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# Some Notes on Savage's Representation Theorem

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# Abstract

Savage's famous representation theorem is based on seven postulates of rational choice. We resolve some loose ends in the literature concerning the relationship between different versions of Savage's axioms. This leads us to the present form of the representation theorem. We also discuss some issues regarding the historical development of Savage's representation theorem.

**Keywords** Representation theorem  $\cdot$  Savage's postulates  $\cdot$  Subjective expected utility

Hartmann (2020) recalls Savage's (1972) postulates of rational choice, i.e., P1–P7, and shows that P3 is redundant, since it is implied by P1, P2, P4, and P7. Hartmann makes some slight mistakes with P5 and P6 (Frahm and Hartmann, 2023). His version of P5, here denoted by  $P5^w$ , is weaker than Savage's P5, whereas his version of P6, henceforth symbolized by  $P6^s$ , is stronger than Savage's P6. However, P5 and P6 are not required to demonstrate that P3 is redundant and thus Hartmann's result remains valid. Nonetheless, it is worth clarifying that P6<sup>s</sup> is implied by P1–P7. Analogously, we show that P5 and P5<sup>w</sup> are equivalent in the presence of P7, which holds true even without P7, given that we accept P1–P3 and focus on simple acts. Fishburn (1970, p. 193) observes that also Savage's P7 can be replaced with a weaker version, in the following denoted by P7<sup>w</sup>. Here, we prove that the equivalence of P7 and P7<sup>w</sup> is already implied by P1, P2, and P6. Our new insights about Savage's postulates of rational choice lead us to the present form of his representation theorem. Finally, we give some remarks on the representation theorem and briefly discuss its historical development.

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## 1 Postulates of rational choice

Let  $S \neq \emptyset$  be some state space and  $X \neq \emptyset$  be a set of consequences. Each  $s \in S$  is called a state (of the world) and every subset of *S* represents an event. A (Savage) act *f* is a mapping from *S* to *X*. The set of all acts is denoted by  $\mathcal{F}$ . A constant act  $f \equiv x \in X$  is an act such that f(s) = x for all  $s \in S$ . Each consequence *x* is considered a constant act, i.e.,  $s \mapsto f(s) = x$  for all  $s \in S$ . Further, " $f_A$ " with  $f \in \mathcal{F}$  and  $A \in 2^S$  is the restriction of *f* to *A*, i.e., the mapping from *A* to *X* created by  $s \mapsto f(s)$  for all  $s \in A$ . Moreover, " $f_A g$ " is the composite act corresponding to *f* on *A* and to *g* on  $A^c$ . Finally, let  $\geq$  be some binary preference relation on  $\mathcal{F}$  and define  $(f \succ g) : \Leftrightarrow \neg(g \ge f)$  as well as  $(f \sim g) : \Leftrightarrow (f \ge g) \land (g \ge f)$ .

Savage's first two postulates are as follows (Savage, 1972):

P1: The preference relation  $\geq$  is complete and transitive. P2: We have

$$f_A g \ge f'_A g \implies f_A g' \ge f'_A g'$$

for all  $f, f', g, g' \in \mathcal{F}$  and  $A \in 2^S$ .

Henceforth, " $f \ge_A g$ " means that  $f_A h \ge g_A h$ , " $f \ge_A g$ " means that  $f_A h \ge g_A h$ , and " $f \sim_A g$ " means that  $f_A h \sim g_A h$  for some  $h \in \mathcal{F}$ . According to P2, the choice of h is arbitrary. Finally, an event A is called *null* if and only if  $f \ge_A g$  for all  $f, g \in \mathcal{F}$ . This leads us to the next postulates:

P3: We have

 $x \ge y \iff x \ge_A y$ 

for all  $x, y \in X$  and each nonnull  $A \in 2^S$ . P4: We have

 $v_A w \ge v_B w \implies x_A y \ge x_B y$ 

for all  $A, B \in 2^S$  and  $v, w, x, y \in X$  such that  $v \succ w$  as well as  $x \succ y$ .

P5: There exist some  $x, y \in X$  such that  $x \succ y$ .

