”, corresponding to  $-2\boldsymbol{\vartheta}'_{-1}\mathbf{z}$ . Given that  $\mathbf{u}_o = \mathbf{K}\mathbf{z}$  and that  $\mathbf{K} = (\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}'\mathbf{\Pi}\mathbf{A})$ , we see that the first term in parentheses is null so that

$$-2\boldsymbol{\vartheta}'_{-1}\mathbf{z} = 2(\mathbf{\Pi}\boldsymbol{\mu} - \boldsymbol{\vartheta})'\mathbf{\Gamma}\mathbf{z}, \text{ i.e. } \boldsymbol{\vartheta}_{-1} = \mathbf{\Gamma}'(\boldsymbol{\vartheta} - \mathbf{\Pi}\boldsymbol{\mu}) \text{ with } \mathbf{\Gamma} = \mathbf{A} + \mathbf{B}\mathbf{K}.$$

Consider hence the sum of the terms “independent of  $\mathbf{z}$ ”, corresponding to  $\nu_{-1}$ . We can write  $\mathbf{u}_e = \mathbf{K}_\mu\boldsymbol{\mu} + \mathbf{K}_\vartheta\boldsymbol{\vartheta}$ , where  $\mathbf{K}_\mu = -(\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})^{-1}\mathbf{B}'\mathbf{\Pi}$  and  $\mathbf{K}_\vartheta = (\mathbf{Q} + \mathbf{B}'\mathbf{\Pi}\mathbf{B})^{-1}\mathbf{B}'$ , and  $\mathbf{B}\mathbf{u}_e + \boldsymbol{\mu} =$

$(\mathbf{I} + \mathbf{BK}_\mu)\boldsymbol{\mu} + \mathbf{BK}_\vartheta\boldsymbol{\vartheta}$ . Substituting out  $\mathbf{u}_e$  and  $\mathbf{Bu}_e + \boldsymbol{\mu}$  and regrouping terms, one finds that

$$\begin{aligned}\nu_{-1} &= \boldsymbol{\mu}' \left( \mathbf{K}'_\mu (\mathbf{Q} + \mathbf{B}'\boldsymbol{\Pi}\mathbf{B})\mathbf{K}_\mu + \boldsymbol{\Pi} + 2\mathbf{K}'_\mu \mathbf{B}'\boldsymbol{\Pi} \right) \boldsymbol{\mu} + \\ &\quad \boldsymbol{\vartheta}' \left( \mathbf{K}'_\vartheta (\mathbf{Q} + \mathbf{B}'\boldsymbol{\Pi}\mathbf{B})\mathbf{K}_\vartheta - 2\mathbf{K}'_\vartheta \mathbf{B}' \right) \boldsymbol{\vartheta} + \\ &\quad 2\boldsymbol{\vartheta}' \left( \mathbf{K}'_\vartheta (\mathbf{Q} + \mathbf{B}'\boldsymbol{\Pi}\mathbf{B})\mathbf{K}_\mu + \mathbf{K}'_\vartheta \mathbf{B}'\boldsymbol{\Pi} - \mathbf{I} - \mathbf{BK}_\mu \right) \boldsymbol{\mu} + \nu.\end{aligned}$$

Substituting out  $\mathbf{K}_\mu$  and  $\mathbf{K}_\vartheta$ , we can show that the three terms in parentheses are equal respectively to  $(\mathbf{I} - \mathbf{BJ}^{-1}\mathbf{B}')\boldsymbol{\Pi}$ ,  $-\mathbf{BJ}^{-1}\mathbf{B}'$  and  $-(\mathbf{I} - \mathbf{BJ}^{-1}\mathbf{B}'\boldsymbol{\Pi})$ , where  $\mathbf{J} = (\mathbf{Q} + \mathbf{B}'\boldsymbol{\Pi}\mathbf{B})$ . This implies that

$$\nu_{-1} = \boldsymbol{\mu}'(\mathbf{I} - \mathbf{BJ}^{-1}\mathbf{B}')\boldsymbol{\Pi}\boldsymbol{\mu} - 2\boldsymbol{\vartheta}'(\mathbf{I} - \mathbf{BJ}^{-1}\mathbf{B}'\boldsymbol{\Pi})\boldsymbol{\mu} - \boldsymbol{\vartheta}'\mathbf{BJ}^{-1}\mathbf{B}'\boldsymbol{\vartheta} + \nu.$$

#### A.14. The Value Function with Deterministic Disturbances to the Plant Equation.

We just combine results in **A.11**, **A.12** and **A.13**. From **A.11** we know that if  $\mathcal{V}_{t+1} = \mathbf{z}'_{t+1}\boldsymbol{\Pi}_{t+1}\mathbf{z}'_{t+1} - 2\boldsymbol{\vartheta}'_{t+1}\mathbf{z}_{t+1} + \nu_{t+1}$ . Then

$$\mathcal{V}_t = \gamma_t + \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \left( c_t + \delta \mathbf{z}'_{t+1} \boldsymbol{\Pi}_{t+1} \mathbf{z}'_{t+1} - 2\delta \boldsymbol{\vartheta}'_{t+1} \mathbf{z}_{t+1} - \frac{1}{\rho} \boldsymbol{\epsilon}'_{t+1} \mathbf{N}^{-1} \boldsymbol{\epsilon}_{t+1} + \delta \nu_{t+1} \right),$$

with  $\gamma_t = -\frac{1}{\rho} \log(\det(\mathbf{I} - \delta\rho\boldsymbol{\Pi}_{t+1}\mathbf{N}))$ . From **A.12** we know that

