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The fundamental theorem of asset pricing with and without transaction costs

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Abstract

We prove a version of the fundamental theorem of asset pricing (FTAP) in continuous time that is based on the strict no-arbitrage condition and that is applicable to both frictionless markets and markets with proportional transaction costs. We consider a market with a single risky asset whose ask price process is higher than or equal to its bid price process. Neither the concatenation property of the set of wealth processes, that is used in the proof of the frictionless FTAP, nor some boundedness property of the trading volume of admissible strategies usually argued within models with a nonvanishing bid–ask spread need to be satisfied in our model.

KEYWORDS

fundamental theorem of asset pricing, no unbounded profit with bounded risk, proportional transaction costs, strict no-arbitrage

1 | INTRODUCTION

In frictionless financial market models with finitely many assets and a single probability measure, the arbitrage theory can be considered to be fully understood in principle. The fundamental theorem of asset pricing (FTAP) by Delbaen and Schachermayer (1998) states that a market model satisfies no free lunch with vanishing risk (NFLVR) iff there exists an equivalent probability measure under which discounted asset prices are σ-martingales. A variant of this theorem by Yan (1998) provides even true martingales by considering a credit line that is a multiple of the sum of all asset prices in the market. For a detailed discussion of the arbitrage theory in frictionless

markets, we refer the reader to Delbaen and Schachermayer (2006) and Eberlein and Kallsen (2019, Subsection 11.7).

The picture for models with proportional transactions costs that generalize frictionless markets by allowing for bid–ask spreads is different. The picture is clear-cut in finite discrete time and for a finite probability space: in a general "currency model," Kabanov and Stricker (2001) show that no-arbitrage (NA) is equivalent to the existence of a so-called consistent price system (CPS), which is a multidimensional martingale under the objective probability measure taking values within the dual of the cone of solvent portfolios at each point in time. In the special case of only one risky asset that we consider in the present paper, a CPS is a pair of an equivalent probability measure and a martingale under this measure that lies between the (discounted) bid and the (discounted) ask price of the risky asset. For infinite probability spaces, this equivalence fails; Schachermayer (2004, Example 3.1) provides an example for an arbitrage-free market, which allows an approximate arbitrage, that is, a nonzero and nonnegative portfolio which is the limit in probability of a sequence of portfolios attainable from zero endowment, and consequently a CPS cannot exist. This raises the obvious question under which stronger no-arbitrage conditions the existence of a CPS can be guaranteed. Schachermayer (2004) introduces the concept of robust no-arbitrage (NA^r)—a no-arbitrage condition which is robust with respect to small changes in the bid–ask spreads. Loosely speaking, if the bid–ask spread (of a pair of assets) does not vanish, there have to exist more favorable bid–ask prices, leading to a smaller spread, such that the modified market still satisfies NA. Schachermayer (2004) shows that NA^r implies that the set of terminal portfolios attainable from zero endowment is closed in probability, and that NA^r is equivalent to the existence of a strictly consistent price system (SCPS), that is, a martingale taking values within the relative interior of the dual of the cone of solvent portfolios at each point in time. For general probability spaces but only one risky asset (in addition to a bank account), it is shown by Grigoriev (2005) that NA already implies the existence of a CPS, although the set of attainable portfolios need not be closed in probability; see also Bayraktar and Zhang (2016) for a different proof that holds in the more general framework of model uncertainty.

Most of the literature on continuous time models is based on the NA^r concept. Guasoni et al. (2010) derive a FTAP for a continuous mid-price process and small deterministic transaction costs. This means that the equivalence between no-arbitrage and the existence of a CPS holds asymptotically for small transaction costs. Guasoni et al. (2012) prove a FTAP under a continuous time extension of NA^r called robust no free lunch with vanishing risk (RNFLVR). The condition states that the bid–ask market has to satisfy NFLVR also for slightly more favorable bid and ask prices. The reduction of the spread is uniformly in time but not uniformly in the scenario. The closedness required for portfolio optimization is shown under similar conditions, see Czichowsky and Schachermayer (2016).

Introduced by Guasoni (2006), another popular sufficient condition to obtain an arbitrage-free model is stickiness. It is formulated in the special case that the investor pays deterministic transaction costs when she buys or sells at a stochastic (mid-)price process. Roughly speaking, the condition states that there are positive probabilities that the mid-price stays in neighborhoods of its starting value. The appeal of the condition is that it is satisfied for many of the stochastic processes usually considered in stochastic modeling (e.g., for fractional Brownian motion). On the other hand, it is of course far away from being necessary.

An alternative to NA^r is the strict no-arbitrage (NA^s) condition introduced by Kabanov et al. (2002). Loosely speaking, a market model satisfies NA^s if any claim which is attainable from zero endowment up to some intermediate time t and can be liquidated in t for sure can also be attained from zero endowment by trading at time t only. Property NA^s alone does not imply the existence

of a CPS, see Schachermayer (2004, Example 3.3) for the existence of an approximate arbitrage under NA^s. For a detailed discussion, we refer to the monograph of Kabanov and Safarian (2009). More recently, Kühn and Molitor (2019) introduced (in discrete time) a variant of NA^s that is called prospective strict no-arbitrage (NA^{ps}). A market model satisfies NA^{ps} if any claim which is attainable from zero endowment by trading up to some intermediate time t and can subsequently be liquidated for sure can also be attained from zero endowment in the subsequent periods (here, "subsequent" is not understood in a strict sense). This means that in contrast to the NA^s criterion, one does not distinguished between a trade that can be realized at time t and a trade from which one knows at time t for sure that it can be realized in the future. NA^{ps} is slightly weaker than NA^r, but it guarantees that the set of terminal portfolios attainable from zero endowment is closed in probability (see Kühn and Molitor, 2019, Theorem 2.6).

The aim of the present paper is to extend Kühn and Molitor (2019) to continuous time. We consider a single risky asset with càdlàg bid and ask price processes that may or may not coincide depending on the scenario and time. An essential preparatory work is the paper by Kühn and Molitor (2022). Under no unbounded profit with bounded risk (NUPBR) for simple strategies, it is shown that there exists a semimartingale price system, that is a semimartingale lying between the bid and ask price processes. Then, Kühn and Molitor (2022) show how these semimartingales can be used to construct gains of general trading strategies that are not necessarily of finite variation. For the continuous time extension of Kühn and Molitor (2019) that we consider in the present paper, it is very natural to merge the NA^{ps} condition with no unbounded profit with bounded risk (NUPBR). The latter is needed for the frictionless FTAP (the combination of NA and NUPBR is equivalent to NFLVR). Very loosely speaking, a market model satisfies the new condition that we call prospective strict no unbounded profit with bounded risk ($NUPBR^{ps}$) if the set of *cost values* of the portfolios that can be liquidated at maximal loss of 1 is bounded in probability. The cost value was introduced by Bayraktar and Yu (2018) as the cost to enter the portfolio position. It is the counterpart of the liquidation value. In the special cases of a discrete time model or a frictionless model, $NUPBR^{ps}$ coincide with NA^{ps} and NUPBR, respectively. As in almost all FTAPs, we prove that the set of attainable terminal portfolios is closed. This property is of independent use.

In plain English, the key idea of our proof is that if a stock position is built up at a time with positive spread (in terms of the "actual" bid and ask prices), there is a positive worst-case risk taken by the investor that cannot be eliminated by smart trading at a later stage. This restricts the amount of shares that can be hold at that time. Of course, the restriction also depends on the current wealth of the investor (whereby we value positions at their cost/purchase price). Once the number of assets is bounded on an interval, a semimartingale price system can be used to show that trading costs cannot explode. The same holds for the total variation of strategies as long as the spread is bounded away from zero. Then, we can apply the stochastic version of Helly's theorem by Campi/Schachermayer, and after passing to forward convex combinations, any sequence of strategies converges pointwise to a finite limit. By contrast, the worst-case risk described above disappears if the spread is zero since then a stock can be liquidated again immediately at the same price as it has been purchased. But, for frictionless intervals, we can apply the (completely different) results used for the proof of the frictionless FTAP by Delbaen and Schachermayer (1998). Here, one directly analyzes the wealth processes (without having a good control over the size of the strategies) and use that the investor can always switch between strategies without transaction costs.

Unfortunately, although the basic intuition described above sounds not too complicated the details are extremely technical. The main difficulties arises from the transition of frictionless periods and periods with friction. This can occur continuously or by jumps that cause different

mathematical difficulties. We nevertheless hope to make the main ideas accessible to a wider readership. The above-mentioned worst-case risk is a new approach to the problem that is also rather different to the arguments used under the RNFLVR condition. The latter use that any transaction leads to costs that are added to the gains coming from a more favorable but still arbitrage-free price system. This means that the starting point is to control the trading volume of the strategies.

The rest of the paper is organized as follows. In Section 2, we introduce the notation and the financial model, discuss the assumptions, and state the main results of the paper (Theorems 2.22 and 2.23). Section 3 is devoted to their proofs. Appendix A consists of auxiliary statements and their proofs that are not directly linked to a financial application.

2 | DEFINITIONS AND MAIN THEOREMS

Throughout the paper, we fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions. The predictable σ-algebra on $\Omega \times [0,T]$ is denoted by $\mathcal{P}$, the set of bounded predictable processes starting at zero by $\mathbf{b}\mathcal{P}$. A stopping time τ is allowed to take the value ∞, but $[\![\tau]\!] := \{(\omega,t) \in \Omega \times [0,T] : t = \tau(\omega)\}$. Especially, we use the notation τ_A, $A \in \mathcal{F}_\tau$, for the stopping time that coincides with τ on A and is infinite otherwise. (In)equalities between stochastic processes are understood "up to evanescence," that is, up to a global P-null set not depending on time. The term $\mathrm{Var}_a^b(X)$ denotes the pathwise variation of a process X on the interval $[a,b]$. In the case that $P(\mathrm{Var}_a^b(X) < \infty) = 1$, $X = X_a + X^\uparrow - X^\downarrow$ is its Jordan–Hahn decomposition into two nondecreasing processes on $[a,b]$ with $X_a^\uparrow = X_a^\downarrow = 0$. A real-valued process X is called làglàd if and only if all paths possess finite left and right limits (but they can have double jumps). We set $\Delta^+X := X_+ - X$ and $\Delta X := \Delta^- X := X - X_-$, where $X_{t+} := \lim_{s\downarrow t} X_s$ and $X_{t-} := \lim_{s\uparrow t} X_s$ (with the convention $X_{T+} := X_T$ and $X_{0-} := X_0$). For a random variable Y, we set $Y^+ := \max(Y,0)$ and $Y^- := \max(-Y,0)$. The standard stochastic integral as defined in Jacod and Shiryaev (2003, Definition III.6.17) is denoted by $\varphi \bullet S$ for $\varphi \in L(S)$ with the convention $\varphi\widetilde{\varphi} \bullet S = (\varphi\widetilde{\varphi}) \bullet S$. It does not cause any ambiguity that by $\varphi \bullet S$ we also denote the integral of an almost simple integrand (cf., e.g., Kühn and Molitor, 2022, Definition 3.15) of the form, for example, $\varphi = 1_{]\!]\tau_1,\tau_2]\!]}$ with respect to a làglàd process S (not necessarily a semimartingale and not even right-continuous). The integral reads $\varphi \bullet S := (S^{\tau_2-} - S^{\tau_1})1_{\{\tau_2>\tau_1\}}$ and analog definitions are canonical (the integral does not allow to "invest" separately in Δ^+S). For làglàd processes X and Y, we define the metric $d_{up}(X,Y) := E(\sup_{t\in[0,T]} |X_t - Y_t| \wedge 1)$ that metrizes the convergence "uniformly in probability." For semimartingales X and Y (that are by definition càdlàg), the Émery metric is defined by $d_{\mathbb{S}}(X,Y) := \sup_{H\in\mathbf{b}\mathcal{P},||H||_\infty\le 1} E(\sup_{t\in[0,T]} |H \bullet (X-Y)_t| \wedge 1)$ that metrizes convergence in the semimartingale topology.

The financial market consists of one risk-free asset or bank account that does not pay interest and one risky asset with bid price $\underline{S}$ and ask price $\overline{S}$ expressed in units of the risk-free asset. We assume that $\underline{S}$ and $\overline{S}$ are adapted càdlàg processes with

$$0 \le \underline{S} \le \overline{S} \quad \text{and} \quad \overline{S}_T > 0. \tag{1}$$

The second condition is made to avoid confusion. Namely, a vanishing ask price would already lead to an arbitrage in the multidimensional sense (see Definition 2.13 below) but not necessarily in the one-dimensional sense, which is considered when the market is frictionless.

Definition 2.1. The *actual bid and ask price processes* are defined as the càdlàg versions of $\underline{X}_t = \operatorname{essinf}_{\mathcal{F}_t} \sup_{u\in[t,T]} \underline{S}_u$ and $\overline{X}_t = \operatorname{esssup}_{\mathcal{F}_t} \inf_{u\in[t,T]} \overline{S}_u$ (whose existence is shown in Proposition A.1 that also provides the precise definition of the conditional essential infimum/supremum). The *actual bid–ask spread* is denoted by $X := \overline{X} - \underline{X}$.

In discrete time transaction costs models, these processes that take certain future trading opportunities into account have already proven to be very useful. For the case of only one risky asset, which we consider in this paper, Sass and Smaga (2014, equation before Lemma 4.1) introduce them by a backward recursion. We leave it as an easy exercise to the reader to prove (by induction on the number of periods) that in (finite) discrete time the processes from Definition 2.1 coincide with the processes in Sass and Smaga (2014). In the general multidimensional Kabanov model, the (discrete time) actual bid and ask price processes correspond to the set-valued processes that are constructed in Rokhlin (2008, p. 95) by a more general backward recursion.

The random variable $\underline{X}_t$ is the highest price at which the investor can liquidate the stock at present or in the future for sure—with the information she has at time t. One has $\underline{X}_t \geq \underline{S}_t$. When the inequality is strict, it is silly to liquidate the position right now. The continuous time counterpart of the NA^{ps} condition from Kühn and Molitor (2019) has to be expressed in terms of $(\underline{X}, \overline{X})$ since it captures trading opportunities in the future.

To work with the processes $\underline{X}$ and $\overline{X}$ is strongly related to freezing a portfolio position as it is done in Guasoni et al. (2012). They start with almost simple strategies and consider associated portfolios that are frozen after a transaction of the original portfolio if a better liquidation price can be achieved for sure in the future. By introducing $\underline{X}$ and $\overline{X}$, we can directly work with general strategies that shortens the proofs. The relation to Guasoni et al. (2012) is discussed further after Definition 2.13.

We decided to pass already now to the actual bid and ask price processes because in the spirit of Guasoni et al. (2012), the admissibility condition has anyhow to be expressed in terms of these processes, and also a meaningful definition of a "frictionless interval" is in terms of the actual bid and ask prices $\underline{X}, \overline{X}$ rather than in terms of $\underline{S}, \overline{S}$ themselves. But, for the motivation of the admissibility condition, we keep the original processes $\underline{S}$ and $\overline{S}$ in mind. We note that the "actual actual bid and ask prices" are just the actual bid and ask prices. For the price processes that are commonly considered, we have that $\underline{X} = \underline{S}$ and $\overline{X} = \overline{S}$.

The arbitrage theory in continuous time frictionless markets is based on the set of general trading strategies, that is, on the set of predictable processes $L(S)$ which are integrable against the semimartingale S modeling the stock price. This set was generalized to models with transaction costs beyond efficient friction by Kühn and Molitor (2022). As in frictionless markets, but in contrast to models with efficient friction, the set contains strategies of infinite variation. Accordingly, the present paper is based on this set and the corresponding self-financing condition. We only outline the definitions that are needed to follow the present paper. **For the rest of the paper, we assume that there exists a semimartingale price system**, that is, a semimartingale S such that

$$\underline{X} \leq S \leq \overline{X}. \tag{2}$$

Remark 2.2. Of course, Equation (2) includes the condition that $\underline{X} \leq \overline{X}$. This condition is violated in a market with $\underline{S} = \overline{S}$ and $\underline{S}$ generated by an unfavorable doubling strategy such that $\underline{S}_0 = 2$ and $\underline{S}_T = 1$. Then, one has $\underline{X} = \underline{S} > \overline{X} = 1$ on a set that is not evanescent. But, the model

is only arbitrage-free with the bank account as numéraire. In the present paper, we work with a numéraire-free no-arbitrage condition that rules out such price processes and perfectly fits to actual bid and ask price processes.

Under a mild NUPBR-condition for simple strategies in the bid–ask model, it is shown in Kühn and Molitor (2022, Theorem 2.7) that a semimartingale price system exists. To motivate Equation (2), we can apply this theorem to $\underline{X}, \overline{X}$. The assumptions we need in the present paper to establish a FTAP are stronger—even if they were only required for simple strategies. Thus, Equation (2) is no further restriction.

For the extension of the self-financing condition to general strategies, also the following assumption is needed.

Assumption 2.3. For every $(\omega, t) \in \Omega \times [0,T)$ with $\overline{X}_t(\omega) = \underline{X}_t(\omega)$, there exists an $\varepsilon > 0$ such that $\overline{X}_s(\omega) = \underline{X}_s(\omega)$ for all $s \in (t, (t+\varepsilon) \wedge T)$ or $\overline{X}_s(\omega) > \underline{X}_s(\omega)$, $\overline{X}_{s-}(\omega) > \underline{X}_{s-}(\omega)$ for all $s \in (t, (t+\varepsilon) \wedge T)$.

This means that each zero of the path $t \mapsto \overline{X}_t(\omega) - \underline{X}_t(\omega)$ is either an inner point from the right of the zero set or a starting point of an excursion away from zero. We note that there starts no excursion of the càdlàg function $f(t) := \sum_{n=1}^{\infty}(2^{-n} - t)1_{(2^{-(n+1)} \le t < 2^{-n})}$ at $t = 0$ since there are zeros of the left limit. Assumption 2.3 incorporating left limits is already needed in the proof of Kühn and Molitor (2022, Lemma 5.1) to ensure that the constructed excursion has a positive length.

Under Assumption 2.3, it is shown in Kühn and Molitor (2022, Lemma 5.1 and Lemma 5.2) that there exist sequences of stopping times $(\tau_1^i)_{i\in\mathbb{N}}$ and $(\sigma_1^i)_{i\in\mathbb{N}}$ that exhaust the set of starting times of the excursions of the spread away from zero and the set of starting times of the frictionless intervals, respectively. Hence, there is a decomposition

$$\begin{aligned}
&\Omega \times [0,T] = \cup_{i\in\mathbb{N}}(\mathcal{I}_i^c \cup \mathcal{I}_i^{fc}) \quad \text{up to evanescence, where}\\
&\mathcal{I}_i^c := [\![(\tau_1^i)_{\{X_{\tau_1^i}=0\}}, \Gamma(\tau_1^i)[\![\setminus [\![(\Gamma(\tau_1^i))_{\{X_{\Gamma(\tau_1^i)-}=0\}}[\![\in \mathcal{P},\\
&\mathcal{I}_i^{fc} := [\![(\sigma_1^i)_{\{X_{\sigma_1^i-}=0\}}[\![\cup [\![\sigma_1^i, \Lambda(\sigma_1^i)[\![\\
&\qquad \cup([\![(\Lambda(\sigma_1^i))_{\{X_{\Lambda(\sigma_1^i)}>0\}}, \Gamma(\Lambda(\sigma_1^i))[\![\setminus [\![(\Gamma(\Lambda(\sigma_1^i)))_{\{X_{\Gamma(\Lambda(\sigma_1^i))-}=0\}}[\![) \in \mathcal{P},\ i \in \mathbb{N}, \qquad (3)
\end{aligned}$$

and the stochastic intervals can be chosen such that they are disjoint. Here, $\Gamma(\tau_1^i) := \inf\{t > \tau_1^i : X_t = 0 \text{ or } X_{t-} = 0\}$ denotes the end time of the excursion starting in τ_1^i, and $\Lambda(\sigma_1^i) := \inf\{t \ge \sigma_1^i : \exists \varepsilon > 0\ \forall s \in (t, (t+\varepsilon) \wedge T)\ X_s > 0\}$ is the starting time of the next excursion after σ_1^i. The interval $\mathcal{I}_i^c$ is with "costs," and the interval $\mathcal{I}_i^{fc}$ starts with a "frictionless" period that is followed by a period with "costs" iff at the end of the frictionless interval, the spread is already positive. By construction, the investor can rebalance her portfolio at the boundaries of $\mathcal{I}_i^c$ and $\mathcal{I}_i^{cf}$ without costs (since the spread can jump away from zero at an *unpredictable* stopping time, this would not be the case if a frictionless period was not sometimes followed by a period with costs). For $X_{\tau_1^i} > 0$, there must exist a frictionless forerunner (possibly only consisting of the single point τ_1^i). Then, the excursion is included in some I_j^{fc}. *For notational convenience, we fix the sequences in Equation* (3) *for the rest of the paper, but we stress that the definitions below do not depend on their choice.*

For a $\varphi \in \mathbf{b}\mathcal{P}$, specifying the number of risky assets the investor holds in her portfolio, Kühn and Molitor (2022, equation before (3.9)) constructed the corresponding self-financing position in the bank account by the $[-\infty, \infty)$-valued predictable process

$$\Pi(\varphi) := \varphi^+ \bullet S - \varphi^- \bullet S' - C^S(\varphi^+) - C^{S'}(-\varphi^-) - \varphi^+ S + \varphi^- S'. \tag{4}$$

Here, S and S' are arbitrary semimartingale price systems. The nondecreasing processes C^S and $C^{S'}$ model accumulated costs that occur when trades are executed at the less favorable bid and ask prices but positions are evaluated with S and S', respectively. The key discovery was that $\Pi(\varphi)$ does not depend on the choice of the semimartingale price system (see Kühn and Molitor, 2022, Corollary 3.22). We consider the construction directly for the actual bid and ask prices $\underline{X}, \overline{X}$ and not for $\underline{S}, \overline{S}$ as in Kühn and Molitor (2022). We apply the cost term C that is $[0, \infty]$-valued separately to φ^+ and $-\varphi^-$ and set $C^{S,S'}(\varphi) := C^S(\varphi^+) + C^{S'}(-\varphi^-)$. This definition makes sense: if one applied C to $\varphi \in \mathbf{b}\mathcal{P}$ with only one semimartingale, one would obtain $C(\varphi^+) + C(-\varphi^-)$ (see Kühn and Molitor, 2022, Step 3 in the proof of Theorem 4.5). The intuition for this is that φ^+ and $-\varphi^-$ never trade in the opposite direction. Thus, executing them through different trading accounts does not yield higher costs. Although the choice of the semimartingale is irrelevant for bounded strategies, we provide more flexibility for the extension to unbounded strategies by allowing different semimartingales for long and short positions. Since in Kühn and Molitor (2022) $S' = S$, one has to check that the proof of Kühn and Molitor (2022, Corollary 3.22) still holds for different semimartingale price systems for the positive and the negative part. But this obviously follows by the above-mentioned decomposition of the costs into $C(\varphi^+)$ and $C(-\varphi^-)$. The process

$$V^{S,S'}(\varphi) := \Pi(\varphi) + \varphi^+ S - \varphi^- S' = \varphi^+ \bullet S - \varphi^- \bullet S' - C^S(\varphi^+) - C^{S'}(-\varphi^-)$$

is the wealth process if long positions are evaluated by S and short positions by S'. The wealth process $V^{S,S'}$ is written as a function of φ only since the increments of the bank account φ^0 result from the self-financing condition. This direct relation exists only for the *increments* of φ and φ^0, which is why the initial values are set to zero. They would have to be modeled separately. With this background information, one can follow the present paper to a large extent without knowing the details of Kühn and Molitor (2022). For a complete understanding, the reader is referred to the construction of the cost term in Kühn and Molitor (2022, Subsections 3.1 and 3.2).

Definition 2.4. Let S be a càdlàg process. The lower and upper predictable envelopes of S, denoted by $\operatorname{essinf}_{\mathcal{F}_-} S$ and $\operatorname{esssup}_{\mathcal{F}_-} S$, are defined as the unique predictable processes such that $(\operatorname{essinf}_{\mathcal{F}_-} S)_\tau = \operatorname{essinf}_{\mathcal{F}_{\tau-}} S_\tau$ and $(\operatorname{esssup}_{\mathcal{F}_-} S)_\tau = \operatorname{esssup}_{\mathcal{F}_{\tau-}} S_\tau$ a.s., respectively, for all predictable stopping times τ (the definition makes sense by Proposition A.2).

Predictable envelopes are needed for a consistent valuation of portfolio positions *after* the portfolio is rebalanced at some time t but *before* prices move at t. Looking at both a semimartingale price system S and its envelope can be seen as a splitting of time. In frictionless markets, these envelopes are not needed since the wealth only changes due to price movements but not due to portfolio rebalancing. The example below that is adapted from Larsson (Preprint) shows that S_- generally does not do the job well enough since S_t can exceed S_{t-} for sure.

Example 2.5 (Example 2.11 in Larsson, Preprint). Let $t_1 \in (0,T)$ and B be a standard Brownian motion. Consider the bid price process $\underline{S}_t = (|B_{t_1}| + B_t - B_{t_1})1_{[t_1,T]}(t)$ that coincides with its actual bid price process, that is, $\underline{X} = \underline{S}$. Nevertheless, $\lim_{t\uparrow t_1} \underline{X}_t$ is zero and differs from $\operatorname{essinf}_{\mathcal{F}_{t_1-}} \underline{X}_{t_1} = |B_{t_1}|$.