P6: Let  $f, g \in \mathcal{F}$  be such that  $f \succ g$  and  $x \in X$ . There exists a partition  $\{A_1, \dots, A_n\}$  of *S* such that  $x_A, f \succ g$  and  $f \succ x_{A,g}$  for  $i = 1, \dots, n$ .

P7: Let  $f, g \in \mathcal{F}$  and  $A \in 2^S$ .

- (i) If  $f(s) \ge_A g$  for all  $s \in A$ , then  $f \ge_A g$ .
- (ii) If  $f \ge_A g(s)$  for all  $s \in A$ , then  $f \ge_A g$ .

P1 and P7 in Hartmann (2020) correspond to Savage's P1 and P7. Hartmann's versions of P2 and P4 use " $\Leftrightarrow$ " instead of " $\Rightarrow$ ." It is obvious that those versions are equivalent to Savage's P2 and P4. Further, Hartmann's version of P3 slightly differs from Savage's P3, too. To be more precise, Hartmann requires that  $(x \ge y) \Leftrightarrow (x_A f \ge y_A f)$  (for all  $x, y \in X, f \in \mathcal{F}$ , and each nonnull  $A \in 2^S$ ), whereas

Savage just postulates that  $(x \ge y) \Leftrightarrow (x \ge_A y)$ . However, in the light of P2, both versions of P3 are equivalent. Finally, as pointed out by Frahm and Hartmann (2023), Hartmann's versions of P5 and P6 essentially differ from Savage's original postulates:

P5<sup>*w*</sup>: There exist some  $f, g \in \mathcal{F}$  such that  $f \succ g$ .

P6<sup>s</sup>: Let  $f, g, h \in \mathcal{F}$  be such that f > g. There exists a partition  $\{A_1, \dots, A_n\}$  of S such that  $h_A f > g$  and  $f > h_A g$  for  $i = 1, \dots, n$ .

Hartmann's  $P5^{w}$  is weaker than Savage's P5 and Hartmann's  $P6^{s}$  is stronger than Savage's P6.

**Observation 1** If P7 is satisfied, then P5 and  $P5^{w}$  are equivalent.

**Proof** It is clear that P5 implies P5<sup>*w*</sup>. Let P5<sup>*w*</sup> and P7 be satisfied and suppose that f > g for some  $f, g \in \mathcal{F}$ . Further, set A = S in P7. From the contrapositive of P7 (i), we conclude that  $f > g(s_1)$  for some  $s_1 \in S$  and from the contrapositive of P7 (ii) it follows that  $f(s_2) > g(s_1)$  for some  $s_2 \in S$ . Hence, P5 and P5<sup>*w*</sup> are equivalent under P7.

Thus, P5 can be replaced with its weaker version  $P5^w$ , given that P7 is required, too. The very idea that a subject can be indifferent among all consequences but not among all acts seems odd. Observation 1 clarifies that such a strange situation can indeed occur only if P7 is violated.

An act is called simple if and only if it has a finite number of consequences on some event *A* with  $A^c$  being null. Savage (1972, Section 5.2) calls such an act a *gamble*, but nowadays the term "simple" seems to be more common and *A* is typically equated with *S* (Fishburn, 1981). However, the latter makes no difference at all, since  $A^c$  is always null. Correspondingly, we call any restriction  $f_A$  with  $A \subset S$  simple if and only if it has a finite number of consequences.

The following proposition states that the subject is indifferent among all simple acts if P1–P3 are satisfied but P5 is violated.<sup>1</sup>

**Proposition 1** If P1–P3 are satisfied but P5 is violated, then  $f \sim g$  for all simple  $f, g \in \mathcal{F}$ .

**Proof** Let P1–P3 be satisfied and P5 be violated, i.e.,  $x \sim y$  for all  $x, y \in X$ . Consider any simple acts f and g. Suppose, without loss of generality, that f and g have a finite number of consequences on S. There exists some partition  $\{A_1, \ldots, A_n\}$  of S such

<sup>&</sup>lt;sup>1</sup> We thank an anonymous reviewer very much for that important hint.

that f and g are constant on  $A_1, \ldots, A_n$ . From Theorem 3 in Savage (1972, p. 26) we conclude that  $f \sim g$ .

Our next observation implies that we may replace P5 with  $P5^{w}$  without explicitly requiring P7, provided that P1–P3 are satisfied and we put our attention to simple acts.

