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On Pendular Voting

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On Pendular Voting

Abstract

"Pendular Voting" presents a novel two-stage voting procedure: A randomly chosen citizen group votes on a proposal replacing the status quo. Depending on the outcome, a "counterproposal" ensues, that is closer to/further away from the status quo than the original proposal. All citizens vote pairwise on the status quo, initial proposal, and counterproposal (majority voting), the middle alternative being the default outcome in case of cyclical collective preferences. We analyze the process on a one-dimensional policy space, allowing for uncertainty about preference distribution. Manipulation may only occur in the first stage, without impacting the final outcome. Pendular Voting can engineer outcomes closer to the median voter's preferences than standard procedures, even with selfish agenda setters.

JEL-Codes: C720, D700, D720.

Keywords: democracy, manipulation, information sharing, referendum.

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1 Introduction

How should democratic procedures be designed if voters have differing preferences and there is aggregate uncertainty about the distribution of these preferences as well as about which alternative is the Condorcet Winner, if it exists?

In such a situation, good democratic procedures should achieve two goals: First, they should facilitate accurate revelation and aggregation of information about the consequences of policy proposals. Second, they should allow adopting policies that concur with the true preferences of the population. For instance, if a Condorcet Winner exists, it should be discovered and then should prevail against any other feasible alternative, including the status quo.

Achieving these objectives in a voting procedure with many participants is difficult for several reasons: First, a large number of policy proposals can be made for a given issue, but in mass voting, only few are put up for a vote. This is the case in the so-called "elite referenda" such as the Brexit referendum, which have increased in the past few decades,[1]. It is also the case in direct democracies like Switzerland, in which at most, *two* proposals from a large set of possible proposals on an issue are put to a vote. Second, there is uncertainty about the underlying distribution of preferences in the electorate, and thus there is ambiguity. Third, once a proposal has been adopted, it is often quite difficult to reverse this policy, even if more information is revealed. This is obvious if the policy involves physical or human capital investments such as infrastructure investments in highways, bridges, public buildings or in the use of environmental resources. This irreversibility extends to many other policies as well. A particularly pertinent example is, again, the Brexit referendum: The delays and procedural complications in implementing Brexit have shown how difficult it is to join *and* to leave a political union, or merely to terminate a set of interconnected, multilateral treaties. Such decisions are only reversible at a high cost. There are many other examples of collective decisions with a high degree

[1] See Lee and Pérez-Velasco (2023)

of irreversibility.[2]

How should we design good voting systems in such environments? In particular, how can we induce information revelation before proposals are made, and can proposals be adjusted in the light of new information before final, irreversible collective decisions are taken?

In this paper, we introduce a new procedure called "Pendular Voting" and examine to which extent it can resolve these issues. The voting procedure works as follows: A selfish agenda-setter chooses a proposal meant to replace a given status quo. In a first stage, only a (small) random sample of the population votes on the proposal. The result of the first stage is made public and may therefore reveal information about the underlying distribution of preferences in the electorate, and in particular about the median voter's most preferred policy. Depending on the outcome of the first stage, a third alternative (next to the proposal and the status quo) is added: This alternative is either closer to or more distant from the status quo. In the model, it is constructed algorithmically, with fixed increments—positive or negative—to the initial proposal, depending on the outcome of the first stage.[3]

After the first-round voting, a second stage takes place: The entire electorate expresses pairwise preferences over the status quo, the initial proposal, and the newly added third alternative, by majority voting and with a suitable default procedure in case of cycles. In particular, if there is a cycle, the middle alternative is chosen.

We investigate the manipulability of this voting procedure by the randomly-selected sample of voters and the possibilities for the agenda-setter to exploit the procedure in a simple model in which citizens have single-peaked preferences over a one-dimensional policy space, but the distribution of preferences is initially unknown and thus, in particular, the median voter's position is unknown. We study how close the outcome will be to the median voter's most preferred policy.

[2]If waiting can reveal the information, then waiting before adopting an irreversible alternative may be desirable (see Gersbach (1993) for a first assessment whether a majority benefits from a "wait and see" choice).

[3]In practice, this proposal could be designed by an expert commission.

We allow groups with the same preferences to coordinate voting and look for perfect Bayesian equilibria and group strategy proofness.

Our main insight is as follows. In the second stage, citizens vote sincerely, and we obtain the Condorcet Winner among the alternatives: status quo, original proposal from the first stage, and second proposal generated after the first stage. In the first stage, strategic voting could occur, but it does not affect the determination of the second proposal and thus does not affect the voting outcome in the second stage either. The reason is that manipulation attempts are countered by manipulation of the groups.

In terms of outcomes, we obtain the following results. When the agenda-setter is benevolent, Pendular Voting engineers outcomes that are closer to the median voter's most preferred policy in all circumstances. When the agenda-setter is purely self-interested in implementing his/her preferred policy, Pendular Voting leads in expectation to better outcomes in terms of distance to the median voter. In order to realize these improvements, the third proposal, constructed after the first round, and thus the increment to the initial proposal, must be chosen appropriately.

This paper makes the following contributions to the literature. First, the concept of Pendular Voting allows to further explore one of the fundamental questions in democracy research: *How can information revelation be accomplished in democratic procedures?* This question has been extensively analyzed in the literature, see for instance Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997). More recent contributions are Bierbrauer and Hellwig (2016) and Britz and Gersbach (2020). When information about an underlying state of nature is dispersed among agents, there are complex incentives which may or may not lead agents to reveal, share, and aggregate their private information through a democratic procedure. In this context, Britz and Gersbach (2020) study a situation where groups of citizens coordinate their messages to strategically misrepresent information before a vote. Moreover, if there are turnout costs, an individual agent may free-ride on the information provided by other agents. Feddersen and Pesendorfer (1997) suggest that if some agents have information of lower quality than others, they may strate-

gically decide to abstain and thus keep such information to themselves. Callander (2008) studies an election where citizens reap benefits from voting for the winning candidate. In the present paper, we investigate how voting processes can be organized when there is not only private information about individual preferences but also about the distribution of these preferences. Ideally, a multi-stage voting process could work as follows: In an initial stage, information is revealed. Then, a proposal is made in light of this information, and finally, a decisive vote takes place.

Third, Pendular Voting may mitigate some of the problems faced by modern democracies, both in direct democracies and representative democracies. In Switzerland's direct democracy, for instance, any citizen can, in principle, propose a change to the constitution. If the proposal is backed by a certain number of citizens in a signature collection process, it must be put to a popular vote.[4] This rule has been in force for more than a century, and there are ongoing discussions about whether the parameters are still appropriate in the times of social media and internet campaigns, as well as with population growth. Pendular Voting might be a way to assess with a random sample of citizens in which direction a proposal should be adjusted; for instance, how an extreme proposal might be moderated.

The Brexit referendum is a good example how Pendular Voting could improve democratic decision-making: While the referendum allowed only two options (leave the European Union vs. remain in the European Union), the voting campaigns showed that there were many more options in reality, such as various forms of a "negotiated" Brexit, a free trade zone, or a "no-deal" Brexit. Looking back, it is clear that Brexit will ultimately occur but that the negotiations on relations with the EU will take quite long. Brexiteers and Remainers cite various opinion polls which deliver contradictory information about the British public's preferences over these options. Applying Pendular Voting could have led to a timely discovery of the underlying preferences, to a more precise and meaningful formulation of the referendum question, and ultimately to greater certainty that the outcome corresponds to the preferences of the population.

[4]For a more detailed discussion of the direct democratic system in Switzerland, see Gersbach et al. (2024).

Introducing the possibility of adding proposals to the model, however, leads to important information manipulation and exploitation risks, which are not examined in Gersbach et al. (2021)'s study of Assessment Voting, that also uses a procedure with so-called "Assessment Groups", but only for binary collective decisions, without any second proposal.

First voters may not express their true preferences in the initial stage. For instance, voters may want to feign support for an extreme policy because they hope that this will lead to a continuation of the status quo. This is similar to the problem of holding "open primaries'. For instance, in the primaries to a US presidential election, supporters of the Democrat Party may want to vote for a very extreme Republican candidate, in the hope that the extreme candidate wins the nomination, but then goes on to lose the election. Second, even if information is truthfully revealed in the first stage, a selfish agenda-setter may exploit this information. S/he may propose an alternative that wins against the status quo and is implemented irreversibly. However, a majority may prefer a large set of alternatives to the proposal that is undertaken. As we will show, with Pendular Voting, we can eliminate, or at least restrain, these attempts and we can ensure that the scheme improves outcomes in terms of the expected distance between the outcome and the median voter's preferred policy.

Our paper is also part of the broadening literature on learning in dynamic collective decisions. Strulovici (2010) examined how long a committee invests in learning until a majority takes a final decision. The duration of learning in committees with heterogeneous members is characterized in Chan et al. (2018).

Our paper is organized as follows: We formally introduce the model setup in Section 2. In Section 3, the Pendular Voting procedure is introduced. We study equilibrium behavior in the second stage of the voting procedure in Section 4. In Section 5, we characterize the extent of information revelation through Pendular Voting. We show that either the Pendular Voting procedure cannot be manipulated or if it does, it has no consequences for the final outcome. We complete this section by characterizing the equilibria under Pendular Voting and by providing conditions where manipulation could occur. In Section

6 and 7, we demonstrate that Pendular Voting leads to welfare gains in expectation, both with a benevolent and with a selfish agenda-setter. In the latter case, the results hold in expectation. In Section 8, we compare Pendular Voting to two-round agenda-setting and voting where the winner of the first round is the status quo in the second round. Section 9 concludes.

2 The Model

A society collectively decides to choose a one-dimensional linear policy $\theta \in [0,1]$.[5] It is convenient to define the notation:

$$\begin{aligned} \alpha^+(\theta) &= \min\{\theta + \mu_2,\ 1\}, \\ \alpha_-(\theta) &= \max\{\theta - \mu_1,\ 0\}, \end{aligned}$$

for some fixed values $1 > \mu_1, \mu_2 > 0$. Later, we will consider μ_1 and μ_2 as parameters of institutional design. They are taken as given in the Pendular Voting procedure and will be considered small in general. In particular, we will assume throughout that $\mu_1 \leq 1 - \mu_2$. This allows us to classify the different possible values of α^+ and α_- into three cases:

- $0 \leq \theta \leq \mu_1$: then $\alpha_- = 0$ and we can equivalently take $\mu_1 = \theta$, with $\alpha^+ = \theta + \mu_2$.
- $\mu_1 \leq \theta \leq 1 - \mu_2$: $\alpha_- = \theta - \mu_1$ and $\alpha^+ = \theta + \mu_2$.
- $1 - \mu_2 \leq \theta \leq 1$: $\alpha_- = \theta - \mu_1$ but $\alpha^+ = 1$ or alternatively $\mu_2 = 1 - \theta$.

With this convention, our definitions of α^+ and α_- can be assumed to be $\alpha^+ = \theta + \mu_2$ and $\alpha_- = \theta - \mu_1$, where μ_2 will become $1 - \theta$ if $1 - \mu_2 \leq \theta \leq 1$ and where μ_1 will become θ if $0 \leq \theta \leq \mu_1$.

Furthermore, we will take the status quo as zero. The society consists of a continuum of citizens with mass one, that is, modeled as a probability space. Each citizen is privately

[5] The model can be extended to a multidimensional policy space in the following way. Once θ is selected, counterproposals are selected on the line $\bar{\theta}0$ and thus the space of admissible policy proposals becomes a one-dimensional policy space.

informed about his/her type, which is some point z in the same space as the policy θ and is also normalized, so that $z \in [0,1]$. It is convenient to refer to a citizen of type z as *citizen* z. Hence, the citizen type is modeled as a random variable Z on the probability space of the population and takes values in $[0,1]$.

Citizen z's utility from policy θ is given by a continuous utility function $u(z,\theta)$ which is symmetric around a single peak at z. That is, we assume, on the one hand, that $u_z : \theta \in [0,1] \mapsto u(z,\theta) \in [0,1]$ has a unique maximum occurring at $\theta = z$ and is such that u_z is strictly increasing for $\theta \leq z$ and strictly decreasing for $z \leq \theta$. On the other hand, we assume that $\mid \theta' - z \mid = \mid \theta'' - z \mid$ implies $u(z,\theta') = u(z,\theta'')$. We denote this condition by saying that u_z are "single-peaked and symmetric".

There is uncertainty at both the individual and aggregate levels, which we model as in Britz and Gersbach (2020). For this, we assume that there is a given distribution of different possible states of nature. Each state of nature determines a distribution of the population, which in our model is reflected in the distribution of the citizen type Z. We denote the random variable describing the state of nature by N and use k to index the different possible states of nature. Thus, there is a family of probability distributions of the citizen type Z associated with each state of nature k. We use f_k and F_k to denote the probability density function and the cumulative distribution function, respectively, of the probability distribution associated with state k. We will assume that the density functions are all non-zero:

$$f_k > 0.$$

Regardless of their type, citizens have a common prior belief: the distribution of the state of nature N and the distribution of the citizen type Z, conditioned on the state of nature k. Z can hence be understood as a random variable with Bayesian distribution, depending on the state of nature.

As we say, for each state of nature k, the citizen type Z follows a distribution F_k, and each distribution has a median that we denote by $\widehat{z}_k$ (note this median is well-defined and given by the equation $F_k(\widehat{z}_k) = 1/2$, in virtue of the distributions being absolutely continuous

with non-zero density function $f_k > 0$). We will also denote by $\widehat{z}$ the random variable associated to the citizen type of the median voter, taking values $\widehat{z} \in [0,1]$ (which is a function of the state of nature $k \in N$).

We will not make further assumptions on the distribution of these variables, which thus remain fairly general. It is only in the last sections (Sections 7 and 8) that we assume additionally that $\widehat{z}$ follows a uniform distribution over the interval $[0,1]$.

Yet, we are always working with absolutely continuous distributions for the state of nature N and the citizen type Z, conditioned on N. Nevertheless, this ensures—except in special cases[6]—that the probability of the event of an exact draw between any two given groups, whose preferences are not exactly equal, is zero. This natural assumption will ensure that we do not have to deal with draws separately in the Pendular Voting procedure.

3 The Pendular Voting Procedure

In this section, we give the formal description of the *Pendular Voting procedure.*

3.1 Preliminaries

An agenda-setter makes a proposal $\overline{\theta} \in [0,1]$. We allow for two different cases: The agenda-setter may be a benevolent social planner who seeks to implement the Condorcet Winner or s/he may pursue his/her own interests. We will study these two cases in Section 8. Since we are looking for ways to improve democratic procedures, we do not consider utilitarian welfare but rather set the goal to implement the Condorcet Winner. That also determines the definition of benevolence of the agenda-setter. We thus work with the following definition of social welfare:

Definition 1. *The* social welfare loss *or henceforth simply* welfare *of a voting procedure is the expected distance between its outcome and the preferred policy of the median voter.*

We will sometimes also speak of the welfare of a voting procedure for a given state of

[6] A special cases is a uniform distribution of types and a proposal $\theta = \frac{1}{2}$.

nature as the distance between the outcome and the median voter's preferred policy.

Since utility functions are single-peaked and symmetric over a one-dimensional policy space, a policy winning in a majority voting contest (i.e., preference by a majority) against another policy is equivalent to the former policy being closer to the median voter's type $\widehat{z}$ than the latter. We hence easily obtain:

Fact 1. *Given two different voting procedures and a fixed state of nature, a welfare increase of the second over the first takes place if and only if the outcome of the second is preferred by a majority to the outcome of the first.*

Another consequence of the previous observation is the fact that the median voter's type $\widehat{z}$ beats every other alternative in a face-to-face majority voting. For clarity, we denote this as:

Definition 2. *A* Condorcet alternative *is a proposal that would not loose against any other proposal in the policy space in a majority voting contest.*

In our case, since the median voter's type is uniquely defined in virtue of the non-zero density function of the population distribution, there is a unique Condorcet alternative. We have:

Fact 2. *Given a continuous set of citizens whose type is given by an absolutely continuous distribution F with nowhere zero density function and assuming the citizens' utility functions are single-peaked and symmetric, then the Condorcet alternative and the median voter's type coincide and are given by*

$$\widehat{z} = F^{-1}\left(\frac{1}{2}\right).$$

3.2 The Pendular Voting game

The Pendular Voting procedure consists of a two-stage game.[7]

[7] We limit ourselves to a two-stage game that generates three binary votes on three alternatives, proposal, counterproposal, and status quo. Mass voting with three alternatives and majority rule already turns out to be complex, as we need a cycle breaker. Further rounds would amplify this complexity and there is no obvious direction how further rounds could improve outcomes.

Stage 1.

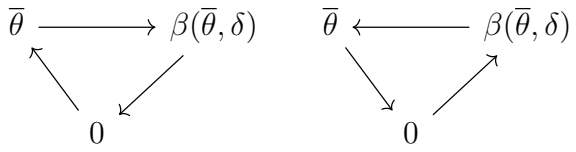
Once the agenda-setter has made the proposal $\overline{\theta}$, a randomly sample of size λ of the population is drawn. We assume that λ is small, but large enough, so that it can be assumed to follow the same distribution as the general population. Each member of the sample group may vote in favor of $\overline{\theta}$ or in favor of the status quo 0. The share of sample group members who vote in favor of $\overline{\theta}$ is denoted by δ. We define

$$\beta(\overline{\theta}, \delta) = \begin{cases} \alpha^{+}(\overline{\theta}) & \text{if} \quad \delta \geq 1/2, \\ \alpha_{-}(\overline{\theta}) & \text{if} \quad \delta < 1/2. \end{cases}$$

Here, $\beta(\overline{\theta}, \delta)$ is an additional proposal determined by votes in the first round.

Stage 2.

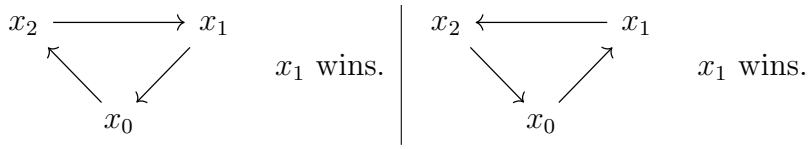
In the second stage, the entire population votes. Each voter is asked to submit his/her pairwise preferences over the three alternatives $\{0, \overline{\theta}, \beta(\overline{\theta}, \delta)\}$. The outcome is then determined as follows: If a majority pairwise prefers any alternative against the other, i.e., if there is a Condorcet Winner, then the outcome is that alternative. The remaining possible outcomes of the voting to be treated are the two "cyclic" or "non-Condorcet" results (where an arrow $x \to y$ means that y wins against x):



To define the outcome in these two cases, we rename $\{0, \overline{\theta}, \beta(\overline{\theta}, \delta)\}$ as x_0, x_1, and x_2, such that $0 = x_0 \leq x_1 \leq x_2$. Then citizens are casting three pairwise votes:

$$\begin{aligned} x_0 &\leftrightarrow x_1, \\ x_0 &\leftrightarrow x_2, \\ x_1 &\leftrightarrow x_2. \end{aligned}$$

In the Pendular Voting procedure, we define the following rule: In a cyclic outcome, the intermediate option x_1 wins, i.e.:



Since we assumed that exact draws occur with probability zero, we do not deal with them in the model.

The crucial feature of the procedure is that the second, decisive voting round is always a three-way ballot including the initial proposal and the status quo. The third alternative on the ballot is determined by the result of the first voting round, that is, random sample group members essentially determine which third alternative to include on the final ballot.

We next describe the equilibrium concept. We look for Bayesian perfect Nash equilibria with the following properties: We assume that citizens with the same preferences vote in the same way and we require that an equilibrium be robust to deviations of groups of voters with the same preferences. Hence, we are looking for Bayesian perfect Nash equilibria among groups of voters.[8]

Allowing groups of citizens to coordinate their deviation is an approach which concurs with recent developments in the literature. The papers by Bierbrauer and Hellwig (2016) as well as Britz and Gersbach (2020) bring together mechanism design and voting games to study public good provision problems. Bierbrauer and Hellwig (2016) show that, under certain robustness conditions, mechanisms which solve such problems must belong to a class they call "voting mechanisms'. Their requirement of robust coalition-proofness is based on the idea that citizens with concurring preferences may coordinate their strategies. Britz and Gersbach (2020) allow citizens with the same preference ranking to coordinate their votes in a democratic mechanism.

