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ORIGINAL PAPER

Decomposed quasiconvex optimization with application to generalized cone problems

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Abstract
We propose a gradient-based method to solve quasiconvex optimization problems through decomposed optimization and prove local superlinear convergence under mild regularity assumptions at the optimal solution. A practical implementation further provides global convergence while maintaining the fast local convergence. In numerical examples from generalized cone programming, the proposed method reduced the number of iterations to 18 to 50% compared to bisection.

Keywords Quasiconvex optimization · Decomposed optimization · Newton's method · Generalized cone problems

1 Introduction

A function $f : C \to \mathbb{R}$ is said to be *quasiconvex* on a convex set $C \subset \mathbb{R}^n$ if and only if f has convex sublevel sets $S_f(t) = \{x \in C \,|\, f(x) \leq t\}$ or, equivalently, there exists a family of convex functions $\phi_t : C \to \mathbb{R}$ such that

$$f(x) \leq t \quad \iff \quad \phi_t(x) \leq 0 \tag{1}$$

for all $(t, x) \in \mathbb{R} \times C$. Several problems in economics, dynamic systems, or machine learning can be cast as optimizing quasiconvex objective and constraint functions [1–3, and references herein]. An important subclass of quasiconvex optimization is the class of generalized cone problems, which include the generalized eigenvalue problem and quasiconvex sum-of-squares programming problems. These problems often arise in the analysis of linear, switching, or polynomial dynamic systems, for example, estimating the maximal decay rate of a stable system or the largest

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invariant sublevel set of a local Lyapunov function [4]. Quasiconvex optimization problems can be solved via interior-point or proximal-point methods [5] as well as subgradient descent [6], but perhaps most popular are bisection techniques [3]. Bisection exhibits robust convergence with guaranteed logarithmic bounds on the number of iterations needed for a specified accuracy. However, these methods rely heavily on a compact confidence interval provided by the user that must contain the optimal value; moreover, its local convergence rate is below the superlinear or even quadratic convergence of gradient-based methods [7].

At the core of the bisection method for quasiconvex programming is a convex optimization problem parametrized in the value t which is feasible if and only if $S_f(t)$ is nonempty. Parametrized optimization problems have been studied extensively, e.g., by [8, 9], to assess continuity and differentiability of the optimal value as a function of the parameter, often through variational analysis [10]. These results play a prominent role in decomposition techniques [11], which solve large-scale optimization problems by decomposition into smaller, more tractable programs.

This letter proposes a decomposed quasiconvex optimization scheme in form of a gradient-based decomposition method for quasiconvex programs.[1] Our main contributions are to prove that minimizing a quasiconvex function is equivalent to finding the unique root of a parametrized optimization problem and to show that the latter is convex for the class of generalized cone problems. Newton's method for solving an equation—rather than optimization—only requires first-order derivatives, which we compute by variational analysis under some regularity assumptions. We then develop a practical algorithm that robustly converges to a desired accuracy in finite time yet provides superlinear and even quadratic convergence rates locally.

The remainder of the paper is organized as follows: In Sect. 2, we formally state the problems of quasiconvex optimization and generalized cone programs as well as briefly describe previous approaches. Section 3 provides the mathematical background. Our decomposed quasiconvex optimization scheme is described in Sect. 4 where we prove wellposedness, local convergence, and applicability to generalized cone problems. We present an implementation in Sect. 5 and analyze termination, accuracy, and complexity of the algorithm. Numerical examples in Sect. 6 demonstrate the usefulness of the approach.

2 Problem statement

Let $C \subset \mathbb{R}^n$ be a convex set and $f : C \to \mathbb{R}$ be a quasiconvex function which is continuous on C; the quasiconvex programming problem is

$$t_\star = \inf f(x) \quad \text{s.t. } x \in X \tag{2}$$

for some nonempty closed convex set $X \subset C$. The set X may be described by convex or quasiconvex constraints (e.g., Equation (2) in [3]).

[1] The term *decomposed quasiconvex optimization* refers to the proposed decomposition method for quasiconvex optimization and is different from the notion of decomposed quasiconvex *functions* in [12].

Assumption 1 The problem in (2) attains an optimal solution $x_\star \in X$, that is, $f(x_\star) = t_\star$.

We introduce generalized cone problems next, which are instances of quasiconvex optimization, and briefly review the two most prominent approaches to solve (2), namely bisection and subgradient descent.

2.1 Generalized cone problems

Generalize cone programs are quasiconvex optimization problems defined by

$$t_\star = \inf_{t\in\mathbb{R},\, x\in X} t \quad \text{subject to} \quad \begin{array}{r} tb(x) - a(x) \in \mathcal{K} \\ b(x) \in \mathcal{K}' \end{array} \tag{3}$$

where $a : \mathbb{R}^n \to \mathbb{R}^m$ and $b : \mathbb{R}^n \to \mathbb{R}^m$ are affine functions, $\mathcal{K} \subset \mathbb{R}^m$ is a closed convex cone, $\mathcal{K}' \subseteq \mathcal{K}$ is a convex subset, and $X \subset \mathbb{R}^n$ is a closed convex set. In the generalized eigenvalue problem, $\mathcal{K}$ and $\mathcal{K}'$ correspond to the cones of positive semidefinite and positive definite matrices, respectively, whereas in quasiconvex sum-of-squares problems, $\mathcal{K} = \mathcal{K}'$ is the cone of sum-of-squares polynomials.

Define the set $C = \{x \in \mathbb{R}^n \mid b(x) \in \mathcal{K}'\}$ as well as the function

$$f : x \mapsto \inf t \text{ s.t. } tb(x) - a(x) \in \mathcal{K} \tag{4}$$

for all $x \in C$, and observe that $\inf_{x\in X\cap C} f(x) = t_\star$ by construction. Denote by $\mathrm{dom} f \subseteq C$ the set of all points $x \in C$ such that (4) attains a finite value. For the remainder of this paper, we assume, without loss of generality, that $X \subseteq C$.

Proposition 1 *The function defined in* (4) *is quasiconvex on* $\mathrm{dom} f$.

The proof follows the ideas of [4, Proof of Theorem 2] for the cone of sum-of-squares polynomials and is given here for completeness.

