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# PAPERS on Economics & Evolution



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# 0702

**Strategic Advertisement with Externalities:  
A New Dynamic Approach**

**by**

**Reinoud Joosten**

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# Strategic advertisement with externalities: a new dynamic approach\*

Reinoud Joosten<sup>†</sup>

April 20, 2007

## Abstract

We model and analyze strategic interaction over time in a duopolistic market. Each period the firms independently and simultaneously choose whether to advertise or not. Advertising increases the own immediate sales, but may also cause an externality, e.g., increase or decrease the immediate sales of the other firm *ceteris paribus*.

There exists also an effect of past advertisement efforts on current sales. The ‘market potential’ of each firm is determined by its own but also by its opponent’s past efforts. A higher effort of either firm leads to an increase of the market potential, however the impact of the own past efforts is always stronger than the impact of the opponent’s past efforts. How much of the market potential materializes as immediate sales, then depends on the current advertisement decisions.

We determine feasible rewards and (subgame perfect) equilibria for the limiting average reward criterion using methods inspired by the repeated-games literature. Uniqueness of equilibrium is by no means guaranteed, but Pareto efficiency may serve very well as a refinement criterion for wide ranges of the advertisement costs.

**JEL-codes:** C72, C73, L13, M31, M37.

**Keywords:** advertising, externalities, average rewards, equilibria.

## 1 Introduction

The aim of this paper is to model and to analyze strategic interaction over time in a duopolistic market in which advertising causes several types of ‘externalities’. For this purpose, we design a so-called game with joint frequency dependent stage payoffs, JFD-game for short, which allows us to model rather complex relationships, and analyze it with modifications of

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techniques traditionally used for infinitely repeated games. JFD-games generalize games with frequency dependent stage payoffs, or FD-games, introduced by Brenner & Witt [2003], and classified and analyzed in Joosten *et al.* [2003]. In an FD-game, current stage payoffs depend on the relative frequencies with which all actions were played in the past. In a JFD-game, current stage payoffs depend on the relative frequencies with which all action pairs were played in the past. Hence, for the former type of games only the ‘marginal’ frequencies matter, in the newer type of games the joint frequencies matter. So, this paper offers two innovations, it brings a novel way of modelling to the advertisement literature, and for this purpose it brings a newly designed game-theoretical approach.

Each period both firms independently and simultaneously first choose whether to advertise or not. The firms know the advertisement decisions and then produce the pertaining Cournot-Nash quantities. If a firm decides to advertise, it pays a fixed amount at the beginning of the period. Hence, for the Cournot competition phase the advertisement costs can be regarded as sunk. It is not our aim to model the Cournot competition explicitly, we are merely interested in its outcome in sales.

Advertising has two effects separated in the time dimension, it affects immediate sales directly and future sales in a cumulative manner (cf., e.g., Friedman [1983]). With respect to the short run effects, advertising increases the own immediate sales given the action of the other firm. Advertising may also cause immediate externalities. Friedman [1983] distinguishes predatory and cooperative advertising. An increase in advertising efforts of one firm leads to a sales decrease of the other in the former type, to an increase in the second type of advertising.

There is also an effect of *current* advertisement on *future* sales. We use the notion of market potential, both with respect to the total market and the individual firms, in order to capture these effects. The *current* market potential of each firm is determined by its own but also by its opponent’s *past* efforts. A higher effort of either firm leads to an increase of the market potentials, but the impact of the own past efforts on the own market potential is always stronger than the impact of the opponent’s past efforts. Advertising is therefore cooperative in its cumulative effects on the market potentials. How much of the market potential materializes as immediate sales, depends on the current advertisement decisions.

Dorfman & Steiner [1954] examine the effects of advertising in a static monopoly and derive necessary conditions for the optimal level of advertising. In a dynamic monopolistic model, Nerlove & Arrow [1962] treat advertisement expenditures similar to investments in a durable good. This durable good is called goodwill which is assumed to influence current sales. Historical investments in advertisement increase the stock of goodwill, but simultaneously goodwill depreciates over time. Nerlove and Arrow derive necessary conditions for optimal advertising, thus generalizing the Dorfman

and Steiner result. Friedman [1983] in turn generalizes the Nerlove-Arrow model to allow oligopolistic competition in advertising and derives necessary conditions for the existence of a noncooperative equilibrium (Nash [1951]).

Our notion of market potential is quite close to goodwill in e.g., Nerlove & Arrow [1962] and Friedman [1983]. The modeling of the changes in time in the former model follows the work of Vidale & Wolfe [1957], though the authors quote Waugh [1959] as a main source of inspiration. Vidale & Wolfe [1957] present an interesting field study giving empirical evidence of the positive effects of past advertising on current sales. Furthermore, once advertising expenditures are stopped, current sales do not collapse, but slowly deteriorate over time. Though Friedman quotes the work of Nerlove and Arrow as a source for the notion of goodwill, his technical treatment of the changes of the level of goodwill in time is inspired by Prescott [1973].

Economics has produced a large body of work on advertising featuring a broad variety of approaches. One source of variety is the modeling of time-related aspects. For instance, is the model static (e.g., Dorfman & Steiner [1954]), or is it dynamic in the sense that the strategic environment may change (e.g., Nerlove & Arrow [1962])? Another source of variety is the market under consideration, e.g., monopoly (Nerlove & Arrow [1962]), oligopoly (Friedman [1983]), leader-follower oligopoly (Kydland [1977]). A third one is possible combinations of advertising with other marketing instruments, e.g., Schmalensee [1978] combines advertising and quality. A fourth one is the entity to be influenced by advertising, for instance sales (e.g., Nerlove & Arrow [1962]) or market shares<sup>1</sup> (e.g., Fershtman [1984]).

Another dimension is based on the distinction by Nelson [1970] between search and experience goods. The characteristics of former kind are known by-and-large before purchase, whereas the characteristics of the latter can be determined only after purchase. Advertising differs for the two types of goods because the information conveyed to the consumers differs. *Informative* advertising provides information on e.g., the price, availability or characteristics of a product; *persuasive* advertising tries to generate consumer interest for a product, often by association or through rather indirect ‘channels’. An example of persuasive advertising would be a famous athlete shown drinking a certain beverage, or eating some kind of cereal. Credence goods (Darby & Karni [1973]) can be regarded as an extreme type of experience good, as it is hard to determine their characteristics even after purchase. The quality of a certain brand of toothpaste can only be determined in the very long run after a visit to a dentist. Informative advertising is directed at search goods, persuasive advertising aims at experience or credence goods.