In addition, we require that an equilibrium be robust to deviations by an entire coalition

[8]We note that the assumption that all citizens with the same preferences vote in the same way is no loss of generality. For formal proofs that deviations of groups of voters with the same preferences are also Nash equilibria if deviations by individual voters are considered, see Tsakas and Xefteris (2023). We also simplify the exposition and assume that exact ties regarding voting outcomes do not occur. This avoids the necessity to deal with ties. One may add the tie-breaking rule that in case of a tie, one of the two involved alternatives is selected randomly, assigning equal probability $\frac{1}{2}$ to each. This would not alter our results.

of groups of citizens and this requirement is applied to both stages. Hence, we require strong equilibria by Aumann (1959). Henceforth, we simply call strong perfect Bayesian equilibria among groups of citihens "equilibria."

4 Second Stage

4.1 The equilibria

In this section, we establish the following claim. At the second stage of the Pendular Voting procedure, there exists an equilibrium in which all groups of citizens find it optimal to vote sincerely, that is, in accordance with their true preferences. We call this the "honest equilibrium". The rules as described in the previous section can be restated as follows.

Definition 3. Cycle-breaker: *If any alternative wins two of the three pairwise votes, then it becomes the outcome of the voting procedure. If each of the three votes is won by a different alternative, i.e. if there is a cyclic outcome, then the intermediate alternative* x_1 *becomes the outcome of the voting procedure.*

We will show that this condition guarantees that there exists an equilibrium in which all citizens vote sincerely. In principle, there are six ways to rank–order the alternatives $\{x_0, x_1, x_2\}$ by some preference order $\succsim$:

$$
\begin{aligned}
x_2 &\succsim x_1 \succsim x_0,\\
x_1 &\succsim x_2 \succsim x_0,\\
x_1 &\succsim x_0 \succsim x_2,\\
x_0 &\succsim x_1 \succsim x_2,\\
x_0 &\succsim x_2 \succsim x_1,\\
x_2 &\succsim x_0 \succsim x_1.
\end{aligned}
$$

Note that the last two preference orders above are inconsistent with our assumption on single-peaked preferences. Hence, we can restrict attention to the following four preference

orders:

$$
\begin{aligned}
x_2 &\succsim x_1 \succsim x_0,\\
x_1 &\succsim x_2 \succsim x_0,\\
x_1 &\succsim x_0 \succsim x_2,\\
x_0 &\succsim x_1 \succsim x_2.
\end{aligned}
$$

Recall that we are assuming that all citizens which share one of these four preference orders can coordinate their votes. Thus, we have to look for profitable deviations by each of the four groups, assuming that the remaining three groups vote sincerely.

Whenever two out of the three proposed policies $0 = x_0 \leq x_1 \leq x_2$ are equal, the pairwise voting degenerates into a binary and final choice. In that case, there is no place for manipulation and all voters will be honest, since the population will then just be divided into two groups and the optimal strategy consists in voting for the preferred option. Hence, we can suppose that all alternatives are distinct, $0 = x_0 < x_1 < x_2$. Now we are ready to show that citizens vote sincerely.

Proposition 1. *In the second stage of the Pendular Voting procedure, there is an equilibrium in which all citizens vote sincerely.*

The proof of Proposition 1 is given in the appendix. From now on, we will assume that the honest equilibrium takes place. Any other potential equilibrium would involve at least two groups voting strategically, but would produce the same outcome.[9]

4.2 Implications

The key implication of the above result is the following. Suppose that the honest equilibrium at the second stage of the Pendular Voting procedure takes place and that the Condorcet alternative qualifies for it. It is then certain that this alternative will also be

[9]In Gersbach et al. (2023)), a more general treatment of this problem is examined. There, the boundaries are identified when a procedure, consisting in three pairwise majority decisions, is either group-strategy-proof or not, and it is assessed which other equilibria might exist and whether these equilibria yield the same outcome.

the outcome of the entire procedure, since it will win any pairwise competition. Furthermore, Theorem 1, combined with our assumption of symmetric, single-peaked preferences $u(z, \theta)$ with peak at z for citizens of type z, allows us to summarize the situation as follows:

Corollary 1. *Consider a state of nature where the median voter is of type $\widehat{z}$. In the honest equilibrium, the outcome of the second round of Pendular Voting is determined as follows:*

- *If the group $x_2 \succsim x_1 \succsim x_0$ is a majority, or equivalently if the median voter prefers x_2 over the rest, or equivalently if $(x_1 + x_2)/2 < \widehat{z}$, then x_2 wins.*
- *If the group $x_0 \succsim x_1 \succsim x_2$ is a majority, or equivalently if the median voter prefers x_0 over the rest, or equivalently if $\widehat{z} < x_1/2$, then x_0 wins.*
- *In any other case, or equivalently if the median voter prefers x_1 over the rest, or equivalently if $x_1/2 < \widehat{z} < (x_1 + x_2)/2$, then x_1 wins.*

In particular, the outcome is always the median voter's preferred policy out of the three proposals, i.e, the Condorcet Winner.

Hence, the question is: How can we ensure that the alternatives present in the second round are as close as possible to the preferred choice of the median voter?

The purpose of the Pendular Voting procedure is to perform better in this respect than a benchmark scenario in which the agenda-setter uses only his/her prior belief to choose a proposal that is then voted upon. We now briefly consider that benchmark scenario.

Proposition 2. *Given a state of nature with median voter's type $\widehat{z}$, if the agenda-setter chooses a proposal $\overline{\theta} < 2\widehat{z}$, then the proposal is accepted by a majority. Otherwise, the status quo prevails.*

Proof of Proposition 2. In line with the arguments used to prove Proposition 1, the reasoning behind this statement is as follows. Under symmetric, single-peaked preferences $u(z, \theta)$, an alternative wins against another in a majority voting contest if and only if the former is closet to the median voter's type $\widehat{z}$ than the latter. Thus, proposal $\overline{\theta}$ wins against the status quo if and only if it is closer to $\widehat{z}$ than 0, i.e., if $|\overline{\theta} - \widehat{z}| < |0 - \widehat{z}| = \widehat{z}$, which is

equivalent to $\overline{\theta} < 2\widehat{z}$.

5 First Stage: Information Revelation through Pendular Voting

5.1 Manipulability in Pendular Voting

We take a proposal $\overline{\theta}$ as given by the agenda-setter, and verify under which conditions the Pendular Voting procedure reliably implements the choice from $\{0, \alpha_-(\overline{\theta}), \overline{\theta}, \alpha^+(\overline{\theta})\}$ which is closest to the median voter. We recall again that we assess social welfare based on the median voter's preference as in Definition 1. This makes sense in such a type of model when discussing democratic procedures: Indeed, a democratic procedure should satisfy a requirement such as "stability to majority voting", which is similar to assessing social welfare based on the median voter's preferences.

For clarity, we consider in the following a given proposal $\overline{\theta}$ and denote the other possible proposals α_- and α^+ correspondingly, so that the four possibilities are $\{0, \alpha_-, \overline{\theta}, \alpha^+\}$. In order to group and classify all the different possible citizens based on their preferences, we define the following groups:

- $Z_3 := \{z \in [0,1] \ : \ (\overline{\theta} + \alpha^+)/2 < z)\}$, with preferences $\alpha^+ \succsim \overline{\theta} \succsim \alpha_- \succsim 0$.
- $Z_2 := \{z \in [0,1] \ : \ (\overline{\theta} + \alpha_-)/2 < z < (\overline{\theta} + \alpha^+)/2)\}$, with preferences $\overline{\theta} \succsim \alpha^+$ and $\overline{\theta} \succsim \alpha_- \succsim 0$. They are further subdivided in:
  - $Z_2^3 := \{z \in Z_2 \ : \ (\alpha_- + \alpha^+)/2 < z\}$, with preferences $\overline{\theta} \succsim \alpha^+ \succsim \alpha_- \succsim 0$.
  - $Z_2^2 := \{z \in Z_2 \ : \ \alpha^+/2 < z < (\alpha_- + \alpha^+)/2\}$, with preferences $\overline{\theta} \succsim \alpha_- \succsim \alpha^+ \succsim 0$.
  - $Z_2^1 := \{z \in Z_2 \ : \ z < \alpha^+/2\}$, with preferences $\overline{\theta} \succsim \alpha_- \succsim 0 \succsim \alpha^+$.
- $Z_1 := \{z \in [0,1] \ : \ \alpha_-/2 < z < (\overline{\theta} + \alpha_-)/2)\}$, with preferences $\alpha_- \succsim \overline{\theta} \succsim \alpha^+$ and $\alpha_- \succsim 0$. They are further subdivided in:
  - $Z_1^3 := \{z \in Z_1 \ : \ \alpha^+/2 < z\}$, with preferences $\alpha_- \succsim \overline{\theta} \succsim \alpha^+ \succsim 0$.

- $Z_1^2 := \{z \in Z_1 \ : \ \overline{\theta}/2 < z < \alpha^+/2\}$ with preferences $\alpha_- \succsim \overline{\theta} \succsim 0 \succsim \alpha^+$.
- $Z_1^1 := \{z \in Z_1 \ : \ z < \overline{\theta}/2\}$, with preferences $\alpha_- \succsim 0 \succsim \overline{\theta} \succsim \alpha^+$.

- $Z_0 := \{z \in [0,1] \ : \ z < \alpha_-/2)\}$, with preferences $0 \succsim \alpha_- \succsim \overline{\theta} \succsim \alpha^+$.

Clearly, this classification is exhaustive and exclusive, except that limiting cases (i.e., $z = \alpha_-/2$) have been excluded. This has no significance, since such limiting cases have zero mass and hence play no role. Note further that the broad classification Z_3, Z_2, Z_1, Z_0, according only to the most preferred alternative, does not suffice, since it would not yield homogeneous groups. Among the groups Z_2 and Z_1, there is space for disagreement between the second and third most preferred options. It is based on those disagreements that the finer subgroups are obtained. However, whenever this subdivision is not relevant, we will argue with the coarser group division, Z_0, Z_1, Z_2 or Z_3.

5.2 The main theorem

In this subsection, we characterize the equilibrium across both stages. To do so, we first consider the following "naive" equilibrium candidate: At the first stage of the voting procedure, $Z_0 \cup Z_1$ vote No and $Z_2 \cup Z_3$ vote Yes. This equilibrium candidate is such that citizens of a type above a certain threshold vote Yes and the others vote No.

We are interested in *manipulation* in the sense that one of the four groups as defined above has an incentive to deviate from the aforementioned equilibrium candidate. Indeed, we are going to claim that Z_1, Z_2, and Z_3 have, in principle, no incentive to make such a deviation, and we examine the conditions under which Z_0 has an incentive to deviate. The examination of the circumstances and of the consequent outcome of the procedure when this manipulation occurs will produce a Bayesian Nash Equilibrium. This equilibrium can be understood as a two-mode equilibrium: if conditions for manipulation do not take place, then all groups find it optimal to vote sincerely in both rounds. If, however, conditions for manipulation are satisfied, in terms of specific relationships between the masses of the different groups, then group Z_0 finds it optimal to deviate in round 1 if the other groups were to vote sincerely. In response to Z_0's deviation, Z_2 and Z_3 will then also deviate, so

as to prevent the members of Z_2 from being successful in their manipulation.

We note that our result does not rely on any assumption about the beliefs of the group members or about the underlying state of nature.

However, we do assume that voting behavior is coordinated within each of the eight subgroups for best responses on the equilibrium path and for deviations. This is a conservative assumption that biases our results in favor of manipulation, and could therefore reduce the benefits of Pendular Voting. As discussed in Subsection 3.2, all equilibria in our paper are also Bayesian perfect Nash equilibria if deviations by individuals are considered.

We obtain:

Theorem 1. *Consider the first stage of the Pendular Voting procedure and define the condition* Manipulation *to occur whenever $Z_0 \cup Z_1^1$ forms a majority and neither Z_0 nor Z_1 is a majority. Assume that the honest equilibrium takes place at the second stage. Then, the following set of strategies at the first stage is a Bayesian Nash Equilibrium:*

$$\textit{If } \text{Not Manipulation} \begin{cases} Z_0 \cup Z_1 \textit{ vote No}, \\ Z_2 \cup Z_3 \textit{ vote Yes}. \end{cases} \qquad \textit{If } \text{Manipulation} \begin{cases} Z_1 \cup Z_2 \cup Z_3 \textit{ vote No}, \\ Z_0 \textit{ votes Yes}. \end{cases}$$

The proof of Theorem 1 is given in the appendix.

Several remarks are in order. First, if manipulation occurs, then Z_0 is not a majority. We thus see that, for the Bayesian Nash Equilibrium described above and under Manipulation conditions, No will win at the first stage and hence α_- will be the outcome. That is, if Manipulation occurs, α_- is guaranteed to be the outcome of the Pendular Voting procedure. Second, this implies that manipulation may happen in equilibrium, but it does not affect which proposal is made in the second round. The manipulation by Z_0 is offset by the manipulation of groups Z_2 and Z_3. These groups fear that if α^+ is chosen as a second proposal, they will end up with the status quo 0 in the second round.

We can also extract as an interesting corollary the fact that:

Corollary 2. *In the Pendular Voting procedure, the only group that could be tempted to*

manipulate is Z_0. It will do so if and only if it believes to be in a state of nature where $Z_0 \cup Z_1^1$ forms a majority but where neither Z_0 nor Z_1 is a majority.

In particular, the strategy where $Z_0 \cup Z_1$ votes No and $Z_2 \cup Z_3$ votes Yes will be a Bayesian Nash Equilibrium if there are no states of nature where $Z_0 \cup Z_1^1$ forms a majority.

Theorem 1 differs from existing results in the literature on strategic voting or truthful mechanisms: The literature typically aims at establishing results on "strategy-proofness" by demonstrating the existence of an equilibrium in which all agents reveal their private information truthfully. Theorem 1, however, does not claim that the Pendular Voting procedure is strategy-proof. Yet, it shows that manipulation—if it occurs—does not affect which second proposal is made after the first stage.

Corollary 2 says that only members of one particular group, called Z_0, may have incentives to vote strategically. Whether or not the members of Z_0 want to manipulate in this way depends on their probabilistic belief about the underlying state of nature. These results, however, hold true regardless of these beliefs.

6 Welfare, Outcome Characterization, and Manipulation

In this section, we dive deep into the properties of Pendular Voting and provide a first round of welfare results.

6.1 The no-welfare-loss result

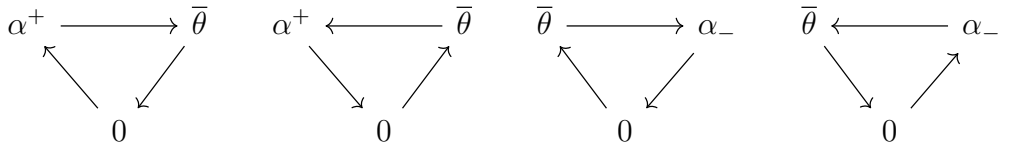
As an intermediate step towards welfare results, we now formulate the *no welfare loss result*.

Recall from the previous section that we are comparing the Pendular Voting procedure to a benchmark procedure in which citizens simply choose between the proposal and the status quo, without any preliminary proposal assessment. A key implication of Theorem 1 is that, whenever the Pendular Voting procedure is manipulated, its outcome is always

α_-. If the Pendular Voting procedure is *not* manipulated, it may still be the case that the outcome of Pendular Voting coincides with that of the benchmark procedure. Finally, there is a case where the Pendular Voting procedure is not manipulated, and yet leads to a different outcome than the benchmark procedure. In that case, the outcome under Pendular Voting is a welfare improvement in the sense that this outcome is preferred by a majority to that of the benchmark procedure:

Proposition 3. *Moving from the benchmark procedure to the Pendular Voting procedure, while holding the proposal constant, never leads to a welfare loss, regardless of the state.*

Proof of Proposition 3. We have to verify that the outcome of Pendular Voting is always either the same outcome or an outcome preferred to the one of the benchmark procedure. The only possible pairwise preferences at the second round that could contradict this corollary are the non-Condorcet, cyclic outcomes. There are four possibilities:



From these four, the only outcome that contradicts the corollary is the first, where 0 is preferred by a majority to the outcome $\overline{\theta}$. However, this is a limit case that occurs with probability zero. To see that, we note that, on the one hand, for 0 to win against $\overline{\theta}$, as before, $Z_0 \cup Z_1^1$ must be a majority. On the other hand, for α^+ to win against 0, $Z_3 \cup Z_2^3 \cup Z_2^2 \cup Z_1^3$ must be a majority. Since they are disjoint, each of them must be exactly equal to half the citizenry, so that by our assumption on limit cases, this occurs with probability zero.

It is important to note that Theorem 1 and Corollary 3 hold, regardless of the prior or posterior beliefs held by any of the citizens.

6.2 Explicit outcome characterization

We now make full use of Theorem 1 and of the corresponding remark to characterize the outcome of the Pendular Voting procedure as follows:

Corollary 3. *Consider the Bayesian Nash Equilibrium of the Pendular Voting procedure given by Theorem 1 for the first stage, in combination with the honest equilibrium of Theorem 1 for the second stage; then:*

- *The outcome is 0 if and only if Z_0 is a majority.*
- *The outcome is α^+ if and only if Z_3 is a majority.*
- *The outcome is α_- if and only if $Z_0 \cup Z_1$ is a majority but Z_0 itself is not a majority.*
- *The outcome is $\overline{\theta}$ if and only if none of the above holds.*

The proof of Corollary 3 is given in the appendix.

As a more convenient formulation of the previous result, we note that under the Bayesian Nash Equilibrium described in Theorem 1, we can explicitly obtain the outcome of the Pendular Voting procedure as a function of the proposed policy $\overline{\theta}$ and the median voter's preferences. We will denote this function by $\Omega_{\overline{\theta}}^{PV} : [0,1] \to [0,1]$, mapping the type of the median voter's type $\widehat{z}$ to the outcome of the Pendular Voting procedure, given that the policy $\overline{\theta}$ was proposed and assuming that the equilibrium of Theorem 1 takes place.

Corollary 4. *The outcome of the Pendular Voting procedure as a function of the median voter's type $\widehat{z}$, given the proposed policy $\overline{\theta}$ and under the equilibrium of Theorem 1, is given by:*

$$\Omega_{\overline{\theta}}^{PV}(\widehat{z}) = \begin{cases} 0, & \text{if} \quad 0 \leq \widehat{z} < \dfrac{\overline{\theta} - \mu_1}{2}, \\ \overline{\theta} - \mu_1, & \text{if} \quad \dfrac{\overline{\theta} - \mu_1}{2} \leq \widehat{z} < \overline{\theta} - \dfrac{\mu_1}{2}, \\ \overline{\theta}, & \text{if} \quad \overline{\theta} - \dfrac{\mu_1}{2} \leq \widehat{z} < \overline{\theta} + \dfrac{\mu_2}{2}, \\ \overline{\theta} + \mu_2, & \text{if} \quad \overline{\theta} + \dfrac{\mu_2}{2} \leq \widehat{z} \leq 1. \end{cases}$$

Proof of Corollay 4. It is a simple matter of translating the conditions of the characterization of the outcome of the Pendular Voting procedure described in the previous result

into inequalities satisfied by $\widehat{z}$ using Corollary 1.

6.3 Welfare improvements by Pendular Voting

We provide some simple facts about the welfare properties of Pendular Voting.

Corollary 5. *The Pendular Voting procedure selects the proposal closest to the median voter's type among the four possible outcomes* $\{0, \overline{\theta} - \mu_1, \overline{\theta}, \overline{\theta} + \mu_2\}$ *under the equilibrium of Theorem 1. In particular, it maximizes social welfare.*

This corollary is very interesting as it describes an emerging property of Pendular Voting: at no point is it prescribed in the procedure rules that a competition for social welfare takes place between the four outcomes, but instead, it arises from the interaction between the two rounds and the voter type distribution. Furthermore, the values of the design parameters μ_1 and μ_2 (which, we recall, are adjusted accordingly if $\overline{\theta}$ approaches either 0 or 1, so that $\mu_1 \leq \overline{\theta}$ and $\mu_2 \leq 1 - \overline{\theta}$) permit to tweak and spread the proposals to cover more policy space.

We compare this with the outcome of the single-round benchmark procedure:

$$\Omega^B_{\overline{\theta}}(\widehat{z}) = \begin{cases} 0, & \text{if} \quad 0 \leq \widehat{z} < \dfrac{\overline{\theta}}{2}, \\ \overline{\theta}, & \text{if} \quad \dfrac{\overline{\theta}}{2} \leq \widehat{z} \leq 1. \end{cases}$$

We can see the functions associated to the outcome of both procedures in Figure 1.