**Observation 2** Let P1–P3 be satisfied and consider only simple acts.

- (i) P5 and  $P5^w$  are equivalent.
- (ii) P7 is satisfied.

**Proof** (i) This is a direct implication of Proposition 1. (ii) Assume that P1–P3 are satisfied. Consider any simple acts f and g. Suppose, without loss of generality, that f and g have a finite number of consequences on S and thus also on any nonempty  $A \in 2^{S}$ . Hence, f possesses some worst consequence x on A, i.e.,  $f(s) \ge x$  for all  $s \in A$ , and from Theorem 3 in Savage (1972, p. 26) we conclude that  $f \ge_A x$ . Now, suppose that  $f(s) \ge_A g$  for all  $s \in A$  and thus  $f \ge_A x \ge_A g$ . P1 and P2 imply that  $f \ge_A g$ . In the case of  $A = \emptyset$ , it is clear that  $f \ge_A g$ , too. Thus, P7 (i) is satisfied. P7 (ii) follows mutatis mutandis.

Observation 2 refers only to simple acts. If we do not focus on simple acts, it can very well happen that  $P5^{w}$  is satisfied without P5 although P1–P4 and P6 hold true.

**Observation 3** The postulates P1–P4, P5<sup>*w*</sup>, and P6 do not imply P5.

For example, suppose that S = X = [0, 1]. Further, let  $\mu$  be the Lebesgue measure and  $\mathcal{L}$  be the Lebesgue  $\sigma$ -algebra on [0, 1]. Consider the functional

$$U: f \mapsto \sup \left\{ \mu(A) : f_A \text{ is simple and } A \in \mathcal{L} \right\}$$

and define  $f \ge g : \Leftrightarrow U(f) \ge U(g)$  for all  $f, g \in \mathcal{F}$ . It can easily be verified that the preference relation  $\ge$  obeys the postulates P1–P4, P5<sup>w</sup>, and P6, but it violates P5 and thus also P7.

Hartmann (2020) mentions in Footnote 4 that Savage's postulate P7 can be replaced with a weaker version, according to Fishburn (1970, p. 193), if we assume that P1, P2, and P6 are satisfied, too. Hence, P3–P5 are not necessary. To be more precise, Fishburn's version of P7 is:

P7<sup>*w*</sup>: Let  $f, g \in \mathcal{F}$  and  $A \in 2^{S}$ .

- (i) If  $f(s) \succ_A g$  for all  $s \in A$ , then  $f \geq_A g$ .
- (ii) If  $f \succ_A g(s)$  for all  $s \in A$ , then  $f \succeq_A g$ .

Here, we provide an axiomatic proof of Hartmann's statement that  $P7^{w}$  is equivalent to P7 under P1, P2, and P6.<sup>2</sup> For this purpose, we need the following lemma.

**Lemma 1** If P2 and P6 are satisfied,  $x \in X$ , and  $f, g \in \mathcal{F}$  are such that  $f \succ_A g$  for some  $A \in 2^S$ , then there exists a partition  $\{A_1, \ldots, A_n\}$  of A such that  $x_{A_i}f \succ_A g$  and  $f \succ_A x_{A_i}g$  for  $i = 1, \ldots, n$ .

**Proof** First of all, note that A must be nonempty if  $f \succ_A g$ . Due to P2,  $f \succ_A g$  is equivalent to  $f' := f_A x \succ g_A x =: g'$  for any  $x \in X$ . P6 guarantees that there exists a partition  $\{B_1, \ldots, B_m\}$  of S such that  $x_{B_i}f' \succ g'$  and  $f' \succ x_{B_i}g'$  for  $i = 1, \ldots, m$ . The acts  $f', g', x_{B_i}f'$ , and  $x_{B_i}g'$  coincide with x on  $A^c$  for  $i = 1, \ldots, m$ . Hence, in order to create the desired partition  $\{A_1, \ldots, A_n\}$ , we can choose the nonempty intersections of A and  $B_1, \ldots, B_m$ .