$$\max_{\boldsymbol{\epsilon}_{t+1}} \left( c_t + \delta \mathbf{z}'_{t+1} \boldsymbol{\Pi}_{t+1} \mathbf{z}'_{t+1} - 2\delta \boldsymbol{\vartheta}'_{t+1} \mathbf{z}_{t+1} - \frac{1}{\rho} \boldsymbol{\epsilon}'_{t+1} \mathbf{N}^{-1} \boldsymbol{\epsilon}_{t+1} + \delta \nu_{t+1} \right) = \mathbf{z}'_{t+1} \tilde{\boldsymbol{\Pi}}_{t+1} \mathbf{z}'_{t+1} - 2\tilde{\boldsymbol{\vartheta}}'_{t+1} \mathbf{z}_{t+1} + \tilde{\nu}_{t+1},$$

where

$$\begin{aligned}\tilde{\boldsymbol{\Pi}}_{t+1} &= ((\delta\boldsymbol{\Pi}_{t+1})^{-1} - \rho\mathbf{N})^{-1}, \\ \tilde{\boldsymbol{\vartheta}}_{t+1} &= ((\delta\boldsymbol{\Pi}_{t+1})^{-1} - \rho\mathbf{N})^{-1} \boldsymbol{\Pi}_{t+1}^{-1} \boldsymbol{\vartheta}_{t+1} = \delta(\mathbf{I} - \delta\rho\boldsymbol{\Pi}_{t+1}\mathbf{N})^{-1} \boldsymbol{\vartheta}_{t+1}, \\ \tilde{\nu}_{t+1} &= \delta\nu_{t+1} + \delta^2_{t+1} \rho \boldsymbol{\vartheta}'_{t+1} \mathbf{N} (\mathbf{I} - \delta\rho\boldsymbol{\Pi}_{t+1}\mathbf{N})^{-1} \boldsymbol{\vartheta}_{t+1}.\end{aligned}$$

Then, from **A.13** we find that  $\mathcal{V}_t = \mathbf{z}'_t \boldsymbol{\Pi}_t \mathbf{z}'_t - 2\boldsymbol{\vartheta}'_t \mathbf{z}_t + \nu_t$ , where

$$\begin{aligned}\boldsymbol{\Pi}_t &= \mathbf{R} + \mathbf{A}' \tilde{\boldsymbol{\Pi}}_{t+1} \mathbf{A} - (\mathbf{S}' + \mathbf{A}' \tilde{\boldsymbol{\Pi}}_{t+1} \mathbf{B})(\mathbf{Q} + \mathbf{B}' \tilde{\boldsymbol{\Pi}}_{t+1} \mathbf{B})^{-1} (\mathbf{S} + \mathbf{B}' \tilde{\boldsymbol{\Pi}}_{t+1} \mathbf{A}), \\ \boldsymbol{\vartheta}_t &= \boldsymbol{\Gamma}'_t (\tilde{\boldsymbol{\vartheta}}_{t+1} - \tilde{\boldsymbol{\Pi}}_{t+1} \boldsymbol{\mu}_{t+1}), \\ \nu_t &= \gamma_t + \tilde{\nu}_{t+1} + \boldsymbol{\mu}'_{t+1} (\mathbf{I} - \mathbf{B} \tilde{\mathbf{J}}_{t+1}^{-1} \mathbf{B}') \tilde{\boldsymbol{\Pi}}_{t+1} \boldsymbol{\mu}_{t+1} - 2\tilde{\boldsymbol{\vartheta}}'_{t+1} (\mathbf{I} - \mathbf{B} \tilde{\mathbf{J}}_{t+1}^{-1} \mathbf{B}' \tilde{\boldsymbol{\Pi}}_{t+1}) \boldsymbol{\mu}_{t+1} - \tilde{\boldsymbol{\vartheta}}'_{t+1} \mathbf{B} \tilde{\mathbf{J}}_{t+1}^{-1} \mathbf{B}' \tilde{\boldsymbol{\vartheta}}_{t+1}\end{aligned}$$

and  $\tilde{\mathbf{J}}_{t+1} = (\mathbf{Q} + \mathbf{B}' \tilde{\boldsymbol{\Pi}}_{t+1} \mathbf{B})$ .

## B. The Optimal Monetary Policy for a Pessimistic Central Bank

### B.1. Optimal Monetary Policy.

In the stationary solution,

$$\begin{aligned}
\tilde{\Pi} &= ((\delta \Pi)^{-1} - \rho \mathbf{N})^{-1} = \delta \Pi (\mathbf{I}_2 - \delta \rho \mathbf{N} \Pi)^{-1} \\
&= \delta \Pi \begin{pmatrix} 1 - \delta \rho \sigma_\pi^2 \pi_1 & -\delta \rho \sigma_\pi^2 \pi_{1,2} \\ -\delta \rho \sigma_y^2 \pi_{1,2} & 1 - \delta \rho \sigma_y^2 \pi_2 \end{pmatrix}^{-1} \\
&= \frac{\delta}{\det(\mathbf{I}_2 - \delta \rho \mathbf{N} \Pi)} \begin{pmatrix} \pi_1 & \pi_{1,2} \\ \pi_{1,2} & \pi_2 \end{pmatrix} \begin{pmatrix} 1 - \delta \rho \sigma_y^2 \pi_2 & \delta \rho \sigma_\pi^2 \pi_{1,2} \\ \delta \rho \sigma_y^2 \pi_{1,2} & 1 - \delta \rho \sigma_\pi^2 \pi_1 \end{pmatrix} \\
&= \frac{\delta}{\det(\mathbf{I}_2 - \delta \rho \mathbf{N} \Pi)} \begin{pmatrix} (1 - \delta \rho \sigma_y^2 \pi_2) \pi_1 + \delta \rho \sigma_y^2 \pi_{1,2}^2 & \pi_{1,2} \\ \pi_{1,2} & (1 - \delta \rho \sigma_\pi^2 \pi_1) \pi_2 + \delta \rho \sigma_\pi^2 \pi_{1,2}^2 \end{pmatrix} \\
&= \frac{\delta}{\det(\mathbf{I}_2 - \delta \rho \mathbf{N} \Pi)} \hat{\Pi}.
\end{aligned}$$

where

$$\det(\mathbf{I}_2 - \delta \rho \mathbf{N} \Pi) = 1 - \delta \rho (\sigma_\pi^2 \pi_1 + \sigma_y^2 \pi_2) + \delta^2 \rho^2 \det(\Pi) \sigma_\pi^2 \sigma_y^2.$$