Of course, we recover the benchmark procedure when $\mu_1 = \mu_2 = 0$: $\Omega^{PV}_{\overline{\theta}}(\widehat{z}) = \Omega^B_{\overline{\theta}}(\widehat{z})$. We thus see how the Pendular Voting procedure makes the outcome of the voting system more flexible and closer to the median voter's preferences. In fact, the following comparison can be made: the outcome of the Pendular Voting and the benchmark procedure coincide, except when:

- $\frac{\overline{\theta} - \mu_1}{2} \leq \widehat{z} \leq \frac{\overline{\theta}}{2}$, where $\Omega^{PV}_{\overline{\theta}}(\widehat{z}) = \overline{\theta} - \mu_1$ while $\Omega^B_{\overline{\theta}}(\widehat{z}) = 0$;
- $\frac{\overline{\theta}}{2} \leq \widehat{z} \leq \overline{\theta} - \frac{\mu_1}{2}$, where $\Omega^{PV}_{\overline{\theta}}(\widehat{z}) = \overline{\theta} - \mu_1$ while $\Omega^B_{\overline{\theta}}(\widehat{z}) = \overline{\theta}$;

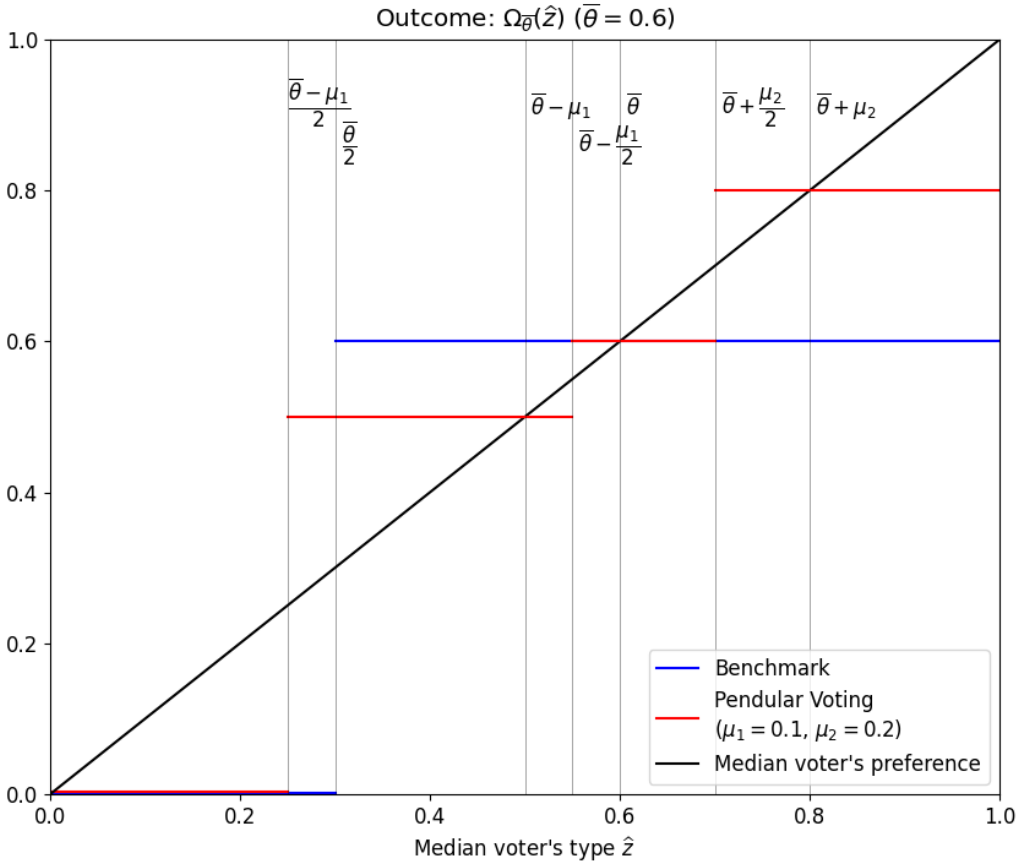


Figure 1: Comparison between the Pendular Voting and the benchmark outcomes as a function of the type of the median voter, for the case $\overline{\theta} = 0.6$, $\mu_1 = 0.1$, $\mu_2 = 0.2$.

- $\overline{\theta} + \frac{\mu_2}{2} \leq \widehat{z} \leq 1$, where $\Omega^{PV}_{\overline{\theta}}(\widehat{z}) = \overline{\theta} + \mu_2$ while $\Omega^{B}_{\overline{\theta}}(\widehat{z}) = \overline{\theta}$.

In all three cases, it can be easily verified that Pendular Voting strictly decreases (for positive μ_i) the distance between the outcome and the median voter's type, i.e., $|\Omega^{PV}_{\overline{\theta}}(\widehat{z}) - \widehat{z}| < |\Omega^{B}_{\overline{\theta}}(\widehat{z}) - \widehat{z}|$, as we also proved by different means in Corollary 3. However, now we can characterize the states where the increase is strictly positive:

Corollary 6. *For every proposed policy $\overline{\theta}$ and with probability 1, Pendular Voting does not decrease social welfare with respect to the single-round benchmark procedure, and strictly increases it whenever $\frac{\overline{\theta}-\mu_1}{2} < \widehat{z} < \overline{\theta} - \frac{\mu_1}{2}$ or $\overline{\theta} + \frac{\mu_2}{2} < \widehat{z} \leq 1$.*

Proof of Corollary 6. We compare $|\Omega^{PV}_{\overline{\theta}}(\widehat{z}) - \widehat{z}| \leq |\Omega^{B}_{\overline{\theta}}(\widehat{z}) - \widehat{z}|$ and note where the inequality

is strict.

In Figure 1 we can see graphically how Pendular Voting manages to approximate the type of the median voter better than the single-round procedure, thanks to the flexibility introduced by the design parameters μ_1, μ_2. In Figure 2, we also see that the distance between the outcome and the median voter's type $|\Omega^A_{\overline{\theta}}(\widehat{z}) - \widehat{z}|$ is, for either $A = PV, B$, piecewise linear and continuous, since whenever there is a change in $\Omega^A_{\overline{\theta}}(\widehat{z})$, the change is symmetric with respect to the median voter's type $\widehat{z}$.

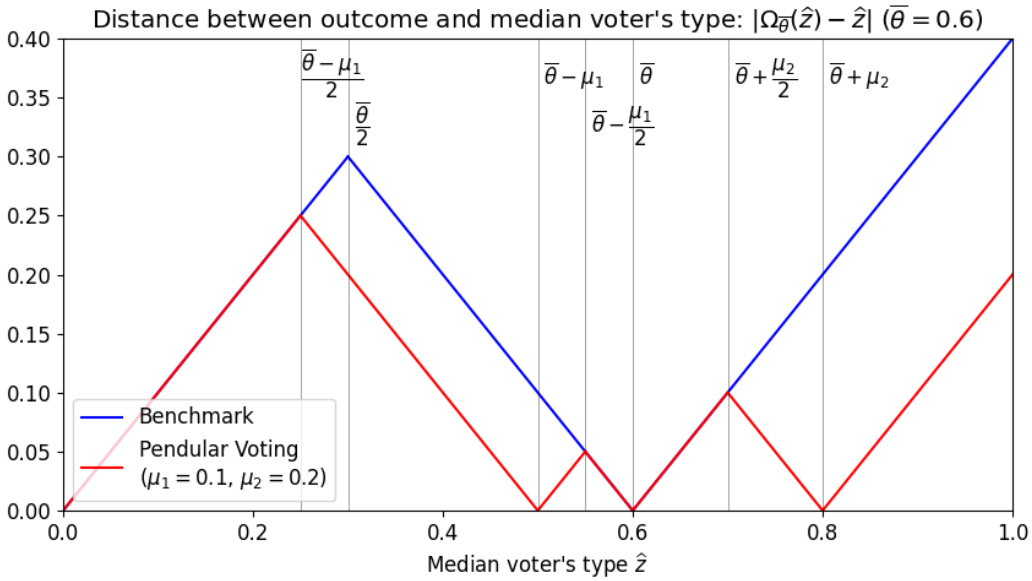


Figure 2: Comparison between the Pendular Voting and the benchmark procedures' social welfare (i.e, distance between the outcome and the median voter's type), for the case $\overline{\theta} = 0.6$, $\mu_1 = 0.1$, $\mu_2 = 0.2$.

6.4 Manipulation and beliefs

The results in the previous subsection are true regardless of the values of the increments μ_1 and μ_2. As said before, we note that the entire discussion so far is independent of the beliefs held by individual groups in the population.

In this subsection, we derive additional results by considering the beliefs of group Z_0 and the occurrence of manipulation. For simplicity, we assume in this subsection that $\mu_1 = \mu_2 = \mu$.

Corollary 2 states that a deviation by group Z_0 can only be profitable if the state of nature is such that all of the following conditions are simultaneously satisfied:

1. $Z_0 \cup Z_1^1$ constitutes a majority, i.e. a majority prefers 0 to $\overline{\theta}$.
2. Z_0 alone does not constitute a majority.
3. Z_1 alone does not constitute a majority.

The complementary cases of these three conditions are expressed by the three inequalities below, assuming to be in a state of nature with median voter of type $\widehat{z}$ and distribution function F.

$$\begin{aligned} \overline{\theta} &\leq 2\widehat{z} \\ \overline{\theta} &\geq 2\widehat{z} + \mu \\ F\left(\overline{\theta} - \frac{1}{2}\mu\right) - F\left(\frac{1}{2}\overline{\theta} - \frac{1}{2}\mu\right) &\geq \frac{1}{2}. \end{aligned} \tag{1}$$

The first inequality describes the case where a majority prefers $\overline{\theta}$ to the status quo, so that in particular, no manipulation is possible. The second describes the case where Z_0 has a majority, while if the third inequality is satisfied, a majority belongs to Z_1. In these cases, manipulation is again impossible. We obtain:

Corollary 7. *Manipulation of the Pendular Voting procedure does not occur if group Z_0 does assign sufficiently low probability that Inequality (1) holds. Moreover, manipulation does not occur if group Z_0 assigns sufficiently low probability to states of nature where $2\widehat{z} < \overline{\theta} < 2\widehat{z} + \mu$.*

One clarifying remark is in order: While we assume that groups of citizens with the same preferences vote in the same way, they are unable to determine their own size. Otherwise, they would know the state of nature. We stress that the assumption that groups can coordinate their votes is "conservative" in nature: If we obtain welfare results that are robust to manipulations by large groups, then they would certainly be robust to manipulations by smaller subsets of these groups.

So far, we have not made any assumptions regarding the distribution of the median voter's type or about the states of nature. A natural assumption is to use first-order dominance ordering. Hence, let us assume that we can index the states of nature either discretely or as a continuum, and then order them in such a way that higher indices of states of nature correspond to a distribution shifted towards 1. That is, for the discrete case with states of nature indexed by $k \in N = 0, 1, \ldots$, we assume that for any $z \in (0, 1)$, we have $F_0(z) > F_1(z) > \ldots$, while for the continuous case where $k \in N = [0, 1]$, we assume that $\frac{d}{dk}F_k(z) < 0$ for $z \in (0, 1)$.

Consider then the case where $\overline{\theta} + \mu \leq 2\widehat{z}_0$, so that by the first-order dominance ordering we just described, we have $\overline{\theta} + \mu \leq 2\widehat{z}_k$ for all states of nature (note that $\frac{d}{dk}F_k(z) < 0$ implies $\frac{d}{dk}\widehat{z}_k > 0$). This means that each of the three proposals α_-, $\overline{\theta}$, and α^+ is preferred to the status quo by a majority, and this is true in *each* state $k \in N$. Therefore, group Z_0 never finds it optimal to vote strategically, and hence, no manipulation is possible.

Corollary 8. *Suppose that $\overline{\theta} + \mu \leq 2\widehat{z}_k$ for every state of nature k. Furthermore, suppose that there are some states $k \in N$ such that $\widehat{z}_k > \overline{\theta} + \mu/2$ occurs with positive probability. Then, moving from the benchmark procedure to the Pendular Voting procedure, while holding the proposal $\overline{\theta}$ constant, leads to a welfare gain with strictly positive probability.*

The two necessary conditions in the corollary above boil down to a requirement that the proposal $\overline{\theta}$, which we are holding fixed here, should not be "too high'. Later on, we will argue that an agenda-setter, regardless of his/her motivation, never has any incentive to make an excessively high proposal in the first place. Hence, the interpretation of the corollary is that switching from the benchmark procedure to the Pendular Voting procedure can indeed be expected to lead to a welfare gain.

One may wonder if the choice between the status quo, the initial proposal, and the upward and downward corrections could not be done more efficiently on a single, four-way ballot paper. There are several problems with this approach, however: First and foremost, our voting rule for the final voting round is designed for the case of a three-way ballot and it does not easily generalize to a four-way ballot. As one can verify, in a four-way ballot, there

is a total of 16 possible outcomes, of which only 8 have a Condorcet Winner. The remaining 8 outcomes, equivalent to the cyclic outcomes in the three-way ballot, are difficult to treat. In particular, the same approach as in the second stage of Pendular Voting, i.e., choosing the middle alternative as the default winner of non-Condorcet outcomes, does not work for four-way ballots, since there are *two* middle alternatives, instead of the previous unique middle alternative, and there is no clear criterion allowing to choose between these two. Second, a four-way ballot would require the elicitation of six, instead of three, pairwise preferences, which can be interpreted as an efficiency loss.

We stress that the arguments behind the main results in this paper do not rely on any notion of probabilistic beliefs by the citizens, except that all states occur with strictly positive probability.

7 The Agenda-setter's Choice

So far, our analysis has led to strong results on welfare gain of the Pendular Voting procedure, relative to the benchmark procedure, under the premise that the proposal made is the same under both procedures. In a nutshell, social welfare increases with the Pendular Voting procedure, compared to the benchmark. The size of the welfare gain depends on μ and goes to zero as μ goes to zero.

After this analysis, a natural follow-up question is this: Suppose that the Pendular Voting procedure is used, and the agenda-setter anticipates the conditions for its manipulability. Would the agenda-setter then want to make the same proposal as in the benchmark procedure? If the agenda-setter does make it indeed, we have shown that a welfare gain can be realized. But what happens if the agenda-setter re-optimizes even his/her original proposal in anticipation of proposal assessment?

7.1 Information about the underlying distribution

We first study the Pendular Voting procedure, assuming that the agenda-setter has access to precise information about the underlying voter type distribution. Since the outcome of

the procedure is determined exclusively by the median voter's type $\widehat{z}$, we simply assume that the agenda-setter has access to the value of $\widehat{z}$. In all that follows we assume that the the equilibrium of Theorem 1 takes place.

We will consider the cases of a benevolent and a selfish agenda-setter, the former aiming to maximize social welfare and the latter aiming to maximize the policy outcome. However, the analysis for the benevolent case is trivial: a benevolent agenda-setter with access to $\widehat{z}$ will simply put forward the proposal $\overline{\theta} := \widehat{z}$. Since $\widehat{z}$ is the Condorcet Winner, this ensures that the outcome will be the median voter's type and thus social welfare is trivially maximized.

The interesting case is that of a selfish agenda-setter. In particular, we assume that the agenda-setter wants to maximize the outcome policy and has a utility function that is linear or affine on $\Omega_{\overline{\theta}}^{PV}(\widehat{z})$.

Lemma 1. *The optimal strategy in Pendular Voting with a selfish agenda-setter aiming to maximize the policy outcome and with knowledge of $\widehat{z}$ is to propose:*

$$\overline{\theta}_{AS}^{PV} = arg\ max_x g(x)$$

where g is the function of three possible arguments given by the association:

$$\begin{aligned} \widehat{z} - \frac{\mu_2}{2} &\mapsto \widehat{z} + \frac{\mu_2}{2} \\ \widehat{z} + \frac{\mu_1}{2} &\mapsto \widehat{z} + \frac{\mu_1}{2} \\ 2\widehat{z} + \mu_1 &\mapsto 2\widehat{z}. \end{aligned}$$

If $0 \leq \widehat{z} \leq \mu_2/2$, then the first association disappears, while if $1 - \mu_2/2 \leq \widehat{z} \leq 1$, then the first association is substituted by

$$\widehat{z} - \frac{\mu_2}{2} \mapsto 1.$$

Additionally, if $(1 - \mu_1)/2 \leq \widehat{z} < 1 - \mu_1/2$, then the last association is substituted by

$$1 \mapsto 1 - \mu_1,$$

and if $1 - \mu_1/2 \leq \widehat{z}$*, then the last two associations are substituted by*

$$1 \mapsto 1.$$

We note that we always have $\mu_2/2 \leq 1 - \mu_2/2$ and $(1 - \mu_1)/2 \leq 1 - \mu_1/2$.

With this lemma, we can now obtain the optimal strategy for a selfish agenda-setter and the final outcome after his/her proposal.

Proposition 4. *The optimal strategy and subsequent outcome of Pendular Voting for a selfish agenda-setter maximizing the policy outcome and with knowledge of* $\widehat{z}$ *are given in Table 1.*

The proof of Proposition 4 is in the appendix. As expected, a selfish agenda-setter always manages to induce an outcome greater than $\widehat{z}$. This must be possible since, with access to the median voter's type $\widehat{z}$, at worst, s/he may always propose the Condorcet Winner and induce an outcome at least equal to or greater than $\widehat{z}$.

Another interesting remark is that in the limit, where the design parameters μ_1 and μ_2 tend to 0, the optimal strategy degenerates to the strategy of a selfish agenda-setter under the benchmark procedure of a single-round with two options, i.e.:

- If $\widehat{z} \leq 1/2$, then $\overline{\theta}_{AS}^{PV} = 2\widehat{z} = \Omega^{PV}_{\overline{\theta}_{AS}^{PV}}(\widehat{z})$.
- If $\widehat{z} \geq 1/2$, then $\overline{\theta}_{AS}^{PV} = 1 = \Omega^{PV}_{\overline{\theta}_{AS}^{B}}(\widehat{z})$.

We finally obtain:

Corollary 9. *The expected social welfare of Pendular Voting with a selfish agenda-setter aiming to maximize the policy outcome and with knowledge of* $\widehat{z}$ *is given by:*

$$\mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}_{AS}^{PV}}(\widehat{z}) - \widehat{z}|] = \begin{cases} \dfrac{1}{8}\left[3\mu_1^2 + 4\mu_1\mu_2 - 4\mu_1 + 2\right], & \text{if} \quad 0 \leq \mu_1 \leq \mu_2, \\ \dfrac{1}{8}\left[7\mu_1^2 - 4\mu_1 + 2\right], & \text{if} \quad \mu_2 \leq \mu_1 \leq 1/2, \\ \dfrac{1}{8}\left[-\mu_1^2 + 4\mu_1\right], & \text{if} \quad 1/2 \leq \mu_1, \end{cases}$$

μ_1	$\widehat{z}$	$\overline{\theta}_{AS}^{PV}$	$\Omega_{\overline{\theta}_{AS}^{PV}}^{PV}(\widehat{z})$
$0 \leq \mu_1 \leq \mu_2$	$\widehat{z} \in \left[0, \frac{\mu_1}{2}\right]$	$\widehat{z} + \frac{\mu_1}{2}$	$\widehat{z} + \frac{\mu_1}{2}$
	$\widehat{z} \in \left[\frac{\mu_1}{2}, \frac{1-\mu_1}{2}\right]$	$2\widehat{z} + \mu_1$	$2\widehat{z}$
	$\widehat{z} \in \left[\frac{1-\mu_1}{2}, 1 - \mu_1 - \frac{\mu_2}{2}\right]$	1	$1 - \mu_1$
	$\widehat{z} \in \left[1 - \mu_1 - \frac{\mu_2}{2}, 1 - \frac{\mu_2}{2}\right]$	$\widehat{z} - \frac{\mu_2}{2}$	$\widehat{z} + \frac{\mu_2}{2}$
	$\widehat{z} \in \left[1 - \frac{\mu_2}{2}, 1\right]$	1	1
$\mu_2 \leq \mu_1 \leq \frac{1}{2}$	$\widehat{z} \in \left[0, \frac{\mu_1}{2}\right]$	$\widehat{z} + \frac{\mu_1}{2}$	$\widehat{z} + \frac{\mu_1}{2}$
	$\widehat{z} \in \left[\frac{\mu_1}{2}, \frac{1-\mu_1}{2}\right]$	$2\widehat{z} + \mu_1$	$2\widehat{z}$
	$\widehat{z} \in \left[\frac{1-\mu_1}{2}, 1 - \frac{3\mu_1}{2}\right]$	1	$1 - \mu_1$
	$\widehat{z} \in \left[1 - \frac{3\mu_1}{2}, 1 - \frac{\mu_1}{2}\right]$	$\widehat{z} + \frac{\mu_1}{2}$	$\widehat{z} + \frac{\mu_1}{2}$
	$\widehat{z} \in \left[1 - \frac{\mu_1}{2}, 1\right]$	1	1
$1/2 \leq \mu_1$	$\widehat{z} \in \left[0, 1 - \frac{\mu_1}{2}\right]$	$\widehat{z} + \frac{\mu_1}{2}$	$\widehat{z} + \frac{\mu_1}{2}$
	$\widehat{z} \in \left[1 - \frac{\mu_1}{2}, 1\right]$	1	1

Table 1: Optimal proposal $\overline{\theta}_{AS}^{PV}$ and outcome $\Omega_{\overline{\theta}_{AS}^{PV}}^{PV}(\widehat{z})$ of Pendular Voting for a selfish agenda-setter maximizing the policy outcome with knowledge of the median voter's type $\widehat{z}$.

and the expected outcome is:

$$\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z})] = \mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z}) - \widehat{z}|] + \frac{1}{2}.$$

In particular, the optimal design parameters in this situation, i.e., those maximizing the expected social welfare, are:

$$\mu_1^* = \frac{2}{7},\ \mu_2^* \in \left[0, \frac{2}{7}\right].$$

With this choice of parameters, the expected outcome and social welfare are

$$\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z})] = \frac{19}{28}, \qquad \mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z}) - \widehat{z}|] = \frac{5}{28}.$$

More details on the calculation of Corollary 9 are given in the appendix.