Proof Without loss of generality, pick any $t \in \mathbb{R}$ such that $S_f(t) \neq \emptyset$; that is, $b(x) \in \mathcal{K}'$ for any $x \in S_f(t)$ and there exists $t_x \leq t$ such that $t_x b(x) - a(x) \in \mathcal{K}$ and hence,

$$tb(x) - a(x) = (t - t_x)b(x) + (t_x b(x) - a(x)) \in \mathcal{K}$$

since $\mathcal{K}$ is a convex cone. To prove that $S_f(t)$ is convex, let $x_1, x_2 \in S_f(t)$ and $\alpha \in (0, 1)$; then

$$\begin{aligned} &tb(\alpha x_1 + (1-\alpha)x_2) - a(\alpha x_1 + (1-\alpha)x_2) \\ &= \alpha(tb(x_1) - a(x_1)) + (1-\alpha)(tb(x_2) - a(x_2)) \in \mathcal{K} \end{aligned}$$

since a and b are affine and $\mathcal{K}$ is convex. Moreover, by a similar argument, $b(\alpha x_1 + (1-\alpha)x_2) = \alpha b(x_1) + (1-\alpha)b(x_2) \in \mathcal{K}'$. Hence, $f(\alpha x_1 + (1-\alpha x_2)) \leq t$, proving convexity of $S_f(t)$.

To conclude the proof, $\operatorname{dom} f$ is the union of all (convex) sets $S_f(t)$ with $t \in \mathbb{R}$, satisfying $S_f(t_1) \subseteq S_f(t_2)$ for any $t_1 \leq t_2$, and therefore a convex set. □

Remark 1 In the proof we established an important property, namely that, for all $(t, x) \in \mathbb{R} \times C$, we have $x \in S_f(t)$ if and only if $tb(x) - a(x) \in \mathcal{K}$.

2.2 Related work

In order to solve (2) or (3), most previous authors either employed bisection or subgradient descent methods.

Bisection methods Let $\hat{t} \in \mathbb{R}$ and recall that $S_f(\hat{t})$ contains all points $x \in C$ such that $f(x) \leq \hat{t}$. Bisection methods make use of the simple observation that $\hat{t}$ is a lower (upper) bound for $t_\star$ if and only if $S_f(\hat{t}) \cap X$ is (not) empty. Suppose now that the optimal value $t_\star$ is contained in the compact interval $\mathcal{I} \subset \mathbb{R}$ and pick $t \in \mathcal{I}$. Since the sets in question are convex, checking nonemptiness of $S_f(t) \cap X$ amounts to a convex feasibility problem. If feasible, we obtain a new, compact interval $\mathcal{I} \cap (-\infty, t]$ which is guaranteed to contain $t_\star$; otherwise, we have that $t_\star \in \mathcal{I} \cap [t, +\infty)$. Choosing suitable test points for t, we obtain a sequence of shrinking intervals that approximate the optimal value $t_\star$ with some accuracy ϵ after a finite number of steps [3]. However, bisection methods are sensitive to the choice of the first interval and do not benefit from a good initial guess for $t_\star$.

Subgradient methods Extending the idea of projected gradient descent to a nondifferential or even semicontinuous functions f, subgradient methods use some initial guess $x_0 \in X$ to obtain an iteration of the form

$$x_{k+1} = \Pi_X(x_k - \alpha_k d_k), \quad d_k \in \tilde{\partial} f(x_k), \|d_k\| = 1, \alpha_k > 0 \tag{5}$$

for all $k \geq 0$, where $\tilde{\partial} f(x)$ is a subdifferential of f at x and $\Pi_X(\cdot)$ denotes the projection onto the closed convex set X. In [6], suitable subdifferentials as well as conditions on the sequence $\{\alpha_k\}_{k \geq 0}$ are derived such that (5) yields a sequence which converges to an optimal solution of (2).

3 Preliminaries

We recall a few results on the sensitivity of optimization problems that we are going to use for our decomposed optimization method.

Normal cone and set-valued maps The normal cone of a (not necessarily convex) set $\Omega \subset \mathbb{R}^m$ at some point $z \in \Omega$ is

$$\mathcal{N}_\Omega(z) = \{v \in \mathbb{R}^m \mid \forall z' \in \Omega, \langle v, z' - z\rangle \leq 0\}$$

and we define $\mathcal{N}_\Omega(z) = \emptyset$ for any $z \notin \Omega$. The normal cone can be considered as a set-valued *mapping*. For a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we define $\operatorname{dom} F = \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and $\operatorname{graph} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\}$. Moreover, F is said to be *Lipschitz continuous* (in a set-valued sense) around $\bar{x} \in \operatorname{dom} F$ if and only if there exist a constant $\ell > 0$ and a neighbourhood $D \subset \operatorname{dom} F$ of $\bar{x}$ such that $F(\cdot)$ has closed values on D and

$$F(x_1) \subset F(x_2) + \ell \mathcal{B}(\|x_1 - x_2\|) \tag{6}$$

for all $x_1, x_2 \in D$, where $\mathcal{B}(r) := \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq r\}$ is a ball in $\mathbb{R}^n$ of radius $r \geq 0$.

Functions, selections, and localizations If F is single-valued on some $D \subset \operatorname{dom} F$, that is, $F(x) = \{f(x)\}$ for all $x \in D$ with $f : \mathbb{R}^n \to \mathbb{R}^m$, then (6) corresponds to the classical notion of Lipschitz continuity of f around $\bar{x}$ for functions. In addition, we say that f is *calm* at $\bar{x}$ if and only if $F : x \mapsto \{f(x)\}$ satisfies (6) for $x_1 \in D$ but $x_2 = \bar{x}$ fixed.

Definition 1 Take $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ and let $D \subset \operatorname{dom} F$ be a neighbourhood of $\bar{x}$; a function $r : D \to \mathbb{R}^m$ is

- a *single-valued localization* of F around $\bar{x}$ for $\bar{y}$ if and only if $\{r(x)\} = F(x) \cap Y$;
- a *selection* of F around $\bar{x}$ for $\bar{y}$ if and only if $\{r(x)\} \cap F(x) \neq \emptyset$;

for all $x \in D$, where $Y \subset \mathbb{R}^m$ is a neighbourhood of $\bar{y}$. [10, p. 77 and p. 125]

Subdifferentials for nonsmooth functions For a function $f : \mathbb{R}^n \to \mathbb{R}$, we define the *Fréchet* subdifferential as

$$\hat{\partial} f(\bar{x}) = \left\{ v \in \mathbb{R}^n \mid \liminf_{x \to \bar{x}} \frac{f(\bar{x}) - f(x) - \langle v, \bar{x} - x\rangle}{\|\bar{x} - x\|} \geq 0 \right\}$$

and the *basic* subdifferential as [13, Definition 1.77]

$$\partial f(\bar{x}) = \{v \in \mathbb{R}^n \mid \exists x_k \to \bar{x}, v_k \to v, \text{ with } f(x_k) \to f(\bar{x}) \text{ and } v_k \in \hat{\partial} f(x_k)\}$$

at $\bar{x} \in \operatorname{dom} f$. The basic subdifferential is equivalent to the Fréchet subdifferential for convex functions [13, Theorem 1.93], satisfies $\partial f(x) = \{\nabla f(x)\}$ if f is strictly differentiable at x [13, Corollary 1.82], and is nonempty for Lipschitz continuous functions in finite-dimensional spaces [13, Corollary 1.81 together with Proposition 1.76].