For dynamic optimal control models of advertisement Sethi [1977] performed a Herculean task by coming up with a classification distinguishing four types. The task may prove to be Sisyphean in this burgeoning field, as

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<sup>1</sup>Relevant if market size is ‘fixed’, e.g., Telser [1962], Schmalensee [1978].

a more recent survey by Feichtinger *et al.* [1994] already features six classes. Three new categories were introduced, categories present in the earlier classification were renamed and expanded, and merely one category survived in its original form. The reader interested in differential games on advertising is referred to Dockner *et al.* [2000] and Jørgensen & Zaccour [2004].

FD-games are stochastic games with finite action spaces and infinite state spaces. The basic idea for FD-games stems from the work of Herrnstein on experimental ‘games against nature’ (cf., Herrnstein [1997]), Brenner & Witt [2003] used frequency-dependent payoffs in a multi-person game. Joosten *et al.* [2003] showed that the analysis of infinitely repeated games (cf., e.g., Van Damme [1991]) can be generalized to this type of games.

Relating this paper to the branches of literature mentioned: in our dynamic deterministic duopolistic leaderless model we restrict ourselves to the effects of cooperative persuasive advertising with long and short run externalities on sales. We assume that the firms wish to maximize the associated average profits over an infinite time-horizon. We determine equilibria for all ‘realistic’ ranges of advertisement costs employing modifications of techniques traditionally used to analyze infinitely repeated games. We find that a continuum of rewards may exist which can be supported by an equilibrium involving ‘threats’.

In the next section, we introduce the advertisement model. Section 3 deals with strategies and rewards in our model, whereas Section 4 deals with the notion of threats and with equilibria. Section 5 deals with Pareto efficiency as a refinement criterion, Section 6 concludes. The Appendix contains issues not fully treated in the main text.

## 2 The rules of the game

The advertisement game is played by two firms (players)  $A$  and  $B$  at discrete moments in time called stages. Each player has two actions and each stage the players independently and simultaneously choose an action. Action 1 for either player denotes ‘advertise’, action 2 denotes ‘not advertise’. We denote the action set of player  $A$  ( $B$ ) by  $J^A = \{1, 2\}$  ( $= J^B$ ) and  $J \equiv J^A \times J^B$ .

The payoffs at stage  $t' \in \mathbb{N}$  of the play depend on the choices of the players at that stage, and on the relative frequencies with which all actions were actually chosen until then. In our model we have two types of externality effects from advertising, an immediate one and one which develops gradually in time. We start by describing the immediate externalities, then we formalize the externalities in time, and finally we connect these effects.

## 2.1 Immediate effects, stage games

The effects of advertising on the immediate sales by the two firms can be expressed by the following **sales matrix**

$$\begin{bmatrix} a^A, a^B & b^A, c^B \\ c^A, b^B & d^A, d^B \end{bmatrix}. \quad (1)$$

The top row (left column) of player  $A$  ( $B$ ) corresponds to action 1, i.e., ‘to advertise’, the bottom row (right column) corresponds to action 2, i.e., ‘not to advertise’. If, e.g., if player  $A$  advertises and player  $B$  does not, then  $A$  has sales  $b^A$  and  $B$  has sales  $c^B$ . The following restrictions are assumed to hold for player  $k$ ,  $k = A, B$ :

$$\begin{aligned} a^k, b^k &> c^k > d^k > 0, \\ a^k + a^{-k} &> b^k + c^{-k} > d^k + d^{-k}, \\ (b^k + c^{-k}) - (d^k + d^{-k}) &> (a^k + a^{-k}) - (b^k + c^{-k}), \end{aligned} \quad (2)$$

where  $\neg k$  denotes player  $k$ ’s opponent. Hence, advertising increases the immediate sales of an agent given the action of the opponent. Observe that if firm  $A$  advertises and  $B$  does not, then  $B$  also gets an increase in current sales relative to the situation in which neither advertises. This increase can be regarded as a positive externality from  $A$ ’s effort on  $B$ ’s immediate sales.

We assume that there exist increasing returns to advertising on total immediate sales, but decreasing marginal returns. Total immediate sales are ranked  $a^k + a^{-k} > b^k + c^{-k} > d^k + d^{-k}$ , so they are highest if both firms advertise, second highest if only one firm advertises, and lowest if no firm advertises. The final inequality in (2) implies that total current sales increase more while moving from the situation in which no firm advertises to the situation in which one firm advertises, than while moving from the latter situation to the one in which both firms advertise. If  $a^k > b^k$ , then advertising is completely cooperative, otherwise it is indeterminate.

Advertising is not for free in general. We assume the costs of advertising to be fixed in time, independent of the size of the sales and identical for both firms. So, the introduction of these costs  $\kappa \geq 0$  defines the game in terms of profits, which yields the **payoff matrix**

$$\begin{bmatrix} a^A - \kappa, a^B - \kappa & b^A - \kappa, c^B \\ c^A, b^B - \kappa & d^A, d^B \end{bmatrix}. \quad (3)$$

Since ‘not advertising’ does not induce any cost, player  $A$ ’s ( $B$ ’s) entries in the bottom row (right column) do not change relative to (1).

**Example 1** *Let the sales matrix be determined by  $a^k = 100$ ,  $b^k = 104$ ,  $c^k = 88$ , and  $d^k = 84$ . Different values of the advertisement costs may alter the character*

of the stage game. For instance, taking  $\kappa = 6$  induces the payoff matrix

$$\begin{bmatrix} 94, 94 & 98, 88 \\ 88, 98 & 84, 84 \end{bmatrix}.$$

Here, advertising is the ‘obvious thing to do’. The payoffs associated with the unique pure Nash equilibrium in which both firms advertise are  $(94, 94)$ . Furthermore, for  $\kappa = 14$ , the stage game of the numerical example is given by

$$\begin{bmatrix} 86, 86 & 90, 88 \\ 88, 90 & 84, 84 \end{bmatrix}.$$

Here, two pure Nash equilibria exist in which one player advertises and the other one does not. Furthermore, a mixed Nash equilibrium exists with symmetric payoffs. For  $\kappa = 25$ , the payoff matrix is given by

$$\begin{bmatrix} 75, 75 & 79, 88 \\ 88, 79 & 84, 84 \end{bmatrix}.$$

Here, not to advertise is a strictly dominant action for both players. Costs are so high that the increase in sales does not make up. ■

## 2.2 Long term effects, market potentials

Another type of externalities accumulates gradually over time. We assume that advertisement at any point in time has two effects in the future. First, the advertisement efforts have a cumulative effect on the way the total market increases and second, the firm showing more cumulative advertisement efforts gets a larger share of this (potentially) expanded market. In order to introduce these externalities, we need several notations.