It is understood that the multiple values of μ_2^* mean that all of them induce the maximum expected social welfare.

As we say, the optimal strategy for the benchmark procedure is to propose $\overline{\theta}^B_{AS} = 2\widehat{z}$ if $\widehat{z} \leq 1/2$ and $\overline{\theta}^B_{AS} = 1$ otherwise; and would give an expected outcome and social welfare:

$$\mathbb{E}_{\widehat{z}}[\Omega^{B}_{\overline{\theta}^{B}_{AS}}(\widehat{z})] = \frac{3}{4}, \qquad \mathbb{E}_{\widehat{z}}[|\Omega^{B}_{\overline{\theta}^{B}_{AS}}(\widehat{z}) - \widehat{z}|] = \frac{1}{4}.$$

We see that Pendular Voting strictly improves social welfare for the optimal design parameters.

7.2 No information about the underlying distribution

In the following, we model the assumption of no information about the underlying voter type distribution by assuming that the median citizen type $\widehat{z} \in [0, 1]$ follows an absolutely continuous distribution, denoting by $G(z) := \mathbb{P}[\widehat{z} \leq z]$ the distribution function.

Proposition 5. *Under the equilibrium introduced in Theorem 1, the expectation of the final outcome of the Pendular Voting procedure* $\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}}(\widehat{z})]$ *as a function of the proposed*

policy $\overline{\theta}$ *is given by*

$$\begin{cases} \mu_2 + \overline{\theta} - \overline{\theta} G\left(\dfrac{\overline{\theta}}{2}\right) - \mu_2 G\left(\overline{\theta} + \dfrac{\mu_2}{2}\right), & \text{if} \quad 0 \leq \overline{\theta} < \mu_1, \\ \mu_2 + \overline{\theta} - \left(\overline{\theta} - \mu_1\right) G\left(\dfrac{\overline{\theta} - \mu_1}{2}\right) - \mu_2 G\left(\overline{\theta} + \dfrac{\mu_2}{2}\right) - \mu_1 G\left(\overline{\theta} - \dfrac{\mu_1}{2}\right), & \text{if} \quad \mu_1 \leq \overline{\theta} < 1 - \mu_2, \\ 1 - \left(\overline{\theta} - \mu_1\right) G\left(\dfrac{\overline{\theta} - \mu_1}{2}\right) - \left(1 - \overline{\theta}\right) G\left(\dfrac{1 + \overline{\theta}}{2}\right) - \mu_1 G\left(\overline{\theta} - \dfrac{\mu_1}{2}\right), & \text{if} \quad 1 - \mu_2 \leq \overline{\theta} \leq 1. \end{cases}$$

The proof of Proposition 5 is given in the appendix. We next apply this formulas to the uniform distribution in order to obtain explicit formulas.

7.3 Uniform distribution

We will obtain concrete formulas for the case of a median voter's type that is uniformly distributed over $[0, 1]$. First, however, we make the observation that the assumption that the median voter's type follows an absolutely continuous distribution with infinitely differentiable density function is not merely natural, but follows easily from the assumption mentioned in Section 6. The uniform distribution is the simplest of such an absolutely continuous distribution.

To see it, consider a continuum of states of nature normalized to $N = [0, 1]$ and ordered so that $\frac{d}{dk} F_k(z) < 0$. If the state of nature k follows an absolutely continuous distribution with infinitely differentiable density function, the monotonocity of the distribution functions of the voter type with respect to k implies that the median voter's type also follows an absolutely continuous distribution, with infinitely differentiable density function (also assuming that F_k all have strictly positive density functions, i.e., that $\frac{d}{dz} F_k(z) > 0$). This can be seen by noting that, in that case, $\frac{d}{dk} \widehat{z}_k > 0$ (recall that the median voter's type is given by $\widehat{z}_k := F_k^{-1}(1/2)$). Hence, we can apply the Inverse Function Theorem to obtain the state of nature in terms of the median voter's type, and thus the differentiable density function of $\widehat{z}$.

Let us suppose now that the median voter's type is uniformly distributed in the interval $[0, 1]$ and let us apply Proposition 5. We obtain:

Proposition 6. *Assume that the median citizen type $\widehat{z}$ is uniformly distributed in $[0,1]$. Then, under the Bayesian Nash Equilibrium of the Pendular Voting procedure introduced in Theorem 1, the expected outcome as a function of the proposed policy $\overline{\theta}$ is*

$$\mathbb{E}_{\widehat{z}}[\Omega_{\overline{\theta}}^{PV}(\widehat{z})] = \begin{cases} -\dfrac{\overline{\theta}^2}{2} + \overline{\theta}\,(1-\mu_2) + \mu_2\left(1 - \dfrac{\mu_2}{2}\right), & \text{if} \quad 0 \leq \overline{\theta} < 1-\mu_2, \\ \dfrac{1}{2}, & \text{if} \quad 1-\mu_2 \leq \overline{\theta} \leq 1. \end{cases}$$

Proof of Proposition 6. We substitute $G(z) = z$ in the formulas obtained in Proposition 5 to obtain:

- $\mu_1 < \overline{\theta} < 1 - \mu_2$:

$$\begin{aligned} \mathbb{E}_{\widehat{z}}[\Omega_{\overline{\theta}}^{PV}(\widehat{z})] &= \mu_2 + \overline{\theta} - (\overline{\theta} - \mu_1)^2/2 - \mu_2\left(\overline{\theta} + \mu_2/2\right) - \mu_1\left(\overline{\theta} - \mu_1/2\right) \\ &= \mu_2 + \overline{\theta} - \overline{\theta}^2/2 - \mu_2\left(\overline{\theta} + \mu_2/2\right). \end{aligned}$$

- $0 \leq \overline{\theta} \leq \mu_1$: we have to substitute $\mu_1 = \overline{\theta}$, and hence nothing changes:

$$\mathbb{E}_{\widehat{z}}[\Omega_{\overline{\theta}}^{PV}(\widehat{z})] = \mu_2 + \overline{\theta} - \overline{\theta}^2/2 - \mu_2\left(\overline{\theta} + \mu_2/2\right).$$

- $1 - \mu_2 \leq \overline{\theta} \leq 1$: substituting $\mu_2 = 1 - \overline{\theta}$:

$$\mathbb{E}_{\widehat{z}}[\Omega_{\overline{\theta}}^{PV}(\widehat{z})] = 1 - \overline{\theta}^2/2 - (1-\overline{\theta})(1+\overline{\theta})/2 = 1/2.$$

We see that the expected outcome is indistinct of the first design parameter μ_1 for a uniformly distributed median voter.

In order to maximize social welfare in the next subsections, we are also interested in the expectation of the distance to the median voter, i.e. in $\mathbb{E}_{\widehat{z}}[|\Omega_{\overline{\theta}}^{PV}(\widehat{z}) - \widehat{z}|]$ and its comparison with $\mathbb{E}_{\widehat{z}}[|\Omega_{\overline{\theta}}^{B}(\widehat{z}) - \widehat{z}|]$. For a uniformly distributed median voter, this is the area in Figure 1 under the graph of $|\Omega_{\overline{\theta}}^{i}(\widehat{z}) - \widehat{z}|$, $i = PV, B$, and the objective is to choose the parameters

μ_1, μ_2 that minimize this area.

Proposition 7. *Assume that the median citizen type $\widehat{z}$ is uniformly distributed in $[0,1]$. Then, under the Bayesian Nash Equilibrium of the Pendular Voting procedure introduced in Theorem 1, the expected distance to the median voter's type as a function of the proposed policy $\overline{\theta}$ is*

$$\mathbb{E}_{\widehat{z}}[|\Omega_{\overline{\theta}}^{PV}(\widehat{z})-\widehat{z}|] = \begin{cases} \left(\frac{\overline{\theta}}{2}\right)^2 + \left(\frac{\mu_2}{2}\right)^2 + \frac{1}{2}\left(1-\overline{\theta}-\mu_2\right)^2, & \text{if} \quad 0 \leq \overline{\theta} < \mu_1, \\ \left(\frac{\overline{\theta}-\mu_1}{2}\right)^2 + \left(\frac{\mu_1}{2}\right)^2 + \left(\frac{\mu_2}{2}\right)^2 + \frac{1}{2}\left(1-\overline{\theta}-\mu_2\right)^2, & \text{if} \quad \mu_1 \leq \overline{\theta} < 1-\mu_2, \\ \left(\frac{\overline{\theta}-\mu_1}{2}\right)^2 + \left(\frac{\mu_1}{2}\right)^2 + \left(\frac{1-\overline{\theta}}{2}\right)^2, & \text{if} \quad 1-\mu_2 \leq \overline{\theta} \leq 1. \end{cases}$$

Proof of Proposition 7. The same procedure as before yields the result, arguing directly to compute and add the area of the triangles in Figure 2.

We can compare this to the benchmark procedure, where

$$\mathbb{E}_{\widehat{z}}[|\Omega_{\overline{\theta}}^{B}(\widehat{z}) - \widehat{z}|] = \left(\frac{\overline{\theta}}{2}\right)^2 + \frac{1}{2}\left(1-\overline{\theta}\right)^2,$$

which coincides with the evaluation at $\mu_1 = \mu_2 = 0$.

7.4 Benevolent agenda-setter and uniform distribution

A benevolent agenda-setter trying to minimize the expected distance between the outcome of Pendular Voting and the median voter, i.e. trying to maximize social welfare, will choose the optimal value $\overline{\theta}_*^{PV}$ of the proposed policy such that $\mathbb{E}_{\widehat{z}}[|\Omega_{\overline{\theta}}^{PV}(\widehat{z}) - \widehat{z}|]$ is minimized for fixed parameters μ_1, μ_2. Obtaining the local minimums of $\mathbb{E}_{\widehat{z}}[|\Omega_{\overline{\theta}}^{PV}(\widehat{z}) - \widehat{z}|]$, as given by Proposition 6, and since $\mu_1 \leq 1-\mu_2$, we always have that $\mu_1 \leq \overline{\theta}_*^{PV} \leq 1-\mu_2$ and is given by

$$\overline{\theta}_*^{PV} = \frac{2+\mu_1-2\mu_2}{3}.$$

A computation shows that value of the expected outcome is

$$\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}^{PV}_*}(\widehat{z})] = \frac{1}{18}(4-\mu_1-\mu_2)(2-\mu_1+\mu_2),$$

and the value of the expected distance to the median voter's type is then

$$\mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}^{PV}_*}(\widehat{z})-\widehat{z}|] = \left(\frac{\mu_1}{2}\right)^2 + \left(\frac{\mu_2}{2}\right)^2 + \frac{3}{2}\left(\frac{1-\mu_1-\mu_2}{3}\right)^2.$$

If we again minimize this expression in terms of the design parameters, one easily finds that the optimal parameters $\mu_1^*,\ \mu_2^*$, i.e., those that maximize social welfare by minimizing the above expression, are:

$$\mu_1^* = \mu_2^* = \frac{2}{7}.$$

Thus, these are the parameters to choose for Pendular Voting if we believe that the median voter's type is uniformly distributed and that the agenda-setter is benevolent. Note in particular that $\mu_1^* \leq 1-\mu_2^*$. With this choice of parameters, we see that

$$\overline{\theta}^{PV}_* = \frac{4}{7},$$

and thus we obtain

$$\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}^{PV}_*}(\widehat{z})] = \frac{24}{49},$$

together with

$$\mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}^{PV}_*}(\widehat{z})-\widehat{z}|] = \frac{1}{14}.$$

If we compare to the benchmark procedure (i.e., $\mu_1=\mu_2=0$), now $\overline{\theta}^B_* = 2/3$,

$$\mathbb{E}_{\widehat{z}}[\Omega^{B}_{\overline{\theta}^{B}_*}(\widehat{z})] = \frac{4}{9}$$

and

$$\mathbb{E}_{\widehat{z}}[|\Omega^{B}_{\overline{\theta}^{B}_*}(\widehat{z})-\widehat{z}|] = \frac{1}{6},$$

we see that Pendular Voting strictly improves the expected social welfare also for a benevolent agenda-setter. We summarize these results in the following proposition:

Corollary 10. *Consider the equilibrium described in Theorem 1 for the Pendular Voting procedure and suppose that the agenda-setter is benevolent and tries to maximize the expected social welfare. Then, the design parameters that maximize social welfare are*

$$\mu_1^* = \mu_2^* = \frac{2}{7}.$$

With this choice of parameters, the benevolent agenda-setter will choose the proposal

$$\overline{\theta}_*^{PV} = \frac{4}{7},$$

and the expected outcome and social welfare are:

$$\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}_*^{PV}}(\widehat{z})] = \frac{24}{49}, \qquad \mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}_*^{PV}}(\widehat{z}) - \widehat{z}|] = \frac{1}{14}.$$

Interestingly, the optimal design parameters almost coincide with those found in the previous subsection to be optimal for a selfish agenda-setter with knowledge of $\widehat{z}$.

7.5 Selfish agenda-setter and uniform distribution

We now look at the case where the agenda-setter is selfish and will act according to his/her own interests. As a tie-breaking rule, when the agenda-setter is indifferent between several proposals that are optimal from his/her perspective, s/he selects the one among these proposals that delivers the highest social welfare.

We assume that the agenda-setter wants to maximize the outcome policy with a linear Bayesian utility function $\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}}(\widehat{z})]$. Then, s/he will propose a policy $\overline{\theta}$ maximizing the above second-order piece-wise polynomial. Noting that the maximum of the first part is attained precisely at $1-\mu_2$, with a value of precisely $1/2$, we see that any choice $\overline{\theta} \in [1-\mu_2, 1]$ will provide the maximum expected outcome, with an expectation $\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}}(\widehat{z})] = 1/2$. Thus, according to our tie-breaking rule, we assume that the agenda-setter will choose $\overline{\theta}^{PV}_{AS} = 1-\mu_2$. If this is the case, the expected distance to the median voter's type is

$$\mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z}) - \widehat{z}|] = \left(\frac{1-\mu_1-\mu_2}{2}\right)^2 + \left(\frac{\mu_1}{2}\right)^2 + \left(\frac{\mu_2}{2}\right)^2.$$

Thus, the optimal parameters μ_1^*, μ_2^*, i.e., those that maximize social welfare by minimizing the above expression, are:

$$\mu_1^* = \mu_2^* = \frac{1}{2},$$

that verify $\mu_1 \leq 1 - \mu_2$. These optimal design parameters are interpreted as follows: since it is always the case that either $\overline{\theta} \leq 1/2$ or $\overline{\theta} \geq 1/2$, then one of the special cases of the definitions where either $\alpha_- = 0$ or $\alpha^+ = 1$ are always active. Depending on which is active, then the possible outcomes are the status quo, $\overline{\theta}$ and $\overline{\theta} \pm 1/2$.

If these were the values of the design parameters, the selfish agenda-setter would propose $\overline{\theta}_{AS}^{PV} = 1/2$, with an expectation $\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}^B_{PV}}(\widehat{z})] = 1/2$ as well as

$$\mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z}) - \widehat{z}|] = \frac{1}{8}.$$

We compare this to the benchmark procedure, where the agenda-setter would choose $\overline{\theta}_{AS}^{B} = 1$ with also an expectation $\mathbb{E}_{\widehat{z}}[\Omega^{B}_{\overline{\theta}^B_{AS}}(\widehat{z})] = 1/2$ but with

$$\mathbb{E}_{\widehat{z}}[|\Omega^{B}_{\overline{\theta}^B_{AS}}(\widehat{z}) - \widehat{z}|] = \frac{1}{4}.$$

We summarize these results in the following proposition:

Corollary 11. *Consider the equilibrium described in Theorem 1 for the Pendular Voting procedure and suppose that the agenda-setter is selfish and tries to maximize the expected outcome. Then, the design parameters that maximize social welfare are*

$$\mu_1^* = \mu_2^* = \frac{1}{2}.$$

With this choice of parameters, the selfish agenda-setter will choose the proposal

$$\overline{\theta}_{AS}^{PV} = \frac{1}{2},$$

and the expected outcome and social welfare are:

$$\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z})] = \frac{1}{2}, \qquad \mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\overline{\theta}^{PV}_{AS}}(\widehat{z}) - \widehat{z}|] = \frac{1}{8}.$$

8 Comparison to Two Rounds of Proposal-making

In this section we compare Pendular Voting to a similar voting procedure in which the citizenry votes between two alternatives that challenge the status quo. The difference with Pendular Voting is the way in which the two challenger alternatives are obtained: these are individually proposed by two different agenda-setters. The two proposals are made consecutively, such that the first agenda-setter makes his/her proposal with no other information but the second agenda-setter makes his/her proposal knowing the first proposal. Then, the voting procedure takes place following the same rules as in the second round of Pendular Voting: each voter submits his/her pairwise preferences over the three alternatives and the winner is the Condorcet Winner if one exists, or the middle alternative if not.

The agenda-setters may or may not have information about the underlying distribution F. We will represent this by the agenda-setter having access to the position of the median voter's type $\widehat{z}$ or not. In the latter, they will instead take decisions modeling it as a random variable as in the previous section. The same procedure will be used by the first agenda-setter to model the second agenda-setter. Additionally, agenda-setters may either have selfish interests or not.

We denote the proposal made by the first agenda-setter by x_{a1} and the proposal made by the second agenda-setter by x_{a2}. We denote the outcome of this single-round procedure with two agenda-setters by $\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z})$, as a function of $\widehat{z}$ for fixed x_{a1} and x_{a2}. If $x_{a1} \leq x_{a2}$, this is given by:

$$\Omega^{2AS}_{(x_{a1},\ x_{a2})} : [0,1] \to [0,1] : z \mapsto \Omega^{2AS}_{(x_{a1},\ x_{a2})}(z) = \begin{cases} 0, & \text{if} \quad z \in [0, x_{a1}/2), \\ x_1, & \text{if} \quad z \in [x_{a1}/2, (x_{a1}+x_{a2})/2), \\ x_2, & \text{if} \quad z \in [(x_{a1}+x_{a2})/2, 1]. \end{cases}$$

Similarly, if $x_{a2} \leq x_{a1}$, the same definition applies after exchanging x_{a1} for x_{a2}. We do not concern ourselves with dealing with exact draws, i.e., $\widehat{z} = x_1/2$ or $\widehat{z} = (x_2 + x_1)/2$, but instead fix the outcome arbitrarily in this zero-probability events.

8.1 Selfish agenda-setters

We first consider the case where both agenda-setters are selfish, i.e., their objective is to induce an outcome that is as close as possible to their respective most preferred policy or utility peak, that we denote by θ_{a1} for the first and θ_{a2} for the second.