If $\underline{X}$ were a frictionless price process, there would be an arbitrage, but as part of a bid–ask model, it can make perfect sense. Now, consider an investor who buys φ_{t_1} stocks at the ask price $\overline{X}_{t_1-}$ with the information $\mathcal{F}_{t_1-}$. The quantity $\overline{X}_{t_1-} - \operatorname{essinf}_{\mathcal{F}_{t_1-}} \underline{X}_{t_1}$ is the minimal worst-case loss per share she takes by building up this position. Neither $\underline{X}_{t_1-}$ nor $\underline{X}_{t_1}$ could provide this information.

The set of unbounded strategies to which Equation (4) is extended is slightly different to Kühn and Molitor (2022). The main difference is that we do not require that the up-convergence of wealth processes holds "globally" over all (countable many) excursions of the spread away from zero. In the special case of a frictionless market, the set of course coincides with the set of integrable processes in the semimartingale sense (see Proposition 2.10).

Definition 2.6. Let $L(\underline{X},\overline{X})$ denote the subset of real-valued, predictable processes φ such that there exists a sequence $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi := \{\psi \in \mathbf{b}\mathcal{P} \,:\, \Pi(\psi) > -\infty\}$ with

(a) $\varphi^n \to \varphi$ pointwise on $\Omega\times[0,T]$ and $(\varphi^n)^+ \le \varphi^+$, $(\varphi^n)^- \le \varphi^-$ for all $n \in \mathbb{N}$,
(b) there exist semimartingales S, S' with $\underline{X} \le S \le \overline{X}$, $\underline{X} \le S' \le \overline{X}$, an optional làglàd wealth process V satisfying

$$\begin{aligned}(1_{J_n} \bullet V - V)1_{\{X=0\}},\ (1_{J_n} \bullet V_- - V_-)1_{\{X_-=0\}} \to 0 \quad &\text{in } d_{up} \text{ as } n\to\infty \text{ for all}\\ &\text{sequences } (J_n)_{n\in\mathbb{N}} \text{ of finite unions of } (I_i^c)_{i\in\mathbb{N}}, (I_i^{fc})_{i\in\mathbb{N}}\\ &\text{with } 1_{J_n} \to 1_{\Omega\times[0,T]} \text{ up to evanescence as } n\to\infty,\end{aligned} \tag{5}$$

and predictable processes ${}^pS$ and ${}^pS'$ with $\operatorname{essinf}_{\mathcal{F}_-} S \le {}^pS, {}^pS' \le \operatorname{esssup}_{\mathcal{F}_-} S'$ such that

$$1_{I_i^c} \bullet V^{S,S'}(\varphi^n) \to 1_{I_i^c} \bullet V, \quad 1_{I_i^{fc}} \bullet V^{S,S'}(\varphi^n) \to 1_{I_i^{fc}} \bullet V \quad \text{in } d_{up} \text{ as } n\to\infty, \tag{6}$$

$$\begin{aligned}&\text{and } \left(1_{I_i^c} \bullet (\Pi(\varphi^n) + (\varphi^n)^+ {}^pS - (\varphi^n)^- {}^pS')\right)_{n\in\mathbb{N}},\\ &\left(1_{I_i^{fc}} \bullet (\Pi(\varphi^n) + (\varphi^n)^+ {}^pS - (\varphi^n)^- {}^pS')\right)_{n\in\mathbb{N}} \text{ are } d_{up}\text{-Cauchy for all } i\in\mathbb{N}.\end{aligned} \tag{7}$$

Furthermore, for all competing sequences $(\widetilde{\varphi}^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ satisfying (a) and for all $i \in \mathbb{N}$, there exists a (deterministic) subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\left(1_{I_i^c} \bullet (V^{S,S'}(\widetilde{\varphi}^{n_k}) - V^{S,S'}(\varphi^{n_k}))\right)^+, \left(1_{I_i^{fc}} \bullet (V^{S,S'}(\widetilde{\varphi}^{n_k}) - V^{S,S'}(\varphi^{n_k}))\right)^+ \to 0, \quad k\to\infty, \tag{8}$$

up to evanescence.

We extend the self-financing operator Π to $L(\underline{X},\overline{X})$ by setting

$$\Pi(\varphi) := V - \varphi^+ S + \varphi^- S', \quad \varphi \in L(\underline{X},\overline{X}). \tag{9}$$

A self-financing strategy is a pair (φ^0,φ) of predictable processes specifying the number of bonds and stocks in the portfolio such that $\varphi - \varphi_0 \in L(\underline{X},\overline{X})$ and $\varphi^0 = \varphi_0^0 + \Pi(\varphi - \varphi_0)$. When the term "bounded strategy" is used, it refers only to the position in the stock, unless otherwise stated.

Proposition 2.7. *$\Pi(\varphi)$ is well-defined, that is, it does not depend on the choice of S, S', and V. In addition, one has that $(\mathbf{b}\mathcal{P})^\Pi \subseteq L(\underline{X},\overline{X})$.*

Proof. (i) Let V^1 and V^2 be wealth processes that result from different approximating strategies with regard to different semimartingales S^1, $(S')^1$ and S^2, $(S')^2$, respectively. From Kühn and Molitor (2022, Proposition 4.2) applied to the strategy $\varphi 1_{J_n}$, it follows that

$$1_{J_n} \bullet V^1 = 1_{J_n} \bullet V^2 + \varphi^+(S^1 - S^2) - \varphi^-((S')^1 - (S')^2) \quad \text{on } J_n \quad \text{for all } J_n \text{ as in (5)}.$$

For this, it is crucial that by construction, at the boundaries of I_i^c and I_i^{fc}, the portfolio can be rebalanced without costs (thus, the choice of the semimartingales does not matter there). By Equations (5) and (6) (the latter is only needed for the last excursion), it follows that $V^1 = V^2 + \varphi^+(S^1 - S^2) - \varphi^-((S')^1 - (S')^2)$ up to evanescence.

(ii) For the second assertion, we take a $\varphi \in (\mathbf{b}\mathcal{P})^\Pi$ and consider it as a constant sequence. The semimartingale price systems are arbitrarily chosen and $V := V^{S,S'}(\varphi) = \varphi^+ \bullet S - \varphi^- \bullet S' - C^S(\varphi^+) - C^{S'}(-\varphi^-)$. Condition (5) holds since $1_{J_n} \bullet (\varphi^+ \bullet S) \to \varphi^+ \bullet S$, $1_{J_n} \bullet (\varphi^- \bullet S') \to \varphi^- \bullet S'$, $1_{J_n} \bullet C^S(\varphi^+) \to C^S(\varphi^+)$, and $1_{J_n} \bullet C^{S'}(-\varphi^-) \to C^{S'}(-\varphi^-)$ converge separately in d_{up} as $n \to \infty$ by the dominated convergence theorem for stochastic integrals (cf., e.g., Cohen and Elliott, 2015, Theorem 12.4.10). For the cost processes that are only làglàd, we refer to the notation and the fact that there are no costs at the boundaries of J_n. Condition (8) is already shown in Kühn and Molitor (2022, Corollary 3.24). □

Remark 2.8. The combination of Equations (5) and (6) does not imply up-convergence of $(V^{S,S'}(\varphi^n))_{n\in\mathbb{N}}$ to V on the whole time interval. Namely, we need not have control over the wealth processes during the excursions uniformly in time.

Remark 2.9. In frictionless markets, the approximating sequence $(\varphi^n)_{n\in\mathbb{N}} \subseteq \mathbf{b}\mathcal{P} = (\mathbf{b}\mathcal{P})^\Pi$ from Definition 2.6 can be chosen such that $\varphi^n \bullet S \geq -1$ for all $n \in \mathbb{N}$ if $\varphi \bullet S \geq -1$. To see this, one considers for arbitrary $\varepsilon > 0$ the stopping times $\tau_n := \inf\{t \geq 0 : \varphi^n \bullet S_t \leq \varphi \bullet S_t - \varepsilon\}$ and uses that $d_{up}(\varphi^n \bullet S, \varphi \bullet S) \to 0$, $(\varphi^n)^+ \leq \varphi^+$, and $(\varphi^n)^- \leq \varphi^-$. Consequently, the approximating bounded (not elementary!) strategies can be chosen to satisfy the admissibility conditions of the unbounded strategy. By condition (7) that allows for a consistent valuation of the portfolio after the trade but before the price movement (cf. Example 2.5), the same property can be shown in the bid–ask model (see Lemma 3.5). Condition (7) is weak in the sense that it only requires minimal consistency for the intermediate valuation of long and short positions by ${}^pS$ and ${}^pS'$, respectively.

Proposition 2.10. *In the special case of a frictionless market, in the sense that* $\underline{X} = \overline{X}$, *condition (6) implies condition (7) with the choice* ${}^{p}S := S_- \vee \operatorname{essinf}_{\mathcal{F}_-} S$ *and* ${}^{p}S' := S'_- \wedge \operatorname{esssup}_{\mathcal{F}_-} S'$, *and we have* $L(\underline{X}, \underline{X}) = L(\underline{X})$.

Proof of Proposition 2.10. It is sufficient to show the first assertion. Then, the second one follows along the lines of the proof of Kühn and Molitor (2022, Proposition 4.3).

Let $S := S' := \underline{X}$ and $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^{\Pi}$ be a sequence such that $\sup_{t\in[0,T]} |(\varphi^n - \varphi^m) \bullet S_t| \to 0$ in probability as $n, m \to \infty$. This implies that $\sup_{t\in[0,T]} |((\varphi_t^n)^+ - (\varphi_t^m)^+)\Delta S_t| + \sup_{t\in[0,T]} |((\varphi_t^n)^- - (\varphi_t^m)^-)\Delta S'_t| \to 0$ in probability. From Equation (A.2), it follows that $\{{}^{p}S > S_-\} \cap \{{}^{p}S > S\}$ and $\{{}^{p}S' < S'_-\} \cap \{{}^{p}S' < S'\}$ are evanescent, and we obtain that $\sup_{t\in[0,T]} |((\varphi_t^n)^+ - (\varphi_t^m)^+)({}^{p}S_t - S_{t-})| + \sup_{t\in[0,T]} |((\varphi_t^n)_t^- - (\varphi_t^m)^-)({}^{p}S'_t - S'_{t-})| \to 0$ in probability. Since $\Pi_t(\varphi^n) + \varphi_t^n S_{t-} = \varphi^n \bullet S_{t-}$ and $\sup_{t\in[0,T]} |(\varphi^n - \varphi^m) \bullet S_{t-}| \to 0$ in probability as $n, m \to \infty$, we are done. □

Our admissibility condition is in the spirit of Guasoni et al. (2012). For a motivation, we refer to Guasoni et al. (2012, Proposition 4.9) and Lemma 3.1 (and the text before the lemma).

Definition 2.11. Let $M \in \mathbb{R}_+$. A self-financing strategy (φ^0, φ) is called M-admissible iff $\varphi^0 + M + (\varphi + M)^+\underline{X} - (\varphi + M)^-\overline{X} \geq 0$. We write $(\varphi^0, \varphi) \in \mathcal{A}^M$. A strategy is admissible iff it lies in $\mathcal{A} := \cup_{M\in\mathbb{R}_+} \mathcal{A}^M$. We denote by $\mathcal{A}_0^M$ and $\mathcal{A}_0$ the corresponding sets of strategies which start at (0,0). The liquidation value process of (φ^0, φ) is defined as $V^{\mathrm{liq}}(\varphi) := \varphi^0 + \varphi^+\underline{X} - \varphi^-\overline{X}$.

The following example shows that in Definition 2.11, the actual bid and ask prices $(\underline{X}, \overline{X})$ cannot be replaced by the original bid and ask prices $(\underline{S}, \overline{S})$ if the set of terminal (liquidation) values that can be achieved by an admissible strategy should be Fatou-closed (cf. Theorem 2.23 for a definition). This is an insight of Guasoni et al. (2012), but we did not find an explicit example in the literature.

Example 2.12 (Admissibility). Let $T = 2$, B be a standard Brownian motion, and U be a random variable that is uniformly distributed on (0,1) and independent of B. Define $m_t := \inf_{0\leq s\leq t} B_s$ and $\tau := \inf\{t \geq 0 : m_t = -U\} \wedge 1$. The original ask price is given by $\overline{S}_t := 3 + B_{t\wedge\tau}$ and the bid price by $\underline{S}_t := 1_{\{t<\tau\}} + (\overline{S}_\tau + 2)/2 1_{\{t\geq\tau\}}$. The filtration is generated by B^τ and augmented by null sets. We have that $\overline{X} = \overline{S}$ and $\underline{X}_t = 2 \cdot 1_{\{t<\tau\}} + (\overline{S}_\tau + 2)/2 1_{\{t\geq\tau\}}$. The only nonsilly investment strategies are to buy stocks before τ has occurred and sell them at time τ (or later). The market satisfies the RNFLVR condition in Guasoni et al. (2012, Definition 5.2(ii)) (one considers the ask and bid price processes $3/4\overline{S}_t + 1/4 \cdot 2$ and $3/2 1_{\{t<\tau\}} + (2/3\overline{S}_\tau + 1/3 \cdot 2)1_{\{t\geq\tau\}}$ that are uniformly in time strictly more favorable than $\overline{S}, \underline{S}$ by $P(-U > -1) = 1$).

We define the function $f(x) := (1-x)^{-1/2} - 1$ for $x \in [0,1)$ and consider the sequence of nondecreasing strategies $\varphi_t^n := f((-m_{t\wedge\tau}) \wedge (1 - 1/n))1_{\{0<t\leq\tau\}}$, $n \in \mathbb{N}$. The stock position is liquidated before T, and the terminal bond position results

$$\varphi_T^{0,n} = f((-m_\tau) \wedge (1 - 1/n))\frac{1 + B_\tau}{2} - \int_0^{(-m_\tau)\wedge(1-1/n)} (1-x)\, df(x) \geq -\int_0^1 (1-x)\, df(x).$$

Following the strategies $(\varphi^n)_{n\in\mathbb{N}}$, stocks are purchased when $\overline{S}$ attains its running minimum (above $3-U$), and the amount of stocks explodes when $\overline{S}$ approaches 2 before τ stops. But, a share purchased at price $2+\varepsilon$ and liquidated at time T cannot produce losses larger than ε. The example was chosen such that $M := \int_0^1 (1-x)\,df(x) < \infty$. This means that $\varphi_T^{0,n} \geq -M \in \mathbb{R}$ for all $n \in \mathbb{N}$. We even have that $(\varphi^{0,n}, \varphi^n)$ is M-admissible in the sense of Definition 2.11. The strategies and their terminal wealth converge pointwise to $\varphi_t^\infty := f(-m_{t\wedge\tau})1_{\{0<t\leq\tau\}}$ and

$$\varphi_T^{0,\infty} = f(-m_\tau)\frac{1+B_\tau}{2} - \int_0^{-m_\tau} (1-x)\,df(x) \geq -\int_0^1 (1-x)\,df(x), \quad \text{respectively.}$$

The limiting strategy $(\varphi^{0,\infty}, \varphi^\infty)$ is M-admissible as well (φ^∞ is not bounded anymore but obviously in $L(\underline{X}, \overline{X})$).

Now, we turn to admissibility in the sense of Definition 2.11 but with $(\underline{X}, \overline{X})$ replaced by $(\underline{S}, \overline{S})$. For each $n \in \mathbb{N}$, we want to determine the minimal $M_n \in \mathbb{R}_+$ such that $(\varphi^{0,n}, \varphi^n)$ is M_n-admissible. The most vulnerable time for the strategy is when B reaches $-1+1/n$ and $U > 1-1/n$. If the stocks were sold at that time, the bond position would be $-\int_0^{1-1/n}(1-x)\,df(x) - f(1-1/n)$. The difference to above is one unit per share. Since the admissibility condition also allows debts in the stock position, we arrive at $M_n = (\int_0^{1-1/n}(1-x)\,df(x) + f(1-1/n))/2 \in \mathbb{R}_+$. By the choice of f, we have that $M_n \to \infty$ as $n \to \infty$. Thus, the sequence is not admissible with regard to a joint $M' \in \mathbb{R}_+$, and the limiting strategy $(\varphi^{0,\infty}\varphi^\infty)$ is not admissible at all.

This already gives a strong hint that one would not have Fatou-closedness of the set of terminal portfolios, which can be achieved by admissible strategies if $(\underline{X}, \overline{X})$ were replaced by $(\underline{S}, \overline{S})$ in Definition 2.11. The reason is that M_n is too large compared to the worst-case risk at maturity. However, it remains to show that the limiting wealth $\varphi_T^{0,\infty}$ cannot be achieved by an admissible strategy different from $(\varphi^{0,\infty}, \varphi^\infty)$. Assume by contradiction that (ψ^0, ψ) is admissible with $(\psi_T^0, \psi_T) = (\varphi_T^{0,\infty}, 0)$. We leave it as an exercise for the reader to show that for each $n \in \mathbb{N}$, ψ has to coincide with φ^n on $\{U \leq 1-1/n\}$. By the minimality of M_n, it follows that (ψ^0, ψ) cannot be M'-admissible for $M' < M_n$. Since $M_n \to \infty$, (ψ^0, ψ) cannot be admissible at all.

For the rest of the paper, we follow the standard convention to assume that

$$\mathcal{F}_T = \mathcal{F}_{T-}, \quad \underline{S}_T = \underline{S}_{T-}, \quad \text{and } \overline{S}_T = \overline{S}_{T-}. \tag{10}$$

For the actual bid and ask prices, this implies that $\underline{X}_T = \underline{X}_{T-}$ and $\overline{X}_T = \overline{X}_{T-}$ as well. The assumption allows to identify (φ_T^0, φ_T) with the terminal portfolio that cannot be rebalanced anymore. It is w.l.o.g. since it just avoids to introduce an additional time after T when the investor can trade at prices $\underline{S}_T, \overline{S}_T$.

Definition 2.13. The market model satisfies the numéraire-free no-arbitrage condition (NA^{nf}) iff there is no admissible strategy (φ^0, φ) with $\varphi_0^0 = \varphi_0 = 0$, $P(\varphi_T^0 \geq 0, \varphi_T \geq 0) = 1$, and $P(\{\varphi_T^0 > 0\} \cup \{\varphi_T > 0\}) > 0$.

The condition is called "numéraire-free" since bounded short positions in both the bond and the stock are allowed. Stating the admissibility condition in terms of $\underline{X}$ and $\overline{X}$ is equivalent to freezing a portfolio position as it is done in Guasoni et al. (2012). On the other hand, freezing a

short position in a stock with a frictionless price modeled by a nonnegative strict local martingale that behaves like a doubling strategy leads to an arbitrage. Thus, the freezing of positions better fits to the numéraire-free arbitrage theory that leads to true martingale price processes.

As discussed in the introduction, we want to merge the conditions NUPBR and NA^{ps} coming from continuous time frictionless models and discrete time transaction costs models, respectively. For this purpose, we consider the *cost value process* introduced by Bayraktar and Yu (2018) as the cost to enter a portfolio position and defined as

$$V^{\mathrm{cost}}(\varphi) := \varphi^0 + \varphi^+\overline{X} - \varphi^-\underline{X} \quad \text{for } (\varphi^0, \varphi) \text{ self-financing with } \varphi^0_0 = \varphi_0 = 0.$$

Definition 2.14. The market model satisfies the *prospective strict no unbounded profit with bounded risk* (NUPBR^{ps}) condition iff

$$\left\{ \sup_{t\in[0,T]} V^{\mathrm{cost}}_t(\varphi) \;:\; (\varphi^0, \varphi) \in \mathcal{A}^1_0 \right\} \quad \text{is bounded in } L^0, \tag{11}$$

and for every sequence $(\varphi^{0,n}, \varphi^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}^1_0$ such that $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ and $(\varphi^{0,n}_T, \varphi^n_T) \to (C^0, C)$ a.s., where (C^0, C) is a maximal element (in the sense that the convergence of 1-admissible strategies cannot hold for a random vector that strictly dominates (C^0, C) with respect to the pointwise order), there exist forward convex combinations $(\lambda_{n,k})_{n\in\mathbb{N},\ k=0,\dots,k_n}$, $k_n \in \mathbb{N}$, that is, $\lambda_{n,k} \in \mathbb{R}_+$ and $\sum_{k=0}^{k_n} \lambda_{n,k} = 1$, such that

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} V^{\mathrm{cost}}_t\Big(\sum_{k=0}^{k_n} \lambda_{n,k}\varphi^{n+k}\Big) < \infty \quad \text{a.s.} \tag{12}$$

Remark 2.15. In the special case that $\underline{X} = \overline{X}$, the condition NUPBR^{ps} coincides with NUPBR (the latter still considered for numéraire-free 1-admissible strategies). Namely, when the running supremum of a frictionless wealth process reaches a prespecified level, the value can be conserved by liquidating the portfolio (for condition (12) in frictionless markets, we refer to the proof of the second assertion of Theorem 2.22). This transfer of the cost value to time T is not possible in models with friction, and the condition has to be stated directly in terms of pathwise suprema. Condition (12) is weaker than assuming L^0-boundedness of the convex hull in Equation (11). The latter would be needed to obtain a finite limit of forward convex combinations from arbitrary sequences of 1-admissible strategies by the L^0-version of Komlós' theorem (see Delbaen and Schachermayer, 1994, Lemma A1.1 and the counterexample in Delbaen and Schachermayer, 1994, Remark 4 in the appendix). Economically, convex combinations of pathwise suprema would not be very meaningful and in general larger than the suprema of mixed strategies considered in Equation (12).

Remark 2.16. From the subadditivity of $C^{S,S'}$, it follows that the set $(\mathbf{b}\mathcal{P})^\Pi$ is convex and $V^{\mathrm{cost}}(\sum_{k=0}^{k_n} \lambda_{n,k}\varphi^{n+k}) \geq \sum_{k=0}^{k_n} \lambda_{n,k} V^{\mathrm{cost}}(\varphi^{n+k})$ for all $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ and $(\lambda_{n,k})_{n\in\mathbb{N},\ k=0,\dots,k_n} \subseteq \mathbb{R}_+$ with $\sum_{k=0}^{k_n} \lambda_{n,k} = 1$. The same holds for V^{liq} and $V^{S,S'}$. On the other hand, in contrast to $L(S)$, the set $L(\underline{X}, \overline{X})$ need not be convex. The reason is that the wealth of a mixed strategy would have to be $+\infty$ (which is excluded) if trading costs cancel by the mixing. We note that for the arbitrage theory, this is no problem: in the frictionless market with price process $S_t = t$, the "arbitrage strategy" $\varphi_t = 1/t$ does not lie in $L(S)$, but NUPBR is already ruled out by bounded strategies.

By convention (10), we could reformulate the NUPBR^{ps} condition by considering only 1-admissible strategies with $\varphi_T^0, \varphi_T \geq -1$. This means that at T, the investor must actually limit her debts, not just be able to do so. Thus, it makes sense to work with the pointwise order than comparing terminal positions in Definition 2.14.

Remark 2.17 (Discrete time). In the case of only one risky asset, that we consider in this paper, the discrete time NA^{ps} condition from Kühn and Molitor (2019, Definition 2.3) reduces to

$$\forall\, \varphi \text{ predictable},\ t = 0, 1, \dots, T \quad (V_t^{\mathrm{liq}}(\varphi) \geq 0 \ \Rightarrow\ V_t^{\mathrm{cost}}(\varphi) = V_t^{\mathrm{liq}}(\varphi) = 0)$$

(this follows by the arguments in the proof of Lemma 3.1). In discrete time, we have that NUPBR$^{ps} \Leftrightarrow$ NA^{ps}. The implication "$\Rightarrow$" follows from the fact that $V_t^{\mathrm{liq}}(\varphi) \geq 0$ implies that $V_u^{\mathrm{liq}}(\varphi) \geq 0$ for $u = 0, 1, \dots, t-1$, where (φ^0, φ) is a discrete time strategy not necessarily admissible ex ante (cf. again the proof of Lemma 3.1). Let us show "$\Leftarrow$." On the set $\{\overline{X}_0 = \operatorname{essinf}_{\mathcal{F}_0} \underline{X}_1\} \in \mathcal{F}_0$, a purchase at time 0 is reversible in the sense of Kühn and Molitor (2019, Definition 3.2). Thus, under NA^{ps}, we have $\overline{X}_1 = \underline{X}_1 = \overline{X}_0$ on $\{\overline{X}_0 = \operatorname{essinf}_{\mathcal{F}_0} \underline{X}_1\}$, and the purchase can be postponed to time 1. On the complement $\{\overline{X}_0 > \operatorname{essinf}_{\mathcal{F}_0} \underline{X}_1\}$, the number of risky assets of a 1-admissible strategy is bounded by $(1 + \operatorname{essinf}_{\mathcal{F}_0} \underline{X}_1)/(\overline{X}_0 - \operatorname{essinf}_{\mathcal{F}_0} \underline{X}_1)$. For a short position in the risky asset, one gets a similar random bound. Applying this argument inductively for $t = 0, 1, 2, \dots$ we obtain the following. For every sequence of 1-admissible strategies $(\varphi^{0,n}, \varphi^n)_{n\in\mathbb{N}}$, there is a corresponding sequence $(\widetilde{\varphi}^{0,n}, \widetilde{\varphi}^n)_{n\in\mathbb{N}}$ with the same liquidation and cost value processes such that $(\widetilde{\varphi}_t^n)_{n\in\mathbb{N}}$ is L^0-bounded for every $t = 1, \dots, T$ (namely, when following the strategies, we only realize the purely nonreversible parts of the orders in the sense of Kühn and Molitor (2019, Lemma 3.3). This means that, for example, at time 0 purchases are only realized on $\{\overline{X}_0 > \operatorname{essinf}_{\mathcal{F}_0} \underline{X}_1\}$). The L^0-boundedness of the adjusted positions yields that NUPBR^{ps} is satisfied.