It is immediate to check that  $\mathbf{B}' \hat{\Pi} \mathbf{B} = \gamma^2 \hat{\pi}_2$ , so that

$$(\mathbf{B}' \tilde{\Pi} \mathbf{B})^{-1} = \frac{1}{\delta} \frac{1}{\gamma^2} \frac{1}{\hat{\pi}_2} \det(\mathbf{I}_2 - \delta \rho \mathbf{N} \Pi), \quad \mathbf{B}(\mathbf{B}' \tilde{\Pi} \mathbf{B})^{-1} \mathbf{B}' = \det(\mathbf{I}_2 - \delta \rho \mathbf{N} \Pi) \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\delta} \frac{1}{\hat{\pi}_2} \end{pmatrix}.$$

So,

$$\mathbf{B}(\mathbf{B}' \tilde{\Pi} \mathbf{B})^{-1} \mathbf{B}' \tilde{\Pi} = \begin{pmatrix} 0 & 0 \\ \frac{1}{\delta} \frac{\hat{\pi}_{1,2}}{\hat{\pi}_2} & 1 \end{pmatrix}, \quad \mathbf{I}_2 - \mathbf{B}(\mathbf{B}' \tilde{\Pi} \mathbf{B})^{-1} \mathbf{B}' \tilde{\Pi} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\delta} \frac{\hat{\pi}_{1,2}}{\hat{\pi}_2} & 0 \end{pmatrix}.$$

In the modified Riccati equation we have

$$\begin{aligned}
\mathbf{\Pi} &= \mathbf{R} + \mathbf{A}'\tilde{\mathbf{\Pi}} \left( \mathbf{I}_2 - \mathbf{B}(\mathbf{B}'\tilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\tilde{\mathbf{\Pi}} \right) \mathbf{A} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} + \frac{\delta}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\mathbf{\Pi})} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \hat{\pi}_1 & \hat{\pi}_{1,2} \\ \hat{\pi}_{1,2} & \hat{\pi}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\hat{\pi}_{1,2}}{\hat{\pi}_2} & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} + \frac{\delta}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\mathbf{\Pi})} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \frac{\det(\hat{\mathbf{\Pi}})}{\hat{\pi}_2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} + \delta \frac{\det(\hat{\mathbf{\Pi}})}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\mathbf{\Pi})} \frac{1}{\hat{\pi}_2} \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{pmatrix}.
\end{aligned}$$

We can define  $W = \frac{1}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\mathbf{\Pi})} \left( \hat{\pi}_1 - \frac{\hat{\pi}_{1,2}^2}{\hat{\pi}_2} \right)$  and conclude that

$$\pi_1 = 1 + \delta W, \quad \pi_{1,2} = \alpha \delta W, \quad \pi_2 = \lambda + \alpha^2 \delta W.$$

Now,

$$\begin{aligned}
\hat{\pi}_1 - \frac{\hat{\pi}_{1,2}^2}{\hat{\pi}_2} &= \pi_1 - \delta\rho \det(\mathbf{\Pi}) \sigma_y^2 - \frac{\pi_{1,2}^2}{\pi_2 - \delta\rho \det(\mathbf{\Pi}) \sigma_\pi^2} \\
&= \frac{(\pi_1 - \delta\rho \det(\mathbf{\Pi}) \sigma_y^2) (\pi_2 - \delta\rho \det(\mathbf{\Pi}) \sigma_\pi^2) - \pi_{1,2}^2}{\pi_2 - \delta\rho \det(\mathbf{\Pi}) \sigma_\pi^2} \\
&= \frac{\det(\mathbf{\Pi}) [1 - \delta\rho(\sigma_\pi^2 \pi_1 + \sigma_y^2 \pi_2) + \delta^2 \rho^2 \det(\mathbf{\Pi}) \sigma_\pi^2 \sigma_y^2]}{\pi_2 - \delta\rho \det(\mathbf{\Pi}) \sigma_\pi^2} \\
&= \frac{\det(\mathbf{\Pi}) \det(\mathbf{I}_2 - \delta\rho\mathbf{N}\mathbf{\Pi})}{\pi_2 - \delta\rho \det(\mathbf{\Pi}) \sigma_\pi^2},
\end{aligned}$$

so that  $W = \frac{\det(\mathbf{\Pi})}{\pi_2 - \delta\rho \det(\mathbf{\Pi}) \sigma_\pi^2}$ . Given the expressions for  $\pi_1$ ,  $\pi_{1,2}$  and  $\pi_2$ , we have that  $\det(\mathbf{\Pi}) = \lambda + \delta(\alpha^2 + \lambda)W$ , so that

$$W = \frac{\lambda + \delta(\alpha^2 + \lambda)W}{\lambda(1 - \delta\rho\sigma_\pi^2) + \delta(\alpha^2 - \delta(\alpha^2 + \lambda)\rho\sigma_\pi^2)W}.$$