Let  $h_t^A = (j_1^A, \dots, j_{t-1}^A)$  be the sequence of actions<sup>2</sup> chosen by player  $A$  until stage  $t' \geq 2$  and let  $h_t^B$  be defined similarly for the other player. Let  $\Delta^{m \times n}$  denote the set of real-numbered non-negative  $m \times n$ -matrices such that all components add up to unity, i.e.,

$$\Delta^{m \times n} = \left\{ z \in \mathbb{R}^{m \times n} \mid z_{ij} \geq 0 \text{ for all } i, j, \text{ and } \sum_{ij} z_{ij} = 1 \right\}.$$

Let matrix  $U(i', j') \in \Delta^{2 \times 2}$  be defined by:

$$U_{ij}(i', j') = \begin{cases} 1 & \text{if } (i, j) = (i', j'), \\ 0 & \text{otherwise.} \end{cases}$$

<sup>2</sup>We assume that each player produces the Cournot quantity in the second phase of the stage in the game that arises from the advertisement decisions taken in the first phase. Hence, we omit the reference to the second phase actions.



Then, let  $q \geq 0$ , and define matrix  $\rho_t \in \Delta^{2 \times 2}$  recursively for  $t \leq t'$  by

$$\begin{aligned}\rho_1 &= \tilde{\rho} \in \Delta^{2 \times 2}, \text{ and} \\ \rho_t &= \frac{q+t-1}{q+t} \rho_{t-1} + \frac{1}{q+t} U(j_{t-1}^A, j_{t-1}^B).\end{aligned}$$

Taking  $q \gg 0$  moderates ‘early’ effects on the stage payoffs. Recall that  $j_{t-1}^A$  denotes the action chosen by  $A$  at stage  $t-1$ . The interpretation of this matrix is that entry  $ij$  of  $\rho_t$  ‘approximates’ the relative frequency with which action pair  $(i, j)$  was used before stage  $t \geq 2$ , as it can be shown that

$$\rho_t = \frac{q+1}{q+t} \tilde{\rho} + \frac{t-1}{q+t} U(h_t^A, h_t^B).$$

Here,  $U(h_t^A, h_t^B) = \frac{1}{t-1} \sum_{k=2}^t U(j_{k-1}^A, j_{k-1}^B)$  for all  $t \geq 2$ . Clearly, the influence of  $\tilde{\rho}$  and  $q$  disappears in the long run.

At stage  $t \in \mathbb{N}$ , the players have chosen action sequences  $h_t^A$  and  $h_t^B$  inducing the matrix  $\rho_t$ . The latter determines the state in which the play is at stage  $t$ .<sup>3</sup> Observe that there exist four possible successor states to any state depending on the action pair chosen at stage  $t$ .

Given  $\rho_t$ , the **market potential of player**  $k$ ,  $k = A, B$  at stage  $t$  is

$$MP_t^k = \mu^k(\rho_t), \quad (4)$$

where  $\mu^k(\cdot)$ ,  $k = A, B$ , is a continuous function from  $\Delta^{2 \times 2}$  to  $\mathbb{R}$ . The market potential of a firm at a certain stage is influenced by the firm’s own advertisement efforts before, but also by the other player’s past advertisement efforts. Own past efforts are always positive ceteris paribus, i.e., the market potential is always higher if the own advertisement efforts have been higher in the past. Also, the own past efforts have a stronger impact on the firm’s market potential than the other firm’s have. Mathematically, this means that for  $\rho_t = \rho$

$$\begin{aligned}\partial \mu^A / \partial \rho_{11} + \partial \mu^A / \partial \rho_{12} &\geq 0, \quad \partial \mu^B / \partial \rho_{11} + \partial \mu^B / \partial \rho_{21} \geq 0, \\ \partial \mu^A / \partial \rho_{11} + \partial \mu^A / \partial \rho_{12} &\geq \partial \mu^A / \partial \rho_{11} + \partial \mu^A / \partial \rho_{21}, \\ \partial \mu^B / \partial \rho_{11} + \partial \mu^B / \partial \rho_{21} &\geq \partial \mu^B / \partial \rho_{11} + \partial \mu^B / \partial \rho_{12}.\end{aligned}$$

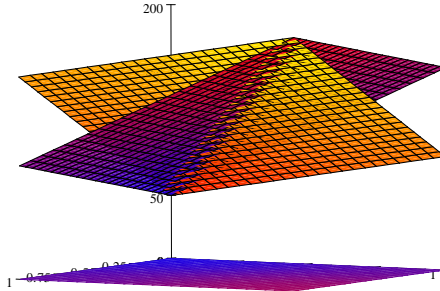
It can be seen easily that this further implies  $\partial \mu^1 / \partial \rho_{12} \geq \partial \mu^1 / \partial \rho_{21}$  as well as  $\partial \mu^2 / \partial \rho_{21} \geq \partial \mu^2 / \partial \rho_{12}$ . Let the **market potential** at stage  $t$  be given by  $MP_t \equiv MP_t^A + MP_t^B$ . Hence, the market potential increases with the rate of advertising by either firm. So, in this sense, advertising has a public good character as in Fershtman [1984].

<sup>3</sup>Slightly more formal, we will denote the state at stage  $t \in \mathbb{N}$  by  $\rho_t$  from now on.

**Example 2** Given  $\rho_t$ , the market potentials of the firms are given by

$$\begin{aligned} MP_t^A &= 50 + 110([\rho_t]_{11} + [\rho_t]_{12}) + 40([\rho_t]_{11} + [\rho_t]_{21}), \\ MP_t^B &= 50 + 110([\rho_t]_{11} + [\rho_t]_{21}) + 40([\rho_t]_{11} + [\rho_t]_{12}). \end{aligned}$$

The interested reader may confirm the above inequalities easily. The impact of the own advertisement efforts on the own market potential is weighted by a factor which is nearly three times the factor connected to the other firm's advertisement efforts. We give a graphical illustration below. For  $\rho_t = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  the market potentials for both firms are equal to 50; for  $\rho_t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  the market potentials are equal to 200. Hence, always advertising by both firms generates an expansion of the market potentials for both firms by 300%. Another interesting extreme case is  $\rho_t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , i.e., player A always advertises and player B never advertises. Then, A has a market potential equal to 160, whereas B has a market potential of 90. Here, the total market potential increases by 150%, but the increase for the player advertising is 220%, and 80% for the other. ■



### 2.3 Combining immediate and long term effects

Now, we intend to be more precise about  $a^k, b^k, c^k, d^k$ ,  $k = A, B$  in (1). For a given state  $\rho_t$  the numbers are fixed indeed, but the stage games vary over time as the play proceeds going from one state to another, and the ‘numbers’ change along. In general, we have numbers  $a_t^k, b_t^k, c_t^k, d_t^k$  connected to the set of states  $\rho_t \in [0, 1]^2$ . The following links the latter numbers to the market

potentials of the respective firms:

$$\begin{aligned} a_t^k &= (1 - \beta) MP_t^k + a_0^k \beta MP_t, \\ b_t^k &= (1 - \beta) MP_t^k + b_0^k \beta MP_t, \\ c_t^k &= (1 - \beta) MP_t^k + c_0^k \beta MP_t, \\ d_t^k &= (1 - \beta) MP_t^k + d_0^k \beta MP_t. \end{aligned}$$

We assume that the inequalities (2) hold for the parameters  $a_0^k, b_0^k, c_0^k, d_0^k$  instead of  $a^k, b^k, c^k, d^k$ , as well as  $a_0^1 + a_0^2 = 1$ ,  $\beta \in [0, 1]$ . Note that for  $\beta = 0$  no strategic interaction on the short run exists. Moreover, the height of the own market potential is exclusively decisive for the current sales. For increasing  $\beta$  the influence of the total market potential increases and then the size of the opponent's market potential increases in importance for the own immediate sales.