Remark 2 If f is a quasiconvex function, other notions of *generalized* subdifferentials have been developed [14–16]; for the parametrized convex problems considered in this paper, however, the basic subdifferential is more suitable.

4 Decomposed quasiconvex optimization

We are going to combine the idea of solving a convex problem for a candidate level $\hat{t}$ with a gradient-based approach. To that extent, consider the function $\psi : (t, x) \mapsto \phi_t(x)$ and recall that $\psi(t, \cdot)$ is a convex function for all $t \in \mathbb{R}$. We introduce the parametrized convex optimization problem

$$\vartheta : t \mapsto \inf_{x \in X} \psi(t, x) \tag{7}$$

which takes values on the extended real line $\mathbb{R} \cup \{-\infty\}$. Indeed, the optimization in (7) may be unbounded in which case we set $\vartheta(t) := -\infty$.

Assumption 2 The problem in (7) obtains an optimal solution for any $t \in \vartheta^{-1}(0)$.

This assumption is satisfied, e.g., if for all $t < t_\star$ there exists a lower bound $\epsilon_t > 0$ such that $\phi_t(x) \geq \epsilon_t$ for all $x \in X$; or if X is a compact set. The following definition extends on the concept of indicator functions in (1).

Definition 2 A *signed metric* for a closed set $\Omega \subset \mathbb{R}^m$ is a function $\lambda : \mathbb{R}^m \to \mathbb{R}$ satisfying $\lambda(z) < 0$ if and only if $z \in \operatorname{int} \Omega$ and $\lambda(z) > 0$ if and only if $z \notin \Omega$ for all $z \in \mathbb{R}^m$.

An example of a signed metric is the signed distance function.[2]

4.1 Properties of parametrized optimization

We derive conditions under which $\vartheta(\cdot)$ is monotonically nonincreasing and has a unique root in $t_\star$.

Lemma 2 *If $\psi(\cdot, x)$ is nonincreasing for all $x \in C$, then the optimal value $\vartheta(\cdot)$ is nonincreasing as well.*

Proof Let $t, t' \in \mathbb{R}$ satisfy $t \leq t'$, suppose that $\psi(\cdot, x)$ is nonincreasing, and $t \in \operatorname{dom} \vartheta$, that is, $\vartheta(t) > -\infty$. By optimality, for any $v > \vartheta(t)$ there exists $x \in X$ such that $\psi(t, x) < v$. Assume now that ϑ is increasing between t and t', that is, $\vartheta(t') > \vartheta(t)$; but then $\psi(t', x) \leq \psi(t, x) < \vartheta(t')$ contradicting that $\vartheta(t')$ is the optimal value.

On the other hand, if $\vartheta(t) = -\infty$, that is, for any $r \in \mathbb{R}$ there exists $x \in X$ such that $\phi_t(x) < r$; then $\psi(t', x) \leq \psi(t, x) < r$ as well and, to complete the proof, $\vartheta(t') = -\infty$ by definition. □

[2] For a closed set $\Omega \subset \mathbb{R}^m$, the signed distance to Ω is defined as

$$\sigma(x;\Omega) = \begin{cases} +\operatorname{dist}(x; \operatorname{bnd} \Omega) & \text{if } x \notin \Omega \\ -\operatorname{dist}(x; \operatorname{bnd} \Omega) & \text{if } x \in \Omega \end{cases}$$

where $\operatorname{dist}(\cdot;\Omega)$ is the classical distance function, and $\sigma(\cdot;\Omega)$ is a convex function if Ω is a convex set.

Note that we have not made any other assumptions about the family ϕ_t other than monotonicity of $\psi(t,\cdot) \equiv \phi_t$. For the following result, which is a major part of our approach, we assume that ψ is a signed metric function for the graph of the sublevel set mapping $S_f : t \mapsto \{x \in C \,|\, f(x) \leq t\}$.

Lemma 3 *Let Assumptions 1 and 2 be satisfied; if $\psi : \mathbb{R} \times C \to \mathbb{R}$ is a signed metric function for* graph S_f*, then $t_\star$ is the optimal value of (2) if and only if the optimal solution of (7) satisfies $\psi(t_\star, x_\star) = 0$.*

Proof Let $x_\star \in \arg\inf_{x \in X} f(x)$ and $t_\star = f(x_\star)$; that is, $\vartheta(t_\star) = \psi(t_\star, x_\star) \leq 0$. Suppose now that $\psi(t_\star, x') < 0$ for some $x' \in X$, then $(t_\star, x') \in \operatorname{int} \operatorname{graph} S_f$ by definition of a signed metric function and hence, there exists $(t', x') \in \operatorname{graph} S_f$ with $t' < t_\star$, that is, $f(x') \leq t' < t_\star$. This contradicts the optimality of $t_\star$.

Let now $\vartheta(t_\star) = \psi(t_\star, x_\star) = 0$ for some $x_\star \in X$ and hence, $f(x_\star) \leq t_\star$. Suppose that $f(x') < t_\star$ for some $x' \in X$ and define $\lambda : (t,x) \mapsto f(x) - t$; then $\lambda(t_\star, x') < 0$ and by upper semicontinuity of f (thus of λ) we have that $\lambda(t,x) < 0$ for all (t, x) around $(t_\star, x')$. Hence, $(t_\star, x') \in \operatorname{int} \operatorname{graph} S_f$, that is, $\psi(t_\star, x') < 0$ in contradiction to the optimality of $x_\star$. □

We later propose a signed metric function $\psi(t,x)$ for generalized cone problems which is convex in x, but whether such a function exists for any quasiconvex function must remain an open question. It should be noted, however, that convexity is merely a convenience when solving the problem in (7) but not needed in the subsequent analysis.

4.2 Decomposed Newton's method

Having established that, if ψ is a signed metric function for graph S_f, solving (2) is equivalent to finding the root of (7), we propose a gradient-based algorithm to solve $\vartheta(t) = 0$ and analyse its convergence to $t_\star$. In order to be able to compute gradients for ϑ we need the following assumption for ψ.

Assumption 3 The mapping $\psi : (t,x) \mapsto \phi_t(x)$ is differentiable around $(t_0, x_0) \in \mathbb{R} \times C$ and the partial derivatives $\nabla_t \psi$ and $\nabla_x \psi$ are Lipschitz continuous around (t_0, x_0).

Note that continuous differentiability (a fortiori, Lipschitz continuity) of ψ together with closedness of X implies that $\vartheta(t)$ is Lipschitz continuous around t_0; this is a common fact in mathematical programming (see, e.g., [10, Exercise 4.6]). Lipschitz continuity, on the other hand, implies that $\vartheta(\cdot)$ is differentiable almost everywhere on a neigbourhood of t_0. The following result from [9] allows us to compute gradients of ϑ. To that extent, define the solution map $\Sigma(t) = \{x \in X \,|\, \psi(t,x) = \vartheta(t)\}$.