8.1.1 Information about the underlying distribution

We first consider a procedure where the agenda-setters have access to the position of the median voter's type $\widehat{z}$. We begin by computing the optimal strategy for the second agenda-setter with this information and given the proposal x_{a1}. For that, we introduce the following function. For fixed $a, b \in [0,1]$ such that $a \leq b$, we define:

$$k_{(a,\ b)} : [0,1] \to [0,1] : x \mapsto k_{(a,\ b)}(x) := \begin{cases} a, & \text{if} \quad x \in [0,a], \\ x, & \text{if} \quad x \in [a,b], \\ b, & \text{if} \quad x \in [b,1]. \end{cases}$$

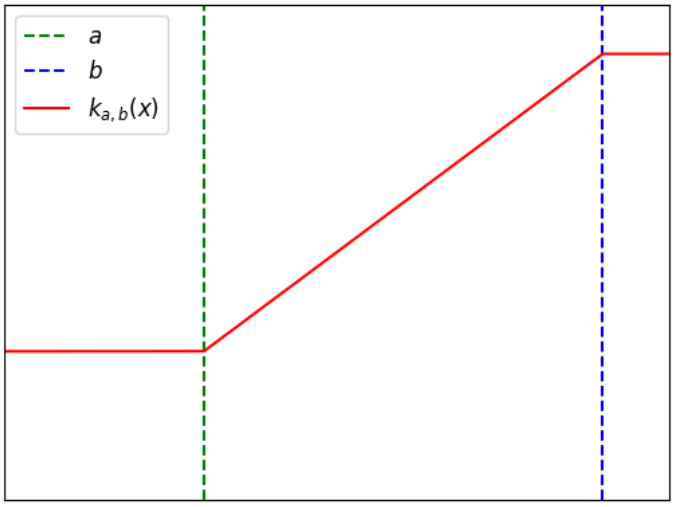


Figure 3: The graph of the function $k_{(a,\ b)}(x)$.

Proposition 8. *The optimal proposal x^*_{a2} for the second agenda-setter, given his/her peak θ_{a2}, the proposal x_{a1} of the first agenda-setter, and the position of the median voter's type $\widehat{z}$, is given by:*

- *If $\widehat{z} \leq 1/2$:*

$$x^*_{a2} = \begin{cases} k_{(x_{a1},\ 2\widehat{z}-x_{a1})}(\theta_{a2}), & \text{if} \quad x_{a1} \in [0, \widehat{z}], \\ k_{(2\widehat{z}-x_{a1},\ x_{a1})}(\theta_{a2}), & \text{if} \quad x_{a1} \in [\widehat{z}, 2\widehat{z}], \\ k_{(0,\ 2\widehat{z})}(\theta_{a2}), & \text{if} \quad x_{a1} \in [2\widehat{z}, 1]. \end{cases}$$

- *If $\widehat{z} \geq 1/2$:*

$$x^*_{a2} = \begin{cases} k_{(x_{a1},\ 1)}(\theta_{a2}), & \text{if} \quad x_{a1} \in [0, 2\widehat{z}-1], \\ k_{(x_{a1},\ 2\widehat{z}-x_{a1})}(\theta_{a2}), & \text{if} \quad x_{a1} \in [2\widehat{z}-1, \widehat{z}], \\ k_{(2\widehat{z}-x_{a1},\ x_{a1})}(\theta_{a2}), & \text{if} \quad x_{a1} \in [\widehat{z}, 1]. \end{cases}$$

Furthermore, after this proposal, the strategy of the second agenda-setter ensures that the outcome coincides with his/her proposal with probability one, i.e.,

$$\Omega^{2AS}_{(x_{a1},\ x^*_{a2})} = x^*_{a2}.$$

Observe that x^*_{a2} is a function of $\widehat{z}$, x_{a1} and θ_{a2}.

Proof of Proposition 8. We show how to argue for the case $\widehat{z} \leq 1/2$ and $0 \leq x_{a1} \leq \widehat{z}$, the other possibilities being entirely analogous. It is just a matter of tracing back the outcome $\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z})$ that is as close as possible to θ_{a2}, based on the proposal x_{a2}. If the proposal is such that $x_{a2} < x_{a1}$ or $x_{a2} > 2z - x_{a1}$, then the outcome will be x_{a1} since it will be closer to $\widehat{z}$ than x_{a2}. On the complementary case, then x_{a2} is closer to $\widehat{z}$ and thus $\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z}) = x_{a2}$.

Hence, if $\theta_{a2} < x_{a1}$, then the closest outcome the second agenda-setter can hope for is given by $x^*_{a2} = x_{a1}$; if $\theta_{a2} > 2z - x_{a1}$, it is the symmetric point $x^*_{a2} = 2z - x_{a1}$, and if $x_{a1} \leq \theta_{a2} \leq 2z - x_{a1}$ then the outcome will be exactly $x^*_{a2} = \theta_{a2}$. The particular case where $\theta_{a2} = 2z - x_{a1}$ and $x^*_{a2} = 2z - x_{a1}$ is a situation where there would be a perfect draw, since both proposals are equidistant to $\widehat{z}$. Here we assume that the outcome is arbitrarily chosen to benefit the second agenda-setter, since otherwise, s/he would choose a proposal $x^*_{a2} := 2z - x_{a1} - \varepsilon$ for $\varepsilon > 0$ arbitrarily small.

The next step is to compute the optimal strategy for the first agenda-setter, based on the known optimal strategy that the second will follow and thus based on the outcome of the procedure as a function now of $\widehat{z}$, θ_{a2} and the proposal x_{a1}, denoted

$$\omega(\theta_{a2},\ x_{a1},\ \widehat{z}) := \Omega^{2AS}_{(x_{a1},\ x^*_{a2})}(\widehat{z}),$$

where, as we said, x^*_{a2} is a function of $\widehat{z}$, x_{a1} and θ_{a2}. We model the lack of information of the first agenda-setter over the peak of the second agenda-setter by assuming that θ_{a2} is uniformly distributed over $[0, 1]$. The optimal proposal x^*_{a1} is then such that

$$x^*_{a1} = \arg\min_{x_{a1}} \mathbb{E}_{\theta_{a2}}[|\omega(\theta_{a2},\ x_{a1},\ \widehat{z}) - \theta_{a1}|],$$

where $\mathbb{E}_{\theta_{a2}}$ denotes the expectation operator over the θ_{a2} probability space. In particular, it is a function of θ_{a1} and of $\widehat{z}$:

$$x^*_{a1} = x^*_{a1}(\theta_{a1},\ \widehat{z}).$$

Of course, this optimal proposal may degenerate into several values of x_{a1} that all attain

the minimum expected distance to the peak θ_{a1}. We have:

Proposition 9. *The optimal proposal x^*_{a1} for the first agenda-setter, given his/her peak θ_{a1}, the position of the median voter's type $\widehat{z}$, and assuming that θ_{a2} is uniformly distributed over $[0,1]$, is given by:*

- *If $\widehat{z} < 1/2$:*

$$x^*_{a1} = \begin{cases} \widehat{z}, & \text{if} \quad \theta_{a1} \in [0, \widehat{z}], \\ \{2\widehat{z} - \theta_{a1}, \theta_{a1}\}, & \text{if} \quad \theta_{a1} \in [\widehat{z}, 2\widehat{z}), \\ \{0\} \cup [2\widehat{z}, 1], & \text{if} \quad \theta_{a1} \in [2\widehat{z}, 1]. \end{cases}$$

- *If $\widehat{z} = 1/2$:*

$$x^*_{a1} = \begin{cases} [\theta_{a1}, 1 - \theta_{a1}], & \text{if} \quad \theta_{a1} \in [0, 1/2], \\ [1 - \theta_{a1}, \theta_{a1}], & \text{if} \quad \theta_{a1} \in [1/2, 1]. \end{cases}$$

- *If $\widehat{z} > 1/2$:*

$$x^*_{a1} = \begin{cases} \theta_{a1}, & \text{if} \quad \theta_{a1} \in [0, 2\widehat{z} - 1], \\ \{\theta_{a1}, 2\widehat{z} - \theta_{a1}\}, & \text{if} \quad \theta_{a1} \in [2\widehat{z} - 1, \widehat{z}], \\ \widehat{z}, & \text{if} \quad \theta_{a1} \in [\widehat{z}, 1]. \end{cases}$$

When several values are provided, it is understood that each of them induces the minimum expected distance to θ_{a1}.

Proof of Proposition 9. As we discussed above, the first agenda-setter's objective is to choose x_{a1} minimizing the function, for fixed θ_{a1} and $\widehat{z}$:

$$E(\theta_{a1}, x_{a1}, \widehat{z}) := \mathbb{E}_{\theta_{a2}}[|\omega(\theta_{a2},\ x_{a1},\ \widehat{z}) - \theta_{a1}|],$$

assuming $\theta_{a2} \sim U([0,1])$. The function $\omega(\theta_{a2},\ x_{a1},\ \widehat{z})$ is given in Proposition 8. For the computation, we first note that for fixed x_{a1} and $\widehat{z}$, we only need to do the following computation:

$$\mathbb{E}_y[|k_{(a,\ b)}(y) - \theta|],$$

for the adequate values of a, b, and θ, and where $y \sim U([0,1])$. A trivial computation

shows:

$$g_{(a,\ b)}(\theta) := \mathbb{E}_y[|k_{(a,\ b)}(y) - \theta|] = \begin{cases} -\theta + \left(b + \dfrac{a^2 - b^2}{2}\right), & \text{if} \quad \theta \in [0, a], \\ \theta^2 - \theta + \left(b - \dfrac{a^2 + b^2}{2}\right), & \text{if} \quad \theta \in [a, b], \\ \theta + \left(-b + \dfrac{-a^2 + b^2}{2}\right), & \text{if} \quad \theta \in [b, 1]. \end{cases}$$

Thus, we can succinctly express the expectation to minimize as:

- If $\widehat{z} \leq 1/2$:

$$E(\theta_{a1}, x_{a1}, \widehat{z}) = \begin{cases} g_{(x_{a1},\ 2\widehat{z} - x_{a1})}(\theta_{a1}), & \text{if} \quad x_{a1} \in [0, \widehat{z}], \\ g_{(2\widehat{z} - x_{a1},\ x_{a1})}(\theta_{a1}), & \text{if} \quad x_{a1} \in [\widehat{z}, 2\widehat{z}], \\ g_{(0,\ 2\widehat{z})}(\theta_{a1}), & \text{if} \quad x_{a1} \in [2\widehat{z}, 1]. \end{cases}$$

- If $\widehat{z} \geq 1/2$:

$$E(\theta_{a1}, x_{a1}, \widehat{z}) = \begin{cases} g_{(x_{a1},\ 1)}(\theta_{a1}), & \text{if} \quad x_{a1} \in [0, 2\widehat{z} - 1], \\ g_{(x_{a1},\ 2\widehat{z} - x_{a1})}(\theta_{a1}), & \text{if} \quad x_{a1} \in [2\widehat{z} - 1, \widehat{z}], \\ g_{(2\widehat{z} - x_{a1},\ x_{a1})}(\theta_{a1}), & \text{if} \quad x_{a1} \in [\widehat{z}, 1]. \end{cases}$$

In order to minimize for fixed $\widehat{z}$ and θ_{a1}, we first have to express the intervals of constant polynomial behavior of the function in terms of θ_{a1}, with x_{a1} as indeterminate.

The result, simplifying the notation of the variables, is shown in Table 2 and represented in Figure 4. In the last column, we indicate whether the function is increasing, decreasing, or constant in that interval. Of course, when $z = 1/2$, some of the increasing and decreasing intervals may become constant, but only then, i.e., otherwise, it is either strictly increasing or decreasing. Now we verify the value of the function at the distinct candidates for the minimum value that arise after evaluating the increasing and decreasing behavior, and obtain the claim.

In Figure 4, one can also see the non-intuitive behavior that the procedure may have: the probability of obtaining a more favourable outcome for the first agenda-setter has a markedly non-monotonic behavior, and at many points, a proposal further away from

z	θ	x	$E(\theta, x, z)$	Growth
$z \leq \frac{1}{2}$	$\theta \in [0, z]$	$x \in [0, \theta]$	$-x^2 - (-2z+1)x + (\theta^2 - \theta + 2z - 2z^2)$	$\searrow$
		$x \in [\theta, z]$	$(2z-1)x + (-\theta + 2z - 2z^2)$	$\searrow$
		$x \in [z, 2z-\theta]$	$(-2z+1)x + (-\theta + 2z^2)$	$\nearrow$
		$x \in [2z-\theta, 2z]$	$-x^2 + (2z+1)x + (\theta^2 - \theta - 2z^2)$	$\nearrow$
		$x \in [2z, 1]$	$(\theta^2 - \theta + 2z - 2z^2)$	$\rightarrow$
	$\theta \in [z, 2z]$	$x \in [0, 2z-\theta]$	$-x^2 + (2z-1)x + (\theta^2 - \theta + 2z - 2z^2)$	$\searrow$
		$x \in [2z-\theta, z]$	$(-2z+1)x + (\theta - 2z + 2z^2)$	$\nearrow$
		$x \in [z, \theta]$	$-(-2z+1)x + (\theta - 2z^2)$	$\searrow$
		$x \in [\theta, 2z]$	$-x^2 + (2z+1)x + (\theta^2 - \theta - 2z^2)$	$\nearrow$
		$x \in [2z, 1]$	$(\theta^2 - \theta + 2z - 2z^2)$	$\rightarrow$
	$\theta \in [2z, 1]$	$x \in [0, z]$	$(-2z+1)x + (\theta - 2z + 2z^2)$	$\nearrow$
		$x \in [z, 2z]$	$(2z-1)x + (\theta - 2z^2)$	$\searrow$
		$x \in [2z, 1]$	$(\theta - 2z + 2z^2)$	$\rightarrow$
$z \geq \frac{1}{2}$	$\theta \in [0, 2z-1]$	$x \in [0, \theta]$	$-x^2/2 + (\theta^2 - \theta + 1/2)$	$\searrow$
		$x \in [\theta, 2z-1]$	$x^2/2 + (-\theta + 1/2)$	$\nearrow$
		$x \in [2z-1, z]$	$(2z-1)x + (-\theta + 2z - 2z^2)$	$\nearrow$
		$x \in [z, 1]$	$(-2z+1)x + (-\theta + 2z^2)$	$\searrow$
	$\theta \in [2z-1, z]$	$x \in [0, 2z-1]$	$-x^2/2 + (\theta^2 - \theta + 1/2)$	$\searrow$
		$x \in [2z-1, \theta]$	$-x^2 + (2z-1)x + (\theta^2 - \theta + 2z - 2z^2)$	$\searrow$
		$x \in [\theta, z]$	$(2z-1)x + (-\theta + 2z - 2z^2)$	$\nearrow$
		$x \in [z, 2z-\theta]$	$(-2z+1)x + (-\theta + 2z^2)$	$\searrow$
		$x \in [2z-\theta, 1]$	$-x^2 + (2z+1)x + (\theta^2 - \theta - 2z^2)$	$\nearrow$
	$\theta \in [z, 1]$	$x \in [0, 2z-1]$	$-x^2/2 + (\theta^2 - \theta + 1/2)$	$\searrow$
		$x \in [2z-1, 2z-\theta]$	$-x^2 + (2z-1)x + (\theta^2 - \theta + 2z - 2z^2)$	$\searrow$
		$x \in [2z-\theta, z]$	$(-2z+1)x + (\theta - 2z + 2z^2)$	$\searrow$
		$x \in [z, \theta]$	$(2z-1)x + (\theta - 2z^2)$	$\nearrow$
		$x \in [\theta, 1]$	$-x^2 + (2z+1)x + (\theta^2 - \theta - 2z^2)$	$\nearrow$

Table 2: Explicit expression $E(\theta, x, z)$ of the expected distance between the outcome and the first agenda-setter's peak θ, given its proposal x and the median voter's type z. The last column shows its growth behavior with respect to x.

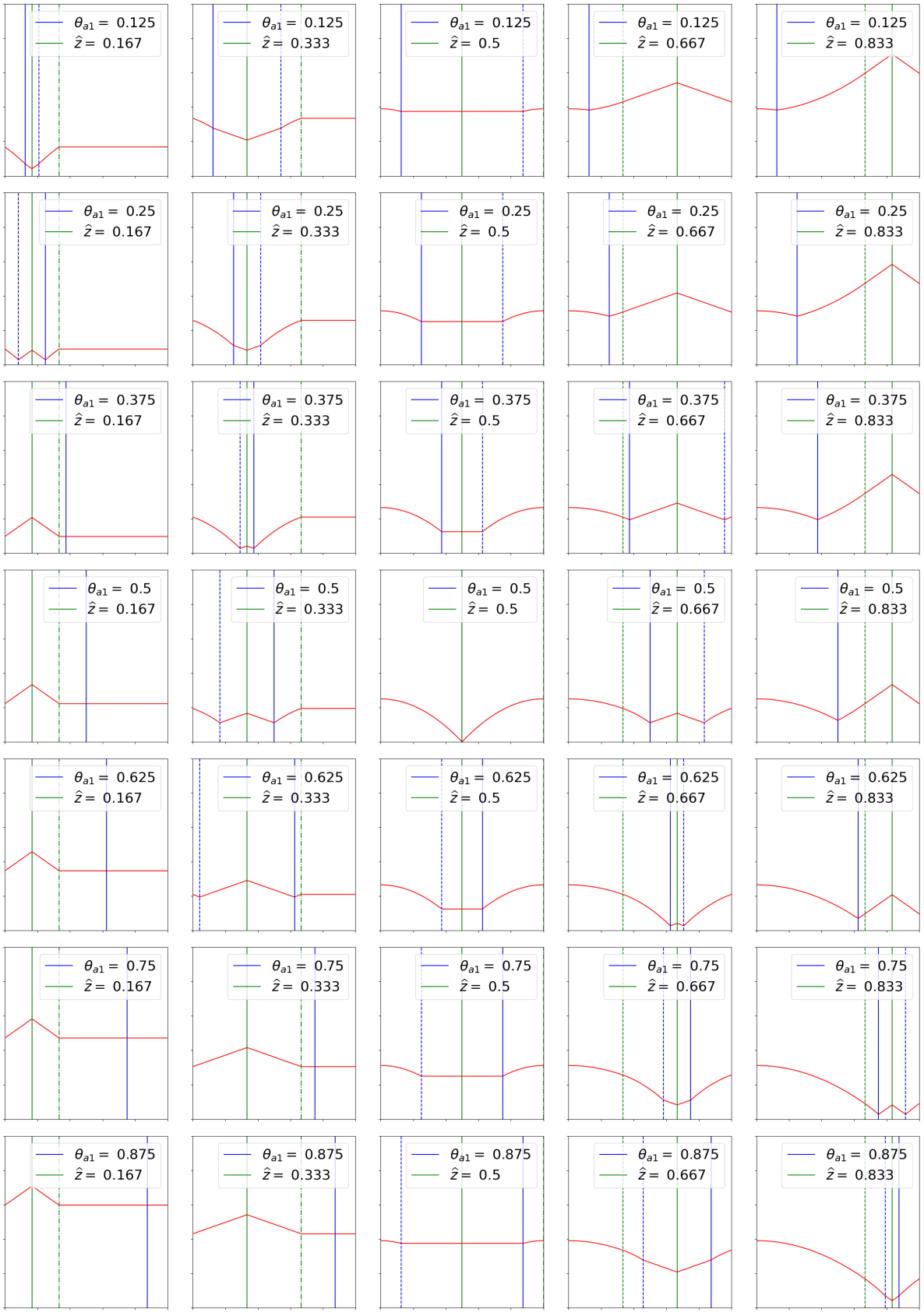


Figure 4: The graph of $E(\theta_{a1}, x, \widehat{z})$ for different values of θ_{a1} and $\widehat{z}$ (red). In blue, we see the value of θ_{a1}, and in dashed blue, the symmetric point $2\widehat{z} - \theta_{a1}$. In green, we see $\widehat{z}$, in dashed green, $2\widehat{z} - 1$, and in dash-point green, $2\widehat{z}$.

his/her peak θ_{a1} leads to better odds of the outcome being closer to the peak.

Combining this result with Proposition 8, we can obtain the strategy that will be played out by both agenda-setters, depending on their peaks:

Corollary 12. *The outcome of the single-round procedure with two agenda-setters with information about $\widehat{z}$, and the optimal strategies of each agenda-setter, is given by:*

$\widehat{z}$	θ_{a1}	x^*_{a1}	$x^*_{a2} = \Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}$
$\widehat{z} < \frac{1}{2}$	$\theta_{a1} \in [0, \widehat{z}]$	$\widehat{z}$	$\widehat{z}$
	$\theta_{a1} \in [\widehat{z}, 2\widehat{z}]$	$\{2\widehat{z} - \theta_{a1}, \theta_{a1}\}$	$k_{(2\widehat{z}-\theta_{a1},\ \theta_{a1})}(\theta_{a2})$
	$\theta_{a1} \in [2\widehat{z}, 1]$	$\{0\} \cup [2\widehat{z}, 1]$	$k_{(0,\ 2\widehat{z})}(\theta_{a2})$
$\widehat{z} = \frac{1}{2}$	$\theta_{a1} \in \left[0, \frac{1}{2}\right]$	$[\theta_{a1}, 1 - \theta_{a1}]$	$k_{(x^*_{a1},\ 1-x^*_{a1})}(\theta_{a2})$
	$\theta_{a1} \in \left[\frac{1}{2}, 1\right]$	$[1 - \theta_{a1}, \theta_{a1}]$	$k_{(1-x^*_{a1},\ x^*_{a1})}(\theta_{a2})$
$\widehat{z} > \frac{1}{2}$	$\theta_{a1} \in [0, 2\widehat{z} - 1]$	θ_{a1}	$k_{(\theta_{a1},\ 1)}(\theta_{a2})$
	$\theta_{a1} \in [2\widehat{z} - 1, \widehat{z}]$	$\{\theta_{a1}, 2\widehat{z} - \theta_{a1}\}$	$k_{(\theta_{a1},\ 2\widehat{z}-\theta_{a1})}(\theta_{a2})$
	$\theta_{a1} \in [\widehat{z}, 1]$	$\widehat{z}$	$\widehat{z}$

We are interested specifically in the case of a first agenda-setter with peak $\theta_{a1} = 1$. This will allow us to compare this procedure to Pendular Voting with a selfish agenda-setter who tries to maximize the outcome as in Section 7. One further identifies $\overline{\theta} \equiv x_{a1}$ to match notations. We compare Pendular Voting's social welfare, under the case with access to the position of $\widehat{z}$, with the expected social welfare of the single-round procedure with two agenda-setters and where the peak of the second agenda-setter is uniformly distributed in $[0, 1]$.