The numbers can be motivated as follows. Suppose that a fraction of  $1 - \beta$  of firm  $k$ 's market potential will be realized as actual sales independent from the advertisement decisions taken by the firms at that stage. Part of the public namely is already committed to purchasing the product from one firm. However, another part of the potential buyers are still not committed to buy from either firm, or are interested in buying but they may be thinking about postponing their purchase. Hence, the total market potential which is at stake, i.e., to be influenced by the advertisement behavior of both firms, is  $\beta MP_t$ . If both firms advertise, then each gets half of the market potential at stake. If only firm  $k$  advertises, then a fraction  $b_0^k$  ( $c_0^k$ ) of the market potential at stake ends up with the (non-)advertising firm as sales, the remaining fraction of  $(1 - b_0^k - c_0^{-k})$  will not materialize as sales. If neither firm advertises, then a fraction  $(1 - d_0^k - d_0^{-k})$  of the market potential at stake will not materialize as sales, the remainder is split equally.

**Example 3** (Continued) *The general expression for the sales game at stage  $t$  given  $\rho_t$  is*

$$\begin{bmatrix} a_t^1, a_t^2 & b_t^1, c_t^2 \\ c_t^1, b_t^2 & d_t^1, d_t^2 \end{bmatrix}$$

where for  $\beta = \frac{1}{5}$ ,  $a_0^k = \frac{1}{2}$ ,  $b_0^k = \frac{3}{5}$ ,  $c_0^k = \frac{1}{5}$ , and  $d_0^k = \frac{1}{10}$ , we have

$$\begin{aligned} a_t^k &= \frac{4}{5} MP_t^k + \frac{1}{10} MP_t, & b_t^k &= \frac{4}{5} MP_t^k + \frac{3}{25} MP_t, \\ c_t^k &= \frac{4}{5} MP_t^k + \frac{1}{25} MP_t, & d_t^k &= \frac{4}{5} MP_t^k + \frac{1}{50} MP_t, \\ MP_t^A &= 50 + 110([\rho_t]_{11} + [\rho_t]_{12}) + 40([\rho_t]_{11} + [\rho_t]_{21}), \\ MP_t^B &= 50 + 110([\rho_t]_{11} + [\rho_t]_{21}) + 40([\rho_t]_{11} + [\rho_t]_{12}), \\ MP_t &= 100 + 150(1 + [\rho_t]_{11} - [\rho_t]_{22}), \\ \rho_1 &= \tilde{\rho} \in \Delta^{2 \times 2}, \text{ and} \\ \rho_t &= \frac{q + t - 1}{q + t} \rho_{t-1} + \frac{1}{q + t} U(j_{t-1}^A, j_{t-1}^B). \end{aligned}$$

To obtain the stage game with respect to the profits, advertisement costs must be subtracted from entries in the first row (column) of player A (B). The effects of advertising on the stage payoffs depend on the costs of advertising, hence, as in Subsection 2.1., the game changes with every value of  $\kappa$ . ■

### 3 What can they do, what can they get?

At stage  $t$ , both players know the current state and the history of play, i.e., the state visited and actions chosen at stage  $u < t$  denoted by  $(\rho_u, j_u^A, j_u^B)$ . A **strategy** prescribes at all stages, for any state and history, a mixed action to be used by a player. The sets of all strategies for A respectively B will be denoted by  $\mathcal{X}^A$  respectively  $\mathcal{X}^B$ , and  $\mathcal{X} \equiv \mathcal{X}^A \times \mathcal{X}^B$ . The payoff to player  $k$ ,  $k = A, B$ , at stage  $t$ , is stochastic and depends on the strategy-pair  $(\pi, \sigma) \in \mathcal{X}$ ; the **expected stage payoff** is denoted by  $R_t^k(\pi, \sigma)$ .

The players receive an infinite stream of stage payoffs during the play, and they are assumed to wish to maximize their average rewards. For a given pair of strategies  $(\pi, \sigma)$ , player  $k$ 's **average reward**,  $k = A, B$ , is given by  $\gamma^k(\pi, \sigma) = \liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T R_t^k(\pi, \sigma)$ ;  $\gamma(\pi, \sigma) \equiv (\gamma^A(\pi, \sigma), \gamma^B(\pi, \sigma))$ .

It may be quite hard to determine the **set of feasible (average) rewards**  $F$ , directly. It is not uncommon in the analysis of repeated or stochastic games to limit the scope of strategies on the one hand, and to focus on rewards on the other. Here, we will do both, we focus on rewards from strategies which are pure and jointly convergent. Then, we extend our analysis to obtain more feasible rewards.

A strategy is **pure**, if at *each* stage a **pure action** is chosen, i.e., the action is chosen with probability 1. The set of pure strategies for player  $k$  is  $\mathcal{P}^k$ , and  $\mathcal{P} \equiv \mathcal{P}^A \times \mathcal{P}^B$ . The strategy pair  $(\pi, \sigma) \in \mathcal{X}$  is **jointly convergent** if and only if  $z^{\pi, \sigma} \in \Delta^{m \times n}$  exists such that for all  $\varepsilon > 0$ :

$$\limsup_{t \rightarrow \infty} \Pr_{\pi, \sigma} \left[ \left| \frac{\#\{j_u^A=i \text{ and } j_u^B=j \mid 1 \leq u \leq t\}}{t} - z_{ij}^{\pi, \sigma} \right| \geq \varepsilon \right] = 0 \text{ for all } (i, j) \in J.$$

where  $\Pr_{\pi, \sigma}$  denotes the probability under strategy-pair  $(\pi, \sigma)$ .  $\mathcal{JC}$  denotes the set of jointly-convergent strategy pairs. Under a pair of jointly-convergent strategies, the relative frequency of each action pair  $(i, j) \in J$  converges with probability 1 to  $z_{ij}^{\pi, \sigma}$  in the terminology of Billingsley [1986, p.274], i.e., this implies  $\lim_{t \rightarrow \infty} E_{\pi, \sigma} \{U(h_t^A, h_t^B)\} = z^{\pi, \sigma}$ . However, this implies also  $\lim_{t \rightarrow \infty} E_{\pi, \sigma} \{\rho_t\} = z^{\pi, \sigma}$ .