Theorem 4 *Let Assumption 3 be satisfied and suppose that $\Sigma(\cdot)$ has a calm selection around $t_0 \in \mathbb{R}$ for $x_0 \in \Sigma(t_0)$; then ϑ is strictly differentiable around t_0 with gradient $\nabla\vartheta(t_0) = \nabla_t\psi(t_0, x_0)$.*

Proof Since X is closed and convex, ψ has Lipschitz-continuous derivatives, Σ has a calm selection around t_0 for x_0, and the parameter t only enters into the objective of (7), the basic subgradient of $\vartheta(t_0)$ satisfies

$$\partial\vartheta(t_0) = \{\xi \mid \exists(\xi, \zeta) \in \partial\psi(t_0, x_0), \zeta + \mathcal{N}_X(x_0) \ni 0\}$$

by virtue of [9, Theorem 3.1].

Since ψ is continuously differentiable, $\partial\psi(t_0, x_0) = \{\nabla\psi(t_0, x_0)\}$ and $\partial\vartheta(t_0)$ contains at most one element. On the other hand, since ϑ is a locally Lipschitz continuous function on a finite-dimensional space, $\partial\vartheta(t_0)$ is nonempty [13, Corollary 1.81 together with Proposition 1.76], that is, $\partial\vartheta(t_0) = \{\xi\}$ where $\xi = \nabla_t\psi(t_0, x_0)$; and $\xi = \nabla\vartheta(t_0)$ by [13, Theorem 3.54], the desired result. □

Combining the results above we can employ Newton's method to solve (2).

Proposition 5 *Let Assumptions 1–3 be satisfied, suppose that $\Sigma(\cdot)$ is Lipschitz continuous (in a set-valued sense)*

around the optimal value $t_\star$ of (2) for some $x_\star \in \Sigma(t_\star)$, and consider the sequence generated by

$$t_{k+1} = t_k - \nabla_t\psi(t_k, x_k)^{-1}\vartheta(t_k), \quad x_k \in \Sigma(t_k) \tag{8}$$

with $t_0 \in \mathbb{R}$ and $k \in \mathbb{N}_0$; if $\nabla_t\psi(t_\star, x_\star) \neq 0$ as well as $\nabla_t\psi(t_k, x_k) \neq 0$ along $\{(t_k, x_k)\}_{k\in\mathbb{N}_0}$ and the initial condition t_0 is sufficiently close to $t_\star$, then $\{t_k\}_{k\in\mathbb{N}_0}$ converges Q-superlinearly to $t_\star$.

Proof Let $\Sigma(\cdot)$ be Lipschitz continuous on some neighbourhood $T \subset \mathbb{R}$, that is, $T \subset \operatorname{dom}\vartheta$; then $\Sigma(t)$ has a calm selection around t for any $x_t \in \Sigma(t)$ and all $t \in T$ and by virtue of Theorem 4, $\vartheta(t)$ is strictly differentiable at t (hence continuously differentiable around $t_\star$) with gradient $\nabla\vartheta(t) = \nabla_t\psi(t, x_t)$. If $\nabla_t\psi(t_k, x_k) \neq 0$ for all $k \in \mathbb{N}_0$, then the sequence $\{(t_k, x_k\}_{k\in\mathbb{N}_0}$ generated by (8) is well defined and, if t_0 is sufficiently close to $t_\star$, that is, $\{t_k\}_{k\in\mathbb{N}_0} \subset T$, corresponds to Newton's method. Noting that $\nabla\vartheta(t_\star) = \nabla_t\psi(t_\star, x_\star) \neq 0$ by assumption, $\{t_k\}_{k\in\mathbb{N}_0}$ converges Q-superlinearly to a root of $\vartheta(\cdot)$ by [17, Theorem 11.2] (also [7, Theorem 2.2(b)]) and hence, the optimal value $t_\star$ (Lemma 3). □

To satisfy the assumptions in Theorem 4, we need $\Sigma(\cdot)$ to have a calm selection around any (t_k, x_k) generated by (8). A sufficient condition is that $\Sigma(\cdot)$ has the *Aubin property* (see [10, Section 4] for a definition) at t_k for x_k for all $k \in \mathbb{N}_0$. If all x_k lie in a neighbourhood of $x_\star$, then it is sufficient for $\Sigma(\cdot)$ to have the Aubin property at $t_\star$ for $x_\star$. This is satisfied if the Karush–Kuhn–Tucker condition

system of (7), viz. $\nabla_x \phi_{t^\star} + \mathcal{N}_X$, is *metrically regular* at $x_\star$ for 0 [10, Theorem 4.3]. Now, if X is a polyhedral set, then metric regularity is equivalent to *strong* metric regularity and implies that $\Sigma(\cdot)$ has a Lipschitz continuous single-valued graphical localization[3] [10, Theorem 9.7 together with Theorem 8.8]. We then obtain the following improvement of Proposition 5.

Proposition 6 *Let Assumptions 1–3 be satisfied; if, in addition to Lipschitz continuity, $\Sigma(\cdot)$ is single-valued around $t_\star$ and $\nabla_t \psi(t_\star, x_\star) \neq 0$, where $\Sigma(t_\star) = \{x_\star\}$, then the sequence generated by (8) has local Q-quadratic convergence.*

Proof Let $\sigma : T \to C$ be a Lipschitz continuous single-valued localization of $\Sigma(\cdot)$ for some neighbourhood $T \subset \mathbb{R}$ of $t_\star$, that is, $\Sigma(t) = \{\sigma(t)\}$ for all $t \in T$; then, by continuity of $\nabla_t \psi$ and σ, there exists a neighbourhood $T' \subseteq T$ of $t_\star$ such that $\nabla_t \psi(t, \sigma(x)) \neq 0$ for all $t \in T'$. Moreover, for all $t \in T$, the gradient $\nabla \vartheta(t) = \nabla_t \psi(t, \sigma(t))$ exists and is Lipschitz continuous at any $t \in T$ by Theorem 4. Following the proof of Proposition 5, the sequence $\{t_k\}_{k \in \mathbb{N}_0}$ converges to the optimal value $t_\star$ and the convergence is Q-quadratic by [17, Theorem 11.2] (also [7, Theorem 2.2(c)]). □

We conclude the analysis with an illustrating example.