Corollary 13. *The optimal strategies that will be played out by each agenda-setter, given that $\theta_{a1} = 1$, that they know the position of $\widehat{z}$, and assuming $\theta_{a2} \sim U([0, 1])$, are:*

$\widehat{z}$	x^*_{a1}	$x^*_{a2} = \Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}$
$\widehat{z} < 1/2$	$\{0\} \cup [2\widehat{z}, 1]$	$k_{(0,\ 2\widehat{z})}(\theta_{a2})$
$\widehat{z} = 1/2$	$[0, 1]$	$k_{(x^*_{a1},\ 1-x^*_{a1})}(\theta_{a2})$
$\widehat{z} > 1/2$	$\widehat{z}$	$\widehat{z}$

Under optimal play, the expected outcome is

$$\mathbb{E}_{\theta_{a2}}[\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}] = \begin{cases} 2\widehat{z}(1-\widehat{z}), & \text{if} \quad \widehat{z} < \dfrac{1}{2}, \\ \widehat{z}, & \text{if} \quad \widehat{z} \geq \dfrac{1}{2}, \end{cases}$$

while the social welfare is

$$\mathbb{E}_{\theta_{a2}}[|\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})} - \widehat{z}|] = \begin{cases} \widehat{z}(1-\widehat{z}), & \text{if} \quad \widehat{z} < \dfrac{1}{2}, \\ 0, & \text{if} \quad \widehat{z} \geq \dfrac{1}{2}. \end{cases}$$

After averaging over $\widehat{z}$ as well, we obtain

$$\mathbb{E}_{\widehat{z},\ \theta_{a2}}[\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}(\widehat{z})] = \frac{13}{24}, \qquad \mathbb{E}_{\widehat{z},\ \theta_{a2}}[|\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}(\widehat{z}) - \widehat{z}|] = \frac{1}{12}.$$

Comparing this result with Corollary 9, we see that Pendular Voting does not necessarily improve social welfare versus a single-round procedure with two agenda-setters. In particular, the expected social welfare was 5/28 for Pendular Voting. This can be interpreted as follows: under perfect knowledge of $\widehat{z}$, the strategies can be fine-tuned to obtain the best outcome feasible given $\widehat{z}$, and this requires the proposals to approximate $\widehat{z}$ to some degree. On the other hand, without any knowledge of $\widehat{z}$, the strategies are blind and disregard the position of $\widehat{z}$, ending up in proposals that are further away from it.

8.1.2 No information about the underlying distribution

In this subsection, we replicate the analysis carried out in the previous subsection, but modeling the lack of information about the distribution of the population by assuming

that the median voter's type $\widehat{z}$ follows a uniform distribution in $[0,1]$, as in Section 7.

We begin by computing the expected distance between the outcome $\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z})$ of the single-round procedure with two agenda-setters and the peak θ_{a2} of the second agenda-setter, from his/her point of view, given the proposal x_{a1} made by the first agenda-setter and as a function of his/her proposal x_{a2}.

Proposition 10. *The expected distance between the outcome $\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z})$ and θ_{a2}, for fixed x_{a1} and x_{a2} and assuming $\widehat{z} \sim U([0,1])$, is given by:*

x_{a2}	θ_{a2}	$\mathbb{E}_{\widehat{z}}[\lvert\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z}) - \theta_{a2}\rvert]$
$x_{a2} \leq x_{a1}$	$\theta_{a2} \in [0, x_{a2}]$	$\theta_{a2}x_{a2} + \left(-\frac{1}{2}x_{a1}^2 + x_{a1} - \theta_{a2}\right)$
	$\theta_{a2} \in [x_{a2}, x_{a1}]$	$(\theta_{a2} - x_{a1})x_{a2} + \left(-\frac{1}{2}x_{a1}^2 + x_{a1} + x_{a1}\theta_{a2} - \theta_{a2}\right)$
	$\theta_{a2} \in [x_{a1}, 1]$	$\left(\frac{1}{2}x_{a1}^2 - x_{a1} + \theta_{a2}\right)$
$x_{a1} \leq x_{a2}$	$\theta_{a2} \in [0, x_{a1}]$	$-\frac{1}{2}x_{a2}^2 + x_{a2} + (\theta_{a2}x_{a1} - \theta_{a2})$
	$\theta_{a2} \in [x_{a1}, x_{a2}]$	$-\frac{1}{2}x_{a2}^2 + x_{a2}(1 - x_{a1} + \theta_{a2}) + (\theta_{a2}x_{a1} - \theta_{a2})$
	$\theta_{a2} \in [x_{a2}, 1]$	$\frac{1}{2}x_{a2}^2 - x_{a2} + (\theta_{a2})$

Proof of Proposition 10. Consider the case $x_{a2} \leq x_{a1}$. We then compute:

$$\mathbb{E}_{\widehat{z}}[\lvert\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z}) - \theta_{a2}\rvert] = \int_0^1 \lvert\Omega^{2AS}_{(x_{a1},\ x_{a2})}(z) - \theta_{a2}\rvert dz.$$

The three cases that appear in the piece-wise expression arise from the three regions where the argument of the absolute value has different sign. From there, it is a simple computation. The case $x_{a1} \leq x_{a2}$ is obtained by swapping x_{a2} for x_{a1}.

Proposition 11. *The optimal proposal for the second agenda-setter, assuming that $\widehat{z}$ is uniformly distributed over $[0,1]$, is to propose its own peak θ_{a2} regardless of the proposal x_{a1} of the first agenda-setter, i.e.:*

$$x^*_{a2} = \theta_{a2}.$$

Proof of Proposition 11. Rewriting the intervals of the piece-wise expression of previous proposition in terms of x_{a2}, we obtain:

θ_{a2}	x_{a2}	$\mathbb{E}_{\widehat{z}}[\lvert\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z}) - \theta_{a2}\rvert]$	Growth
$\theta_{a2} \leq x_{a1}$	$x_{a2} \in [0, \theta_{a2}]$	$-(x_{a1} - \theta_{a2})x_{a2} + \left(-\frac{1}{2}x_{a1}^2 + x_{a1} + x_{a1}\theta_{a2} - \theta_{a2}\right)$	$\searrow$
	$x_{a2} \in [\theta_{a2}, x_{a1}]$	$\theta_{a2}x_{a2} + \left(-\frac{1}{2}x_{a1}^2 + x_{a1} - \theta_{a2}\right)$	$\nearrow$
	$x_{a2} \in [x_{a1}, 1]$	$-\frac{1}{2}x_{a2}^2 + x_{a2} + (\theta_{a2}x_{a1} - \theta_{a2})$	$\nearrow$
$x_{a1} \leq \theta_{a2}$	$x_{a2} \in [0, x_{a1}]$	$\left(\frac{1}{2}x_{a1}^2 - x_{a1} + \theta_{a2}\right)$	$\rightarrow$
	$x_{a2} \in [x_{a1}, \theta_{a2}]$	$\frac{1}{2}x_{a2}^2 - x_{a2} + (\theta_{a2})$	$\searrow$
	$x_{a2} \in [\theta_{a2}, 1]$	$-\frac{1}{2}x_{a2}^2 + x_{a2}(1 - x_{a1} + \theta_{a2}) + (\theta_{a2}x_{a1} - \theta_{a2})$	$\nearrow$

The last column describes the behavior of the expression in that region, which immediately allows to identify the unique value that minimizes it, arguing as in the proof of Proposition 9 (the same convention for the upward or downward pointing arrows is used, where again the function is strictly monotone except in the boundary values, where it might not be).

Corollary 14. *The optimal strategy for the single-round procedure with two agenda-setters, assuming that $\widehat{z}$ is uniformly distributed over $[0, 1]$, is for each agenda-setter to propose his/her own peak:*

$$x_{a1}^* = \theta_{a1}, \qquad x_{a2}^* = \theta_{a2}.$$

Proof of Corollary 14. To find the optimal strategy for the first agenda-setter, we need to compute the expectation of the distance between his/her peak θ_{a1} and the outcome, depending on the proposal x_{a1} and the optimal response x_{a2}^*, i.e., $\mathbb{E}_{\widehat{z},\ \theta_{a2}}[\lvert\Omega^{2AS}_{(x_{a1},\ x_{a2}^*)}(\widehat{z}) - \theta_{a1}\rvert]$. Furthermore, since $x_{a2}^* = \theta_{a2}$, the expression to minimize can be written as:

$$\mathbb{E}_{\widehat{z},\ \theta_{a2}}[\lvert\Omega^{2AS}_{(x_{a1},\ \theta_{a2})}(\widehat{z}) - \theta_{a1}\rvert] = \mathbb{E}_{\theta_{a2}}\left[\mathbb{E}_{\widehat{z}}[\lvert\Omega^{2AS}_{(x_{a1},\ \theta_{a2})}(\widehat{z}) - \theta_{a1}\rvert]\right].$$

Furthermore, from the result of the previous proposition we see that setting $x^*_{a1} := \theta_{a1}$ will minimize, for every θ_{a2}, the integrand $\mathbb{E}_{\widehat{z}}[|\Omega^{2AS}_{(x_{a1},\ \theta_{a2})}(\widehat{z}) - \theta_{a1}|]$, regarded as a function of x_{a1}, θ_{a2} and θ_{a1}, for fixed θ_{a2} and θ_{a1}. Since the minimum $x^*_{a1} := \theta_{a1}$ does not depend on θ_{a2}, we can simply evaluate the expression at $x^*_{a1} := \theta_{a1}$ before taking the expectation over θ_{a2}, and this directly yields the argument, also minimizing the averaged expression $\mathbb{E}_{\widehat{z},\ \theta_{a2}}[|\Omega^{2AS}_{(x_{a1},\ \theta_{a2})}(\widehat{z}) - \theta_{a1}|]$. This is due to the fact that if for some function $f : \mathbb{R}^2 \to \mathbb{R}$, we have $f(x_0, y) \leq f(x, y)$, for all x and y, then $\mathbb{E}_Y[f(x_0, Y)] \leq \mathbb{E}_Y[f(x, Y)]$ for all x and any random variable Y, such that both integrals exist.

We conclude by making the comparison with Pendular Voting. As before, we take $\theta_{a1} = 1$, i.e., the agenda-setter maximizes the policy outcome. After each agenda-setter makes his/her optimal proposals, i.e., their peaks, according to Corollary 14, the outcome is $\Omega^{2AS}_{(1,\ \theta_{a2})}(\widehat{z})$:

- If $\widehat{z} \leq 1/2$:
$$\Omega^{2AS}_{(1,\ \theta_{a2})}(\widehat{z}) = \begin{cases} \theta_{a2}, & \text{if} \quad \theta_{a2} \in [0, 2\widehat{z}], \\ 0, & \text{if} \quad \theta_{a2} \in [2\widehat{z}, 1]. \end{cases}$$

- If $\widehat{z} \geq 1/2$:
$$\Omega^{2AS}_{(1,\ \theta_{a2})}(\widehat{z}) = \begin{cases} 1, & \text{if} \quad \theta_{a2} \in [0, 2\widehat{z} - 1], \\ \theta_{a2} & \text{if} \quad \theta_{a2} \in [2\widehat{z} - 1, 1]. \end{cases}$$

To be able to compare this with Pendular Voting, we take the expectation over θ_{a2}, noting that $\hat{z}$ is assumed to be uniformly distributed, to find:

$$\mathbb{E}_{\theta_{a2}}[\Omega^{2AS}_{(1,\ \theta_{a2})}(\widehat{z})] = \begin{cases} 2\widehat{z}^2, & \text{if} \quad \widehat{z} \leq \dfrac{1}{2}, \\ -2\widehat{z}^2 + 4\widehat{z} - 1, & \text{if} \quad \widehat{z} \geq \dfrac{1}{2}, \end{cases}$$

and

$$\mathbb{E}_{\theta_{a2}}[|\Omega^{2AS}_{(1,\ \theta_{a2})}(\widehat{z}) - \widehat{z}|] = -\widehat{z}^2 + \widehat{z}.$$

Similarly, one computes:

$$\mathbb{E}_{\widehat{z},\ \theta_{a2}}[\Omega^{2AS}_{(1,\ \theta_{a2})}(\widehat{z})] = \frac{1}{2},$$

and

$$\mathbb{E}_{\widehat{z},\ \theta_{a2}}[|\Omega^{2AS}_{(1,\ \theta_{a2})}(\widehat{z}) - \widehat{z}|] = \frac{1}{6}.$$

Combining everything, we obtain:

Corollary 15. *The optimal strategies that will be played out by each agenda-setter, given that* $\theta_{a1} = 1$ *and assuming that* $\widehat{z} \sim U([0,1])$ *and also* $\theta_{a2} \sim U([0,1])$*, are* $x^*_{a1} = 1$ *and* $x^*_{a2} = \theta_{a2}$*. Under optimal play, the expected outcome and social welfare are:*

$$\mathbb{E}_{\theta_{a2}}[\Omega^{2AS}_{(1,\ \theta_{a2})}] = \begin{cases} 2\widehat{z}, & \text{if} \quad \widehat{z} < \dfrac{1}{2}, \\ -\widehat{z}^2 + 4\widehat{z} - 1, & \text{if} \quad \widehat{z} \geq \dfrac{1}{2}, \end{cases} \qquad \mathbb{E}_{\theta_{a2}}[|\Omega^{2AS}_{(1,\ \theta_{a2})} - \widehat{z}|] = \widehat{z}(1-\widehat{z}),$$

as well as

$$\mathbb{E}_{\widehat{z},\ \theta_{a2}}[\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}(\widehat{z})] = \frac{1}{2}, \qquad \mathbb{E}_{\widehat{z},\ \theta_{a2}}[|\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}(\widehat{z}) - \widehat{z}|] = \frac{1}{6}.$$

Compared to Corollary 11, we see that Pendular Voting performs better on average, with an expected social welfare of 1/8 for the optimal design parameters. It is interesting to note that while social welfare in Pendular Voting is protected for values of $\widehat{z}$ around 1/2, the opposite occurs now.

8.2 Benevolent versus selfish agenda-setter

We now consider the case where the first agenda-setter is selfish and tries to induce the outcome closest as possible to his/her peak θ_{a1}, but the second agenda-setter, on the contrary, is altruistic or benevolent and tries to maximize social welfare with the information from the first outcome in the first round.

In a real setting, the benevolent agenda-setter may take the form of an appointed expert committee whose objective is to counterbalance new proposals and regulate the procedure to maximize social welfare and avoid manipulation of the procedure by a potentially selfish agenda-setter.

8.2.1 Information about the underlying distribution

As we discussed in Subsection 7.1, this is a trivial case: a benevolent agenda-setter with access to $\widehat{z}$ proposes $x_{a2} := \widehat{z}$, so that the outcome is automatically $\widehat{z}$ and social welfare is maximal.

8.2.2 No information about the underlying distribution

We begin by computing the expected social welfare given the two proposals of the agenda-setters, and then optimizing it for fixed x_{a1} by choosing the appropriate x_{a2}. To obtain a unique solution, we introduce a tie breaking rule. Assume that if there are multiple solutions leading to the same social welfare, then the benevolent agenda-setter will choose the one which is closer the expected Condorcet winner $\mathbb{E}[\hat{z}] = 1/2$.

Proposition 12. *The expected social welfare of the single-round procedure with two agenda-setters is given by:*

$$\mathbb{E}_{\widehat{z}}[|\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z}) - \widehat{z}|] = \begin{cases} \dfrac{x_{a1}^2}{2} + \dfrac{3x_{a2}^2}{4} - \dfrac{x_{a1}x_{a2}}{2} - x_{a2} + \dfrac{1}{2}, & \text{if} \quad x_{a1} \leq x_{a2}, \\ \dfrac{x_{a2}^2}{2} + \dfrac{3x_{a1}^2}{4} - \dfrac{x_{a1}x_{a2}}{2} - x_{a1} + \dfrac{1}{2}, & \text{if} \quad x_{a2} \leq x_{a1}. \end{cases}$$

Thus, the optimal strategy for a benevolent second agenda-setter is to propose:

$$x^*_{a2} = \begin{cases} \dfrac{2 + x_{a1}}{3}, & \text{if} \quad x_{a1} \in [0, y^*), \\ \dfrac{x_{a1}}{2}, & \text{if} \quad x_{a1} \in [y^*, 1], \end{cases}$$

where we make the definition

$$y^* := \frac{2(4 - \sqrt{6})}{5} \simeq 0.620.$$

Proof of Proposition 12. The case where $x_{a2} \leq x_{a1}$ follows from the case $x_{a1} \leq x_{a2}$ by exchanging the proposals. For the latter, looking at the area between $\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z})$ and the

graph of $f(\widehat{z}) = \widehat{z}$, one computes:

$$\mathbb{E}_{\widehat{z}}[|\Omega^{2AS}_{(x_{a1},\ x_{a2})}(\widehat{z}) - \widehat{z}|] = \int_0^1 |\Omega^{2AS}_{(x_{a1},\ x_{a2})}(z) - z|dz = \frac{x_{a1}^2}{4} + \frac{(x_{a1} - x_{a2})^2}{4} + \frac{(1 - x_{a2})}{2}$$
$$= \frac{x_{a1}^2}{2} + \frac{3x_{a2}^2}{4} - \frac{x_{a1}x_{a2}}{2} - x_{a2} + \frac{1}{2}.$$

The next step is to find the minimum of this function, regarded as a polynomial in x_{a2}. One sees that the minimum of the polynomial of the region where $x_{a1} \leq x_{a2}$ is $x_{a1}/2$, within that same region, while the local minimum of the polynomial of the region where $x_{a2} \leq x_{a1}$ is $(2 + x_{a1})/3$, again within the same region. Thus, we only need to compare the value at each of these local minimums, that is:

$$\mathbb{E}_{\widehat{z}}[|\Omega^{2AS}_{(x_{a1},\ x_{a1}/2)}(\widehat{z}) - \widehat{z}|] = \frac{5}{8}x_{a1}^2 - x_{a1} + \frac{1}{2},$$

versus

$$\mathbb{E}_{\widehat{z}}[|\Omega^{2AS}_{(x_{a1},\ (2+x_{a1})/3)}(\widehat{z}) - \widehat{z}|] = \frac{5}{12}x_{a1}^2 - \frac{1}{3}x_{a1} + \frac{1}{6}.$$

The former is strictly greater than the latter if and only if

$$\frac{5}{2}x_{a1}^2 - 8x_{a1} + 4 > 0.$$

The roots of this polynomial are

$$\frac{2(4 \pm \sqrt{6})}{5},$$

and since $2(4 + \sqrt{6})/5 > 1$, we conclude that this inequality is satisfied if and only if $x_{a1} \leq y^* := 2(4 - \sqrt{6})/5$.

This result can be interpreted by saying that a benevolent agenda-setter will make proposals in the direction opposite to the proposal made by the first agenda-setter, with the objective of covering more policy space and thus improving social welfare on average and given his/her lack of knowledge of $\widehat{z}$.

With the optimal strategy of the second agenda-setter being known, the first agenda-setter will act accordingly to minimize the expected distance to his/her peak θ_{a1}.

Proposition 13. *Assuming optimal play by a benevolent agenda-setter in a single-round procedure and that $\widehat{z} \sim U([0,1])$, the optimal proposal of the first agenda-setter with peak θ_{a1} is given by:*

$$x^*_{a1} = \begin{cases} y^*, & \text{if} \quad \theta_{a1} \in [0, y^*], \\ \theta_{a1}, & \text{if} \quad \theta_{a1} \in \left[y^*, \frac{2}{3}\right], \\ \{\theta_{a1},\ 2\theta_{a1} - 3\}, & \text{if} \quad \theta_{a1} \in \left[\frac{2}{3}, \frac{2+y^*}{3}\right], \\ \theta_{a1}, & \text{if} \quad \theta_{a1} \in \left[\frac{2+y^*}{3}, 1\right], \end{cases}$$

where two values symbolize that both are optimal.