Rearranging we find that

$$\delta \left( \alpha^2 - \delta(\alpha^2 + \lambda)\rho\sigma_\pi^2 \right) W^2 - \left( \delta(\alpha^2 + \lambda) - \lambda + \delta\lambda\rho\sigma_\pi^2 \right) W - \lambda = 0$$

whose roots are

$$W_{\pm} = \frac{\delta(\alpha^2 + \lambda) - \lambda(1 - \delta\rho\sigma_{\pi}^2) \pm \Delta^{1/2}}{2\delta(\alpha^2 - \delta(\alpha^2 + \lambda)\rho\sigma_{\pi}^2)}, \quad \text{where}$$

$\Delta = \left(\delta(\alpha^2 + \lambda) - \lambda(1 - \delta\rho\sigma_{\pi}^2)\right)^2 + 4\delta\lambda(\alpha^2 - \delta(\alpha^2 + \lambda)\rho\sigma_{\pi}^2)$ . For  $\rho = 0$ ,  $\Delta = \left(\delta\alpha^2 - (1 - \delta)\lambda\right)^2 + 4\alpha^2\delta\lambda$ , while

$$W_{\pm} = \frac{1}{2} \left(1 - \frac{(1 - \delta)\lambda \pm \Delta^{1/2}}{\alpha^2\delta}\right) = \frac{1}{2} \left(1 - \frac{(1 - \delta)\lambda}{\alpha^2\delta} \pm \sqrt{\left(1 + \frac{(1 - \delta)}{\alpha^2\delta}\right)^2 + \frac{4\lambda}{\alpha^2}}\right).$$

Only the positive root will be coherent with the conditions that the matrix  $\tilde{\mathbf{\Pi}}$  is positive definite. This means that there is no indeterminacy in the stationary solution. To determine  $\mathbf{K}$  consider that

$$\begin{aligned} \mathbf{B}'\tilde{\mathbf{\Pi}}\mathbf{A} &= \frac{\delta}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\mathbf{\Pi})} (0 \quad -\gamma) \begin{pmatrix} \hat{\pi}_1 & \hat{\pi}_{1,2} \\ \hat{\pi}_{1,2} & \hat{\pi}_2 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\ &= -\frac{\delta\gamma}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\mathbf{\Pi})} (\hat{\pi}_{1,2} \quad \alpha\hat{\pi}_{1,2} + \beta\hat{\pi}_2). \end{aligned}$$

Given that  $\mathbf{K} = -(\mathbf{B}'\tilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\tilde{\mathbf{\Pi}}\mathbf{A}$ , we find that

$$\mathbf{K} = \frac{1}{\gamma} \begin{pmatrix} \hat{\pi}_{1,2} & \alpha\hat{\pi}_{1,2} + \beta \\ \hat{\pi}_2 & \end{pmatrix}.$$

Finally, since  $\hat{\pi}_{1,2} = \pi_{1,2} = \alpha\delta W$  and  $\hat{\pi}_2 = \pi_2 - \delta\det(\mathbf{\Pi})\rho\sigma_{\pi}^2 = \lambda + \alpha^2\delta W - \delta(\lambda + \delta(\alpha^2 + \lambda)W)\rho\sigma_{\pi}^2$ , we find that

$$\mathbf{K} = \frac{1}{\gamma} \left( \frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \delta(\lambda + \delta(\alpha^2 + \lambda)W)\rho\sigma_{\pi}^2} \quad \beta + \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \delta(\lambda + \delta(\alpha^2 + \lambda)W)\rho\sigma_{\pi}^2} \right).$$

To reach a minimum  $\delta\mathbf{\Pi}_{t+1} - (1/\rho)\mathbf{N}^{-1}$  must be negative definite. This corresponds to the double condition that

$$\delta\pi_1 - \frac{1}{\rho} \frac{1}{\sigma_{\pi}^2} < 0, \quad \left(\delta\pi_1 - \frac{1}{\rho} \frac{1}{\sigma_{\pi}^2}\right) \left(\delta\pi_2 - \frac{1}{\rho} \frac{1}{\sigma_y^2}\right) - \delta^2 \pi_{1,2} > 0.$$

## B.2. The Value Function and the Inflation Forecast.

Given the plant equation for  $\pi_t$  we immediately see that  $\pi_{t+1|t} = \pi_t + \alpha y_t$ . Then, consider that

$$\begin{aligned}
\mathbf{z}'_t \mathbf{\Pi} \mathbf{z}_t &= (\pi_t \ y_t) \begin{pmatrix} 1 + \delta W & \alpha \delta W \\ \alpha + \delta W & \lambda + \alpha^2 \delta W \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= (\pi_t \ y_t) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + (\pi_t \ y_t) \begin{pmatrix} \delta W & \alpha \delta W \\ \alpha \delta W & \alpha^2 \delta W \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= \pi_t^2 + \lambda y_t^2 + \delta W (\pi_t \ y_t) \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \begin{pmatrix} 1 & \alpha \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= \pi_t^2 + \lambda y_t^2 + \delta W (\pi_t + \alpha y_t)^2.
\end{aligned}$$

Immediately it follows that

$$\exp\left(\frac{\rho}{2} \mathbf{v}_t\right) = \exp\left(\frac{\rho}{2} [\nu + \pi_t^2 + \lambda y_t^2 + \delta W \pi_{t+1|t}^2]\right).$$

## B.3. Inflation and Output Gap Forecast.

Since  $\pi_{t+1|t} = \pi_t + \alpha y_t$ , we find that

$$r_t = \frac{1}{\gamma} \left( \beta y_t + \frac{\alpha \delta W}{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2} \pi_{t+1|t} \right)$$

Inserting this into the plant equation for output gap, we find that

$$y_{t+1|t} = - \frac{\alpha \delta W}{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2} \pi_{t+1|t}.$$