Example 1 The function $f(x) = \sqrt{|x|}$ is quasiconvex on $\mathbb{R}$ with signed metric function $\psi(t, x) = |x| - t|t|$ for graph S_f. The problem (7) obtains its optimum at the origin for all $t \in \mathbb{R}$, hence $\vartheta(t) \equiv \psi(t, 0)$ and $\vartheta'(t) = -2|t|$. Newton's method converges to the approximation $t_\star < 1 \times 10^{-3}$ with accuracy below 1×10^{-3} in 12 iterations. For comparison, a subgradient descent algorithm, using $f'(x) = (2x)^{-1}\sqrt{|x|}$ if $x \neq 0$ and $\partial f(0) \ni 0$ else, with backtracking took 51 iterations to converge to the approximation $f(x_\mathrm{f}) < 0.125$ with same accuracy. (Both methods were initialized at $x_0 = 10$ and $t_0 = \sqrt{x_0}$, respectively.)

4.3 A signed metric for generalized cones

We propose to solve generalized cone programming problems using the gradient-based approach presented in the previous section. In particular, we introduce a signed metric function for (3) that has favourable properties for Newton's method. Namely, define

$$\psi_{\mathcal{K}} : (t, x) \mapsto \lambda_{\mathcal{K}}(tb(x) - a(x)) \tag{9}$$

for all $x \in C$, that is, $b(x) \in \mathcal{K}'$; where $\lambda_{\mathcal{K}} : \mathbb{R}^m \to \mathbb{R}$ is a continuous convex signed metric function for $\mathcal{K}$. Recall that $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$.

[3] Note that since $\Sigma(\cdot)$ has convex values, a single-valued graphical localization is equivalent to $\Sigma(\cdot)$ being unique.

Assumption 4 The problem in (3) obtains an optimal solution $(t_\star, x_\star) \in \mathbb{R} \times X$ and there exists $\mu \in \mathcal{K}^*$ with $\inf_{(t,x)\in\mathbb{R}\times X} t - \mu(tb(x) - a(x)) = t_\star$.

Assumption 4 is the classical assumption of strong duality, which is equivalent to the existence of a Karush–Kuhn–Tucker (KKT) solution [18, Proposition 3.3].

Proposition 7 *For any $(t, x) \in \mathbb{R} \times C$, the function $\psi_\mathcal{K}(t, x)$ is continuous in t, convex in x, and nonnegative if and only if $x \in S_f(t)$. Moreover, if either b is surjective or $\mathcal{K}' \subseteq \operatorname{int}\mathcal{K}$, then $\psi_\mathcal{K}$ is a signed metric for* graph S_f.

Proof Convexity and continuity of $\psi_\mathcal{K}(t, x)$ are obvious. Take $t \in \mathbb{R}$ and $x \in C$, that is, $b(x) \in \mathcal{K}'$. If $x \notin S_f(t)$, then $t'b(x) - a(x) \notin \mathcal{K}$ for all $t' \leq t$ and hence, $\lambda_\mathcal{K}(tb(x) - a(x)) > 0$ since $\lambda_\mathcal{K}$ is a signed metric function for $\mathcal{K}$. On the other hand, suppose that $x \in S_f(t)$, that is, $tb(x) - a(x) \in \mathcal{K}$ and $\lambda_\mathcal{K}(tb(x) - a(x)) \leq 0$.

Moreover, if $(t, x) \in \operatorname{int}$ graph S_f, then $t'b(x') - a(x') \in \mathcal{K}$ and $b(x') \in \mathcal{K}'$ for all (t', x') around (t, x), and thus $b(x) \in \operatorname{int}\mathcal{K}$ if either b is surjective or $\mathcal{K}' \subseteq \operatorname{int}\mathcal{K}$. Pick some $\tau > 0$ and any $\kappa \in \mathbb{R}^m$ sufficiently close to 0 such that $(t - \tau)b(x) - a(x) \in \mathcal{K}$ and $b(x) + \kappa/\tau \in \mathcal{K}$. Then

$$\mathcal{K} \ni (t - \tau)b(x) - a(x) + \tau(b(x) + \kappa/\tau) = tb(x) - a(x) + \kappa$$

since $\mathcal{K}$ is a convex cone; in other words, $tb(x) - a(x) \in \operatorname{int}\mathcal{K}$ and $\psi(t, x) < 0$. We have already shown that $\psi(t, x) > 0$ for all $(t, x) \notin$ graph $S_f = \cup_{t\in\mathbb{R}} S_f(t)$; continuity of $\psi_\mathcal{K}$ also implies that $\psi_\mathcal{K}(t, x) = 0$ for all $(t, x) \in$ bnd graph S_f, completing the proof that $\psi_\mathcal{K}$ is a signed metric function for graph S_f. □

A possible choice for the function $\lambda_\mathcal{K}$ is the *scalarization function*

$$\lambda_{\mathcal{K},\xi} : z \mapsto \inf\{r \in \mathbb{R} \mid z + r\xi \in \mathcal{K}\} \tag{10}$$

for some $\xi \in \operatorname{int}\mathcal{K}$. The properties of scalarization functions are extensively discussed in [19, Chapter 5.2.2]. If $\mathcal{K}$ is proper, that is, $\mathcal{K}$ is a closed convex cone satisfying $\mathcal{K} \cap -\mathcal{K} = \{0\}$ and $\mathcal{K} - \mathcal{K} = \mathbb{R}^m$, then $\lambda_{\mathcal{K},\xi}$ is well defined [19, Corollary 5.2.8] and maps each $z \in \mathbb{R}^m$ to a finite signed 'distance' between z and the boundary of $\mathcal{K}$. We show that the signed metric function (9) together with (10), denoted by $\psi_{\mathcal{K},\xi}$, satisfies Assumption 2 as well as the assumptions in Lemma 2.

Lemma 8 *For any $t < t_\star$, the function $\psi_{\mathcal{K},\xi}(t, \cdot)$ satisfies $\inf_{x\in X} \psi_{\mathcal{K},\xi}(t, x) > 0$.*

Proof By means of contradiction, assume there exists $t' < t_\star$ together with a sequence $\{x_k \in X\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \psi_{\mathcal{K},\xi}(t', x_k) = 0$; that is, by (10), there exists $z_k \in \mathcal{K}$ and $r_k \to 0$ satisfying $z_k = t'b(x_k) - a(x_k) + r_k\xi$ for all $k \in \mathbb{N}$. Hence, for any $\mu \in \mathcal{K}^*$ there exists $\kappa \in \mathbb{N}$ such that $t' - \mu(t'b(x_k) - a(x_k)) = t' - \mu(z_k) + r_k\mu(\xi) \leq t' + r_k\mu(\xi) < t_\star$ for all $k \geq \kappa$, noting that μ is nonnegative on $\mathcal{K}$, contradicting Assumption 4. □

Lemma 9 *For any $x \in X$, the function $\psi_{\mathcal{K},\xi}(\cdot, x)$ is nonincreasing.*

Proof Take $(t,x) \in \mathbb{R} \times X$ and denote $z = tb(x) - a(x) + \psi_{\mathcal{K},\xi}(t,x)\xi$; since $z \in \mathcal{K}$ and $b(x) \in \mathcal{K}' \subseteq \mathcal{K}$, we have that $z + \tau b(x) \in \mathcal{K}$ and hence,

$$\psi_{\mathcal{K},\xi}(t+\tau,x) = \inf\{r \in \mathbb{R} \mid (t+\tau)b(x) - a(x) + r\xi \in \mathcal{K}\} \leq \psi_{\mathcal{K},\xi}(t,x)$$

for all $\tau \geq 0$, the desired result. □

Moreover, since $\lambda_{\mathcal{K},\xi}$ is a convex (a fortiori, absolute continuous) function, it is differentiable almost everywhere. We obtain the following characterization of the gradient, if existing, from the dual theory of convex cone programs.