Assumption 2.18. Let τ be a stopping time such that on $\{\tau < \infty\}$, there starts an excursion of the actual spread $X := \overline{X} - \underline{X}$ away from zero (cf. Equation 3). Then, there exist a stopping time σ with $\sigma \geq \tau$ and $\sigma > \tau$ on $\{\tau < \infty,\ X_\tau = 0\}$ and probability measures Q^1 and Q^2 equivalent to P such that $\overline{X}^\sigma - \overline{X}^\tau$ is a Q^1-supermartingale and $\underline{X}^\sigma - \underline{X}^\tau$ is a Q^2-submartingale. Let L^1 and L^2 be the stochastic logarithm of the corresponding density process Z^{Q^1} and Z^{Q^2}, respectively, that is, $Z^{Q^i} = 1 + Z_-^{Q^i} \bullet L^i$. For each $A \in \mathcal{P}$, we define Z^A by $Z^A = 1 + Z_-^A 1_A \bullet L^1 + Z_-^A 1_{(\Omega\times[0,T])\setminus A} \bullet L^2$ and assume that Z^A, $A \in \mathcal{P}$, are true martingales defining probability measures Q^A, for which we, in turn, assume that they are uniformly equivalent to P, that is,

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall A \in \mathcal{P}\ \forall B \in \mathcal{F} \quad (P(B) \leq \delta\ \Rightarrow Q^A(B) \leq \varepsilon) \tag{13}$$

$$\text{and} \quad \forall \varepsilon > 0\ \exists \delta > 0\ \forall A \in \mathcal{P}\ \forall B \in \mathcal{F} \quad (Q^A(B) \leq \delta\ \Rightarrow P(B) \leq \varepsilon). \tag{14}$$

Let $\Gamma(\tau)$ be the end time of the excursion. Analogously to above, there exist a stopping time $\widetilde{\sigma}$ with $\widetilde{\sigma} \leq \Gamma(\tau)$ and $\widetilde{\sigma} < \Gamma(\tau)$ on $\{\Gamma(\tau) < \infty,\ X_{\Gamma(\tau)-} = 0\}$ and probability measures $\widetilde{Q}^1$ and $\widetilde{Q}^2$ equivalent to P such that $\overline{X}^{\Gamma(\tau)} 1_{\{X_{\Gamma(\tau)-} > 0\}} + \overline{X}^{\Gamma(\tau)-} 1_{\{X_{\Gamma(\tau)-} = 0\}} - \overline{X}^{\widetilde{\sigma}}$ is a $\widetilde{Q}^1$-submartingale and $\underline{X}^{\Gamma(\tau)} 1_{\{X_{\Gamma(\tau)-} > 0\}} + \underline{X}^{\Gamma(\tau)-} 1_{\{X_{\Gamma(\tau)-} = 0\}} - \underline{X}^{\widetilde{\sigma}}$ is a $\widetilde{Q}^2$-supermartingale. The pair $(\widetilde{Q}^1, \widetilde{Q}^2)$ satisfies the same pasting conditions as (Q^1, Q^2) from above.

Remark 2.19. First of all, it should be noted that Assumptions 2.3 and 2.18 are automatically satisfied if the efficient friction condition in the sense of $P(\inf_{t\in[0,T]}(\overline{X}_t - \underline{X}_t) > 0) = 1$ holds. Thus, the assumptions are weaker than those in the previous literature. In addition, they are automatically satisfied if the spread can only move away from zero or come back by jumps.

But unfortunately, the situation is extremely complicated when this happens continuously. Already to construct self-financing portfolios, Kühn and Molitor (2022) need Assumption 2.3 that rules out Brownian local time behavior, and under which the starting times of excursions of the spread away from zero are stopping times. Assumption 2.18 is for beginning and end, and it goes beyond Assumption 2.3. It requires that for an arbitrarily short random duration at the beginning of an excursion, the market would be arbitrage-free even if purchases could be liquidated at the ask price and short positions could be closed at the bid price. This can be regarded as a local tightening of the NUPBR^{ps} condition at the starting time of an excursion: The process V^{cost} values a purchased position at the higher ask price as long as it is in the portfolio. In the fictitious frictionless market described above, the position can even be liquidated at the ask price, that is, the cost value can be realized. With short sells, it is the other way round, that is, the fictitious frictionless market consists of different price processes for long and short positions. Since long and short positions cannot be hold simultanously, there is a nonconvex trading constraint, and separation theorems are not applicable. The condition is mirrored at the end of an excursion.

Remark 2.20. Condition (13) is equivalent to the condition that $\{Z_T^A : A \in \mathcal{P}\}$ is uniformly integrable. By the Neyman–Pearson lemma, condition (14) is equivalent to $\inf\{q_\varepsilon(Z_T^A) : A \in \mathcal{P}\} > 0$ for all $\varepsilon > 0$, where $q_\varepsilon(Z_T^A)$ denotes the right ε-quantile of the distribution of Z_T^A under P. For price processes of the form $\overline{X}_t = \overline{S}_t = B_t + \overline{\mu}t$ and $\underline{X}_t = \underline{S}_t = B_t + \underline{\mu}t$ with a standard Brownian motion B and $\overline{\mu} \geq \underline{\mu}$, Assumption 2.18 can easily be verified by using Novikov's condition.

Definition 2.21. A consistent price system (CPS) is a pair (S, Q) such that Q is a probability measure equivalent to P and S is a Q-martingale with $\underline{X} \leq S \leq \overline{X}$ (and thus a fortiori $\underline{S} \leq S \leq \overline{S}$).

Theorem 2.22. *If the market model satisfies Assumption 2.3, Assumption 2.18,* NA^{nf}*, and* NUPBR^{ps}*, then there exists a CPS. Conversely, if (S, Q) is a CPS, then the bid–ask model with bid price S and ask price S satisfies Assumption 2.3, Assumption 2.18,* NA^{nf}*, and* NUPBR^{ps}.

Theorem 2.23. *Under Assumption 2.3, Assumption 2.18,* NA^{nf}*, and* NUPBR^{ps}*, the cone $C_0 := \{(\varphi_T^0, \varphi_T) : (\varphi^0, \varphi) \in \mathcal{A}\} - L^0(\Omega, \mathcal{F}, P; \mathbb{R}^2_+)$ is Fatou-closed in the sense that for every $M \in \mathbb{R}_+$, every sequence $(C^{0,n}, C^n)_{n\in\mathbb{N}} \subseteq C_0$ with $C^{0,n}, C^n \geq -M$ for all $n \in \mathbb{N}$, and every $(C^0, C) \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ with $(C^{0,n}, C^n) \to (C^0, C)$ a.s. as $n \to \infty$, there exists a $(\varphi^0, \varphi) \in \mathcal{A}$ with $(\varphi_T^0, \varphi_T) \geq (C^0, C)$ a.s.*

3 | PROOF OF THEOREMS 2.22 AND 2.23

The following lemma corresponds to Guasoni et al. (2012, Proposition 4.9). Intuitively, it states that at any intermediate time, the credit line (in terms of bonds and stocks) required for the trading strategy (that is followed so far) is minimized by freezing the portfolio and close the stock position at the best price that can be achieved for sure now or in the future. Put differently, consider an investor who has built up a, say, positive stock position at some time t. Then, to minimize

her worst-case risk, she just has to sell the stocks at price $\underline{X}_t$, and complicated dynamic trading strategies cannot improve the result.

Lemma 3.1. *Let $M \in \mathbb{R}_+$. Assume that the model satisfies NA^{nf}. Let (φ^0, φ) be an admissible strategy with $P(\varphi_T^0 \geq -M, \varphi_T \geq -M) = 1$. Then, (φ^0, φ) is M-admissible.*

Proof. It is sufficient to show the following seemingly weaker implication: for all $t_0 \in (0,T)$ and for all admissible strategies (φ^0, φ) with $\varphi_{t_0}^0 = 0$, $\varphi_{t_0} = 1$, and $\varphi_T = 0$, we have $\operatorname{essinf}_{\mathcal{F}_{t_0}} \varphi_T^0 \leq \underline{X}_{t_0}$ (by symmetry, the arguments for an initial stock position below $-M$ are the same).

Since we already passed to the actual prices, we prefer not to use the processes $\underline{S}$ and $\overline{S}$ anymore (even though this may be more appealing at this place). Instead, we observe that $\underline{X}_t = \operatorname{essinf}_{\mathcal{F}_t} \sup_{u\in[t,T]} \underline{X}_u$ and $\overline{X}_t = \operatorname{esssup}_{\mathcal{F}_t} \inf_{u\in[t,T]} \overline{X}_u$. W.l.o.g. let $\mathcal{F}_{t_0}$ be P-trivial. Assume by contradiction that there exists $\varepsilon > 0$ such that $P(\varphi_T^0 \geq \underline{X}_{t_0} + \varepsilon) = 1$. The interesting case is that $\overline{X}_{t_0} \geq \underline{X}_{t_0} + \varepsilon$ since otherwise already $(\varphi^0 - \overline{X}_{t_0}, \varphi)1_{]\!]t_0,T]\!]}$ would be an arbitrage, but we do not have to make a case differentiation. Consider the stopping time

$$\tau := \inf\{t \geq t_0 : (\overline{X}_t \leq \underline{X}_{t_0} + 2\varepsilon/3 \text{ or } \varphi_t \leq 0) \text{ and } \underline{X}_u \leq \underline{X}_{t_0} + \varepsilon/3 \quad \forall u \in (t_0, t)\}. \tag{15}$$

We have that $P(\tau < \infty) > 0$ since $P(\varphi_T = 0,\ \sup_{u\in[t_0,T]} \underline{X}_u \leq \underline{X}_{t_0} + \varepsilon/3) > 0$ by the definition of the conditional essential infimum. In addition, before time $\tau(\omega) < \infty$, the ask price is above $\underline{X}_{t_0} + 2\varepsilon/3$ and the bid price below $\underline{X}_{t_0} + \varepsilon/3$. At time $\tau(\omega) < \infty$, either the investor can buy the stock at a better price than ever before or she short-sells the stock for the first time. Under the contradiction assumption, we can switch from strategy 0 to strategy φ at time τ and obtain an arbitrage. To formalize this, let us show several inequalities, at first only in the case that (φ^0, φ) is almost simple. Here, S and S' are arbitrary semimartingale price systems. The first inequality reads

$$\overline{X}_{t_0-} + V_\tau^{S,S'}(\varphi 1_{]\!]t_0,T]\!]}) + \varphi_\tau(\overline{X}_\tau - S_\tau) \leq \underline{X}_{t_0} + 2\varepsilon/3 \text{ on } \{\tau < \infty, \overline{X}_\tau \leq \underline{X}_{t_0} + 2\varepsilon/3, \varphi_\tau > 0\}. \tag{16}$$

The LHS are the costs to build up the position $(\varphi_\tau^0, \varphi_\tau)$ at time τ (note that $V_0^{S,S'}(\cdot) = 0$ and $\overline{X}_{t_0-} + V_{t_0}^{S,S'}(\varphi 1_{]\!]t_0,T]\!]})$ is the wealth of (φ^0, φ) at time t_0). For almost simple strategies, inequality (16) follows from $\overline{X}_t \geq \underline{X}_{t_0} + 2\varepsilon/3 \geq \overline{X}_\tau$ and $\underline{X}_{t_0} + \varepsilon/3 \geq \underline{X}_t$ for all $t \in [t_0, \tau)$, which means that the initial stock position cannot be liquidated at a better price than $\underline{X}_{t_0} + \varepsilon/3$ and further purchases reduce the cost value. Next, we have again for (φ^0, φ) almost simple

$$\overline{X}_{t_0-} + V_\tau^{S,S'}(\varphi 1_{]\!]t_0,T]\!]}) + \varphi_\tau(\underline{X}_\tau - S_\tau) \leq \underline{X}_{t_0} + \varepsilon/3 \text{ on } \{\tau < \infty, \overline{X}_\tau > \underline{X}_{t_0} + 2\varepsilon/3, \varphi_\tau > 0\}. \tag{17}$$

The estimate is for the case that τ is triggered by a nonpositive stock position but the infimum is not attained. Since the long position is liquidated immediately afterwards at the lower bid price, we do not need to have control over $\overline{X}_\tau$, but $\underline{X}_\tau \leq \underline{X}_{t_0} + \varepsilon/3$ holds on $\{\tau < \infty\}$. Finally, one has for almost simple strategies

$$\overline{X}_{t_0-} + V_{\tau-}^{S,S'}(\varphi 1_{]\!]t_0,T]\!]}) + \varphi_{\tau-}(\underline{X}_{\tau-} - S_{\tau-}) \leq \underline{X}_{t_0} + \varepsilon/3 \text{ on } \{\tau < \infty,\ \varphi_\tau \leq 0\}, \tag{18}$$

where the left limit $\varphi_{\tau-}$ also exists for $\varphi \in L(\underline{X}, \overline{X})$ since $\overline{X} - \underline{X} \geq \varepsilon/3$ on $[t_0, \tau)$ if $\tau < \infty$, see (Kühn and Molitor, 2022, Proposition 3.3) and Proposition 3.2(a).

Let us show how Equations (16)–(18) can successively be extended to strategies from $(\mathbf{b}\mathcal{P})^{\Pi}$ and $L(\underline{X}, \overline{X})$. Let $\varphi \in (\mathbf{b}\mathcal{P})^{\Pi}$ and τ from (15) belongs to this strategy. Since $\overline{X}_- - \underline{X}_-$ is away from zero up to time τ on $\{\tau < \infty\}$, φ can be approximated by almost simple strategies uniformly in probability in the sense of Kühn and Molitor (2022, proof of Proposition 3.17). The approximation can be adjusted such that the almost simple strategies take the value zero if φ or its right limit takes the value zero. We observe that τ need not coincide with the stopping times in Equation (15) for the almost simple strategies. But, Equations (16)–(18) still hold with the different stopping time since we only need that the almost simple strategy does not become negative before the stopping time. Thus, the inequalities carry over to φ by convergence of $V^{S,S'}$ uniformly in probability and pointwise convergence of the strategies and their left limits at τ (see Kühn and Molitor (2022, Theorem 3.19(ii) and Proposition 3.17)). Now, let $\varphi \in L(\underline{X}, \overline{X})$. From the proof of Lemma 3.3, it follows that the approximating sequence $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^{\Pi}$ in Definition 2.6 can be chosen such that $(\varphi^n)^+ \wedge 1 = \varphi^+ \wedge 1$ for all $n \in \mathbb{N}$. This means that the associated τ is the same for φ^n and φ and the inequalities carry over to φ by Equation (6) and the fact that $\varphi^n_{\tau-} \to \varphi_{\tau-}$ on $\{\underline{X}_{\tau-} < S_{\tau-}\}$ by Proposition 3.2(b) after passing to a deterministic subsequence.

Putting inequalities (16)–(18) together implies that the self-financing strategy (ψ^0, ψ) with $\psi := \varphi 1_{[\![\tau_{\{\varphi_\tau \leq 0\}}, \tau]\!] \cup [\![\tau, T]\!]}$ is admissible and satisfies $\psi_T = \varphi_T = 0$, $\psi^0_T = 0$ on $\{\tau = \infty\}$, and $\psi^0_T \geq \varphi^0_T - \underline{X}_{t_0} - 2\varepsilon/3$ on $\{\tau < \infty\}$. Under the contradiction assumption, one has $\varphi^0_T - \underline{X}_{t_0} - 2\varepsilon/3 \geq \varepsilon/3$. As observed above, we have $P(\tau < \infty) > 0$. Thus, (ψ^0, ψ) must be an arbitrage, which is a contradiction. □

The following proposition describes jumps of general wealth processes at points with positive spread. Here, strategies have to be of finite variation.

Proposition 3.2. *(a) Let $\varphi \in L(\underline{X}, \overline{X})$. On $\{\overline{X} > \underline{X}\}$, the paths of φ are of finite variation in right-hand neighborhoods; consequently, the right-hand limit of φ exists, $\Delta^+\varphi$ is finite, and $\Delta^+ V^{S,S'}(\varphi) = (S - \overline{X})(\Delta^+\varphi^+)^+ + (\underline{X} - S)(\Delta^+\varphi^+)^- + (\underline{X} - S')(\Delta^+\varphi^-)^+ + (S' - \overline{X})(\Delta^+\varphi^-)^-$ up to evanescence. Analogously, on $\{\overline{X}_- > \underline{X}_-\}$, the paths of φ are of finite variation in left-hand neighborhoods, the left-hand limit of φ exists, $\Delta^-\varphi$ is finite, and $\Delta^- V^{S,S'}(\varphi) = \varphi^+\Delta S - \varphi^-\Delta S' + (S_- - \overline{X}_-)(\Delta^-\varphi^+)^+ + (\underline{X}_- - S_-)(\Delta^-\varphi^+)^- + (\underline{X}_- - S'_-)(\Delta^-\varphi^-)^+ + (S'_- - \overline{X}_-)(\Delta^-\varphi^-)^-$ up to evanescence.*

(b) Let $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^{\Pi}$ be an "optimal" approximating sequence of φ in the sense of Definition 2.6. Then, one has that $(\Delta^+(\varphi^{n_k})^+)^+ \to (\Delta^+\varphi^+)^+$ on $\{\overline{X} > S\} \cup \{\overline{X} > \underline{X},\ (\Delta^+\varphi^+)^+ = 0\}$ and $(\Delta^-(\varphi^{n_k})^+)^+ \to (\Delta^-\varphi^+)^+$ on $\{\overline{X}_- > S_-\} \cup \{\overline{X}_- > \underline{X}_-,\ (\Delta^-\varphi^+)^+ = 0\}$ up to evanescence as $k \to \infty$ for some (deterministic) subsequence $(n_k)_{k\in\mathbb{N}}$. The analogous statements for all combinations of negative/positive parts of the strategies and their jumps hold as well (cf. (a)).

Proof. Ad (a). Let $(\omega, t_1) \in \Omega \times [0, T)$ with $\overline{X}_{t_1}(\omega) > \underline{X}_{t_1}(\omega)$ and, omitting ω in the notation, $t_2 := \inf\{t > t_1 : \overline{X}_t \leq 2/3\overline{X}_{t_1} + 1/3\underline{X}_{t_1} \text{ or } \underline{X}_t \geq 1/3\overline{X}_{t_1} + 2/3\underline{X}_{t_1}\} \wedge T$. By the right-continuity of $\overline{X}$ and $\underline{X}$, we have that $t_2 > t_1$. For an almost simple strategy φ, elementary calculations show the estimate

$$V^{S,S'}_{t_2}(\varphi) - V^{S,S'}_{t_1}(\varphi) \leq -(\overline{X}_{t_1} - \underline{X}_{t_1})/6\mathrm{Var}^{t_2}_{t_1}(\varphi) + (|\varphi_{t_1}| + |\varphi_{t_2}|)(\sup_{t\in[t_1,t_2]} \overline{X}_t - \inf_{t\in[t_1,t_2]} \underline{X}_t). \quad (19)$$

(Having the "last in first out principle" in mind, we argue as follows: A stock position which is both built up and liquidated between t_1 and t_2 cause a loss higher than $(\overline{X}_{t_2} - \underline{X}_{t_1})/3$. Gains from shares which are trades at most once can be estimated very roughly since their trading volume is bounded by $|\varphi_{t_1}| + |\varphi_{t_2}|$).

It remains to extend (19) successively to $(\mathbf{b}\mathcal{P})^{\Pi}$ and to $L(\underline{X}, \overline{X})$. This follows by convergence up to evanescence along subsequences (cf. Kühn and Molitor, 2022, Theorem 3.19(ii)) and Definition 2.6 and by the fact that the variation of the pointwise limiting strategy is dominated by the lim inf of the variation processes. The arguments for the left-hand neighborhood are the same. For a strategy path of finite variation the equations in (a) easily follow from the definition of the cost term (cf. Kühn and Molitor, 2022, Definition 3.2 and the proof of (A.3)).

Ad (b). Convergence in probability implies almost sure convergence along a subsequence. Thus, by a diagonalization argument, we can find a (deterministic) subsequence $(n_k)_{k\in\mathbb{N}}$ (not depending on i) such that for each $i \in \mathbb{N}$, the convergence in Equation (6) holds uniformly in time for almost all ω. By part (a), we know that on the set $\{\overline{X} > \underline{X}\}$, the right-hand limits of the strategies exist and $(S - \overline{X})(\Delta^+(\varphi^{n_k})^+)^+ + (\underline{X} - S)(\Delta^+(\varphi^{n_k})^+)^- + (\underline{X} - S')(\Delta^+(\varphi^{n_k})^-)^+ + (S' - \overline{X})(\Delta^+(\varphi^{n_k})^-)^- \to (S - \overline{X})(\Delta^+\varphi^+)^+ + (\underline{X} - S)(\Delta^+\varphi^+)^- + (\underline{X} - S')(\Delta^+\varphi^-)^+ + (S' - \overline{X})(\Delta^+\varphi^-)^-$ up to evanescence by the above-mentioned uniform convergence of the wealth processes on a single excursion. There remains the problem that $\Delta^+\varphi^{n_k}$ can have the opposite sign of $\Delta^+\varphi$ although the jumps of the cost terms are similar.

We fix an $\omega \in \Omega$ outside the P-null sets from above and omit it in the notation. Let $t_1 \in [0, T)$. To save space, we only write down the case that $\varphi_{t_1} \geq 0, \overline{X}_{t_1} > S_{t_1}$, and $\Delta^+\varphi_{t_1} > 0$. The other cases follow analogously. Now, assume by contradiction that, in addition, there exist an $\varepsilon > 0$ and a (random) subsequence $(k_l)_{l\in\mathbb{N}}$ with

$$|(\Delta^+(\varphi_{t_1}^{n_{k_l}})^+)^+ - (\Delta^+\varphi_{t_1}^+)^+| \geq \varepsilon \quad \text{for all } l \in \mathbb{N}. \tag{20}$$

Define

$$t_2 := \inf\{t > t_1 : \varphi_t \leq \varphi_{t_1+} - \Delta^+\varphi_{t_1}/2 \quad \text{or} \quad V_t^{S,S'}(\varphi) \leq V_{t_1+}^{S,S'}(\varphi) - \Delta^+\varphi_{t_1}(\overline{X}_{t_1} - S_{t_1})/7$$
$$\text{or} \quad \overline{X}_t \leq 2/3\overline{X}_{t_1} + 1/3S_{t_1} \quad \text{or} \quad S_t \geq 1/3\overline{X}_{t_1} + 2/3S_{t_1}\} \wedge T.$$

By $\varphi_{t_1}^{n_k} \to \varphi_{t_1}$, $\Delta^+V_{t_1}^{S,S'}(\varphi^{n_k}) \to \Delta^+V_{t_1}^{S,S'}(\varphi)$, and $\Delta^+V_{t_1}^{S,S'}(\varphi) < 0$, we must have that $\Delta^+\varphi_{t_1}^{n_{k_l}} < 0$ for all l large enough (depending on ω). Indeed, if $\Delta^+\varphi_{t_1}^{n_{k_l}}$ had the same sign as $\Delta^+\varphi_{t_1}$, divergent absolute values would be contrary to convergent jumps of the cost term. By $\varphi_{(t_1+t_2)/2}^{n_k} \to \varphi_{(t_1+t_2)/2}$ and the construction of t_2, the loss of wealth that $(\varphi^{n_{k_l}})_{l\in\mathbb{N}}$ necessarily produces on $(t_1, (t_1 + t_2)/2]$ to get closer to φ again can be estimated from below by $1/6(\overline{X}_{t_1} - S_{t_1})\Delta^+\varphi_{t_1}$ for $l \to \infty$. Since this dominates the possible losses of φ *after* t_1+, we arrive at a contradiction to $V_{t_1}^{S,S'}(\varphi^{n_k}) \to V_{t_1}^{S,S'}(\varphi)$ and $V_{(t_1+t_2)/2}^{S,S'}(\varphi^{n_k}) \to V_{(t_1+t_2)/2}^{S,S'}(\varphi)$ as $k \to \infty$. This means that if $(\varphi^{n_{k_l}})_{l\in\mathbb{N}}$ traded in the opposite direction of φ, it would have to compensate for this promptly, which would lead to *additional* transaction costs. The assertion for the left-hand jump follows analogously. □

By definition of $\varphi \in L(\underline{X}, \overline{X})$, its wealth process can be approximated by wealth processes $V^{S,S'}(\varphi^n)$ with bounded strategies φ^n, $n \in \mathbb{N}$, satisfying $(\varphi^n)^+ \leq \varphi^+$, $(\varphi^n)^- \leq \varphi^-$, and $\varphi^n \to \varphi$. However, φ^n need not be M-admissible if φ is M-admissible. Lemma 3.5 shows that we can

choose the approximating bounded strategies such that they are M-admissible. The following two lemmas prepare Lemma 3.5. Since the investor has a credit line of M stocks, the "safest short-term strategy" is to hold $-M$ stocks. This means that for $M > 0$, the inequalities $(\varphi^n)^+ \leq \varphi^+$ and $(\varphi^n)^- \leq \varphi^-$ do not imply that φ^n is "safer than φ in the short term." The following lemma overcomes this problem.