Since  $\pi_{t+2|t} = \pi_{t+1|t} + \alpha y_{t+1|t}$  and  $\pi_{t+1|t} = - \frac{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2}{\alpha \delta W} y_{t+1|t}$ , we conclude that

$$\pi_{t+2|t} = - \frac{1}{\alpha \delta W} (\lambda - \theta \rho \sigma_\pi^2) y_{t+1|t} = \left( \frac{\lambda - \theta \rho \sigma_\pi^2}{\alpha^2 \delta W + \lambda - \theta \rho \sigma_\pi^2} \right) \pi_{t+1|t}.$$

#### B.4. Unconditional Variance of Inflation, Output Gap and Short-term Interest Rate.

By definition, considering that  $\kappa_y = \beta/\gamma + \alpha\kappa_\pi$ ,

$$\begin{aligned}\Gamma &= \mathbf{A} + \mathbf{B}\mathbf{K} = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 \\ -\gamma \end{pmatrix} \begin{pmatrix} \kappa_\pi & \frac{\beta}{\gamma} + \alpha\kappa_\pi \end{pmatrix} \\ &= \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\gamma\kappa_\pi & -\beta - \alpha\gamma\kappa_\pi \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -\gamma\kappa_\pi & -\alpha\gamma\kappa_\pi \end{pmatrix}.\end{aligned}$$

So,

$$\mathbf{I}_2 - \Gamma = \begin{pmatrix} 0 & -\alpha \\ \gamma\kappa_\pi & 1 + \alpha\gamma\kappa_\pi \end{pmatrix} \quad \text{so that } \Lambda = (\mathbf{I}_2 - \Gamma)^{-1} = \frac{1}{\alpha\gamma\kappa_\pi} \begin{pmatrix} 1 + \alpha\gamma\kappa_\pi & \alpha \\ -\gamma\kappa_\pi & 0 \end{pmatrix}.$$

Now,  $\text{Var}[\mathbf{z}_t] = \Lambda\mathbf{N}\Lambda'$ . Consider that

$$\Lambda\mathbf{N} = \frac{1}{\alpha\gamma\kappa_\pi} \begin{pmatrix} 1 + \alpha\gamma\kappa_\pi & \alpha \\ -\gamma\kappa_\pi & 0 \end{pmatrix} \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} = \frac{1}{\alpha\gamma\kappa_\pi} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)\sigma_\pi^2 & \alpha\sigma_y^2 \\ -\gamma\kappa_\pi\sigma_\pi^2 & 0 \end{pmatrix}$$

so that

$$\begin{aligned}\Lambda\mathbf{N}\Lambda' &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)\sigma_\pi^2 & \alpha\sigma_y^2 \\ -\gamma\kappa_\pi\sigma_\pi^2 & 0 \end{pmatrix} \begin{pmatrix} 1 + \alpha\gamma\kappa_\pi & -\gamma\kappa_\pi \\ \alpha & 0 \end{pmatrix} \\ &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2\sigma_\pi^2 + \alpha^2\sigma_y^2 & -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\sigma_\pi^2 \\ -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\sigma_\pi^2 & \gamma^2\kappa_\pi^2\sigma_\pi^2 \end{pmatrix},\end{aligned}$$

ie.

$$\text{Var}[\pi_t] = \frac{(1 + \alpha\gamma\kappa_\pi)^2}{(\alpha\gamma\kappa_\pi)^2}\sigma_\pi^2 + \frac{1}{(\gamma\kappa_\pi)^2}\sigma_y^2, \quad \text{Var}[y_t] = \frac{1}{\alpha^2}\sigma_\pi^2.$$

Finally,  $\text{Var}[r_t] = \mathbf{K}\text{Var}[\mathbf{z}_t]\mathbf{K}'$ . Consider that

$$\begin{aligned}\text{Var}[\mathbf{z}_t]\mathbf{K}' &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2\sigma_\pi^2 + \alpha^2\sigma_y^2 & -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\sigma_\pi^2 \\ -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\sigma_\pi^2 & \gamma^2\kappa_\pi^2\sigma_\pi^2 \end{pmatrix} \begin{pmatrix} \kappa_\pi \\ \frac{\beta}{\gamma} + \alpha\kappa_\pi \end{pmatrix} \\ &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2\kappa_\pi\sigma_\pi^2 + \alpha^2\kappa_\pi\sigma_y^2 - \gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\left(\frac{\beta}{\gamma} + \alpha\kappa_\pi\right)\sigma_\pi^2 \\ -\gamma\kappa_\pi^2(1 + \alpha\gamma\kappa_\pi)\sigma_\pi^2 + \gamma^2\kappa_\pi^2\left(\frac{\beta}{\gamma} + \alpha\kappa_\pi\right)\sigma_\pi^2 \end{pmatrix},\end{aligned}$$

while

$$\begin{aligned}
\mathbf{K}\text{Var}[\mathbf{z}_t]\mathbf{K}' &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} \kappa_\pi & \frac{\beta}{\gamma} + \alpha\kappa_\pi \end{pmatrix} \times \\
&\quad \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2 \kappa_\pi \sigma_\pi^2 + \alpha^2 \kappa_\pi \sigma_y^2 - \gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi) \left(\frac{\beta}{\gamma} + \alpha\kappa_\pi\right) \sigma_\pi^2 \\ -\gamma\kappa_\pi^2(1 + \alpha\gamma\kappa_\pi) \sigma_\pi^2 + \gamma^2 \kappa_\pi^2 \left(\frac{\beta}{\gamma} + \alpha\kappa_\pi\right) \sigma_\pi^2 \end{pmatrix} \\
&= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \left( \alpha^2 \kappa_\pi^2 \sigma_y^2 + \left[ (1 + \alpha\gamma\kappa_\pi)^2 \kappa_\pi^2 - 2(1 + \alpha\gamma\kappa_\pi) \gamma \kappa_\pi^2 \left(\frac{\beta}{\gamma} + \alpha\kappa_\pi\right) \right. \right. \\
&\quad \left. \left. + \gamma^2 \kappa_\pi^2 \left(\frac{\beta}{\gamma} + \alpha\kappa_\pi\right)^2 \right] \sigma_\pi^2 \right) \\
&= \frac{1}{\gamma^2} \sigma_y^2 + \frac{1}{(\alpha\gamma)^2} \left[ (1 + \alpha\gamma\kappa_\pi) - \gamma \left(\frac{\beta}{\gamma} + \alpha\kappa_\pi\right) \right]^2 \sigma_\pi^2 \\
&= \frac{1}{\gamma^2} \left[ \sigma_y^2 + \left(\frac{1 - \beta}{\alpha}\right)^2 \sigma_\pi^2 \right].
\end{aligned}$$