Proof of Proposition 13. With the result of the previous proposition, we can compute the expected distance between θ_{a1} and the outcome after the benevolent agenda-setter reacts to the proposal x_{a1}. We obtain:

x_{a1}	θ_{a1}	$\mathbb{E}_{\widehat{z}}[\lvert\Omega^{2AS}_{(x_{a1},\ x^*_{a2})}(\widehat{z}) - \theta_{a1}\rvert]$
$x_{a1} \leq y^*$	$\theta_{a1} \in [0, x_{a1}]$	$\frac{1}{2}\theta_{a1}x_{a1} + \frac{1}{6}(x_{a1} - \theta_{a1})(2 + x_{a1}) + \frac{2}{9}(2 + x_{a1} - 3\theta_{a1})(1 - x_{a1})$
	$\theta_{a1} \in \left[x_{a1}, \frac{2+x_{a1}}{3}\right]$	$\frac{1}{2}\theta_{a1}x_{a1} - \frac{1}{6}(x_{a1} - \theta_{a1})(2 + x_{a1}) + \frac{2}{9}(2 + x_{a1} - 3\theta_{a1})(1 - x_{a1})$
	$\theta_{a1} \in \left[\frac{2+x_{a1}}{3}, 1\right]$	$\frac{1}{2}\theta_{a1}x_{a1} - \frac{1}{6}(x_{a1} - \theta_{a1})(2 + x_{a1}) - \frac{2}{9}(2 + x_{a1} - 3\theta_{a1})(1 - x_{a1})$
$x_{a1} \geq y^*$	$\theta_{a1} \in \left[0, \frac{x_{a1}}{2}\right]$	$\frac{1}{4}\theta_{a1}x_{a1} + \frac{1}{4}(x_{a1} - 2\theta_{a1})x_{a1} + \frac{1}{4}(x_{a1} - \theta_{a1})(4 - 3x_{a1})$
	$\theta_{a1} \in \left[\frac{x_{a1}}{2}, x_{a1}\right]$	$\frac{1}{4}\theta_{a1}x_{a1} - \frac{1}{4}(x_{a1} - 2\theta_{a1})x_{a1} + \frac{1}{4}(x_{a1} - \theta_{a1})(4 - 3x_{a1})$
	$\theta_{a1} \in [x_{a1}, 1]$	$\frac{1}{4}\theta_{a1}x_{a1} - \frac{1}{4}(x_{a1} - 2\theta_{a1})x_{a1} - \frac{1}{4}(x_{a1} - \theta_{a1})(4 - 3x_{a1})$

In order to minimize this quantity over x_{a1}, we first have to express the intervals of constant polynomial behavior in terms of θ_{a1}.

The result is shown in Table 3 and represented in Figure 5. In the last column, we indicate whether the function is increasing, decreasing, or neither but instead concave, in that particular interval. As in the proof of Proposition 9, this allows to find the minimum points and obtain the claim.

In particular, unlike the case where the agenda-setter has access to $\widehat{z}$, in this case a selfish agenda-setter aiming to maximize the policy outcome will propose $x^*_{a1} = 1$, against which the second benevolent agenda-setter reacts proposing $x^*_{a2} = 1/2$. This situation coincides with the Pendular Voting procedure with optimal design parameters for a selfish agenda-setter with no information about $\widehat{z}$. The outcome of both is the preferred alternative out of $\{0, 1/2, 1\}$, and thus:

Corollary 16. *Consider a single-round procedure where $\widehat{z} \sim U([0,1])$ and where the second agenda-setter aims to maximize social welfare. Then, the optimal proposal for a selfish first agenda-setter trying to maximize the outcome is $x^*_{a1} = 1$, that of the benevolent agenda-setter is $x^*_{a2} = 1/2$, and the expected outcome and social welfare are :*

$$\mathbb{E}_{\widehat{z}}[\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}(\widehat{z})] = \frac{1}{2}, \qquad \mathbb{E}_{\widehat{z}}[|\Omega^{2AS}_{(x^*_{a1},\ x^*_{a2})}(\widehat{z}) - \widehat{z}|] = \frac{1}{8}.$$

In particular, Pendular Voting with a benevolent agenda-setter performs better, with a social welfare of

$$\mathbb{E}_{\widehat{z}}[|\Omega^{PV}_{\theta^{PV}_{AS}}(\widehat{z}) - \widehat{z}|] = \frac{1}{14}.$$

9 Discussion and Conclusion

We have provided a first analysis of Pendular Voting. The paper is connected with the voting literature in general, and also with an emerging strand of literature on new forms of democracy. The insights in this paper further could, on the one hand, help provide a theoretical foundation for the concept of a "counterproposal" (or "second proposal") for referenda in representative democracies or direct democracies. When a policy proposal to change the constitution is put to a popular vote, Pendular Voting induces a second

θ_{a1}	x_{a1}	$\mathbb{E}_{\widehat{z}}[\lvert\Omega^{2AS}_{(x_{a1},\ x^*_{a2})}(\widehat{z}) - \theta_{a1}\rvert]$	Growth
$\theta_{a1} \in \left[0, \frac{y^*}{2}\right]$	$x_{a1} \in [0, \theta_{a1}]$	$\frac{1}{18}\left[-7x_{a1}^2 + x_{a1}(24\theta_{a1} - 10) + (-6\theta_{a1} + 8)\right]$	$\searrow$
	$x_{a1} \in [\theta_{a1}, y^*)$	$\frac{1}{18}\left[-x_{a1}^2 + x_{a1}(2 + 18\theta_{a1}) + (-18\theta_{a1} + 8)\right]$	$\nearrow$
	$x_{a1} \in [y^*, 1]$	$\frac{1}{2}\left[-x_{a1}^2 + x_{a1}(\theta_{a1} + 2) - 2\theta_{a1}\right]$	$\nearrow$
$\theta_{a1} \in \left[\frac{y^*}{2}, \frac{1}{2}\right]$	$x_{a1} \in [0, \theta_{a1}]$	$\frac{1}{18}\left[-7x_{a1}^2 + x_{a1}(24\theta_{a1} - 10) + (-6\theta_{a1} + 8)\right]$	$\searrow$
	$x_{a1} \in [\theta_{a1}, y^*)$	$\frac{1}{18}\left[-x_{a1}^2 + x_{a1}(2 + 18\theta_{a1}) + (-18\theta_{a1} + 8)\right]$	$\nearrow$
	$x_{a1} \in [y^*, 2\theta_{a1}]$	$\frac{1}{2}\left[-2x_{a1}^2 + x_{a1}(3\theta_{a1} + 2) - 2\theta_{a1}\right]$	Concave
	$x_{a1} \in [2\theta_{a1}, 1]$	$\frac{1}{2}\left[-x_{a1}^2 + x_{a1}(\theta_{a1} + 2) - 2\theta_{a1}\right]$	$\nearrow$
$\theta_{a1} \in \left[\frac{1}{2}, y^*\right]$	$x_{a1} \in [0, \theta_{a1}]$	$\frac{1}{18}\left[-7x_{a1}^2 + x_{a1}(24\theta_{a1} - 10) + (-6\theta_{a1} + 8)\right]$	Concave
	$x_{a1} \in [\theta_{a1}, y^*)$	$\frac{1}{18}\left[-x_{a1}^2 + x_{a1}(2 + 18\theta_{a1}) + (-18\theta_{a1} + 8)\right]$	$\nearrow$
	$x_{a1} \in [y^*, 1]$	$\frac{1}{2}\left[-2x_{a1}^2 + x_{a1}(3\theta_{a1} + 2) - 2\theta_{a1}\right]$	Concave
$\theta_{a1} \in \left[y^*, \frac{2}{3}\right]$	$x_{a1} \in [0, y^*)$	$\frac{1}{18}\left[-7x_{a1}^2 + x_{a1}(24\theta_{a1} - 10) + (-6\theta_{a1} + 8)\right]$	Concave
	$x_{a1} \in [y^*, \theta_{a1}]$	$\frac{1}{2}\left[x_{a1}^2 - 2x_{a1} + 2\theta_{a1}\right]$	$\searrow$
	$x_{a1} \in [\theta_{a1}, 1]$	$\frac{1}{2}\left[-2x_{a1}^2 + x_{a1}(3\theta_{a1} + 2) - 2\theta_{a1}\right]$	Concave
$\theta_{a1} \in \left[\frac{2}{3}, \frac{2 + y^*}{3}\right]$	$x_{a1} \in [0, 3\theta_{a1} - 2]$	$\frac{1}{18}\left[x_{a1}^2 - 2x_{a1} + (18\theta_{a1} - 8)\right]$	$\searrow$
	$x_{a1} \in [3\theta_{a1} - 2, y^*)$	$\frac{1}{18}\left[-7x_{a1}^2 + x_{a1}(24\theta_{a1} - 10) + (-6\theta_{a1} + 8)\right]$	Concave
	$x_{a1} \in [y^*, \theta_{a1}]$	$\frac{1}{2}\left[x_{a1}^2 - 2x_{a1} + 2\theta_{a1}\right]$	$\searrow$
	$x_{a1} \in [\theta_{a1}, 1]$	$\frac{1}{2}\left[-2x_{a1}^2 + x_{a1}(3\theta_{a1} + 2) - 2\theta_{a1}\right]$	$\nearrow$
$\theta_{a1} \in \left[\frac{2 + y^*}{3}, 1\right]$	$x_{a1} \in [0, y^*)$	$\frac{1}{18}\left[x_{a1}^2 - 2x_{a1} + (18\theta_{a1} - 8)\right]$	$\searrow$
	$x_{a1} \in [y^*, \theta_{a1}]$	$\frac{1}{2}\left[x_{a1}^2 - 2x_{a1} + 2\theta_{a1}\right]$	$\searrow$
	$x_{a1} \in [\theta_{a1}, 1]$	$\frac{1}{2}\left[-2x_{a1}^2 + x_{a1}(3\theta_{a1} + 2) - 2\theta_{a1}\right]$	$\nearrow$

Table 3: Explicit value of the expected distance between the outcome and θ_{a1}, given the proposal x_{a1}. The last column shows its growth behavior with respect to x_{a1}.

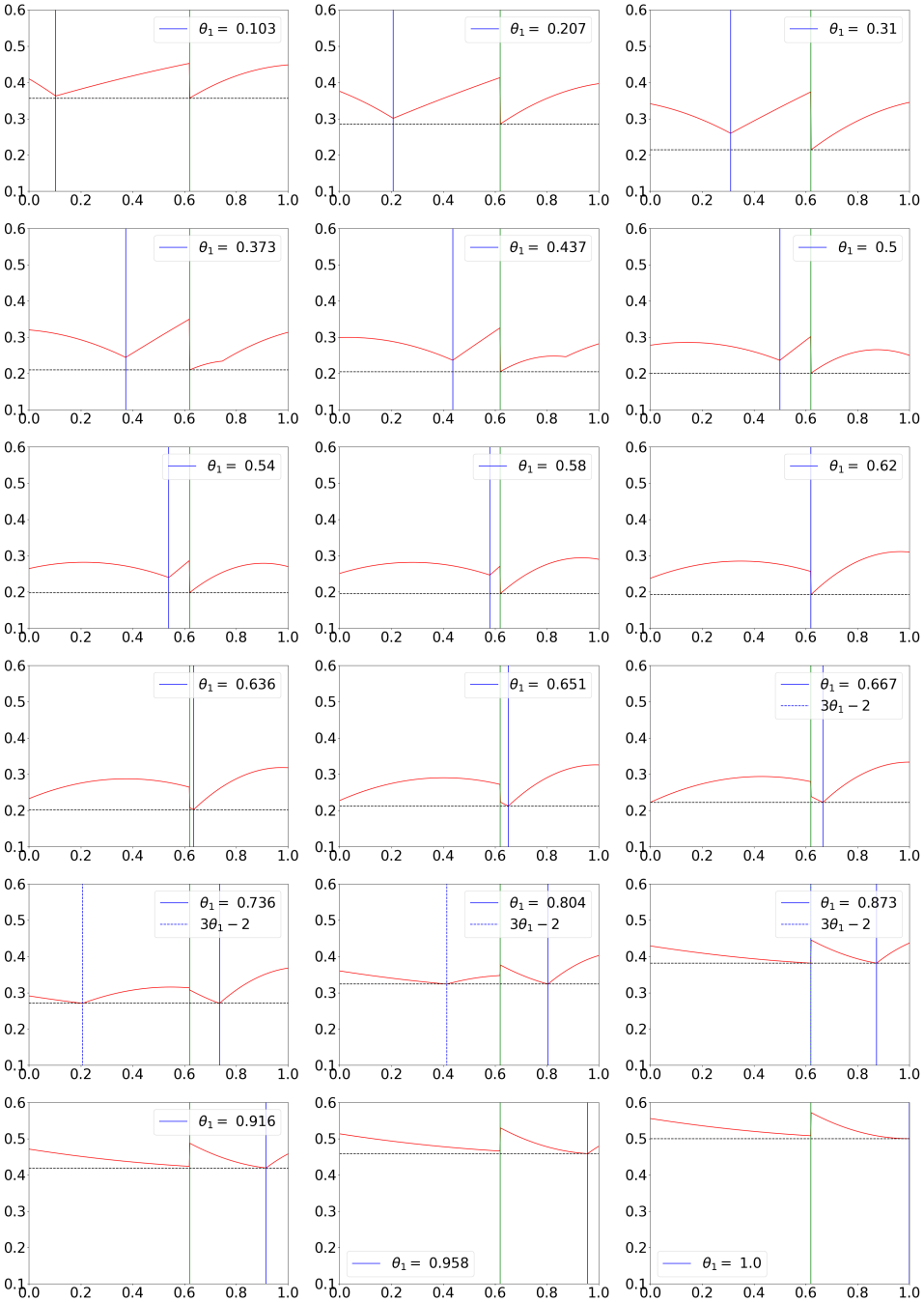


Figure 5: The graph of $\mathbb{E}_{\widehat{z}}[|\Omega^{2AS}_{(x_{a1},\ x^*_{a2})}(\widehat{z}) - \theta_{a1}|]$ for different values of θ_{a1} (red). In blue, we see the value of θ_{a1}, in green y^*, and in a horizontal dashed black line, we see the minimum value. For the relevant case, the value of $3\theta_{a1} - 2$ is represented by a dashed blue line. Each of the 6 row represents, correspondingly, examples of each of the 6 cases described in Table 3: $\theta_{a1} \in [0, y^*/2]$, $\theta_{a1} \in [y^*/2, 1/2]$, $\theta_{a1} \in [1/2, y^*]$, $\theta_{a1} \in [y^*, 2/3]$, $\theta_{a1} \in [2/3, (2 + y^*)/3]$, and $\theta_{a1} \in [(2 + y^*)/3, 1]$.

proposal—more moderate or strengthened, compared to the initial proposal—that opens up the possibility for a citizenry to select an alternative that is closer to the (potentially unknown) median voter's position.

Several issues can be taken up in future research about Pendular Voting. For instance, there are institutional issues as to who should construct the counterproposal. While the model prescribes an algorithm that constructs the counterproposal, in practice, an expert commission, a citizen council or a subcommittee of the parliament might be appropriate to take over the task. Moreover, while in most cases, a metric to determine moderate or more extreme counterproposals, compared to the status quo, is available, there may also be cases in which this definition is more subtle. Yet, even if the policy space is n-dimensional ($n > 1$), once a proposal is made, taking the line from the status quo to the proposal produces a one-dimensional policy space on which a counterproposal can be constructed, and the Pendular Voting procedure and our results can be applied.

Moreover, numerous further issues deserve thorough investigation. For instance, one might consider alternative ways to construct the three alternatives. When the initial proposal obtains a large support, one could construct two new proposals and discard the status quo. Moreover, if the initial proposal obtains only a small share of votes, the process may be terminated in order to save the information and administrative costs associated to the execution of mass voting processes, if the chance that the status quo wins is close to one. Finally, one might also apply Pendular Voting for committee decisions when the size of the group is smaller. In such cases, the entire group votes in both stages and the first stage signals whether the initial proposal should be moderated or strengthened.

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Appendix

Proof of Lemma 1. We recall from Corollary 4 that the outcome of Pendular Voting is given by:

$$\Omega_{\overline{\theta}}^{PV}(\widehat{z}) = \begin{cases} 0, & \text{if} \quad 0 \leq \widehat{z} < \dfrac{\overline{\theta} - \mu_1}{2}, \\ \overline{\theta} - \mu_1, & \text{if} \quad \dfrac{\overline{\theta} - \mu_1}{2} \leq \widehat{z} < \overline{\theta} - \dfrac{\mu_1}{2}, \\ \overline{\theta}, & \text{if} \quad \overline{\theta} - \dfrac{\mu_1}{2} \leq \widehat{z} < \overline{\theta} + \dfrac{\mu_2}{2}, \\ \overline{\theta} + \mu_2, & \text{if} \quad \overline{\theta} + \dfrac{\mu_2}{2} \leq \widehat{z} \leq 1. \end{cases}$$

If we rewrite this expression with $\overline{\theta}$ as the main variable, we obtain:

$$\Omega_{\overline{\theta}}^{PV}(\widehat{z}) = \begin{cases} \overline{\theta} + \mu_2, & \text{if} \quad 0 \leq \overline{\theta} \leq \widehat{z} - \dfrac{\mu_2}{2}, \\ \overline{\theta}, & \text{if} \quad \widehat{z} - \dfrac{\mu_2}{2} < \overline{\theta} \leq \widehat{z} + \dfrac{\mu_1}{2}, \\ \overline{\theta} - \mu_1, & \text{if} \quad \widehat{z} + \dfrac{\mu_1}{2} < \overline{\theta} \leq 2\widehat{z} + \mu_1, \\ 0, & \text{if} \quad 2\widehat{z} + \mu_1 < \overline{\theta} \leq 1, \end{cases}$$

where if $\widehat{z} \leq \mu_2/2$, the first case disappears, and similarly if $\widehat{z} \geq (1 - \mu_1)/2$ or $\widehat{z} \geq 1 - \mu_1/2$, the last or two last cases disappear, correspondingly.

Thus, if the agenda-setter knows in advance the position of the median voter's type $\widehat{z}$, s/he will propose the argument that maximizes $\Omega_{\overline{\theta}}^{PV}(\widehat{z})$. Since $\Omega_{\overline{\theta}}^{PV}(\widehat{z})$ is piece-wise linear with slope 1 or 0, it is a matter of finding the argument that maximizes the correspondence:

$$\begin{aligned} \widehat{z} - \frac{\mu_2}{2} &\mapsto \widehat{z} + \frac{\mu_2}{2}, \\ \widehat{z} + \frac{\mu_1}{2} &\mapsto \widehat{z} + \frac{\mu_1}{2}, \\ 2\widehat{z} + \mu_1 &\mapsto 2\widehat{z}. \end{aligned}$$

This covers the case where all intervals in the piece-wise definition of $\Omega_{\overline{\theta}}^{PV}(\widehat{z})$ are none-empty or not a single point. If $\widehat{z} \leq \mu_2/2$, then the first interval $[0, \widehat{z} - \mu_2/2]$ becomes trivial, and thus we do not consider the first association $\widehat{z} - \mu_2/2 \mapsto \widehat{z} + \mu_2/2$. Similarly, if $(1 - \mu_1)/2 \leq \widehat{z} < 1 - \mu_1/2$, then the last interval $[2\widehat{z} + \mu_1, 1]$ is cut short, so that the maximum value of this segment is given by

$$1 \mapsto 1 - \mu_1,$$

and if $1 - \mu_1/2 \leq \widehat{z}$, then the last two associations are substituted by the single one

$$1 \mapsto 1.$$

The additional case where $\widehat{z} \geq 1 - \mu_2/2$ comes from the definition of α^+, since then $\alpha^+ = \widehat{z} + \mu_2/2$

would be greater than 1 and thus $\alpha^+ = 1$. A similar phenomenon occurs if $\widehat{z} \leq \mu_1/2$ since then $\alpha_- \leq 0$ for $\overline{\theta} = \widehat{z} + \mu_1/2$, but this case is not relevant when maximizing the outcome.

Proof of Proposition 4. These explicit functional expressions are obtained by verifying which of the values in the formula for $\overline{\theta}_{AS}^{PV}$ given in the previous lemma maximize the function g at each interval of definition of $\Omega_{\overline{\theta}}^{PV}(\widehat{z})$.