Lemma 3.3. *Let $\varphi \in L(\underline{X}, \overline{X})$ and $M \in \mathbb{R}_+$. Then, the approximating sequence $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ in Definition 2.6 can be chosen such that*

$$(\varphi^n)^- \wedge M = \varphi^- \wedge M \tag{21}$$

and thus $(\varphi^n + M)^+ \leq (\varphi + M)^+$ for all $n \in \mathbb{N}$.

We note that, by contrast, the inequality $(\varphi^n + M)^- \leq (\varphi + M)^-$ already follows from $(\varphi^n)^- \leq \varphi$.

Proof of Lemma 3.3. *Step 1*: Let $\varphi \in L(\underline{X}, \overline{X})$ and $M \in \mathbb{R}_+$. By definition, there exists a sequence $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ with $(\varphi^n)^+ \leq \varphi^+$, $(\varphi^n)^- \leq \varphi^-$, $\varphi^n \to \varphi$, and semimartingale price systems S, S' such that Equations (5)–(7) hold and Equation (8) is satisfied for all competing sequences. For any $\psi \in (\mathbf{b}\mathcal{P})^\Pi$, we have that

$$C^{S'}(-\psi^-) = C^{S'}(-(\psi^- \wedge M)) + C^{S'}(-(\psi + M)^-) \tag{22}$$

since the strategies $-(\psi^- \wedge M)$ and $-(\psi + M)^-$ never trade in the opposite direction (formally, one checks it for almost simple strategies and apply the approximation in Kühn and Molitor, 2022, proof of Theorem 3.19). Consequently, we can decompose: $V^{S,S'}(\varphi^n) = V^{S,S'}((\varphi^n)^+) + V^{S,S'}(-((\varphi^n)^- \wedge M)) + V^{S,S'}(-(\varphi^n + M)^-)$ and consider the alternative pointwise approximation $\widetilde{\varphi}^n := \varphi^n 1_{\{\varphi\geq 0\}} - (\varphi^- \wedge M) - (\varphi^n + M)^- = \varphi^n \wedge ((-M) \vee \varphi)$, $n \in \mathbb{N}$. Again by Equation (22), we have that $V^{S,S'}(\widetilde{\varphi}^n) = V^{S,S'}((\varphi^n)^+) + V^{S,S'}(-(\varphi^- \wedge M)) + V^{S,S'}(-(\varphi^n + M)^-)$ and thus $V^{S,S'}(\widetilde{\varphi}^n) - V^{S,S'}(\varphi^n) = V^{S,S'}(-(\varphi^- \wedge M)) - V^{S,S'}(-((\varphi^n)^- \wedge M))$. By the Fatou-type estimate in Kühn and Molitor (2022, Corollary 3.24), there exists a (deterministic) subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$(V^{S,S'}(-(\varphi^- \wedge M)) - V^{S,S'}(-((\varphi^{n_k})^- \wedge M)))^- \to 0 \quad \text{up to evanescence as } k \to \infty. \tag{23}$$

Now, we turn to a single interval $\mathcal{I}_i^c$ (the same for $\mathcal{I}_i^{fc}$) and define $Y^n := 1_{\mathcal{I}_i^c} \bullet (V^{S,S'}(-(\varphi^- \wedge M)) - V^{S,S'}(-((\varphi^n)^- \wedge M)))$. Equation (23) already implies that, with $(\varphi^n)_{n\in\mathbb{N}}$, also $(\widetilde{\varphi}^n)_{n\in\mathbb{N}}$ is better than all competing sequences in the sense of Equation (8). By Equation (8), there exists a further subsequence $(k_l)_{l\in\mathbb{N}}$ such that $(1_{\mathcal{I}_i^c} \bullet (V^{S,S'}(\widetilde{\varphi}^{n_{k_l}}) - V^{S,S'}(\varphi^{n_{k_l}})))^+ \to 0$, and together with Equation (23), we arrive at

$$1_{\mathcal{I}_i^c} \bullet Y^{n_{k_l}} \to 0 \quad \text{up to evanescence as } l \to \infty. \tag{24}$$

We continue by observing that

$$Y^n = 1_{\mathcal{I}_i^c} \bullet ((\varphi^n)^- \wedge M - (\varphi^- \wedge M)) \bullet S' - 1_{\mathcal{I}_i^c} \bullet C^{S'}(-(\varphi^- \wedge M)) + 1_{\mathcal{I}_i^c} \bullet C^{S'}(-((\varphi^n)^- \wedge M))$$

and define $\widetilde{Y}^n := 1_{\mathcal{I}_i^c} \bullet C^{S'}(-((\varphi^n)^- \wedge M)) - 1_{\mathcal{I}_i^c} \bullet C^{S'}(-(\varphi^- \wedge M))$. There exists a further (deterministic) subsequence $(l_j)_{j\in\mathbb{N}}$ such that

$$\sup_{t\in[0,T]} |((\varphi^{n_{k_{l_j}}})^- \wedge M - (\varphi^- \wedge M)) \bullet S'_t| \to 0, \quad \text{a.s.} \tag{25}$$

(cf. Jacod and Shiryaev, 2003, Theorem I.4.31(iii)) and Proposition 3.2(b) holds true for $(n_{k_{l_j}})_{j\in\mathbb{N}}$. Putting Equations (24) and (25) together, we have

$$\widetilde{Y}^{n_{k_{l_j}}} \to 0 \text{ up to evanescence as } j \to \infty. \tag{26}$$

Step 2: It remains to show that $(\widetilde{\varphi}^{n_{k_{l_j}}})_{j\in\mathbb{N}}$ satisfies Equations (6) and (7), where the limiting wealth process is of course the same as for the approximating strategies $(\varphi^n)_{n\in\mathbb{N}}$ we started with. By Equation (25), it is sufficient to show that $\sup_{t\in[0,T]} |\widetilde{Y}_t^{n_{k_{l_j}}}| \to 0$ a.s. as $j \to \infty$. W.l.o.g. $n_{k_{l_j}} = j$. We assume by contradiction that there exists $A \in \mathcal{F}_T$ with $P(A) > 0$ such that on A, the sequence $(\sup_{t\in[0,T]} |\widetilde{Y}_t^n|)_{n\in\mathbb{N}}$ does not tend to 0, but the convergence in Equation (26) holds. We fix an $\omega \in A$ that is omitted in the following notation. There exists $\varepsilon > 0$ such that $\sup_{t\in[0,T]} |\widetilde{Y}_t^n| > \varepsilon$ for infinitely many n (depending on ω). Define $\sigma_n := \inf\{t \geq 0 : |\widetilde{Y}_t^n| > \varepsilon\}$. The sequence $(\sigma_n)_{n\in\mathbb{N}}$ has an accumulation point in $[0,T]$ denoted by σ. By Equation (26), we have that

$$\widetilde{Y}_\sigma^n \to 0 \quad \text{as } n \to \infty. \tag{27}$$

At first, we consider the behavior of $\widetilde{Y}^n$ in a right-hand neighborhood of σ. By $\varphi_\sigma^n \to \varphi_\sigma$ and Proposition 3.2(b), considering the cases $\Delta^+\varphi_\sigma^- < 0$, $\Delta^+\varphi_\sigma^- > 0$, and $\Delta^+\varphi_\sigma^- = 0$, we obtain that $(S'_\sigma - \overline{X}_\sigma)((\varphi_{\sigma+}^n)^- \wedge M - ((\varphi_\sigma^n)^- \wedge M))^- + (\underline{X}_\sigma - S'_\sigma)((\varphi_{\sigma+}^n)^- \wedge M - ((\varphi_\sigma^n)^- \wedge M))^+$ converges to $(S'_\sigma - \overline{X}_\sigma)(\varphi_{\sigma+}^- \wedge M - (\varphi_\sigma^- \wedge M))^- + (\underline{X}_\sigma - S'_\sigma)(\varphi_{\sigma+}^- \wedge M - (\varphi_\sigma^- \wedge M))^+$. This means that $\Delta^+ V_\sigma^{S,S'}(-((\varphi^n)^- \wedge M)) \to \Delta^+ V_\sigma^{S,S'}(-(\varphi^- \wedge M))$. By the right-continuity of stochastic integrals and by Equation (27), we have that $C_{\sigma+}^{S'}(-((\varphi^n)^- \wedge M)) \to C_{\sigma+}^{S'}(-(\varphi^- \wedge M))$. In addition, there exists $\sigma' > \sigma$ such that the limit cost term $C_{\sigma'}^{S'}(-(\varphi^- \wedge M))$ is bounded from above by $C_{\sigma+}^{S'}(-(\varphi^- \wedge M)) + \varepsilon/2$. By $C_{\sigma'}^{S'}(-((\varphi^n)^- \wedge M)) \to C_{\sigma'}^{S'}(-(\varphi^- \wedge M))$ as $n \to \infty$, Equation (27), and by the monotonicity of the cost terms, we obtain $\sup_{n\geq n_0,\, t\in[\sigma,\sigma']} |C_t^{S'}(-((\varphi^n)^- \wedge M)) - C_t^{S'}(-(\varphi^- \wedge M))| \leq \varepsilon$ for n_0 large enough. The left-hand neighborhood of σ can be handled in the same way. We arrive at a contradiction to $\sigma_n \to \sigma$. This shows that $\sup_{t\in[0,T]} |\widetilde{Y}_t^n| \to 0$ a.s. and thus Equation (6). Since the differences of the cost terms and the semimartingale gains converge separately in d_{up}, we can argue as in Proposition 2.10 to derive Equation (7) with the same process ${}^p S'$ that does the job for $(\varphi^n)_{n\in\mathbb{N}}$. □

Lemma 3.4. *For $\psi \in L(\underline{X}, \overline{X})$, we define the predictable process*

$$L(\psi) := \Pi(\psi) + M + (\psi + M)^+ \text{essinf}_{\mathcal{F}_-} \underline{X} - (\psi + M)^- \text{esssup}_{\mathcal{F}_-} \overline{X}$$

(that models the liquidation value of $(\Pi(\psi) + M, \psi + M)$ after the portfolio is rebalanced but before the prices jump at some time t). Then,

$$(\Pi(\psi), \psi) \text{ is } M\text{-admissible} \quad \Rightarrow \quad L(\psi) \geq 0$$

Proof. We suppose otherwise. Then, by a section theorem for predictable sets (see, e.g., He et al., 1992, Theorem 4.8), there would exist a predictable stopping time τ with $P(\tau < \infty) > 0$ and $L_\tau(\psi) < 0$ on $\{\tau < \infty\}$. This implies $P(\Pi_\tau(\psi) + M + (\psi_\tau + M)^+\underline{X}_\tau - (\psi_\tau + M)^-\overline{X}_\tau < 0) > 0$, a contradiction to the M-admissibility of $(\Pi(\psi), \psi)$. □

Lemma 3.5. *Let $\varphi \in L(\underline{X}, \overline{X})$ such that $(\Pi(\varphi), \varphi)$ is M-admissible for some $M \in \mathbb{R}_+$. Then, the approximating sequence $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ in Definition 2.6 can be chosen such that $(\Pi(\varphi^n), \varphi^n)$ are also M-admissible for all $n \in \mathbb{N}$.*

Proof of Lemma 3.5. Let $\varphi \in L(\underline{X}, \overline{X})$ such that $(\Pi(\varphi), \varphi)$ is M-admissible. By Lemma 3.3, we can and do choose an approximating sequence $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ such that Equation (21) holds. The approximation holds with regard to the semimartingale price systems S and S' and the associated predictable processes by ${}^pS$ and ${}^pS'$. We write $\varphi^{0,n} := \Pi(\varphi^n), \varphi^0 := \Pi(\varphi), V^{S,S'}(\varphi) := V$ and introduce the predictable processes ${}^pV^n := \varphi^{0,n} + (\varphi^n)^+ {}^pS - (\varphi^n)^- {}^pS'$, ${}^pV := \varphi^0 + \varphi^+ {}^pS - \varphi^- {}^pS'$, $L^n := L(\varphi^n)$, and $L := L(\varphi)$.

Step 1: In the first step, we prove the lemma for the special case that φ invests only during finitely many intervals $\mathcal{I}_i^c$ and $\mathcal{I}_i^{fc}$. Let $\delta > 0$. In this special case, Equations (6) and (7) imply that $P(\sup_{t\in[0,T]}(|V_t^{S,S'}(\varphi^n) - V_t^{S,S'}(\varphi)| + |{}^pV_t^n - {}^pV_t|) > \delta) \le \delta$ for n large enough. We define $\tau := \inf\{t \ge 0 : V_t^{S,S'}(\varphi^n) < V_t^{S,S'}(\varphi) - \delta \text{ or } {}^pV_t^n < {}^pV_t - \delta\}$ and $A := \{{}^pV_\tau^n \ge {}^pV_\tau - \delta\} \in \mathcal{F}_{\tau-}$. Since τ is the debut of a progressive set, it is a stopping time (see e.g., Cohen and Elliott, 2015, Theorem 7.3.4). Consequently, $[\![0,\tau[\![\cup [\![\tau_A]\!] = [\![0,\tau]\!] \cap \{{}^pV^n \ge {}^pV - \delta\} \in \mathcal{P}$, which allows us to consider the strategy

$$\widetilde{\varphi}^n := \varphi^n 1_{[\![0,\tau[\![\cup [\![\tau_A]\!]}.$$

At first, we compare liquidation values strictly before τ. By Equation (21), it is easy to check that

$$\begin{aligned}
&\varphi^{0,n} + M + (\varphi^n + M)^+\underline{X} - (\varphi^n + M)^-\overline{X} - \left(\varphi^0 + M + (\varphi + M)^+\underline{X} - (\varphi + M)^-\overline{X}\right)\\
&= V^{S,S'}(\varphi^n) - V^{S,S'}(\varphi) + (\varphi^+ - (\varphi^n)^+)(S - \underline{X}) + (\varphi^- - (\varphi^n)^-)(\overline{X} - S')\\
&\ge V^{S,S'}(\varphi^n) - V^{S,S'}(\varphi) \ge -\delta \qquad \text{on } [\![0,\tau[\![\qquad (28)
\end{aligned}$$

and thus, by the M-admissibility of φ,

$$\varphi^{0,n} + M + (\varphi^n + M)^+\underline{X} - (\varphi^n + M)^-\overline{X} \ge -\delta \qquad \text{on } [\![0,\tau[\![. \qquad (29)$$

We proceed by analyzing the liquidation value of the portfolio $(\Pi(\widetilde{\varphi}^n) + M, \widetilde{\varphi}^n + M)$ at τ if the event $\Omega \setminus A$ occurs. This means that long and short positions in the stock are liquidated at the prices $\underline{X}_{\tau-}$ and $\overline{X}_{\tau-}$, respectively. The paths of the processes $\varphi^{0,n}$ and φ^n must be of finite variation in a left-hand neighborhood of τ if $\overline{X}_{\tau-} > \underline{X}_{\tau-}$ (see Proposition 3.2(a)). Consequently, on $(\Omega \setminus A) \cap \{\overline{X}_{\tau-} > \underline{X}_{\tau-}\}$, one receives $(\varphi^n_{\tau-} + M)^+\underline{X}_{\tau-} - (\varphi^n_{\tau-} + M)^-\overline{X}_{\tau-} \ge -(\varphi^{0,n}_{\tau-} + M + \delta)$ by Equation (29). On $(\Omega \setminus A) \cap \{\overline{X}_{\tau-} = \underline{X}_{\tau-}\}$, we use that $V_t^{S,S'}(\varphi^n + M) + M \ge -\delta$ for all $t < \tau$ by Equation (29), $V_t^{S,S'}(\varphi^n + M) \to V_{\tau-}^{S,S'}(\varphi^n + M)$ as $t \uparrow \tau$, and $V_{\tau-}^{S,S'}(\varphi^n + M) + M = V_{\tau-}^{S,S'}(\varphi^n +$

$M) + M + \Delta^- C_\tau(\varphi^n + M) = \varphi_\tau^{0,n} + M + (\varphi_\tau^n + M)^+ \underline{X}_{\tau-} - (\varphi_\tau^n + M)^- \overline{X}_{\tau-}$. This yields that

$$\widetilde{\varphi}_\tau^{0,n} + M = \varphi_\tau^{0,n} + M + (\varphi_\tau^n + M)^+ \underline{X}_{\tau-} - (\varphi_\tau^n + M)^- \overline{X}_{\tau-} \geq -\delta \quad \text{on } \Omega \setminus A. \tag{30}$$

Finally, we analyze the liquidation on the event A, that is, we have to show that $\varphi^{0,n} + M + (\varphi^n + M)^+ \underline{X} - (\varphi^n + M)^- \overline{X} \geq 0$ on the set $[\![\tau_A]\!]$. Analogously to Equation (28), again using Equation (21), we estimate

$$\begin{aligned} L^n - L &= {}^pV^n - {}^pV + (\varphi^+ - (\varphi^n)^+)({}^pS - \operatorname{essinf}_{\mathcal{F}_-} \underline{X}) + (\varphi^- - (\varphi^n)^-)(\operatorname{esssup}_{\mathcal{F}_-} \overline{X} - {}^pS') \\ &\geq {}^pV^n - {}^pV \quad \text{on } [\![\tau_A]\!]. \end{aligned} \tag{31}$$

By Lemma 3.4, we have that $L \geq 0$ up to evanescence. Together with Equation (31), we obtain that

$$L^n \geq L - \delta \geq -\delta \quad \text{on } [\![\tau_A]\!]. \tag{32}$$

In order to replace the predictable lower bound L^n of the liquidation process by the process itself, we pointwise distinguish the two cases below. Note that τ is in general not predictable and thus, $(\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_\tau$ need not coincide with $\operatorname{essinf}_{\mathcal{F}_{\tau-}} \underline{X}_\tau$.

Case 1: $\underline{X}_\tau \geq (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_\tau$ if $\varphi_\tau + M \geq 0$ (and $\overline{X}_\tau \leq (\operatorname{esssup}_{\mathcal{F}_-} \overline{X})_\tau$ if $\varphi_\tau + M < 0$). This means that the liquidation value becomes higher. Inequation (32) yields

$$\begin{aligned} &\varphi^{0,n} + M + (\varphi^n + M)^+ \underline{X} - (\varphi^n + M)^- \overline{X} \\ &\geq L^n \geq -\delta \quad \text{on } [\![\tau_A]\!] \cap (\{\varphi + M \geq 0,\ \underline{X} \geq \operatorname{essinf}_{\mathcal{F}_-} \underline{X}\} \cup \{\varphi + M < 0,\ \overline{X} \leq \operatorname{esssup}_{\mathcal{F}_-} \overline{X}\}). \end{aligned} \tag{33}$$

Case 2: $\underline{X}_\tau < (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_\tau$ if $\varphi_\tau + M \geq 0$ (and $\overline{X}_\tau > (\operatorname{esssup}_{\mathcal{F}_-} \overline{X})_\tau$ if $\varphi_\tau + M < 0$). This case can only occur if τ is an unpredictable stopping time with $\underline{X}_\tau < \underline{X}_{\tau-}$ (or $\overline{X}_\tau > \overline{X}_{\tau-}$). The liquidation value becomes smaller than its predictable lower bound, but the decrease is smaller than that of the limiting strategy:

$$\begin{aligned} &\varphi^{0,n} + M + (\varphi^n + M)^+ \underline{X} - (\varphi^n + M)^- \overline{X} \\ &= \varphi^0 + M + (\varphi + M)^+ \underline{X} - (\varphi + M)^- \overline{X} + L^n - L \\ &\quad + ((\varphi^n)^+ - \varphi^+)(\underline{X} - \operatorname{essinf}_{\mathcal{F}_-} \underline{X}) + (\varphi^- - (\varphi^n)^-)(\overline{X} - \operatorname{esssup}_{\mathcal{F}_-} \overline{X}) \\ &\geq \varphi^0 + M + (\varphi + M)^+ \underline{X} - (\varphi + M)^- \overline{X} + L^n - L \\ &\geq -\delta \quad \text{on } [\![\tau_A]\!] \cap (\{\varphi + M \geq 0,\ \underline{X} < \operatorname{essinf}_{\mathcal{F}_-} \underline{X}\} \cup \{\varphi + M < 0,\ \overline{X} > \operatorname{esssup}_{\mathcal{F}_-} \overline{X}\}), \end{aligned} \tag{34}$$

where the equality holds by Equation (21) and the second inequality holds because (φ^0, φ) is M-admissible.

Putting together, the strategy $(\Pi(\widetilde{\varphi}^n), \widetilde{\varphi}^n)$ is $(M + \delta)$-admissible. By compression and since $\delta > 0$ is arbitrary, we find an approximating sequence (with a suitable null sequence $(\delta_n)_{n \in \mathbb{N}}$) that is also M-admissible.

Step 2: We now proceed to the general case. Again, let $\delta > 0$. By Equation (5), one can find a finite union J of intervals $(I_i^c)_{i\in\mathbb{N}}$, $(I_i^{fc})_{i\in\mathbb{N}}$ such that $P(\sup_{t\in[0,T]} |1_J \bullet V_t - V_t|1_{\{X_t=0\}} \vee |1_J \bullet V_{t-} - V_{t-}|1_{\{X_{t-}=0\}} > \delta) \leq \delta$. Given $\delta > 0$, we modify the approximating strategies such that they satisfy $\varphi^n := -M$ on $(\Omega \times [0,T]) \setminus J$. By construction of I_i^c, I_i^{cf}, there are no trading costs at the boundaries of the intervals. In addition, by the semimartingale property of S', J can be chosen close enough to $\Omega \times [0,T]$ such that $d_{\mathbb{S}}(-M1_{(\Omega\times[0,T])\setminus J} \bullet S', 0) \leq \delta$. With these considerations in mind, the proof is analog to Step 1. We note that on $(\Omega \times [0,T]) \setminus J$, the strategies φ^n need not be liquidated since they just hold $-M$ stocks. □

Lemma 3.6. *Let $\varphi^n, \varphi \in L(\underline{X}, \overline{X})$ for all $n \in \mathbb{N}$ and let S, S' be semimartingale price systems. If φ^n is M-admissible for all $n \in \mathbb{N}$, $V^{S,S'}(\varphi^n) \to V^{S,S'}(\varphi)$ uniformly in probability, and $\varphi^n \to \varphi$ up to evanescence on $\{\overline{X} > \underline{X}\}$, then φ is M-admissible as well.*

Proof. For any $\psi \in L(\underline{X}, \overline{X})$, we define the process $A(\psi) := \Pi(\psi) + M + (\psi + M)^+\underline{X} - (\psi + M)^-\overline{X} - V^{S,S'}(\psi)$ and rewrite it to

$$A(\psi) = M + 1_{\{\psi\geq 0\}}(\psi(\underline{X} - S) + M\underline{X}) + 1_{\{-M<\psi<0\}}(\psi(\underline{X} - S') + M\underline{X})$$
$$+1_{\{\psi\leq -M\}}(\psi(\overline{X} - S') + M\overline{X}).$$

Consequently, $\varphi^n \to \varphi$ implies that $A(\varphi^n) \to A(\varphi)$ pointwise. Since, $\Pi(\varphi^n) + M + (\varphi^n + M)^+\underline{X} - (\varphi^n + M)^-\overline{X} \geq 0$ for all $n \in \mathbb{N}$ and $V^{S,S'}(\varphi^n) \to V^{S,S'}(\varphi)$ uniformly in probability, the assertion follows. □

For the rest of the section, we fix an $M \in \mathbb{R}_+$ and a sequence

$$(\varphi^{0,n}, \varphi^n)_{n\in\mathbb{N}} \subseteq \mathcal{A} \text{ such that } \varphi_T^{0,n}, \varphi_T^n \geq -M \text{ for all } n \in \mathbb{N} \text{ and } (\varphi_T^{0,n}, \varphi_T^n) \to (C^0, C) \text{ a.s.}, \quad (35)$$

where $(C^0, C) \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ is a *maximal element* in the sense that Equation (35) cannot hold for a random vector that strictly dominates (C^0, C) in the pointwise order. For example, $(2, -1)$ does not dominate (0,0) even if $\overline{S}_T = 1$. By Equation (11), maximal elements exist following the arguments in Delbaen and Schachermayer (1994, Lemma 4.3).

To prove Theorem 2.23, we have to show that there exists an M-admissible strategy with terminal value (C^0, C), but there is still a long way to go. We refer to Delbaen and Schachermayer (1994, Remark 4.4) for an in-depth discussion for the need of maximal elements and also for the argument why it is sufficient to consider sequences as in Equation (35) to prove Theorem 2.23. We note that for these considerations, it does not make any difference that we consider a two-dimensional framework here. Under convention (10), we can interpret $(\varphi_T^{0,n}, \varphi_T^n)$ as the position after the market has closed. Thus, positions in different "currencies" are analog to wealth in different scenarios.