### B.5. Optimal Monetary Policy with Imperfect State Observation.

In the stationary solution, we find that

$$\check{\mathbf{z}}_t = (\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})^{-1} \hat{\mathbf{z}}_t, \quad \text{where}$$

$$(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})^{-1} = \frac{1}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \begin{pmatrix} 1 - \rho\sigma_y^2 \pi_2 & \rho\sigma_\pi^2 \pi_{1,2} \\ \rho\sigma_y^2 \pi_{1,2} & 1 - \rho\sigma_\pi^2 \pi_1 \end{pmatrix},$$

so that

$$\begin{aligned}
\check{\pi}_t &= \left( \frac{1 - \rho\sigma_y^2 \pi_2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{\pi}_t + \frac{\rho\sigma_\pi^2 \pi_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{y}_t \right), \\
\check{y}_t &= \left( \frac{\rho\sigma_y^2 \pi_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{\pi}_t + \frac{1 - \rho\sigma_\pi^2 \pi_1}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{y}_t \right).
\end{aligned}$$

Given that

$$\begin{aligned}
\frac{1 - \rho\sigma_y^2 \pi_2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} &= 1 + \frac{\pi_1 - \det(\mathbf{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2 - \rho\mathbf{\Pi})} \rho\sigma_\pi^2, \\
\frac{1 - \rho\sigma_\pi^2 \pi_1}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} &= 1 + \frac{\pi_2 - \det(\mathbf{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \rho\sigma_y^2,
\end{aligned}$$

we conclude that the MSE is

$$\begin{aligned}\check{\pi}_t &= \hat{\pi}_t + \left( \frac{\pi_1 - \det(\mathbf{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{\pi}_t + \frac{\pi_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{y}_t \right) \rho\sigma_\pi^2, \\ \check{y}_t &= \hat{y}_t + \left( \frac{\pi_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{\pi}_t + \frac{\pi_2 - \det(\mathbf{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} \hat{y}_t \right) \rho\sigma_y^2.\end{aligned}$$

### B.6. Unconditional Variance of Inflation, Output Gap and Short-term Interest Rate Under Imperfect State Observation.

By definition  $\Psi = \mathbf{B}\mathbf{K}_I$ . As  $\mathbf{B} = (0, -\gamma)$ , we can write

$$\Psi = \begin{pmatrix} 0 & 0 \\ -\gamma\kappa_\pi^I & -\gamma\kappa_y^I \end{pmatrix}.$$

This implies that

$$\mathbf{I}_2 - \Psi = \begin{pmatrix} 1 & 0 \\ \gamma\kappa_\pi^I & 1 + \gamma\kappa_y^I \end{pmatrix} \quad \text{and} \quad (\mathbf{I}_2 - \Psi)^{-1} = \frac{1}{1 + \gamma\kappa_y^I} \begin{pmatrix} 1 + \gamma\kappa_y^I & 0 \\ -\gamma\kappa_\pi^I & 1 \end{pmatrix},$$

while

$$\Phi = (\mathbf{I}_2 - \Psi)^{-1} \mathbf{A} = \frac{1}{1 + \gamma\kappa_y^I} \begin{pmatrix} 1 + \gamma\kappa_y^I & 0 \\ -\gamma\kappa_\pi^I & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \frac{-\gamma\kappa_\pi^I}{1 + \gamma\kappa_y^I} & \frac{\beta - \alpha\gamma\kappa_\pi^I}{1 + \gamma\kappa_y^I} \end{pmatrix}.$$

Consider that  $\Psi\Phi = \Psi(\mathbf{I}_2 - \Psi)^{-1}\mathbf{A}$  and that

$$\mathbf{A} - \Psi\Phi = [\mathbf{I}_2 + \Psi(\mathbf{I}_2 - \Psi)^{-1}]\mathbf{A}.$$

For any square matrix  $\mathbf{M}$ ,

$$\mathbf{I} - \mathbf{M}(\mathbf{I} + \mathbf{M})^{-1} = \mathbf{I} + \mathbf{M}.$$

Taking  $\mathbf{M} = -\Psi$ ,

$$\mathbf{I}_2 + \Psi(\mathbf{I}_2 - \Psi)^{-1} = (\mathbf{I}_2 - \Psi)^{-1}.$$

This implies that

$$\mathbf{I}_2 - \mathbf{A} - \Psi\Phi = \mathbf{I}_2 - (\mathbf{I}_2 - \Psi)^{-1}\mathbf{A} = \mathbf{I}_2 - \Phi.$$

It follows that

$$\mathbf{I}_2 - \mathbf{A} - \Psi\Phi = \frac{1}{1 + \gamma\kappa_y^I} \begin{pmatrix} 0 & -\alpha(1 + \gamma\kappa_y^I) \\ \gamma\kappa_\pi^I & 1 - \beta + \alpha\gamma\kappa_\pi^I + \gamma\kappa_y^I \end{pmatrix}$$

and

$$\Lambda_I = (\mathbf{I}_2 - \mathbf{A} - \Psi \Phi)^{-1} = \begin{pmatrix} 1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I} & \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} \\ -\frac{1}{\alpha} & 0 \end{pmatrix}.$$