The **set of jointly-convergent pure-strategy rewards** is given by

$$P^{\mathcal{JC}} \equiv cl \left\{ (x^1, x^2) \in \mathbb{R}^2 \mid \exists_{(\pi, \sigma) \in \mathcal{P} \cap \mathcal{JC}} : (\gamma^k(\pi, \sigma), \gamma^k(\pi, \sigma)) = (x^1, x^2) \right\},$$

where  $cl S$  is the closure of the set  $S$ . The interpretation is that for any pair of rewards in this set, we can find a pair of jointly-convergent pure strategies that yield rewards arbitrarily close to the original pair of rewards.

The set  $P^{\mathcal{J}\mathcal{C}}$  can be determined rather conveniently, as we will show now. For a jointly-convergent pair of strategies  $(\pi, \sigma)$ , we have that the expected market potentials for the firms and the total market potential converge to respectively  $\mu^A(z^{\pi, \sigma})$ ,  $\mu^B(z^{\pi, \sigma})$  and  $\mu^A(z^{\pi, \sigma}) + \mu^B(z^{\pi, \sigma})$  because the functions involved are continuous (cf., e.g., Billingsley [1986]). The ‘expected long run stage game’ for this pair of jointly convergent strategies  $(\pi, \sigma)$  is

$$\left[ \begin{array}{cc} a_{\infty}^A(\pi, \sigma) - \kappa, a_{\infty}^B(\pi, \sigma) - \kappa & b_{\infty}^A(\pi, \sigma) - \kappa, c_{\infty}^B(\pi, \sigma) \\ c_{\infty}^A(\pi, \sigma), b_{\infty}^B(\pi, \sigma) - \kappa & d_{\infty}^A(\pi, \sigma), d_{\infty}^B(\pi, \sigma) \end{array} \right]$$

where,  $a_{\infty}^A(\pi, \sigma) = (1 - \beta)\mu_t^A(z^{\pi, \sigma}) + a_0^k\beta(\mu_t^A(z^{\pi, \sigma}) + \mu_t^B(z^{\pi, \sigma}))$  and the other parameters are defined analogously. Furthermore, under  $(\pi, \sigma)$  the relative frequency of action pair (1, 1) being chosen is  $z_{11}^{\pi, \sigma}$  and each time the players receive an expected payoff  $(a_{\infty}^A(\pi, \sigma) - \kappa, a_{\infty}^B(\pi, \sigma) - \kappa)$  in the long run. The following notion is then to be interpreted as the sum of all long-run expected payoffs connected to the action pairs chosen weighted by the relative frequencies of those action pairs occurring in the long run. Let

$$\begin{aligned} \varphi(z^{\pi, \sigma}) \equiv & z_{11}^{\pi, \sigma} (a_{\infty}^A(\pi, \sigma) - \kappa, a_{\infty}^B(\pi, \sigma) - \kappa) + z_{12}^{\pi, \sigma} (b_{\infty}^A(\pi, \sigma) - \kappa, c_{\infty}^B(\pi, \sigma)) \\ & + z_{21}^{\pi, \sigma} (c_{\infty}^A(\pi, \sigma), b_{\infty}^B(\pi, \sigma) - \kappa) + z_{22}^{\pi, \sigma} (d_{\infty}^A(\pi, \sigma), d_{\infty}^B(\pi, \sigma)). \end{aligned}$$

Alternatively, the players receive a long run average payoff equal to  $\varphi(z^{\pi, \sigma})$ . This finalizes the argument that for this jointly-convergent pair of strategies  $(\pi, \sigma)$ , we have  $\gamma(\pi, \sigma) = \varphi(z^{\pi, \sigma})$ .

The following result, illustrated in Figures 2 and 3, can be found in Joosten *et al.* [2003] for FD-games. Related ideas were designed for the analysis of repeated games with vanishing actions (cf., Joosten [1996, 2005a], Schoenmakers *et al.* [2002]). Let  $CP^{\mathcal{J}\mathcal{C}}$  denote the convex hull of  $P^{\mathcal{J}\mathcal{C}}$ .

**Theorem 4** *For any FD-game, we have  $P^{\mathcal{J}\mathcal{C}} = \bigcup_{z \in \Delta^{m \times n}} \varphi(z)$ . Moreover, each pair of rewards in  $CP^{\mathcal{J}\mathcal{C}}$  is feasible.*

From the formulation of Theorem 4 it may not be immediately apparent, but an implication is that the set of feasible rewards can be visualized rather elegantly. For this purpose, several algorithms have been designed, involving the computation of a pair of feasible rewards for a significant number of ‘frequency-matrices’  $z \in \Delta^{m \times n}$ . To obtain Figures 1 and 2, we have used such a program.

## 4 What is ‘rational’ to do?

The strategy pair  $(\pi^*, \sigma^*)$  is an **equilibrium**, if no player can improve by unilateral deviation, i.e.,  $\gamma^A(\pi^*, \sigma^*) \geq \gamma^A(\pi, \sigma^*)$ ,  $\gamma^B(\pi^*, \sigma^*) \geq \gamma^B(\pi^*, \sigma)$  for all  $\pi \in \mathcal{X}^A, \sigma \in \mathcal{X}^B$ . An equilibrium is called **subgame perfect** if for

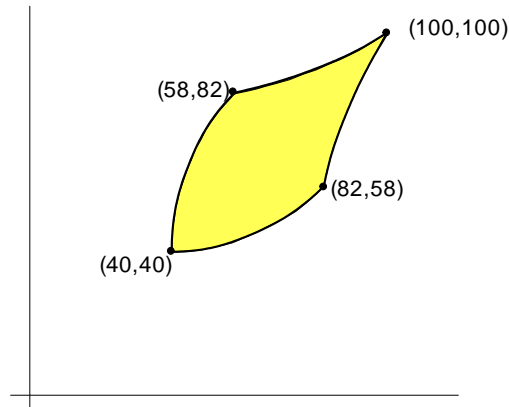


Figure 1: The jointly-convergent pure-strategy rewards for  $\kappa = 100$ . By Theorem 4, all rewards in the convex hull of this set are feasible, as well.

each possible state and possible history (even unreached states and histories) the subsequent play corresponds to an equilibrium, i.e., no player can improve by deviating unilaterally from then on.