By Lemma 3.1, the sequence in Equation (35) has to be M-admissible. In addition, for each $n \in \mathbb{N}$, there is a sequence of bounded processes $(\varphi^{n,m})_{m\in\mathbb{N}}$ that approximate φ^n in the sense of Definition 2.6. By Lemma 3.5, the self-financing strategies $(\varphi^{0,n,m}, \varphi^{n,m})$ with $\varphi^{0,n,m} := \Pi(\varphi^{n,m})$ can be chosen to be M-admissible as well. Conditions (5) and (6) in Definition 2.6 imply that $V_T^{S,S'}(\varphi^{n,m}) \to V_T$ in probability as $m \to \infty$, which means that $\varphi_T^{0,n,m} + (\varphi_T^{n,m})^+S_T - (\varphi_T^{n,m})^-S'_T \to \varphi_T^{0,n} + (\varphi_T^n)^+S_T - (\varphi_T^n)^-S'_T$ in probability. By $\varphi_T^{n,m} \to \varphi_T^n$ pointwise as $m \to \infty$, we get $\varphi_T^{0,n,m} \to$

$\varphi_T^{0,n}$ in probability as $m \to \infty$ for all $n \in \mathbb{N}$. Since the convergence in probability is metrizable, there exists a subsequence $(m_n)_{n\in\mathbb{N}}$ with $(\varphi_T^{0,n,m_n}, \varphi_T^{n,m_n}) \to (C^0, C)$ in probability as $n \to \infty$ (one can choose m_n such that $P(|\varphi_T^{0,n,m_n} - \varphi_T^{0,n}| > 1/n \text{ or } |\varphi_T^{n,m_n} - \varphi_T^{n}| > 1/n) \leq 1/n$). By passing to a subsequence, we can also get almost sure convergence. This allows us to assume w.l.o.g. that there exists a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that already the sequence in Equation (35) satisfies

$$|\varphi^n| \leq a_n \quad \text{and} \quad (\varphi^{0,n}, \varphi^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}^M, \quad n \in \mathbb{N}. \tag{36}$$

Of course, a_n may explode as $n \to \infty$ but for the analysis at the boundaries of the intervals with friction, we want to argue with bounded strategies.

Note 3.7. Let $\widetilde{S}$ be a semimartingale price system. Then, there exist semimartingale price systems S and S' such that for every starting time of an excursion τ, there exist stopping times σ and $\widetilde{\sigma}$ satisfying the conditions from Assumption 2.18 with

$$(S, S')1_{[\![\tau,\Gamma(\tau)[\![} = (\overline{X}, \underline{X})1_{[\![\tau,\sigma[\![} + (\widetilde{S}, \widetilde{S})1_{[\![\sigma,\widetilde{\sigma}[\![} + (\underline{X}, \overline{X})1_{[\![\widetilde{\sigma},\Gamma(\tau)[\![}, \tag{37}$$

Proof. There are at most countably many excursions. Since the processes $\overline{X}, \underline{X}$, and $\widetilde{S}$ are càdlàg, one can choose σ and $\widetilde{\sigma}$ close enough to τ and $\Gamma(\tau)$, respectively, such that the correction terms that occurs by replacing the semimartingale by Q^A-super- and submartingales at the boundaries are arbitrarily small in terms of the metric $d_{\mathbb{S}}$. □

For the rest of the section, we fix the semimartingale price systems S and S' to evaluate long and short positions in the risky asset, respectively, and construct a limiting strategy of the sequence in Equation (35)/(36) using these semimartingales in Definition 2.6(b). This implies that if the spread cannot move away from zero continuously or return to zero continuously, then one can take any semimartingale price system to construct the limiting strategy.

3.1 | Fictitious dormant market

In the proof of the frictionless FTAP, the concatenation property of the set of wealth processes plays a crucial role. This property is not available in models with transaction costs since on $\{X_t > 0\}$, one cannot switch between two strategies without additional costs. On the other hand, frictionless markets are included in our model. To prove Theorem 2.23, we proceed in two steps. In the *first step*, we consider a fictitious market that behaves similar to a frictionless market and satisfies the concatenation property. When the bid–ask spread is positive, the market is dormant, but at other times, one can switch between the strategies. More precisely, the gains that are obtained during an excursion of the spread away from zero enter in the wealth processes but one cannot switch from one strategy to another strategy during this time. Then, in the *second step*, the limiting strategy is constructed separately for each excursion, and properties (6)–(8) are verified accordingly. We stress that the second step is the main new step. However, the first step that is developed in this subsection is not only needed to capture the frictionless intervals but also to paste the limiting strategies of the countably many excursions together and obtain a suitable limiting strategy along the whole interval.

Intuitively, the dormant model coincides with the original model if $X_- = 0$, and it is dormant during excursions of the spread away from zero. When an excursion ends and a new frictionless

interval begins, the model is restarted with the current frictionless wealth (including gains from trades during the excursions). At the starting time of an excursion, two different cases can occur: Either the spread is still zero. Then, we investor can still switch to another strategy using the information that an excursion has just started. Or, the spread jumps away from zero. Then, the investor is already in the midst of an excursion, and in the fictitious model, the gains at the jump time and the gains during the rest of the excursion cannot be separated from each other. The easiest way to model this is to allow *in the first case* for double jumps of the wealth process at the starting time of an excursion: the left jump models movements triggered by a possible synchronous jump of the semimartingale price systems, and the right jump models the anticipated gains during the excursion. By contrast, if the spread jumps away from zero, the fictitious wealth already takes the value after the excursion and remains constant up to the end of the excursion. Formally, we introduce the generalized time change $(\tau_t, D_t)_{t\in[0,T]}$ by

$$\tau_t := \begin{cases} t & \text{if } X_t = 0 \\ \Gamma(\tau_1^i) \wedge T & \text{if } X_t > 0 \text{ and for } i \in \mathbb{N} \text{ with } t \in [\tau_1^i, \Gamma(\tau_1^i)) \end{cases}$$

and $D_t := \{X_t > 0\} \cap (\cup_{i\in\mathbb{N}}\{\tau_1^i \le t < \Gamma(\tau_1^i) \le T,\, X_{\Gamma(\tau_1^i)-} = 0\})$. We observe that $(\tau_t)_{t\in[0,T]}$ is a nondecreasing family of stopping times (not necessarily right-continuous). It is shown in Kühn and Molitor (2022, proof of Lemma 5.2) that $\Gamma(\tau_1^i)_{\{X_{\Gamma(\tau_1^i)-}=0\}}$ is a predictable stopping time. Together with $\{X_t > 0\} = \{\tau_t > t\} \in \mathcal{F}_{\tau_t-}$ for $t \in [0,T)$, this implies that $D_t \in \mathcal{F}_{\tau_t-}$ for all $t \in [0,T]$, using that $D_T = \emptyset$. We introduce the not necessarily right-continuous filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ by

$$\widetilde{\mathcal{F}}_t := \sigma(\mathcal{F}_{\tau_t-} \cup ((\Omega \setminus D_t) \cap \mathcal{F}_{\tau_t})), \quad t \in [0,T].$$

For any $\psi \in (\mathbf{b}\mathcal{P})^\Pi$ with wealth process $V^{S,S'}(\psi) := \psi^+ \bullet S - \psi^- \bullet S' - C^S(\psi^+) - C^{S'}(-\psi^-)$, the *dormant wealth process* $\widetilde{V}^\psi$ is defined as

$$\widetilde{V}_t^\psi := V_{\tau_t-}^{S,S'}(\psi)1_{D_t} + V_{\tau_t}^{S,S'}(\psi)1_{\Omega\setminus D_t}, \quad t \in [0,T]. \tag{38}$$

For the bounded strategies above, we write $\widetilde{V}^n := \widetilde{V}^{\varphi^n}$, $n \in \mathbb{N}$. We note that the definition of $\widetilde{V}_t^\psi$ on $\{\tau_t = T\}$ can depend on the choice of (S, S'), but these semimartingale price systems are already fixed. The paths of the process $\widetilde{V}^\psi$ are làglàd but can have double jumps. The integration theory for làglàd integrators is comparatively little developed (see the monograph by Abdelghani and Melnikov, 2022, for a survey). An integration theory that is tailor-made for trading models with làglàd price processes (not trading strategies!) is introduced in Kühn and Stroh (2009). The key idea is that one can separately invest in the part of the right jumps of the asset price process that can be overlapped by countably many stopping times, that is, that is "accessible."

Remark 3.8. It is quite natural to model the dormant market with double jumps since this allows to keep the time domain $[0,T]$. However, a reader who prefers to follow the arguments of this subsection within the standard framework of càdlàg integrators may introduce at time $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$ an additional time shift of 2^{-i}: the movement in the original model is paused briefly, and the right jump turns into a left jump taking place at a later point in time. Formally, this corresponds to the generalized time change $((\tau\circ\tau')_{t+}, D'_t)_{t\in[0,T]}$ with $\tau'_t := \inf\{u \ge 0 : u + \sum_{i=1}^\infty 2^{-i}1_{\{\tau_1^i<u,\, X_{\tau_1^i}=0\}} \ge t\}$ and $D'_t := \{X_{\tau'_t} > 0\} \cap \{\tau'_t < T\} \cap \{X_{((\tau\circ\tau')_{t+})-} = 0\}$ for $t \in [0, T + \sum_{i=1}^\infty 2^{-i}1_{\{\tau_1^i<T,\, X_{\tau_1^i}=0\}}]$.

Lemma 3.9 (cf. Corollary 10.12 in Jacod (1979)). *For every $\psi \in (\mathbf{b}\mathcal{P})^{\Pi}$, the process $\widetilde{V}^{\psi}$ is an optional semimartingale (see, e.g., Kühn and Stroh, 2009, Definition 2.5) under the filtration $\widetilde{\mathbb{F}}$.*

Proof. We have to start with some preparatory work on stopping times regarding the two filtrations. Let σ be an $\widetilde{\mathbb{F}}$-stopping time.

Step 1: Let us show that τ_σ is an $(\mathcal{F}_t)_{t\in[0,T]}$-stopping time. By construction of $(\tau_t)_{t\in[0,T]}$, one has

$$\{\tau_\sigma \le t\} = (\{X_t = 0\} \cap \{\sigma \le t\}) \cup \cup_{i\in\mathbb{N}} \big(\{\tau_1^i \le t\} \cap \{\sigma < \tau_1^i\}\big), \quad t \in [0,T]. \tag{39}$$

In addition, $\{q < \tau_1^i\} \in \mathcal{F}_{\tau_1^i}$ and $\{q < \tau_1^i\} \cap \mathcal{F}_{\tau_q} \subseteq \mathcal{F}_{\tau_1^i}$ imply that

$$\{\sigma < \tau_1^i\} = \cup_{q\in\mathbb{Q}\cap[0,T]} \big(\{\sigma < q\} \cap \{q < \tau_1^i\}\big) \in \mathcal{F}_{\tau_1^i}. \tag{40}$$

By Equation (39), $\{X_t = 0\} \cap \widetilde{\mathcal{F}}_t \subseteq \mathcal{F}_t$, and Equation (40), one obtains that $\{\tau_\sigma \le t\} \in \mathcal{F}_t$.

Step 2: Let us show that $(\tau_\sigma)_{D_\sigma}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stopping time, where $D_\sigma := \{\omega \in \Omega : \omega \in D_{\sigma(\omega)}\}$. Since $\Gamma(\tau_1^i)_{\{X_{\Gamma(\tau_1^i)-}=0\}}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$- predictable stopping time (see Kühn and Molitor, 2022, proof of Lemma 5.2), we have that $\{\tau_1^i \le \sigma < \Gamma(\tau_1^i) \wedge T\} \cap \{X_{\Gamma(\tau_1^i)-} = 0\} \in \mathcal{F}_{\Gamma(\tau_1^i)-}$ and $[\![\Gamma(\tau_1^i)_{\{\tau_1^i\le\sigma<\Gamma(\tau_1^i)\wedge T\}\cap\{X_{\Gamma(\tau_1^i)-}=0\}}, T]\!] \in \mathcal{P}$ (cf., e.g., Jacod and Shiryaev, 2003, Proposition I.2.10). By $[\![(\tau_\sigma)_{D_\sigma}, T]\!] = \cup_{i\in\mathbb{N}} [\![\Gamma(\tau_1^i)_{\{\tau_1^i\le\sigma<\Gamma(\tau_1^i)\wedge T\}\cap\{X_{\Gamma(\tau_1^i)-}=0\}}, T]\!]$ we are done.

Step 3: Now, we come to the main part of the proof. Since the cost terms C^S, $C^{S'}$ are nondecreasing adapted processes, $V^{S,S'}(\psi)$ is an optional semimartingale under $(\mathcal{F}_t)_{t\in[0,T]}$ which even possesses a local martingale part M with càdlàg paths. We can adapt the arguments of Jacod (1979, Theorem 10.10). It is sufficient to analyze the time-changed process $\widetilde{M}_t := M_{\tau_t-}1_{D_t} + M_{\tau_t}1_{\Omega\setminus D_t}$, $t \in [0,T]$, whereas the time-changed finite variation process is obviously of finite variation.

Let $(T_k)_{k\in\mathbb{N}}$ be a localizing sequence of $(\mathcal{F}_t)_{t\in[0,T]}$-stopping times such that M^{T_k}, $k \in \mathbb{N}$, are martingales under $(\mathcal{F}_t)_{t\in[0,T]}$. Let us show that the time-changed stopped process $\widetilde{M}_t^k := M^{T_k}_{\tau_t-}1_{D_t} + M^{T_k}_{\tau_t}1_{\Omega\setminus D_t}$, $t \in [0,T]$, is an optional martingale under $\widetilde{\mathbb{F}}$. For this, let σ be an arbitrary $\widetilde{\mathbb{F}}$-stopping time. By Steps 1 and 2, we have that $[\![(\tau_\sigma)_{D_\sigma}]\!] \cup]\!]\tau_\sigma, T]\!] \in \mathcal{P}$ and thus $E(\widetilde{M}_T^k - \widetilde{M}_\sigma^k) = E(1_{[\![(\tau_\sigma)_{D_\sigma}]\!]\cup]\!]\tau_\sigma,T]\!]} \bullet M_T^{T_k}) = 0$, using that $\tau_T = T$ and $D_T = \emptyset$.

Now, consider $S_k := \tau_{T_k}$, $k \in \mathbb{N}$, which is a localizing sequence of $\widetilde{\mathbb{F}}$-stopping times by $\mathcal{F}_t \subseteq \widetilde{\mathcal{F}}_t$ and Step 1. Since $\widetilde{M} = \widetilde{M}^k$ on $[\![0, \sup\{s \ge 0 : \tau_s < T_k\}[\![$ and $\widetilde{M}$ is constant on $]\!]\sup\{s \ge 0 : \tau_s < T_k\}, \tau_{T_k}]\!]$, the process $\widetilde{M}$ is an optional semimartingale under $\widetilde{\mathbb{F}}$, and the assertion of the lemma follows (we note that one has no control over the left and right jump at $\sup\{s \ge 0 : \tau_s < T_k\}$, which is in general not even an $\widetilde{\mathbb{F}}$-stopping time, thus one cannot conclude that $\widetilde{M}$ is an optional local martingale). □

Lemma3.10 ([cf. Lemma 4.5 in Delbaen and Schachermayer (1994)]). *Let* NA^{nf} *and* NUPBR^{ps} *be satisfied. Then,*

$$\sup_{t\in[0,T]} |\widetilde{V}_t^n - \widetilde{V}_t^m| \to 0 \quad \textit{in probability as } n, m \to \infty.$$

This means that at the times the bid–ask spread vanishes, the wealth processes have to be close together.

Proof. The proof is analogous to that in the frictionless model by Delbaen and Schachermayer (1994, Lemma 4.5), but a bit more technical since we have to deal with processes that have double jumps. By Equation (35), we already know that $(\widetilde{V}^n_T)_{n\in\mathbb{N}}$ is a Cauchy sequence in probability. Now, assume by contradiction that there exists $\varepsilon > 0$ and subsequences $(n_k, m_k)_{k\in\mathbb{N}}$ with $P(\sup_{t\in[0,T),\tau_t<T}(\widetilde{V}^{n_k}_t - \widetilde{V}^{m_k}_t) > \varepsilon) \geq \varepsilon$ for all $k \in \mathbb{N}$ (note convention (10)). Let $\widetilde{T}_k := \inf\{t \geq 0 : \tau_t < T,\ \widetilde{V}^{n_k}_t - \widetilde{V}^{m_k}_t \geq \varepsilon\}$. Define $T_k := \widetilde{T}_k$ on $\{\widetilde{T}_k < \infty\} \cap \{X_{\widetilde{T}_k} = 0\} \cap \{\widetilde{V}^{n_k}_{\widetilde{T}_k} - \widetilde{V}^{m_k}_{\widetilde{T}_k} \geq \varepsilon\}$, $T_k := \Gamma(\widetilde{T}_k)$ elsewhere on $\{\widetilde{T}_k < \infty\}$, and $T_k := \infty$ on $\{\widetilde{T}_k = \infty\}$. Consider $\psi^k := \varphi^{n_k} 1_{[\![0,T_k]\!]\setminus[\![(T_k)_A]\!]} + \varphi^{m_k} 1_{[\![(T_k)_A]\!]\cup]\!]T_k,T]\!]}$ with $A := \{\widetilde{T}_k < \infty\} \cap (\{X_{\widetilde{T}_k} > 0\} \cup \{\widetilde{V}^{n_k}_{\widetilde{T}_k} - \widetilde{V}^{m_k}_{\widetilde{T}_k} < \varepsilon\}) \cap \{X_{\Gamma(\widetilde{T}_k)-} = 0\}$. The process ψ^k is predictable in the original model and thus generates a strategy. If $\widetilde{T}_k$ coincides with some τ^i_1, there are two cases. In the case that $X_{\widetilde{T}_k} = 0$ and $\widetilde{V}^{n_k}_{\widetilde{T}_k} - \widetilde{V}^{m_k}_{\widetilde{T}_k} \geq \varepsilon$ one switches from φ^{n_k} to φ^{m_k} before the excursion starts (using the information $\mathcal{F}_{\tau^i_1}$). Otherwise, one still follows the strategy φ^{n_k} during the excursion and switches to φ^{m_k} at its end. In the fictitious model, the information about the excursion is already available at τ^i_1 and the model is dormant afterwards. We arrive at $\psi^k_T = \varphi^{m_k}_T$, $\Pi_T(\psi^k) \geq \Pi_T(\varphi^{m_k}) + \varepsilon$ on $\{\widetilde{T}_k < \infty\}$ and $\psi^k_T = \varphi^{n_k}_T$, $\Pi_T(\psi^k) = \Pi_T(\varphi^{n_k})$ on $\{\widetilde{T}_k = \infty\}$ with $P(\widetilde{T}_k < \infty) \geq \varepsilon$. Now, we pass to forward convex combination of $(\psi^k)_{k\in\mathbb{N}}$ (cf. Delbaen and Schachermayer, 1994, Lemma A1.1). Since $(\Pi(\psi^k), \psi^k) \in \mathcal{A}^M$ for all $k \in \mathbb{N}$ and the self-financing operator Π is concave on $(\mathbf{b}\mathcal{P})^\Pi$, we arrive at a contradiction to the maximality of (C^0, C) in the above sense. □

Lemma 3.11. *Let Y be an $\widetilde{\mathbb{F}}$-optional semimartingale whose right jumps only take place at the stopping times $(\tau^i_1)_{\{X_{\tau^i_1}=0\}}$, $i \in \mathbb{N}$. Then, the sequence $\left(\sum_{i=1}^k \Delta^+ Y_{\tau^i_1} 1_{\{X_{\tau^i_1}=0\}} 1_{]\!]\tau^i_1,T]\!]}\right)_{k\in\mathbb{N}}$ is d_{up}-Cauchy, and the limiting process $Y^g := \sum_{i=1}^\infty \Delta^+ Y_{\tau^i_1} 1_{\{X_{\tau^i_1}=0\}} 1_{]\!]\tau^i_1,T]\!]}$ is a left-continuous $\widetilde{\mathbb{F}}$-optional semimartingale. The process $Y^r := Y - Y^g$ is a càdlàg $\widetilde{\mathbb{F}}$-semimartingale.*

Proof. By the continuity of the integral in Kühn and Stroh (2009, Theorem 3.5), applied to the optional semimartingale Y and the sequence of optional integrands $H^{2,k} := \sum_{i=1}^k 1_{[\![(\tau^i_1)_{\{X_{\tau^i_1}=0\}}]\!]}$, $k \in \mathbb{N}$, the above sequence is d_{up}-Cauchy, and its limit Y^g is the integral of the pointwise limit of $(H^{2,k})_{k\in\mathbb{N}}$ with respect to the integrator Y. Then, the integral Y^g is an optional semimartingale by Galtchouk (1985, Theorem 2.3) and the arguments at the end of Step 2 in the proof of Kühn and Stroh (2009, Theorem 3.5) (the former is only for the case that the integrator is a locally square integrable martingale). By construction, the process Y^r has no right jumps. □

Next, we proceed towards convergence with respect to the semimartingale topology. Since the processes are not càdlàg but can have right jumps, the Émery metric has to be adjusted. To avoid the introduction of too much notation, we write down $\widetilde{d}_{\mathbb{S}}$ only for $\widetilde{\mathbb{F}}$-optional semimartingales Y^1, Y^2 whose right jumps only take place at the stopping times $(\tau^i_1)_{\{X_{\tau^i_1}=0\}}$, $i \in \mathbb{N}$, using the decomposition in Lemma 3.11:

$$\widetilde{d}_{\mathbb{S}}(Y^1, Y^2) := \sup_{H \text{ } \widetilde{\mathbb{F}}\text{-predictable with } |H|\leq 1,\ G_i \in L^0(\mathcal{F}_{\tau^i_1}),\ |G_i|\leq 1} E\Big(\sup_{t\in[0,T]} |H \bullet ((Y^1)^r - (Y^2)^r)_t + \sum_{i=1}^\infty G_i 1_{\{\tau^i_1 < t,\ X_{\tau^i_1}=0\}} (\Delta^+ Y^1_{\tau^i_1} - \Delta^+ Y^2_{\tau^i_1})| \wedge 1\Big). \tag{41}$$

The integral in Equation (41) can be seen as a standard stochastic integral under $\widetilde{\mathbb{F}}$ or $\widetilde{\mathbb{F}}^+$ (the filtrations generate the same predictable sets). The sum in Equation (41) converges uniformly in probability by the same arguments as in the proof of Lemma 3.11.

Remark 3.12. With the integral in Kühn and Stroh (2009, Theorem 3.5), the definition of $\widetilde{d}_{\mathbb{S}}$ for arbitrary optional semimartingales is canonical. For the purposes of the present paper, however, it is sufficient to consider optional semimartingales whose right jumps only take place at the stopping times $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$, $i \in \mathbb{N}$. By Lemma 3.9, the dormant wealth processes $\widetilde{V}^{\psi}$ satisfy this property. This special situation is much simpler since right jumps can then be treated as left jumps that take place at separate times (cf. Remark 3.8).

Lemma 3.13 ([cf. Lemmas 4.7–4.11 in Delbaen and Schachermayer (1994)]). *Let* NA^{nf} *and* NUPBR^{ps} *be satisfied. Then, there exist forward convex combinations* $(\lambda_{n,k})_{n\in\mathbb{N},\ k=0,\dots,k_n}$, $k_n \in \mathbb{N}$, *that is,* $\lambda_{n,k} \in \mathbb{R}_+$ *and* $\sum_{k=0}^{k_n} \lambda_{n,k} = 1$, *such that for* $\psi^n := \sum_{k=0}^{k_n} \lambda_{n,k}\varphi^{n+k}$, $(\widetilde{V}^{\psi^n})_{n\in\mathbb{N}}$ *is a* $\widetilde{d}_{\mathbb{S}}$*-Cauchy sequence. In addition, there exists an* $\widetilde{\mathbb{F}}$*-optional semimartingale* $\widetilde{V}$ *whose right jumps only take place at the stopping times* $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$, $i \in \mathbb{N}$, *such that* $\widetilde{d}_{\mathbb{S}}(\widetilde{V}^{\psi^n}, \widetilde{V}) \to 0$ *as* $n \to \infty$.

As soon as the lemma is proven, we pass to these forward convex combinations and use that $\widetilde{d}_{\mathbb{S}}(\widetilde{V}^n, \widetilde{V}) \to 0$ as $n \to \infty$. By concavity of Π, the properties of $(\varphi^{0,n}, \varphi^n)_{n\in\mathbb{N}}$ remain valid.

Proof of Lemma 3.13. By the time change τ' from Remark 3.8, that pauses the movement at the stopping times $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$, $i \in \mathbb{N}$, $\widetilde{\mathbb{F}}$-optional semimartingales Y^1, Y^2 whose right jumps only take place at the times $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$, $i \in \mathbb{N}$, can be transformed into the processes $(Y^j)'_t := Y_{\tau'_{t+}}$, $t \in [0,T]$, $j \in \{0,1\}$, that are càdlàg semimartingales under the right-continuous filtration $\mathcal{F}'_t := \bigcap_{s>t} \widetilde{\mathcal{F}}_{\tau'_s}$ for $t \in [0,T)$ and $\mathcal{F}'_T := \widetilde{\mathcal{F}}_{\tau'_T}$. By construction of τ', it follows that $d_{\mathbb{S}}((Y^1)', (Y^2)') = \widetilde{d}_{\mathbb{S}}(Y^1, Y^2)$ (cf., for example, the proof of He et al., 1992, Theorem 5.55). This makes the results below for càdlàg semimartingales directly applicable to our setting.