We have that

$$\Lambda_I \mathbf{N} = \begin{pmatrix} 1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I} & \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} \\ -\frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I}\right) \sigma_\pi^2 & \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} \sigma_y^2 \\ -\frac{1}{\alpha} \sigma_\pi^2 & 0 \end{pmatrix}$$

and that

$$\begin{aligned} \text{Var}^I[\mathbf{z}_t] &= \Lambda_I \mathbf{N} \Lambda_I' = \begin{pmatrix} \left(1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I}\right) \sigma_\pi^2 & \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} \sigma_y^2 \\ -\frac{1}{\alpha} \sigma_\pi^2 & 0 \end{pmatrix} \begin{pmatrix} 1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I} & -\frac{1}{\alpha} \\ \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \left(1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I}\right)^2 \sigma_\pi^2 + \left(\frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I}\right)^2 \sigma_y^2 & -\frac{1}{\alpha} \left(1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I}\right) \sigma_\pi^2 \\ -\frac{1}{\alpha} \left(1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I}\right) \sigma_\pi^2 & \frac{1}{\alpha^2} \sigma_\pi^2 \end{pmatrix}, \end{aligned}$$

from which we immediately conclude that  $\text{Var}^I[y_t] = \text{Var}[y_t] = (1/\alpha^2)\sigma_\pi^2$ . For the unconditional variance of the short-term interest rate, consider first that

$$\begin{aligned} \Phi \Lambda_I &= \begin{pmatrix} 1 & \alpha \\ \frac{-\gamma \kappa_\pi^I}{1 + \gamma \kappa_y^I} & \frac{\beta - \alpha \gamma \kappa_\pi^I}{1 + \gamma \kappa_y^I} \end{pmatrix} \begin{pmatrix} 1 + \frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I} & \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} \\ -\frac{1}{\alpha} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I}\right) & \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} \\ -\frac{1}{\alpha} & -1 \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{aligned} \mathbf{K}_I \Phi \Lambda_I &= \begin{pmatrix} \kappa_\pi^I & \kappa_y^I \end{pmatrix} \begin{pmatrix} \left(\frac{1 + \gamma \kappa_y^I - \beta}{\alpha \gamma \kappa_\pi^I}\right) & \frac{1 + \gamma \kappa_y^I}{\gamma \kappa_\pi^I} \\ -\frac{1}{\alpha} & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\alpha \gamma} (1 - \beta) & \frac{1}{\gamma} \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbf{K}_I \Phi \Lambda_I \mathbf{N} &= \begin{pmatrix} \frac{1}{\alpha \gamma} (1 - \beta) & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\alpha \gamma} (1 - \beta) \sigma_\pi^2 & \frac{1}{\gamma} \sigma_y^2 \end{pmatrix} \text{ and that} \end{aligned}$$

$$\begin{aligned}
\text{Var}^I[r_t] &= \mathbf{K}_I \Phi \Lambda_I \mathbf{N} \Lambda_I' \Phi' \mathbf{K}_I \\
&= \begin{pmatrix} \frac{1}{\alpha\gamma} (1 - \beta) \sigma_\pi^2 & \frac{1}{\gamma} \sigma_y^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha\gamma} (1 - \beta) \\ \frac{1}{\gamma} \end{pmatrix} \\
&= \frac{1}{(\alpha\gamma)^2} (1 - \beta)^2 \sigma_\pi^2 + \frac{1}{\gamma^2} \sigma_y^2,
\end{aligned}$$

so that  $\text{Var}^I[r_t] = \text{Var}[r_t]$ .

### B.7. Optimal Monetary Policy with a Positive First-best Inflation Rate.

For  $\boldsymbol{\mu}_t = \boldsymbol{\mu}$ ,

$$\boldsymbol{\vartheta}_t = \boldsymbol{\Gamma}'_t \tilde{\boldsymbol{\Pi}}_{t+1} (\boldsymbol{\Pi}_{t+1}^{-1} \boldsymbol{\vartheta}_{t+1} - \boldsymbol{\mu}).$$

In steady state,

$$\begin{aligned}
\boldsymbol{\vartheta} &= \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} (\boldsymbol{\Pi}^{-1} \boldsymbol{\vartheta} - \boldsymbol{\mu}) \\
&= \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\Pi}^{-1} \boldsymbol{\vartheta} - \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\mu}, \quad \text{so that} \\
&= -(\mathbf{I} - \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\Pi}^{-1})^{-1} \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\mu} \\
&= -[(\boldsymbol{\Pi} - \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}}) \boldsymbol{\Pi}^{-1}]^{-1} \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\mu} \\
&= -\boldsymbol{\Pi} (\boldsymbol{\Pi} - \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}})^{-1} \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\mu}.
\end{aligned}$$