Proposition 10 *Take $\bar{z} \in \mathbb{R}^m$; the dual problem*

$$\sup_{-y \in \mathcal{K}^*} \langle y, \bar{z} \rangle \quad \text{s.t. } 1 + \langle y, \xi \rangle = 0 \tag{11}$$

where $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$, has a unique solution $\bar{y}$ if and only if the gradient $\nabla \lambda_{\mathcal{K},\xi}(\bar{z})$ of the scalarization function (10) exists and equals $\bar{y}$.

Proof Let $\bar{r} = \lambda_{\mathcal{K},\xi}(\bar{z})$; since the optimization problem in (10) is feasible and $\xi \in \mathcal{K}$, Slater's condition is satisfied and the KKT conditions

$$\begin{aligned} 1 + \langle y, \xi \rangle &= 0 \\ \langle y, \bar{z} + \bar{r}\xi \rangle &= 0 \end{aligned}$$

have a solution $\bar{y} \in -\mathcal{K}^*$. Rewriting yields $\langle \bar{y}, \bar{z} \rangle = \bar{r}$ and hence, $\bar{y}$ is an optimal solution of (11). We claim that any $y \in \mathbb{R}^m$ is an optimal solution of (11) if and only if y is a subgradient of $\lambda_{\mathcal{K},\xi}(\bar{z})$, that is,

$$\langle y, z - \bar{z} \rangle \leq \lambda_{\mathcal{K},\xi}(z) - \bar{r}$$

for all $z \in \mathbb{R}^m$. Suppose y is a subgradient, then $\langle y, \pm\xi \rangle \leq \lambda_{\mathcal{K},\xi}(\bar{z} \pm \xi) - \bar{r} \leq \pm 1$, where we have used that $\bar{z} + \bar{r}\xi \in \mathcal{K}$, and $\langle y, \kappa \rangle \leq \lambda_{\mathcal{K},\xi}(\bar{z} + \kappa) - \bar{r} \leq 0$ for all $\kappa \in \mathcal{K}$, since $\mathcal{K}$ is a convex cone. Moreover, $\langle y, -\bar{z} \rangle \leq \bar{r}$ and hence, y is an optimal solution of (11). On the other hand, take $z \in \mathbb{R}^m$ and $r = \lambda_{\mathcal{K},\xi}(z)$; then

$$\langle y, z - \bar{z} \rangle = \langle y, z + r\xi \rangle - r\langle y, \xi \rangle - \langle y, \bar{z} \rangle \leq r - \bar{r}$$

if y is an optimal solution of (11), since $z + r\xi \in \mathcal{K}$ and $y \in -\mathcal{K}^*$. To conclude the proof, note that $\nabla \lambda_{\mathcal{K},\xi} = \bar{y}$ if and only if $\bar{y}$ is the unique subgradient. □

For a characterization of the subgradients for arbitrary optimization problems with affine and convex cone constraints, see [8, Theorem 2]. In particular, the KKT conditions of both (10) and (11) can be written as

$$h(\bar{z}, (r,y)) + \mathcal{N}_{\mathbb{R} \times \mathcal{K}^*}((r,y)) \ni 0 \tag{12}$$

with $h(\bar{z}) : (r,y) \mapsto (1 + \langle y, \xi \rangle, \bar{z} + r\xi)$ being an affinely parametrized, affine map. For $\lambda_{\mathcal{K},\xi}$ to be continuously differentiable around $z_\star := t_\star b(x_\star) - a(x_\star)$, we need

that the solution (11) is unique and Lipschitz continuous for $\bar{z}$ around $z_\star$, which is the case if $h(z_\star, \cdot) + \mathcal{N}_{\mathbb{R}\times\mathcal{K}^*}$ is strongly regular at $(r_\star, y_\star)$ for 0. If $\mathcal{K}$ is a polyhedral cone, then strong regularity corresponds to $z_\star$ lying in exactly one of the generating half-spaces.

5 Implementation

We propose a practical algorithm to solve quasiconvex optimization algorithms using decomposed optimization and signed metric functions. To that extent, we assume that a signed metric function ψ for graph S_f and some initial guess $t^{(0)}$ is given by the user. The algorithm is then generating a sequence $(t^{(1)}, t^{(2)}, \ldots)$ until some convergence condition is met. Now, as mentioned above, the optimization problem in (7) might be unbounded for some $t^{(k)}$ and we do not obtain any sensitivity information. In order to choose a new point in that case, we introduce a confidence interval $I = (a, b) \subset \mathbb{R}$ which might be initially unbounded (that is, $a, b \in \mathbb{R} \cup \{\pm\infty\}$). If the solution $\vartheta(t^{(k)})$ of (7) is negative (including $\vartheta(t^{(k)}) = -\infty$) for any $k \in \mathbb{N}$, we update the lower bound of I to be equal to $t^{(k)}$; if the solution is positive, we update the upper bound accordingly. This ensures that $t_\star$ remains an element of I. However, when (7) is unbounded or the Newton step yields a point t_+ outside I, we instead perform a bisection to find a new estimate. The full procedure is described by Algorithm 1. In the following analysis, we denote the size of the confidence interval by $|I| = b - a$, where $|I| = \infty$ if the interval is unbounded.

Algorithm 1 Quasiconvex programming by decomposed optimization.