Based on the analogon of Lemma 3.10, it is shown in Delbaen and Schachermayer (1994, proofs of Lemmas 4.7 to 4.11) that after passing to forward convex combinations, the convergence holds in the stronger Émery topology (when reading one must abstract from the fact that the semimartingales are stochastic integrals). The proofs are reformulated in Kabanov (1997) in a more abstract setting. It seems to be easier to adapt the arguments in Kabanov (1997) to our setting, which is what we want to do in the following. We define the following set of $\mathbb{F}$-optional semimartingales:

$$\mathcal{X} := \{\widetilde{V}^{\psi} : \psi \in (\mathbf{b}\mathcal{P})^{\Pi} \text{ such that } (\Pi(\psi), \psi) \text{ is } M\text{-admissible}\}.$$

This means that for $Y \in \mathcal{X}$, Y_T is the gain of an M-admissible strategy in the original model if on $\{X_T > 0\}$, terminal stock positions are evaluated by (S, S'). We list key properties of the set $\mathcal{X}$:

(i) By the estimate $V_T^{S,S'}(\psi) \le V_T^{\mathrm{cost}}(\psi)$ and condition (11), the set $\{Y_T : Y \in \mathcal{X}\}$ is L^0-bounded.
(ii) For any $Y^1, Y^2 \in \mathcal{X}$, $\lambda \in [0,1]$, there exists $Y^3 \in \mathcal{X}$ with $Y^3 \ge \lambda Y^1 + (1-\lambda)Y^2$ (this holds since the wealth process is concave in the strategy).
(iii) Let $Y^1, Y^2 \in \mathcal{X}$ and $(Y^1)^r, (Y^2)^r$ defined as in Lemma 3.11. For any [0,1]-valued $\widetilde{\mathbb{F}}$-predictable processes H^1, H^2 with $H^1H^2 = 0$ and $G_i^1, G_i^2 \in L^0(\mathcal{F}_{\tau_1^i}; [0,1])$, $i \in \mathbb{N}$, with $G_i^1 G_i^2 = 0$, the

process

$$H^1 \bullet (Y^1)^r + \sum_{i=1}^{\infty} G_i^1 1_{\{X_{\tau_1^i}=0\}} 1_{\rrbracket \tau_1^i, T \rrbracket} \Delta^+ Y^1_{\tau_1^i} + H^2 \bullet (Y^2)^r + \sum_{i=1}^{\infty} G_i^2 1_{\{X_{\tau_1^i}=0\}} 1_{\rrbracket \tau_1^i, T \rrbracket} \Delta^+ Y^2_{\tau_1^i}$$

is the dormant wealth process of a self-financing strategy (not necessarily M-admissible).

(iv) In the dormant market, one can consider the numéraire $N_t := (1 + S_{\tau_t -})1_{D_t} + (1 + S_{\tau_t})1_{\Omega \setminus D_t}$, $t \in [0,T]$, that is the sum of 1 and the time-changed semimartingale price system S. For an M-admissible strategy (ψ^0, ψ) with $\psi \in (\mathbf{b}\mathcal{P})^{\Pi}$, we have that $\widetilde{V}_t^{\psi}/N_t \geq -M$ on $\{\tau_t < T\}$ for all $t \in [0,T)$ (for stopping at $\tau_t -$ cf. Lemma 3.4 combined with NA^{nf}). We refer to the term of an "allowable strategy" introduced in (Yan, 1998, Definition 2.4).

With these four properties of $\mathcal{X}$, one obtains a $\widetilde{d}_{\mathbb{S}}$-Cauchy sequence along the lines of Kabanov (1997, proofs of Lemmas 2.3–2.8 and Lemma 3.3). Let us only describe the minor adjustments that are needed. By considering the dormant market, one switches from one strategy to another strategy only at frictionless points. Here, one can express the difference of frictionless wealth as multiple of N (cf. item (iv)) and control it as in Kabanov (1997). Then, during an excursion of the spread away from zero, the number of stocks of a concatenated strategy coincides with some ψ, with (ψ^0, ψ) admissible, and the position in the bank account coincides with ψ^0 shifted by the difference of frictionless wealth before the excursion. This means that the difference of frictionless wealth coming from past trades is invested in the bank account, which allows to estimate the constant M' with which the concatenated strategy is M'-admissible. On the technical level, the arguments in the proof of Lemma 3.10 have to be repeated to concatenate strategies, using that $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$, $i \in \mathbb{N}$, are stopping times and the left limit process N_- is locally bounded. In that proof, it can also be seen how the maximality of (C^0, C) is used.

To prove the second assertion, we use the fact that the space of semimartingales is complete with regard to the Émery topology (see Émery, 1979, Theorem 1 for càdlàg semimartingales). Along the lines of the proof of Émery (1979, Theorem 1), one can show that there exists a limiting $\widetilde{\mathbb{F}}$-optional semimartingale $\widetilde{V}$ that shares with $\widetilde{V}^{\psi}$ the property that right jumps only take place at $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$, $i \in \mathbb{N}$. We stress that $\widetilde{V}$ does in general not lie in $\mathcal{X}$, which is generated by bounded strategies. □

Remark 3.14. In a more recent article, Cuchiero and Teichmann (2015, Theorem 3.3(ii)) show that in the case of a frictionless market, Lemma 3.13 holds without passing to (further) forward convex combinations. We leave it as an open problem if their arguments can also be adapted to our setting. For the proof of Theorem 2.23, this question is not crucial, and the proofs of Kabanov (1997) can be more easily adapted.

Proof of Theorem 2.23. So far, we made well-known results on frictionless markets accessible to our model by considering a fictitious dormant market that ignores the problems that occur by positive bid–ask spreads. We now turn to the construction of the limiting strategy when the spread does not vanish. The random vector (C^0, C) and the approximating sequence $(\varphi^{0,n}, \varphi^n)_{n \in \mathbb{N}}$ are still from Equation (35), and we have to show that (C^0, C) is the terminal position of an M-admissible strategy. We can assume that $(\varphi^{0,n}, \varphi^n)_{n \in \mathbb{N}}$ satisfies Equation (36).

Step 1: By Equation (12), the second part of the NUPBR^{ps} condition, we can assume w.l.o.g. that

$$\sup_{t\in[0,T]}\sup_{n\in\mathbb{N}} V_t^{\text{cost}}(\varphi^n) < \infty. \tag{42}$$

If this does not already hold for the original sequence of cost value processes, we pass to forward convex combinations. Equation (42) allows us to define the finite process $A_t := \sup_{s\in[0,t]}\sup_{n\in\mathbb{N}} V_s^{\text{cost}}(\varphi^n)$ that dominates all cost processes. Putting this together with the M-admissibility of φ^n, the later means that $\varphi^{0,n} + M + (\varphi^n + M)^+\underline{X} - (\varphi^n + M)^-\overline{X} \geq 0$, we can control the size of the strategies. Namely, we get

$$|\varphi^n|(\overline{X} - \underline{X}) \leq A + M + M\overline{X}, \quad \forall n \in \mathbb{N}. \tag{43}$$

Estimate (43) is used for the case that positions are built up during a frictionless interval but the spread jumps away from zero. For the case that the portfolio is rebalanced under a positive spread, we need another estimate. The finite process A is prelocally bounded, that is, there exists a sequence of stopping times $(T_m)_{m\in\mathbb{N}}$ with $P(T_m = \infty) \to 1$ for $m \to \infty$ and $V^{\text{cost}}(\varphi^n)1_{[\![0,T_m[\![} \leq m$ for all $n \in \mathbb{N}$. Let us show that

$$\left(\varphi^{0,n} + (\varphi^n)^+\overline{X}_- - (\varphi^n)^-\underline{X}_-\right)1_{[\![0,T_m]\!]} \leq m, \quad \forall n, m \in \mathbb{N}. \tag{44}$$

Intuitively, this means that trading cannot increase the cost value of a portfolio. We have to prove that $\varphi_s^{0,n} + (\varphi_s^n)^+\overline{X}_s - (\varphi_s^n)^-\underline{X}_s \leq m$ for all $s < t$ implies that $\varphi_t^{0,n} + (\varphi_t^n)^+\overline{X}_{t-} - (\varphi_t^n)^-\underline{X}_{t-} \leq m$. On the set $\{\overline{X}_{t-} > \underline{X}_{t-}\}$, φ_s^n converges to φ_{t-}^n, and the variation of φ^n on $[s,t)$ vanishes as $s \uparrow t$ by Proposition 3.2(a). This implies that

$$\varphi_{t-}^{0,n} + (\varphi_{t-}^n)^+\overline{X}_{t-} - (\varphi_{t-}^n)^-\underline{X}_{t-} \leq m. \tag{45}$$

In addition, the LHS of Equation (45) dominates $\varphi_t^{0,n} + (\varphi_t^n)^+\overline{X}_{t-} - (\varphi_t^n)^-\underline{X}_{t-}$. On $\{\overline{X}_{t-} = \underline{X}_{t-}\}$, we can use that $V_s^{S,S'}(\varphi^n) \leq \varphi_s^{0,n} + (\varphi_s^n)^+\overline{X}_s - (\varphi_s^n)^-\underline{X}_s \leq m$ for all $s < t$ and $V_s^{S,S'}(\varphi^n) \to V_{t-}^{S,S'}(\varphi^n) = \varphi_t^{0,n} + (\varphi_t^n)^+\overline{X}_{t-} - (\varphi_t^n)^-\underline{X}_{t-}$.

We can and do choose the sequence $(T_m)_{m\in\mathbb{N}}$ from above such that one has, in addition, that

$$\overline{X}_-1_{[\![0_{\{T_m>0\}}]\!]\cup[\![0,T_m]\!]} \leq m, \quad \forall m \in \mathbb{N}.$$

From Equation (44), we subtract the inequality

$$\varphi^{0,n} + M + (\varphi^n + M)^+\text{essinf}_{\mathcal{F}_-}\underline{X} - (\varphi^n + M)^-\text{esssup}_{\mathcal{F}_-}\overline{X} \geq 0$$

that holds by Lemma 3.4 and obtain the estimate

$$\max\{(\varphi^n)^+(\overline{X}_- - \text{essinf}_{\mathcal{F}_-}\underline{X}), (\varphi^n)^-(\text{esssup}_{\mathcal{F}_-}\overline{X} - \underline{X}_-)\} \leq m + M + Mm \quad \text{on } [\![0,T_m]\!].$$

We arrive at

$$|\varphi^n|\hat{X} \leq m + M + Mm \quad \text{on }]\!]0, T_m]\!],$$
$$\text{where } \hat{X} := \min(\overline{X}_- - \operatorname{essinf}_{\mathcal{F}_-} \underline{X}, \operatorname{esssup}_{\mathcal{F}_-} \overline{X} - \underline{X}_-). \tag{46}$$

Estimate (46) is crucial to control the position in the risky asset during an excursion of the spread away from zero.

Step 2: Let $\tau := \tau_1^i$ be the starting time of an excursion (cf. Equation 3). The spread at τ can be zero or positive. Let us show that the end of an excursion can be rewritten as follows:

$$\begin{aligned}\Gamma(\tau) &:= \inf\{t > \tau : \overline{X}_t = \underline{X}_t \text{ or } \overline{X}_{t-} = \underline{X}_{t-}\} \\ &= \inf\{t > \tau : \overline{X}_t = \underline{X}_t \text{ or } \overline{X}_{t-} = \underline{X}_{t-} \text{ or } \overline{X}_{t-} = (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_t \text{ or } (\operatorname{esssup}_{\mathcal{F}_-} \overline{X})_t = \underline{X}_{t-}\}.\end{aligned} \tag{47}$$

Let τ_1 be a predictable stopping time with $\overline{X}_{\tau_1-} = (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_{\tau_1}$ on $\{\tau_1 < \infty\}$. Since $\underline{X}_{\tau_1} \geq (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_{\tau_1}$, the NA^{nf} condition implies that $\underline{X}_{\tau_1} = \overline{X}_{\tau_1-}$. This means that a long stock position built up at time τ_1- can be liquidated for sure at time τ_1. Consequently, we must have that $\overline{X}_{\tau_1} = \underline{X}_{\tau_1}$ on $\{\tau_1 < \infty\}$ since otherwise the sequence $\psi^n := n1_{]\!]\tau_1]\!]}$, $n \in \mathbb{N}$, would violate the NUPBR^{ps} condition. This means that at $t = \tau_1$, the first condition $\overline{X}_t = \underline{X}_t$ is satisfied as well.

To complete the proof, we define the debut $\tau_2 := \inf\{t > \tau : \overline{X}_{t-} = (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_t\}$ that is a (not necessarily predictable) stopping time with $]\!](\tau_2)_{\{\overline{X}_{\tau_2-} = (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_{\tau_2}\}}]\!] \in \mathcal{P}$. We have to show that $P(\tau_2 \geq \Gamma(\tau)) = 1$. Assume by contradiction that there exists an $\varepsilon > 0$ such that $P(\tau_2 + \varepsilon < \Gamma(\tau)) > 0$. By a section theorem for predictable sets (see, e.g., He et al., 1992, Theorem 4.8) applied to the predictable set $\{\overline{X}_- = \operatorname{essinf}_{\mathcal{F}_-} \underline{X}\} \cap]\!]\tau_2, (\tau_2 + \varepsilon) \wedge \Gamma(\tau)]\!]$, there exists a predictable stopping time τ_3 with $P(\tau_3 < \infty, \tau_2 + \varepsilon < \Gamma(\tau)) > 0$, $\tau_2 \leq \tau_3 \leq \tau_2 + \varepsilon$ and $\overline{X}_{\tau_3-} = (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_{\tau_3}$ on $\{\tau_3 < \infty, \tau_2 + \varepsilon < \Gamma(\tau)\}$. Above, we have shown that this implies that $\overline{X}_{\tau_3} = \underline{X}_{\tau_3}$ on $\{\tau_3 < \infty, \tau_2 + \varepsilon < \Gamma(\tau)\}$—a contradiction to the definition of $\Gamma(\tau)$. By exactly the same arguments, we get rid of the condition $(\operatorname{esssup}_{\mathcal{F}_-} \overline{X})_t = \underline{X}_{t-}$ in Equation (47).

In the following, we construct a double sequence of stopping times $(\tau^{1,N}, \tau^{2,N})_{N\in\mathbb{N}}$ with which one can exhaust the excursion while keeping the spread away from zero. We set $\tau^{1,N} := (\tau + 1/N 1_{\{\overline{X}_\tau = \underline{X}_\tau\}}) \wedge \sigma$. Since the stopping time $(\Gamma(\tau))_{\{\overline{X}_{\Gamma(\tau)-} = \underline{X}_{\Gamma(\tau)-}\}}$ is predictable, it possesses an announcing sequence. Thus, there exists a sequence $(\tau^{2,N})_{N\in\mathbb{N}}$ with $\widetilde{\sigma} \leq \tau^{2,N} \leq \Gamma(\tau)$, $\tau^{2,N} < \Gamma(\tau)$ on $\{\overline{X}_{\Gamma(\tau)-} = \underline{X}_{\Gamma(\tau)-}\}$, and $\tau^{2,N} \to \Gamma(\tau)$ a.s. as $N \to \infty$, where $\widetilde{\sigma}$ comes from Assumption 2.18. For fixed $N \in \mathbb{N}$, the event $\{\tau^{2,N} < \tau^{1,N}\}$ can have positive probability, but we have

$$]\!]\tau, \Gamma(\tau)]\!] \setminus]\!](\Gamma(\tau))_{\{\overline{X}_{\Gamma(\tau)-} = \underline{X}_{\Gamma(\tau)-}\}}]\!] = \cup_{N\in\mathbb{N}}]\!]\tau^{1,N}, \tau^{2,N}]\!]. \tag{48}$$

For all $N \in \mathbb{N}$, we define

$$\eta^N := \inf_{t \in [\tau^{1,N}_{\{\overline{X}_\tau = \underline{X}_\tau\}}] \cup (\tau^{1,N}, \tau^{2,N}) \cup [\tau^{2,N}_{\{\tau^{2,N} < \Gamma(\tau)\}}]} \min\Big\{\overline{X}_{t-} - (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_t, (\operatorname{esssup}_{\mathcal{F}_-} \overline{X})_t - \underline{X}_{t-}, \overline{X}_{t-} - \underline{X}_{t-}\Big\} \tag{49}$$

with the convention that $\inf \emptyset := \infty$. Let us show that

$$\eta^N > 0 \quad \text{a.s.} \tag{50}$$

The NA^{nf} condition and a section theorem for predictable sets (see, e.g., He et al., 1992, Theorem 4.8) imply that $\eta^N \geq 0$ a.s. Now, fix some $N \in \mathbb{N}$ and define

$$B_n := \left\{ \inf_{t \in [\tau^{1,N}_{\{\overline{X}_\tau = \underline{X}_\tau\}}, \tau^{2,N}] \cup (\tau^{1,N}, \tau^{2,N}]} (\overline{X}_{t-} - (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_t) \leq 1/n \right\}, \quad n \in \mathbb{N}.$$

In contrast to Equation (49), $t = \Gamma(\tau)$ is included in the infimum that makes B_n predictable. Let $\varepsilon > 0$. Again by a section theorem for predictable sets, there exists a sequence of predictable stopping times $(\sigma_n)_{n\in\mathbb{N}}$ such that $P(B_n \cap \{\sigma_n < \infty\}) \geq P(B_n) - \varepsilon 2^{-n}$, $(\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_{\sigma_n} \geq \overline{X}_{\sigma_n -} - 1/n$, $\tau^{1,N} \leq \sigma_n \leq \tau^{2,N}$ on $\{\sigma_n < \infty\}$, and $\tau^{1,N} < \sigma_n$ on $\{\overline{X}_\tau > \underline{X}_\tau\}$. In addition, σ_n can be chosen such that it does not exceed the debut of B_n by more than 2^{-n}. It follows that $\underline{X}_{\sigma_n} \geq \overline{X}_{\sigma_n -} - 1/n$ on $\{\sigma_n < \infty\}$ which means that the strategies $\psi^n := n 1_{[\![\sigma_n]\!]}$, $n \in \mathbb{N}$, are 1-admissible. The NUPBR^{ps} condition implies that the sequence $(n(\overline{X}_{\sigma_n} - \overline{X}_{\sigma_n -}) 1_{\{\sigma_n < \infty\}})_{n\in\mathbb{N}}$ and thus $(n(\overline{X}_{\sigma_n} - \underline{X}_{\sigma_n}) 1_{\{\sigma_n < \infty\}})_{n\in\mathbb{N}}$ is L^0-bounded. The latter implies that $\overline{X}_{\sigma_n} - \underline{X}_{\sigma_n}$ converges to zero in probability on $B := \cap_{n\in\mathbb{N}}(B_n \cap \{\sigma_n < \infty\})$ as $n \to \infty$. Consequently, there exists a (deterministic) subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\overline{X}_{\sigma_{n_k}} - \underline{X}_{\sigma_{n_k}} \to 0$ on B a.s. as $k \to \infty$. First, we observe that on $\{\tau^{2,N} < \Gamma(\tau)\}$, the bid–ask spread is bounded away from zero and thus $B \subseteq \{\tau^{2,N} = \Gamma(\tau)\}$ a.s. On the other hand, on $\{\tau^{2,N} = \Gamma(\tau)\}$, we have that $\overline{X}_{\Gamma(\tau)-} > \underline{X}_{\Gamma(\tau)-}$. Putting together, we obtain that $B \subseteq \{\sigma_{n_k} = \Gamma(\tau) \text{ for all but finitely many } k\}$ a.s. Since the stopping times are close to the debuts of B_{n_k} and $\varepsilon > 0$ was arbitrary, we arrive at Equation (50) by symmetry.

We conclude this step with a remark. Following Example 2.5, it can happen that $\overline{X}_{\Gamma(\tau)-} = (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_{\Gamma(\tau)}$ but $\overline{X}_{\Gamma(\tau)-} > \underline{X}_{\Gamma(\tau)-}$. In this case, the point $\Gamma(\tau)$ is still considered to be part of the regime with friction. However, on $\{\overline{X}_{\Gamma(\tau)-} = (\operatorname{essinf}_{\mathcal{F}_-} \underline{X})_{\Gamma(\tau)}\}$, we must have anyway that $\overline{X}_{\Gamma(\tau)-} = \underline{X}_{\Gamma(\tau)} = \overline{X}_{\Gamma(\tau)}$, and there are no investment opportunities between $\Gamma(\tau)-$ and $\Gamma(\tau)$.

Step 3a: In the following, we consider an interval $\mathcal{I}^c_i$, $i \in \mathbb{N}$, whose left endpoint $(\tau^i_1)_{\{X_{\tau^i_1} = 0\}}$ is the starting time of an excursion of the spread X away from zero but at which the spread is still zero. This is the most tricky case in the proof since we do not have an upper bound for the number of stocks of M-admissible strategies on $\{X_{\tau^i_1} = 0\}$. Assumption 2.18 is needed to show that with $n \to \infty$ the cost value processes $V^{\text{cost}}(\varphi^n)$ cannot increase significantly "closer and closer" to $(\tau^i_1)_{\{X_{\tau^i_1} = 0\}}$.

Let $\tau := (\tau^i_1)_{\{X_{\tau^i_1} = 0\}}$ be accompanied by the measures Q^A, $A \in \mathcal{P}$, from Assumption 2.18. In addition, we fix some $m \in \mathbb{N}$ and $\varepsilon \in (0, 1)$. By Proposition A.3(b), there exists $\gamma \in (0, \varepsilon)$ such that for every $A \in \mathcal{P}$ and every Q^A-supermartingale Y on $[0, T]$ with $Y_0 = 0$ and $Y \geq -m - M$, the following implication holds:

$$Q^A\Big(\sup_{t\in[0,T]} |Y_t| > \gamma\Big) \leq \gamma \quad \Rightarrow \quad d_{\mathbb{S}}(Y, 0) \leq \varepsilon^2, \tag{51}$$

where $d_{\mathbb{S}}$ denotes the Émery distance under P (the proposition is applied under the measures Q^A, but by Equation 14, $d_{\mathbb{S}}$ is small if the Émery distance under Q^A is small). Next, there exists $\delta \in (0, \gamma/3)$ such that for every $A \in \mathcal{P}$ and every Q^A-supermartingale Y with $Y_0 = 0$ and $Y \geq -m - M$

$$P(Y_T < -3\delta) \leq 3\delta \quad \Rightarrow \quad P(\sup_{t\in[0,T]} |Y_t| > \gamma) \vee Q^A(\sup_{t\in[0,T]} |Y_t| > \gamma) \leq \gamma. \tag{52}$$

Indeed, by Equation (13)/(14), we can switch between the measures P and Q^A, and one has that $E_{Q^A}(Y_T^-) \leq a + (m+M)Q^A(Y_T < -a)$ for all $a \in \mathbb{R}_+$. Since at every stopping time, the expected loss of a supermartingale exceeds the expected gain, and the former is maximal at maturity, implication (52) follows by considering the stopping times $\inf\{t \geq 0 : Y_t > \gamma\}$ and $\inf\{t \geq 0 : Y_t < -\gamma\}$.