For  $\mathbf{Q} = \mathbf{0}$ ,  $u_t = \mathbf{K} \mathbf{z}_t + (\mathbf{B}' \tilde{\boldsymbol{\Pi}} \mathbf{B})^{-1} \mathbf{B}' \tilde{\boldsymbol{\Pi}} (\boldsymbol{\Pi}^{-1} \boldsymbol{\vartheta} - \boldsymbol{\mu})$ , where

$$\begin{aligned}
\boldsymbol{\Pi}^{-1} \boldsymbol{\vartheta} - \boldsymbol{\mu} &= -(\boldsymbol{\Pi} - \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}})^{-1} \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\mu} - \boldsymbol{\mu} \\
&= -[\mathbf{I} - (\boldsymbol{\Pi} - \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}})^{-1} \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}}] \boldsymbol{\mu},
\end{aligned}$$

so that  $u_t = \mathbf{K} \mathbf{z}_t + u_e$ , where  $u_e = -(\mathbf{B}' \tilde{\boldsymbol{\Pi}} \mathbf{B})^{-1} \mathbf{B}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\mu} - (\mathbf{B}' \tilde{\boldsymbol{\Pi}} \mathbf{B})^{-1} \mathbf{B}' \tilde{\boldsymbol{\Pi}} (\boldsymbol{\Pi} - \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}})^{-1} \boldsymbol{\Gamma}' \tilde{\boldsymbol{\Pi}} \boldsymbol{\mu}$ . Now,

$$(\mathbf{B}' \tilde{\boldsymbol{\Pi}} \mathbf{B})^{-1} = \frac{1}{\delta} \frac{1}{\gamma^2} \frac{1}{\hat{\pi}_2} \det(\mathbf{I}_2 - \delta \rho \mathbf{N} \boldsymbol{\Pi}),$$

while

$$\begin{aligned}
\mathbf{B}' \tilde{\boldsymbol{\Pi}} &= (0 \quad -\gamma) \frac{\delta}{\det(\mathbf{I}_2 - \delta \rho \mathbf{N} \boldsymbol{\Pi})} \begin{pmatrix} \hat{\pi}_1 & \hat{\pi}_{1,2} \\ \hat{\pi}_{1,2} & \hat{\pi}_2 \end{pmatrix} \\
&= \frac{\delta}{\det(\mathbf{I}_2 - \delta \rho \mathbf{N} \boldsymbol{\Pi})} \begin{pmatrix} -\gamma \hat{\pi}_{1,2} & -\gamma \hat{\pi}_2 \end{pmatrix},
\end{aligned}$$

so that

$$-(\mathbf{B}'\tilde{\Pi}\mathbf{B})^{-1}\mathbf{B}'\tilde{\Pi} = \frac{1}{\gamma} \begin{pmatrix} \hat{\pi}_{1,2} & \\ \hat{\pi}_2 & 1 \end{pmatrix} = \begin{pmatrix} \kappa_\pi & \frac{1}{\gamma} \end{pmatrix},$$

in that  $\kappa_\pi = \hat{\pi}_{1,2}/(\gamma\hat{\pi}_2)$ . So, for  $\boldsymbol{\mu} = \begin{pmatrix} 0 \\ \gamma\pi^* \end{pmatrix}$ ,  $-(\mathbf{B}'\tilde{\Pi}\mathbf{B})^{-1}\mathbf{B}'\tilde{\Pi}\boldsymbol{\mu} = \pi^*$ . In addition, consider that

$$\begin{aligned} \Gamma'\tilde{\Pi} &= \frac{\delta}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\Pi)} \begin{pmatrix} 1 & -\gamma\kappa_\pi \\ \alpha & -\alpha\gamma\kappa_\pi \end{pmatrix} \begin{pmatrix} \hat{\pi}_1 & \hat{\pi}_{1,2} \\ \hat{\pi}_{1,2} & \hat{\pi}_2 \end{pmatrix} \\ &= \frac{\delta}{\det(\mathbf{I}_2 - \delta\rho\mathbf{N}\Pi)} \begin{pmatrix} \hat{\pi}_1 - \gamma\kappa_\pi\hat{\pi}_{1,2} & \hat{\pi}_{1,2} - \gamma\kappa_\pi\hat{\pi}_2 \\ \alpha(\hat{\pi}_1 - \gamma\kappa_\pi\hat{\pi}_{1,2}) & \alpha(\hat{\pi}_{1,2} - \gamma\kappa_\pi\hat{\pi}_2) \end{pmatrix}. \end{aligned}$$

Since  $\kappa_\pi = \hat{\pi}_{1,2}/(\gamma\hat{\pi}_2)$ ,  $\Gamma'\tilde{\Pi} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ . Then,

$$\Gamma'\tilde{\Pi}\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and hence}$$

$-(\mathbf{B}'\tilde{\Pi}\mathbf{B})^{-1}\mathbf{B}'\tilde{\Pi}(\Pi - \Gamma'\tilde{\Pi})^{-1}\Gamma'\tilde{\Pi}\boldsymbol{\mu} = 0$ . We conclude that  $u_e = \pi^*$  and that  $\boldsymbol{\vartheta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . This means that

$$v_t = \kappa_\pi s_t + \kappa_y y_t + \pi^* \text{ and } F(\mathbf{z}_t) = \mathbf{z}'_t \Pi \mathbf{z}_t.$$

### B.8. ML Estimate for the State Vector $\mathbf{z}_t$ with a Positive First-best Inflation Rate.

Given the plant equation,  $\hat{\mathbf{z}}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}\mathbf{u}_{t-1} + \boldsymbol{\mu}_t$ , where  $\mathbf{z}'_t = (\pi_t - \pi^* \ y_t)$  and  $\boldsymbol{\mu}'_t = (0 \ \gamma\pi^*)$ . Given  $\mathbf{A}$  and  $\mathbf{B}$  we have that

$$\begin{aligned} \widehat{\pi_t - \pi^*} &= \pi_{t-1} - \pi^* + \alpha y_t, \\ \hat{y}_t &= \beta y_{t-1} - \gamma v_{t-1} + \gamma \pi^*. \end{aligned}$$

Since  $v_t = r_t + \pi^*$ , we conclude that

$$\begin{aligned} \hat{\pi}_t &= \pi_{t-1} + \alpha y_t, \\ \hat{y}_t &= \beta y_{t-1} - \gamma r_{t-1}. \end{aligned}$$