In the construction of equilibria for repeated games, ‘threats’ play an important role. A threat specifies the conditions under which one player will punish the other, as well as the subsequent measures. We call  $v = (v^A, v^B)$  the **threat point**, where  $v^A = \min_{\sigma \in \mathcal{X}^B} \max_{\pi \in \mathcal{X}^A} \gamma^A(\pi, \sigma)$ , and  $v^B = \min_{\pi \in \mathcal{X}^A} \max_{\sigma \in \mathcal{X}^B} \gamma^B(\pi, \sigma)$ . So,  $v^A$  is the highest amount  $A$  can get if  $B$  tries to minimize his average payoffs. Under a pair of **individually rational** rewards each player receives at least his threat-point reward.

To present the general idea of the result of Joosten *et al.* [2003] to come, we adopt terms from Hart [1985], Forges [1986] and Aumann & Maschler [1995]. First, there is a ‘master plan’ which is followed by each player as long as the other does too; then there are ‘punishments’ which come into effect if a deviation from the master plan occurs. The master plan is a sequence of ‘intra-play communications’ between the players, the purpose of which is to decide by which equilibrium the play is to continue. The outcome of the communication period is determined by a ‘jointly controlled lottery’, i.e., at each stage of the communication period the players randomize with equal probability on both actions; at the end of the communication period one sequence of pairs of action choices materializes.

Detection of deviation from the master plan *after* the communication period is easy as both players use pure actions on the equilibrium path from then on. Deviation *during* the communication period by using an *alternative randomization* on the actions is impossible to detect. However, it can be shown that no alternative unilateral randomization yields a higher reward.

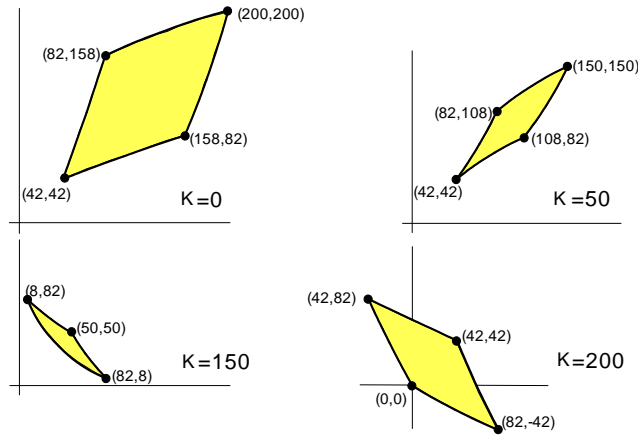


Figure 2: Jointly-convergent pure-strategy rewards for different  $\kappa$ .

So, the outcome of the procedure is an equilibrium. For more details, we refer to Joosten *et al.* [2003]. We restate here the major result which applies to general games with frequency-dependent stage payoffs.

**Theorem 5** *Each pair of rewards in the convex hull of all individually-rational pure-strategy rewards can be supported by an equilibrium. Moreover, each pair of rewards in the convex hull of all pure-strategy rewards giving each player strictly more than the threat-point reward, can be supported by a subgame-perfect equilibrium.*

The following is visualized in Figure 4 and illustrated in Example 7.

**Corollary 6** *Let  $E' = \{(x, y) \in P^{\mathcal{J}C} \mid (x, y) \geq v\}$  be the set of all individually rational jointly-convergent pure-strategy rewards. Then, each pair of rewards in the convex hull of  $E'$  can be supported by an equilibrium. Moreover, all rewards in  $E'$  giving A strictly more than  $v^A$  and B strictly more than  $v^B$  can be supported by a subgame-perfect equilibrium.*

**Example 7** (Continued) *For the threat points we have the following:*

$$v = \begin{cases} (158 - \kappa, 158 - \kappa) & \text{for } 0 \leq \kappa \leq 116, \\ (42, 42) & \text{for } \kappa \geq 116. \end{cases}$$

So, for  $\kappa = 100$ ,  $E' = \{(x, y) \in P^{\mathcal{J}C} \mid (x, y) \geq (58, 58)\}$ .

*To obtain any reward in  $E'$  an equilibrium in pure strategies with threats suffices. Any deviation from the equilibrium path will be punished, hence no player has an incentive to deviate unilaterally from the equilibrium path.*

*Observe that  $E'$  is not convex. We will now show how a pair of rewards in conv*

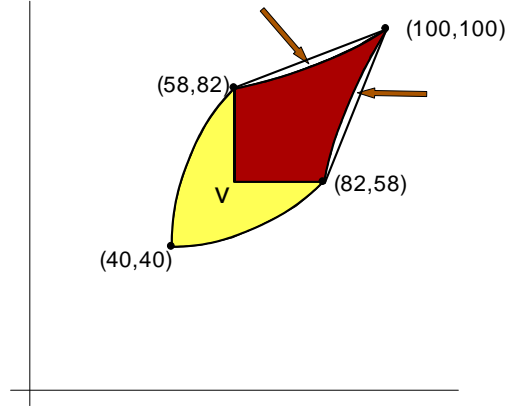


Figure 3: The set of jointly-convergent pure-strategy equilibrium rewards  $E'$ , is shown in red. The arrows point at  $\text{conv } E' \setminus E'$  ( $\kappa = 100$ ).

$E'$  can be obtained which does not belong to  $E'$ . Observe that  $\frac{1}{4}(100, 100) + \frac{3}{4}(58, 82) \notin E'$ . Let  $(\pi, \sigma)$  be given by

$$\begin{aligned} \pi_t = \sigma_t &= \left(\frac{1}{2}, \frac{1}{2}\right) && \text{for } t = 1, 2 \\ \pi_t = \sigma_t &= 1 && \text{for } t \geq 3 \quad \text{if } j_1^A = j_1^B \text{ and } j_2^A = j_2^B \\ \pi_t &= 1, \sigma_t = 2 && \text{for } t \geq 3 \quad \text{otherwise.} \end{aligned}$$

At the first two stages, each player randomizes with equal probability on both actions. Then, the play continues with probability  $\frac{1}{4}$  with  $((1, 1), (1, 1), (1, 1), \dots)$  and with the complementary probability with  $((1, 2), (1, 2), (1, 2), \dots)$ , because the event  $j_1^A = j_1^B$  and  $j_2^A = j_2^B$  has probability  $\frac{1}{4}$ . Furthermore, the first sequence gives long run average payoffs of  $(100, 100)$  and the second one  $(58, 82)$ . Hence,  $\gamma(\pi, \sigma)$  yields rewards  $\frac{1}{4}(100, 100) + \frac{3}{4}(58, 82)$ . Here, the communication phase has length 2, the jointly controlled lottery consists of the randomization in the first two periods as presented. The play proceeds according to the outcome determined by the jointly controlled lottery. Similarly all convex combinations of  $(58, 82)$  and  $(100, 100)$  being multiples of  $\frac{1}{4}$  can be obtained. Moreover, for a communication period consisting of three periods we can generate all multiples of  $\frac{1}{8}$ ; in general, for  $T$  periods one gets all multiples of  $\frac{1}{2^T}$ .