By Lemma 3.10 and $\widetilde{V}_\tau^n = V_\tau^{S,S'}(\varphi^n) = V_\tau^{\mathrm{cost}}(\varphi^n)$ on $\{\tau < \infty\}$ for all $n \in \mathbb{N}$, we have that $P(\tau < \infty,\ |V_\tau^{\mathrm{cost}}(\varphi^{n_1}) - V_\tau^{\mathrm{cost}}(\varphi^{n_2})| > \delta) \leq \delta$ for all n_1, n_2 large enough. Since the converging sequence $(\varphi_T^{0,n}, \varphi_T^n)_{n\in\mathbb{N}}$ is maximal, there exists $n^\varepsilon \in \mathbb{N}$ such that for all $n_1, n_2 \geq n^\varepsilon$, there exists *no* M-admissible (ψ^0, ψ) with $(\psi_T^0, \psi_T) \geq (\varphi_T^{0,n_1} \wedge \varphi_T^{0,n_2}, \varphi_T^{n_1} \wedge \varphi_T^{n_2})$ and $P(\psi_T^0 \geq \varphi_T^{0,n_1} + \delta) \geq \delta$ (cf. the end of the proof of Lemma 3.10). By $|\varphi^{n^\varepsilon}| \leq a_{n^\varepsilon}$ (cf. Equation 36), we find an $N^\varepsilon \in \mathbb{N}$ such that $P(\sigma > \tau,\ \inf_{t\in[\tau,\tau^{1,N^\varepsilon}]} V_t^{\mathrm{liq}}(\varphi^{n^\varepsilon}) - V_\tau^{\mathrm{cost}}(\varphi^{n^\varepsilon}) < \delta) \leq \delta$ and $P(\tau < \sigma \leq \tau^{1,N^\varepsilon}) \leq \varepsilon$. Let us show that

$$P(\inf_{t\in[\tau,\tau^{1,N^\varepsilon}]} (V_t^{\mathrm{cost}}(\varphi^n) - V_t^{\mathrm{liq}}(\varphi^{n^\varepsilon})) < -\delta) \leq \delta \quad \text{for all } n \geq n^\varepsilon. \tag{53}$$

We suppose otherwise. Then, one can switch from φ^{n^ε} to φ^n at a stopping time $\widetilde{\tau}_n$ with $P(\widetilde{\tau}_n < \infty) > \delta$ and $\tau \leq \widetilde{\tau}_n \leq \tau^{1,N^\varepsilon}$, $V_{\widetilde{\tau}_n}^{\mathrm{cost}}(\varphi^n) - V_{\widetilde{\tau}_n}^{\mathrm{liq}}(\varphi^{n^\varepsilon}) < -\delta$ on $\{\widetilde{\tau}_n < \infty\}$ (such a stopping time exists by a section theorem for optional sets, see, e.g., He et al. (1992, Theorem 4.7)). This generates a superior strategy (ψ^0, ψ) of the type described above that is a contradiction. Putting together we obtain that

$$P(\inf_{t\in[\tau,\tau^{1,N^\varepsilon}]} (V_t^{\mathrm{cost}}(\varphi^n) - V_\tau^{\mathrm{cost}}(\varphi^n)) < -3\delta) \leq 3\delta \quad \text{for all } n \geq n^\varepsilon. \tag{54}$$

But, Equation (54) implies that

$$P(\inf_{t\in[\tau,\tau^{1,N^\varepsilon}]} ((\varphi^n)^+ 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \overline{X}_t - (\varphi^n)^- 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \underline{X}_t) < -3\delta) \leq 3\delta. \tag{55}$$

The processes $1_{\{T_m>\tau\}}((\varphi^n)^+ 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \overline{X} - (\varphi^n)^- 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \underline{X})$, $n \in \mathbb{N}$, are bounded from below by $-m - M$. By Assumption 2.18, the nth process is a supermartingale with respect to the measure Q^{A_n} with $A_n := \{\varphi^n \geq 0\}$. Thus, we can derive from Equations (55) and (52) that

$$Q^{A_n}(1_{\{T_m>\tau\}} \sup_{t\in[\tau,\tau^{1,N^\varepsilon}]} |(\varphi^n)^+ 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \overline{X}_t - (\varphi^n)^- 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \underline{X}_t| > \gamma) \leq \gamma. \tag{56}$$

From Equation (51), it follows that

$$d_{\mathbb{S}}(1_{\{T_m>\tau\}}((\varphi^n)^+ 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \overline{X} - (\varphi^n)^- 1_{\rrbracket\tau,\tau^{1,N^\varepsilon}\rrbracket} \bullet \underline{X}), 0) \leq \varepsilon^2 \quad \text{for all } n \geq n^\varepsilon \tag{57}$$

and $d_{\mathbb{S}}$ taken with respect to P. By $P(\tau < \sigma \leq \tau^{1,N^\varepsilon}) \leq \varepsilon$, $(\overline{X}, \underline{X})$ coincides with (S, S') on $[\![\tau, \tau^{1,N^\varepsilon}]\!]$ with high probability (cf. Note 3.7). Together with Equation (54), $3\delta \leq \varepsilon$, and Equation (57), we conclude that

$$P(1_{\{T_m > \tau\}} \sup_{t \in [\tau, \tau^{1,N^\varepsilon}]} |V_t^{S,S'}(\varphi^n) - V_\tau^{S,S'}(\varphi^n)| > \varepsilon) \leq 3\varepsilon \quad \text{for all } n \geq n^\varepsilon. \tag{58}$$

Step 3b: Let $\Gamma(\tau)$ be the end time of the excursion. The numbers ε and n^ε are still given by Step 3a. Analogously to τ^{1,N^ε}, again using Equation (36), we find an N' such that $P(\tau^{2,N'} < \Gamma(\tau),\ \sup_{t \in [\tau^{2,N'}, \Gamma(\tau))} V_t^{\text{cost}}(\varphi^{n^\varepsilon}) - V_{\Gamma(\tau)}^{\text{liq}}(\varphi^{n^\varepsilon}) 1_{\{X_{\Gamma(\tau)-} > 0\}} - V_{\Gamma(\tau)-}^{\text{liq}}(\varphi^{n^\varepsilon}) 1_{\{X_{\Gamma(\tau)-} = 0\}} > \delta) \leq \delta$. Since Step 3a is a fortiori true if we reduce $\tau^{1,N}$, we can assume that $N' = N^\varepsilon$. By similar arguments as for Equation (53), we get

$$P(\sup_{t \in [\tau^{2,N^\varepsilon}, \Gamma(\tau))} (V_t^{\text{liq}}(\varphi^n) - V_t^{\text{cost}}(\varphi^{n^\varepsilon})) > \delta) \leq \delta \quad \text{for all } n \geq n^\varepsilon. \tag{59}$$

If Equation (59) did not hold, then one could switch from φ^n to φ^{n^ε} and improve the terminal position. Economically, this means that it could be anticipated if the sequence of strategies performed too bad at the foreseeable end of the excursion, and one would switch to φ^{n^ε} before that happens. By contrast, at the beginning of the excursion, one would switch from φ^{n^ε} to φ^n *after* a bad performance of φ^n, compare Equation (53). Consequently, towards the end of the excursion, the liquidation values of the sequence of strategies can be controlled, instead of the cost values as at the beginning of the excursion. Putting together, we obtain

$$P(V_{\Gamma(\tau)}^{\text{liq}}(\varphi^n) 1_{\{X_{\Gamma(\tau)-} > 0\}} + V_{\Gamma(\tau)-}^{\text{liq}}(\varphi^n) 1_{\{X_{\Gamma(\tau)-} = 0\}} - \sup_{t \in [\tau^{2,N^\varepsilon}, \Gamma(\tau))} V_t^{\text{liq}}(\varphi^n) < -3\delta) \leq 3\delta \tag{60}$$

for all $n \geq n^\varepsilon$. Since the cost term C is nondecreasing, Equation (60) implies that

$$\begin{aligned} &P((\varphi^n)^+ 1_{]\!]\tau^{2,N^\varepsilon}, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-} = 0\}}]\!]} \bullet \underline{X}_T \\ &\quad - (\varphi^n)^- 1_{]\!]\tau^{2,N^\varepsilon}, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-} = 0\}}]\!]} \bullet \overline{X}_T < -3\delta) \leq 3\delta \qquad \text{for all } n \geq n^\varepsilon. \end{aligned} \tag{61}$$

The processes

$$1_{\{T_m > \tau^{2,N^\varepsilon}\}}((\varphi^n)^+ 1_{]\!]\tau^{2,N^\varepsilon}, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-} = 0\}}]\!]} \bullet \underline{X} - (\varphi^n)^- 1_{]\!]\tau^{2,N^\varepsilon}, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-} = 0\}}]\!]} \bullet \overline{X}),$$

$n \in \mathbb{N}$, are bounded from below by $-m - M$, and the nth process is a supermartingale with respect to the measure $\widetilde{Q}^{A_n}$ with $A_n := \{\varphi^n \geq 0\}$. Thus, we can derive from Equations (61) and (52) that

$$\begin{aligned} \widetilde{Q}^{A_n}(1_{\{T_m > \tau^{2,N^\varepsilon}\}} \sup_{t \in [\tau^{2,N^\varepsilon}, \Gamma(\tau)]} &|(\varphi^n)^+ 1_{]\!]\tau^{2,N^\varepsilon}, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-} = 0\}}]\!]} \bullet \underline{X}_t \\ &- (\varphi^n)^- 1_{]\!]\tau^{2,N^\varepsilon}, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-} = 0\}}]\!]} \bullet \overline{X}_t| > \gamma) \leq \gamma \quad \text{for all } n \geq n^\varepsilon. \end{aligned} \tag{62}$$

From Equations (62) and (51), it follows that

$$d_{\mathbb{S}}(1_{\{T_m>\tau^{2,N^\varepsilon}\}}((\varphi^n)^+1_{[\![\tau^{2,N^\varepsilon},\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]}\bullet \underline{X}- (\varphi^n)^-1_{[\![\tau^{2,N^\varepsilon},\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]}\bullet \overline{X}),0)\le\varepsilon^2 \quad \text{for all } n\ge n^\varepsilon \tag{63}$$

and $d_{\mathbb{S}}$ taken with respect to P. Since $\tau^{2,N^\varepsilon}\ge\widetilde{\sigma}$, we have that $(\underline{X},\overline{X})=(S,S')$ on $[\![\tau^{2,N^\varepsilon},\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]$, compare Note 3.7. Together with Equations (60) and (63), we arrive at

$$P(1_{\{T_m>\tau^{2,N^\varepsilon}\}}\sup_{t\in[\tau^{2,N^\varepsilon},\Gamma(\tau))}|V_t^{S,S'}(\varphi^n)-V_{\tau^{2,N^\varepsilon}}^{S,S'}(\varphi^n)|\vee(1_{\{X_{\Gamma(\tau)-}>0\}}|V_{\Gamma(\tau)}^{\mathrm{liq}}(\varphi^n)-V_{\tau^{2,N^\varepsilon}}^{\mathrm{liq}}(\varphi^n)|) > \varepsilon)\le 2\varepsilon \quad \text{for all } n\ge n^\varepsilon. \tag{64}$$

Step 4: Now, we fix $m,N\in\mathbb{N}$ and show how a limiting strategy can be constructed on the stochastic interval $[\![\tau^{1,N},\tau^{2,N}\wedge T_m]\!]$ defined in Equation (48). By Equation (46)/(49), we have

$$|\varphi^n|1_{[\![\tau^{1,N}\wedge T_m,\tau^{2,N}\wedge T_m]\!]}\le(m+M+Mm)/\eta^N=:Y_{m,N}\quad\forall n\in\mathbb{N}, \tag{65}$$

and by Equation (50), $Y_{m,N}$ is a finite random variable (not necessarily bounded). First we observe that by the semimartingale property of S and S', the set

$$\{\psi^+1_{[\![\tau^{1,N},\tau^{2,N}]\!]}\bullet S_T-\psi^-1_{[\![\tau^{1,N},\tau^{2,N}]\!]}\bullet S'_T\ :\ \psi \text{ is a predictable process with } |\psi|\le Y_{m,N}\} \tag{66}$$

is L^0-bounded (namely, for a given probability of error, $Y_{m,N}$ can be estimated by a constant). This implies that the set

$$\mathrm{conv}(C^{S,S'}(\varphi^n,[\tau^{1,N},\tau^{2,N}\wedge T_m]);n\in\mathbb{N}) \quad \text{is also } L^0\text{-bounded.} \tag{67}$$

Note that since L^0 is not locally convex, the L^0-boundedness of $(C^{S,S'}(\varphi^n,[\tau^{1,N},\tau^{2,N}]))_{n\in\mathbb{N}}$ would be potentially weaker. However, we can consider a convex combination of strategies $\varphi^n,\varphi^{n+1},\dots,\varphi^{n+k}$ but executing them through different trading accounts. This means that we do not benefit from the subadditivity of $C^{S,S'}$. Since the resulting strategies are still M-admissible, Equation (67) follows from Equations (65) and (66).

The processes $\underline{X}$, $\overline{X}$, S, and S' are càdlàg, and the paths of $\overline{X}-\underline{X}$ are bounded away from zero on $[\tau^{1,N},\tau^{2,N})$. The latter holds since $\tau^{2,N}<\Gamma(\tau)$ on $\{\overline{X}_{\Gamma(\tau)-}=\underline{X}_{\Gamma(\tau)-}\}$. Consequently, we have that $\overline{X}-S\ge(\overline{X}_{\tau^{1,N}}-\underline{X}_{\tau^{1,N}})/3>0$ or $S-\underline{X}\ge(\overline{X}_{\tau^{1,N}}-\underline{X}_{\tau^{1,N}})/3>0$ on an interval with positive random length, and after finitely many analogous estimates, we arrive at $\tau^{2,n}$. The same holds for $\overline{X}-S'$ and $S'-\underline{X}$. By Equation (67) and Kühn and Molitor (2022, Proposition 3.3), this implies that φ^n, $n\in\mathbb{N}$ are processes of finite variation on $[\![\tau^{1,N},\tau^{2,N}]\!]$. Furthermore, by $|(\varphi^n)^\uparrow_t-(\varphi^n)^\uparrow_s-((\varphi^n)^\downarrow_t-(\varphi^n)^\downarrow_s)|\le 2Y_{m,N}$ for all $\tau^{1,N}\le s\le t\le\tau^{2,N}\wedge T_m$ and $n\in\mathbb{N}$ (cf. Equation (65)), it implies that $\mathrm{conv}((\varphi^n)^\uparrow_{\tau^{2,N}\wedge T_m}-(\varphi^n)^\uparrow_{\tau^{1,N}\wedge T_m};n\in\mathbb{N})$ and $\mathrm{conv}((\varphi^n)^\downarrow_{\tau^{2,N}\wedge T_m}-(\varphi^n)^\downarrow_{\tau^{1,N}\wedge T_m};n\in\mathbb{N})$ are L^0-bounded, too.

Now, we can proceed as in Schachermayer (2014, proof of Theorem 3.4). It is a stochastic version of Helly's classic theorem that shows the existence of a converging subsequence of monotone functions on the real line. We repeat only the results that are needed in the present paper. On

$[\![\tau^{1,N}, \tau^{2,N} \wedge T_m]\!]$, there exists a predictable process φ such that after passing to forward convex combinations, $\varphi^n \to \varphi$ up to evanescence. If one applies the same construction for a larger pair (N, m) (and thus on a larger subinterval of the excursion), the strategy coincides with φ on the smaller subinterval up to evanescence. By $P(T_m = \infty) \to 1$ as $m \to \infty$ and Equation (48), we arrive at a predictable process φ with

$$\varphi^n \to \varphi \quad \text{on }]\!]\tau, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!] \quad \text{up to evanescence}, \quad n \to \infty, \tag{68}$$

after passing to joint forward convex combinations using a diagonalization argument.

Step 5: In this step, we want to show that $\varphi^+ 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \in L(S)$ and $\varphi^- 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \in L(S')$, where the semimartingales S and S' are defined in Note 3.7. Let $\varepsilon > 0$. We choose $m \in \mathbb{N}$ such that $P(T_m < \infty) \leq \varepsilon^2$. The stopping times τ^{1,N^ε}, τ^{2,N^ε} are from Step 2. They actually also depend on m. Beforehand, we observe that the $d_{\mathbb{S}}$-distance of two semimartingales is bounded from above by the probability that their paths do not coincide. Then, by Equation (57) and the triangle inequality of the metric $d_{\mathbb{S}}$, we obtain

$$d_{\mathbb{S}}((\varphi^{n_1})^+ 1_{]\!]\tau,\tau^{1,N^\varepsilon}]\!]} \bullet \overline{X} - (\varphi^{n_1})^- 1_{]\!]\tau,\tau^{1,N^\varepsilon}]\!]} \bullet \underline{X}, (\varphi^{n_2})^+ 1_{]\!]\tau,\tau^{1,N^\varepsilon}]\!]} \bullet \overline{X} - (\varphi^{n_2})^- 1_{]\!]\tau,\tau^{1,N^\varepsilon}]\!]} \bullet \underline{X}) \leq 3\varepsilon^2$$

for all $n_1, n_2 \geq n^\varepsilon$. By Equation (63), we have the analog estimate at the end of the excursion. Since $\varphi^n 1_{]\!]\tau^{1,N^\varepsilon}\wedge T_m, \tau^{2,N^\varepsilon}\wedge T_m]\!]} \leq Y_{m,N^\varepsilon}$ for all $n \in \mathbb{N}$, all strategies are bounded by $y \in \mathbb{R}_+$ on $]\!]\tau^{1,N^\varepsilon}, \tau^{2,N^\varepsilon}]\!]$ outside the event $\{T_m < \infty\} \cup \{Y_{m,N^\varepsilon} > y\}$ that does not depend on n and has smaller probability than $2\varepsilon^2$ for y large enough (with a corresponding bound of the effect on $d_{\mathbb{S}}$). Consequently, we can argue with the dominated convergence theorem for stochastic integrals (cf., for example, Cohen and Elliott, 2015, Theorem 12.4.10) and the pointwise convergence (68) to deduce that

$$\begin{aligned} &d_{\mathbb{S}}((\varphi^{n_1})^+ 1_{]\!]\tau^{1,N^\varepsilon},\tau^{2,N^\varepsilon}]\!]} \bullet S - (\varphi^{n_1})^- 1_{]\!]\tau^{1,N^\varepsilon},\tau^{2,N^\varepsilon}]\!]} \bullet S', \\ &\quad (\varphi^{n_2})^+ 1_{]\!]\tau^{1,N^\varepsilon},\tau^{2,N^\varepsilon}]\!]} \bullet S - (\varphi^{n_2})^- 1_{]\!]\tau^{1,N^\varepsilon},\tau^{2,N^\varepsilon}]\!]} \bullet S') \leq 3\varepsilon^2 \end{aligned}$$

for n_1, n_2 large enough. Since $\varepsilon > 0$ was arbitrary and $(\varphi^n)^+(\varphi^n)^- = 0$, we conclude that the sequences

$$((\varphi^n)^+ 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \bullet S)_{n\in\mathbb{N}}, \quad ((\varphi^n)^- 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \bullet S')_{n\in\mathbb{N}}$$

and a fortiori the sequences

$$(((\varphi^n)^+ \wedge \varphi^+) 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \bullet S)_{n\in\mathbb{N}}, \; (((\varphi^n)^- \wedge \varphi^-) 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \bullet S')_{n\in\mathbb{N}}$$

are $d_{\mathbb{S}}$-Cauchy. Therefore, together with Equation (68) and $(\varphi^n)^+ \wedge \varphi^+ \leq \varphi^+$, $(\varphi^n)^- \wedge \varphi^- \leq \varphi^-$, we are in the position to apply Chou et al. (1980) (see also Kühn and Molitor, 2022, Note 4.4) and arrive at

$$\varphi^+ 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \in L(S) \quad \text{and} \quad \varphi^- 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \in L(S'). \tag{69}$$

Step 6: Let us show that $\varphi 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \in L(\underline{X},\overline{X})$ (this states that φ, which is not yet globally defined, satisfies the "local" properties (6)–(8) of Definition 2.6 on the interval $\mathcal{I}_i^c$). As a candidate for the "optimal" sequence $(\psi^N)_{N\in\mathbb{N}} \subseteq \mathbf{b}\mathcal{P}$, we take

$$\psi^N := \operatorname{median}(-y_N, \varphi, y_N) 1_{]\!]\tau^{1,N},\tau^{2,N}\wedge T_{m_N}]\!]} \in \mathbf{b}\mathcal{P},$$

where $m_N \in \mathbb{N}$ is large enough such that $P(T_{m_N} < \infty) \le 1/N$ and $y_N \in \mathbb{R}_+$ is large enough such that $P(Y_{m_N,N} > y_N) \le 1/N$.

Let $\varepsilon > 0$ and m large enough such that $P(T_m < \infty) \le \varepsilon$. By Equations (57) and (58), we have $P(T_m = \infty,\ C^{S,S'}_{\tau^{1,N^\varepsilon}}(\varphi^n) - C^{S,S'}_{\tau^{1,N}}(\varphi^n) > 2\varepsilon) \le 4\varepsilon$ for all $n \ge n_\varepsilon$ and $N \ge N^\varepsilon$. By the Fatou-type estimate in Kühn and Molitor (2022, Proposition 3.11), the cost term of the limiting strategy cannot be higher in the sense that

$$P(T_m = \infty,\ Y_{m,N} \le y_N,\ C^{S,S'}_{\tau^{1,N^\varepsilon}}(\varphi) - C^{S,S'}_{\tau^{1,N}}(\varphi) > 3\varepsilon) \le 4\varepsilon \quad \text{for all } N \ge N^\varepsilon. \tag{70}$$

By Equation (69) and again by the dominated convergence theorem for stochastic integrals, we find $N' \ge N^\varepsilon$ such that $d_{\mathbb{S}}(\varphi^+ 1_{]\!]\tau,\tau^{1,N'}]\!]} \bullet S - \varphi^- 1_{]\!]\tau,\tau^{1,N^\varepsilon}]\!]} \bullet S', 0) \le \varepsilon^2$. Adding up the increments, we get that

$$P(T_m = \infty,\ Y_{m,N} \le y_N,\ \sup_{t\in[\tau^{1,N},\tau^{1,N'}]} |V^{S,S'}_t(\varphi) - V^{S,S'}_{\tau^{1,N}}(\varphi)| > 4\varepsilon) \le 5\varepsilon \quad \text{for all } N \ge N'. \tag{71}$$

By Equations (63) and (64), the analog estimate holds for the end of the excursion. Since $S = \overline{X}$, $S' = \underline{X}$ on $]\!]\tau,\sigma]\!]$, the strategy ψ^N does not produce trading costs at time $\tau^{1,N}$, and we have that $(V^{S,S'}(\psi^N))_{N\in\mathbb{N}}$ is an up-Cauchy sequence. Since trading gains and trading costs converge separately, condition (7) follows from Equation (6) by the arguments in the proof of Proposition 2.10.

Now let $(\widetilde{\psi}^N)_{N\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ be a competing sequence with $(\widetilde{\psi}^N)^+ \le \varphi^+$, $(\widetilde{\psi}^N)^- \le \varphi^-$, and $\widetilde{\psi}^N \to \varphi$ pointwise. Since, by Step 5, φ^+, φ^- are integrable with respect to the semimartingale price systems, it follows again from the dominated convergence theorem for stochastic integrals that $d_{\mathbb{S}}((\widetilde{\psi}^N)^+ \bullet S - (\widetilde{\psi}^N)^- \bullet S', \varphi^+ \bullet S - \varphi^- \bullet S') \to 0$ as $N \to \infty$ and thus

$$d_{\mathbb{S}}((\widetilde{\psi}^N)^+ \bullet S - (\widetilde{\psi}^N)^- \bullet S', (\psi^N)^+ \bullet S - (\psi^N)^- \bullet S') \to 0, \quad N \to \infty. \tag{72}$$

On the other hand, again by Kühn and Molitor (2022, Proposition 3.11), the cost term of the competing sequence can be estimated from below by

$$\liminf_{N\to\infty} C^{S,S'}(\widetilde{\psi}^N) \ge C^{S,S'}(\psi^{N'}) - C^{S,S'}_{\tau^{1,N'}}(\psi^{N'}) \quad \text{for every } N' \in \mathbb{N}. \tag{73}$$

By Equation (70), $C^{S,S'}_{\tau^{1,N'}}(\psi^{N'}) \to 0$ in probability as $N' \to \infty$. Thus, putting Equations (72) and (73) together yields (8), and we arrive at

$$\varphi 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \in L(\underline{X},\overline{X}) \quad \text{for } \tau = (\tau_1^i)_{\{X_{\tau_1^i}=0\}},\ i \in \mathbb{N}. \tag{74}$$

We define the wealth process starting in zero at time $(\tau_1^i)_{\{X_{\tau_1^i}=0\}}$ as

$$V^{1,i} := \operatorname*{up-lim}_{N\to\infty} V^{S,S'}(\psi^N) \quad \text{on } \mathcal{I}_i^c =]\!](\tau_1^i)_{\{X_{\tau_1^i}=0\}}, \Gamma(\tau_1^i)]\!] \setminus [\![(\Gamma(\tau_1^i))_{\{X_{\Gamma(\tau_1^i)-}=0\}}]\!], \quad i \in \mathbb{N} \tag{75}$$

(it corresponds to $1_{\mathcal{I}_i^c} \bullet V$, but the global wealth process V in Definition 2.6 is not yet defined).

Step 7: By Equation (43), we have that for fixed $i \in \mathbb{N}$, $(\varphi^n_{\Lambda(\sigma_1^i)} 1_{\{X_{\Lambda(\sigma_1^i)}>0\}})_{n\in\mathbb{N}}$ is L^0-bounded. After passing once again to forward convex combinations, it converges a.s. to some $\mathcal{F}_{\Lambda(\sigma_1^i)}$-measurable finite random variable ψ^i. By a diagonalization argument, one finds a joint sequence for all $i \in \mathbb{N}$ (cf. the paragraph at the end of the proof). With this, one can argue as in Step 4, but starting directly at the beginning of the excursion. Step 3a is not needed, and the arguments in Steps 5 and 6 only become simpler. Consequently, we obtain a predictable process φ that satisfies

$$(\varphi - \psi^i) 1_{]\!]\tau,\Gamma(\tau)]\!]\setminus[\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]} \in L(\underline{X}, \overline{X}) \quad \text{where } \tau := (\Lambda(\sigma_1^i))_{\{X_{\Lambda(\sigma_1^i)}>0\}},\ i \in \mathbb{N}. \tag{76}$$

On $]\!]\tau, \Gamma(\tau)]\!] \setminus [\![(\Gamma(\tau))_{\{X_{\Gamma(\tau)-}=0\}}]\!]$, the limiting wealth process $V^{2,i}$ is defined as in Equation (75). Observe that the convergence of the strategies at the starting time of the excursion on $\{X_\tau > 0\}$ follows from an analysis of the frictionless intervals whereas at the end the behavior can be explained by considering only the excursion itself.