Require: Initial guess $t^{(0)} \in \mathbb{R}$ and $I_0 = (a_0, b_0) \subset \mathbb{R}$ with $t^{(0)}, t_\star \in I_0$
Ensure: The value $t^{(K)}$ is close to $t_\star$ for some $K \in \mathbb{N}_0$

for $k \in \mathbb{N}_0$ **do**
  Solve (7) to obtain $\vartheta(t^{(k)})$ with optimal solution $\bar{x}_k \in \mathbb{R}^n$
  if $\vartheta(t^{(k)}) > 0$ **then**
    Set $a_{k+1} = t^{(k)}$ and $b_{k+1} = b_k$
  else
    Set $a_{k+1} = a_k$ and $b_{k+1} = t^{(k)}$
  end if
  Let $D_k = \mathrm{sat}(\nabla_t \psi(t^{(k)}, \bar{x}_k); D_{\max})$
  Compute $t_+ = t^{(k)} - D_k^{-1} \vartheta(t^{(k)})$
  if $\vartheta(t^{(k)}) = -\infty$ or $t_+ \notin I_{k+1}$ **then**
    Set $t_{k+1} = (a_{k+1} + b_{k+1})/2$
  else if $\vartheta(t^{(k)} \leq 0$ **then**
    Set $t_{k+1} = \max\{t_+, (a_{k+1} + b_{k+1})/2\}$
  else
    Set $t_{k+1} = \min\{t_+, (a_{k+1} + b_{k+1})/2\}$
  end if
  if $|t_{k+1} - t_k| < \epsilon_1$ and $|\vartheta(t^{(k)})| < \epsilon_2$ **then**
    Terminate and **return** $t^{(k)}$
  end if
end for

Termination Algorithm 1 has a combined termination condition: the first condition, size of the last step, is reminiscent of bisection algorithms where the step length corresponds to half the length of the confidence interval; the second condition, a measure of constraint satisfaction, prevents the algorithm from converging to an infeasible point. Unfortunately, termination of the algorithm in this form cannot be guaranteed. Indeed, even with the addition of a confidence interval, only local convergence of Algorithm 1 holds under the assumptions of Propositions 5 and 6. It is well known that Newton's method is susceptible to limit cycles if initialized too far away from the optimal solution [7, Example 2.16]. We prevent such behaviour by restricting the step length via the update rule

$$t_{k+1} = \begin{cases} \min\{t_+, (a_{k+1} + b_{k+1})/2\} & \text{if } \vartheta(t^{(k)}) > 0 \\ \max\{t_+, (a_{k+1} + b_{k+1})/2\} & \text{if } \vartheta(t^{(k)}) \leq 0 \end{cases}$$

if $t_+ \in I$ and $a_{k+1}, b_{k+1} \in \mathbb{R}$. Thus, it is ensured that the confidence interval shrinks if Newton's method would otherwise circle the optimal solution. On the other hand, since the optimal value of (7) is not differentiable everywhere, a large (estimated) gradient may lead to a small step despite $\vartheta(t^{(k)})$ being not zero. Here, we saturate[4] the absolute value of the gradient $\nabla_t \psi(t^{(k)}, \bar{x}_k)$ by some $D_{\max} > 0$ to ensure that $|t_{k+1} - t_k| \to 0$ only if $\vartheta(t^{(k)}) \to 0$.

Accuracy By Lemma 3, the optimal value $t_\star$ is the unique root of $\vartheta(\cdot)$; hence, there exists a monotonically increasing function $\Delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq}$ and a neighbourhood $T \subset \mathbb{R}$ of $t_\star$ satisfying $|\vartheta(t)| \geq \Delta(|t - t_\star|)$ for all $t \in T$. The algorithm terminates after $K \in \mathbb{N}_0$ iterations only if

$$|t^{(K)} - t_\star| < \Delta^{-1}(\epsilon_2)$$

as a result of the second termination condition. In addition, if $t^{(K+1)}$ is the result of a Newton step, then the first termination condition yields

$$|t^{(K)} - t_\star| < D_K \Delta^{-1}(\epsilon_1)$$

whereas, in the case of a bisection (that is, $|I_{K+1}| < \infty$), we have that

$$|t^{(K)} - t_\star| \leq |I_{K+1}| < 2\epsilon_1$$

since $t^{(K)}, t_\star \in I_{K+1}$ by construction and $|t^{(K+1)} - t^{(K)}| = |I_{K+1}|/2$. In particular, under the assumptions of Propositions 5 and 6, the solution of (7) is differentiable around $t_\star$

[4] The saturation function is commonly defined as

$$\mathrm{sat}(r;D) = \begin{cases} +D & \text{if } r > +D \\ r & \text{if } r \in [-D, +D] \\ -D & \text{if } r < -D \end{cases}$$

for any $r \in \mathbb{R}$ some constant $D > 0$.

with nonzero gradient, that is, the mapping Δ may be chosen as $\Delta : \tau \mapsto (D_K - \epsilon_D)\tau$ with $\epsilon_D > 0$. In this case, we obtain that

$$|t^{(K)} - t_\star| < \frac{1}{1 - D_K^{-1}\epsilon_D}\epsilon_1$$

from the Newton step and the first termination condition.

Complexity Suppose now that $\vartheta(\cdot)$ is nonincreasing (see Lemma 2) and note that thus, for any $\tau > 0$ there exists a constant $d \in (0, D_{\max}/2]$ satisfying $|\vartheta(t)| \geq d|t - t_\star|$ for all $t \in \mathbb{R}$ with $|t - t_\star| > \tau$; moreover, assume that the interval I_0 is bounded (that is, $|I_0| < \infty$). Hence, for all $k \geq 0$,

$$\frac{d}{D_{\max}}|t^{(k)} - t_\star| \leq |t^{(k+1)} - t^{(k)}| \leq \frac{|I_{k+1}|}{2}$$

by the revised update rule and construction of I_{k+1}. In addition, as $t^{(k+1)}$ is a boundary point of I_{k+2}, we have that

$$\begin{aligned}|I_{k+2}| &\leq \max\{|t^{(k+1)} - t^{(k)}|, |I_{k+1}| - |t^{(k+1)} - t^{(k)}|\} \\ &\leq \max\{\frac{|I_{k+1}|}{2}, |I_{k+1}| - \frac{d}{D_{\max}}|t^{(k)} - t_\star|\} \\ &\leq |I_{k+1}| - \frac{d}{D_{\max}}|t^{(k)} - t_\star|\end{aligned}$$

where the last inequality follows from $|t^{(k)} - t_\star| \leq |I_{k+1}|$. We claim that for any $\tau > 0$ there exists $K \geq 0$ such that $|t^{(K)} - t_\star| \leq \tau$; for otherwise,

$$\tau < |t^{(K+1)} - t_\star| \leq |I_{K+1}| \leq |I_1| - K\frac{d\tau}{D_{\max}}$$

for all $K \geq 0$, which contradicts that $d\tau/D_{\max} > 0$ and $|I_1| < \infty$. Reordering this inequality yields the upper bound

$$K \leq \frac{D_{\max}}{d\tau}(|I_1| - \tau) \leq \frac{D_{\max}}{d\tau}(\max\{b_0 - t^{(0)}, t^{(0)} - a_0\} - \tau)$$

for the number of iterations that are needed to reach τ-distance of the optimal solution. On the other hand, if the confidence interval remains unbounded, that is, either $t^{(k)} \leq t_\star$ or $t^{(k)} \geq t_\star$ for all $k \in \mathbb{N}_0$, we can use similar arguments to obtain the upper bound

$$K \leq \frac{D_{\max}}{d\tau}(|t^{(0)} - t_\star| - \tau)$$

for any $\tau > 0$. Clearly, these convergence rates are inferior to the logarithmic bounds for bisection; however, under the assumptions of Propositions 5 and 6, we only

need to come 'close enough' of the optimal value to obtain superlinear or quadratic convergence.