**Observation 4** If P1, P2, and P6 are satisfied, then P7 and P7<sup>w</sup> are equivalent.

**Proof** It is clear that P7 implies P7<sup>w</sup>. Let P1, P2, P6, and P7<sup>w</sup> be satisfied and consider some  $f, g \in \mathcal{F}$  and  $A \in 2^S$ . We already have  $f \ge_A g$  if the event A is null. Thus, let us assume that A is nonnull. Suppose that  $f(s) \ge_A g$  for all  $s \in A$  and choose any  $s_0 \in A$ . Assume that  $g >_A f$ . Due to Lemma 1, there exists a partition  $\{A_1, \ldots, A_n\}$  of A such that  $f(s) \ge_A g >_A f(s_0)_A f$  for all  $s \in A$  and  $i = 1, \ldots, n$ . P1 and P2 imply that  $f(s) >_A f(s_0)_A f$  for all  $s \in A$  and  $i = 1, \ldots, n$ . Now, P7<sup>w</sup> (i) guarantees that  $f \ge_A f(s_0)_A f$ , i.e.,  $f \ge_A f(s_0)$ , for  $i = 1, \ldots, n$ . From Theorem 2 in Savage (1972, p. 24) it follows that  $f \ge_A f(s_0) \ge_A g$ , i.e.,  $f \ge_A g$ , which contradicts our initial assumption that  $g >_A f$ . This means that  $f \ge_A g$ , which completes the proof of P7 (i). The proof of P7 (ii) follows analogously by using P7<sup>w</sup> (ii). Thus, P7<sup>w</sup> implies P7 given P1, P2, and P6. We conclude that P7 and P7<sup>w</sup> are equivalent under these circumstances.

### 2 The representation theorem

Savage (1954, Section 5.3) first applied P1–P6 to derive a subjective expected-utility (SEU) representation of  $\geq$  for all *simple* acts, without using P7 at that stage.<sup>3</sup> Observation 2 guarantees that P5 can be replaced with its weaker version P5<sup>w</sup> and that P7 is redundant, given that all acts that are taken into consideration are simple. P3 is not redundant given Savage's first six postulates of rational choice and so we do not generalize this part of his work. Savage (1954, Section 5.4) next added P7 to P1–P6 in order to extend his SEU representation of  $\geq$  to *all* acts, but he did not notice that P1–P7 imply bounded utility. This was pointed out later on by Fishburn (1970, Section 14.1) and recognized also by Savage (1972, p. 80). Wakker (1993b,

<sup>&</sup>lt;sup>2</sup> Wakker (1970) gives an alternative proof in an unpublished note.

<sup>&</sup>lt;sup>3</sup> Abdellaoui and Wakker (2020) provide a simplified proof.

Theorem 2.17) shows how Savage's representation theorem can be extended to unbounded utility.

We conclude that P5 is implied by P5<sup>w</sup> and P7. Further, P7 is implied by P1, P2, P6, and P7<sup>w</sup>. Finally, according to Hartmann (2020, Theorem 1), P1, P2, P4, and P7 imply P3. Hence, in the presence of P1, P2, P4, and P6, we can eliminate P3 and substitute P7 with Fishburn's weaker P7<sup>w</sup> as well as P5 with Hartmann's weaker P5<sup>w</sup>. This leads us to the following representation theorem, which can be considered the main result of Savage's theory of rational choice.

**Theorem 1** (Representation theorem) *The preference relation*  $\geq$  *satisfies the postulates P1, P2, P4, P5<sup>w</sup>, P6, and P7<sup>w</sup> if and only if there exists a convex-ranged probability measure P on 2<sup>s</sup> and a real-valued, nonconstant, and bounded utility function u on X such that* 

$$f \ge g \quad \Longleftrightarrow \quad \int_{S} u(f) \, dP \ge \int_{S} u(g) \, dP$$

for all  $f,g \in \mathcal{F}$ , in which case P is unique and u is unique up to positive affine transformations.