To show that  $\gamma(\pi, \sigma)$  is an equilibrium reward, we construct  $(\pi^e, \sigma^e)$  such that on the equilibrium path the play is exactly identical to the play induced by  $(\pi, \sigma)$ :

$$\begin{aligned} \pi_t^e &= \begin{cases} 2 & \text{if } j_{t'}^B \neq \sigma_{t'} \text{ for any } t' = 3, \dots, t-1 \\ \pi_t & \text{otherwise.} \end{cases} \\ \sigma_t^e &= \begin{cases} 2 & \text{if } j_{t'}^A \neq \pi_{t'} \text{ for any } t' = 3, \dots, t-1 \\ \sigma_t & \text{otherwise.} \end{cases} \end{aligned}$$



Then,  $\gamma(\pi^e, \sigma^e) = \gamma(\pi, \sigma) = \frac{1}{4}(100, 100) + \frac{3}{4}(58, 82)$ . Firm B ‘threatens’ to play action 2 forever if firm A deviates after  $t = 2$  from  $\pi$ , inducing average payoffs of  $v^A < \gamma^A(\pi^e, \sigma^e)$  to firm A. A similar statement holds for a unilateral deviation by the other firm. No unilateral deviation during the communication period can be detected, but any alternative unilateral randomization yields the same probabilities of continued play for  $t \geq 3$ . Hence  $(\pi^e, \sigma^e)$  is an equilibrium. ■

## 5 Pareto efficiency

Appreciation for Folk Theorems as presented in the previous section, varies widely among the profession. Commenting on ‘standard’ approaches on repeated games Gintis [2000] states (p.129): ‘By explaining practically anything, the model in fact explains nothing’. Gintis then expresses a preference for Pareto efficiency as a refinement criterion, as well as for more realistic punishments in case of unilateral deviation from an equilibrium path, which should be more forgiving, e.g., allow for repair. Contrastingly, Osborne & Rubinstein [1994] point out that equilibria of the infinitely repeated game exist which are Pareto-superior to any equilibrium of the one-shot game.

Joosten [2004] showed that rewards from certain subgame perfect equilibria which are ‘forgiving’, almost entirely coincide with the set of rewards from jointly-convergent pure-strategy equilibria. An implication is that the equilibrium rewards which fulfill the properties desired by Gintis discussed above, is the set of Pareto-efficient equilibrium rewards, which may then be obtained by a pair of ‘forgiving’ strategies instead of ‘grim trigger’ strategies.

**Example 8** (Continued) Let  $PE = \{(x, y) \in \text{conv } E' \mid \nexists_{(x', y') \in \text{conv } E'} (x', y') > (x, y)\}$  denote the set of Pareto-efficient rewards in the convex hull of  $E'$ . Hence, for any pair of rewards in  $PE$  it holds that there exists no alternative pair of rewards in  $\text{conv } E'$  such that both firms are better off.

We introduce a couple of notations, let for given  $\kappa \in [0, 158]$

$$\begin{aligned} \vartheta^\kappa &\equiv \min \left\{ 82, \left( \frac{4520 - 21\kappa}{2} - \left( \frac{244 - \kappa}{6} \right) \sqrt{3417 - 12\kappa} \right) \right\}, \\ M^\kappa &\equiv \max\{200 - \kappa, 42\}, \\ P^+ &\equiv \text{bd}(\text{conv } E') \setminus (\text{conv}\{v, (v^A, \vartheta^\kappa)\} \cup \text{conv}\{v, (\vartheta^\kappa, v^B)\}). \end{aligned}$$

Here,  $\text{bd}(S)$  denotes the boundary of the set  $S$ . Then, we have

$$\begin{aligned} PE &= P^+ && \text{for } \kappa \in [118\frac{1}{2}, 158], \\ PE &= \{(M^\kappa, M^\kappa)\} && \text{otherwise.} \end{aligned}$$

So, the criterion of Pareto efficiency reduces the number of equilibria significantly.<sup>4</sup> If  $\kappa \in [118\frac{1}{2}, 158]$ , certain boundary elements of  $\text{conv } E'$  are Pareto efficient.

<sup>4</sup>See also the Appendix.

Otherwise, there is a unique Pareto efficient element in  $\text{conv } E'$ . Figure 4 visualizes the first case. Note that for  $\kappa \in [118\frac{1}{2}, 158]$  individually rational rewards exist which Pareto dominate Pareto-efficient equilibrium rewards in  $E'$ . ■

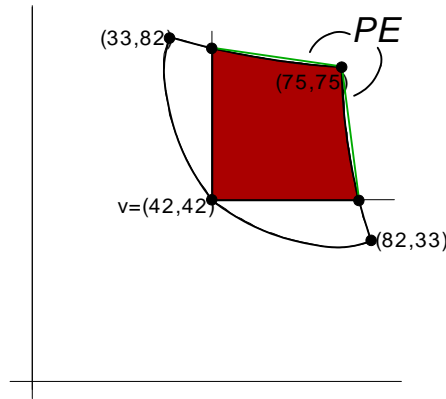


Figure 4:  $E'$  is indicated in red,  $PE \subset \text{bd}(\text{conv } E')$  is shown as a green kinked line ( $\kappa = 125$ ).

The stimulating idea of the focal point introduced by Schelling [1960] might be useful to reduce the abundance of equilibria. The literature is quite inconclusive about which direction to take even in ‘simpler’ games, see e.g., Janssen [1998, 2001], Sugden [1995] and Bacharach [1993]. The unique symmetric Pareto-efficient equilibrium in the example above, has one of the necessary properties of a focal point namely salience, also referred to as prominence or conspicuousness.

## 6 Conclusion

We have formulated a new dynamic model of advertising in very general terms. A broad variety of long and short term externalities can be modeled by altering the (restrictions on the) parameters chosen. We have analyzed one family of models rather completely, a full classification and analysis of the *general* setting must be reserved for later. Distinction in this family is made by the height of the advertisement costs which are assumed fixed and equal for both agents. The analysis shows that three subcases can be discerned. For high advertisement costs, only one equilibrium exists namely never to advertise at all. For low advertisement costs, the unique Pareto efficient equilibrium is to advertise always, and a continuum of equilibria exists which are not Pareto efficient. For intermediate costs, a continuum of equilibria exists and a continuum of Pareto efficient equilibria exists. Our

analysis precisely determines the boundary values for the different cases, as well as the (Pareto-efficient) equilibrium rewards.