6 Numerical examples

We present three examples for quasiconvex sum-of-squares programming that arise in the analysis of dynamical systems. Details on the examples can be found in [4]. The implementation of Algorithm 1 is based on the sum-of-squares toolbox MULTIPOLY/SOSOPT [20] and will be compared to the toolbox's bisection method `gsosopt`. If not otherwise noted, the confidence intervals for both Algorithm 1 and `gsosopt` were initialized as $I_0 = [-50, 0]$ and tolerances[5] were chosen as $\epsilon_1 = 1 \times 10^{-3}$ and $\epsilon_2 = 1 \times 10^{-6}$. All computations were performed with a 2.8 GHz quad-core Intel Core i7 processor with 16 GB of memory using SEDUMI for the low-level semidefinite programs [21]. Source code is available at [22].

Decay rate bound A lower bound $\lambda = -t/2$ on the maximum decay rate of an exponentially stable, polynomial dynamic system can be computed by means of the quasiconvex sum-of-squares program

$$\inf_{t\in\mathbb{R},\, u\in\mathbb{R}^3} t \quad \text{subject to} \quad \begin{array}{l} tV(u) - J_x V(u) f \in \mathbb{R}[x] \\ V(u) \in (\mathbb{R}[x] + l_x) \end{array} \tag{13}$$

with indeterminate variables $x = (x_1, x_2)$, where $V : u \mapsto u_1 x_1^2 + u_2 x_1 x_2 + u_3 x_2^2$ is a parametrized Lyapunov function candidate, the linear operator $J_x : \mathbb{R}[x] \to \mathbb{R}[x]^2$ returns the Jacobian matrix of $V(u)$, and the polynomial $l_x = x_1^2 + x_2^2$ enforces positive definiteness of the Lyapunov function. The system dynamics are given by

$$f = \begin{bmatrix} -0.125x_1^3 - 1.125x_1x_2^2 + 0.75x_2^3 + 0.75x_1^2 + 1.5x_1x_2 + 0.75x_2^2 - 4x_1 + 5x_2 \\ -0.375x_1^2x_2 + 0.75x_1x_2^2 - 0.875x_2^3 + 0.25x_1^2 + 0.5x_1x_2 + 0.25x_2^2 - x_1 - 2x_2 \end{bmatrix}$$

for this example. If $t^{(0)}$ is initialized at the center of the confidence interval, Algorithm 1 returns $t^{(\text{fin})} = -3.8560$ after 8 iterations, whereas `gsosopt` terminated after 16 iterations with the interval $I_{\text{fin}} = [-3.8567, -3.8559]$. Computation times for the semidefinite subproblems are compatible (approximately 0.02 s per iteration). To investigate the effects of an initial guess closer to the optimal value, we have chosen $t^{(0)} = -4$; here, Algorithm 1 took only 4 iterations whereas `gsosopt` still took 13 iterations. On the other hand, it appears that Problem (7) for (13) is unbounded for any initial guess *larger* than the optimal value; yet, with an initial value such as $t^{(0)} = -3$, our method still terminates faster than `gsosopt` with 11 instead of 17 iterations. Unlike `gsosopt`, our method does not require a bounded confidence interval to begin with; setting $I_0 = (-\inf, +\inf)$ and $t^{(0)} = -10$, the algorithm terminates within 7 iterations.

[5] `gsosopt` terminates if the confidence interval has length ϵ_1 or smaller.

Table 1 Details of solving the quasiconvex problem (14)

$t^{(0)}$	I_0	Algorithm 1		`gsosopt`	
		Iterations	Time	Iterations	Time
−25	[−500]	7	3.8s	16	5.6s
−10	”	5	3.0s	15	5.7s
−5	”	4	2.4s	14	5.7s
−2.5	”	3	2.0s	13	5.4s
−2	”	3	2.0s	17	6.1s
−5	(− inf, 0)	4	2.3s	–	–
−2	”	3	1.8s	–	–
−5	(− inf, + inf)	7	4.2s	–	–
−2	”	3	1.9s	–	–

Local stability analysis Suppose a polynomial $V \in \mathbb{R}[x]$ is known to be a local Lyapunov function for a polynomial dynamic system; then solving the quasiconvex sum-of-squares program

$$\inf_{t\in\mathbb{R},\, u\in\mathbb{R}^8} t \quad \text{subject to} \quad \begin{aligned} ts(u) + (Vs(u) - J_x Vf - \varepsilon l_x) &\in \mathbb{R}[x] \\ s(u) &\in \mathbb{R}[x] \end{aligned} \tag{14}$$

where $s(\cdot)$ is a parametrized sum-of-squares multiplier (see [4, Section IV.B] for details), yields a lower bound $r = -t$ on the largest sublevel set of V contained in the stable domain. We take

$$f = \begin{bmatrix} -x_2 \\ x_1 + (x_1^2 - 1)x_2 \end{bmatrix}, \quad V = 1.5x_1^2 - 1x_1x_2 + x_2^2$$

for this example. We have used both Algorithm 1 and `gsosopt` to solve (14) with different choices for $t^{(0)}$ and I_0; the results were $t^{(\text{fin})} = -2.3045$ and $I_{\text{fin}} = [-2.3048, -2.3041]$, respectively. Details of the computations are given in Table 1. Note how an initial guess close to the optimal value but near the boundary of the confidence interval leads to a large number of iterations of `gsosopt`.

Region of attraction estimation A popular strategy to estimate the region of attraction of a locally stable, polynomial dynamic system is to alternate between solving (14) and updating the candidate Lyapunov function V via a convex sum-of-squares program. In such a scheme, the quasiconvex sum-of-squares program is solved repeatedly. If the changes in V are small enough, the next optimal value will be close to the previous, suggesting warmstarting. We have performed 20 rounds of an alternating scheme for the dynamics in the previous example; Algorithm 1, if initialized in the previous solution, took on average 6.5 iterations compared to 10 iterations per solution with a default initial guess. Consistent with the results in Table 1, on the

other hand, `gsosopt` performed worse under warmstarting (17 instead of 16 iterations per solution) as the optimal value decreases.

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Data availibility The source code for Algorithm 1, numerical examples, and libraries for MATLAB are available from [22].

Declarations

Conflict of interest There are no conflict of interest to declare that are relevant to this work.

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