This means that  $\geq$  has a SEU representation for all acts if and only if it satisfies the axiomatic system P1, P2, P4, P5<sup>w</sup>, P6, and P7<sup>w</sup>, which is weaker than P1–P7.<sup>4</sup> Further, Harju et al. (2023) demonstrate that any proper subset of P1–P6 and P7<sup>w</sup> other than P1, P2, P4–P6, and P7<sup>w</sup> is insufficient for that SEU representation of  $\geq$ . For this reason, the axioms P1, P2, P4–P6, and P7<sup>w</sup> are independent, which holds true after any weakening that leads to the SEU representation of  $\geq$  for all acts. Hence, the axioms P1, P2, P4, P5<sup>w</sup>, P6, and P7<sup>w</sup> are independent, too.

The representation theorem requires the probability measure P to be *convex* ranged, not only nonatomic: A probability measure P is said to be

- nonatomic if and only if  $P(A) > 0 \Rightarrow \exists B \subset A : P(A) > P(B) > 0$  but
- convex ranged if and only if  $P(A) > p > 0 \Rightarrow \exists B \subset A : P(B) = p$ .

If *P* is  $\sigma$ -additive, both conditions are equivalent, but if it is not  $\sigma$ -additive, the latter condition turns out to be stronger than the former (Wakker, 1993a). Savage (1972, p. 41) notes Ulam's (1930) famous proof that every convex ranged and  $\sigma$ -additive probability measure that is defined on all subsets of the unit interval is inconsistent with the continuum hypothesis. More generally, Stinchcombe (1997, Footnote 3) points out that the same holds true if the probability measure is defined on all subsets of any infinite set (Sierpinski, 1956). Hence, Savage's probability measure *P* cannot be  $\sigma$ -additive if the continuum hypothesis is true (Wakker, 1993a). In any case, the integrals in Theorem 1 do not require that *P* is

<sup>&</sup>lt;sup>4</sup> We prove in the Appendix that P1–P5, P5<sup>w</sup>, P6<sup>s</sup>, P6, P7, and P7<sup>w</sup> are implied by the SEU representation of  $\geq$ .

 $\sigma$ -additive. This is already mentioned by Savage (1972, Appendix 1) and elaborated in great detail by Fishburn (1970, Section 10.3).

Frahm and Hartmann (2023) mention that P5 and P6 are incorrectly cited by Hartmann (2020), but this does not affect the main result, namely that P3 is redundant if we aim at the SEU representation of  $\geq$  for all acts. Nonetheless, it is worth clarifying that Hartmann's stronger version of P6, i.e., P6<sup>s</sup>, is implied by P1–P7. Even under the weaker postulates P1, P2, P4, P5<sup>w</sup>, and P7<sup>w</sup>, which are used by Theorem 1, P6 and P6<sup>s</sup> turn out to be equivalent.

**Observation 5** If P1, P2, P4, P5<sup>w</sup>, and P7<sup>w</sup> are satisfied, then P6 and P6<sup>s</sup> are equivalent.

**Proof** It is clear that P6<sup>s</sup> implies P6. Let P1, P2, P4, P5<sup>w</sup>, P6, and P7<sup>w</sup> be satisfied. According to Theorem 1, it follows that  $\geq$  possesses the SEU representation for all acts. Let  $f, g, h \in \mathcal{F}$  be such that f > g, which means that  $\int_{S} u(f) dP > \int_{S} u(g) dP$ , and define

$$\delta := \int_{S} u(f) \, dP - \int_{S} u(g) \, dP > 0.$$

Further, let *a* be a lower and b > a an upper bound of the utility function *u*. Hence, for each  $A \in 2^S$ , the difference between  $\int_A u(h) dP + \int_{A^c} u(g) dP$  and  $\int_S u(g) dP$  is at most (b - a)P(A). The same holds true for the difference between  $\int_S u(f) dP$  and  $\int_A u(h) dP + \int_{A^c} u(f) dP$ . Since the probability measure *P* is convex ranged, there exists a partition  $\{A_1, \ldots, A_n\}$  of *S* such that  $P(A_i) < \delta/(b - a)$ , i.e.,  $(b - a)P(A_i) < \delta$ , for  $i = 1, \ldots, n$ . Thus,

$$\left(\int_{A_i} u(h) dP + \int_{A_i^c} u(g) dP\right) - \int_S u(g) dP < \delta = \int_S u(f) dP - \int_S u(g) dP$$

and

$$\int_{S} u(f) dP - \left( \int_{A_i} u(h) dP + \int_{A_i^c} u(f) dP \right) < \delta = \int_{S} u(f) dP - \int_{S} u(g) dP$$