We have made concessions to reality to obtain first results on advertisement games with frequency-dependent stage payoffs. We modeled advertising as an either-or decision, not taking into account that various budgets might be attributed to it. Our model is easily generalized as any one-period budget in a finite range with a smallest monetary unit can be modeled as a separate action. Vidale & Wolfe [1957] described the interaction of advertising and sales using a simple differential equation in terms of three parameters, the sales decay constant, the saturation level, and the response constant. Some of the phenomena these parameters are meant to capture, are present in our model, albeit implicitly. Further research must reveal whether such empirical findings can be approximated to a higher degree. Extending the model to allow an  $n$ -firm advertisement game is also high on the agenda.

The class of FD-games is rather young, hence the tools for analysis are far from mature yet. Large parts of the approach, most importantly Theorem 5 and its corollary, seem generalizable, but a comprehensive formal generalization is still pending. Obviously, visualizing the relevant sets of rewards will be difficult for  $n \geq 3$ .

A crucial step in our approach is finding the set of jointly-convergent pure-strategy rewards, another one is determining the threat point. The first step is depends crucially on the functions which determine the average payoffs being continuous on the relevant domains of the stochastic variables involved. Unfortunately, there exists no general theory on (finding) threat points in FD-games, yet.

## 7 Appendix

**Proof of the claim made in Example 7.** We intend to show that  $v^A = \max\{158 - \kappa, 42\}$ , divided into two steps. We show that the amount which Firm A can guarantee itself, is identical to the amount to which Firm B can keep Firm A. Now, Firm A can always guarantee itself  $\max\{158 - \kappa, 42\}$  by using

$$\pi = \begin{cases} (1, 1, 1, \dots) & \text{if } \kappa \leq 116, \\ (2, 2, 2, \dots) & \text{if } \kappa \geq 116. \end{cases}$$

If namely  $\kappa \leq 116$ , then  $\pi$  induces long term stage payoffs determined by

$$[153 - \kappa + 47\rho_t^B, 158 - \kappa + 50\rho_t^B]$$

depending on firm  $B$ 's actions. Let  $\sigma$  be firm  $B$ 's strategy,  $\underline{\rho} = \liminf_{t \rightarrow \infty} \rho_t^B$ , and  $\bar{\rho} = \limsup_{t \rightarrow \infty} \rho_t^B$  where  $\rho_t^B$  corresponds to  $\sigma$ . Then in the long run

$$\begin{aligned} [153 - \kappa + 47\underline{\rho}, 158 - \kappa + 50\underline{\rho}] &\leq [153 - \kappa + 47\rho_t^B, 158 - \kappa + 50\rho_t^B] \\ &\leq [153 - \kappa + 47\bar{\rho}, 158 - \kappa + 50\bar{\rho}] \end{aligned}$$

Observe that  $x < y$  implies that for  $t \rightarrow \infty$ ,  $x\rho + y(1 - \rho) \geq x\rho_t^B + y(1 - \rho_t^B) \geq x\bar{\rho} + y(1 - \bar{\rho})$ . So, for large  $t$ :

$$\begin{aligned} & (153 - \kappa + 47\rho)\rho + (158 - \kappa + 50\rho)(1 - \rho) \\ \geq & (153 - \kappa + 47\rho_t^B)\rho_t^B + (158 - \kappa + 50\rho_t^B)(1 - \rho_t^B) \\ \geq & (153 - \kappa + 47\bar{\rho})\bar{\rho} + (158 - \kappa + 50\bar{\rho})(1 - \bar{\rho}) \end{aligned}$$

Minimizing the upper bound occurs for  $\underline{\rho} = 0$ , minimizing the lower bound occurs for  $\bar{\rho} = 0$ . Hence, firm A can guarantee itself at least  $158 - \kappa$ , and it can be kept at that amount by firm B if  $\sigma$  satisfies  $\underline{\rho} = \bar{\rho} = 0$ . (The case  $\kappa \geq 116$  is similar.) Firm B can keep Firm A's rewards at  $\max\{158 - \kappa, 42\}$  by using  $\sigma = (2, 2, 2, \dots)$ . Let  $\pi$  be firm A's strategy,  $\underline{\rho} = \liminf_{t \rightarrow \infty} \rho_t^A$  and  $\bar{\rho} = \limsup_{t \rightarrow \infty} \rho_t^A$  where  $\rho_t^A$  corresponds to  $\sigma$ . Then, Firm A faces long term stage payoffs of

$$\left[ \begin{array}{c} 52 + 106\rho_t^A - \kappa \\ 42 + 91\rho_t^A \end{array} \right] \leq \left[ \begin{array}{c} 52 + 106\bar{\rho} - \kappa \\ 42 + 91\bar{\rho} \end{array} \right].$$

Observe that  $x > y$  implies that for  $t \rightarrow \infty$ ,  $x\rho_t^B + y(1 - \rho_t^B) \leq x\bar{\rho} + y(1 - \bar{\rho})$ . Hence, A's long term average stage payoffs are smaller than or equal to  $(52 + 106\bar{\rho} - \kappa)\bar{\rho} + (42 + 91\bar{\rho})(1 - \bar{\rho}) = 15\bar{\rho}^2 + (101 - \kappa)\bar{\rho} + 42$ . The latter takes its minimum at  $\bar{\rho}^* = \frac{\kappa - 101}{30}$ . It follows easily that for  $\bar{\rho}^* \leq \frac{1}{2}$  the maximizing element in the range  $[0, 1]$  is  $\bar{\rho} = 1$  and the associated rewards are  $158 - \kappa$ , and for  $\bar{\rho}^* \geq \frac{1}{2}$  the maximizing element in this range is  $\bar{\rho} = 0$  and the associated rewards are 42. Note that  $\bar{\rho}^* = \frac{1}{2}$  precisely for  $\kappa = 116$ . ■

**Formulas from Example 8.** We used the following insights.

- (1) The set of Pareto-efficient elements of  $\text{conv } E'$  is nonempty and is a subset of the boundary of  $\text{conv } E'$ .
- (2)  $\text{bd}(\text{conv } E') \setminus P^+$  does not contain a Pareto-efficient element.
- (3)  $M^\kappa = \vartheta^\kappa$  for  $\kappa = 118\frac{1}{2}$ .
- (4) For  $\kappa \in [0, 118\frac{1}{2}]$  it holds that  $M^\kappa \geq \vartheta^\kappa$ . Hence, if  $(x, y) \in P^+$ , then  $(x, y) \leq (M^\kappa, M^\kappa)$ .
- (5) For  $\kappa \in [118\frac{1}{2}, 158]$  it holds that  $M^\kappa \leq \vartheta^\kappa$ . Hence, for any  $(x, y) \in P^+ \setminus \{(\vartheta^\kappa, v^B)\}$  we have that  $\varepsilon_1, \varepsilon_2 > 0$  exist such that  $(x + \varepsilon_1, y - \varepsilon_2) \in P^+$ . (The same argument can be shown in the other direction.) This implies that  $P^+$  does not contain an element that Pareto dominates all other elements of  $P^+$ . ■

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