Step 8: Let $\mathcal{I}_i^f := [\![(\sigma_1^i)_{\{X_{\sigma_1^i-}=0\}}]\!] \cup]\!]\sigma_1^i, \Lambda(\sigma_1^i)]\!]$ be a frictionless interval. It is $(\mathcal{F}_t)_{t\in[0,T]}$-predictable and thus a fortiori $\widetilde{\mathbb{F}}$-predictable, and we have that

$$\begin{aligned} 1_{\mathcal{I}_i^f} \bullet (\widetilde{V}^n)^r &= (\varphi^n)^+ 1_{\mathcal{I}_i^f} \bullet S - (\varphi^n)^- 1_{\mathcal{I}_i^f} \bullet S' \\ &+ 1_{(\cdot \geq \Lambda(\sigma_1^i))} 1_{\{X_{\Lambda(\sigma_1^i)}>0\}} (\Delta \widetilde{V}^n_{\Lambda(\sigma_1^i)} - (\varphi^n_{\Lambda(\sigma_1^i)})^+ \Delta S_{\Lambda(\sigma_1^i)} + (\varphi^n_{\Lambda(\sigma_1^i)})^- \Delta S'_{\Lambda(\sigma_1^i)}), i \in \mathbb{N}, \end{aligned} \tag{77}$$

where $\widetilde{V}^n$ is defined in Equation (38) and $(\widetilde{V}^n)^r$ is its "càdlàg part" in the sense of Lemma 3.11. By Lemma 3.13, $(1_{\mathcal{I}_i^f} \bullet (\widetilde{V}^n)^r)_{n\in\mathbb{N}}$ is $\widetilde{d}_{\mathbb{S}}$-Cauchy and $(\Delta \widetilde{V}^n_{\Lambda(\sigma_1^i)})_{n\in\mathbb{N}}$ is Cauchy with regard to the convergence in probability. Together with the convergence of $(\varphi^n_{\Lambda(\sigma_1^i)} 1_{\{X_{\Lambda(\sigma_1^i)}>0\}})_{n\in\mathbb{N}}$ established in Step 7, we conclude with Equation (77) that $((\varphi^n)^+ 1_{\mathcal{I}_i^f} \bullet S - (\varphi^n)^- 1_{\mathcal{I}_i^f} \bullet S')_{n\in\mathbb{N}}$ is $\widetilde{d}_{\mathbb{S}}$-Cauchy (and since S and S' are càdlàg $(\mathcal{F}_t)_{t\in[0,T]}$-semimartingales, the sequence is also $d_{\mathbb{S}}$-Cauchy). This means that although at the end of the frictionless interval, the spread can be positive, we have established a corresponding Cauchy sequence in a purely frictionless market. By Mémin's theorem (see Mémin, 1980, Theorem V.4), there exists a predictable process φ such that $\varphi^+ 1_{\mathcal{I}_i^f} \in L(S)$, $\varphi^- 1_{\mathcal{I}_i^f} \in L(S')$, and

$$d_{\mathbb{S}}((\varphi^n)^+ 1_{\mathcal{I}_i^f} \bullet S - (\varphi^n)^- 1_{\mathcal{I}_i^f} \bullet S', \varphi^+ 1_{\mathcal{I}_i^f} \bullet S - \varphi^- 1_{\mathcal{I}_i^f} \bullet S') \to 0 \quad n \to \infty. \tag{78}$$

In fact, it only follows the existence of a limiting strategy in $L((S, S'))$, the set of two-dimensional predictable processes which are integrable with respect to (S, S'). But, by $(\varphi^n)^+(\varphi^n)^- = 0$, the proof (that argues with $\mathcal{M}^2 \otimes \mathcal{A}$) shows the stronger assertion above. We can and do assume that

$$\varphi = \lim_{n\to\infty} \varphi^n \quad \text{on the set } \{\lim_{n\to\infty} \varphi^n \text{ exists in } \mathbb{R}\} \cap \mathcal{I}_i^f \in \mathcal{P}. \tag{79}$$

To see this, define $\widetilde{\varphi} := \lim_{n\to\infty} \varphi^n$ if this limit exists in $\mathbb{R}$ and $\widetilde{\varphi} := \varphi$ otherwise. Then, using that $\{\widetilde{\varphi} \neq \varphi\} = \cup_{k\in\mathbb{N}}\{|\varphi^n| \leq k \ \forall n \in \mathbb{N}\} \cap \{\widetilde{\varphi} \neq \varphi\}$ it follows that the process $1_{\{\widetilde{\varphi}^+ \neq \varphi^+\}} \bullet S - 1_{\{\widetilde{\varphi}^- \neq \varphi^-\}} \bullet S'$ must be evanescent by Equation (78) and the dominated convergence theorem for stochastic integrals.

Step 9: Putting together the partial constructions of the previous steps, we have a candidate for a global limiting strategy. Summing up:

$$\varphi := \begin{cases} \text{according to Equation (78)} & \text{on } \{X_- = 0\} \\ \text{according to Equation (74)} & \text{on } [\![(\tau_1^i)_{\{X_{\tau_1^i}=0\}}, \Gamma(\tau_1^i)]\!] \setminus [\![(\Gamma(\tau_1^i))_{\{X_{\Gamma(\tau_1^i)-}=0\}}]\!],\ i \in \mathbb{N} \\ \text{according to Equation (76)} & \text{on } [\![(\Lambda(\sigma_1^i))_{\{X_{\Lambda(\sigma_1^i)}>0\}}, \Gamma(\Lambda(\tau_1^i))]\!] \setminus [\![(\Gamma(\Lambda(\tau_1^i)))_{\{X_{\Gamma(\Lambda(\tau_1^i))-}=0\}}]\!],\ i \in \mathbb{N}. \end{cases} \tag{80}$$

We note that by Equation (79), one has that $\psi^i = \varphi_{\Lambda(\sigma_1^i)} 1_{\{X_{\Lambda(\tau_1^i)}>0\}}$ a.s. for all $i \in \mathbb{N}$ in Equation (76).

Using the process $\widetilde{V}$ from Lemma 3.13 that is the semimartingale limit in the coarser $\widetilde{\mathbb{F}}$-model and the processes from Equation (75), we define the $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process V by

$$V_t := \widetilde{V}_{\tau_1^i} + V_t^{1,i} \quad \text{for } t \in (\tau_1^i, \Gamma(\tau_1^i)),\ X_{\tau_1^i} = 0$$

$$V_t := \widetilde{V}_{\Lambda(\sigma_1^i)-} + \varphi^+_{\Lambda(\sigma_1^i)} \Delta S_{\Lambda(\sigma_1^i)} - \varphi^-_{\Lambda(\sigma_1^i)} \Delta S'_{\Lambda(\sigma_1^i)} + V_t^{2,i} \text{ for } t \in [\Lambda(\sigma_1^i), \Gamma(\Lambda(\sigma_1^i))), X_{\Lambda(\sigma_1^i)} > 0$$

$$V_t := \widetilde{V}_t \quad \text{otherwise.}$$

By Steps 6 and 7, we have that V satisfies (6), (7), and the corresponding "optimal" sequences satisfy Equation (8). Let $\widetilde{V}^r$ be the "càdlàg part" of $\widetilde{V}$ as defined in Lemma 3.11. Since $1_{\mathcal{I}_i^c} \bullet V = \Delta^+ \widetilde{V}_{\tau_1^i} 1_{\{X_{\tau_1^i}=0\}} 1_{[\![\tau_1^i, T]\!]}$ and $1_{\mathcal{I}_i^{fc}} \bullet V = 1_{\mathcal{I}_i^{fc}} \bullet \widetilde{V}^r$ both on $\{X = 0\}$, and $\mathcal{I}_i^{fc}$ is $\widetilde{\mathbb{F}}$-predictable, it follows from the continuity of the integral with respect to $\widetilde{V}$ (see Lemma 3.11 and its proof) that V satisfies Equation (5). We arrive at $\varphi \in L(\underline{X}, \overline{X})$.

Step 10: Finally, we observe that by Equation (68) and $\varphi^n_{\Lambda(\sigma_1^i)} \to \varphi_{\Lambda(\sigma_1^i)}$ a.s. on $\{X_{\Lambda(\sigma_1^i)} > 0\}$, Lemma 3.6 is applicable, and thus $(\Pi(\varphi), \varphi)$ is M-admissible. This completes the proof.

To construct φ, it was necessary to pass to forward convex combinations of the sequence $(\varphi^n)_{n\in\mathbb{N}}$ introduced in Equation (35) at several places in the proof. Let us look at the whole picture.

Once we pass to forward convex combinations in Lemma 3.13. Once again we do it for the cost value processes. Then, we pass to these combinations for each excursion (Step 4) and each end time of a frictionless interval (Step 7). For the latter two, one again applies a diagonalization argument to obtain a joint sequence of forward convex combinations. □

We reformulate the implication $(i) \Rightarrow (ii)$ of Delbaen and Schachermayer (1994, Theorem 4.2) to make it directly applicable to our two-dimensional setting:

Theorem 3.15. *If a cone $C_0 \subseteq L^0(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ is Fatou-closed (in the sense of Theorem 2.23), then $C := C_0 \cap L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ is $\sigma(L^\infty, L^1)$-closed with respect to the measure space $\widetilde{\Omega} := \Omega \times \{0, 1\}$, $\widetilde{\mathcal{F}} := \mathcal{F} \otimes 2^{\{0,1\}}$, $\mu := P \otimes (\delta_0 + \delta_1)$, where δ_0, δ_1 denote Dirac measures.*

Theorem 3.16 (2-dimensional version of Kreps-Yan). *Let C be a $\sigma(L^\infty, L^1)$-closed convex cone in $L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ containing $L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2_-)$ and such that $C \cap L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2_+) = \{0\}$. Then,*

there exists $(G^0, G) \in L^1(\Omega, \mathcal{F}, P; (\mathbb{R}_+ \setminus \{0\})^2)$ *with* $E(G^0) = 1$ *such that* $E(C^0 G^0 + CG) \leq 0$ *for all* $(C^0, C) \in \mathcal{C}$.

Of course, Theorem 3.16 is completely standard. One may again consider the measure space $\widetilde{\Omega} := \Omega \times \{0,1\}, \widetilde{\mathcal{F}} := \mathcal{F} \otimes 2^{\{0,1\}}, \mu := P \otimes (\delta_0 + \delta_1)$ and apply the one-dimensional proof of "⇒" in Delbaen and Schachermayer (2006, Theorem 5.2.2) (noting that only G^0 is normalized).

Proof of Theorem 2.22. (i) By Theorems 2.23, 3.15, and 3.16, there exists $(G^0, G) \in L^1(\Omega, \mathcal{F}, P; (\mathbb{R}_+ \setminus \{0\})^2)$ with $E(\varphi_T^0 G^0 + \varphi_T G) \leq 0$ for all $(\varphi^0, \varphi) \in \mathcal{A}$. For the convenience of the reader, we repeat and adjust the arguments in Schachermayer (2014, Definition 4.1 and Proposition 4.2). One defines càdlàg P-martingales by $Z_t^0 = E(G^0|\mathcal{F}_t)$, $Z_t = E(G|\mathcal{F}_t)$, and sets $dQ/dP := Z_T^0 >$ a.s., $S := Z/Z^0$. By construction, S is a (true) Q-martingale. Let us show that $\underline{X} \leq S \leq \overline{X}$. Assume by contradiction that there exists a stopping time τ with $P(\tau < \infty) > 0$ and $Z_\tau/Z_\tau^0 > \overline{X}_\tau$ on $\{\tau < \infty\}$. We consider the strategy $(\varphi^0, \varphi) := (-(\overline{X}_\tau \wedge 1), 1 \wedge (1/\overline{X}_\tau))1_{]\!]\tau, T]\!]}$ that is bounded in both components and satisfies

$$E(\varphi_T^0 Z_T^0 + \varphi_T Z_T) = E((\varphi_{\tau+}^0 Z_\tau^0 + \varphi_{\tau+} Z_\tau)1_{\{\tau<\infty\}}) > 0,$$

a contradiction. Analogously, for $Z_\tau/Z_\tau^0 < \underline{X}_\tau$, one considers the bounded strategy $(\varphi^0, \varphi) := (\underline{X}_\tau \wedge 1, -(1 \wedge (1/\underline{X}_\tau)))1_{]\!]\tau, T]\!]}$.

(ii) Let $Q \sim P$, S be a Q-martingale, and $\underline{S} = \overline{S} = S$. The martingale property implies that $\underline{X} = \overline{X} = S$. Thus, Assumptions 2.3 and 2.18 are automatically satisfied. It remains to show that the model with bid price S and ask price S satisfies NA^{nf} and NUPBR^{ps}. The 1-admissibility condition boils down to $\varphi^0 + \varphi S \geq -(1 + S)$. By Kallsen (2004, Lemma 3.3 and Proposition 3.1), $V = \varphi \bullet S$ has to be a Q-supermartingale for any admissible strategy. This means that for every $(\varphi^0, \varphi) \in \mathcal{A}$ either $P(\varphi_T^0 + \varphi_T S_T = 0) = 1$ or $P(\varphi_T^0 + \varphi_T S_T < 0) > 0$. Together with the condition $P(S_T > 0) = 1$, compare Equation (1), we obtain NA^{nf}.

For $(\varphi^0, \varphi) \in \mathcal{A}_0^1$ and $a \in \mathbb{R}_+$, we define $\tau_a := \inf\{t \geq 0 : V_t(\varphi) > a\}$. Since S is a Q-martingale and V is a Q-supermartingale, one has

$$aQ(\tau_a < \infty) \leq E_Q(V_{\tau_a \wedge T}(\varphi)^+) \leq E_Q(V_{\tau_a \wedge T}(\varphi)^-) \leq 1 + E_Q(S_{\tau_a \wedge T}) = 1 + S_0.$$

Since the RHS does not depend on φ, the market satisfies Equation (11). Since we have a frictionless market satisfying NFLVR with numéraire $1 + S$, we can apply (Delbaen and Schachermayer, 1994, Proposition 3.2 and Lemma 4.5) and any "maximal" sequence of wealth processes is d_{up}-Cauchy. This implies condition (12), and we arrive at NUPBR^{ps}. □

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APPENDIX A

Proposition A.1. *There exists a càdlàg process* $\underline{X}$ *with* $\underline{X}_t = \operatorname{essinf}_{\mathcal{F}_t} \sup_{u\in[t,T]} \underline{S}_u$ *a.s. for all* $t \in [0,T]$, *where*

$$\operatorname{essinf}_{\mathcal{F}_t} \sup_{u\in[t,T]} \underline{S}_u := \operatorname{esssup}\{Z : \Omega \to \overline{\mathbb{R}} : \ Z \text{ is } \mathcal{F}_t\text{-measurable and } Z \le \sup_{u\in[t,T]} \underline{S}_u \text{ a.s.}\}. \quad \text{(A.1)}$$

Proof. Let D be a dense countable subset of $[0,T]$ with $T \in D$. Let $\widetilde{X}_t$, $t \in D$, be versions of $\operatorname{essinf}_{\mathcal{F}_t} \sup_{u\in[t,T]} \underline{S}_u$ (for existence and uniqueness up to null sets cf., e.g., He et al. (1992, Definition 1.12 and Theorem 1.13)) and

$$A := \{\widetilde{X}_{t_1} \le \widetilde{X}_{t_2} + \sup_{u\in[t_1,t_2]} \underline{S}_u - \underline{S}_{t_2} \quad \text{for all } t_1, t_2 \in D \text{ with } t_1 < t_2\}.$$

For every $\mathcal{F}_{t_1}$-measurable random variable Z with $Z \le \sup_{u\in[t_1,T]} \underline{S}_u$, the random variable $Z' := Z + \underline{S}_{t_2} - \sup_{u\in[t_1,t_2]} \underline{S}_u$ is an $\mathcal{F}_{t_2}$-measurable lower bound of $\sup_{u\in[t_2,T]} \underline{S}_u$. This implies that $P(A) = 1$, and by the usual conditions, we can define the adapted process $\underline{X}$ (up to evanescence) by

$$\underline{X}_t(\omega) := \begin{cases} \lim_{q\in D, q>t, q\to t} \widetilde{X}_q(\omega) & \text{for } \omega \in A \text{ and } t < T \\ \underline{S}_t(\omega) & \text{otherwise.} \end{cases}$$

Namely, for $\omega \in A$, the above limit exists by the right-continuity of $\underline{S}$. The existence of finite left limits carries over from $\underline{S}$ to $\underline{X}$, and the process $\underline{X}$ is càdlàg. It remains to show that for every $t \in [0,T)$, $\underline{X}_t$ is a version of $\operatorname{essinf}_{\mathcal{F}_t} \sup_{u\in[t,T]} \underline{S}_u =: \widehat{X}_t$. The estimate $P(\widehat{X}_t \le \underline{X}_t) = 1$ follows from the same arguments which lead to $P(A) = 1$ combined with the right-continuity of $\underline{S}$. For the opposite estimate, we follow an argument from Larsson (Preprint, Lemma 2.8(iv)): for a sequence $(q_n)_{n\in\mathbb{N}} \subseteq D$ with $q_n > t$ and $q_n \to t$, one has $\inf_{n\in\mathbb{N}} \operatorname{essinf}_{\mathcal{F}_{q_n}} \sup_{u\in[t,T]} \underline{S}_u = \widehat{X}_t$ a.s., since $\cap_{n\in\mathbb{N}} \mathcal{F}_{q_n} = \mathcal{F}_t$. Together with $\widetilde{X}_{q_n} \le \operatorname{essinf}_{\mathcal{F}_{q_n}} \sup_{u\in[t,T]} \underline{S}_u$, we arrive at $P(\widehat{X}_t = \underline{X}_t) = 1$. □

Proposition A.2. *Let X be a càdlàg process. Then, there exists a unique (up to evanescence) predictable process Y with $Y_\tau = \operatorname{essinf}_{\mathcal{F}_{\tau-}} X_\tau$ a.s. for all predictable stopping times τ (where $\operatorname{essinf}_{\mathcal{F}_{\tau-}}$ ... is defined analogously to (A.1) with the σ-algebra $\mathcal{F}_{\tau-}$).*

Proof. The set $\{\Delta X \neq 0\}$ is thin, that is, there exists a sequence of stopping times with $\{\Delta X \neq 0\} = \bigcup_{n\in\mathbb{N}} [\![T_n]\!]$ (cf., e.g., He et al., 1992, Theorem 3.32). For each $n \in \mathbb{N}$, let $(\sigma_{n,m})_{m\in\mathbb{N}}$ be a maximal sequence of predictable stopping times that access the accessible part of the graph of T_n (in the sense of He et al., 1992, proof of Theorem 4.20). This means that $[\![T_n]\!] \setminus \bigcup_{m\in\mathbb{N}} [\![\sigma_{n,m}]\!]$ cannot be overlapped by the graph of a predictable stopping time with positive probability. We pass to a single sequence $(\sigma_k)_{k\in\mathbb{N}}$ and obtain the same property for $\{\Delta X \neq 0\} \setminus \bigcup_{k\in\mathbb{N}} [\![\sigma_k]\!]$. We can and do choose the sequence such that $P(\sigma_{k_1} = \sigma_{k_2} < \infty) = 0$ for all $k_1 \neq k_2$. Let us define the process

$$Y := \begin{cases} X_- & \text{on } (\Omega \times [0,T]) \setminus \bigcup_{k\in\mathbb{N}} [\![\sigma_k]\!] \\ \operatorname{essinf}_{\mathcal{F}_{\sigma_k-}} X_{\sigma_k} & \text{on } [\![\sigma_k]\!] \text{ for some } k \in \mathbb{N} \end{cases} \tag{A.2}$$

that is obviously predictable. Now, let τ be a predictable stopping time. We have that

$$Y_\tau = X_{\tau-} 1_{\{\tau \neq \sigma_k \ \forall k\in\mathbb{N}\}} + \sum_{k\in\mathbb{N}} \operatorname{essinf}_{\mathcal{F}_{\sigma_k-}} X_{\sigma_k} 1_{\{\tau=\sigma_k\}} \quad \text{a.s.} \tag{A.3}$$

In Equation (A.3), $X_{\tau-}$ can be replaced by X_τ or $\operatorname{essinf}_{\mathcal{F}_{\tau-}} X_\tau$ since the sequence $(\sigma_k)_{k\in\mathbb{N}}$ is maximal which implies that X does not jump on this set with positive probability. On $\{\tau = \sigma_k\} \in \mathcal{P}$, we have that $\operatorname{essinf}_{\mathcal{F}_{\sigma_k-}} X_{\sigma_k} = \operatorname{essinf}_{\mathcal{F}_{\tau-}} X_\tau$ a.s. (we leave it to the reader to check this "local property" of essinf). Together, we obtain that Y satisfies the properties of the proposition. Almost-sure-uniqueness along predictable stopping times follows in the same way. Then, an application of a section theorem for predictable sets (see, e.g., He et al., 1992, Theorem 4.8) shows uniqueness of the predictable process up to evanescence. □

Proposition A.3. *(a) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and Y be a supermartingale starting at zero with $-1 \le Y \le \varepsilon_1$ and $P(Y_T < -\varepsilon_2) \le \varepsilon_3$. Then, the Doob–Meyer decomposition $Y = M - A$ (i.e., M is a martingale, and A is a nondecreasing, predictable process with $M_0 = A_0 = 0$) satisfies $E(A_T^2) \le (\varepsilon_1 + 1)(\varepsilon_2 + \varepsilon_3)$ and $E(M_T^2) \le \varepsilon_1^2 + \varepsilon_2 + \varepsilon_2^2 + 2\varepsilon_3 + 3\varepsilon_1\varepsilon_2 + 3\varepsilon_1\varepsilon_3$, that is, the second moments are of order* $\max(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

(b) There exists a $C \in \mathbb{R}_+$ such that for all $\varepsilon > 0$ and all supermartingales $\widetilde{Y}$ starting at zero with $-1 \le \widetilde{Y}$ and $P(\sup_{t\in[0,T]} |\widetilde{Y}_t| > \varepsilon) \le \varepsilon$, one has $d_{\mathbb{S}}(\widetilde{Y}, 0) \le C\sqrt{\varepsilon}$.

Proof. *Step 1*: We show assertion (a) for the discrete time Doob(-Meyer) decomposition along a finite deterministic grid, w.l.o.g. $0, 1, \dots, T-1, T$ with $T \in \mathbb{N}$. For the first moment of A_T, we have $E(A_T) = E(-Y_T) \leq \varepsilon_2 + \varepsilon_3$. Following the proof of Meyer (1972, Theorem II.45), we use this for an estimate of the second moment:

$$\begin{aligned} E(A_T^2) &\leq 2E\Big(\sum_{s=1}^{T}\sum_{t=s}^{T}(A_s - A_{s-1})(A_t - A_{t-1})\Big) = 2\sum_{s=1}^{T} E((A_s - A_{s-1})(A_T - A_{s-1})) \\ &= 2\sum_{s=1}^{T} E((A_s - A_{s-1})E(A_T - A_{s-1} \mid \mathcal{F}_{s-1})) \\ &= \sum_{s=1}^{T} E((A_s - A_{s-1})E(Y_T - Y_{s-1} \mid \mathcal{F}_{s-1})) \\ &\leq \sum_{s=1}^{T} E((A_s - A_{s-1})(\varepsilon_1 + 1)) = (\varepsilon_1 + 1)E(A_T) \leq (\varepsilon_1 + 1)(\varepsilon_2 + \varepsilon_3), \end{aligned}$$

where for the second equation, it is used that $A_s - A_{s-1}$ is $\mathcal{F}_{s-1}$-measurable. This yields $E(M_T^2) \leq E((Y_T^-)^2) + E((A_T + \varepsilon_1)^2) \leq \varepsilon_2^2 + \varepsilon_3 + (\varepsilon_1 + 1)(\varepsilon_2 + \varepsilon_3) + 2(\varepsilon_2 + \varepsilon_3)\varepsilon_1 + \varepsilon_1^2$.

Step 2: The continuous time extension follows by an inspection of the proof of Beiglböck et al. (2012, Theorem 1.1) in which the mesh of the grid in Step 1 tends to zero. The arguments are easier than in the original proof since we know from the estimate in Step 1 that the sequence of terminal values of the discrete time martingales is $L^2(P)$-bounded. Thus, it is sufficient to apply the Komlós theorem for Hilbert spaces, and one obtains L^2-convergence to the terminal value of the continuous time martingale part (cf. Beiglböck et al., 2012, Equation 6). This implies L^2-convergence of the terminal values of the drift parts, and the estimates from Step 1 also hold for the continuous time Doob–Meyer decomposition. This yields (a).

Step 3: Let $\xi := \inf\{t > 0 : \widetilde{Y}_t > \varepsilon\}$. Consider the (pre-)stopped process $Y_t := \widetilde{Y}_t 1_{(t<\xi)} + \widetilde{Y}_{\xi-} 1_{(t \geq \xi)}$. Since $\widetilde{Y}$ is a supermartingale and $\Delta\widetilde{Y}_\xi \geq 0$, Y has to be a supermartingale as well. The process Y is bounded from above by ε and we have that $P(Y_T < -\varepsilon) \leq P(\sup_{t\in[0,T]} |\widetilde{Y}_t| > \varepsilon) \leq \varepsilon$. This allows us to apply part (a) with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$ to Y, and we obtain for its Doob–Meyer decomposition that $E(M_T^2) \leq 3\varepsilon + 8\varepsilon^2$ and $E(A_T^2) \leq 2\varepsilon + 2\varepsilon^2$. By the corollary to Protter (2004, Theorem 24 of Chapter IV), we have $\sup_{H\in\mathbf{b}\mathcal{P}, ||H||_\infty \leq 1} \sqrt{E(\sup_{t\in[0,T]} |H \bullet Y_t|^2)} \leq 3\sqrt{E(M_T^2)} + 3\sqrt{E(A_T^2)}$, which implies that $d_{\mathbb{S}}(Y, 0) \leq \sqrt{27\varepsilon + 72\varepsilon^2} + \sqrt{18\varepsilon + 18\varepsilon^2}$. Since the paths of $\widetilde{Y}$ and Y coincide at least with probability $1 - \varepsilon$, we have that $d_{\mathbb{S}}(\widetilde{Y}, Y) \leq \varepsilon$, and (b) follows with the triangle inequality of $d_{\mathbb{S}}$. □