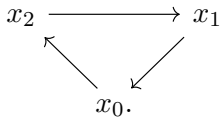
For this, it is useful to note that since $\mu_1 + \mu_2 \leq 1$ from our initial model assumptions, we always have $\mu_2/2 \leq (1-\mu_1)/2$, and if $\mu_1 \geq 1/2$, it must also be that $\mu_2 \leq 1/2$. Additionally, one has $\mu_1/2 \leq (1-\mu_1)/2$ if and only if $\mu_1 \leq 1/2$ and if and only if $(1-\mu_1)/2 \leq 1 - 3\mu_1/2$.

Proof of Proposition 1. We will prove that out of the four groups considered above, each finds it optimal to be honest in the second stage of the Pendular Voting procedure under the assumption that the other three groups also vote honestly. We first observe that we can assume that, at all times, the group under consideration is not a majority, since otherwise, it wins by being honest and the result follows automatically.

1. Group $x_2 \succsim x_1 \succsim x_0$: If this group is not a majority, x_2 looses against x_1, so x_2 never wins. This group will not lie between x_0 and x_1, since it can only hurt its own interests to do so: either there will be a cycle, in which case x_1 wins, or x_2 looses against both x_1 and x_0, in which case $x_1 \leftrightarrow x_0$ decides the final outcome. Thus, all groups are honest in $x_1 \leftrightarrow x_0$ and the only possibility that x_0 wins against x_1 is that the group with preferences $x_0 \succsim x_1 \succsim x_2$ is a majority, in which case, again, its voting is irrelevant.

2. Group $x_1 \succsim x_2 \succsim x_0$: Again, we assume that all groups are a minority. In that case, it is not convenient for the group at hand to lie between x_1 and x_0. To see this, we note first that, in the (maybe dishonest) scenario where x_0 wins over x_1, the group would not lie on the other pairwise choices: lying between x_0 and x_2 could only make x_0 win, and lying between x_2 and x_1 could only make x_2 win or play no role.

   Thence, if this group is dishonest when voting between x_1 and x_0, then it is honest in the remaining choices. Since the other groups are not a majority either, x_1 wins over x_2 and the only possible advantageous outcome of having been situated between x_0 and x_1 would have been the clockwise cycle

   If this occurs, having said the truth would either have made no difference or changed the outcome to one where x_1 wins again. Hence lying didn't accomplish anything.

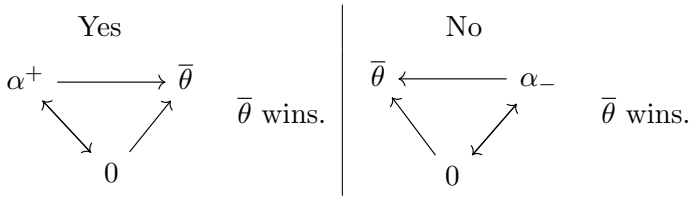
In this way, we can assume that x_1 wins over x_0. Since the group with preferences $x_2 \succsim x_1 \succsim x_0$ is a minority, the group at hand simply tells the truth as to its preference between x_2 and x_1 to make the latter win. (In this case, the voting between x_2 and x_0 plays no role, so being honest is also optimal here).

3. Group $x_1 \succsim x_0 \succsim x_2$: the argument is entirely symmetric to that of the previous group, after exchanging x_0 and x_2.

4. Group $x_0 \succsim x_1 \succsim x_2$: again, the argument is entirely symmetric to that of the first group, after exchanging x_0 and x_2.

Proof of Theorem 1. We first consider the strategy profile where groups Z_0 and Z_1 vote No and groups Z_2 and Z_3 vote Yes, as in the case of *Not Manipulation*. Then, we verify whether any group has a profitable deviation from that strategy profile. We show first that groups Z_3, Z_2, and Z_1 have no such deviation. Then, we show that group Z_0 may have an incentive to deviate, and identify the conditions under which such manipulation is optimal.

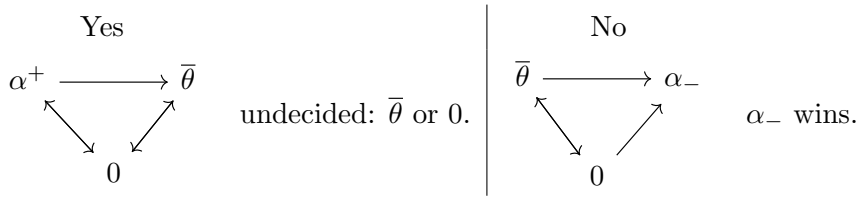
1. Consider Z_3's voting decision in the first round. In states where Z_3 itself is a majority, Z_3 can have its most preferred option α^+ if and only if it qualifies for the second round. Thus, voting Yes at the first stage is optimal for Z_3. We will show that Z_3's choice is inconsequential in any state of nature where Z_3 is not a majority. It is straightforward that Z_3's choice is inconsequential in those states where $Z_0 \cup Z_1$ has a majority. Now consider states where neither Z_3 nor $Z_0 \cup Z_1$ builds a majority. Then, α^+ could never win against $\overline{\theta}$ in the second round. But since $Z_2 \cup Z_3$ is a majority, neither 0 nor α_- could win against $\overline{\theta}$ in the second round either. So $\overline{\theta}$ is the outcome, regardless of what happens in the first voting round, and so group Z_3 has no incentive to deviate. In other words, under the assumption that neither of $Z_0 \cup Z_1, Z_2, Z_3$ has a majority, by voting Yes or No in the first stage, Z_3 can contribute to one of the two following possible outcomes:



Since both result in $\overline{\theta}$ winning, Z_3's vote is irrelevant and there is no incentive to manipulate.

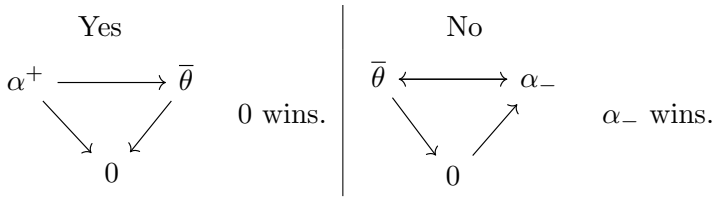
2. Consider Z_2's voting decision in the first round. The same argument as the one used for Z_3 can be now used. Again we can rule out the cases where either of $Z_0 \cup Z_1, Z_2, Z_3$ has a majority, and hence we face the same two possibilities as above. Hence again Z_2's vote is irrelevant and there is no incentive for Z_2 to deviate.

3. Consider Z_1's voting decision in the first round. It is inconsequential in all states in which either $Z_2 \cup Z_3$ or Z_0 is a majority, and trivial if Z_1 is a majority, so we consider the complementary case. Suppose that Z_1 votes No. Then α_- qualifies for the second round. Since $Z_2 \cup Z_3$ is a minority, $\overline{\theta}$ cannot win the second round. Since Z_0 is also a minority, 0 cannot win either, and the outcome is α_-. Now suppose Z_1 switches from No to Yes and is able to sway the outcome of the first round to Yes. Then, α^+ qualifies for the second round. Again because $Z_2 \cup Z_3$ is a minority, α^+ cannot win in the second round, and thus the outcome is either 0 or $\overline{\theta}$. But Z_1 prefers α_- to 0 or $\overline{\theta}$. Hence, whatever the state, Z_1 cannot gain from the deviation. As before, the two possible outcomes depending on the first stage are:



4. Consider Z_0's voting decision at the first stage. We assume again that $Z_2 \cup Z_3, Z_1$ and Z_0 are not a majority. Then, the two possible outcomes are the ones depicted above, the same as for 3. If No wins the first stage, then, reasoning as above, the outcome is α_-. If, however, Z_0 switches from No to Yes, then α^+ qualifies for the second round and again, because $Z_2 \cup Z_3$ is a minority, α^+ cannot win in the second round, so the outcome must be either 0 or $\overline{\theta}$. If a majority prefers $\overline{\theta}$ to 0, then $\overline{\theta}$ is the Condorcet Winner and it wins. The groups that prefer $\overline{\theta}$ to 0 are Z_3, Z_2, Z_1^3 and Z_1^2, while Z_1^1 and Z_0 prefer 0. Hence, if Z_3, Z_2, Z_1^3, Z_1^2 combined are a majority, Z_0 will vote No to avoid that $\overline{\theta}$ wins. Finally, the case where $Z_1^1 \cup Z_0$ is a majority remains. In that case, the outcome will again be $\overline{\theta}$ unless 0 is the Condorcet Winner, winning against α^+. But the groups that prefer 0 to α^+ are Z_0, Z_1^1, Z_1^2 and Z_2^1. Since we are in the case where $Z_1^1 \cup Z_0$ is a majority, 0 beats α^+ and thus 0 is the final outcome. This means that Z_0 finds it optimal to deviate if no group is a majority and if $Z_1^1 \cup Z_0$ is a majority, i.e., given the *Manipulation* conditions as defined above (and of course assuming that the other groups vote according to the "naive" equilibrium).

In this way, we see that manipulation can occur at the first round and that the proposed strategy where Z_0 and Z_1 vote No and Z_2 and Z_3 vote Yes does not provide any equilibrium. However, we have characterized the states where manipulation occurs, and we can now repeat the analysis carried out under these circumstances. Indeed, assume that *Manipulation* conditions hold: $Z_0 \cup Z_1^1$ forms a majority but neither Z_0 nor Z_1 is a majority. Then, as before, the only possible outcomes are:

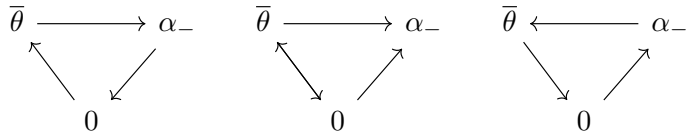


Hence, conditioned on $Z_0 \cup Z_1^1$ being a majority but neither Z_0 nor Z_1 being one, 0 always wins if Yes wins in the first round and α_- wins if No does.

This allows us to complete the previous set of strategies to the case where *Manipulation* conditions hold. The equilibrium is given by choosing the optimal strategy for each group under *Manipulation* conditions: between $0 \leftrightarrow \alpha_-$, the only group preferring 0 is Z_0; thus, it is optimal for Z_0 to vote Yes and for the rest, $Z_1 \cup Z_2 \cup Z_3$, to vote No.

Proof of Corollay 3. We analyze each possible outcome:

- α^+ will be the outcome if and only if Yes is the result of the first stage and α^+ is the Condorcet Winner, which can only occur if Z_3 is a majority (recall Corollary 1).
- 0 will be the outcome if and only if it is the Condorcet Winner, and we analyze the case where Yes and No is the outcome at the first stage separately. If No wins, 0 is preferred to α_- if and only if Z_0 is a majority. If Yes wins, then 0 must win over $\overline{\theta}$, so that $Z_0 \cup Z_1^1$ must be a majority. If Z_0 or Z_1 were a majority, they would have voted No, so we are in the manipulation conditions. But then, 0 is never the outcome, and hence the only option is that Z_0 is a majority.
- α_- will be the outcome if and only if No wins and, in the second stage, we have one of the following outcomes:



  The first cycle will never happen, since it requires that 0 wins over α_- and hence that Z_0 is a majority, in which case 0 would win. The other cycle will only happen if both disjoint subgroups $Z_0 \cup Z_1^1$ and $Z_3 \cup Z_2$ are majorities, so that we can discard it, as it will almost never occur, as we argued before. Hence α_- only wins under the second outcome, i.e. if it is the Condorcet Winner.

  If manipulation conditions hold (a particular case of $Z_0 \cup Z_1$ being a majority but not Z_0), α_- is the outcome. If not, the outcome where α_- is the Condorcet Winner occurs at the conditions that $Z_0 \cup Z_1$ is a majority (so that No wins; thus α_- wins automatically against $\overline{\theta}$) and Z_0 is not the majority (so that α_- wins against 0). All in all, all the cases where

α_- is the final outcome are summed up as: $Z_0 \cup Z_1$ is a majority but Z_0 is not a majority on its own.

Proof of Proposition 5. It is a direct computation based on the outcome function obtained in Corollary 4, in combination with the definitions of the parameters $\alpha^+ = \min\{\overline{\theta} + \mu_2,\ 1\}$, $\alpha_- = \max\{\overline{\theta} - \mu_1, 0\}$. We first verify which values the max and min in these expressions take, and then obtain the expected outcome, computing the probability that each possible outcome wins. Since $\mu_1 \leq 1 - \mu_2$, the three cases to consider are:

- $0 \leq \overline{\theta} \leq \mu_1$: then $\alpha_- = 0$ and we can equivalently take $\mu_1 = \overline{\theta}$, with $\alpha^+ = \overline{\theta} + \mu_2$,
- $\mu_1 < \overline{\theta} < 1 - \mu_2$: $\alpha_- = \overline{\theta} - \mu_1$ and $\alpha^+ = \overline{\theta} + \mu_2$,
- $1 - \mu_2 \leq \overline{\theta} \leq 1$: $\alpha_- = \overline{\theta} - \mu_1$ but $\alpha^+ = 1$ or alternatively $\mu_2 = 1 - \overline{\theta}$.

Hence, we can just assume that $\alpha^+ = \overline{\theta} + \mu_2$ and $\alpha_- = \overline{\theta} - \mu_1$ and choose μ_i adequately in each case. We want to compute:

$$\begin{aligned}
\mathbb{E}_{\widehat{z}}[\Omega^{PV}_{\overline{\theta}}(\widehat{z})] &= \alpha^+\mathbb{P}[\alpha^+ \text{ wins}] + \overline{\theta}\mathbb{P}[\overline{\theta} \text{ wins}] + \alpha_-\mathbb{P}[\alpha_- \text{ wins}] + 0\mathbb{P}[0 \text{ wins}] \\
&= (\overline{\theta} + \mu_2)\mathbb{P}[\alpha^+ \text{ wins}] + \overline{\theta}\mathbb{P}[\overline{\theta} \text{ wins}] + (\overline{\theta} - \mu_1)\mathbb{P}[\alpha_- \text{ wins}] \\
&= \overline{\theta}\mathbb{P}[0 \text{ does not win}] + \mu_2\mathbb{P}[\alpha^+ \text{ wins}] - \mu_1\mathbb{P}[\alpha_- \text{ wins}] \\
&= \overline{\theta}(1 - \mathbb{P}[0 \text{ wins}]) + \mu_2\mathbb{P}[\alpha^+ \text{ wins}] - \mu_1\mathbb{P}[\alpha_- \text{ wins}].
\end{aligned}$$

By Corollary 4, these probabilities are:

- 0 wins when Z_0 is a majority:

$$\mathbb{P}[0 \text{ wins}] = \mathbb{P}[Z_0 \text{ is majority}] = \mathbb{P}[\widehat{z} \leq \alpha_-/2] = G((\overline{\theta} - \mu_1)/2).$$

- α^+ wins when Z_3 is a majority:

$$\mathbb{P}[\alpha^+ \text{ wins}] = \mathbb{P}[Z_3 \text{ is majority}] = \mathbb{P}[\widehat{z} \geq (\overline{\theta} + \alpha_+)/2] = 1 - G(\overline{\theta} + \mu_2/2).$$

- α_- wins when $Z_0 \cup Z_1$ is a majority but not Z_0:

$$\begin{aligned}\mathbb{P}[\alpha_- \text{ wins}] &= \mathbb{P}[Z_0 \cup Z_1 \text{ is majority but not } Z_0] \\ &= \mathbb{P}[Z_0 \cup Z_1 \text{ is majority}] - \mathbb{P}[Z_0 \text{ is not majority}] \\ &= \mathbb{P}[\widehat{z} \leq (\overline{\theta} + \alpha_-)/2] - \mathbb{P}[\alpha_-/2 \leq \widehat{z}] \\ &= G(\overline{\theta} - \mu_1/2) - G((\overline{\theta} - \mu_1)/2).\end{aligned}$$

Combining everything, we obtain:

- $\mu_1 \leq \overline{\theta} \leq 1 - \mu_2$:

$$\begin{aligned}\mathbb{E}_{\widehat{z}}[\Omega_{\overline{\theta}}^{PV}(\widehat{z})] &= \overline{\theta}(1 - G((\overline{\theta} - \mu_1)/2)) + \mu_2(1 - G(\overline{\theta} + \mu_2/2)) \\ &\quad - \mu_1(G(\overline{\theta} - \mu_1/2) - G((\overline{\theta} - \mu_1)/2)) \\ &= \mu_2 + \overline{\theta} - \left(\overline{\theta} - \mu_1\right) G\left((\overline{\theta} - \mu_1)/2\right) - \mu_2 G\left(\overline{\theta} + \mu_2/2\right) - \mu_1 G\left(\overline{\theta} - \mu_1/2\right).\end{aligned}$$

- $0 \leq \overline{\theta} \leq \mu_1$: substituting $\mu_1 = \overline{\theta}$:

$$\mathbb{E}_{\widehat{z}}[\Omega_{\overline{\theta}}^{PV}(\widehat{z})] = \mu_2 + \overline{\theta} - \mu_2 G\left(\overline{\theta} + \mu_2/2\right) - \overline{\theta} G\left(\overline{\theta}/2\right).$$

- $1 - \mu_2 \leq \overline{\theta} \leq 1$: substituting $\mu_2 = 1 - \overline{\theta}$:

$$\begin{aligned}\mathbb{E}_{\widehat{z}}[\Omega_{\overline{\theta}}^{PV}(\widehat{z})] = 1 &- \left(\overline{\theta} - \mu_1\right) G\left((\overline{\theta} - \mu_1)/2\right) \\ &- (1 - \overline{\theta}) G\left((1 + \overline{\theta})/2\right) - \mu_1 G\left(\overline{\theta} - \mu_1/2\right).\end{aligned}$$

Proof of Corollary 9.

We provide a detailed characterization of all possible welfare realizations.

- If $0 \leq \mu_1 \leq \mu_2$:

$$|\Omega_{\overline{\theta}_{AS}^{PV}}^{PV}(\widehat{z}) - \widehat{z}| = \begin{cases} \dfrac{\mu_1}{2}, & \text{if} \quad \widehat{z} \in \left[0, \dfrac{\mu_1}{2}\right], \\ \widehat{z}, & \text{if} \quad \widehat{z} \in \left[\dfrac{\mu_1}{2}, \dfrac{1-\mu_1}{2}\right], \\ 1 - \mu_1 - \widehat{z}, & \text{if} \quad \widehat{z} \in \left[\dfrac{1-\mu_1}{2}, 1 - \mu_1 - \dfrac{\mu_2}{2}\right], \\ \dfrac{\mu_2}{2}, & \text{if} \quad \widehat{z} \in \left[1 - \mu_1 - \dfrac{\mu_2}{2}, 1 - \dfrac{\mu_2}{2}\right], \\ 1 - \widehat{z}, & \text{if} \quad \widehat{z} \in \left[1 - \dfrac{\mu_2}{2}, 1\right]. \end{cases}$$

- If $\mu_2 \leq \mu_1 \leq 1/2$:

$$|\Omega^{PV}_{\widehat{\theta}^{PV}_{AS}}(\widehat{z}) - \widehat{z}| = \begin{cases} \dfrac{\mu_1}{2}, & \text{if} \quad \widehat{z} \in \left[0, \dfrac{\mu_1}{2}\right], \\ \widehat{z}, & \text{if} \quad \widehat{z} \in \left[\dfrac{\mu_1}{2}, \dfrac{1-\mu_1}{2}\right], \\ 1-\mu_1-\widehat{z}, & \text{if} \quad \widehat{z} \in \left[\dfrac{1-\mu_1}{2}, 1-\dfrac{3\mu_1}{2}\right], \\ \dfrac{\mu_1}{2}, & \text{if} \quad \widehat{z} \in \left[1-\dfrac{3\mu_1}{2}, 1-\dfrac{\mu_2}{2}\right], \\ 1-\widehat{z}, & \text{if} \quad \widehat{z} \in \left[1-\dfrac{\mu_2}{2}, 1\right]. \end{cases}$$

- If $1/2 \leq \mu_1$:

$$|\Omega^{PV}_{\widehat{\theta}^{PV}_{AS}}(\widehat{z}) - \widehat{z}| = \begin{cases} \dfrac{\mu_1}{2}, & \text{if} \quad \widehat{z} \in \left[0, 1-\dfrac{\mu_1}{2}\right], \\ 1-\widehat{z}, & \text{if} \quad \widehat{z} \in \left[1-\dfrac{\mu_1}{2}, 1\right]. \end{cases}$$

With these realizations, we can calculate the expected welfare and obtain the formulas of the corollary.