for i = 1, ..., n. This means that

$$\int_{A_i} u(h) dP + \int_{A_i^c} u(g) dP < \int_S u(f) dP \quad \text{and} \quad \int_S u(g) dP < \int_{A_i} u(h) dP + \int_{A_i^c} u(f) dP$$

for i = 1, ..., n, i.e.,  $h_{A_i}f > g$  and  $f > h_{A_i}g$  for i = 1, ..., n. We conclude that P6 implies P6<sup>s</sup>, given that P1, P2, P4, P5<sup>w</sup>, and P7<sup>w</sup> are satisfied. To sum up, P6 and P6<sup>s</sup> are equivalent under these circumstances.

# Appendix

Now, we prove that all postulates of rational choice follow from the SEU representation of  $\geq$  for all acts. Throughout this appendix, we assume that *P* and *u* satisfy all properties described by Theorem 1. In particular, the utility function *u* is bounded.

P1: Since *u* is bounded, each  $f \in \mathcal{F}$  possesses some expected utility  $\int_S u(f) dP$ . Hence,  $\geq$  is complete. Moreover, we have

$$\int_{S} u(f) dP \ge \int_{S} u(g) dP \ge \int_{S} u(h) dP \implies \int_{S} u(f) dP \ge \int_{S} u(h) dP$$

for all  $f, g, h \in \mathcal{F}$ . Thus,  $\geq$  is also transitive.

P2: Consider any  $f, f', g, g' \in \mathcal{F}$  and  $A \in 2^S$ . Suppose that  $\int_S u(f_A g) dP \ge \int_S u(f'_A g) dP$ , i.e.,

$$\int_{A} u(f) dP + \int_{A^c} u(g) dP \ge \int_{A} u(f') dP + \int_{A^c} u(g) dP.$$

This means that  $\int_A u(f) dP \ge \int_A u(f') dP$  and thus

$$\int_{A} u(f) dP + \int_{A^c} u(g') dP \ge \int_{A} u(f') dP + \int_{A^c} u(g') dP,$$

i.e.,  $\int_{S} u(f_Ag') dP \ge \int_{S} u(f'_Ag') dP$ . Put another way,  $f_Ag \ge f'_Ag \Rightarrow f_Ag' \ge f'_Ag'$ . P3: We have  $u(x) \ge u(y) \Leftrightarrow u(x)P(A) \ge u(y)P(A)$  for all  $x, y \in X$  and  $A \in 2^S$  with

P(A) > 0. This means that  $x \ge y \Leftrightarrow x \ge_A y$  whenever A is nonnull.

P4: Let  $v, w, x, y \in X$  be such that u(v) > u(w), u(x) > u(y), and

$$u(v)P(A) + u(w)P(A^c) \ge u(v)P(B) + u(w)P(B^c)$$

for any  $A, B \in 2^S$ . It follows that  $P(A) \ge P(B)$  and thus

 $u(x)P(A) + u(y)P(A^{c}) \ge u(x)P(B) + u(y)P(B^{c}).$ 

This means that  $v_A w \ge v_B w \Rightarrow x_A y \ge x_B y$ . P5: Since *u* is nonconstant, there exist some  $x, y \in X$  such that u(x) > u(y), i.e., x > y

P5<sup>w</sup>: This is a direct implication of P5.

P6: This is a direct implication of  $P6^s$ .

P6<sup>s</sup>: See Observation 5, whose proof is based on the SEU representation of  $\geq$  for all acts.

$$\left(\inf_{A} u(f)\right) P(A) \ge \int_{A} u(g) dP \implies \int_{A} u(f) dP \ge \int_{A} u(g) dP$$

We conclude that  $f \ge_A g$  if  $f(s) \ge_A g$  for all  $s \in A$ . Analogously, it follows that  $f \ge_A g$  if  $f \ge_A g(s)$  for all  $s \in A$ . Hence, P7 is satisfied.

P7<sup>*w*</sup>: This is a direct implication of P7.

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**Data Availability** This work is not based on any empirical or simulated data. Hence, there is no dataset necessary or available in order to interpret, replicate, and build upon the findings reported here.

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